WEIGHTED BOND POSETS AND A NEW CHROMATIC SYMMETRIC FUNCTION

RAFAEL S. GONZÁLEZ D'LEÓN* AND MICHELLE L. WACHS**

Abstract. The classical bond lattice of a graph was used in a formula of Whitney to compute the chromatic polynomial. In this paper, we use weighted versions of the bond lattice, to introduce and study a new polynomial and a new symmetric function invariant of a graph. Examples of the new polynomial invariant include the classical Narayana polynomials (for the path graph), the tree-Eulerian polynomials (for the complete graph), and the binomial-Eulerian polynomials (for the star graph). These examples suggest an interesting connection to h-polynomials of general graph-associahedra. We prove that for any chordal graph, our polynomial graph invariant is γ-positive.

Multiweighted bond posets yield a symmetric function graph invariant that is an analog of the chromatic polynomial. The highest degree homogeneous component of this symmetric function is of particular interest. The parking function symmetric function introduced by Haiman arises as an example, as do symmetric functions studied by the first author in connection with multibracketed Lie algebras and with colored exterior algebras. The γ-positivity result mentioned above is a specialization of an e-positivity result for the highest degree homogeneous component, which we prove for any chordal graph using the theory of lexicographic shellability. We conjecture that this symmetric function is Schur-log-concave, which specializes to Huh's log-concavity theorem for the chromatic polynomial.

Contents

Date: August 7, 2026.

Key words and phrases. weighted partitions, bond lattice, chromatic polynomial, graph associahedra, gamma-positivity and unimodality, symmetric functions, lexicographic shellability.

* Partially supported by an AMS-Simons Research Enhancement Grant for Primarily Undergraduate Institution Faculty and by the program of postdoctoral fellowships of Minciencias (Colombian Ministry of Science).

**Partially supported by NSF grants DMS 1502606 and DMS 2207337.

1. Introduction

In [9] Björner and Welker introduce the notion of Rees product of posets, which is a combinatorial analog of the Rees construction for semigroup algebras. A particular example that they consider can be viewed as a weighted version of the Boolean algebra $\mathcal{B}_n$, i.e., the lattice of subsets of the set $[n] := \{1, 2, \dots, n\}$. A *weighted set* B^v is a finite set B together with a weight $v \in \{0, 1, \dots, |B|-1\}$ attached (the empty set does not get any weight). The *weighted Boolean algebra* $\mathcal{WB}_n$ consists of the weighted subsets of $[n]$ with a certain order relation that reduces to the usual inclusion order on subsets when the weights are ignored.

The poset $\mathcal{WB}_n$ is pure (or graded) of length n, but is not bounded. Its minimum element is

$$\hat{0} = \emptyset$$

and its maximal elements are

$$[n]^0, [n]^1, \dots, [n]^{n-1}.$$

The Hasse diagram of $\mathcal{WB}_3$ is shown in Figure 1.1, where the commas and the set brackets have been omitted.

Shareshian and Wachs [37] prove that $\mathcal{WB}_n \cup \{\hat{1}\}$, where $\hat{1}$ is an attached maximum element, is EL-shellable and that the classical Eulerian polynomial $A_n(t)$ arises when one applies the Möbius function $\mu_{\mathcal{WB}_n}$ to the maximal intervals of $\mathcal{WB}_n$. More precisely they prove

$$(-1)^n \sum_{j=0}^{n-1} \mu_{\mathcal{WB}_n}(\hat{0}, [n]^j)\, t^j = A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)}, \tag{1.1}$$

where $\mathfrak{S}_n$ is the symmetric group on $[n]$ and des is the descent statistic defined by

$$\operatorname{des}(\sigma) := |\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}|.$$

A weighted version of the partition lattice Π_n, analogous to the weighted Boolean algebra $\mathcal{WB}_n$, was proposed by Dotsenko and Khoroshkin in [13] as a tool for studying the free Lie algebra with two compatible

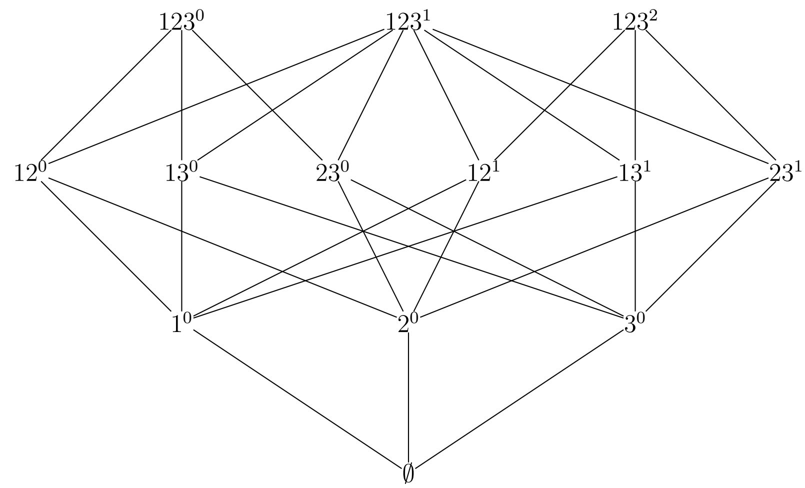

FIGURE 1.1. Weighted Boolean algebra $\mathcal{WB}_3$

brackets. In [22] González D'León and Wachs carry out a study of this poset and its connection to the free Lie algebra with two compatible brackets.

A *weighted partition* of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where

- $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$
- each $B_i^{v_i}$ is a weighted set, i.e., $v_i \in \{0, 1, 2, ..., |B_i|-1\}$ for all i.

The *poset of weighted partitions* $\mathcal{W}\Pi_n$ is the set of weighted partitions of $[n]$ with a certain order relation that reduces to the usual refinement order on partitions when the weights are ignored.[1] [2] The poset $\mathcal{W}\Pi_n$ is pure of length $n-1$, but is not bounded. Its minimum element is

$$\hat{0} := \{\{1\}^0, \{2\}^0, \dots, \{n\}^0\} \tag{1.2}$$

and it has n maximal elements

$$\{[n]^0\},\ \{[n]^1\}, \dots, \{[n]^{n-1}\}. \tag{1.3}$$

As is customary for ordinary partitions, we sometimes use the notation $B_1^{v_1}|B_2^{v_2}|\cdots|B_k^{v_k}$ with set brackets omitted to denote the weighted partition $\{B_1^{v_1}, \dots, B_k^{v_k}\}$. The Hasse diagram of $\mathcal{W}\Pi_3$ is shown in Figure 1.2.

In [22] González D'León and Wachs prove that $\mathcal{W}\Pi_n \cup \{\hat{1}\}$ is EL-shellable and that the *tree-Eulerian polynomial* (cf. [20]), studied by Gessel and Seo [19, Theorem 9.1], arises when one applies the Möbius

[1]Precise definitions are given in Section 2.

[2]The notation $\mathcal{W}\Pi_n$ used in this paper is different from that of [22] where the notation Π_n^w is used instead.

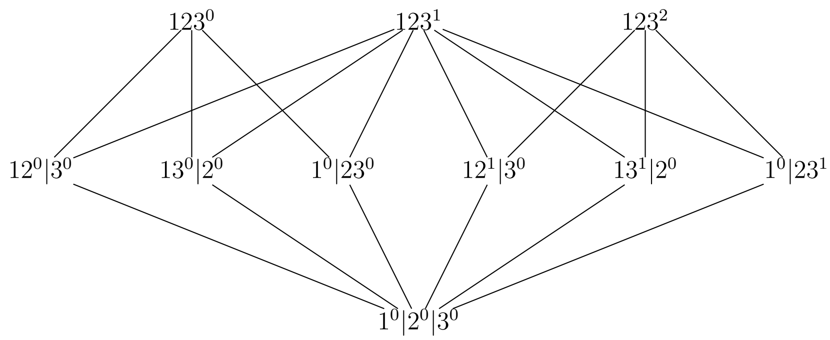

FIGURE 1.2. Weighted partition poset $\mathcal{W}\Pi_3$

function $\mu_{\mathcal{W}\Pi_n}$ to the maximal intervals of $\mathcal{W}\Pi_n$. More precisely, they prove

$$\sum_{j=0}^{n-1} \mu_{\mathcal{W}\Pi_n}(\hat{0}, \{[n]^j\})\, t^j = (-1)^{n-1} \sum_{T \in \mathcal{T}_n} t^{\operatorname{des}(T)},$$

where $\mathcal{T}_n$ is the set of (nonplanar) rooted trees on node set $[n]$ and $\operatorname{des}(T)$ is the *descent* statistic on $T \in \mathcal{T}_n$ that gives the number of pairs (p, c) for which p is the parent of c in T and $p > c$.

In this paper we introduce a weighted version of the bond lattice of a graph, which generalizes the weighted partition poset. The *bond lattice* Π_G of a graph G on vertex set $[n]$ is the induced subposet of Π_n consisting of the partitions whose blocks induce connected subgraphs of G. For the path graph $P_3 = ([3], \{\{1,2\}, \{2,3\}\})$, the Hasse diagram of Π_{P_3} is given in Figure 1.3. Note that the partition $13|2$ does not belong to Π_{P_3} since the induced subgraph $P_3|_{\{1,3\}}$ is not connected.

Bond lattices play an important role in the study of chromatic polynomials. Indeed, a formula of Whitney [47] for the chromatic polynomial $\chi_G(t)$ of G is given by

$$\chi_G(t) = \sum_{\pi \in \Pi_G} \mu_{\Pi_G}(\hat{0}, \pi) t^{|\pi|}. \tag{1.4}$$

Through the weighted version of the bond lattice, we introduce and study a different polynomial graph invariant.

Let G be a graph on vertex set $[n]$. We define the *weighted bond poset* $\mathcal{W}\Pi_G$ of G to be the induced subposet of $\mathcal{W}\Pi_n$ consisting of weighted partitions $\boldsymbol{\pi}$ whose underlying partitions π are in Π_G. Note that for the complete graph K_n, we have

$$\mathcal{W}\Pi_{K_n} = \mathcal{W}\Pi_n.$$

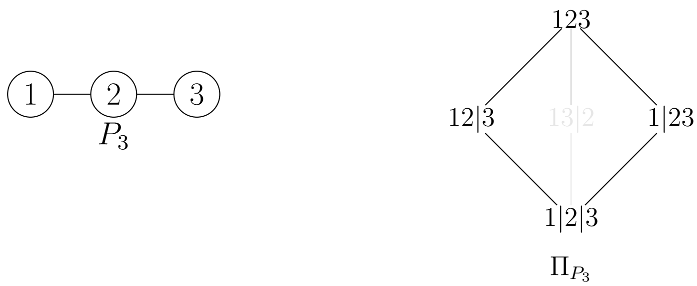

FIGURE 1.3. Bond lattice of P_3.

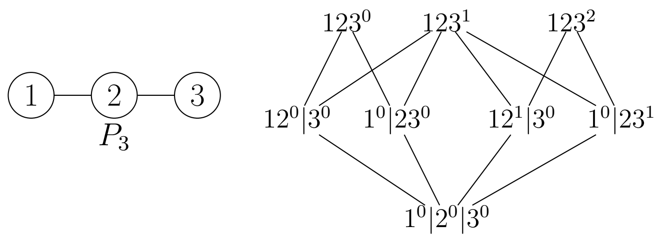

FIGURE 1.4. Weighted bond poset $\mathcal{W}\Pi_{P_3}$

The weighted bond poset $\mathcal{W}\Pi_G$ is pure, but not bounded. Its minimum element is the same as that for $\mathcal{W}\Pi_n$, given in (1.2). If G is connected, its maximal elements are also the same as those for $\mathcal{W}\Pi_n$, which are given in (1.3). For the path graph P_3, the Hasse diagram of $\mathcal{W}\Pi_{P_3}$ is given in Figure 1.4.

Now let G be a connected graph on vertex set $[n]$ and define the *Möbius polynomial* of G to be

$$\mu_G(t) := \sum_{j=0}^{n-1} \mu_{\mathcal{W}\Pi_G}(\hat{0}, \{[n]^j\})t^j.$$

For graphs G that are not connected, the sum is taken over the maximal elements of $\mathcal{W}\Pi_G$. The constant term $\mu_G(0)$ is the Möbius invariant $\mu(\Pi_G)$ of the bond poset Π_G. From Figures 1.2 and 1.4, we see that

$$\mu_{K_3}(t) = 2 + 5t + 2t^2 \quad \text{and} \quad \mu_{P_3}(t) = 1 + 3t + t^2. \tag{1.5}$$

We show that if G is a path graph on $[n]$ then $(-1)^{n-1}\mu_G(t)$ is a Narayana polynomial and if G is the star graph on $[n]$ then $(-1)^{n-1}\mu_G(t)$ is a polynomial arising in the work of Postnikov, Reiner, and Williams [34] on graph associahedra, called the binomial-Eulerian polynomial in [38].[3] As was discussed in [34], the Narayana polynomials and the

[3] Definitions are given in Sections 3.2 and 3.3.

binomial-Eulerian polynomials are h-polynomials of certain graph associahedra. The Möbius polynomials of these examples motivated the following conjecture, which appeared in an earlier version of this paper and has since been proved by Avila, Carrillo, and the first author [2, 3].

Conjecture 1.1. *Let G be a graph with n vertices and k connected components. Let $\mathcal{P}_G$ be the graph associahedron associated with G and let $h_{\mathcal{P}_G}(t)$ denote its h-polynomial. Then*

$$(-1)^{n-k}\mu_G(t) - h_{\mathcal{P}_G}(t) \succeq 0,$$

with equality if G is a forest, where $p(t) \succeq 0$ indicates that all the coefficients of the polynomial $p(t)$ are nonnegative.

In [34] Postnikov, Reiner, and Williams establish the important γ-positivity property for $h_{A_G}(t)$ when G is a *chordal graph* in order to confirm, for this particular class of polytopes, a well known conjecture of Gal [17], which asserts that every simple flag polytope is γ-positive. The γ-positivity property implies unimodality (and palindromicity), which was established by Stanley [40] for all convex polytopes. In this paper we prove the analogous result for $\mu_G(t)$.

Theorem 1.2. *If G is a chordal graph with n vertices and k connected components then $(-1)^{n-k}\mu_G(t)$ is γ-positive.*

Our proof of Theorem 1.2 relies on an EL-labeling of the weighted bond poset of a chordal graph and the theory of lexicographic shellability. Previously this technique was used in [37, 32, 23] to establish γ-positivity of variations and generalizations of $(-1)^n \sum_{j=0}^{n} \mu_{\mathcal{WB}_n}(\hat{0}, [n]^j)t^j$ and in [22] to establish γ-positivity of $(-1)^{n-1}\mu_{K_n}(t)$.

Our results hold in greater generality. In [21] González D'León considers a multiweighted generalization of the weighted partition poset $\mathcal{W}\Pi_n$ in order to study the multilinear component of the free Lie algebra with more than two compatible brackets. In [23] he also introduces a multiweighted generalization of a variant of the weighted Boolean algebra $\mathcal{WB}_n$ in order to study the multilinear component of colored exterior algebras. In this paper we consider an analogous multiweighted generalization of the weighted bond posets $\mathcal{W}\Pi_G$. This yields a symmetric function generalization $M_G(\mathbf{x})$ of the Möbius polynomial $\mu_G(t)$, which we call the *Möbius symmetric function* of G. Indeed, the specialization $M_G(\mathbf{x})(1, t, 0, \dots)$ yields $\mu_G(t)$. The specialization $M_G(\mathbf{x})(1, 0, 0, \dots) = \mu_G(0)$ yields the Möbius invariant $\mu(\Pi_G)$.

When G is the complete graph K_n, we have that $(-1)^{n-1}M_G(\mathbf{x})$ is equal to the symmetric function studied in [21], whose coefficients give the dimensions of components of the free Lie algebra with multiple compatible brackets. When G is the star graph with n vertices, $(-1)^{n-1}M_G(\mathbf{x})$ is equal to the symmetric function studied in [23], whose coefficients give the dimensions of components of the colored exterior algebra. We show that when G is equal to the path graph with n vertices, $(-1)^{n-1}\omega M_G(\mathbf{x})$ is the parking function symmetric function of Haiman [28], where ω is the standard involution on the ring of symmetric functions taking the complete homogeneous symmetric function h_n to the elementary symmetric function e_n.

We also obtain a multiweighted generalization of Theorem 1.2 whose proof relies on an EL-labeling of the multiweighted bond poset of a chordal graph.

Theorem 1.3. *If G is a chordal graph with n vertices and k connected components then the symmetric function $(-1)^{n-k}M_G(\mathbf{x})$ is e-positive, in the sense that when expanded in the basis of elementary symmetric functions $\{e_\lambda : \lambda \vdash n-1\}$, the coefficients are nonnegative.*

We conjecture that Theorem 1.3 is true for all graphs.

A symmetric function generalization of the chromatic polynomial can also be obtained from the multiweighted bond posets. The maximal degree component of this symmetric function is $M_G(\mathbf{x})$. For chordal graphs G, we establish a property that we call alternating e-positivity and we conjecture that it holds for all graphs. We also present a conjecture asserting that this symmetric function has a property that we call Schur-log-concavity. This conjecture generalizes Huh's celebrated log-concavity result for the chromatic polynomial [31].

The paper is organized as follows. Sections 2 - 4 deal with the Möbius polynomial $\mu_G(t)$ and Sections 5 - 7 deal with the symmetric function generalizations of the Möbius polynomial and the chromatic polynomial.

In Section 2, we present the definitions and basic properties of the weighted bond lattice and Möbius polynomial of a graph. In Section 3, we present recurrence relations which enable us to show that $(-1)^{n-1}\mu_G(t)$ is the tree-Eulerian polynomial when G is the complete graph, the Narayana polynomial when G is the path graph, and the binomial-Eulerian polynomial when G is the star graph. We also discuss in Section 3 the connection with graph associahedra. In Section 4, we present γ-positivity results for Möbius polynomials of chordal graphs, which are analogous to those of Postnikov, Reiner and Williams [34] for h-polynomials of graph associahedra of chordal graphs. Proofs of some of the results in Sections 3 and 4 are deferred to Sections 5 and 6 where more general results are proved.

In Section 5, we introduce the multiweighted generalization of the weighted bond poset and establish recurrence relations for the Möbius symmetric functions, which enable us to derive the relationship between the Möbius symmetric function of the path graph and the parking function symmetric function. Section 6 deals with EL-shellability of the multiweighted bond poset and its e-positivity implications, which include Theorem 1.3. The symmetric function generalization of the chromatic polynomial is discussed in Section 7.

2. Definitions and basic properties

Let Π_S denote the lattice of partitions of the finite set S partially ordered by refinement, that is $\pi \leq \pi'$ in Π_S if blocks of π are merged to obtain blocks of π'. See [44] for basic information, terminology and notation on partially ordered sets.

A *weighted partition* of a set S is a set $\boldsymbol{\pi} = \{B_1^{v_1}, B_2^{v_2}, ..., B_k^{v_k}\}$ of weighted blocks such that $\{B_1, B_2, ..., B_k\}$ is a partition of S and $v_i \in \{0, 1, 2, ..., |B_i| - 1\}$ for all $i \in [k]$. We refer to $\pi = \{B_1, B_2, ..., B_k\}$ as the *underlying partition* of $\boldsymbol{\pi}$ and v_i as the weight of the block B_i. The *poset of weighted partitions* $\mathcal{W}\Pi_S$ is the set of weighted partitions of S with order relation given by

$$\{A_1^{w_1}, A_2^{w_2}, ..., A_j^{w_j}\} \leq \{B_1^{v_1}, B_2^{v_2}, ..., B_k^{v_k}\}$$

if the following conditions hold:

- $\{A_1, A_2, ..., A_j\} \leq \{B_1, B_2, ..., B_k\}$ in Π_S
- if $B_m = A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_l}$ then $v_m - (w_{i_1} + w_{i_2} + ... + w_{i_l}) \in \{0, 1, ..., l-1\}$.

Equivalently, the covering relation $\lessdot$ is given by

$$\{A_1^{w_1}, A_2^{w_2}, ..., A_j^{w_j}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_k^{v_k}\}$$

if the following conditions hold:

- $\{A_1, A_2, \ldots, A_j\} \lessdot \{B_1, B_2, \ldots, B_k\}$ in Π_S
- if $B_m = A_h \cup A_i$, where $h \neq i$, then $v_m - (w_h + w_i) \in \{0, 1\}$
- if $B_m = A_i$ then $v_m = w_i$.

For each $n \in \mathbb{Z}_{>0}$, let $\Pi_n := \Pi_{[n]}$ and $\mathcal{W}\Pi_n := \mathcal{W}\Pi_{[n]}$, where $[n] := \{1, 2, \ldots, n\}$. Recall that the Hasse diagram of $\mathcal{W}\Pi_3$ (using the notation $B_1^{w_1}|B_2^{w_2}|, \ldots, |B_k^{w_k}$ with set brackets omitted to denote the weighted partition $\{B_1^{w_1}, \ldots, B_k^{w_k}\}$) is given in Figure 1.2.

Let $G = (V, E)$ be a finite graph. The *bond lattice* Π_G of G is the induced subposet of Π_V consisting of the partitions of V whose blocks induce connected subgraphs of G. The *weighted bond poset* $\mathcal{W}\Pi_G$ of G is defined to be the induced subposet of $\mathcal{W}\Pi_V$ consisting of all weighted partitions whose underlying partitions are in Π_G. Recall that the Hasse diagrams of Π_{P_3} and $\mathcal{W}\Pi_{P_3}$ are given in Figure 1.3 and 1.4, respectively.

Let K_n be the complete graph on vertex set $[n]$. Since

$$\mathcal{W}\Pi_{K_n} = \mathcal{W}\Pi_n,$$

the weighted bond poset of a graph is a more general structure than the weighted partition poset.

For $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ in $\mathcal{W}\Pi_G$, let

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}]_G = \{\boldsymbol{\pi} \in \mathcal{W}\Pi_G : \boldsymbol{\alpha} \le \boldsymbol{\pi} \le \boldsymbol{\beta}\}.$$

When $G = (V, E)$ is connected, for convenience, we will drop the set brackets for the maximal elements of $\mathcal{W}\Pi_G$. That is, we write the weighted partition $\{V^w\}$ as V^w and the maximal interval $[\hat{0}, \{V^w\}]_G$ as $[\hat{0}, V^w]_G$

A poset is said to be *bounded* if it has a minimum element, which is denoted $\hat{0}$, and a maximum element, which is denoted $\hat{1}$. A poset is said to be *pure* (or *graded*) if all its maximal chains have the same length, where the *length* of a chain is one less than the number of elements in the chain. The length $\ell(P)$ of a pure poset P is the common length of its maximal chains.

The bond lattice Π_G of any connected graph G is pure and bounded, while the weighted bond poset $\mathcal{W}\Pi_G$ is pure but not bounded even when G is connected. Indeed if $V = \{v_1, v_2, \ldots, v_n\}$ then $\mathcal{W}\Pi_G$ has minimum element

$$\hat{0} := \{\{v_1\}^0, \{v_2\}^0, \ldots, \{v_n\}^0\}$$

and if G is connected, $\mathcal{W}\Pi_G$ it has n maximal elements

$$V^0, V^1, \ldots, V^{n-1},$$

but no maximum element unless $n = 1$. If $G = (V, E)$ has connected components $G_i = (V_i, E_i)$, where $i \in [k]$, then the maximal elements of $\mathcal{W}\Pi_G$ are all the weighted partitions whose underlying partition is $\{V_1, \ldots, V_k\}$. Note that

$$\ell(\mathcal{W}\Pi_G) = \ell(\Pi_G) = |V| - k.$$

We have the following observation whose straightforward proof is left to the reader.

Proposition 2.1. *Let G be a graph whose connected components are $G_1, \ldots, G_k$. Then the map*

$$\varphi : \mathcal{W}\Pi_{G_1} \times \cdots \times \mathcal{W}\Pi_{G_k} \to \mathcal{W}\Pi_G$$

defined by $\varphi(\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_k) = \boldsymbol{\pi}_1 \cup \cdots \cup \boldsymbol{\pi}_k$ is a poset isomorphism.

Recall that the *Möbius function* of a poset P is defined recursively on intervals $[x, y]$ of P by

$$\mu_P(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\displaystyle\sum_{x \le z < y} \mu_P(x,z) & \text{if } x < y. \end{cases} \tag{2.1}$$

Define the *Möbius polynomial* of a graph G by

$$\mu_G(t) := \sum_{\boldsymbol{\pi} \in \mathrm{Max}(\mathcal{W}\Pi_G)} \mu_{\mathcal{W}\Pi_G}(\hat{0}, \boldsymbol{\pi}) t^{w(\boldsymbol{\pi})},$$

where $\mathrm{Max}(P)$ is the set of maximal elements of a poset P, and $w(\boldsymbol{\pi})$ is the sum of the weights of the blocks of $\boldsymbol{\pi}$ for any weighted partition $\boldsymbol{\pi}$. If G is a connected graph on vertex set V then

$$\mu_G(t) = \sum_{j=0}^{|V|-1} \mu_{\mathcal{W}\Pi_G}(\hat{0}, V^j)\, t^j. \tag{2.2}$$

The following proposition, which is a consequence of Proposition 2.1 and of the multiplicative property of the Möbius function (see [44, Proposition 3.8.2]) tells us that we can focus on Möbius polynomials of connected graphs.

Proposition 2.2. *Let G be a graph whose connected components are $G_1, \dots, G_k$. Then*

$$\mu_G(t) = \prod_{i=1}^{k} \mu_{G_i}(t).$$

Proof. For each $i \in [k]$, let V_i be the vertex set of G_i. The maximal elements of $\mathcal{W}\Pi_G$ are of the form $\{V_1^{j_1}, V_2^{j_2}, \dots, V_k^{j_k}\}$, where $0 \le j_i \le |V_i| - 1$ for each i. It follows that

$$\mu_G(t) = \sum_{j_1=0}^{|V_1|-1} \sum_{j_2=0}^{|V_2|-1} \cdots \sum_{j_k=0}^{|V_k|-1} \mu_{\mathcal{W}\Pi_G}(\hat{0}, \{V_1^{j_1}, V_2^{j_2}, \dots, V_k^{j_k}\}) t^{j_1+j_2+\cdots+j_k}.$$

By Proposition 2.1

$$\mu_{\mathcal{W}\Pi_G}(\hat{0}, \{V_1^{j_1}, \dots, V_k^{j_k}\}) = \mu_{\times_{i=1}^{k} \mathcal{W}\Pi_{G_i}}((\hat{0}_1, \cdots, \hat{0}_k), (\{V_1^{j_1}\}, \dots, \{V_k^{j_k}\})),$$

where $\hat{0}_i$ is the minimum element of $\mathcal{W}\Pi_{G_i}$. Now by the multiplicative property of the Möbius function (see [44, Proposition 3.8.2]),

$$\mu_{\times_{i=1}^{k} \mathcal{W}\Pi_{G_i}}((\hat{0}_1, \cdots, \hat{0}_k), (\{V_1^{j_1}\}, \dots, \{V_k^{j_k}\})) = \prod_{i=1}^{k} \mu_{\mathcal{W}\Pi_{G_i}}(\hat{0}_i, \{V_i^{j_i}\}).$$

Putting these equations together yields,

$$\begin{aligned}\mu_G(t) &= \sum_{j_1=0}^{|V_1|-1} \sum_{j_2=0}^{|V_2|-1} \cdots \sum_{j_k=0}^{|V_k|-1} \left(\prod_{i=1}^{k} \mu_{\mathcal{W}\Pi_{G_i}}(\hat{0}_i, \{V_i^{j_i}\}) \right) t^{j_1+j_2+\cdots+j_k} \\ &= \prod_{i=1}^{k} \sum_{j_i=0}^{|V_i|-1} \mu_{\mathcal{W}\Pi_{G_i}}(\hat{0}, V_i^{j_i}) t^{j_i} \\ &= \prod_{i=1}^{k} \mu_{G_i}(t). \qquad \square\end{aligned}$$

Proposition 2.3. *Let G be a connected graph on vertex set V. Then for all $j = 0, \ldots, |V| - 1$,*

$$[\hat{0}, V^j]_G \cong [\hat{0}, V^{|V|-1-j}]_G,$$

and both intervals $[\hat{0}, V^0]_G$ and $[\hat{0}, V^{|V|-1}]_G$ are isomorphic to the bond lattice Π_G.

Proof. It is not difficult to see that the map from $[\hat{0}, V^j]_G$ to $[\hat{0}, V^{|V|-1-j}]_G$ which takes $\{B_1^{w_1}, \ldots, B_k^{w_k}\}$ to $\{B_1^{|B_1|-1-w_1}, \ldots, B_k^{|B_k|-1-w_k}\}$ is a well-defined poset isomorphism. Since all the blocks of the weighted partitions in $[\hat{0}, V^0]_G$ have weight 0, the map that removes these weights is an isomorphism from $[\hat{0}, V^0]_G$ to Π_G. $\square$

Proposition 2.4. *Let G be a graph with n vertices and k connected components. Then*

1. *$\mu_G(t)$ has degree $n - k$.*
2. *$\mu_G(0)$ is equal to the Möbius invariant $\mu(\Pi_G)$ of the bond lattice Π_G.*
3. *$\mu_G(t)$ is palindromic in the sense that the coefficient of t^j equals the coefficient of t^{n-k-j} for all $j = 0, \ldots, n - k$.*

Proof. Item (1) for connected graphs follows from Equation (2.2), Proposition 2.3, and the fact that the Möbius invariant of a geometric lattice is never 0; see [44, Exercise 3.100b]. Indeed, by (2.2), the degree of $\mu_G(t)$ is at most $n - 1$ and by Proposition 2.3, the coefficient of t^{n-1} is $\mu(\Pi_G)$. Thus since Π_G is a geometric lattice, the coefficient of t^{n-1} is not 0. For graphs that are not connected, item (1) follows from the connected case and Proposition 2.2.

Item (2) for connected graphs follows from Proposition 2.3. To prove item (2) in general, we first note that Proposition 2.2 implies that $\mu_G(0) = \prod_{i=1}^{k} \mu_{G_i}(0)$, where the G_i are the connected components of G. Since $\Pi_G \cong \Pi_{G_1} \times \cdots \times \Pi_{G_k}$, item (2) now follows from the multiplicative property of the Möbius invariant and the connected case.

To prove item (3), we use Propositions 2.2 and 2.3 and the fact that products of palindromic polynomials are palindromic. □

3. Recurrence relations and examples

We present recurrence relations for the Möbius polynomials, which can be used to study specific examples.

3.1. Recurrence relations for the Möbius polynomials.

Let $G = (V, E)$. For $B \subseteq V$, the *restriction* $G|_B$ is defined to be the subgraph of G induced by the vertex set B. For $\pi \in \Pi_G$, the *contraction* G/π is defined to be the graph whose vertices are the blocks of π and whose edge set is

$$\{\{A, B\} \subseteq \pi \mid A \neq B,\ \{a, b\} \in E \text{ for some } a \in A \text{ and } b \in B\}.$$

Note that if $\pi \in \Pi_G$ then the restriction $G|_B$ is connected for all $B \in \pi$ and the contraction G/π is connected if and only if G is connected.

Although the following result can be proved directly by simply using the recursive definition of Möbius function for the first recurrence relation (3.1) and the dual version of the definition of Möbius function for the second recurrence relation (3.2), we prove it as a special case of Theorem 5.7 below.

Theorem 3.1. *Let G be a connected graph with more than one vertex. Then $\mu_G(t)$ can be computed recursively by either*

$$\mu_G(t) = -\sum_{\pi \in \Pi_G \setminus \{\hat{1}\}} [|\pi|]_t \prod_{B \in \pi} \mu_{G|_B}(t), \tag{3.1}$$

or

$$\mu_G(t) = -\sum_{\pi \in \Pi_G \setminus \{\hat{0}\}} \mu_{G/\pi}(t) \prod_{B \in \pi} [|B|]_t, \tag{3.2}$$

where $[k]_t := 1 + t + \cdots + t^{k-1}$ is the t-analog of k and the initial condition is given by $\mu_G(t) = 1$ when $G = (\{\bullet\}, \emptyset)$, is the graph with one vertex.

The recurrence relation (3.2) in the case of the complete graph

$$K_n := ([n], \{\{i, j\} : 1 \leq i < j < n\})$$

and the compositional formula (see [43, Theorem 5.1.4]) were used in [22] to obtain the exponential generating function formula

$$\sum_{n \geq 1} \mu_{K_n}(t) \frac{y^n}{n!} = \left(\sum_{n \geq 1} [n]_t \frac{y^n}{n!} \right)^{\langle -1 \rangle} = \left(\frac{e^{ty} - e^y}{t - 1} \right)^{\langle -1 \rangle}, \tag{3.3}$$

where $(\bullet)^{\langle -1\rangle}$ denotes the inverse with respect to composition (substitution) of power series. This and [14, Equation (10)] were used in [22] to obtain the formula,

$$(-1)^{n-1}\mu_{K_n}(t) = \prod_{i=1}^{n-1}((n-i)+it). \tag{3.4}$$

A nice consequence of this formula is

$$(-1)^{n-1}\mu_{K_n}(1) = n^{n-1},$$

which by Cayley's formula is the number of (nonplanar) rooted trees on node set $[n]$. Equation 3.4 and a refinement of Cayley's formula obtained by Gessel and Seo [19, Theorem 9.1] yield

$$(-1)^{n-1}\mu_{K_n}(t) = \sum_{T\in\mathcal{T}_n} t^{\mathrm{des}(T)}, \tag{3.5}$$

where $\mathcal{T}_n$ is the set of (nonplanar) rooted trees on node set $[n]$ and $\mathrm{des}(T)$ is the number of pairs (p,c), where p is the parent of c in T and $p > c$. We call the polynomial on the right hand side of (3.5), the *tree-Eulerian polynomial*; cf. [20].

3.2. **The path graph and Narayana polynomials.** Define the *path graph* to be

$$P_n := ([n], \{\{i,i+1\} : i \in [n-1]\}).$$

In this section, we apply the recurrence relations of Theorem 3.1 to the path graphs to show that the signless Möbius polynomials for these graphs are equal to the well known Narayana polynomials.

The Narayana numbers are defined by

$$N(n,j) := \frac{1}{n}\binom{n}{j}\binom{n}{j+1} \tag{3.6}$$

and the Narayana polynomials by

$$N_n(t) := \sum_{j=0}^{n-1} N(n,j)t^j.$$

The Narayana numbers refine the Catalan numbers

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Indeed, by Vandermonde's identity

$$N_n(1) = C_n.$$

Theorem 3.2. *For $n \geq 1$,*

$$(-1)^{n-1}\mu_{P_n}(t) = N_n(t).$$

Consequently,

$$(-1)^{n-1}\mu_{P_n}(1) = C_n.$$

Proof. The recurrence relations of Theorem 3.1 applied to $G = P_n$ can be expressed as

$$\sum_{k=1}^{n}[k]_t \sum_{(n_1,\dots,n_k)\vDash n} \prod_{i=1}^{k} \mu_{P_{n_i}}(t) = \delta_{n,1}$$

and

$$\sum_{k=1}^{n} \mu_{P_k}(t) \sum_{(n_1,\dots,n_k)\vDash n} \prod_{i=1}^{k}[n_i]_t = \delta_{n,1},$$

where $\vDash$ denotes "is a composition of". Both equations are equivalent to the generating function formula

(3.7)

$$\sum_{n\geq 1}(-1)^{n-1}\mu_{P_n}(t)y^n = \left(\sum_{n\geq 1}(-1)^{n-1}[n]_t\, y^n\right)^{\langle -1\rangle} = \left(\frac{y}{(1+y)(1+ty)}\right)^{\langle -1\rangle},$$

where again $(\bullet)^{\langle -1\rangle}$ denotes the compositional inverse.[4] Hence, by the Lagrange inversion formula (see [43, Theorem 5.4.2]), $n(-1)^{n-1}\mu_{P_n}(t)$ is equal to the coefficient of y^{n-1} in $((1+y)(1+ty))^n$. Since

$$\begin{aligned}((1+y)(1+ty))^n &= \sum_{j=0}^{n}\binom{n}{j}y^j\sum_{j=0}^{n}\binom{n}{j}(ty)^j \\ &= \sum_{k\geq 0}\left(\sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-j}t^j\right)y^k,\end{aligned}$$

the coefficient of y^{n-1} is

$$\sum_{j=0}^{n-1}\binom{n}{j}\binom{n}{n-1-j}t^j = \sum_{j=0}^{n-1}\binom{n}{j}\binom{n}{j+1}t^j.$$

This means that

$$(-1)^{n-1}\mu_{P_n}(t) = \frac{1}{n}\sum_{j=0}^{n-1}\binom{n}{j}\binom{n}{j+1}t^j,$$

[4] It is known that $\frac{y}{(1+y)(1+ty)}$ is the compositional inverse of the generating function for the Narayana polynomials. For the sake of completeness we are including this proof.

which by (3.6) yields the desired result. □

3.3. **The star graph and binomial-Eulerian polynomials.** Define the *star graph* to be

$$St_n := ([n], \{\{1, i\} : i \in \{2, \dots, n\}\}).$$

In this section, we use the recurrence relations of Theorem 3.1 applied to the star graphs to show that the signless Möbius polynomials for these graphs are equal to a variant of the Eulerian polynomials.

Recall that the classical Eulerian polynomials are defined by

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)}, \tag{3.8}$$

where

$$\operatorname{des}(\sigma) := |\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}|.$$

Following [38], we refer to the polynomial

$$\tilde{A}_n(t) := 1 + t \sum_{m=1}^{n} \binom{n}{m} A_m(t)$$

as the *binomial-Eulerian polynomial* of degree n. We refer to the coefficient $\tilde{a}_{n,j}$ of t^j in $\tilde{A}_n(t)$ as a binomial-Eulerian number. The binomial-Eulerian polynomials arose in the work of Postinkov, Reiner and Williams [34] as the h-polynomial of a polytope that they called the stellohedron. A q-analog and a symmetric function analog were studied by Shareshian and Wachs [38] and further studied by González D'León [23].

Just as the Eulerian numbers enumerate permutations with a given number of descents, the binomial-Eulerian numbers enumerate partial permutations with a given number of descents, where a partial permutation of $[n]$ is a permutation of a nonempty subset of $[n]$. For example, when $n = 3$, the partial permutations of $[n]$ written in one line notation are:

$$1, 2, 3, 12, 21, 13, 31, 23, 32, 123, 132, 213, 231, 312, 321.$$

Let $\tilde{\mathfrak{S}}_n$ be the set of partial permutations of $[n]$. The descent number of a partial permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ is defined in the usual way as $\operatorname{des}(\sigma) = |\{i \in [m-1] : \sigma_i > \sigma_{i+1}\}|$. For the partial permutations listed above the respective descent numbers are:

$$0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 2.$$

The following combinatorial characterization of $\tilde{A}_n(t)$ is straightforward to prove:

$$\tilde{A}_n(t) = 1 + \sum_{\sigma \in \tilde{\mathfrak{S}}_n} t^{\mathrm{des}(\sigma)+1}. \tag{3.9}$$

It follows from this and the list of descent numbers above that

$$\tilde{A}_3(t) = 1 + 7t + 7t^2 + t^3.$$

Theorem 3.3. *For $n \geq 1$,*

$$(-1)^{n-1}\mu_{St_n}(t) = \tilde{A}_{n-1}(t).$$

Consequently

$$(-1)^{n-1}\mu_{St_n}(1) = 1 + \sum_{m=1}^{n-1}(n-1)_m,$$

where $(n-1)_m$ is the falling factorial $(n-1)(n-2)\dots(n-m)$.

Proof. The recurrence relations of Theorem 3.1 for $G = St_n$ reduce to

$$\sum_{k=0}^{n-1}\binom{n-1}{k}[k+1]_t\ \mu_{St_{n-k}}(t) = \delta_{n,1}.$$

This is equivalent to the exponential generating function formula

$$\sum_{n\geq 0}\mu_{St_{n+1}}(t)\frac{y^n}{n!} = \left(\sum_{n\geq 0}[n+1]_t\frac{y^n}{n!}\right)^{-1} = \frac{t-1}{te^{ty}-e^y}. \tag{3.10}$$

Replacing y by $-y$ yields

$$\begin{aligned}\sum_{n\geq 0}(-1)^n\mu_{St_{n+1}}(t)\frac{y^n}{n!} &= \frac{t-1}{te^{-ty}-e^{-y}}\\ &= \frac{(t-1)e^{ty}}{t-e^{(t-1)y}}.\end{aligned} \tag{3.11}$$

It is straightforward using Euler's exponential generating formula

$$\sum_{n\geq 1}A_n(t)\frac{y^n}{n!} = \frac{e^y-e^{ty}}{e^{ty}-te^y}, \tag{3.12}$$

to show that the formula given in (3.11) also holds for the exponential generating function for $\tilde{A}_n(t)$. Indeed we have

$$\begin{aligned} \sum_{n\geq 0} \tilde{A}_n(t)\frac{y^n}{n!} &= \sum_{n\geq 0}\left(1+t\sum_{m=1}^{n}\binom{n}{m}A_m(t)\right)\frac{y^n}{n!} \\ &= e^y + t\sum_{n\geq 1} A_n(t)\frac{y^n}{n!}\sum_{n\geq 0}\frac{y^n}{n!} \\ &= e^y + t\frac{e^y - e^{ty}}{e^{ty}-te^y}e^y \\ &= \frac{(t-1)e^{ty}}{t-e^{(t-1)y}} \end{aligned} \tag{3.13}$$

with the second to last equality following from (3.12) and the last equality following from straightforward manipulation. Comparing this with (3.11) yields the desired result. □

Remark 3.4. A q-analog and symmetric function analog of the exponential generating function formula given in (3.13) appears in [38, Propositions 4.3 and 3.3] respectively.

3.4. **A connection with graph associahedra.** The h-polynomial of a *simple*[5] d-dimensional polytope $\mathcal{P}$ is defined by

$$h_{\mathcal{P}}(t) = \sum_{i=0}^{d} f_i(t-1)^i, \tag{3.14}$$

where f_i is the number of i dimensional faces of $\mathcal{P}$.

It is well known that the Eulerian polynomials and the Narayana polynomials are h-polynomials of simple polytopes known as the permutohedron and the associahedron, respectively. It is shown in [34] that the binomial-Eulerian polynomials are h-polynomials of simple polytopes as well. Hence by Theorems 3.2 and 3.3, respectively, $(-1)^{n-1}\mu_G(t)$ is the h-polynomial of a simple polytope when G is the path graph P_n or the star graph St_n. It is natural to ask if there are any other graphs G for which this is the case; in this section we discuss this question.

Graph associahedra are a family of simple polytopes that were studied by Carr and Devadoss [11] and that form part of the larger class of generalized permutahedra studied by Postnikov in [35]. Several ocurrances of equivalent families have appeared in the literature as well. In particular, the dual simplicial complexes of graph associahedra are included in the family of nested set complexes studied by De Concini

[5]Here we will be following the convention in [34], in which the h-polynomial is defined for simple polytopes rather than, as is often done, for simplicial polytopes.

and Procesi in [12]. We should also mention that in recent years a great many papers have appeared with further connections between generalized permutahedra and other areas of mathematics. A paper we would like to mention is that of Aguiar and Ardila [1], in which a beautiful structure of a Hopf monoid on generalized permutahedra is described, for which the restriction to graph associahedra yields a known Hopf monoid on graphs.

We follow the notation and terminology used in [35, 34]. A *building set* on a finite set V is a collection $\mathcal{B} \subseteq 2^V$ satisfying:

- $\{v\} \in \mathcal{B}$ for all $v \in V$.
- If $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$ then $B \cup B' \in \mathcal{B}$.

Given a graph G on vertex set V, one can define the building set,

$$\mathcal{B}(G) := \{\emptyset \neq B \subseteq V \mid G|_B \text{ is connected}\}.$$

The *nestohedron* $\mathcal{P}_\mathcal{B}$ associated to a building set $\mathcal{B}$ on $V \subseteq [n]$ is the polytope defined as the Minkowski sum

$$\mathcal{P}_\mathcal{B} := \sum_{B \in \mathcal{B}} \Delta_B,$$

where the *Minkowski sum* of two subsets $S, T \in \mathbb{R}^n$ is defined as the subset $S + T := \{x + y \mid x \in S \text{ and } y \in T\} \subseteq \mathbb{R}^n$ and Δ_B is the simplex in $\mathbb{R}^n$ obtained as the convex hull of the standard unit vectors in $\{\mathbf{e}_i \mid i \in B\}$. In [35] it is shown that the nestohedron $\mathcal{P}_\mathcal{B}$ is a simple polytope.

Nestohedra of the form $\mathcal{P}_G := \mathcal{P}_{\mathcal{B}(G)}$, where G is a graph on $V \subseteq [n]$, are called *graph-associahedra* in [11]. It follows from [35, Theorem 7.11] that if $G_1, \dots, G_k$ are the connected components of G then

$$h_{\mathcal{P}_G}(t) = h_{\mathcal{P}_{G_1}}(t) \cdots h_{\mathcal{P}_{G_k}}(t). \tag{3.15}$$

The graph associahedron for the complete graph is the standard permutohedron and the graph associahedron for the path graph is the standard associahedron; see [34, Section 10]. The graph associahedron for the star graph is called the *stellohedron* in [34, Section 10.4]. As was mentioned above, it is well known that the h-polynomials for the permutohedron and the associahedron are the Eulerian and Narayana polynomials, respectively. Thus

$$\begin{aligned} h_{\mathcal{P}_{K_n}}(t) &= A_n(t) \\ h_{\mathcal{P}_{P_n}}(t) &= N_n(t). \end{aligned} \tag{3.16}$$

In [34, Section 10.4], it is shown that,

$$h_{\mathcal{P}_{St_n}}(t) = \tilde{A}_{n-1}(t).$$

It follows from these equations and Theorems 3.2 and 3.3 that

$$(-1)^{n-k}\mu_G(t) = h_{\mathcal{P}_G}(t)$$

is true for the special cases that G is the path graph and the star graph. By (3.15) and Proposition 2.2, it is also true for forests G whose trees are path graphs or star graphs.

In an earlier version of this paper the authors conjectured the following. For a polynomial $p(t) = \sum_{i=0}^{d} a_i t^i$, we write $p(t) \succeq 0$ if all its coefficients are nonnegative.

Conjecture 3.5. *For any graph G with n vertices and k connected components we have that*

$$(-1)^{n-k}\mu_G(t) - h_{\mathcal{P}_G}(t) \succeq 0, \tag{3.17}$$

with equality if G is a forest.

More recently, the first author together with Avila and Carrillo were able to prove this conjecture in [2, 3] and generalize it to a theorem for general building sets.

Furthermore, we observe in the following proposition that if equality in (3.17) holds then G is necessarily a forest.

Proposition 3.6. *Let G be a graph with n vertices and k connected components. If $(-1)^{n-k}\mu_G(t)$ is equal to the h-polynomial of a simple polytope then G is a forest.*

Proof. The h-polytope of every simple polytope has constant term equal to 1. Recall from Proposition 2.4 that the constant term $\mu_G(0)$ of $\mu_G(t)$ is equal to the Möbius invariant $\mu(\Pi_G)$. The result now follows from the known fact that $(-1)^{n-k}\mu(\Pi_G) = 1$ if and only if G is a forest. One can prove this by using Whitney's Broken Circuit Theorem [47], which tells us that $(-1)^{n-k}\mu(\Pi_G)$ is equal to the number of spanning forests of G that contain no broken circuits. (A broken circuit is a subgraph of G obtained by removing the smallest edge, under a fixed total ordering of the edges of G, from a cycle of G.) $\square$

By Theorem 3.1, the following recurrence relations are consequences of Conjecture 3.5 (now proved in [2, 3] as noted above).

Corollary 3.7. *Let T be a tree on node set $[n]$. If $n > 1$ then*

$$h_{\mathcal{P}_T}(t) = \sum_{\pi \in \Pi_T \setminus \{\hat{1}\}} (-1)^{|\pi|}[|\pi|]_t \prod_{B\in\pi} h_{\mathcal{P}_{T|_B}}(t) \tag{3.18}$$

and

$$h_{\mathcal{P}_T}(t) = \sum_{\pi \in \Pi_T \setminus \{\hat{0}\}} (-1)^{n-|\pi|-1} h_{\mathcal{P}_{T/\pi}}(t) \prod_{B\in\pi} [|B|]_t. \tag{3.19}$$

Remark 3.8. In [35, Theorem 7.11], Postnikov gives a recurrence for $h_{\mathcal{P}_{\mathcal{B}}}(t)$, where $\mathcal{B}$ is a building set. If one applies this to $\mathcal{B}(T)$, where T is a tree, one gets a recurrence relation that is different from those in the conjectured Corollary 3.7. Indeed the recurrence relation in [35] for trees T is

$$h_{\mathcal{P}_T}(t) = \sum_{U \subsetneq [n]} (t-1)^{n-|U|-1} \prod_{R \in T|_U} h_{\mathcal{P}_R}(t),$$

where the forest $T|_U$ is viewed as a set of trees.

4. γ-positivity for chordal graphs

In this section we discuss a property of polynomials in $\mathbb{R}[t]$ called γ-positivity. Postnikov, Reiner, and Williams [34] established this property for h-polynomials of graph associahedra of chordal graphs and here we do the same for Möbius polynomials. We review their γ-positivity result in Section 4.1 for the sake of comparison with our analogous result presented in Section 4.2.

Let $p(t) = \sum_{j=0}^{d} a_j t^j \in \mathbb{R}[t]$. Recall that $p(t)$ is said to be *palindromic* if $a_j = a_{d-j}$ for all j, and $p(t)$ is said to be *unimodal* if

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_c \geq \cdots \geq a_{d-1} \geq a_d \geq 0, \tag{4.1}$$

for some $c \in \{0, 1, \ldots, d\}$. Note that if $p(t)$ is palindromic and unimodal and $a_0 > 0$ (which means d is the degree of $p(t)$) then all the coefficients of $p(t)$ are positive, denoted $p(t) \succ 0$.

It is straightforward to see that $p(t)$ is palindromic if and only if

$$p(t) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_j t^j (1+t)^{d-2j}$$

for some $\gamma_j \in \mathbb{R}$. The coefficients γ_j are commonly called γ-*coefficients*. If $\gamma_j \geq 0$ for all j then the polynomial $p(t)$ is said to be γ-*positive*.[6] It is well known and straightforward to prove that if $p(t)$ is γ-positive then $p(t)$ is both palindromic and unimodal.

A celebrated result of Stanley [40] asserts that the h-polynomial of any convex simple polytope is unimodal as well as palindromic (which was already well-known). Interest in γ-positivity stems in part from Gal's conjecture [17] which in the case of convex polytopes asserts that the h-polynomial of any simple flag polytope is γ-positive.

A graph is said to be *chordal* if it does not have any induced cycles of length greater than 3. Chordal graphs are exactly those that admit a *(reverse) perfect elimination order (PEO)*, i.e., a total order of the

[6] This is often called γ-*nonnegative*.

vertices $v_1, v_2, \ldots, v_n$ such that for every i the set of neighbors v_j of v_i, such that $j < i$, form a clique, see [34, Section 9.2].

We will say that a graph G on vertex set $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{Z}_{>0}$ is a *perfectly labeled chordal graph* if the natural order $v_1 < v_2 < \cdots < v_n$ is a PEO for G. Note that every chordal graph is isomorphic to a perfectly labeled chordal graph. The complete graph on $[n]$ is a perfectly labeled chordal graph, as is every increasing tree on $[n]$, where a tree on $[n]$ is *increasing* if when rooted at 1, every vertex is smaller than its children.

4.1. **h-polynomials of chordal graph associahedra.** In [34], Postnikov, Reiner, and Williams establish γ-positivity of the h-polynomial of the graph associahedron of any chordal graph, thereby confirming Gal's conjecture for this class of flag simple polytopes.[7] They do this by giving a combinatorial characterization of the coefficients and the γ-coefficients of the h-polynomial in terms of a certain set of permutations. We review this result in this section.

For any graph G on vertex set $[n]$, a *G-permutation*[8] is a permutation $\sigma \in \mathfrak{S}_n$ such that for every $i \in [n]$, the elements $\sigma(i)$ and $\min\{\sigma(1), \ldots, \sigma(i)\}$ are in the same connected component of $G|_{\{\sigma(1),\ldots,\sigma(i)\}}$. Let $\mathfrak{S}(G)$ be the set of G-permutations and let $\widehat{\mathfrak{S}}(G) := \mathfrak{S}(G) \cap \widehat{\mathfrak{S}}_n$, where $\widehat{\mathfrak{S}}_n$ is the subset of $\mathfrak{S}_n$ consisting of permutations σ with no consecutive ascents; that is for all $i \in [n-2]$, if $\sigma(i) < \sigma(i+1)$ then $\sigma(i+1) > \sigma(i+2)$.[9] For $\sigma \in \mathfrak{S}_n$, let

$$\operatorname{asc}(\sigma) = |\{i \in [n-1] : \sigma(i) < \sigma(i+1)\}|.$$

Theorem 4.1 (Postnikov, Reiner, Williams [34])**.** *For $n \geq 1$ and G a connected perfectly labeled chordal graph on $[n]$, we have that*

$$h_{\mathcal{P}_G}(t) = \sum_{\sigma \in \mathfrak{S}(G)} t^{\operatorname{asc}(\sigma)} \tag{4.2}$$

and

$$h_{\mathcal{P}_G}(t) = \sum_{\substack{\sigma \in \widehat{\mathfrak{S}}(G) \\ \sigma(n-1) > \sigma(n)}} t^{\operatorname{asc}(\sigma)} (1+t)^{n-1-2\operatorname{asc}(\sigma)}. \tag{4.3}$$

Now by (3.15) and the fact that products of γ-positive polynomials are γ-positive we have the following consequence.

[7] They do this more generally for chordal nestohedron.

[8] Note that max is used instead of min in [34]. Hence, the roles of descents and ascents are exchanged.

[9] Note that in [34], $\widehat{\mathfrak{S}}_n$ denotes a smaller subset in which a final descent is also required.

Corollary 4.2. *If G is a chordal graph then $h_{\mathcal{P}_G}(t)$ is γ-positive.*

Example 4.3. (1) For $G = K_n$, we have $\mathfrak{S}(G) = \mathfrak{S}_n$. Thus, in this case, (4.2) reduces to the combinatorial definition of the Eulerian polynomial given in (3.8), and (4.3) reduces to a well known formula of Foata and Schützenberger [16, Theorem 5.6] for the Eulerian polynomials.

(2) For $G = P_n$, we have that $\sigma \in \mathfrak{S}(G)$ if and only if σ avoids the pattern 132, i.e., if $1 \leq i < j < k \leq n$ and $\sigma(i) < \sigma(k)$ then $\sigma(j) < \sigma(k)$. Thus, in this case, (4.2) and (4.3) reduce to well known combinatorial formulas for the Narayana polynomials in terms of descent numbers of pattern avoiding permutations; see [33, Sections 2.3 and 4.3].

(3) For $G = St_{n+1}$, we have that $\sigma \in \mathfrak{S}(G)$ if and only if

$$\sigma(1) > \sigma(2) > \cdots > \sigma(k) = 1$$

for some $k \in [n+1]$. Thus, in this case, (4.2) reduces to a formula equivalent to the combinatorial characterization of the binomial-Eulerian polynomials given in (3.9), and (4.3) reduces to the formula

$$\tilde{A}_n(t) = \sum_{\substack{\sigma \in \widehat{\mathfrak{S}}_{n+1} \\ \sigma(n) > \sigma(n+1) \\ \sigma(1) > \sigma(2) > \cdots > \sigma(k) = 1}} t^{\operatorname{asc}(\sigma)} (1+t)^{n-1-2\operatorname{asc}(\sigma)}. \tag{4.4}$$

4.2. **Möbius polynomials of chordal graphs.** In this subsection we present results for Möbius polynomials that are analogous to the results for h-polynomials reviewed in Section 4.1.

By *binary tree* we mean a planar rooted tree in which every internal node has a left child and a right child. A *leaf-labeled binary tree* is a binary tree whose leaves are labeled with distinct labels. If A is the set of leaf labels of the binary tree T we say that T is a binary tree on leaf set A. Let $\mathcal{BT}_n$ denote the set of binary trees on leaf set $[n]$.

For each node x of $T \in \mathcal{BT}_n$, let

- A_x be the subset of $[n]$ consisting of the leaf labels of the subtree rooted at x
- $L(x)$ be the left child of x
- $R(x)$ be the right child of x.

We say that $T \in \mathcal{BT}_n$ is *normalized* if

$$\min A_{L(x)} < \min A_{R(x)}$$

for all internal nodes x of T. Let

$$\mathcal{N}_n = \{T \in \mathcal{BT}_n : T \text{ is normalized}\}.$$

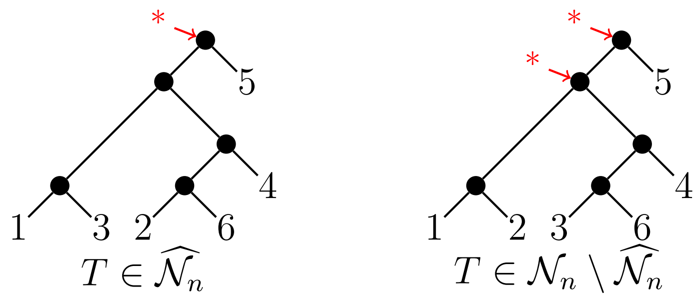

FIGURE 4.1. Example of a normalized tree that is in $\widehat{\mathcal{N}_n}$ and one that is not. The red $*$ indicate nodes that are not Lyndon.

We say that an internal node x of $T \in \mathcal{N}_n$ is a *Lyndon node* if its left child is a leaf or

$$\min A_{R(L(x))} > \min A_{R(x)}. \tag{4.5}$$

We refer to these nodes as Lyndon nodes because the classical Lyndon trees are the normalized binary trees in which all internal nodes are Lyndon. Now let

$$\widehat{\mathcal{N}_n} = \{T \in \mathcal{N}_n : \forall x \in \mathbf{int}(T),\ x \text{ is not Lyndon} \implies L(x) \text{ is Lyndon}\},$$

where $\mathbf{int}(T)$ is the set of internal nodes of T. See Figure 4.1 for an example of a tree in $\widehat{\mathcal{N}_n}$ and a normalized tree not in $\widehat{\mathcal{N}_n}$.

Let G be a graph on vertex set $[n]$. Define

$$\mathcal{N}_G := \{T \in \mathcal{N}_n : G|_{A_x} \text{ is connected } \forall x \in \mathbf{int}(T)\}$$

and

$$\widehat{\mathcal{N}_G} := \mathcal{N}_G \cap \widehat{\mathcal{N}_n}.$$

The proof of the following theorem is deferred to Section 6.3, where the more general Theorem 6.18 is proved.

Theorem 4.4. *Let G be a connected perfectly labeled chordal graph on vertex set $[n]$. Then*

$$(-1)^{n-1}\mu_G(t) = \sum_{T \in \widehat{\mathcal{N}_G}} t^{m(T)}(1+t)^{n-1-2m(T)},$$

where $m(T)$ is the number of nonLyndon internal nodes of T.

Example 4.5. Let G be P_4 with the additional edge $\{1,3\}$. Since G is a perfectly labeled chordal graph, we can apply Theorem 4.4. In

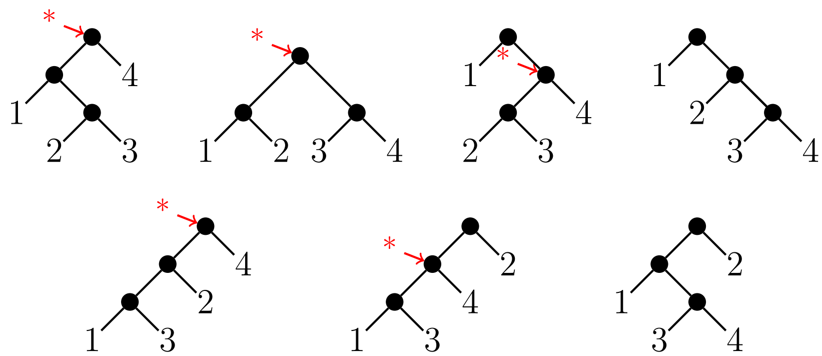

FIGURE 4.2. Trees in $\widehat{\mathcal{N}_G}$ for G in Example 4.5.

Figure 4.2, the trees of $\widehat{\mathcal{N}_G}$ are listed and their nonLyndon nodes are indicated. By Theorem 4.4, we have that

$$-\mu_G(t) = 2(1+t)^3 + 5t(1+t).$$

By eliminating the trees that are not in $\mathcal{N}_{P_4}$, we see that

$$-\mu_{P_4}(t) = (1+t)^3 + 3t(1+t) = N_4(t).$$

Corollary 4.6. *Let G be a chordal graph with n vertices and k connected components. Then $(-1)^{n-k}\mu_G(t)$ is γ-positive.*

Proof. This follows from Theorem 4.4, Proposition 2.2, and the fact that products of γ-positive polynomials are γ-positive. □

Corollary 4.7. *Let G and H be perfectly labeled chordal graphs with n vertices and k connected components. If H is a subgraph of G then $(-1)^{n-k}(\mu_G(t) - \mu_H(t))$ is γ-positive.*

Proof. In the connected case, this result follows immediately from Theorem 4.4, since clearly $\widehat{\mathcal{N}_H} \subseteq \widehat{\mathcal{N}_G}$. In the general case we also use Proposition 2.2. Details are given in the proof of the more general Corollary 6.20. □

Corollary 4.8. *Let G and H be perfectly labeled chordal graphs with n vertices and k connected components. Then*

$$(-1)^{n-k}\mu_G(t) \succ 0. \tag{4.6}$$

If H is a subgraph of G then

$$(-1)^{n-k}(\mu_G(t) - \mu_H(t)) \succ 0. \tag{4.7}$$

Proof. Equation (4.6) follows from Corollary 4.6 and the the fact that $\mu_G(0) \neq 0$, cf. Proposition 2.4.

Equation (4.7) follows from Corollary 4.7 and the the fact that $\mu_G(0) - \mu_H(0) \neq 0$. Through a similar argument to the one used in proof of Proposition 3.6, one can prove this fact by using Whitney's Broken Circuit Theorem [47]. Indeed, removing one edge that does not increase the number of connected components decreases the number of spanning forests that are free of broken circuits. □

Example 4.9. Let G be the graph in Example 4.5 and let $H = P_4$. We have

$$\begin{aligned}-(\mu_G(t) - \mu_H(t)) &= 2(1+t)^3 + 5t(1+t) - ((1+t)^3 + 3t(1+t)) \\ &= (1+t)^3 + 2t(1+t),\end{aligned}$$

which confirms Corollary 4.7 for these graphs.

Example 4.10. We apply Theorem 4.4 to the path graphs. A binary tree T on leaf set $[n]$ is in $\mathcal{N}_{P_n}$ if and only if the leaves of T are labeled from left to right in increasing order. Consequently, the only Lyndon nodes of T are the internal nodes whose left child is a leaf. Since there can be only one leaf labeling for the trees in $\mathcal{N}_{P_n}$, we can view these trees as unlabeled binary trees. Thus by Theorems 3.2 and 4.4, we get the following combinatorial interpretation of the Narayana polynomials

$$N_n(t) = \sum_T t^{m(T)}(1+t)^{n-1-2m(T)},$$

where the sum is taken over the unlabeled binary trees T with n leaves in which the left child of every left child is a leaf, and $m(T)$ is the number of internal nodes of T that are left children.

Example 4.11. We apply Theorem 4.4 to the star graphs St_{n+1}. We will represent a binary tree T as $T_L \wedge T_R$, where T_L is the left subtree of T and T_R is the right subtree of T. A binary tree T on leaf set $[n+1]$ is in $\mathcal{N}_{St_{n+1}}$ if and only if T is of the form

$$(\dots((\sigma(1) \wedge \sigma(2)) \wedge \sigma(3)) \wedge \cdots) \wedge \sigma(n+1),$$

where $\sigma \in \mathfrak{S}_{n+1}$ and $\sigma(1) = 1$. (See Figure 4.3.) Moreover, x is a Lyndon node of T if and only if $R(x) = \sigma(2)$, or $R(x) = \sigma(i)$ where $3 \leq i \leq n+1$ and $\sigma(i-1) > \sigma(i)$.

Let σ_T be the permutation in $\mathfrak{S}_n$ defined by letting $\sigma_T(i) = \sigma(i+1) - 1$ for all $i \in [n-1]$. It follows that $T \in \widehat{\mathcal{N}_{St_{n+1}}}$ if and only if σ_T has no consecutive ascents. Thus by Theorems 3.3 and 4.4,

$$\tilde{A}_n(t) = \sum_{\sigma \in \widehat{\mathfrak{S}}_n} t^{\operatorname{asc}(\sigma)}(1+t)^{n-2\operatorname{asc}(\sigma)}, \tag{4.8}$$

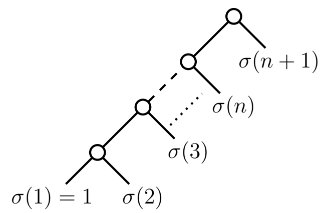

FIGURE 4.3. Trees in $\mathcal{N}_{St_{n+1}}$

where $\widehat{\mathfrak{S}}_n$ is the set of permutations in $\mathfrak{S}_n$ with no consecutive ascents and $\mathrm{asc}(\sigma)$ is the number of ascents of σ. This combinatorial description of the γ-coefficients of $\tilde{A}_n(t)$ was previously obtained in [38], along with a q-analog and a symmetric function analog.

Note that the Postnikov-Reiner-Williams formula (4.4) yields a different combinatorial description of the γ-coefficients. Equating the two descriptions yields

$$|\{\sigma \in \widehat{\mathfrak{S}}_n \,|\, \mathrm{asc}(\sigma) = j\}| =$$

$$|\{\sigma \in \widehat{\mathfrak{S}}_{n+1} | \sigma(n) > \sigma(n{+}1), \sigma(i) > \sigma(i{+}1) \forall i \in [\sigma^{-1}(1){-}1], \mathrm{asc}(\sigma) = j\}|.$$

A nice bijective proof of this was obtained by Ellzey [15].

We observe that (4.6) (without the chordality assumption) is a consequence of Conjecture 3.5, which as was noted above was proved in [2, 3]. Indeed this follows from the fact that the h-polynomial of any convex simple polytope has positive coefficients. We think it likely that other results discussed above also hold without the chordality assumption. Item (1) of the following conjecture has been tested by computer for connected graphs with up to 7 vertices.

Conjecture 4.12. *Let G be a graph with n vertices and k connected components. Then*

(1) $(-1)^{n-k}\mu_G(t)$ *is γ-positive.*
(2) *If H is a subgraph of G with n vertices and k connected components then*

$$(-1)^{n-k}(\mu_G(t) - \mu_H(t))$$

is γ-positive.

It is well known that all real-rooted palindromic polynomials with nonnegative real coefficients are γ-positive; see [10]. Since by Proposition 2.4, $\mu_G(t)$ is palindromic for all G, we conjecture the following.

Conjecture 4.13. *Let G be a graph with n vertices and k connected components. Then*

(1) *$(-1)^{n-k}\mu_G(t)$ is real-rooted.*
(2) *If H is a subgraph of G with n vertices and k connected components then*
$$(-1)^{n-k}(\mu_G(t) - \mu_H(t))$$
is real-rooted.

We have verified (1) by computer for all connected graphs with up to 7 vertices. It follows from (3.4) that (1) holds for the complete graph. It is known that the Narayana polynomials are real-rooted (see [33, Exercise 4.7]) and more recently, Haglund and Zang [27] established real-rootedness, for the binomial-Eulerian polynomials. Avila, Carrillo, and the first author prove (1) in the case of the cycle graph in [2, 3]. Hence by Theorems 3.2 and 3.3, and Proposition 2.2, item (1) is valid for all graphs whose connected components are complete graphs, path graphs, star graphs, or cycle graphs.

Next we consider the polynomial $(-1)^{n-k}\mu_G(t) - h_{\mathcal{P}_G}(t)$ of Conjecture 3.5. (As noted above this conjecture has now been proved in [2, 3].) Palindromicity of this polynomial follows from the fact that $\mu_G(t)$ and $h_{\mathcal{P}_G}(t)$ are both palindromic with the same center of symmetry. Note that this polynomial fails to be γ-positive already for $G = K_3$. Indeed, $\mu_{K_3}(t) - h_{\mathcal{P}_{K_3}}(t) = 1 + t + t^2$. However, note that the weaker unimodality property still holds for K_3. In fact, as we see in the next result, it holds for all complete graphs.

Theorem 4.14. *For all $n \geq 1$, the polynomial $(-1)^{n-1}\mu_{K_n}(t) - h_{\mathcal{P}_{K_n}}(t)$ is palindromic and unimodal. Moreover, $(-1)^{n-1}\mu_{K_n}(t) - h_{\mathcal{P}_{K_n}}(t) \succ 0$.*

Proof. By (3.5), (3.8) and (3.16), we have
$$(-1)^{n-k}\mu_{K_n}(t) - h_{\mathcal{P}_{K_n}}(t) = \sum_{T \in \mathcal{UT}_n} \sum_{w \in L(T)} t^{\mathrm{des}(T,w)} - \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)},$$
where $\mathcal{UT}_n$ is the set of unlabeled nonplanar rooted trees with n nodes and $L(T)$ is the set of labelings of T. Since permutations can be viewed as labeled linear nonplanar rooted trees, we have that
$$(-1)^{n-k}\mu_{K_n}(t) - h_{\mathcal{P}_{K_n}}(t) = \sum_{T \in \overline{\mathcal{UT}}_n} \sum_{w \in L(T)} t^{\mathrm{des}(T,w)},$$
where $\overline{\mathcal{UT}}_n$ is the set of *nonlinear* unlabeled nonplanar rooted trees with n nodes. Now for each $T \in \mathcal{UT}_n$, we claim that $\sum_{w \in L(T)} t^{\mathrm{des}(T,w)}$ is palindromic and unimodal. Indeed, this follows from [25, Lemma 2.2 and Theorem 2.3], which gives the claimed result for rooted *planar* trees. Since the polynomial $\sum_{w \in L(T)} t^{\mathrm{des}(T,w)}$ when T is planar is a constant multiple of the polynomial when T is nonplanar, we can conclude

that our claim holds. Since every tree has both an increasing and a decreasing labeling, the polynomial $\sum_{w\in L(T)} t^{\mathrm{des}(T,w)}$ has strictly positive constant term and degree $n-1$ for all $T \in \mathcal{UT}_n$. The result now follows from the fact that the sum of palindromic unimodal polynomials with the same center of symmetry is palindromic and unimodal. □

The result for K_n leads us to make the following conjecture, which is stronger than Conjecture 3.5 and is still open. We have checked this by computer for all graphs with up to $n = 7$ vertices.

Conjecture 4.15. *For any graph G with n vertices and k connected components, the palindromic polynomial $(-1)^{n-k}\mu_G(t) - h_{\mathcal{P}_G}(t)$ is unimodal, and all its coefficients are positive.*

The following even stronger conjecture has also been checked by computer for all graphs with up to $n = 7$ vertices.

Conjecture 4.16. *For a graph G with n vertices and k connected components, the palindromic polynomial $(-1)^{n-k}\mu_G(t) - h_{\mathcal{P}_G}(t)$ is log-concave, in the sense of (7.5), and all its coefficients are positive.*

5. Multiweighted bond posets and Möbius symmetric functions

In [21] González D'León considers a generalization of the weighted partition poset based on the observation that a weighted block B^u of a weighted partition can be written as $B^{(u,|B|-1-u)}$ thereby using weights that are pairs of nonnegative integers rather than single nonnegative integers. To merge two bi-weighted blocks, $B_1^{(\mu(1),\mu(2))}$ and $B_2^{(\nu(1),\nu(2))}$, now means to form either the bi-weighted block $(B_1 \cup B_2)^{(\mu(1)+\nu(1)+1,\mu(2)+\nu(2))}$ or the biweighted block $(B_1 \cup B_2)^{(\mu(1)+\nu(1),\mu(2)+\nu(2)+1)}$. By using weights that are r-tuples instead of pairs, one gets the more general r-weighted partition poset studied in [21]. In this section we generalize the notion of weighted bond posets to r-weighted bond posets. This leads to a symmetric function generalization of the Möbius polynomial $\mu_G(t)$ and symmetric function generalizations of the results for $\mu_G(t)$ discussed in earlier sections.

5.1. **Preliminaries.** For $r > 0$, a *weak r-composition* of an integer $m \geq 0$ is an r-tuple $\nu = (\nu(1), \mu(2), \dots, \nu(r))$ of nonnegative integers whose sum $\sum_{i=1}^r \nu(i)$ equals m. We denote by $\mathrm{WComp}_{m,r}$ the set of weak r-compositions of m. For $\mu = (\mu(1), \dots, \mu(r)) \in \mathrm{WComp}_{m,r}$ and $\nu = (\nu(1), \dots, \nu(r)) \in \mathrm{WComp}_{n,r}$, let

$$\mu + \nu := (\mu(1) + \nu(1), \dots, \mu(r) + \nu(r)) \in \mathrm{WComp}_{m+n,r}.$$

We say $\mu \geq \nu$ if

$$\mu - \nu := (\mu(1) - \nu(1), \ldots, \mu(r) - \nu(r)) \in \mathrm{WComp}_{m-n,r}.$$

Let $\mathbf{0}$ denote the r-tuple of 0's and for each $i = 1, \ldots, r$, let $\mathbf{e_i}$ denote the r-tuple with a 1 in the ith component and 0's elsewhere.

By padding with 0's we can view weak r-compositions of m as s-compositions of m for $s > r$. Thus we can say

$$\mathrm{WComp}_{m,1} \subset \mathrm{WComp}_{m,2} \subset \mathrm{WComp}_{m,3} \subset \cdots,$$

and define

$$\mathrm{WComp}_m := \mathrm{WComp}_{m,\infty} := \bigcup_{r \geq 1} \mathrm{WComp}_{m,r}.$$

We can view WComp_m as the set of infinite sequences of nonnegative integers whose sum is m. We call these infinite sequences *weak* ∞-*compositions* of m or just *weak compositions* of m.

For $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, an r-*weighted partition* of a set S is defined to be a set of the form $\boldsymbol{\pi} = \{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_k^{\nu_k}\}$, where $\{B_1, B_2, ..., B_k\}$ is a partition of S and $\nu_i \in \mathrm{WComp}_{|B_i|-1,r}$ for all $i \in [k]$. In what follows, it will be convenient to let $[\infty]$ denote $\mathbb{Z}_{>0}$.

The *poset of* r-*weighted partitions* Π_S^r is the set of r-weighted partitions of S with order relation defined by

$$\{A_1^{\mu_1}, A_2^{\mu_2}, ..., A_j^{\mu_j}\} \leq \{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_k^{\nu_k}\}$$

if the following conditions hold:

- $\{A_1, A_2, ..., A_j\} \leq \{B_1, B_2, ..., B_k\}$ in Π_S
- if $B_m = A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_l}$ then

$$\nu_m \geq \mu_{i_1} + \mu_{i_2} + ... + \mu_{i_l}.$$

The covering relation $\lessdot$ is given by

$$\{A_1^{\mu_1}, A_2^{\mu_2}, ..., A_j^{\mu_j}\} \lessdot \{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_k^{\nu_k}\}$$

if and only if the following conditions hold:

- $\{A_1, A_2, \ldots, A_j\} \lessdot \{B_1, B_2, \ldots, B_k\}$ in Π_S
- if $B_m = A_h \cup A_i$, where $h \neq i$, then $\nu_m - (\mu_h + \mu_i) = \mathbf{e_s}$ for some $s \in [r]$.
- if $B_m = A_i$ then $\nu_m = \mu_i$.

Note that when $r = 1$, the r-weighted partition poset Π_S^r is isomorphic to Π_S and when $r = 2$, the r-weighted partition poset Π_S^r is isomorphic to the weighted partition poset $\mathcal{W}\Pi_S$. Indeed, the map taking the biweighted partition $B_1^{\nu_1}|\cdots|B_k^{\nu_k}$ to the single weighted partition $B_1^{\nu_1(1)}|\cdots|B_k^{\nu_k(1)}$ is an isomorphism from Π_S^2 to $\mathcal{W}\Pi_S$.

Let $\Pi_n^r := \Pi_{[n]}^r$. See Figure 5.1 for the Hasse diagram of Π_3^3.

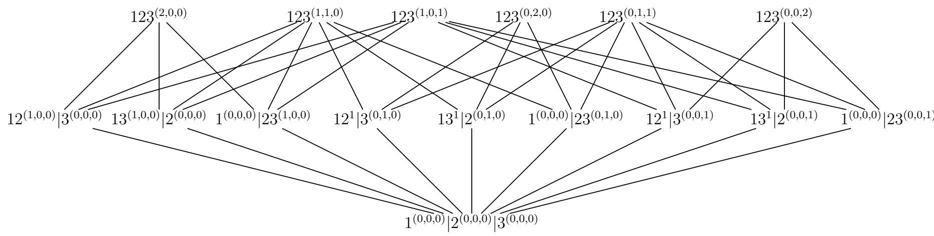

FIGURE 5.1. The poset of 3-weighted partitions Π_3^3

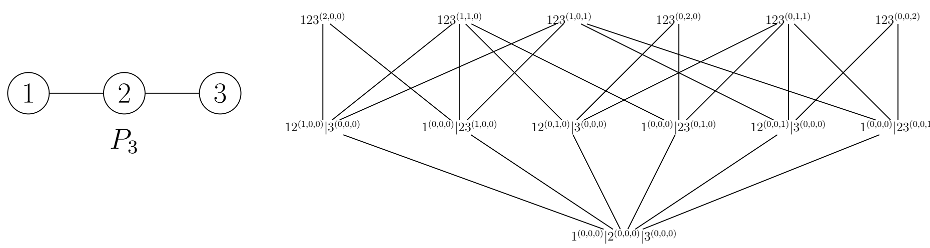

FIGURE 5.2. The 3-weighted bond poset $\Pi_{P_3}^3$

Remark 5.1. In [21] González D'León obtains results connecting the r-weighted partition poset Π_n^r to the free Lie algebra with r compatible brackets. These results generalize classical results in the $r = 1$ case, and results of both authors [22] in the $r = 2$ case.

Given a graph $G = (V, E)$ and $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, we define the *$r$-weighted bond poset* Π_G^r of G to be the induced subposet of Π_V^r consisting of r-weighted partitions $\{B_1^{\nu_1}, \dots, B_k^{\nu_k}\}$ whose underlying partition $\{B_1, \dots, B_k\}$ is in Π_G, that is, for each i, the induced subgraph $G|_{B_i}$ is connected. Note that Π_G^r is finite if and only if r is finite or $E = \emptyset$.

Clearly, for all r,

$$\Pi_{K_n}^r = \Pi_n^r.$$

Note that for all graphs G,

$$\Pi_G^1 \simeq \Pi_G \quad \text{and} \quad \Pi_G^2 \simeq \mathcal{W}\Pi_G.$$

See Figure 5.2 for the Hasse diagram of $\Pi_{P_3}^3$.

Just as for the $r = 2$ case discused in Section 2, for $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ in Π_G^r, let $[\boldsymbol{\alpha}, \boldsymbol{\beta}]_G$ denote the closed interval $\{\boldsymbol{\pi} \in \Pi_G^r : \boldsymbol{\alpha} \leq \boldsymbol{\pi} \leq \boldsymbol{\beta}\}$. When $G = (V, E)$ is connected, again we drop the set brackets for the maximal elements $\{V^\nu\}$ of Π_G^r.

For each graph $G = (V, E)$ with k connected components, and $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, the poset Π_G^r is pure of length $|V| - k$. It has minimum element $\hat{0} := v_1^{\mathbf{0}} | \cdots | v_n^{\mathbf{0}}$, where $V = \{v_1, \dots, v_n\}$, and if G is connected,

Π_G^r has maximal element $\{V^\nu\}$ (which we write as V^ν) for each $\nu \in \mathrm{WComp}_{n-1,r}$.

Proposition 2.1 generalizes to r-weighted bond posets Π_G^r.

Proposition 5.2. *Let G be a graph whose connected components are $G_1, \dots, G_k$. Then for all $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, the map*

$$\varphi : \Pi_{G_1}^r \times \cdots \times \Pi_{G_k}^r \to \Pi_G^r$$

defined by $\varphi(\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_k) = \boldsymbol{\pi}_1 \cup \cdots \cup \boldsymbol{\pi}_k$ is a poset isomorphism.

The following result is straightforward to prove.

Proposition 5.3. *Let G be a graph on vertex set V and let $r, s \in \mathbb{Z}_{>0} \cup \{\infty\}$.*

(1) *If $r < s$ then each maximal interval of Π_G^r is isomorphic to a maximal interval of Π_G^s*
(2) *Suppose $\alpha \in \mathrm{WComp}_{n-1,r}$ and $\beta \in \mathrm{WComp}_{n-1,s}$ have the same multiset of nonzero entries. If G is connected then the maximal intervals $[\hat{0}, V^\alpha]_G$ and $[\hat{0}, V^\beta]_G$ are isomorphic.*

5.2. **Möbius symmetric functions.** We are now ready to define the graph invariant that generalizes the Möbius polynomial defined in Section 2.

For any graph G, the *Möbius symmetric function* of G in infinitely many variables $\mathbf{x} = (x_1, x_2, \dots)$ is defined by

$$M_G(\mathbf{x}) := \sum_{\boldsymbol{\pi} \in \mathrm{Max}(\Pi_G^\infty)} \mu_{\Pi_G^\infty}(\hat{0}, \boldsymbol{\pi}) \mathbf{x}^{w(\boldsymbol{\pi})},$$

where, $\mathbf{x}^\nu = x_1^{\nu(1)} x_2^{\nu(2)} \cdots$ for $\nu \in \mathrm{WComp}_{m,\infty}$, and $w(\boldsymbol{\pi})$ is the sum of the weights of the blocks of $\boldsymbol{\pi}$. (Recall $\mathrm{Max}(P)$ is the set of maximal elements of a poset P.) It follows from Proposition 5.3 that $M_G(\mathbf{x})$ is indeed a symmetric function. Moreover, if G has n vertices and k connected components then $M_G(\mathbf{x})$ is a homogeneous symmetric function of degree $n - k$. This is in fact the highest degree homogeneous component of the symmetric function defined in Equation (7.1) which, as we observe in Section 7, is a symmetric function analog of the chromatic polynomial of G.

Note that if G is a connected graph on vertex set V then

$$M_G(\mathbf{x}) = \sum_{\nu \in \mathrm{WComp}_{|V|-1}} \mu_{\Pi_G^\infty}(\hat{0}, V^\nu) \mathbf{x}^\nu.$$

Moreover, it follows from Proposition 5.3 that by viewing partitions of m as weak compositions of m (by padding with zeros), we have for connected G,

$$M_G(\mathbf{x}) = \sum_{\lambda \vdash |V|-1} \mu_{\Pi_G^\infty}(\hat{0}, V^\lambda) m_\lambda(\mathbf{x}),$$

where m_λ is the monomial symmetric function indexed by the partition λ.

It also follows from Proposition 5.3 that for all r,

$$M_G(x_1, \dots, x_r, 0, 0, \dots) = \sum_{\boldsymbol{\pi} \in \mathrm{Max}(\Pi_G^r)} \mu_{\Pi_G^r}(\hat{0}, \boldsymbol{\pi}) x_1^{w(\boldsymbol{\pi})(1)} \cdots x_r^{w(\boldsymbol{\pi})(r)}.$$

For each r, define the *Möbius symmetric polynomial* $M_G(x_1, \dots, x_r)$ in r variables to be $M_G(x_1, \dots, x_r, 0, 0, \dots)$.

Note that $M_G(1)$ is the Möbius invariant $\mu(\Pi_G)$ of the bond lattice Π_G and $M_G(1, t)$ is the Möbius polynomial $\mu_G(t)$. From Figure 5.1, one can see that

$$M_{K_3}(x_1, x_2, x_3) = 2(x_1^2 + x_2^2 + x_3^2) + 5(x_1x_2 + x_1x_3 + x_2x_3)$$

and from Figure 5.2, one can see that

$$M_{P_3}(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2) + 3(x_1x_2 + x_1x_3 + x_2x_3).$$

The isomorphism given in Proposition 5.2 yields the following generalization of Proposition 2.2.

Proposition 5.4. *Let G be a graph whose connected components are $G_1, \dots, G_k$. Then*

$$M_G(\mathbf{x}) = \prod_{i=1}^{k} M_{G_i}(\mathbf{x}).$$

5.3. **Recurrence relations for Möbius symmetric functions.** The recursive structure of the weighted partition poset is described in the following propositions, whose straightforward proofs are left to the reader. Recall the definitions of $G|B$ and G/π given in Section 3.1. Equation (5.1) below is a consequence of the multiplicative property of the Möbius function (see [44, Proposition 3.8.2]).

Proposition 5.5. *Let G be a graph and let $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. If*

$$\boldsymbol{\pi} = \{B_1^{\nu_1}, \dots, B_k^{\nu_k}\} \in \Pi_G^r,$$

then

$$[\hat{0}, \boldsymbol{\pi}]_G \cong [\hat{0}, B_1^{\nu_1}]_{G|_{B_1}} \times \cdots \times [\hat{0}, B_k^{\nu_k}]_{G|_{B_k}}.$$

Consequently,

$$\mu_{\Pi_G^r}(\hat{0}, \boldsymbol{\pi}) = \prod_{i=1}^{k} \mu_{\Pi^r_{G|_{B_i}}}(\hat{0}, B_i^{\nu_i}). \tag{5.1}$$

For $\boldsymbol{\pi} \in \Pi_G^r$, let $U_G^r(\boldsymbol{\pi})$ be the upper order ideal (order filter) generated by $\boldsymbol{\pi}$. For connected G on vertex set V and $\nu \in \mathrm{WComp}_{|V|-1,r}$, the maximal elements of $U_G^r(\boldsymbol{\pi})$ are of the form V^ν, where

$$\nu - w(\boldsymbol{\pi}) \in \mathrm{WComp}_{|\boldsymbol{\pi}|-1,r}. \tag{5.2}$$

Proposition 5.6. *Let G be a connnected graph on vertex set V, let $r \in \mathbb{Z}_{>0} \cup \{\infty\}$ and let $\boldsymbol{\pi} = \{B_1^{\nu_1}, \ldots, B_k^{\nu_k}\} \in \Pi_G^r$. Then*

$$U_G(\boldsymbol{\pi}) \cong \Pi^r_{G/\{B_1,\ldots,B_k\}}.$$

Moreover, for all $\nu \in \mathrm{WComp}_{|V|-1,r}$ such that $\nu - w(\boldsymbol{\pi}) \in \mathrm{WComp}_{k-1,r}$, we have

$$[\boldsymbol{\pi}, V^\nu]_G \cong [\hat{0}, \{\{B_1, \ldots, B_k\}^{\nu - w(\boldsymbol{\pi})}\}]_{G/\{B_1,\ldots,B_k\}}.$$

Consequently,

$$\mu_{\Pi_G^r}(\boldsymbol{\pi}, V^\nu) = \mu_{\Pi^r_{G/\{B_1,\ldots,B_k\}}}(\hat{0}, \{\{B_1, \ldots, B_k\}^{\nu - w(\boldsymbol{\pi})}\}).$$

These recursive properties yield the following recurrence relations for the Möbius symmetric functions, which reduce to those for the Möbius polynomials given in Theorem 3.1. Indeed, $M_H(1,t) = \mu_H(t)$ for all graphs H and $h_m(1,t) = [m+1]_t$ for all $m \geq 0$.

Theorem 5.7. *Let G be a connected graph with more than one vertex. Then*

$$M_G(\mathbf{x}) = -\sum_{\pi \in \Pi_G \setminus \{\hat{1}\}} h_{|\pi|-1}(\mathbf{x}) \prod_{B \in \pi} M_{G|_B}(\mathbf{x}), \tag{5.3}$$

and

$$M_G(\mathbf{x}) = -\sum_{\pi \in \Pi_G \setminus \{\hat{0}\}} M_{G/\pi}(\mathbf{x}) \prod_{B \in \pi} h_{|B|-1}(\mathbf{x}), \tag{5.4}$$

where $h_n(\mathbf{x})$ is the complete homogenous symmetric function of degree n.

Proof. Let V be the vertex set of G.

By the recursive definition of the Möbius function, for $\nu \in \mathrm{WComp}_{|V|-1}$,

$$\sum_{\boldsymbol{\pi} \in [\hat{0}, V^\nu]_G} \mu_{\Pi_G^\infty}(\hat{0}, \boldsymbol{\pi}) = 0.$$

Therefore by (5.2),

$$
\begin{aligned}
0 &= \sum_{\nu \in \mathrm{WComp}_{|V|-1}} \mathbf{x}^{\nu} \sum_{\boldsymbol{\pi} \in [\hat{0}, V^{\nu}]_G} \mu_{\Pi_G^{\infty}}(\hat{0}, \boldsymbol{\pi}) \\
&= \sum_{\boldsymbol{\pi} \in \Pi_G^{\infty}} \mu_{\Pi_G^{\infty}}(\hat{0}, \boldsymbol{\pi}) \mathbf{x}^{w(\boldsymbol{\pi})} \sum_{\eta \in \mathrm{WComp}_{|\boldsymbol{\pi}|-1}} \mathbf{x}^{\eta} \\
&= \sum_{\boldsymbol{\pi} \in \Pi_G^{\infty}} \mu_{\Pi_G^{\infty}}(\hat{0}, \boldsymbol{\pi}) \mathbf{x}^{w(\boldsymbol{\pi})} \, h_{|\boldsymbol{\pi}|-1}(\mathbf{x}).
\end{aligned}
$$

Now by (5.1),

$$
\begin{aligned}
0 &= \sum_{k=1}^{n} h_{k-1}(\mathbf{x}) \sum_{\{B_1,\dots,B_k\} \in \Pi_G} \sum_{\substack{\nu_1,\nu_2,\dots,\nu_k \\ \nu_i \in \mathrm{WComp}_{|B_i|-1}}} \prod_{i=1}^{k} \mu_{\Pi_{G|_{B_i}}^{\infty}}(\hat{0}, B_i^{\nu_i}) \mathbf{x}^{\nu_i} \\
&= \sum_{k=1}^{n} h_{k-1}(\mathbf{x}) \sum_{\{B_1,\dots,B_k\} \in \Pi_G} \prod_{i=1}^{k} \sum_{\nu \in \mathrm{WComp}_{|B_i|-1}} \mu_{\Pi_{G|_{B_i}}^{\infty}}(\hat{0}, B_i^{\nu}) \mathbf{x}^{\nu} \\
&= \sum_{k=1}^{n} h_{k-1}(\mathbf{x}) \sum_{\{B_1,\dots,B_k\} \in \Pi_G} \prod_{i=1}^{k} M_{G|_{B_i}}(\mathbf{x}),
\end{aligned}
$$

which proves (5.3).

By the dual version of the recursive definition of the Möbius function, and (5.2),

$$
\begin{aligned}
0 &= \sum_{\nu \in \mathrm{WComp}_{|V|-1}} \mathbf{x}^{\nu} \sum_{\boldsymbol{\pi} \in [\hat{0}, V^{\nu}]_G} \mu_{\Pi_G^{\infty}}(\boldsymbol{\pi}, V^{\nu}) \\
&= \sum_{\boldsymbol{\pi} \in \Pi_G^{\infty}} \mathbf{x}^{w(\boldsymbol{\pi})} \sum_{\nu \in \mathrm{WComp}_{|\boldsymbol{\pi}|-1}} \mu_{\Pi_G^{\infty}}(\boldsymbol{\pi}, V^{\nu + w(\boldsymbol{\pi})}) \mathbf{x}^{\nu}.
\end{aligned}
$$

Now by Proposition 5.6,

$$
\begin{aligned}
0 &= \sum_{k=1}^{n} \sum_{\pi = \{B_1,\dots,B_k\} \in \Pi_G} \sum_{\substack{\nu_1,\nu_2,\dots,\nu_k \\ \nu_i \in \mathrm{WComp}_{|B_i|-1}}} \mathbf{x}^{\sum_{i=1}^{k} \nu_i} \sum_{\nu \in \mathrm{WComp}_{k-1}} \mu_{\Pi_{G/\pi}^{\infty}}(\hat{0}, V(G/\pi)^{\nu}) \mathbf{x}^{\nu} \\
&= \sum_{\pi \in \Pi_G} \prod_{B \in \pi} \sum_{\nu \in \mathrm{WComp}_{|B|-1}} \mathbf{x}^{\nu} \; M_{G/\pi}(\mathbf{x}),
\end{aligned}
$$

which proves (5.4). □

5.4. **Examples.** In [21], González D'León uses the complete graph case of the recurrence relations of Theorem 5.7 to obtain the exponential generating function formula

$$\sum_{n\geq 1} M_{K_n}(\mathbf{x})\frac{y^n}{n!} = \left(\sum_{n\geq 1} h_{n-1}(\mathbf{x})\frac{y^n}{n!}\right)^{\langle -1\rangle},$$

which is a symmetric function generalization of (3.3). In [21] it is shown that the coefficient of $\mathbf{x}^\nu$ in $(-1)^{n-1}M_{K_n}(\mathbf{x})$ is the dimension of a subspace of the multilineal component, associated with ν, of the free Lie algebra with multiple compatible brackets.

The recurrence relations of Theorem 5.7 in the case of the star graph reduce to

$$\sum_{k=0}^{n-1} \binom{n-1}{k} h_k(\mathbf{x})\, M_{St_{n-k}}(\mathbf{x}) = \delta_{n,1}.$$

This was used by González D'León [23] to obtain the exponential generating function formula

$$\sum_{n\geq 0} M_{St_{n+1}}(\mathbf{x})\frac{y^n}{n!} = \left(\sum_{n\geq 0} h_n(\mathbf{x})\frac{y^n}{n!}\right)^{-1}, \tag{5.5}$$

which is a symmetric function generalization of (3.10). It is shown in [23] that the coefficient of $\mathbf{x}^\nu$ in $(-1)^{n-1}M_{St_n}(\mathbf{x})$ is the dimension of a subspace of the multilinear component, associated with ν, of the colored exterior algebra.

Remark 5.8. The r-weighted bond poset $\Pi^r_{St_n}$ appeared in the guise of the r-weighted Boolean algebra in the work of González D'León [23] on colored exterior algebras. For each $\boldsymbol{\pi} \in \Pi^r_{St_n}$, let B^ν be the weighted block of $\boldsymbol{\pi}$ that contains 1. Now remove 1 from B and reduce the remaining elements by 1 to get the set $\bar{B}$. The map taking $\boldsymbol{\pi}$ to $\bar{B}^\nu$ defines an isomorphism from $\Pi^r_{St_n}$ to the r-weighted Boolean algebra consisting of weighted sets S^ν, where $S \subseteq [n-1]$ and $\nu \in \mathrm{WComp}_{|S|,r}$. Note that in the $r = 2$ case, this r-weighted Boolean algebra is a variant of the weighted Boolean algebra $\mathcal{WB}_n$ mentioned in the Introduction.

Next we turn to the path graph P_n. By using the recurrence relations of Theorem 5.7, we will show that the Möbius symmetric functions for the path graphs are related to a class of symmetric functions known as parking function symmetric functions.

A *parking function* on $[n]$ is a sequence $(p_1, \dots, p_n) \in [n]^n$ whose weakly increasing rearrangement $p_{\sigma(1)} \leq \cdots \leq p_{\sigma(n)}$ satisfies $p_{\sigma(i)} \leq i$ for all $i \in [n]$; see [43, Exercise 5.49]. Let $PF_n(\mathbf{x})$ denote the Frobenius characteristic of the permutation representation associated with the action of $\mathfrak{S}_n$ on the set of parking functions on $[n]$, where a permutation acts by permuting the entries of the parking function. The symmetric function $PF_n(\mathbf{x})$ is known as the *parking function symmetric function* and was first considered in this context by Haiman [28]. In [42] Stanley observed that

$$PF_n(\mathbf{x}) = \text{ coeff. of } y^n \text{ in } \frac{1}{n+1}\left(\sum_{j\geq 0} h_j(\mathbf{x})y^j\right)^{n+1}. \tag{5.6}$$

Expansions of $\omega PF_n(\mathbf{x})$ in the monomial symmetric function basis $\{m_\lambda : \lambda \vdash n\}$ and in the elementary symmetric function basis $\{e_\lambda : \lambda \vdash n\}$ can be found in [42] as well, where ω is the involution on the ring of symmetric functions that takes h_n to e_n. Given $\pi \in \Pi_n$, let $\lambda(\pi)$ be the partition of n given by

$$\lambda(\pi) = (|B_1|, |B_2|, \dots, |B_k|), \tag{5.7}$$

where $B_1, \dots, B_k$ are the blocks of π listed in weakly decreasing order of size. For all $n \geq 1$,

$$\omega PF_n(\mathbf{x}) = \frac{1}{n+1}\sum_{\lambda\vdash n}\prod_i \binom{n+1}{\lambda(i)} m_\lambda(\mathbf{x}) \tag{5.8}$$

and

$$\omega PF_n(\mathbf{x}) = \sum_{\pi\in\mathcal{NC}_n} e_{\lambda(\pi)}(\mathbf{x}), \tag{5.9}$$

where $\mathcal{NC}_n$ is the set of noncrossing partitions of $[n]$.

Theorem 5.9. *For all $n \geq 1$,*

$$(-1)^{n-1}M_{P_n}(\mathbf{x}) = \omega PF_{n-1}(\mathbf{x}).$$

Proof. The recurrence relations of Theorem 5.7 in the case of $G = P_n$ reduce to

$$\sum_{k=1}^{n} h_{k-1}(\mathbf{x}) \sum_{(n_1,\dots,n_k)\models n} \prod_{i=1}^{k} M_{P_{n_i}}(\mathbf{x}) = \delta_{n,1}$$

and

$$\sum_{k=1}^{n} M_{P_k}(\mathbf{x}) \sum_{(n_1,\dots,n_k)\models n} \prod_{i=1}^{k} h_{n_i-1}(\mathbf{x}) = \delta_{n,1}.$$

Both equations are equivalent to the generating function formula

$$\sum_{n\geq 1}(-1)^{n-1}M_{P_n}(\mathbf{x})y^n = \left(\sum_{n\geq 1}(-1)^{n-1}h_{n-1}(\mathbf{x})\, y^n\right)^{\langle -1\rangle} = \left(\frac{y}{\prod_{i\geq 1}(1+x_iy)}\right)^{\langle -1\rangle},$$

which is a symmetric function analog of (3.7). Following the proof of [42, Proposition 2.2 (b)], by the Lagrange inversion formula (see [43, Theorem 5.4.2]), we thus have that $M_{P_n}(\mathbf{x})$ is equal to the coefficient of y^{n-1} in

$$\frac{1}{n}\left(\prod_{i\geq 1}(1+x_iy)\right)^n = \frac{1}{n}\left(\sum_{j\geq 0} e_j(\mathbf{x})y^j\right)^n.$$

By (5.6), this coefficient is precisely $\omega PF_{n-1}(\mathbf{x})$, which yields the desired result. $\square$

We have the following consequence of Theorem 5.9 and Equation (5.8).

Corollary 5.10. *For all $r, n \geq 1$,*

$$(-1)^{n-1}M_{P_n}(x_1,\ldots,x_r) = \sum_{\nu\in \mathrm{WComp}_{n-1,r}} \frac{1}{n}\prod_{i=1}^{r}\binom{n}{\nu(i)} x_1^{\nu(1)}\cdots x_r^{\nu(r)}.$$

The coefficients $N_\nu := \frac{1}{n}\prod_{i=1}^r \binom{n}{\nu(i)}$ are sometimes called *generalized Fuss-Narayana numbers*, in part, because they refine the Fuss-Narayana numbers $N_r(n,j) := \frac{1}{n}\binom{n}{j}\binom{(r-1)n}{n-j-1}$, which in turn refine the Fuss-Catalan numbers $C_{n,r} := \frac{1}{n}\binom{rn}{n-1}$. Indeed Vandermonde's identity yields,

$$\sum_{\substack{\nu\in\mathrm{WComp}_{n-1,r}\\ \nu(r)=j}} N_\nu = N_r(n,j)$$

and

$$\sum_{j=0}^{n-1} N_r(n,j) = C_{n,r}.$$

It therefore follows from Corollary 5.10 that

$$(-1)^{n-1}M_{P_n}(\overbrace{1,1,\ldots,1}^{r}) = C_{n,r}.$$

Note that $N_r(n,j)$ is the usual Narayana number $N(n,j)$ and $C_{n,r}$ is the usual Catalan number C_n when $r=2$. Thus Corollary 5.10 reduces to Theorem 3.2 when $r=2$.

Another consequence of Theorem 5.9 is that .$(-1)^{n-1}M_{P_n}(\mathbf{x})$ is e-positive, that is when expanded in the basis of elementary symmetric functions, all the coefficients are nonnegative. Indeed, by (5.9),

$$(-1)^{n-1}M_{P_n}(\mathbf{x}) = \sum_{\pi\in\mathcal{NC}_{n-1}} e_{\lambda(\pi)}(\mathbf{x}). \tag{5.10}$$

In the next section we establish e-positivity for all chordal graphs and conjecture that it holds for all graphs.

6. EL-shellability, e-positivity, and γ-positivity

In this section, we prove e-positivity results for the Möbius symmetric functions. These results reduce to the γ-positivity results for the Möbius polynomials discussed in Section 4.2. Our proofs rely on the theory of lexicographic shellability, which was first introduced by Björner in [4] and further devoloped by Björner and Wachs in [5, 6, 7, 8]. We briefly review the basics in Section 6.1. See [46] for further information on this topic.

6.1. Review: EL labeling of the multiweighted partition poset.

An *edge-labeling* of a finite bounded poset P is a function $\lambda : \mathcal{E}(P) \to \Lambda$ from the set $\mathcal{E}(P)$ of cover relations of P to some other poset Λ. An edge labeling $\lambda : \mathcal{E}(P) \to \Lambda$ of P is said to be an *EL-labeling* if it satisfies the following conditions:

(1) In every closed interval $[x, y]$ of P there is a unique *(strictly) increasing maximal chain*, i.e., a maximal chain $\mathbf{c} := (x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_\ell = y)$ such that

$$\lambda(x_0 \lessdot x_1) < \lambda(x_1 \lessdot x_2) < \cdots < \lambda(x_{\ell-1} \lessdot x_\ell).$$

(2) The *word of labels*

$$\lambda(\mathbf{c}) := \lambda(x_0 \lessdot x_1)\lambda(x_1 \lessdot x_2) \cdots \lambda(x_{\ell-1} \lessdot x_\ell)$$

associated to the unique increasing maximal chain $\mathbf{c}$ comes, in the lexicographic order, before the word of labels of any other maximal chain in the interval.

If P admits an EL-labeling, we say that it is *edge lexicographically shellable (EL-shellable)*. Note that the EL-labeling restricts to an EL-labeling of every closed interval of P. A maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_\ell$ of a closed interval of P is said to be *ascent-free* if its word of labels has no ascents, that is, for all $i \in [\ell - 1]$ we have $\lambda(x_{i-1} \lessdot x_i) \not< \lambda(x_i \lessdot x_{i+1})$.

For any maximal chain $\mathbf{c}$ of a closed interval $[x, y]$, let $\bar{\mathbf{c}} := \mathbf{c} - \{x, y\}$. A chain $x_0 < x_1 < \cdots < x_\ell$ of length ℓ in P will be referred to as an *ℓ-chain*.

EL-shellability has strong topological and algebraic implications for the order complex $\Delta(x, y)$ of each open interval (x, y) of the poset. This will be discussed in Section 6.5, while here we only need its connection with the Möbius function as given in the following result.

Theorem 6.1 (Stanley [39], cf. Björner [4, Theorem 2.7])**.** *Let λ be an EL-labeling of a pure bounded finite poset P. Then for every $x \leq_P y$ in P,*

$$\mu(x,y) = (-1)^{\ell(x,y)}|\{\bar{\mathbf{c}} \mid \mathbf{c} \textit{ is an ascent-free maximal chain of } [x,y]\}|.$$

Remark 6.2. This theorem is implicit in [39], where a less stringent labeling, now known as an ER-labeling (or R-labeling), was used. EL-shellabillity was introduced later in [4] in order to obtain stronger topological conclusions; see Theorem 6.27. In both papers the increasing maximal chains were only required to be weakly increasing and the ascents were defined as weak ascents. The theorem is valid either way; see [46, Remark 3.25].

Recall that the poset Π_G^r is pure, but not bounded. Hence we consider the *augmented r-weighted bond poset* $\widehat{\Pi_G^r} := \Pi_G^r \cup \{\hat{1}\}$, which is pure and bounded. We now present an EL-labeling of $\widehat{\Pi_{K_n}^r} = \widehat{\Pi_n^r}$ obtained by González D'León and Wachs [22] in the $r = 2$ case and by González D'León [21] for general $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. Note that the definition of EL-labeling makes sense not only for finite bounded posets, but also for *infinite* bounded posets in which all maximal chains are finite. Thus we can extend the definition of EL-labeling to such posets, which include the poset $\widehat{\Pi_n^\infty}$. Note also that all the closed intervals of Π_n^∞ are finite. So the restriction of any EL-labeling of $\widehat{\Pi_n^\infty}$ to any closed interval of Π_n^∞ is an EL-labeling in the traditional sense.

For fixed $r \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $a \in [n]$, let

$$\Gamma_{a,r} = \{(a,b,j) \mid a < b \leq n+1 \text{ and } j \in [r]\}$$

with order relation given by $(a,b,j) \leq (a,b',j')$ whenever $b \leq b'$ and $j \leq j'$. (Here $[\infty] := \mathbb{Z}_{>0}$.) We define then $\Lambda_{n,r}$ as the ordinal sum

$$\Lambda_{n,r} = \Gamma_{1,r} \oplus \Gamma_{2,r} \oplus \cdots \oplus \Gamma_{n,r}.$$

See Figure 6.1 for the Hasse diagrams of $\Lambda_{3,2}$ and $\Lambda_{2,3}$.

Recall that if $\boldsymbol{\alpha} \lessdot \boldsymbol{\beta}$ in Π_n^r then $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\alpha}$ by merging two r-weighted blocks A^μ and B^ν to obtain the r-weighted block $(A \cup B)^{\mathbf{e}_j+\mu+\nu}$, where $j \in [r]$. Assume $\min A < \min B$ and let

$$\lambda(\boldsymbol{\alpha} \lessdot \boldsymbol{\beta}) = (\min A, \min B, j).$$

For all $\nu \in \mathrm{WComp}_{n-1,r}$, let

$$\lambda([n]^\nu \lessdot \hat{1}) = (1, n+1, 1).$$

This defines an edge labeling $\lambda : \mathcal{E}(\widehat{\Pi_n^r}) \to \Lambda_{n,r}$.

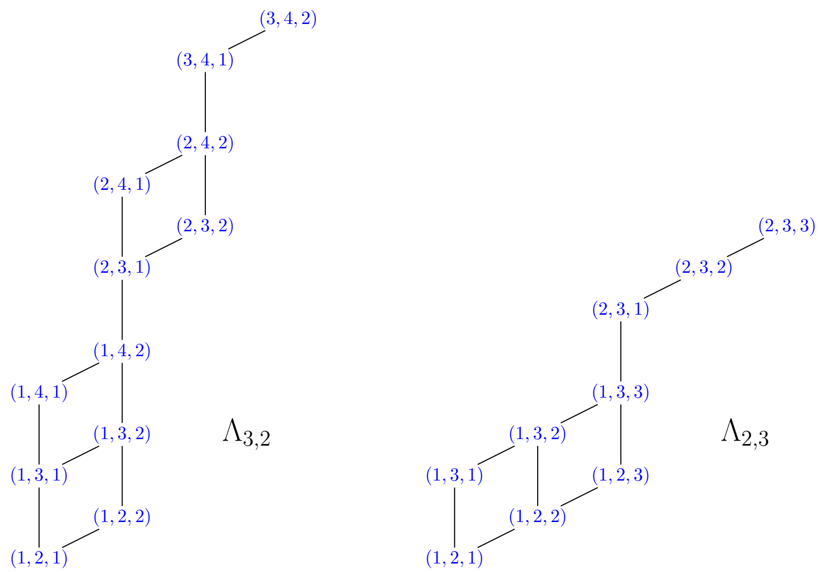

FIGURE 6.1. Posets of labels $\Lambda_{3,2}$ and $\Lambda_{2,3}$

Theorem 6.3 ([21], [22] r=2 case)**.** *For all $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, the edge labeling $\lambda : \mathcal{E}(\widehat{\Pi_n^r}) \to \Lambda_{n,r}$ defined above is an EL-labeling of $\widehat{\Pi_n^r}$.*

Given any chain $\mathbf{c}$ of Π_n^r, the underlying partitions of the weighted partitions of $\mathbf{c}$ form a chain c in Π_n, which we call the *underlying chain* of $\mathbf{c}$. We now describe the underlying chain of the unique increasing maximal chain of each interval of Π_n^r under the EL-labeling λ.

Definition 6.4. Let $\alpha < \beta \in \Pi_n$ and let

$$B_1, \dots, B_m$$

be the blocks of β, listed in increasing order of their minimum elements. For each $i = 1, \dots, m$, let

$$A_{i,1}, \dots, A_{i,k_i}$$

be the blocks of α that are merged to form the block B_i, also listed in increasing order of their minimums. Now define the *fundamental chain* $c(\alpha, \beta)$ of the interval $[\alpha, \beta]$ to be the concatenation of the chains $c_0, c_1, \dots, c_m$, where c_0 is the chain consisting only of the partition α and for each $i \geq 1$, the chain c_i is

$$(A_{i,1} \cup A_{i,2}) \lessdot_{\Pi_n} (\bigcup_{j=1}^{3} A_{i,j}) \lessdot_{\Pi_n} \cdots \lessdot_{\Pi_n} (\bigcup_{j=1}^{k_i} A_{i,j}) = B_i,$$

where only the newly merged block is listed. In other words, c_i is the chain

$$\pi_{i,1} \lessdot_{\Pi_n} \pi_{i,2} \lessdot_{\Pi_n} \cdots \lessdot_{\Pi_n} \pi_{i,k_i-1}$$

where for each $p \in [k_i - 1]$, the partition $\pi_{i,p}$ is equal to

$$\{B_1, \dots, B_{i-1}\} \cup \{\bigcup_{j=1}^{p+1} A_{i,j}, A_{i,p+2}, \dots, A_{i,k_i}\} \cup \bigcup_{j=i+1}^{m} \{A_{j,1}, \dots, A_{j,k_j}\}.$$

The following result is extracted from the proof of Theorem 6.3 given in [21, 22].

Lemma 6.5. *Let* $\boldsymbol{\alpha} < \boldsymbol{\beta} \in \widehat{\Pi_n^r}$.

(1) *If* $\boldsymbol{\beta} \neq \hat{1}$ *then the underlying chain of the unique increasing maximal chain of* $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ *is the fundamental chain* $c(\alpha, \beta)$, *where* α *is the underlying partition of* $\boldsymbol{\alpha}$ *and* β *is the underlying partition of* $\boldsymbol{\beta}$.
(2) *Suppose* $\boldsymbol{\beta} = \hat{1}$. *Then the unique increasing maximal chain of* $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ *is the concatenation of the unique increasing maximal chain of* $[\boldsymbol{\alpha}, [n]^{w(\boldsymbol{\alpha})+(|\boldsymbol{\alpha}|-1)\mathbf{e}_1}]$ *and* $\hat{1}$, *where* $w(\boldsymbol{\alpha})$ *is the sum of the weights of the blocks of* $\boldsymbol{\alpha}$.

6.2. **Review: Lyndon-colored trees and ascent-free chains.** In this section we present the description given in [21, 22] of the ascent-free maximal chains of each maximal interval $[\hat{0}, [n]^\nu]$ of Π_n^r under the EL-labeling λ of Theorem 6.3 in terms of certain leaf labeled binary trees with colored internal nodes. This, by Corollary 6.1, gives a combinatorial description of the coefficients of the Möbius symmetric function.

Recall that for each r, each maximal interval $[\hat{0}, [n]^\nu]$ of Π_n^r can be viewed as a maximal interval of Π_n^∞ by viewing the weak r-compositions as weak ∞-compositions. Hence we can restrict ourselves to $r = \infty$ in what follows.

Recall the discussion of leaf-labeled binary trees given in Section 4.2. Now we add *color* to the internal nodes of these trees. Let $\mathbf{int}(T)$ be the set of internal nodes of a binary tree T. A coloring of a binary tree T is a function $\mathbf{clr} : \mathbf{int}(T) \to \mathbb{Z}_{>0}$ that assigns a positive integer or *color* to every internal node of T. For each colored leaf-labeled binary tree T, define the *color type* of T to be the weak ∞-composition $\nu(T) = (\nu(1), \nu(2), \cdots)$, where $\nu(i)$ is the number of internal nodes of T that have color i. For each $\nu \in \mathrm{WComp}_{n-1}$, let $\mathcal{CT}_{n,\nu}$ denote the set of colored binary trees on leaf set $[n]$ that have color type ν.

Recall the discussion of normalized leaf-labeled binary tree and Lyndon nodes in Section 4.2. A *Lyndon-coloring* of a normalized leaf-labeled binary tree T is a coloring $\mathbf{clr}$ that satisfies: for all internal

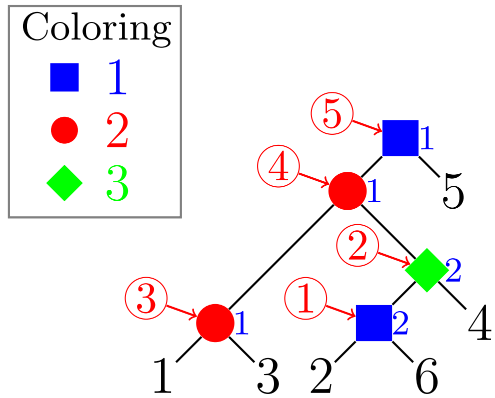

FIGURE 6.2. Example of a Lyndon-colored binary tree in $\mathcal{L}yn_{6,(2,2,1)}$

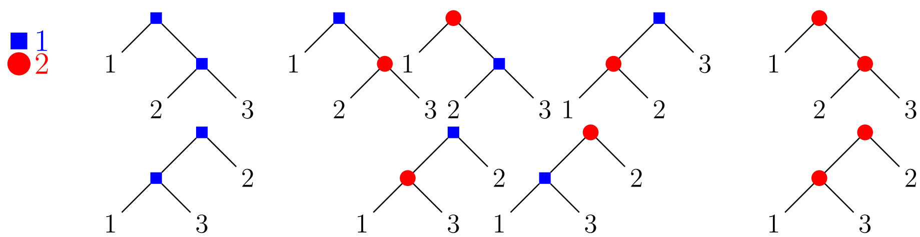

FIGURE 6.3. The Lyndon-colored trees on leaf set [3] with colors in [2].

nodes x of T whose left child $L(x)$ is not a leaf,

$$\min A_{R(L(x))} < \min A_{R(x)} \implies \mathbf{clr}(L(x)) > \mathbf{clr}(x).$$

In other words, each nonLyndon internal node must be assigned a color that is less than the color assigned to its left child. Thus only Lyndon trees can have a monochromatic Lyndon-coloring. For $\nu \in \mathrm{WComp}_{n-1}$, let $\mathcal{L}yn_{n,\nu}$ denote the set of Lyndon-colored normalized binary trees on leaf set $[n]$ that have color type ν.

An example of a Lyndon-colored normalized binary tree in $\mathcal{L}yn_{6,(2,2,1)}$ is given in Figure 6.2. The small number next to each internal node x is $\min A_x$ and the circled numbers give an ordering of the internal nodes which will be explained below. In Figure 6.3, all the Lyndon-colored normalized binary trees on leaf set [3] with colors in [2] are listed. From this we see that

$$|\mathcal{L}yn_{3,(2,0)}| = 2 \quad \text{and} \quad |\mathcal{L}yn_{3,(1,1)}| = 5 \quad \text{and} \quad |\mathcal{L}yn_{3,(0,2)}| = 2$$

Binary trees on leaf set $[n]$ are used in [45] to encode maximal chains of $\Pi_n = \Pi_n^1$ and bicolored binary trees on leaf set $[n]$ are used in [22]

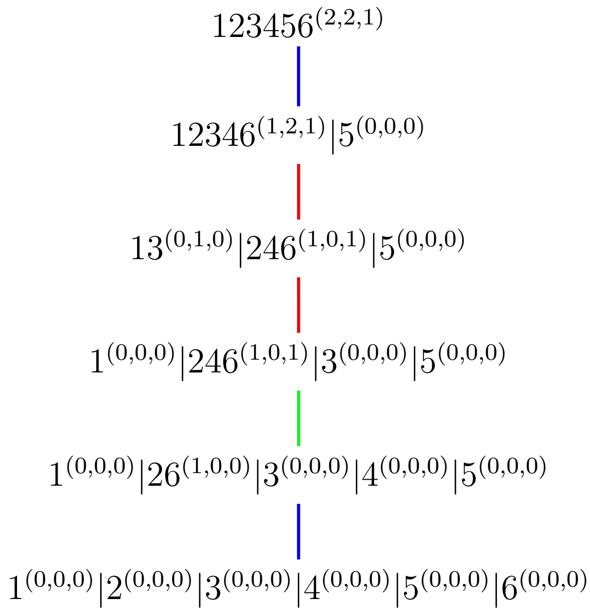

FIGURE 6.4. The chain $c(\mathbf{T},\tau)$ for the colored leaf-labeled tree $\mathbf{T}$ and linear extension τ of Figure 6.2.

to encode maximal chains of $\mathcal{W}\Pi_n = \Pi_n^2$. The general r-colored binary trees on leaf set $[n]$ are used in [21] to encode maximal chains of Π_n^r. Now we describe how this is done.

We call an ordering of the internal nodes of a binary tree T in which each node appears before its parent, a *linear extension* of T. For each $\mathbf{T} \in \mathcal{CT}_{n,\nu}$ and linear extension $\tau = (x_1, \dots, x_{n-1})$ of $\mathbf{T}$, we can construct a maximal chain

$$c(\mathbf{T},\tau) = (\hat{0} = \boldsymbol{\pi}_0 \lessdot \boldsymbol{\pi}_1 \lessdot \cdots \lessdot \boldsymbol{\pi}_{n-1} = [n]^\nu)$$

of $[\hat{0}, [n]^\nu]$ such that for all $i \in [n-1]$, the weighted partition $\boldsymbol{\pi}_i$ is obtained from the weighted partition $\boldsymbol{\pi}_{i-1}$ by merging the weighted blocks $A^\alpha_{L(x_i)}$ and $A^\beta_{R(x_i)}$ of $\boldsymbol{\pi}_{i-1}$ to obtain the weighted block $A_{x_i}^{\alpha+\beta+\mathbf{e}_{\mathbf{clr}(x_i)}}$ of $\boldsymbol{\pi}_i$. In Figure 6.4 we illustrate the chain $c(\mathbf{T},\tau)$ associated to the colored leaf-labeled tree $\mathbf{T}$ of Figure 6.2 and the linear extension τ given by the circled numbers.

Note that every maximal chain of $[\hat{0}, [n]^\nu]$ can be represented as $c(\mathbf{T},\tau)$ for some $\mathbf{T} \in \mathcal{CT}_{n,\nu}$ and linear extension τ of $\mathbf{T}$. However, the pair $(\mathbf{T},\tau)$ is not unique unless we restrict to normalized trees in $\mathcal{CT}_{n,\nu}$. It is not difficult to see that for each binary tree T on leaf set

$[n]$, there is a unique linear extension $(x_1, \ldots, x_{n-1})$ of T such that

$$\min A_{x_1} \geq \min A_{x_2} \geq \cdots \geq \min A_{x_{n-1}}.$$

We will call this linear extension the *decreasing-minimum* linear extension of T and denote it by τ_T. The linear extension depicted by the circled numbers in Figure 6.2 is the decreasing-minimum linear extension τ_T of the colored tree T in the figure.

Theorem 6.6 ([21], r=2 case [22])**.** *For all $\nu \in \mathrm{WComp}_{n-1,r}$,*

$$\{c(\mathbf{T}, \tau_{\mathbf{T}}) \mid \mathbf{T} \in \mathcal{L}yn_{n,\nu}\}$$

is the set of ascent-free maximal chains of the interval $[\hat{0}, [n]^\nu]$ under the EL-labeling of $\widehat{\Pi_n^\infty}$ given inTheorem 6.3.

Now by Corollary 6.1, we have the following result.

Corollary 6.7 ([21])**.** *For all $n \geq 1$,*

$$(-1)^{n-1} M_{K_n}(\mathbf{x}) = \sum_{\nu \in \mathrm{WComp}_{n-1}} |\mathcal{L}yn_{n,\nu}| \mathbf{x}^\nu.$$

From this corollary, one can show that $(-1)^{n-1} M_{K_n}(\mathbf{x})$ is e-positive. Details are given in a more general setting in Section 6.3. From Figure 6.3, one can see that

$$M_{K_3}(x_1, x_2) = 2x_1^2 + 5x_1x_2 + 2x_2^2 = e_2(x_1, x_2) + 2e_{1,1}(x_1, x_2)$$

6.3. **Generalization to chordal graphs.** In this section, we generalize the results of the previous section to multiweighted bond posets of chordal graphs.

Recall the definitions of PEO and *perfectly labeled chordal graph* from Section 4. The following observation gives an equivalent definition of a PEO.

Proposition 6.8. *An ordering $v_1, v_2, \cdots, v_n$ of the vertices of a graph G is a PEO if and only if whenever $i < j < k$, and $\{v_i, v_k\}$ and $\{v_j, v_k\}$ are edges of G then $\{v_i, v_j\}$ must also be an edge of G, i.e., there are no induced subgraphs that are paths of the form $v_i - v_k - v_j$.*

Corollary 6.9. *Let G be a perfectly labeled chordal graph and let H be an induced subgraph. Then H is a perfectly labeled chordal graph.*

Let G be a graph on vertex set $\{v_1 < \cdots < v_n\} \subseteq \mathbb{Z}_{>0}$. We say that a sequence $v_{j_1}, v_{j_2}, \ldots, v_{j_m}$ is an *increasing path* from v_{j_1} to v_{j_m} if $j_1 < j_2 < \cdots < j_m$ and $\{v_{j_i}, v_{j_{i+1}}\}$ is an edge of G for all $i \in [m-1]$.

Lemma 6.10. *Let G be a connected perfectly labeled chordal graph on vertex set $\{v_1 < \cdots < v_n\} \subseteq \mathbb{Z}_{>0}$. Then for any $j \in [n]$ there is an increasing path from v_1 to v_j.*

Proof. Using the fact that any induced subgraph of a perfectly labeled chordal graph is a perfectly labeled chordal graph (Corollary 6.9), we will proceed by induction on the number n of vertices of G. In graphs with one vertex the statement is trivial, so we assume that the lemma is true for any graph with less than n vertices and let G be a graph with n vertices. Suppose there are no increasing paths from v_1 to v_j. Note then in particular, $j > 1$ and $\{v_1, v_j\}$ cannot be an edge of G. If the graph $G - \{v_j\}$ has more than one component, we can consider the induced subgraph formed by v_j and the component of $G - \{v_j\}$ that contains v_1. Since this is a connected graph with less than n vertices, we will find an increasing path from v_1 to v_j, which gives a contradiction. This implies that $G - \{v_j\}$ has to be connected.

Let v_k be the neighbor of v_j in G with k as small as possible. We know that $k > 1$ but it must happen that $k > j$, otherwise if $k < j$ we could find an increasing path p from v_1 to v_k in $G - \{v_j\}$ and then p concatenated with v_j is an increasing path from v_1 to v_j in G. So let now p be an increasing path in $G - \{v_j\}$ from v_1 to v_k and let $\{v_i, v_k\}$ be its last edge. Since p is increasing $i < k$. Taking into account the minimality of k, the vertices v_i, v_j, v_k provide a violation to the condition of a PEO given in Proposition 6.8, which gives us a contradiction. Hence there must be an increasing path from v_1 to v_j for all $j \in [n]$. □

We will need the following lemma to prove that r-weighted bond posets of chordal graphs are EL-shellable.

Lemma 6.11. *Let G be a connected perfectly labeled chordal graph on $[n]$. Let $B_1, B_2, \ldots, B_k$ be a collection of disjoint subsets of $[n]$ such that*

- $\min B_1 < \min B_2 < \cdots < \min B_k$
- $G|_{B_j}$ *is connected for all* j
- $G|_{B_1 \cup B_2 \cup \cdots \cup B_k}$ *is connected.*

Then for every $s \leq k$, the subgraph $G|_{B_1 \cup B_2 \cup \cdots \cup B_s}$ is connected.

Proof. We proceed by induction on s. If $s = 1$ we have from the hypothesis that $G|_{B_1}$ is connected; so let us assume that $G|_{B_1 \cup B_2 \cup \cdots \cup B_{s-1}}$ is connected. Since $G|_{B_1 \cup B_2 \cup \cdots \cup B_k}$ is connected we know from Corollary 6.9 and Lemma 6.10 that there is an increasing path p from $\min B_1$ to $\min B_s$. Let $\{j, \min B_s\}$ be the last edge of p. Since p is increasing we have that $j < \min B_s$ and hence, by the fact that the B_i's are listed in increasing order of their minimums, we have that $j \in B_1 \cup \cdots \cup B_{s-1}$. Hence $G|_{B_1 \cup \cdots \cup B_s}$ is connected. □

We are now ready to prove that the augmented r-weighted bond posets of chordal graphs are EL-shellable. Let G be a connected graph on $[n]$ and $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. Note that each maximal chain of $\widehat{\Pi_G^r}$ is a maximal chain of $\widehat{\Pi_n^r}$. Hence any edge labeling of $\widehat{\Pi_n^r}$ can be restricted to an edge labeling of $\widehat{\Pi_G^r}$.

Theorem 6.12. *Let G be a connected perfectly labeled chordal graph on $[n]$ and let $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. Then the restriction to $\mathcal{E}(\widehat{\Pi_G^r})$ of the EL labeling $\lambda : \mathcal{E}(\widehat{\Pi_n^r}) \to \Lambda_{n,r}$ given in Theorem 6.3 is an EL-labeling of $\widehat{\Pi_G^r}$.*

Proof. Let $\boldsymbol{\alpha} < \boldsymbol{\beta}$ in $\widehat{\Pi_G^r}$. One need only show that the unique increasing maximal chain of the interval $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ in $\widehat{\Pi_n^r}$ is also a chain $\widehat{\Pi_G^r}$. The result will then follow from Theorem 6.3.

Suppose $\boldsymbol{\beta} \neq \hat{1}$ and let α and β be the underlying partitions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. Let $A_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq k_i$, be as in Definition 6.4. Since $\alpha, \beta \in \Pi_G$, each induced subgraph $G|_{A_{i,j}}$ is connected as is $G|_{\bigcup_{j=1}^{k_i} A_{i,j}}$. It therefore follows from Lemma 6.11 that the induced subgraph $G|_{\bigcup_{j=1}^{s} A_{i,j}}$ is connected for all $i = 1, \dots, m$ and $s = 1, \dots, k_i$. Hence, the chains c_i of Definition 6.4 are in Π_G, which means that the fundamental chain $c(\alpha, \beta)$ of $[\alpha, \beta]$ (i.e, the concatenation of $c_0, c_1, \dots, c_m$) is as well. Now by Lemma 6.5 (1), the unique increasing maximal chain of the interval $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ of $\widehat{\Pi_n^r}$ is also a chain in $\widehat{\Pi_G^r}$.

Now suppose $\beta = \hat{1}$. Let $\nu \in \mathrm{WComp}_{n-1,r}$ and let $\mathbf{c}_\nu$ be the unique increasing maximal chain of $[\boldsymbol{\alpha}, [n]^\nu]$ in $\widehat{\Pi_n^r}$. By the previous case, $\mathbf{c}_\nu$ is also in $\widehat{\Pi_G^r}$. Hence so is the concatenation of $\mathbf{c}_\nu$ with $\hat{1}$. This means that the unique increasing maximal chain of $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ in $\widehat{\Pi_n^r}$, as identified by Lemma 6.5 (2), is also in $\widehat{\Pi_G^r}$. □

Corollary 6.13. *If G is a chordal graph then all closed intervals of Π_G^r are EL-shellable for all $r \in \mathbb{Z}_{>0} \cup \{\infty\}$.*

Proof. We only need to prove this for maximal closed intervals. Since G is chordal, its connected components $G_1, \dots, G_k$ are chordal as well. Since every chordal graph can be perfectly labeled, each $\widehat{\Pi_{G_i}^r}$ is EL-shellable by Theorem 6.12. It follows that each maximal closed interval of $\Pi_{G_i}^r$ is EL-shellable. By Proposition 5.2 each maximal closed interval of Π_G^r is isomorphic to a product $P_1 \times \cdots \times P_k$, where each P_i

is a maximal closed interval of $\Pi^r_{G_i}$. The result now follows from [4, Theorem 4.3]. □

Question 6.14. *Is $\widehat{\Pi^r_G}$ EL-shellable for every graph G and for all $r \in \mathbb{Z}_{>0} \cup \{\infty\}$?*

We have the following generalization of Theorem 6.6.

Theorem 6.15. *Let G be a connected perfectly labeled chordal graph on $[n]$ and let $\nu \in \mathrm{WComp}_{n-1}$. Then the set of ascent-free maximal chains of the interval $[\hat{0}, [n]^\nu]_G$ for the EL-labeling of $\widehat{\Pi^\infty_G}$ given in Theorem 6.12 is*

$$\{c(\mathbf{T}, \tau_T) \mid \mathbf{T} \in \mathcal{L}yn_{G,\nu}\},$$

where

$$\mathcal{L}yn_{G,\nu} := \{\mathbf{T} \in \mathcal{L}yn_{n,\nu} : G|_{A_x} \text{ is connected for all nodes } x \text{ of } \mathbf{T}\}.$$

Proof. Since the EL-labeling of $\widehat{\Pi^\infty_G}$ is the restriction of the EL-labeling of $\widehat{\Pi^\infty_{K_n}}$, the ascent-free chains of any interval $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ of $\widehat{\Pi^\infty_G}$ are the ascent-free chains of the interval $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ of $\widehat{\Pi^\infty_{K_n}}$ that are also in $\widehat{\Pi^\infty_G}$. The ascent-free maximal chains of the interval $[\hat{0}, [n]^\nu]_{K_n}$, as given in Theorem 6.6, have the form $c(\mathbf{T}, \tau_T)$, where $\mathbf{T} \in \mathcal{L}yn_{n,\nu}$. Clearly, the chain $c(\mathbf{T}, \tau_{\mathbf{T}})$ is in $\widehat{\Pi^\infty_G}$ if and only if $G|_{A_x}$ is connected for all nodes x of $\mathbf{T}$. Thus the ascent-free chains of $[\hat{0}, [n]^\nu]$ are precisely the chains $c(\mathbf{T}, \tau_{\mathbf{T}})$ for which $\mathbf{T} \in \mathcal{L}yn_{G,\nu}$. □

Now by Corollary 6.1 we have the following consequence.

Corollary 6.16. *Let G be a connected perfectly labeled chordal graph on $[n]$. Then*

$$(-1)^{n-1} M_G(\mathbf{x}) = \sum_{\nu \in \mathrm{WComp}(n-1)} |\mathcal{L}yn_{G,\nu}| \mathbf{x}^\nu.$$

Example 6.17. For $G = P_3$, the only trees of Figure 6.3 that are in $\mathcal{L}yn_{G,\nu}$ for some ν are the ones in the first row. Hence by Corollary 6.16,

$$M_{P_3}(x_1, x_2) = x_1^2 + 3x_1x_2 + x_2^2 = e_2(x_1, x_2) + e_{1,1}(x_1, x_2).$$

Given a normalized binary tree T on node set $[n]$, define the *Lyndon set partition* $\pi(T)$ to be the finest partition of the set $\mathbf{int}(T)$ that satisfies if $x \in \mathbf{int}(T)$ is not a Lyndon node, then x and $L(x)$ are in the same block of π. The *Lyndon type* $\lambda(T)$ of T is the partition of $n-1$ whose parts are the sizes of the blocks of the Lyndon set partition $\pi(T)$; that is

$$\lambda(T) = \lambda(\pi(T)), \tag{6.1}$$

where $\lambda(\pi)$ is the number partition associated with the set partition π, as in (5.7). For example, for the tree T of Figure 6.2 (with colors ignored) we have that $\lambda(T) = (2,1,1,1)$.

Theorem 6.18. *Let G be a connected perfectly labeled chordal graph on $[n]$. Then*

$$(-1)^{n-1}M_G(\mathbf{x}) = \sum_{T\in\mathcal{N}_G} e_{\lambda(T)}(\mathbf{x}),$$

where $\mathcal{N}_G$ is the set of normalized binary trees T on leaf set $[n]$ for which $G|_{A_x}$ is connected for all nodes x of T.

Consequently, if G is a connected chordal graph then $(-1)^{n-1}M_G(\mathbf{x})$ is e-positive.

Proof. Let $T \in \mathcal{N}_G$. Each block of $\pi(T)$ forms a path $y_1, y_2, \dots, y_k$ of internal nodes of T such that for each $i \in [k-1]$, $L(y_i) = y_{i+1}$ and y_i is nonLyndon. It follows that a coloring map $\mathbf{clr} : \mathbf{int}(T) \to \mathbb{Z}_{>0}$ is a Lyndon-coloring if and only if

$$\mathbf{clr}(y_1) < \mathbf{clr}(y_2) < \cdots < \mathbf{clr}(y_k)$$

for all blocks.

Hence,

$$\sum_{\substack{\mathbf{T}\in\mathcal{L}yn_G\\ \mathbf{T}=(T,\mathbf{clr})}} \mathbf{x}^{\nu(\mathbf{T})} = e_{\lambda(T)}(\mathbf{x}), \tag{6.2}$$

where

$$\mathcal{L}yn_G := \bigcup_{\nu\in\mathrm{WComp}_{n-1}} \mathcal{L}yn_{G,\nu}$$

and T is the underlying uncolored tree of $\mathbf{T}$.

From Corollary 6.16 and Equation (6.2) we have

$$\begin{aligned}(-1)^{n-1}M_G(\mathbf{x}) &= \sum_{\nu\in\mathrm{WComp}_{n-1}} |\mathcal{L}yn_{G,\nu}|\mathbf{x}^{\nu} \\ &= \sum_{\mathbf{T}\in\mathcal{L}yn_G} \mathbf{x}^{\nu(\mathbf{T})} \\ &= \sum_{T\in\mathcal{N}_G} \sum_{\substack{\mathbf{T}\in\mathcal{L}yn_G\\ \mathbf{T}=(T,\mathbf{clr})}} \mathbf{x}^{\nu(\mathbf{T})} \\ &= \sum_{T\in\mathcal{N}_G} e_{\lambda(T)}(\mathbf{x}).\end{aligned}$$

□

Proof of Theorem 4.4. This is a special case of Theorem 6.18. Indeed, we use the facts that $\mu_G(t) = M_G(1,t)$ and

$$e_\lambda(1,t) = \begin{cases} t^{m_2(\lambda)}(1+t)^{m_1(\lambda)} & \text{if } \lambda \text{ has no parts greater than 2} \\ 0 & \text{otherwise,} \end{cases}$$

where $m_i(\lambda)$ is the number of parts of λ that are equal to i. Clearly for $T \in \mathcal{N}_G$, the partition $\lambda(T)$ has no parts greater than 2 if and only if $T \in \widehat{\mathcal{N}_G}$. In this case $m_2(\lambda(T))$ equals the number $m(T)$ of nonLyndon nodes of T and $m_1(\lambda(T)) = n-1-2m_2(\lambda(T))$. Hence the result follows from Theorem 6.18. □

Corollary 6.19. *Let G be a chordal graph with n vertices and k connected components. Then the symmetric function $(-1)^{n-k}M_G(\mathbf{x})$ is e-positive.*

Proof. This follows from Theorem 6.18, Proposition 5.4, and the fact that products of e-positive symmetric functions are e-positive. □

Corollary 6.20. *Let G and H be perfectly labeled chordal graphs with n vertices and k connected components. If H is a subgraph of G then*

$$(-1)^{n-k}(M_G(\mathbf{x}) - M_H(\mathbf{x}))$$

is e-positive.

Proof. Suppose G is connected. Then H must be as well since it has the same number of connected components as G. Clearly $\mathcal{N}_H \subseteq \mathcal{N}_G$. Hence by Theorem 6.18

$$(-1)^{n-1}(M_G(\mathbf{x}) - M_H(\mathbf{x})) = \sum_{T\in\mathcal{N}_G-\mathcal{N}_H} e_{\lambda(T)}(\mathbf{x}).$$

Now suppose G has connected components $G_i = (V_i, E_i)$, where $i \in [k]$. Then H must have connected components $H_i = (V_i, E_i')$ where $E_i' \subseteq E_i$. It follows from Theorem 6.18 and Corollary 6.20 in the connected case that all $(-1)^{|V_i|-1}M_{G_i}(\mathbf{x})$, $(-1)^{|V_i|-1}M_{H_i}(\mathbf{x})$, and $(-1)^{|V_i|-1}(M_{G_i}(\mathbf{x}) - M_{H_i}(\mathbf{x}))$ are e-positive. By Proposition 5.4,

$$(-1)^{n-k}(M_G(\mathbf{x})-M_H(\mathbf{x})) = \prod_{i=1}^{k}(-1)^{|V_i|-1}M_{G_i}(\mathbf{x})-\prod_{i=1}^{k}(-1)^{|V_i|-1}M_{H_i}(\mathbf{x}).$$

We now use the fact that if f_i, g_i, where $i \in [k]$, are symmetric functions and the f_i's, g_i's and $f_i - g_i$'s are e-positive then $\prod_{i=1}^k f_i - \prod_{i=1}^k g_i$ is e-positive. This follows by induction from the $k = 2$ case, which holds because

$$f_1f_2 - g_1g_2 = (f_1 - g_1)f_2 + (f_2 - g_2)g_1$$

and products and sums of e-positive symmetric functions are e-positive. Thus $(-1)^{n-k}(M_G(\mathbf{x}) - M_H(\mathbf{x}))$ is e-positive. $\square$

Conjecture 6.21. *Let G be a graph with n vertices and k connected components. Then*

(1) $(-1)^{n-k}M_G(\mathbf{x})$ *is e-positive.*
(2) *If H is a subgraph of G with n vertices and k connected components then*
$$(-1)^{n-k}(M_G(\mathbf{x}) - M_H(\mathbf{x}))$$
is e-positive.

We have verified (1) for this conjecture computationally for all graphs with up to 7 vertices.

6.4. **Examples.** Let $\mathcal{UT}_{n,r}$ denote the set of unlabeled (i.e. no leaf labels) r-ary trees with n internal nodes. For each $\nu \in \mathrm{WComp}_{n,r}$, let $\mathcal{UT}_{\nu,r}$ be the set of r-ary planar trees T such that, for each i, $\nu(i)$ is the number of internal nodes of T whose ith child (from left to right) is an internal node.

Theorem 6.22. *For all $r, n \geq 1$,*
$$(-1)^{n-1}M_{P_n}(x_1, \dots, x_r) = \sum_{\nu \in \mathrm{WComp}_{n,r}} |\mathcal{UT}_{\nu,r}|\ x_1^{\nu(1)} \cdots x_r^{\nu(r)}.$$

Proof. For any graph G on $[n]$, let $\mathcal{L}yn_{G,r} = \bigcup_{\nu \in \mathrm{WComp}_{n-1,r}} \mathcal{L}yn_{G,\nu}$. The result follows from Corollary 6.16 and the direct bijection between $\mathcal{L}yn_{P_n,r}$ and $\mathcal{UT}_{n,r}$ given below. By the discussion in Example 4.10, we can view the trees in $\mathcal{L}yn_{P_n,r}$ as *unlabeled* binary trees whose internal nodes are colored with colors in $[r]$. We represent such a binary tree T as $T = T_L \overset{c}{\wedge} T_R$, where T_L is the colored left subtree of T, T_R is the colored right subtree of T, and c is the color of the root of T. Then the (unlabeled) tree $T \in \mathcal{L}yn_{P_n,r}$ if and only if
$$T = (\dots(T_0 \overset{c_1}{\wedge} T_1) \overset{c_2}{\wedge} \cdots) \overset{c_k}{\wedge} T_k,$$
where T_0 is a single (unlabeled) leaf, each (unlabeled) $T_i \in \mathcal{L}yn_{P_{n_i},r}$, for some n_i, and $r \geq c_1 > c_2 > \cdots > c_k \geq 1$. Now let $\psi(T)$ be the r-ary tree defined recursively by letting the ith subtree of the root of $\psi(T)$ be $\psi(T_j)$ if $i = c_j$ and be a leaf if $i \in [r] - \{c_1, \dots, c_k\}$. This defines a map from $\mathcal{L}yn_{P_n,r}$ to $\mathcal{UT}_{n,r}$, which is easily seen to be a bijection. Also, for each $\nu \in \mathrm{WComp}_{n,r}$, we have $T \in \mathcal{L}yn_{P_n,\nu}$ if and only if $\psi(T) \in \mathcal{UT}_{\nu,r}$. Hence $|\mathcal{L}yn_{P_n,\nu}| = |\mathcal{UT}_{\nu,r}|$. $\square$

Remark 6.23. Note that Corollary 5.10 and Theorem 6.22 imply that $|\mathcal{UT}_{\nu,r}|$ is equal to the generalized Fuss-Narayana number N_ν for all $\nu \in \mathrm{WComp}_{n-1,r}$. This is a refinement of the well known result that the Fuss-Catalan numbers count r-ary trees.

Remark 6.24. An alternative proof of Theorem 5.9 can be given that uses Theorem 6.18 and a bijection from $\mathcal{N}_{P_n}$ to the set of noncrossing partitions $\mathcal{NC}_n$. By the discussion in Example 4.10, we can ignore the leaf labels and view the trees in $\mathcal{N}_{P_n}$ as trees in $\mathcal{UT}_{n,2}$. The bijection is as follows. For each $T \in \mathcal{UT}_{n,2}$, label its internal nodes in inorder (i.e. left subtree labeled in inorder first, then root, then right subtree labeled in inorder) and let $\pi(T)$ be the partition of $[n]$ whose blocks are the maximal left paths of labeled internal nodes of T, that is, the blocks of $\pi(T)$ are of the form $\{x_1, \dots, x_m\}$, where $L(x_i) = x_{i+1}$ for all $i \in [m-1]$, x_1 is not a left child and the left child of x_m is a leaf. We leave it to the reader to check that this defines a bijection $\pi : \mathcal{N}_{P_n} = \mathcal{UT}_{n,2} \to \mathcal{NC}_n$. Note that the Lyndon type $\lambda(T)$ of each $T \in \mathcal{N}_{P_n}$ is equal to the block type $\lambda(\pi(T))$. Hence π is a bijection from $\mathcal{N}_{P_n}$ to $\mathcal{NC}_n$ that takes Lyndon type to block type. Theorem 5.9 now follows from (5.9) and Theorem 6.18.

Next we apply Theorem 6.18 to the star graph St_n. For each subset $S = \{s_1, s_2, \dots, s_{k-1}\}_<$ of $[n-1]$, the composition of n associated with S is defined by,

$$\mathrm{co}(S) = (s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_{k-1}).$$

Given a composition ν with k parts, let e_ν be the elementary symmetric function $e_{\nu(1)} e_{\nu(2)} \cdots e_{\nu(k)}$. Recall, the *descent set* $\mathrm{Des}(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is defined to be

$$\mathrm{Des}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}.$$

Theorem 6.25 ([23]). *For all $n \geq 1$,*

$$(-1)^n M_{St_{n+1}}(\mathbf{x}) = \sum_{\sigma \in \mathfrak{S}_n} e_{\mathrm{co}(\mathrm{Des}(\sigma))}(\mathbf{x}).$$

Proof. Consider the bijection, described in Example 4.11, from $\mathcal{N}_{St_{n+1}}$ to $\mathfrak{S}_n$ that takes T to σ_T. Also recall the definition of the Lyndon-set partition $\pi(T)$ used to define the Lyndon type as $\lambda(T) = \lambda(\pi(T))$ in (6.1).

Note that for all $T \in \mathcal{N}_{St_{n+1}}$, each block of the partition $\pi(T)$ is in bijective correspondence with a maximal ascending run of the permutation σ_T, where a maximal ascending run of a permutation $\sigma \in \mathfrak{S}_n$ is a set of the form $\{\sigma(s_i + 1), \sigma(s_i + 2), \dots, \sigma(s_{i+1})\}$, where

$\{s_0, s_1, \dots, s_k\}_< = \mathrm{Des}(\sigma) \cup \{0, n\}$. It follows that the Lyndon type $\lambda(T)$ is the weakly decreasing rearrangement of $\mathrm{co}(\mathrm{Des}(\sigma_T))$. Hence for all $T \in \mathcal{N}_{St_{n+1}}$, we have $e_{\lambda(T)} = e_{\mathrm{co}(\mathrm{Des}(\sigma_T))}$. The result now follows from Theorem 6.18. □

Note that since $\tilde{A}_n(t) = (-1)^n \mu_{St_{n+1}}(t) = (-1)^n M_{St_{n+1}}(1, t)$, Equation (4.8) is a consequence of Theorem 6.25.

Remark 6.26. Let $\Lambda^n_{\mathbb{Q}}$ be the vector space of homogeneous symmetric functions of degree n over $\mathbb{Q}$. Let $\eta : \Lambda^n_{\mathbb{Q}} \to \mathbb{Q}[t]$ be the homomorphism defined on the basis $\{e_\lambda : \lambda \vdash n\}$ by $\eta(e_\lambda) = t^{\ell(\lambda)-1}$.

(1) In [21] González D'León shows that $(-1)^{n-1}\eta(M_{K_n}(\mathbf{x}))$ is equal to a polynomial whose coefficients are called *second order Eulerian numbers* in [26]. Stanley and Gessel [18] give a combinatorial interpretation of these polynomials in terms of descent number of certain multiset permutations that they call *Stirling permutations.* In [24] González D'León gives a combinatorial interpretation of $(-1)^{n-1}M_{K_n}(\mathbf{x})$ in terms of Stirling permutations, which, under the specialization η, reduces to the Stanley-Gessel interpretation of the second order Eulerian polynomials.

(2) It follows from Equation (5.10) and a well-known interpretation of the Narayana polynomials as $N_n(t) = \sum_{\pi \in \mathcal{NC}_n} t^{|\pi|-1}$ that

$$(-1)^{n-1}\eta(M_{P_n}(\mathbf{x})) = N_{n-1}(t).$$

(3) It follows from Theorem 6.25 that

$$(-1)^{n-1}\eta(M_{St_n}(\mathbf{x})) = A_{n-1}(t),$$

which appeared in [24].

6.5. **Topological consequences.** Although not needed in the rest of the paper, we discuss some topological consequences of the EL-labeling of Theorem 6.12. First we recall the well-known Philip Hall's theorem stating that given any open interval (x, y) of a pure bounded finite poset P, the value of the Möbius function $\mu_P(x, y)$ is equal to the reduced Euler characteristic of the order complex $\Delta(x, y)$ of the interval (x, y), where the order complex of a poset is the simplicial complex whose faces are the chains of the poset. An EL-labeling has the following further topological and homological implications for the order complex $\Delta(x, y)$.

Theorem 6.27 (Björner [4], Björner and Wachs [7, Theorem 5.9], cf. Wachs [46, Theorem 3.2.4])**.** *Let λ be an EL-labeling of a pure bounded finite poset P. Then for each open interval (x, y) of P we have that,*

(1) *the order complex* $\Delta(x,y)$ *has the homotopy type of a wedge of*

$$|\{\mathbf{c} \mid \mathbf{c} \text{ is an ascent-free maximal chain of } [x,y]\}|$$

$\ell(x,y)$*-spheres, and hence the poset* P *is* Cohen-Macaulay*, i.e.,* $\widetilde{H}^d(\Delta(x,y)) = 0$ *for every* $d \neq \ell(x,y)$*;*

(2) *the set*

$$\{\bar{\mathbf{c}} \mid \mathbf{c} \text{ is an ascent-free maximal chain of } [x,y]\}$$

forms a basis for the reduced cohomology $\widetilde{H}^{\ell(x,y)}(\Delta(x,y))$*.*

The following are therefore topological consequences of Theorems 6.12 and 6.15. These consequences were proved in [21] ($r = 2$ case in [22]) for $G = K_n$.

Theorem 6.28. *Let* G *be a connected perfectly labeled chordal graph on* $[n]$ *and let* $r \in \mathbb{Z}_{>0} \cup \{\infty\}$*. Then* $\widehat{\Pi_G^r}$ *is Cohen-Macaulay.*

Theorem 6.29. *Let* G *be a connected perfectly labeled chordal graph on* $[n]$ *and let* $\nu \in \mathrm{WComp}_{n-1,r}$*, where* $r \in \mathbb{Z}_{>0} \cup \{\infty\}$*. Then the set*

$$\{\bar{c}(\mathbf{T}, \rho_T) \mid \mathbf{T} \in \mathcal{L}yn_{G,\nu}\},$$

where $\bar{c}(\mathbf{T}, \rho_T) := c(\mathbf{T}, \rho_T) \setminus \{\hat{0}, [n]^\nu\}$ *and* ρ_T *is any linear extension of* T*, is a basis for top cohomology* $\widetilde{H}^{n-3}(\Delta(\hat{0}, [n]^\nu))$ *of the order complex* $\Delta(\hat{0}, [n]^\nu)$ *of the open interval* $(\hat{0}, [n]^\nu)$ *of* Π_G^r*.*

Proof. It follows from Theorems 6.27 and 6.15 that

$$\{\bar{c}(\mathbf{T}, \tau_T) \mid \mathbf{T} \in \mathcal{L}yn_{G,\nu}\},$$

where τ_T is the decreasing-minimum linear extension, is a basis for top cohomology of the open interval $(\hat{0}, [n]^\nu)$. We claim that if τ and τ' are any two linear extensions of the internal nodes of $\mathbf{T} \in \mathcal{L}yn_{G,\nu}$, then in the top cohomology group of the order complex of the interval $(\hat{0}, [n]^\nu)$ of Π_G^r we have that

$$\bar{c}(\mathbf{T}, \tau) = \pm\bar{c}(\mathbf{T}, \tau').$$

Indeed, this was proved in [21] for $G = K_n$ and the proof goes through for any connected perfectly labeled chordal graph G. It follows that the decreasing-minimum linear extension τ_T can be replaced by any linear extension ρ_T. □

7. A symmetric function analog of the chromatic polynomial

We consider a symmetric function analog of the characteristic polynomial of a bond lattice. Thanks to Whitney's formula in Equation (1.4), this is a symmetric function analog of the chromatic polynomial of a graph, which is different from Stanley's well-known chromatic symmetric function [41].

7.1. **Preliminaries.** Given a pure (or ranked) poset P, the *characteristic polynomial* of a pure poset P with a minimum element $\hat{0}$ is defined as

$$\mathrm{ch}_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{\rho(x)},$$

where $\rho(x)$ is the rank of x, i.e., the length of the interval $[\hat{0}, x]$.

For any graph G, let

$$\Psi_G(\mathbf{x}) = \sum_{\boldsymbol{\pi} \in \Pi_G^\infty} \mu_{\Pi_G^\infty}(\hat{0}, \boldsymbol{\pi}) \mathbf{x}^{w(\boldsymbol{\pi})} \tag{7.1}$$

Equivalently,

$$\Psi_G(\mathbf{x}) = \sum_{\pi \in \Pi_G} M_{G|_\pi}(\mathbf{x})$$

where $G|_\pi$ is the subgraph of G whose connected components are of the form $G|_B$ where $B \in \pi$. For example,

$$\Psi_{P_3}(\mathbf{x}) = 1 - 2e_1(\mathbf{x}) + e_{1,1}(\mathbf{x}) + e_2(\mathbf{x}) \tag{7.2}$$

The specialization $\Psi_G(\overbrace{t, t, \dots, t}^{r}, 0, 0, \dots)$ is equal to the characteristic polynomial of the r-weighted bond poset Π_G^r. By (1.4), the chromatic polynomial $\chi_G(t)$ is obtained from $\Psi_G(t, 0, 0, \dots)$ by reversing its coefficients. That is

$$\Psi_G(t, 0, 0, \dots) = t^n \chi_G(t^{-1}),$$

where n is the number of vertices of G. Thus we may think of $\Psi_G(\mathbf{x})$ as a symmetric function analog of the chromatic polynomial.

There is a well-known classical formula for the characteristic polynomial of $\Pi_{K_n} = \Pi_n$ given by,

$$\Psi_{K_n}(t, 0, 0, 0, \dots) = (t-1)(t-2) \cdots (t-n+1). \tag{7.3}$$

The formula for the characteristic polynomial of $\Pi_{K_n}^2 = \mathcal{W}\Pi_{K_n}$ was given in [22, Theorem 2.8],

$$\Psi_{K_n}(t, t, 0, 0, \dots) = (t-n)^{n-1}. \tag{7.4}$$

Question 7.1. *Are there nice formulas for* $\Psi_{K_n}(\overbrace{t, t, \dots, t}^{r}, 0, 0, \dots)$ *when* $r > 2$*?*

7.2. **Alternating e-positivity and Schur-log-concavity.** The next theorem generalizes, in the case of chordal graphs, the fact that the characteristic polynomials of bond lattices alternate in sign. Given a symmetric function $f(\mathbf{x})$, let $f(\mathbf{x})|_{\deg d}$ denote the degree d homogeneous component of $f(\mathbf{x})$. We say that $f(\mathbf{x})$ is *alternating e-positive* if $(-1)^d f(\mathbf{x})|_{\deg d}$ is e-positive for all $d \geq 0$.

Theorem 7.2. *If G is a chordal graph then $\Psi_G(\mathbf{x})$ is alternating e-positive.*

Proof. Let n be the number of vertices of G. Note that $G|_\pi$ has n vertices and $|\pi|$ connected components for all $\pi \in \Pi_G$. Since $M_{G|_\pi}(\mathbf{x})$ is homogeneous of degree $n - |\pi|$, for all d we have

$$\Psi_G(\mathbf{x})|_{\deg d} = \sum_{\substack{\pi \in \Pi_G \\ |\pi| = n-d}} M_{G|_\pi}(\mathbf{x}).$$

Since G is chordal, each $G|_\pi$ is chordal. We can therefore apply Corollary 6.19 to each $G|_\pi$. Thus, if $|\pi| = n - d$ then $(-1)^d M_{G|_\pi}(\mathbf{x})$ is e-positive. It follows that each $(-1)^d \Psi_G(\mathbf{x})|_{\deg d}$ is e-positive giving the desired result. □

A similar argument shows that Conjecture 6.21 (1) is equivalent to the conjecture that Theorem 7.2 is true for any graph.

A polynomial $\sum_{j=0}^d a_j t^j \in \mathbb{R}[t]$ is said to be *log-concave* if for all $i \in [d-1]$,

$$a_i^2 \geq a_{i-1} a_{i+1}. \tag{7.5}$$

For sequences of positive real numbers, log-concavity implies unimodality (see [10, Lemma 1.1]).

A longstanding conjecture of Hoggar [29], proved in 2012 by Huh [30], asserts that for any graph G, the chromatic polynomial $\chi_G(t)$ is log-concave. Since the coefficients of the chromatic polynomial alternate in sign, Huh's result also settles a conjecture of Read [36] stating that $(-1)^n \chi_G(-t)$ is unimodal.

Note that log-concavity of $\chi_G(t)$ is equivalent to log-concavity of $\Psi_G(t, 0, 0, \dots)$ since the sequences of coefficients are the reverse of each other. This leads to the following conjectural generalization. We will say that a symmetric function $f(\mathbf{x})$ is *Schur-log-concave* if for all $d \geq 1$,

$$(f(\mathbf{x})|_{\deg d})^2 - f(\mathbf{x})|_{\deg d-1} f(\mathbf{x})|_{\deg d+1}$$

is Schur-positive (i.e., when expanded in the Schur basis $\{s_\lambda(\mathbf{x}) : \lambda \vdash 2d\}$, the coefficients are nonnegative). The following conjecture has been shown computationally for all connected graphs G with at most 7 vertices.

Conjecture 7.3. *For any graph G, the symmetric function $\Psi_G(\mathbf{x})$ is Schur-log-concave.*

Example 7.4. Checking the Schur-log-concavity condition at $d = 1$ for $\Psi_{P_3}(\mathbf{x})$ from Equation (7.2) yields

$$(-2e_1(\mathbf{x}))^2 - (1)(e_{1,1}(\mathbf{x}) + e_2(\mathbf{x})) = 3s_{1,1}(\mathbf{x}) + 2s_2(\mathbf{x}),$$

which is Schur-positive.

Note that Schur-log-concavity of $\Psi_G(\mathbf{x})$ implies log-concavity of the characteristic polynomial $\Psi_G(t, \cdots, t, 0, 0, \dots)$ of Π_G^r for all r. Huh's result (i.e. the r=1 case) is therefore a specialization of Conjecture 7.3 for $\Psi_G(\mathbf{x})$. For $r = 2$ and $G = K_n$, log-concavity of $\Psi_{K_n}(t, t, 0, 0, \dots)$ follows from (7.4) and the fact that real-rooted polynomials with nonnegative coefficients are log-concave (see [10, Lemma 1.1]).

In a follow-up paper, we discuss Conjecture 7.3 further, as well as a related symmetric function generalization of the rank polynomial.

Acknowledgements

R. S. González D'León would like to thank Universidad Sergio Arboleda and the program of postdoctoral fellowships of Minciencias (Colombian Ministry of Science) since part of this project was completed with their support. He is also thankful for the partial support for this research provided by an AMS-Simons Research Enhancement Grant for Primarily Undergraduate Institution Faculty at Loyola University Chicago.

References

[1] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra. *Mem. Amer. Math. Soc.*, 289(1437):vi+119, 2023.

[2] N. Avila. The μ-polynomials of graph associahedra. Bachelor's thesis, Universidad Sergio Arboleda, 2022.

[3] N. Avila, S.A. Carrillo, and R. S. González D'León. The μ-polynomials of nestohedra. *In preparation*, 2026.

[4] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.

[5] Anders Björner and Michelle Wachs. Bruhat order of Coxeter groups and shellability. *Adv. in Math.*, 43(1):87–100, 1982.

[6] Anders Björner and Michelle Wachs. On lexicographically shellable posets. *Trans. Amer. Math. Soc.*, 277(1):323–341, 1983.

[7] Anders Björner and Michelle L. Wachs. Shellable nonpure complexes and posets. I. *Trans. Amer. Math. Soc.*, 348(4):1299–1327, 1996.

[8] Anders Björner and Michelle L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.
[9] Anders Björner and Volkmar Welker. Segre and Rees products of posets, with ring-theoretic applications. *J. Pure Appl. Algebra*, 198(1-3):43–55, 2005.
[10] Petter Brändén. Unimodality, log-concavity, real-rootedness and beyond. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 437–483. CRC Press, Boca Raton, FL, 2015.
[11] Michael P. Carr and Satyan L. Devadoss. Coxeter complexes and graph-associahedra. *Topology Appl.*, 153(12):2155–2168, 2006.
[12] Corrado De Concini and Claudio Procesi. Wonderful models of subspace arrangements. *Selecta Mathematica*, 1(3):459–494, 1995.
[13] V. V. Dotsenko and A. S. Khoroshkin. Character formulas for the operad of a pair of compatible brackets and for the bi-Hamiltonian operad. *Funktsional. Anal. i Prilozhen.*, 41(1):1–22, 96, 2007.
[14] Brian Drake. *An inversion theorem for labeled trees and some limits of areas under lattice paths*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Brandeis University.
[15] Brittney Ellzey. personal communication.
[16] Dominique Foata and Marcel-P. Schützenberger. *Théorie géométrique des polynômes eulériens*. Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin-New York, 1970.
[17] Swiatoslaw R Gal. Real root conjecture fails for five-and higher-dimensional spheres. *Discrete & Computational Geometry*, 34(2):269–284, 2005.
[18] Ira Gessel and Richard P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.
[19] Ira M. Gessel and Seunghyun Seo. A refinement of Cayley's formula for trees. *Electron. J. Combin.*, 11(2):Research Paper 27, 23 pp. (electronic), 2004/06.
[20] Rafael S. González D'León. A Note on the γ-Coefficients of the Tree Eulerian Polynomial. *The Electronic Journal of Combinatorics*, 23(1):P1–20, 2016.
[21] Rafael S. González D'León. On the free Lie algebra with multiple brackets. *Adv. in Appl. Math.*, 79:37–97, 2016.
[22] Rafael S. González D'León and Michelle L. Wachs. On the (co)homology of the poset of weighted partitions. *Trans. Amer. Math. Soc.*, 368(10):6779–6818, 2016.
[23] Rafael S González D'León. The colored symmetric and exterior algebras. *Journal of Algebra*, 496:187–215, 2018.
[24] Rafael S González D'León. A family of symmetric functions associated with stirling permutations. *Journal of Combinatorics*, 10(4):675–709, 2019.
[25] Amy Grady and Svetlana Poznanović. Tree descent polynomials: unimodality and central limit theorem. *Ann. Comb.*, 24(1):109–117, 2020.
[26] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1989. A foundation for computer science.
[27] James Haglund and Philip B. Zhang. Real-rootedness of variations of Eulerian polynomials. *Adv. in Appl. Math.*, 109:38–54, 2019.
[28] Mark D. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3(1):17–76, 1994.

[29] S. G. Hoggar. Chromatic polynomials and logarithmic concavity. *J. Combinatorial Theory Ser. B*, 16:248–254, 1974.
[30] June Huh. Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. *J. Amer. Math. Soc.*, 25(3):907–927, 2012.
[31] June Huh. Combinatorial applications of the Hodge-Riemann relations. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*, pages 3093–3111. World Sci. Publ., Hackensack, NJ, 2018.
[32] Svante Linusson, John Shareshian, and Michelle L. Wachs. Rees products and lexicographic shellability. *J. Comb.*, 3(3):243–276, 2012.
[33] T. Kyle Petersen. *Eulerian numbers*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2015. With a foreword by Richard Stanley.
[34] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Doc. Math*, 13(207-273):51, 2008.
[35] Alexander Postnikov. Permutohedra, associahedra, and beyond. *International Mathematics Research Notices*, 2009(6):1026–1106, 2009.
[36] Ronald C. Read. An introduction to chromatic polynomials. *J. Combinatorial Theory*, 4:52–71, 1968.
[37] John Shareshian and Michelle L. Wachs. Poset homology of Rees products, and q-Eulerian polynomials. *Electron. J. Combin.*, 16(2, Special volume in honor of Anders Bjorner):Research Paper 20, 29, 2009.
[38] John Shareshian and Michelle L. Wachs. Gamma-positivity of variations of Eulerian polynomials. *J. Comb.*, 11(1):1–33, 2020.
[39] Richard P. Stanley. Finite lattices and Jordan-Hölder sets. *Algebra Universalis*, 4(1):361–371, 1974.
[40] Richard P. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.*, 35(3):236–238, 1980.
[41] Richard P Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111(1):166–194, 1995.
[42] Richard P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2):Research Paper 20, approx. 14, 1997. The Wilf Festschrift (Philadelphia, PA, 1996).
[43] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
[44] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
[45] Michelle L. Wachs. On the (co)homology of the partition lattice and the free Lie algebra. *Discrete Math.*, 193(1-3):287–319, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).
[46] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.
[47] Hassler Whitney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, 38(8):572–579, 1932.

(R. S. González D'León) Department of Mathematics and Statistics, Loyola University of Chicago, Chicago, USA
Email address: rgonzalezdleon@luc.edu
URL: https://dleon.combinatoria.co/

(M. L. Wachs) Department of Mathematics, University of Miami, Coral Gables, Florida, USA
Email address: wachs@math.miami.edu
URL: http://www.math.miami.edu/~wachs/