

HYPERBOLICITY OF NODAL HYPERSURFACES

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ABSTRACT. We show that a nodal hypersurface X in \mathbb{P}^3 of degree d with a sufficiently large number l of nodes, $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$, is algebraically quasi-hyperbolic, i.e. X can only have finitely many rational and elliptic curves. Our results use the theory of symmetric differentials and algebraic foliations and give a very striking example of the jumping of the number of symmetric differentials in families.

0. INTRODUCTION

A compact analytic variety X is said to be Kobayashi hyperbolic (Brody) if it does not contain any entire curves, i.e. there are no nonconstant holomorphic mappings $f : \mathbb{C} \rightarrow X$.

Conjecture. (*Kobayashi*) *The general hypersurface X of \mathbb{P}^3 of degree ≥ 5 is Kobayashi hyperbolic.*

We are more interested in an algebraic version of this conjecture (see below) but we recall that the results of McQuillan [Mc98] ($d \geq 36$) and Demailly and El Goul [DeEl00] ($d \geq 21$) which state that a very general hypersurface of degree $d \geq 21$ in \mathbb{P}^3 is Kobayashi hyperbolic (a condition holds for a very generic hypersurface of degree d if it holds for all hypersurfaces not belonging to some countable union of subvarieties of the parameter space of hypersurfaces of degree d). See also [SiYe96] and [EG03] for the related problem of the hyperbolicity of the complement of an hypersurface in \mathbb{P}^2 .

The algebraic version of the Kobayashi's conjecture states that the general hypersurface X of \mathbb{P}^3 of degree ≥ 5 does not contain any rational or elliptic curves, i.e. X is algebraically hyperbolic. The algebraic version is, a priori, weaker but Green and Griffiths [GrGr80] formulated a conjecture which reduces the Kobayashi's hyperbolicity of X to the algebraic hyperbolicity of X . More generally and in a broader direction,

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the first author and separately Green and Griffiths formulated that a surface X of general type contains only finitely many rational and elliptic curves, i.e. X is algebraically quasi-hyperbolic (this conjecture was first explicitly written by Lang [La86]). If X is a variety of general type then there is a jet-bundle $J_k(X)$ over X with a proper subvariety $S \subset J_k(X)$ so that the curves in X are divided into the class of regular curves and the class of special curves. The class of regular curves $C \subset X$ consists of the ones whose lifting $j_k(C')$ into $J_k(X)$ is not contained in S (C' denotes the normalization of C). The class of the special curves consists of the ones whose lifting $j_k(C') \subset S$. The special curves satisfy differential equations of order k defined by S . Now for regular curves we have a linear estimate on genus viz degree with respect to K_X . Namely $g(C') \geq c(X)K_X.C$, $c(X)$ is a constant depending on X . For special curves there isn't such estimate. But if we consider special curves of X lifting a surface component of S then we have boundedness results for curves of any fixed genus coming from the theory of foliations [Jo78].

On the algebraic version of the Kobayashi's conjecture, there are the Clemens's [Cl86] and Xu's [Xu94] results giving that a very general hypersurface in \mathbb{P}^3 of degree ≥ 5 does not contain rational or elliptic curves. For higher dimensions, Ein [Ein88] (see also [Vo96]) showed that a very general hypersurface on \mathbb{P}^{n+1} of degree $\geq 2n + 2$ does not contain any submanifold that is not of general type.

In this paper, we analyze how the number of nodes in a hypersurface X of \mathbb{P}^3 influences the presence of symmetric differentials on the minimal resolution Y of X (a symmetric differential on Y is a global section of $S^m\Omega_Y^1$). The existence of symmetric differentials on a surface Y imposes constraints on the genus of curves on Y . Moreover, if a surface of general type Y satisfies $h^0(Y, S^m\Omega_Y^1) \uparrow m^3$ then Y must be algebraically quasi-hyperbolic (see section 3 of [Bo77] and [De79]). We remark that this approach to algebraic hyperbolicity has the advantage that it can also be used as an ingredient to obtain analytic hyperbolicity.

The pivotal result in this paper is that the minimal resolution Y of a nodal hypersurface X in \mathbb{P}^3 of degree d with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes satisfies $h^0(Y, S^m\Omega_Y^1) \uparrow m^3$. Hence, Y and X are algebraically quasi-hyperbolic. Recall, that the literature on nodal hypersurfaces gives that there are nodal hypersurfaces of degree d with number of nodes $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ for degrees $d \geq 6$ (see [Mi83]). The existence of symmetric differentials on the minimal resolutions of nodal hypersurfaces has some striking consequences. It is well known that the smooth hypersurfaces in \mathbb{P}^3 have no symmetric differentials ([Br71], [Bo78] and [Sa79]). Our result brings the theory of symmetric differentials back as a tool to obtain (algebraic/analytical) hyperbolicity properties of hypersurfaces. Also, we get a simple example of the jumping of the number of symmetric differentials in families. Atiyah [At52] showed that a family of smooth hypersurfaces in \mathbb{P}^3 specializing to a nodal hypersurface has a simultaneous resolution. Consider the simultaneous resolution of a family whose general member is a smooth hypersurface and whose special member is a nodal hypersurface with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes. Our results imply that the resulting family has a general member with no symmetric differentials but whose special member has plenty of symmetric differentials. We finally point out that a byproduct of this paper

is a source of examples of special hypersurfaces that are algebraically quasi-hyperbolic. In the smooth case there are some constructions of special hyperbolic hypersurfaces in \mathbb{P}^3 , see for example [ShZa00].

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1. PRELIMINARIES

Let E be a vector bundle of rank r on a smooth surface X . The Hirzebruch-Riemann-Roch theorem expresses the Euler characteristic $\chi(X, E) = \sum (-1)^i \dim H^i(X, \mathcal{O}(E))$ in terms of topological invariants $c_1(E)$, $c_2(E)$, $K = -c_1(T_X)$ and $\chi = c_2(T_X) \in H^*(X, \mathbb{Z})$:

$$\chi(X, E) = ch(E)Td(X)[X] = \left(-\frac{1}{2}c_1(E) \cdot K + \frac{1}{2}c_1^2(E) - c_2(E) + \frac{r}{12}(K^2 + \chi)\right)[X] \quad (1.1)$$

We are interested in the asymptotic Hirzebruch-Riemann-Roch formula for $S^m E$ and in particular its leading term:

$$\chi(X, S^m E) = \frac{m^{r+1}}{3!}(c_1^2(E) - c_2(E)) + O(m^r) \quad (1.2)$$

which depends on the bundle invariants and does not depend on the surface's Chern classes. We are interested in the symmetric products of the sheaf of differentials Ω_X^1 . So, actually in this case, the invariants of the bundle Ω_X^1 are invariants of the surface X .

Our results require the analysis of the case where the smooth surface Y is the minimal resolution of a surface X with quotient (nodal) singularities. In particular, a key factor in our study is the contribution of the singularities of X to the Euler characteristic $\chi(Y, S^m \Omega_Y^1)$. We used the general framework developed in the work of Wahl [Wa93] and Blache [Bl96]. In their work, they considered a local Riemann-Roch with local Chern classes [Wa 93], [Bl96] and the connection between the local and the global Riemann-Roch [Bl96].

Let (X, x) be the germ of a normal surface singularity and (Y, E) be a good resolution of (X, x) , i.e the exceptional divisor E over x consists of smooth curves intersecting transversely. Wahl defined local Chern classes, $c_i(\tilde{\mathcal{F}})_{\text{loc}}$, of locally free sheaves $\tilde{\mathcal{F}}$ of rank 2 on Y (Wahl also calls them, relative Chern classes since they can be viewed as classes in $H^{2i}(Y, \partial Y, \mathbb{Q})$). Let $\tilde{\mathcal{F}}$ be a locally free sheaf of rank 2 then the Chern numbers are given by:

a) $c_1^2(\tilde{\mathcal{F}})_{\text{loc}} = [\det \tilde{\mathcal{F}}]^2 \in \mathbb{Q}$

where $[\det \tilde{\mathcal{F}}]$ is the unique divisor $[\det \tilde{\mathcal{F}}] = \sum a_i E_i$, E_i are the irreducible components of E , satisfying $(\sum a_i E_i)E_j = \deg_{E_j} \det \tilde{\mathcal{F}}$ for all j .

$$\text{b) } c_2(\tilde{\mathcal{F}})_{\text{loc}} = \inf \left\{ \frac{1}{\deg f} [\mathcal{L}] \cdot [\mathcal{M}] \right\}$$

where the infimum is taken over all quadruples $[W, f : (W, F) \rightarrow (Y, E), \mathcal{L}, \mathcal{M}]$ such that f is a proper surjective generically finite map with $f^{-1}(E) = F$ and \mathcal{L}, \mathcal{M} are line bundles fitting an exact sequence $0 \rightarrow \mathcal{L} \rightarrow f^* \tilde{\mathcal{F}} \rightarrow \mathcal{M} \rightarrow 0$.

In [Wa93] it was defined a modified Euler characteristic of a locally free sheaf $\tilde{\mathcal{F}}$ on Y :

$$\chi'(\tilde{\mathcal{F}})_x = \dim[H^0(Y \setminus E, \tilde{\mathcal{F}})/H^0(Y, \tilde{\mathcal{F}})] + h^1(Y, \tilde{\mathcal{F}}) \quad (1.3)$$

Wahl conjectured that for a rank 2 bundle $\tilde{\mathcal{F}}$ the following asymptotic local Riemann-Roch formula holds:

$$\chi'(S^m \tilde{\mathcal{F}})_x = -\frac{m^3}{6}(c_1^2(\tilde{\mathcal{F}})_{\text{loc}} - c_2(\tilde{\mathcal{F}})_{\text{loc}}) + O(m^2) \quad (1.4)$$

using the local Chern numbers of $\tilde{\mathcal{F}}$. This conjecture holds in the case (X, x) is a nodal singularity [Wa93] and more generally when (X, x) is any quotient singularity [La00].

Recall that X is an orbifold if all $x \in X$ have a neighborhood $U_x \subset X$ such that: there is an open neighborhood of $0 \in \mathbb{C}^n$ with a small finite subgroup $G_x \subset GL_n(\mathbb{C})$ such that $U_x = V/G_x$. Hence an orbifold has only quotient singularities. Consider the germ $(X, x) = (Y, y)/G_x$ with $(Y, y) = (\mathbb{C}^n, 0)$, the projection $\pi : (Y, y) \rightarrow (X, x)$ is called the local smoothing covering of X at x . A coherent sheaf \mathcal{S} on an orbifold X is locally V -free if every $x \in X$ has a neighborhood (U_x, x) with a local smoothing $\pi : (V, y) \rightarrow (U_x, x)$ and a free sheaf $\tilde{\mathcal{S}}$ such that $\mathcal{S}|_{U_x} \simeq \pi_*^{G_x}(\tilde{\mathcal{S}})$ [B196].

The interplay between the local and global Hirzebruch-Riemann-Roch formula for orbifolds is described in [B196]. Let X be a projective orbifold with isolated singularities and $\sigma : Y \rightarrow X$ a good resolution of X . Let \mathcal{F} be a locally V -free sheaf on X and $\tilde{\mathcal{F}}$ be a coherent sheaf on Y such that $\mathcal{F} \simeq (\sigma_* \tilde{\mathcal{F}})^{\vee\vee}$, then:

$$\chi(X, \mathcal{F}) = \chi(Y, \tilde{\mathcal{F}}) + \sum_{x \in \text{Sing} X} \chi'(\tilde{\mathcal{F}})_x \quad (1.5)$$

2. SYMMETRIC TENSORS ON NODAL HYPERSURFACES

The goal of this section is to show that the resolutions Y of nodal hypersurfaces $X \subset \mathbb{P}^3$ containing a sufficiently large number of nodes have symmetric differentials. This result contrasts with the, well known, fact that smooth hypersurfaces $X \subset \mathbb{P}^3$ have no symmetric differentials, see (2.5). Atiyah's result on simultaneous resolution of nodal surfaces [At58] states that Y lies in a smooth family $\omega : X_t \rightarrow \Delta$ whose central fiber $X_0 = Y$ and the other members X_t are smooth hypersurfaces of the same degree as X . So as a consequence of this section, we have perhaps the most striking example of a smooth family of projective manifolds with jumps on the dimension of the spaces of symmetric differentials. In section 3, we use the existence of symmetric differentials to obtain the algebraic quasi-hyperbolicity of nodal hypersurfaces.

Let X be an hypersurface of degree d in \mathbb{P}^n . We first recall what is known for smooth hypersurfaces. There are two key exact sequences for determining the properties of the sheaf of differentials Ω_X^1 for a smooth X :

$$0 \rightarrow \Omega_{\mathbb{P}^n}^1|_X \rightarrow \bigoplus^{n+1} \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow 0 \quad (2.1)$$

$$0 \rightarrow \mathcal{O}_X(-d) \rightarrow \Omega_{\mathbb{P}^n}^1|_X \rightarrow \Omega_X^1 \rightarrow 0 \quad (2.2)$$

Our main interest is on hypersurfaces in \mathbb{P}^3 . For this case, the Chern character identity for the exact sequence (2.1) gives $ch(\Omega_{\mathbb{P}^3}^1|_X) = ch(\bigoplus^4 \mathcal{O}_X(-1)) = (1 + c_1(\mathcal{O}_X(-1)))^4$. The exact sequence (2.2) gives $ch(\Omega_X^1)ch(\mathcal{O}_X(-d)) = ch(\Omega_{\mathbb{P}^3}^1|_X)$. Hence (noting that $c_1(\mathcal{O}_X(1))c_1(\mathcal{O}_X(1)) = d$):

$$c_1(\Omega_X^1) = (d - 4)c_1(\mathcal{O}_X(1))$$

$$c_2(\Omega_X^1) = c_2(\Omega_{\mathbb{P}^3}^1|_X) - c_1(\Omega_X^1)c_1(\mathcal{O}_X(-d)) = d(d^2 - 4d + 6) \quad (2.3)$$

The formula (1.2) then gives the asymptotic behavior of the Euler characteristic $\chi(X, S^m \Omega_X^1)$ on X :

$$h^0(X, S^m \Omega_X^1) - h^1(X, S^m \Omega_X^1) + h^2(X, S^m \Omega_X^1) = \frac{1}{6}(10d - 4d^2)m^3 + O(m^2) \quad (2.4)$$

As is well known, if X is a smooth hypersurface in \mathbb{P}^n then it has no symmetric differentials, i.e:

$$H^0(X, S^m \Omega_X^1) = 0 \quad (2.5)$$

There are at least 3 different proofs of this result ([Br71], [Bo78] and [Sa79]). Sakai's version is the shortest to describe. Consider the spectral sequence: $E_1^{pq} = H^q(X, \wedge^{-p} N_{X/\mathbb{P}^3}^\vee \otimes S^{m+p} \Omega_{\mathbb{P}^n}^1|_X) \implies H^{p+q}(X, S^m \Omega_X^1)$ ($-m \leq p \leq 0$). The vanishing (2.5) follows from $H^i(X, S^m \Omega_{\mathbb{P}^n}^1|_X \otimes \mathcal{O}_X(t)) = 0$ for $i = 0, 1$ and $m \geq t + 2$.

Let X be a nodal hypersurface in \mathbb{P}^3 , i.e. its singularities are ordinary double points. Denote by Y the minimal resolution of X with resolution map $\sigma : Y \rightarrow X$. It follows from Atiyah's result on simultaneous resolution of nodal surfaces [At58] that Y lies in a smooth family $\omega : X_t \rightarrow \Delta$ whose central fiber $X_0 = Y$ and the other members X_t are smooth hypersurfaces of the same degree as X . In fact, these X_t come, after base change, from a family of hypersurfaces whose central fiber is X . This implies that every coherent sheaf \mathcal{F} on the family X_t that is flat over Δ has constant Euler characteristic $\chi(X_t, \mathcal{F}_t)$ in the family. In particular, one gets:

$$\chi(Y, S^m \Omega_Y^1) = \chi(X_t, S^m \Omega_{X_t}^1) = \frac{1}{6}(10d - 4d^2)m^3 + O(m^2) \quad (2.4')$$

Let X be a nodal hypersurface in \mathbb{P}^3 with l nodes and Y be the minimal resolution of X with resolution map $\sigma : Y \rightarrow X$ and exceptional locus $E = \amalg_i E_i$ where the E_i are (-2) -rational curves. The nodal hypersurface X is an orbifold and Ω_X^1 , the reflexive sheaf extending the locally free sheaf $\Omega_X^1|_{X_{\text{reg}}}$, is a locally V-free sheaf on X . On the resolution Y there are two natural sheaves of differentials to consider Ω_Y^1 and $\Omega_Y^1(\log E)$. It is clear that $(\sigma_* \Omega_Y^1)^{\vee\vee} = \Omega_X^1$ and $(\sigma_* \Omega_Y^1(\log E))^{\vee\vee} = \Omega_X^1$.

Recalling that the class of reflexive sheaves is not closed under tensor product operations we define $\hat{S}^m \Omega_X^1 \equiv (S^m \Omega_X^1)^{\vee\vee}$ (note that $(\sigma_* S^m \Omega_Y^1)^{\vee\vee} = (\sigma_* S^m \Omega_Y^1(\log E))^{\vee\vee} = \hat{S}^m \Omega_X^1$). Formula (1.5) gives the relation between the Euler characteristics of both sheaves $S^m \Omega_Y^1$ and $S^m \Omega_Y^1(\log E)$ on Y and the Euler characteristic of $\hat{S}^m \Omega_X^1$ on X .

$$\chi(X, \hat{S}^m \Omega_X^1) = \chi(Y, S^m \Omega_Y^1) + \sum_{x_i \in \text{Sing} X} \chi'(S^m \Omega_Y^1)_{x_i} \quad (2.6)$$

$$\chi(X, \hat{S}^m \Omega_X^1) = \chi(Y, S^m \Omega_Y^1(\log E)) + \sum_{x_i \in \text{Sing} X} \chi'(S^m \Omega_Y^1(\log E))_{x_i} \quad (2.7)$$

Lemma 2.1. *Let (Y, E) be a good resolution of the germ (X, x) of a nodal surface singularity. Then the asymptotic local Euler characteristic for $S^m \Omega_Y^1$ and $S^m \Omega_Y^1(\log E)$ is given by the formula:*

$$\chi'(S^m \Omega_Y^1)_x = \chi'(S^m \Omega_Y^1(\log E))_x = \frac{1}{4}m^3 + O(m^2) \quad (2.8)$$

Proof. To find the leading terms of $\chi'(S^m\Omega_Y^1)_x$ and $\chi'(S^m\Omega_Y^1(\log E))_x$, we recall what is known about their respective local Chern numbers [Wahl93]:

$$\begin{aligned} c_1^2(\Omega_Y^1)_{\text{loc}} &= 0 & c_1^2(\Omega_Y^1(\log E))_{\text{loc}} &= (c_1(\Omega_Y^1)_{\text{loc}} + E)^2 = -2 \\ c_2(\Omega_Y^1(\log E))_{\text{loc}} &= -\frac{1}{2} & c_2(\Omega_Y^1)_{\text{loc}} &= c_2(\Omega_Y^1(\log E))_{\text{loc}} + \chi_{\text{top}}(E) = \frac{3}{2} \end{aligned} \quad (2.9)$$

The key facts, for arriving to the above numbers, are that the nodes are quotient singularities with group \mathbb{Z}_2 and that the following exact sequence holds:

$$0 \rightarrow \Omega_Y^1 \rightarrow \Omega_Y^1(\log E) \rightarrow \mathcal{O}_E \rightarrow 0 \quad (2.10)$$

The formula (1.4) and the local Chern numbers (2.9) provide us with the desired equality (2.8). □

The modified local Euler characteristic $\chi'(S^m\Omega_Y^1)_x$ is composed of two terms $\dim[H^0(Y \setminus E, \tilde{\mathcal{F}})/H^0(Y, \tilde{\mathcal{F}})]$ and $h^1(Y, \tilde{\mathcal{F}})$, see formula (1.3). The following calculation clarifies each of the two terms for a nodal singularity.

Lemma 2.2. *Let (Y, E) be a good resolution of the germ (X, x) of a nodal surface singularity. Then:*

$$\dim[H^0(Y \setminus E, S^m\Omega_Y^1)/H^0(Y, S^m\Omega_Y^1)] = \frac{1}{4}m^3 + O(m^2) \quad (2.11)$$

and:

$$h^1(Y, S^m\Omega_Y^1) < Cm^2 \quad (2.12)$$

Proof. Let $\sigma : (Y, E) \rightarrow (X, x)$ be the minimal resolution of the nodal singularity. The germ of a nodal singularity (X, x) has a local smoothing covering $\pi : (X', x') \rightarrow (X, x)$ with group \mathbb{Z}_2 (i.e. (X, x) is the quotient of a smooth germ (X', x') by \mathbb{Z}_2 with fixed point set x'). Denote by $f : (Y', E') \rightarrow (X', x')$ the blow up of (X', x') at x' and $g : (Y', E') \rightarrow (Y, E)$ the double covering ramified at E making $\pi \circ f = \sigma \circ g : (Y', E') \rightarrow (X, x)$. There are natural bijections between $H^0(Y \setminus E, S^m\Omega_Y^1)$, $H^0(X \setminus x, S^m\Omega_X^1)$ and $[H^0(X' \setminus x', S^m\Omega_{X'}^1)]^{\mathbb{Z}_2}$. The following then holds:

$$\dim[H^0(Y \setminus E, S^m\Omega_Y^1)/H^0(Y, S^m\Omega_Y^1)] = \dim[f^*[H^0(X' \setminus x', S^m\Omega_{X'}^1)]^{\mathbb{Z}_2}/g^*H^0(Y, S^m\Omega_Y^1)]$$

Below we give a coordinate chart approach to find this dimension.

Consider the blow up map $f : (\mathbb{C}^2, u, v) \rightarrow (\mathbb{C}^2, z_1, z_2)$ given by $(u, v) \rightarrow (u, uv)$. Consider also the double ramified covering $g : (\mathbb{C}^2, u, v) \rightarrow (\mathbb{C}^2, x, y)$ given by $(u, v) \rightarrow (u^2, v)$.

The relations between the differentials in the different coordinate charts \mathbb{C}^2 are:

$$f^* dz_1 = du \quad f^* dz_2 = udv + vdu$$

and

$$(2.13)$$

$$g^* dx = 2udu \quad g^* dy = dv$$

We may write the pullback by f of a symmetric differential monomial on (\mathbb{C}^2, z_1, z_2) as:

$$f^*(dz_1^{m_1} dz_2^{m_2}) = \sum_{i=0}^{m_2} \binom{m_2}{i} u^{m_2-i} v^i dv^{m_2-i} du^{m_1+i}$$

So in any such pullback of order m ($m_1 + m_2 = m$) there is a term in du^m with no power of u in the coefficient (and all other terms have du with an order smaller than m). On the other hand, by (2.13) a symmetric differential $\omega = u^{i_1} v^{i_2} du^{m_1} dv^{m_2}$ is a pullback by g only if $i_1 \geq m_1$. This and (2.13) imply that $f^*(h(z_1, z_2) dz_1^{m_1} dz_2^{m_2})$ with $m_1 + m_2 = m$ is also a pullback by g only if the Taylor expansion of h has all terms with combined order in z_1 and z_2 to be greater or equal to m (hence all terms of $f^*(h(z_1, z_2) dz_1^{m_1} dz_2^{m_2})$ will have the coefficients with u of order greater or equal to m).

Now there are $m + 1$ symmetric differential monomials of order m in z_1 and z_2 . The number of all monomials in z_1 and z_2 of degree less than m is $1 + 2 + \dots + m$. Hence the number symmetric differentials of the form $\omega = z_1^{i_1} z_2^{i_2} dz_1^{m_1} dz_2^{m_2}$ having $m_1 + m_2 = m$ such that $f^*\omega$ is not a pullback by g , i.e. $i_1 + i_2 < m$, is $\frac{1}{2}m^3 \bmod m^2$. Recall that the goal is to find $\dim[f^*[H^0(X' \setminus x', S^m \Omega_{X'}^1)]^{\mathbb{Z}_2} / g^* H^0(Y, S^m \Omega_Y^1)]$. Hence we should only count the symmetric differentials that are \mathbb{Z}_2 invariant, i.e. $i_1 + i_2 = m \bmod 2$, which gives $\frac{1}{4}m^3 \bmod m^2$. From $\dim[H^0(Y \setminus E, S^m \Omega_Y^1) / H^0(Y, S^m \Omega_Y^1)] = \dim[f^*[H^0(X' \setminus x', S^m \Omega_{X'}^1)]^{\mathbb{Z}_2} / g^* H^0(Y, S^m \Omega_Y^1)] = \frac{1}{4}m^3 \bmod m^2$, lemma 2.1 and formula (1.3) it follows $h^1(Y, S^m \Omega_Y^1) < Cm^2$.

□

Proposition 2.3. *Let X be a surface with nodal singularities whose resolution Y is a surface of general type. Then for $m > 2$:*

$$h^2(X, \hat{S}^m \Omega_X^1) = 0 \tag{2.14}$$

Proof. From Serre duality for reflexive sheaves on normal surfaces, one obtains:

$$h^2(X, \hat{S}^m \Omega_X^1) = h^0(X, ((\hat{S}^m \Omega_X^1)^\vee \otimes \omega_X)^{\vee\vee})$$

The normality property of the reflexive sheaf $((\hat{S}^m \Omega_X^1)^\vee \otimes \omega_X)^{\vee\vee}$ plus the reflexive nature of $(\hat{S}^m \Omega_X^1)^\vee \otimes \omega_X$ on X_{reg} give:

$$h^0(X, ((\hat{S}^m \Omega_X^1)^\vee \otimes \omega_X)^{\vee\vee}) = h^0(X_{reg}, (S^m \Omega_X^1)^\vee \otimes \omega_X)$$

From the isomorphism $\Omega_X^1{}^\vee \simeq \Omega_X^1 \otimes \omega_X$ on X_{reg} , one derives the following equality:

$$h^2(X, \hat{S}^m \Omega_X^1) = h^0(X_{reg}, S^m \Omega_X^1 \otimes \omega_X^{(1-m)}) \quad (2.15)$$

Consider the the auxiliary vector bundles $S^{2l} \Omega_X^1 \otimes \omega_X^{-l}$ on X_{reg} . Suppose there is an $m > 2$ such that $h^2(X, \hat{S}^m \Omega_X^1) > 0$ then $h^0(X_{reg}, S^m \Omega_X^1 \otimes \omega_X^{(1-m)}) > 0$ by (2.15). Let $l = qm$, $s \in H^0(X_{reg}, S^m \Omega_X^1 \otimes \omega_X^{(1-m)})$ be a nontrivial section and $w \in H^0(X_{reg}, \omega_X^{(m-2)q})$ then $s^{2q} \otimes w \in H^0(X_{reg}, S^{2l} \Omega_X^1 \otimes \omega_X^{-l})$. Since $m > 2$ this implies that if $q \gg 0$ then $S^{2l} \Omega_X^1 \otimes \omega_X^{-l}$ has a nontrivial section vanishing at some point of X_{reg} .

The existence of a nontrivial section of $S^{2l} \Omega_X^1 \otimes \omega_X^{-l}$ vanishing at some point of X_{reg} implies unstability properties of Ω_X^1 on X_{reg} . Applying the stability theory of Bogomolov (and Mumford), it follows that there is a line bundle $L \subset \Omega_X^1$ on X_{reg} such that the line bundle $(L^2 \otimes \omega_X^{-1})^k$ has nontrivial sections for some k ([Bo78] or [De79]). The bigness of K_X on X_{reg} finally implies that:

$$h^0(X_{reg}, L^k) = Ck^2 + O(k) \quad (2.16)$$

Before proceeding we need the following lemma:

Lemma 2.4. *Let X be a projective surface with nodal singularities. Then there is a smoothing cover of X . More precisely, there is smooth projective surface S with a finite map $f : S \rightarrow X$. In particular, the pre-image of the singular points consists of a finite set of smooth points.*

Proof. The proof follows from the construction in theorem 2.5 of [KaBo98]. More precisely, for any surface X with a finite number of quotient singularities $\{x_1, \dots, x_l\}$ there is a projective variety W with a holomorphic surjection $\pi : W \rightarrow X$. The fibration given by π is a \mathbb{P}^n -bundle over $X \setminus \text{Sing}(X)$ and has fibers \mathbb{P}^n/G_{x_i} over the quotient singularity x_i . The group G_{x_i} is the local fundamental group at the x_i , G_{x_i} acts on $\mathbb{P}^n \times U$, where U is a neighborhood of the x_i in X . The action of the G_{x_i} on \mathbb{P}^n is faithful and projective and the action on U is the action producing the quotient singularity x_i . The variety W has singularities in codimension ≥ 3 , hence the intersection of a general codimension n plane with W is our smooth S and $f = \pi|_S$. \square

By the previous lemma, there is a smoothing cover of X , $f : S \rightarrow X$. Denote by T the pre-image on S of the set of nodal points of X . Let $df : f^*\Omega_X^1 \rightarrow \Omega_S^1$ be the differential map. Consider the rank one subsheaf of $\Omega_S^1|_{S \setminus T}$ on $S \setminus T$ which is the image sheaf $\mathcal{S} = df(L)$. Since T has codimension 2 on S , one can extend \mathcal{S} to rank one coherent subsheaf on S of Ω_S^1 . Let $L' \subset \Omega_S^1$ be the line bundle which is the saturation of \mathcal{S} on Ω_S^1 . The growth of the number of sections of L'^k described in (2.16) implies that:

$$h^0(S, L'^k) \geq Ck^2 + O(k) \quad (2.17)$$

On the other hand one has the general result stating that, on any complex surface Y , if $F \subset \Omega_Y^1$ is a line bundle, then growth of $h^0(S, F^k)$ is at most linear on k , which gives the desired contradiction. \square

So far the results we have gathered can be used to show the existence of symmetric differentials, i.e. sections of $H^0(X, \hat{S}^m \Omega_X^1)$, on nodal hypersurfaces $X \subset \mathbb{P}^3$ with enough nodes.

Corollary 2.5. *Let X be a nodal hypersurface of degree d and l nodes in \mathbb{P}^3 . If $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ then :*

$$h^0(X, \hat{S}^m \Omega_X^1) \geq [\frac{l}{4} + \frac{1}{6}(10d - 4d^2)]m^3 + O(m^2)$$

Proof. Let Y be the minimal resolution of X and $\{x_i\}_{i=1, \dots, l}$ be the l nodes of X . The Riemann-Roch formula (2.6), the formula (2.4') and the leading term of the local Euler Characteristics $\chi(Y, S^m \Omega_Y^1)_{x_i}$ found in lemma 2.1 give:

$$\chi(X, \hat{S}^m \Omega_X^1) = [\frac{1}{6}(10d - 4d^2) + \frac{l}{4}]m^3 + O(m^2)$$

Hence if $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ the Euler characteristic $\chi(X, \hat{S}^m \Omega_X^1) > 0$ and grows with m as m^3 . The result follows since $h^2(X, \hat{S}^m \Omega_X^1) = 0$ by proposition 2.3. \square

The goal of this section is not yet achieved. The sections of $H^0(X, \hat{S}^m \Omega_X^1)$ do not give, a priori, symmetric differentials on the resolution Y . They correspond to sections of $H^0(Y \setminus E, S^m \Omega_Y^1)$. We would like to point out, that we have immediately a bit more. It follows from a result of Miyaoka ([Mi83], see also corollary 4.7 of [Wa93]) that for quotient singularities:

$$h^0(Y \setminus E, S^m \Omega_Y^1) = h^0(Y, S^m \Omega_Y^1(\log E))$$

Therefore if X has enough nodes then Y has symmetric log-differentials. In the next result, we go farther and show the existence of the desired symmetric differentials on Y .

Theorem 2.6. *Let X be a nodal hypersurface of degree d and l nodes in \mathbb{P}^3 and Y its minimal resolution. If $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$, then Y has symmetric differentials. More precisely:*

$$h^0(Y, S^m \Omega_Y^1) \geq [\frac{l}{4} + \frac{1}{6}(10d - 4d^2)]m^3 + O(m^2) \quad (2.18)$$

There are nodal hypersurfaces of degree $d \geq 6$ with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes.

Proof. Consider the exact sequence :

$$0 \rightarrow S^m \Omega_Y^1 \rightarrow S^m \Omega_Y^1(\log E) \rightarrow \mathcal{Q} \rightarrow 0 \quad (2.19)$$

where the sheaf $\mathcal{Q} = S^m \Omega_Y^1(\log E)/S^m \Omega_Y^1$ is supported on the exceptional locus E .

To start with, we want to find the asymptotic growth of $h^0(Y, \mathcal{Q})$. Let (U_i, E_i) be the germ of a neighborhood of an exceptional curve E_i . The long cohomology sequence associated with (2.19) on U_i and $h^1(U_i, S^m \Omega_Y^1) = O(m^2)$, (2.12), give that $h^0(U_i, \mathcal{Q}) = \dim(H^0(U_i, S^m \Omega_Y^1(\log E_i))/H^0(U_i, S^m \Omega_Y^1)) + O(m^2)$.

To proceed we recall a result of Miyaoka (see corollary 4.7 of [Wa93] or [Mi83]) stating that for germs (U, E) of resolutions of quotient singularities:

$$H^0(U \setminus E, S^m \Omega_U^1) = H^0(U, S^m \Omega_U^1(\log E)) \quad (2.20)$$

Combining all the above with (1.3) and lemma 2.2, it follows that:

$$h^0(E_i, \mathcal{Q}) = \frac{1}{4}m^3 + O(m^2) \quad (2.21)$$

and

$$h^0(Y, \mathcal{Q}) = \frac{l}{4}m^3 + O(m^2) \quad (2.22)$$

Next, we want to find a lower bound for $h^1(Y, S^m \Omega_Y^1)$. Consider the exact sequence (2.19) tensored with the line bundle $\mathcal{O}((1-m)K_Y)$ and use the isomorphism $\mathcal{O}(K_Y)|_{U_i} \simeq \mathcal{O}_{U_i}$ for the neighborhood germ (U_i, E_i) of a (-2) -curve to obtain:

$$0 \rightarrow S^m \Omega_Y^1 \otimes \mathcal{O}((1-m)K_Y) \rightarrow S^m \Omega_Y^1(\log E) \otimes \mathcal{O}((1-m)K_Y) \rightarrow \mathcal{Q} \rightarrow 0 \quad (2.23)$$

The associated long cohomology sequence gives:

$$H^0(Y, S^m \Omega_Y^1(\log E) \otimes \mathcal{O}((1-m)K_Y)) \rightarrow H^0(Y, \mathcal{Q}) \rightarrow H^1(Y, S^m \Omega_Y^1 \otimes \mathcal{O}((1-m)K_Y))$$

Serre duality gives $h^1(Y, S^m \Omega_Y^1) = h^1(Y, S^m \Omega_Y^1 \otimes \mathcal{O}((1-m)K_Y))$. Hence if we show that $h^0(Y, S^m \Omega_Y^1(\log E) \otimes \mathcal{O}((1-m)K_Y)) = 0$, it follows that:

$$h^1(Y, S^m \Omega_Y^1) \geq \frac{l}{4} m^3 + O(m^2) \quad (2.24)$$

The inequality (2.18) then follows from proposition 2.3 and the Euler characteristic $\chi(Y, S^m \Omega_Y^1) = \frac{1}{6}(10d - 4d^2)m^3 + O(m^2)$ being constant along a flat family.

The equality (2.20) implies that $H^0(Y, S^m \Omega_Y^1(\log E) \otimes \mathcal{O}((1-m)K_Y)) = H^0(Y \setminus E, S^m \Omega_Y^1 \otimes \mathcal{O}((1-m)K_Y))$. On the other hand, it follows from proposition 2.3 and its proof that $h^0(Y \setminus E, S^m \Omega_Y^1 \otimes \mathcal{O}((1-m)K_Y)) = h^0(X_{reg}, S^m \Omega_X^1 \otimes \mathcal{O}((1-m)K_X)) = h^2(X, \hat{S}^m \Omega_X^1) = 0$.

Segre showed in [Se47] that the maximum number of nodes $l(d)$ allowed in an hypersurface of degree d satisfies $l(d) \geq \frac{1}{4}d^2(d-1)$. The lower bound for $l(d)$ of Segre gives that for $d \geq 9$ then there are nodal hypersurfaces with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes. The cases $d = 6, 7, 8$ follow from other specific lower bounds already found, see table in [Mi83]. For the case $d = 5$ there are not enough nodes, the maximum number of nodes is 31 [Be83] and we needed 33. \square

Remark: As was mentioned in the introduction of this section, the theorem 2.6 gives us an example of the jumping of the number of symmetric differentials on a smooth family of projective manifolds. This example, a family of smooth hypersurfaces X_t of degree $d \geq 6$ in \mathbb{P}^3 specializing to the resolution Y of a nodal hypersurface with enough nodes, is perhaps the most striking in its consequences and simplicity.

3. HYPERBOLICITY OF NODAL HYPERSURFACES

We prove in this section that the nodal hypersurfaces $X \subset \mathbb{P}^3$ whose number of nodes are sufficiently large when compared with their degrees are algebraically quasi-hyperbolic. The main tools are the existence of symmetric differentials proved in section 2 and the restrictions that they impose on curves with a fixed geometric genus in X .

A nontrivial symmetric differential $\omega \in H^0(Y, S^m \Omega_Y^1)$ defines a multi-foliation on Y . Recall that there is a natural bijection between $H^0(Y, S^m \Omega_Y^1)$ and $H^0(\mathbb{P}(\Omega_Y^1), \mathcal{O}(m))$. If $\omega \in H^0(Y, S^m \Omega_Y^1)$ and $y \in Y$ then $\omega(y)$ defines naturally a polynomial of order m on $\mathbb{P}(\Omega_Y^1)_y \simeq \mathbb{P}^1$. The zeroes of $\omega(y)$ in $\mathbb{P}(\Omega_Y^1)_y$ determine the directions that the local leaves, of the multi-foliation defined by ω , can take at y . Let C be a smooth curve and $f : C \rightarrow Y$ be holomorphic map which is bimeromorphic onto its image. The curve $f(C)$ is a leaf of the multi-foliation defined by ω if the differential pullback $f^* \omega \in H^0(C, (\Omega_C^1)^{\otimes m})$ is trivial. There is another way to describe the condition of being a leaf of ω . Consider the lift of the map $f : C \rightarrow Y$ to a map $t_f : C \rightarrow \mathbb{P}(\Omega_Y^1)$.

The curve $f(C)$ is a leaf if $t_f(C) \subset (s_\omega)_0$, where $(s_\omega)_0$ is the zero locus of the section $s_\omega \in H^0(\mathbb{P}(\Omega_Y^1), \mathcal{O}(m))$ corresponding to ω .

Theorem 3.1. *Let X be a nodal hypersurface of \mathbb{P}^3 with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes, then X is algebraically quasi-hyperbolic.*

Proof. The result follows if we show that the minimal resolution Y of X is algebraically quasi-hyperbolic. Let $\phi_m : \mathbb{P}(\Omega_Y^1) \rightarrow \mathbb{P}^l$ be the rational map defined by the linear system $|\mathcal{O}(m)|$. Denote by $Z_m \subset \mathbb{P}(\Omega_Y^1)$ the locus consisting of the union of the positive dimensional fibers of ϕ and the base locus B_m of $|\mathcal{O}(m)|$. The condition on the number of nodes implies, by the theorem 2.6, that $H^0(Y, S^m \Omega_Y^1) \uparrow m^3$. Hence the locus Z_m consists of a finite union of irreducible components. Let C be a smooth curve and $f : C \rightarrow Y$ be holomorphic map which is bimeromorphic onto its image. The curve $f(C)$ is said to be irregular if its lift $t_f(C) \subset Z_m$, otherwise it is called regular. The set of irregular curves can be broken into two groups: the curves that are leaves of all multi-foliations defined by the elements of $H^0(Y, S^m \Omega_Y^1)$ (i.e. the ones whose lift lies on the base locus B_m); the curves whose lift lie on the positive dimensional fibers of ϕ .

Let C' be a regular curve with normalization $f : C \rightarrow C' \subset Y$. The regularity of C' implies that there is a symmetric differential $\omega \in H^0(Y, S^m \Omega_Y^1)$ such that the differential pullback $f^* \omega \in H^0(C, (\Omega_C^1)^{\otimes m})$ is nontrivial but vanishes somewhere. Hence $\deg_C(\Omega_C^1)^{\otimes m} > 0$ and C can not be rational or elliptic.

If C' is irregular then its lift lies on Z_m . The number of curves that lift into irreducible components of Z_m that are not horizontal and 2-dimensional is clearly finite. The finiteness of the number of curves that lift to the 2-dimensional horizontal components of Z_m is more delicate. These components have a naturally defined foliation on them which permits the use of the theory of foliations. In fact, the theory of algebraic foliations and Y being of general type give that there are at most finitely many rational and elliptic curves lifting to any of the 2-dimensional horizontal components of Z_m , which finishes the proof (for more details see [Bo77] and [Jo78] and also [De79]). \square

REFERENCES

- [At58] **M.Atiya**, *On analytic surfaces with double points*, Proc. Roy. Soc. London. Ser. A **247** (1958), 237-244.
- [Be79] **A.Beuville**, *Sur le nombre de points doubles d'une surface dans \mathbb{P}^3 ($\mu(5) = 31$)*, *Journées de Geometrie Algebrique d'Angers, Juillet 1979*, Sijthoff & Noordhoff, Alphen ann den Rijn, Germantown, Md **247** (1980), 207-215.
- [Bl96] **R.Blache**, *Chern classes and Hirzebruch-Riemann-Roch theorem for coherent sheaves on complex projective orbifolds with isolated singularities*, Math. Z. **222** (1996), 7-57.
- [Bo77] **F.Bogomolov**, *Families of curves on a surface of general type*, Soviet Math. Dokl. **18** (1977), 1294-1927.
- [Bo78] **F.Bogomolov**, *Holomorphic tensors and vector bundles*, Izvestya A.N. USSR **42** (6) (1978), 1227 - 1287 (English translation Math USSR Izvestya 1979 v 13 (1) p 499 -544).
- [Br71] **P.Bruckmann**, *Tensor differential forms on algebraic varieties*, Izvestya A.N. USSR **35** (1971), (English translation Math USSR Izvestya 1979 v 13 (1) p 499 -544).
- [Cl86] **H.Clemens**, *Curves on generic hypersurfaces*, Ann. Sci. ENS **19** (1986), 629-636.
- [De79] **M.Deschamps**, *Courbes de genre geometrique borne sur une surface de type general (d'apres F. Bogomolov)*, Lect. Notes in Math. (Springer Verlag) **710** (1979), 233-247.

- [DeEl00] **P. Demailly; J. El Goul**, *Hyperbolicity of generic surfaces of high degree in projective 3-space*, Amer. J. Math. **122** (2000), 515-546.
- [Ei88] **L.Ein**, *Subvarieties of generic complete intersections*, Inv. Math. **94** (1988), 163-169.
- [El03] **El Goul**, *Logarithmic jets and hyperbolicity*, Osaka J. Math. **40** (2003), 469-491.
- [GrGr79] **M. Green; P.Griffiths**, *Two applications of algebraic geometry to entire holomorphic maps 3-space*, The Chern Symposium 1979 (Springer Verlag) (1980), 41-74.
- [Jo78] **J-P, Jouanolou**, *Hypersurfaces solutions d'une equation de Pfaff analytique*, Math. Annalen **232** (1978), 239-248.
- [KaBo98] **L.Katzarkov; F.Bogomolov**, *Complex projective surfaces and infinite groups*, G.A.F.A. **8** (1998), 243-272.
- [La86] **S.Lang**, *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. Z. **14** (1986), 159-205.
- [La00] **A.Langer**, *Chern classes of reflexive sheaves on normal surfaces*, Math. Z. **235** (2000), 591-614.
- [Mc98] **M.McQuillan**, *Diophantine approximations and foliations*, Publ. IHES **87** (1998), 121-174.
- [Mi3] **Y.Miyaoka**, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), 159-171.
- [Sa79] **F.Sakai**, *Symmetric powers of the cotangent bundle and classification of algebraic varieties*, Lect. Notes in Math. (Springer Verlag) **732** (1979), 545-563.
- [Se47] **B.Segre**, *Sul massimo numero di nodi delle superficie di dato ordine*, Bull. U.M.I **2** (1947), 204-212.
- [ShZa00] **B. Shiffman; M. Zaidenberg**, *Two classes of hyperbolic surfaces in \mathbb{P}^3* , Int. J. Math. **11** (2000), 65-101.
- [SiYe96] **Y-T.Siu; S.K.Yeung**, *Hyperbolicity of the complement of a generic smooth of high degree in the complex projectiveplane*, Inv. Math. **124** (1996), 573-618.
- [Vo96] **C.Voisin**, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom. **44** (1996), 200-213.
- [Wa93] **J. Wahl**, *Second Chern class and Riemann-Roch for vector bundles on resolutions of surface singularities.*, Math. Ann. **295** (1993), 81-110.
- [Xu94] **G.Xu**, *Subvarieties of general hypersurfaces in projective space*, J. Diff. Geom. **39** (1994), 139-172.

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