ON THE (CO)HOMOLOGY OF THE POSET OF WEIGHTED PARTITIONS

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Abstract. We consider the poset of weighted partitions Π_n^w, introduced by Dotsenko and Khoroshkin in their study of a certain pair of dual operads. The maximal intervals of Π_n^w provide a generalization of the lattice Π_n of partitions, which we show possesses many of the well-known properties of Π_n. In particular, we prove these intervals are EL-shellable, we show that the Möbius invariant of each maximal interval is given up to sign by the number of rooted trees on node set $\{1, 2, \ldots, n\}$ having a fixed number of descents, we find combinatorial bases for homology and cohomology, and we give an explicit sign twisted $\mathfrak{S}_n$-module isomorphism from cohomology to the multilinear component of the free Lie algebra with two compatible brackets. We also show that the characteristic polynomial of Π_n^w has a nice factorization analogous to that of Π_n.

Contents

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1. Introduction

We recall some combinatorial, topological and representation theoretic properties of the lattice Π_n of partitions of the set $[n] := \{1, 2, \ldots, n\}$ ordered by refinement.[1] The Möbius invariant of Π_n is given by

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!,$$

and the characteristic polynomial by

$$\chi_{\Pi_n}(x) = (x-1)(x-2)\ldots(x-n+1)$$

(see [27, Example 3.10.4]). It was proved by Björner [5], using an edge labeling of Stanley [24], that Π_n is EL-shellable; consequently the order complex $\Delta(\overline{\Pi}_n)$ of the proper part $\overline{\Pi}_n$ of the partition lattice Π_n has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$. Various nice bases for the homology and cohomology of the partition lattice have been introduced and studied; see [31] for a discussion of these bases.

The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this action induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\tilde{H}_{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ and on the unique nonvanishing simplicial cohomology $\tilde{H}^{n-3}(\overline{\Pi}_n)$. Joyal [19] observed that a formula of Stanley and Hanlon (see [25]) for the character of this representation is a sign twisted version of an earlier formula of Brandt [9] for the character of the representation of $\mathfrak{S}_n$ on the multilinear component $\mathcal{L}ie(n)$ of the free Lie algebra on n generators. Hence the following $\mathfrak{S}_n$-module isomorphism holds,

$$\tilde{H}_{n-3}(\overline{\Pi}_n) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n) \otimes \mathrm{sgn}_n, \tag{1.1}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$. Joyal [19] gave a proof of the isomorphism using his theory of species. The first purely combinatorial proof was obtained by Barcelo [2] who provided a bijection between known bases for the two $\mathfrak{S}_n$-modules (Björner's NBC basis for $\tilde{H}_{n-3}(\overline{\Pi}_n)$ and the Lyndon basis for $\mathcal{L}ie(n)$) and analyzed the representation matrices for these bases. Later Wachs [31] gave a more general combinatorial proof by providing a natural bijection between generating sets of $\tilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n)$, which revealed the strong connection between the two $\mathfrak{S}_n$-modules.

In this paper we explore analogous properties for a weighted version of Π_n, introduced by Dotsenko and Khoroshkin [11] in their study of Koszulness of certain quadratic binary operads. A weighted partition of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$

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[1] The poset terminology used here is defined in Section 2.

and $v_i \in \{0, 1, 2, ..., |B_i| - 1\}$ for all i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with order relation given by $\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_t}\} \leq \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ if the following conditions hold:

- $\{A_1, A_2, ..., A_s\} \leq \{B_1, B_2, ..., B_t\}$ in Π_n
- if $B_k = A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_l}$ then $v_k - (w_{i_1} + w_{i_2} + ... + w_{i_l}) \in \{0, 1, ..., l-1\}$.

Equivalently, we can define the covering relation by

$$\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$$

if the following conditions hold:

- $\{A_1, A_2, \ldots, A_s\} \lessdot \{B_1, B_2, \ldots, B_t\}$ in Π_n
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$
- if $B_k = A_i$ then $v_k = w_i$.

In Figure 1 below the set brackets and commas have been omitted.

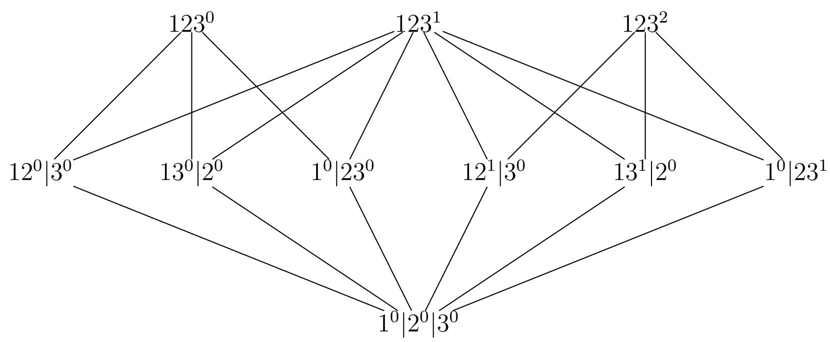


FIGURE 1. Weighted partition poset for $n = 3$

The poset Π_n^w has a minimum element

$$\hat{0} := \{\{1\}^0, \{2\}^0, \ldots, \{n\}^0\}$$

and n maximal elements

$$\{[n]^0\},\ \{[n]^1\}, \ldots, \{[n]^{n-1}\}.$$

We write each maximal element $\{[n]^i\}$ as $[n]^i$. Note that for all i, the maximal intervals $[\hat{0}, [n]^i]$ and $[\hat{0}, [n]^{n-1-i}]$ are isomorphic to each other, and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to Π_n.

The basic properties of Π_n mentioned above have nice weighted analogs for the intervals $[\hat{0}, [n]^i]$. For instance, the $\mathfrak{S}_n$-module isomorphism (1.1) can be generalized. Let $\mathcal{L}ie_2(n)$ be the multilinear component of the free Lie algebra on n generators with two compatible brackets (defined in Section 4.1) and let $\mathcal{L}ie_2(n,i)$ be the component of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with i brackets of one type and $n-1-i$ brackets of the other type. The symmetric group acts naturally on each $\mathcal{L}ie_2(n,i)$ and on each open interval $(\hat{0},[n]^i)$. It follows from operad theoretic results of Vallette [30] and Dotsenko-Khoroshkin [12] that the following $\mathfrak{S}_n$- module isomorphism holds:

$$\tilde{H}_{n-3}((\hat{0},[n]^i)) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,i) \otimes \mathrm{sgn}_n\,. \tag{1.2}$$

Note that this reduces to (1.1) when $i=0$ or $i=n-1$. The character of each $\mathfrak{S}_n$-module $\mathcal{L}ie_2(n,i)$ was computed by Dotsenko and Khoroshkin [11].

In [20] Liu proves a conjecture of Feigin that $\dim \mathcal{L}ie_2(n) = n^{n-1}$ by constructing a combinatorial basis for $\mathcal{L}ie_2(n)$ indexed by rooted trees on node set $[n]$. An operad theoretic proof of Feigin's conjecture was obtained by Dotsenko and Khoroshkin [11], but with a gap pointed out in [28] and corrected in [12]. In fact, Liu and Dotsenko-Khoroshkin obtain the following refinement of Feigin's conjecture

$$\sum_{i=0}^{n-1} \dim \mathcal{L}ie_2(n,i) t^i = \prod_{j=1}^{n-1}((n-j)+jt). \tag{1.3}$$

Since, as was proved by Drake [13], the right hand side of (1.3) is equal to the generating function for rooted trees on node set $[n]$ according to the number of descents of the tree, it follows that for each i, the dimension of $\mathcal{L}ie_2(n,i)$ equals the number of rooted trees on node set $[n]$ with i descents. (Drake's result is a refinement of the well-known result that the number of trees on node set $[n]$ is n^{n-1}.)

In this paper we give an alternative proof of (1.2) by presenting an explicit bijection between natural generating sets of $\tilde{H}^{n-3}((\hat{0},[n]^i))$ and $\mathcal{L}ie_2(n,i)$, which reveals the connection between these modules and generalizes the bijection that Wachs [31] used to prove (1.1). With (1.2), we take a different path to proving the Liu and Dotsenko-Khoroshkin formula (1.3), one that employs poset theoretic techniques.

We prove that the augmented poset of weighted partitions

$$\widehat{\Pi_n^w} := \Pi_n^w \cup \{\hat{1}\}$$

is EL-shellable by providing an interesting weighted analog of the Björner-Stanley EL-labeling of Π_n. In fact our labeling restricts to the Björner-Stanley EL-labeling on the intervals $[\hat{0},[n]^0]$ and $[\hat{0},[n]^{n-1}]$. A consequence of shellability is that $\widehat{\Pi_n^w}$ is Cohen-Macaulay, which implies a result of Dotsenko and Khoroshkin [12], obtained through operad theory, that all maximal intervals $[\hat{0},[n]^i]$ of Π_n^w are Cohen-Macaulay. (Two prior attempts [11, 28] to establish Cohen-Macaulayness of $[\hat{0}, [n]^i]$ are discussed in Remark 3.8.) The ascent-free chains of our EL-labeling provide a generalization of the Lyndon basis for cohomology of $\overline{\Pi}_n$ (i.e. the basis for cohomology that corresponds to the classical Lyndon basis for $\mathcal{L}ie(n)$).

Direct computation of the Möbius function of Π_n^w, which exploits the recursive nature of Π_n^w and makes use of the compositional formula, shows that

$$(-1)^{n-1} \sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) t^i$$

equals the right hand side of (1.3). From this computation and the fact that $\widehat{\Pi_n^w}$ is EL-shellable (and thus the maximal intervals of Π_n^w are Cohen-Macaulay), we conclude that

$$\sum_{i=0}^{n-1} \operatorname{rank} \tilde{H}_{n-3}((\hat{0}, [n]^i)) t^i = \prod_{j=1}^{n-1} ((n-j) + jt). \tag{1.4}$$

The Liu and Dotsenko-Khoroshkin formula (1.3) is a consequence of this and (1.2).

By (1.4) and Drake's result mentioned above, the rank of $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ is equal to the number of rooted trees on $[n]$ with i descents. We construct a nice combinatorial basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ consisting of fundamental cycles indexed by such rooted trees, which generalizes Björner's NBC basis for $\tilde{H}_{n-3}(\overline{\Pi}_n)$. Our proof that these fundamental cycles form a basis relies on Liu's [20] generalization for $\mathcal{L}ie_2(n, i)$ of the classical Lyndon basis for $\mathcal{L}ie(n)$ and our bijective proof of (1.2). Indeed, our bijection enables us to transfer bases for $\mathcal{L}ie_2(n, i)$ to bases for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and vice verse. We first transfer Liu's generalization of the Lyndon basis to $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and then use the natural pairing between homology and cohomology to prove that our proposed homology basis is indeed a basis. (We also obtain an alternative proof that Liu's generalization of the Lyndon basis is a basis along the way.) By transferring the basis for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ that comes from the ascent-free chains of our EL-labeling to $\mathcal{L}ie_2(n, i)$, we obtain a different generalization of the Lyndon basis that has a somewhat simpler description than that of Liu's generalized Lyndon basis.

The paper is organized as follows: In Section 2 we derive basic properties of the weighted partition lattice, which include the formula for the Möbius function of Π_n^w mentioned above. We also show that the Möbius invariant of the augmented poset of weighted partitions $\widehat{\Pi_n^w}$ is given by

$$\mu_{\widehat{\Pi_n^w}}(\hat{0},\hat{1}) = (-1)^n(n-1)^{n-1}$$

and the characteristic polynomial factors nicely as

$$\chi_{\Pi_n^w}(x) = (x-n)^{n-1}.$$

The Whitney numbers of the first and second kind are also discussed.

Section 3 contains our results on EL-shellability of the augmented poset of weighted partitions and its topological consequences.

In Section 4 we give a presentation of the cohomology of the maximal open intervals $(\hat{0},[n]^i)$ in terms of maximal chains associated with labeled bicolored binary trees. This presentation enables us to use a natural bijection between generating sets of $\tilde{H}^{n-3}((\hat{0},[n]^i))$ and $\mathcal{L}ie_2(n,i)$ to establish the $\mathfrak{S}_n$-module isomorphism (1.2). Bases for cohomology and for homology of $(\hat{0},[n]^i)$ are discussed in Section 5. We also construct bases for cohomology of the full poset $\Pi_n^w \setminus \{\hat{0}\}$.

By extending the technique of Section 4, we prove in Section 6 that Whitney homology of Π_n^w tensored with the sign representation is isomorphic to the multilinear component of the exterior algebra of the doubly bracketed free Lie algebra on n generators. In Section 7 we mention related results that will appear in forthcoming papers.

2. Basic properties

For poset terminology not defined here see [27], [33]. For $u \le v$ in a poset P, the open interval $\{w \in P : u < w < v\}$ is denoted by (u,v) and the closed interval $\{w \in P : u \le w \le v\}$ by $[u,v]$. A poset is said to be *bounded* if it has a minimum element $\hat{0}$ and a maximum element $\hat{1}$. For a bounded poset P, we define the *proper part* of P as $\overline{P} := P\setminus\{\hat{0},\hat{1}\}$. A poset is said to be *pure* (or ranked) if all its maximal chains have the same length, where the length of a chain $s_0 < s_1 < \cdots < s_n$ is n. The *length* $l(P)$ of a poset P is the length of its longest chain. For a poset P with a minimum element $\hat{0}$, the rank function $\rho : P \to \mathbb{N}$ is defined by $\rho(s) = l([\hat{0},s])$. The rank generating function $\mathcal{F}_P(x)$ is defined by $\mathcal{F}_P(x) = \sum_{u\in P} x^{\rho(u)}$.

2.1. **The rank generating function.**

It is easy to see that the weighted partition poset Π_n^w is pure of length $n-1$ and has minimum element $\hat{0} = \{\{1\}^0,\dots,\{n\}^0\}$. For each $\alpha \in \Pi_n^w$, we have $\rho(\alpha) = n - |\alpha|$.

Proposition 2.1. *For all $n \ge 1$, the rank generating function is given by*

$$\mathcal{F}_{\Pi_n^w}(x) = \sum_{k=0}^{n-1} \binom{n}{k}(n-k)^k x^k.$$

Proof. Let $R_n(k) = \{\alpha \in \Pi_n^w \,|\, \rho(\alpha) = k\}$. We need to show

$$|R_n(k)| = \binom{n}{n-k}(n-k)^k. \tag{2.1}$$

A weighted partition in $R_n(k)$ can be viewed as a partition of $[n]$ into $n-k$ blocks, with one element of each block marked (or distinguished). To choose such a partition, we first choose the $n-k$ marked elements. There are $\binom{n}{n-k}$ ways to choose these elements and place them in $n-k$ distinct blocks. To each of the remaining k elements we allocate one of these $n-k$ blocks. We can do this in $(n-k)^k$ ways. Hence (2.1) holds. □

2.2. **The Möbius function.** For $\alpha = \{A_1^{w_1}, \dots, A_k^{w_k}\} \in \Pi_n^w$, let $w(\alpha) = \sum_{i=1}^k w_i$. The following observations will be used to compute the Möbius function of the weighted partition poset.

Proposition 2.2. *For all* $\alpha = \{A_1^{w_1}, \dots, A_k^{w_k}\} \in \Pi_n^w$*,*

(1) $[\alpha, \hat{1}]$ *and* $\widehat{\Pi_k^w}$ *are isomorphic posets,*
(2) $[\alpha, [n]^i]$ *and* $[\hat{0}, [|\alpha|]^{i-w(\alpha)}]$ *are isomorphic posets for* $w(\alpha) \le i \le n-1$*,*
(3) $[\hat{0}, \alpha]$ *and* $[\hat{0}, [|A_1|]^{w_1}] \times \cdots \times [\hat{0}, [|A_k|]^{w_k}]$ *are isomorphic posets.*

For a bounded poset P, let μ_P denote its Möbius function. We will use the recursive definition of the Möbius function and the compositional formula to derive the following result.

Proposition 2.3. *For all* $n \ge 1$*,*

$$\sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) t^i = (-1)^{n-1} \prod_{i=1}^{n-1}((n-i) + it). \tag{2.2}$$

Consequently,

$$\sum_{i=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} n^{n-1}.$$

Proof. By the recursive definition of the Möbius function we have that

$$\sum_{i=0}^{n-1} t^i \sum_{\hat{0} \le \alpha \le [n]^i} \mu_{\Pi_n^w}(\alpha, [n]^i) = \delta_{n,1}.$$

Proposition 2.2 implies $\mu_{\Pi_n^w}(\alpha, [n]^i) = \mu_{\Pi_{|\alpha|}^w}(\hat{0}, [|\alpha|]^j)$, where $j = i - w(\alpha)$. Note also that $\hat{0} \le \alpha \le [n]^i$ if and only if $w(\alpha) \le i$ and $i - w(\alpha) \le$

$|\alpha| - 1$. Hence,

$$\begin{aligned}
\delta_{n,1} &= \sum_{\alpha \in \Pi_n^w} t^{w(\alpha)} \sum_{i=w(\alpha)}^{w(\alpha)+|\alpha|-1} \mu_{\Pi_n^w}(\alpha, [n]^i) t^{i-w(\alpha)} \\
&= \sum_{\alpha \in \Pi_n^w} t^{w(\alpha)} \sum_{j=0}^{|\alpha|-1} \mu_{\Pi_{|\alpha|}^w}(\hat{0}, [|\alpha|]^j) t^j \\
&= \sum_{\pi \in \Pi_n} \left(\prod_{B \in \pi} (t^{|B|-1} + t^{|B|-2} + \cdots + 1) \right) \sum_{j=0}^{|\pi|-1} \mu_{\Pi_{|\pi|}^w}(\hat{0}, [|\pi|]^j) t^j \\
&= \sum_{\pi \in \Pi_n} \left(\prod_{B \in \pi} \frac{t^{|B|} - 1}{t - 1} \right) \sum_{j=0}^{|\pi|-1} \mu_{\Pi_{|\pi|}^w}(\hat{0}, [|\pi|]^j) t^j.
\end{aligned}$$

This implies by the compositional formula (see [26, Theorem 5.1.4]) that

$$U(x) = \sum_{n \geq 1} \frac{t^n - 1}{t - 1} \frac{x^n}{n!} = \frac{e^{tx} - e^x}{t - 1}$$

and

$$W(x) = \sum_{n \geq 1} \sum_{j=0}^{n-1} \mu_{\Pi_n^w}(\hat{0}, [n]^j) t^j \frac{x^n}{n!}$$

are compositional inverses.

It follows from [14, Theorem 5.1] that the compositional inverse of $U(x)$ is given by

$$\sum_{n \geq 1} (-1)^{n-1} \prod_{i=1}^{n-1} ((n-i) + it) \frac{x^n}{n!}.$$

(See [13, Eq. (10)].) This yields (2.2). □

Let T be a rooted tree on node set $[n]$. A *descent* of T is a node x that has a smaller label than its parent $p_T(x)$. We call the edge $\{x, p_T(x)\}$ a *descent edge*. We denote by $\mathcal{T}_{n,i}$ the set of rooted trees on node set $[n]$ with exactly i descents. In [13] Drake proves that

$$\sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{2.3}$$

The following result is a consequence of this and Proposition 2.3.

Corollary 2.4. *For all $n \geq 1$ and $i \in \{0, 1, \ldots, n-1\}$,*

$$\mu_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} |\mathcal{T}_{n,i}|.$$

We can use Proposition 2.2 and Corollary 2.4 to compute the Möbius function on other intervals. A rooted forest on node set $[n]$ is a set of rooted trees whose node sets form a partition of $[n]$. We associate a weighted partition $\alpha(F)$ with each rooted forest $F = \{T_1, \dots, T_k\}$ on node set $[n]$, by letting $\alpha(F) = \{A_1^{w_1}, \dots, A_k^{w_k}\}$ where A_i is the node set of T_i and w_i is the number of descents of T_i. For lower intervals we obtain the following generalization of Corollary 2.4.

Corollary 2.5. *For all* $\alpha \in \Pi_n^w$,

$$\mu_{\Pi_n^w}(\hat{0}, \alpha) = (-1)^{n-|\alpha|} |\{F \in \mathcal{F}_n : \alpha(F) = \alpha\}|,$$

where $\mathcal{F}_n$ *is the set of rooted forests on node set* $[n]$.

Next we consider the full poset $\widehat{\Pi_n^w}$. To compute its Möbius invariant we will make use of Abel's identity (see [26, Ex. 5.31 c]),

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k}. \tag{2.4}$$

Proposition 2.6.

$$\mu_{\widehat{\Pi_n^w}}(\hat{0}, \hat{1}) = (-1)^n (n-1)^{n-1}.$$

Proof. We proceed by induction on n. If $n = 1$ then

$$\mu_{\widehat{\Pi_1^w}}(\hat{0}, \hat{1}) = -1 = (-1)^1 (1-1)^{1-1}$$

since $\widehat{\Pi_1^w}$ is the chain of length 1.

Let $n \geq 1$ and let $\alpha \in \Pi_n^w \setminus \{\hat{0}\}$. Since the interval $[\alpha, \hat{1}]$ in $\widehat{\Pi_n^w}$ is isomorphic to $\widehat{\Pi_{|\alpha|}^w}$ (cf. Proposition 2.2), we can assume by induction that

$$\mu_{\widehat{\Pi_n^w}}(\alpha, \hat{1}) = (-1)^{|\alpha|} (|\alpha| - 1)^{|\alpha|-1}.$$

Hence by the recursive definition of the Möbius function we have,

$$\begin{aligned} \mu_{\widehat{\Pi_n^w}}(\hat{0}, \hat{1}) &= -\sum_{\alpha \in \widehat{\Pi_n^w} \setminus \hat{0}} \mu_{\widehat{\Pi_n^w}}(\alpha, \hat{1}) \\ &= -1 - \sum_{k=1}^{n-1} \sum_{\substack{\alpha \in \Pi_n^w \\ |\alpha| = k}} \mu_{\widehat{\Pi_n^w}}(\alpha, \hat{1}) \\ &= -1 - \sum_{k=1}^{n-1} \sum_{\substack{\alpha \in \Pi_n^w \\ |\alpha| = k}} (-1)^k (k-1)^{k-1} \end{aligned}$$

$$= -1 - \sum_{k=1}^{n-1} \binom{n}{k} k^{n-k} (-1)^k (k-1)^{k-1} \quad \text{(by (2.1))}$$

$$= -1 + \sum_{k=0}^{n} \binom{n}{k} k^{n-k} (1-k)^{k-1} - (1-n)^{n-1}. \tag{2.5}$$

By setting $x = 1, y = 0, z = 1$ in Abel's identity (2.4), we get

$$1 = \sum_{k=0}^{n} \binom{n}{k} (1-k)^{k-1} k^{n-k}.$$

Substituting this into (2.5) yields the result. □

Remark 2.7. In Section 2.3 we compute the characteristic polynomial of Π_n^w and use it to give a second proof of Proposition 2.6.

2.3. **The characteristic polynomial.** Recall that the characteristic polynomial of Π_n factors nicely. We prove that the same is true for Π_n^w.

Theorem 2.8. *For all $n \geq 1$, the characteristic polynomial of Π_n^w is given by*

$$\chi_{\Pi_n^w}(x) := \sum_{\alpha \in \Pi_n^w} \mu_{\Pi_n^w}(\hat{0}, \alpha) x^{n-1-\rho(\alpha)} = (x-n)^{n-1}.$$

We will need the following result.

Proposition 2.9 (see [26, Proposition 5.3.2])**.** *Let $\mathcal{F}_n^k$ be the number of rooted forests on node set $[n]$ with k rooted trees. Then*

$$|\mathcal{F}_n^k| = \binom{n-1}{k-1} n^{n-k}.$$

Proof of Theorem 2.8. We have

$$\begin{aligned}
\chi_{\Pi_n^w}(x) &= \sum_{\alpha \in \Pi_n^w} \mu(\hat{0}, \alpha) x^{|\alpha|-1} \\
&= \sum_{k=1}^{n} \sum_{\substack{\alpha \in \Pi_n^w \\ |\alpha| = k}} \mu(\hat{0}, \alpha) x^{k-1} \\
&= \sum_{k=1}^{n} (-1)^{n-k} |\mathcal{F}_n^k| x^{k-1} \quad \text{(by Corollary 2.5)} \\
&= \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} n^{n-k} x^{k-1} \quad \text{(by Proposition 2.9)}
\end{aligned}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-n)^{n-1-k} x^k$$
$$= (x-n)^{n-1}.$$

□

Theorem 2.8 yields an easier way to calculate $\mu_{\widehat{\Pi_n^w}}(\hat{0},\hat{1})$.

Second proof of Proposition 2.6 . By the recursive definition of Möbius function,

$$\begin{aligned} \mu_{\widehat{\Pi_n^w}}(\hat{0},\hat{1}) &= -\sum_{\alpha\in\Pi_n^w} \mu(\hat{0},\alpha) \\ &= -\chi_{\Pi_n^w}(1) \\ &= -(1-n)^{n-1} \\ &= (-1)^n(n-1)^{n-1}. \end{aligned}$$

□

2.4. **Whitney numbers and uniformity.** Let P be a pure poset of length n with minimum element $\hat{0}$. Recall that the *Whitney number of the first kind* $w_k(P)$ is the coefficient of x^{n-k} in the characteristic polynomial $\chi_P(x)$ and the *Whitney number of the second kind* $W_k(P)$ is the coefficient of x^k in the rank generating function $\mathcal{F}_P(x)$; see [27]. It follows from Theorem 2.8 and Proposition 2.1, respectively, that

$$\begin{aligned} w_k(\Pi_n^w) &= (-1)^k \binom{n-1}{k} n^k \\ W_k(\Pi_n^w) &= \binom{n}{k} (n-k)^k. \end{aligned} \tag{2.6}$$

For the partition lattice Π_n, the Whitney numbers of the first and second kind are the Stirling numbers of the first and second kind. It is well-known that the Stirling numbers of the first kind and second kind form inverse matrices, cf., [27, Proposition 1.9.1 a]. This can be viewed as a consequence of a property of the partition lattice called uniformity [27, Ex. 3.130]. We observe in this section that Π_n^w is also uniform and discuss a Whitney number consequence.

A pure poset P of length l with minimum element $\hat{0}$ and with rank function ρ, is said to be *uniform* if there is a family of posets $\{P_i : 0 \leq i \leq l\}$ such that for all $x \in P$, the upper order ideal $I_x := \{y \in P : x \leq y\}$ is isomorphic to P_i, where $i = l - \rho(x)$. We refer to $(P_0, \dots, P_l)$ as the associated *uniform sequence*. It follows from Proposition 2.2 that $P = \Pi_n^w$ is uniform with $P_i = \Pi_{i+1}^w$ for $i = 0, \dots, n-1$. We will use the following variant of [27, Exercise 3.130(a)] whose proof is left to the reader. (A weighted version of this is proved in [15].)

Proposition 2.10. *Let P be a uniform poset of length l, with associated uniform sequence $(P_0, \dots, P_l)$. Then the matrices $[w_{i-j}(P_i)]_{0\le i,j\le l}$ and $[W_{i-j}(P_i)]_{0\le i,j\le l}$ are inverses of each other.*

From the uniformity of Π_n^w and (2.6), we have the following consequence of Proposition 2.10.

Corollary 2.11. *The matrices $A = [(-1)^{i-j}\binom{i-1}{j-1} i^{i-j}]_{1\le i,j\le n}$ and $B = [\binom{i}{j} j^{i-j}]_{1\le i,j\le n}$ are inverses of each other.*

This result is not new and an equivalent dual version (conjugated by the matrix $[(-1)^j\delta_{i,j}]_{1\le i,j\le n}$) was already obtained by Sagan in [22], also by using essentially Proposition 2.10, but with a completely different poset. So we can consider this to be a new proof of that result (see also [18]).

Chapoton and Vallette [10] consider another poset that is quite similar to the poset of weighted partitions, namely the poset of pointed partitions. A pointed partition of $[n]$ is a partition of $[n]$ in which one element of each block is distinguished. The covering relation is given by

$$\{(A_1,a_1),(A_2,a_2),...,(A_s,a_s)\} \lessdot \{(B_1,b_1),(B_2,b_2),...,(B_t,b_t)\},$$

where a_i is the distinguished element of A_i and b_i is the distinguished element of B_i for each i, if the following conditions hold:

- $\{A_1, A_2, \dots, A_s\} \lessdot \{B_1, B_2, \dots, B_t\}$ in Π_n
- if $B_k = A_i \cup A_j$, where $i \ne j$, then $b_k \in \{a_i, a_j\}$
- if $B_k = A_i$ then $b_k = a_i$.

Let Π_n^p be the poset of pointed partitions of $[n]$. It is easy to see that there is a rank preserving bijection between Π_n^w and Π_n^p. It follows that both posets have the same Whitney numbers of the second kind. Since both posets are uniform, it follows from Proposition 2.10 that both posets have the same Whitney numbers of the first kind and thus the same characteristic polynomial. The following result of Chapoton and Vallette [10] is therefore equivalent to Theorem 2.8.

Corollary 2.12 (Chapoton and Vallette [10])**.** *For all $n \ge 1$, the characteristic polynomial of Π_n^p is given by*

$$\chi_{\Pi_n^p}(x) = (x-n)^{n-1}. \tag{2.7}$$

Consequently,

$$\mu_{\widehat{\Pi_n^p}}(\hat{0},\hat{1}) = (-1)^n(n-1)^{n-1}.$$

One can also compute the Möbius function for all intervals of Π_n^p from (2.7). Indeed, since all n maximal intervals are isomorphic to each other, the Möbius invariant can be obtained from (2.7) by setting $x = 0$ and then dividing by n. This yields for all i,

$$(-1)^n \mu_{\widehat{\Pi_n^p}}(\hat{0}, ([n], i)) = n^{n-2},$$

which is the number of trees on node set $[n]$. The Möbius function on other intervals can be computed from this since all intervals of Π_n^p are isomorphic to products of maximal intervals of "smaller" posets of pointed partitions.

3. Homotopy type of the poset of weighted partitions

In this section we use EL-shellability to determine the homotopy type of the intervals of $\widehat{\Pi_n^w}$ and to show that $\widehat{\Pi_n^w}$ is Cohen-Macaulay, extending a result of Dotsenko and Khoroshkin [12], in which operad theory is used to prove that all intervals of Π_n^w are Cohen-Macaulay. Some prior attempts to establish shellability of the maximal intervals are discussed in Remark 3.8.

3.1. **EL-shellability.**

After reviewing some basic facts from the theory of lexicographic shellability (cf. [5], [7], [8], [33]), we will present our main results on lexicographic shellability of the poset of weighted partitions.

An *edge labeling* of a bounded poset P is a map $\lambda : \mathcal{E}(P) \to \Lambda$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of P, i.e., the covering relations $x \lessdot y$ of P, and Λ is some poset. Given an edge labeling $\lambda : \mathcal{E}(P) \to \Lambda$, one can associate a label word

$$\lambda(c) = \lambda(x_0, x_1)\lambda(x_1, x_2) \cdots \lambda(x_{t-1}, x_t)$$

with each maximal chain $c = (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{t-1} \lessdot x_t = \hat{1})$. We say that c is *increasing* if its label word $\lambda(c)$ is *strictly* increasing. That is, c is increasing if

$$\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{t-1}, x_t).$$

We say that c is *ascent-free* (or decreasing, falling) if its label word $\lambda(c)$ has no ascents, i.e. $\lambda(x_i, x_{i+1}) \not< \lambda(x_{i+1}, x_{i+2})$, for all $i = 0, \dots, t-2$. We can partially order the maximal chains lexicographically by using the lexicographic order on the corresponding label words. Any edge labeling λ of P restricts to an edge labeling of each closed interval $[x, y]$ of P. So we may refer to increasing and ascent-free maximal chains of $[x, y]$, and lexicographic order of maximal chains of $[x, y]$.

Definition 3.1. Let P be a bounded poset. *An edge-lexicographical labeling* (EL-labeling, for short) of P is an edge labeling such that in each closed interval $[x, y]$ of P, there is a unique increasing maximal chain, and this chain lexicographically precedes all other maximal chains of $[x, y]$. A poset that admits an EL-labeling is said to be *EL-shellable*.

Note that if P is EL-shellable then so is every closed interval of P.

A classical EL-labeling for the partition lattice Π_n is obtained as follows. Let $\Lambda = \{(i, j) \in [n-1] \times [n] : i < j\}$ with lexicographic order as the order relation on Λ. If $x \lessdot y$ in Π_n then y is obtained from x by merging two blocks A and B, where $\min A < \min B$. Let $\lambda(x, y) = (\min A, \min B)$. This defines a map $\lambda : \mathcal{E}(\Pi_n) \to \Lambda$. By viewing Λ as the set of atoms of Π_n, one sees that this labeling is a special case of an edge labeling for geometric lattices, which first appeared in Stanley [24] and was one of Björner's [5] initial examples of an EL-labeling.

We now generalize the Björner-Stanley EL-labeling of Π_n to the weighted partition lattice. For each $a \in [n]$, let $\Gamma_a := \{(a, b)^u : a < b \leq n+1,\ u \in \{0, 1\}\}$. We partially order Γ_a by letting $(a, b)^u \leq (a, c)^v$ if $b \leq c$ and $u \leq v$. Note that Γ_a is isomorphic to the direct product of the chain $a+1 < a+2 < \cdots < n+1$ and the chain $0 < 1$. Now define Λ_n to be the ordinal sum $\Lambda_n := \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n$. (See Figure 2b.)

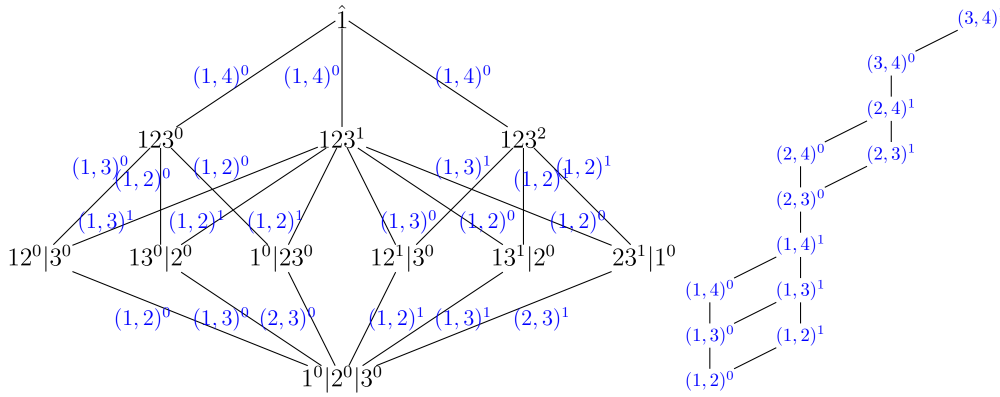
If $x \lessdot y$ in Π_n^w then y is obtained from x by merging two blocks A and B, where $\min A < \min B$, and assigning weight $u + w_A + w_B$ to the resulting block $A \cup B$, where $u \in \{0, 1\}$, and w_A, w_B are the respective weights of A and B in the weighted partition x. Let

$$\lambda(x \lessdot y) = (\min A, \min B)^u.$$

This defines a map $\lambda : \mathcal{E}(\Pi_n^w) \to \Lambda_n$. We extend this map to $\lambda : \mathcal{E}(\widehat{\Pi_n^w}) \to \Lambda_n$ by letting $\lambda([n]^i \lessdot \hat{1}) = (1, n+1)^0$, for all $i = 0, \ldots, n-1$. (See Figure 2a.) Note that when λ is restricted to the intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$, which are both isomorphic to Π_n, the labeling reduces to the Björner-Stanley EL-labeling of Π_n.

Theorem 3.2. *The labeling $\lambda : \mathcal{E}(\widehat{\Pi_n^w}) \to \Lambda_n$ defined above is an EL-labeling of $\widehat{\Pi_n^w}$.*

Proof. We need to show that in every closed interval of $\widehat{\Pi_n^w}$ there is a unique increasing chain (from bottom to top), which is also lexicographically first. Let ρ denote the rank function of $\widehat{\Pi_n^w}$. We divide the proof into 4 cases:



(A) Labeling λ (B) Λ_3

FIGURE 2. EL-labeling of the poset $\widehat{\Pi_3^w}$

(1) *Intervals of the form* $[\hat{0}, [n]^r]$. Since, from bottom to top, the last step of merging two blocks includes a block that contains 1, all of the maximal chains have a final label of the form $(1, m)^u$, and so any increasing maximal chain has to have label word $(1, 2)^{u_1}(1, 3)^{u_2} \cdots (1, n)^{u_{n-1}}$ with $u_i = 0$ for $i \leq n - 1 - r$ and $u_i = 1$ for $i > n - 1 - r$. This label word is lexicographically first and the only chain with this label word is (listing only the nonsingleton blocks)

$$\hat{0} \lessdot 12^{u_1} \lessdot 123^{u_1+u_2} \lessdot \cdots \lessdot 123 \cdots n^r.$$

(2) *Intervals of the form* $[\hat{0}, \alpha]$ *for* $\rho(\alpha) < n - 1$. Let $A_1^{u_1}, \ldots, A_k^{u_k}$ be the weighted blocks of α, where $\min A_i < \min A_j$ if $i < j$. For each i, let $m_i = \min A_i$. By the previous case, in each of the posets $[\hat{0}, A_i^{u_i}]$ there is only one increasing manner of merging the blocks, and the labels of the increasing chain belong to the label set Γ_{m_i}. The increasing chain is also lexicographically first. Consider the maximal chain of $[\hat{0}, \alpha]$ obtained by first merging the blocks of the increasing chain in $[\hat{0}, A_1^{u_1}]$, then the ones in the increasing chain in $[\hat{0}, A_2^{u_2}]$, and so on. The constructed chain is still increasing since the labels in Γ_{m_i} are less than the labels in $\Gamma_{m_{i+1}}$ for each $i = 1, \ldots, k - 1$. It is not difficult to see that this is the only increasing chain of $[\hat{0}, \alpha]$ and that it is lexicographically first.

(3) *The interval* $[\hat{0}, \hat{1}]$. An increasing chain c of this interval must be of the form $c' \cup \{\hat{1}\}$, where c' is the unique increasing chain of some interval $[\hat{0}, [n]^r]$. By Case 1, the label word of c' ends

in $(1,n)^u$ for some u. For c to be increasing, u must be 0. But $u = 0$ only in the interval $[\hat{0}, [n]^0]$. Hence the unique increasing chain of $[\hat{0}, [n]^0]$ concatenated with $\hat{1}$ is the only increasing chain of $[\hat{0}, \hat{1}]$. It is clearly lexicographically first.

(4) *Intervals of the form $[\alpha, \beta]$ for $\alpha \neq \hat{0}$.* We extend the definition of Π_n^w to Π_S^w, where S is an arbitrary finite set of positive integers, by considering partitions of S rather than $[n]$. We also extend the definition of the labeling λ to $\widehat{\Pi_S^w}$. Now we can identify the interval $[\alpha, \hat{1}]$ with $\widehat{\Pi_S^w}$, where S is the set of minimum elements of the blocks of α, by replacing each block A of α by its minimum element and subtracting the weight of A from the weight of the block containing A in each weighted partition of $[\alpha, \hat{1}]$. This isomorphism preserves the labeling and so the three previous cases show that there is a unique increasing chain in $[\alpha, \beta]$ that is also lexicographically first.

□

3.2. **Topological consequences.** When we attribute a topological property to a poset P, we are really attributing the property to the order complex $\Delta(P)$, which is defined to be the simplicial complex whose faces are the chains of P. For instance, by $\tilde{H}_r(P; \mathbf{k})$ and $\tilde{H}^r(P; \mathbf{k})$ we mean, respectively, reduced simplicial homology and cohomology of the order complex $\Delta(P)$, taken over $\mathbf{k}$, where $\mathbf{k}$ is an arbitrary field or the ring of integers $\mathbb{Z}$. (We will usually omit the $\mathbf{k}$ and write just $\tilde{H}_r(P)$ and $\tilde{H}^r(P)$.) For a brief review of the homology and cohomology of posets, see the appendix (Section A).

The fundamental link between lexicographic shellability and topology is given is the following result. Recall that the proper part $\overline{P}$ of a bounded poset P is defined by $\overline{P} := P \setminus \{\hat{0}, \hat{1}\}$. Hence, if c is a maximal chain of P then $\bar{c}$ denotes the maximal chain of $\overline{P}$ given by $c \setminus \{\hat{0}, \hat{1}\}$.

Theorem 3.3 (Björner and Wachs [8]). *Let λ be an EL-labeling of a bounded poset P. Then for all $x < y$ in P,*

(1) *the open interval (x, y) is homotopy equivalent to a wedge of spheres, where for each $r \in \mathbb{N}$ the number of spheres of dimension r is the number of ascent-free maximal chains of the closed interval $[x, y]$ of length $r + 2$.*
(2) *the set*

$$\{\bar{c} : c \text{ is an ascent-free maximal chain of } [x, y] \text{ of length } r + 2\}$$

forms a basis for cohomology $\tilde{H}^r((x, y))$, for all r.

Since the Möbius invariant of a bounded poset P equals the reduced Euler characteristic of the order complex of $\overline{P}$, the Euler-Poincaré formula implies the following corollary.

Corollary 3.4. *Let P be a pure EL-shellable poset of length n. Then*

(1) *$\overline{P}$ has the homotopy type of a wedge of spheres all of dimension $n-2$, where the number of spheres is $|\mu_P(\hat{0},\hat{1})|$.*
(2) *P is Cohen-Macaulay, which means that $\tilde{H}_i((x,y)) = 0$ for all $x < y$ in P and $i < l([x,y]) - 2$.*

In [12] Dotsenko and Khoroshkin use operad theory to prove that all intervals of Π_n^w are Cohen-Macaulay. The following extension of their result is a consequence of Theorem 3.2.

Corollary 3.5. *The poset $\widehat{\Pi_n^w}$ is Cohen-Macaulay.*

Now by Theorem 3.2, Proposition 2.6 and Corollary 2.4 we have,

Theorem 3.6. *For all $n \geq 1$,*

(1) *$\Pi_n^w \setminus \{\hat{0}\}$ has the homotopy type of a wedge of $(n-1)^{n-1}$ spheres of dimension $n-2$,*
(2) *$(\hat{0},[n]^i)$ has the homotopy type of a wedge of $|\mathcal{T}_{n,i}|$ spheres of dimension $n-3$ for all $i \in \{0,1,\dots,n-1\}$.*

It follows from Theorem 3.6 (and Proposition A.1 in the appendix) that top cohomology $\tilde{H}^{n-2}(\Pi_n^w \setminus \hat{0})$ and $\tilde{H}^{n-3}((\hat{0},[n]^i))$ are free $\mathbf{k}$-modules, which are isomorphic to the corresponding top homology modules, that is

$$\tilde{H}^{n-2}(\Pi_n^w \setminus \hat{0}) \simeq \tilde{H}_{n-2}(\Pi_n^w \setminus \hat{0})$$

and

$$\tilde{H}^{n-3}((\hat{0},[n]^i)) \simeq \tilde{H}_{n-3}((\hat{0},[n]^i))$$

for $0 \leq i \leq n-1$. Moreover, we have the following result.

Corollary 3.7. *For $0 \leq i \leq n-1$,*

$$\begin{aligned} \operatorname{rank} \tilde{H}_{n-2}(\Pi_n^w \setminus \hat{0}) &= (n-1)^{n-1} \\ \operatorname{rank} \tilde{H}_{n-3}((\hat{0},[n]^i)) &= |\mathcal{T}_{n,i}| \\ \operatorname{rank} \bigoplus_{i=0}^{n-1} \tilde{H}_{n-3}((\hat{0},[n]^i)) &= n^{n-1}. \end{aligned}$$

Remark 3.8. In a prior attempt to establish Cohen-Macaulayness of each maximal interval $[\hat{0},[n]^i]$ of Π_n^w, it is argued in [11] that the intervals are totally semimodular and hence CL-shellable[2]. In [28] it is noted

[2]CL-shellability is a property more general the EL-shellability, which also implies Cohen-Macaulaynes; see [7], [8] or [33]

that this is not the case and a proposed recursive atom ordering[3] of each maximal interval $[\hat{0},[n]^i]$ is given in order to establish CL-shellability. In [28, Proof of Proposition 3.9] it is claimed that given any linear ordering $\{i_1,j_1\},\{i_2,j_2\},\cdots,\{i_m,j_m\}$ of the atoms of Π_n (the singleton blocks have been omitted), the linear ordering

$$\{i_1,j_1\}^0,\{i_1,j_1\}^1,\{i_2,j_2\}^0,\{i_2,j_2\}^1\cdots\{i_m,j_m\}^0,\{i_m,j_m\}^1 \tag{3.1}$$

satisfies the criteria for being a recursive atom ordering of $[\hat{0},[n]^i]$, where $1\leq i\leq n-2$. We note here that one of the requisite conditions in the definition of recursive atom ordering fails to hold when $n=4$ and $i=2$. Indeed, assume (without loss of generality) that the first two atoms in the atom ordering of $[\hat{0},[4]^2]$ given in (3.1) are $\{1,2\}^0$ and $\{1,2\}^1$. Then the atoms of the interval $[\{1,2\}^1,[4]^2]$ that cover $\{1,2\}^0$ are $\{1,2,3\}^1$ and $\{1,2,4\}^1$. So by the definition of recursive atom ordering one of these covers must come first in any recursive atom ordering of $[\{1,2\}^1,[4]^2]$ and the other must come second. But this contradicts the form of (3.1) applied to the interval $[\{1,2\}^1,[4]^2]$ which requires the atom $\{1,2,3\}^2$ to immediately follow the atom $\{1,2,3\}^1$ and the atom $\{1,2,4\}^2$ to immediately follow the atom $\{1,2,4\}^1$. The proof of Proposition 3.9 of [28] breaks down in the second from last paragraph.

4. Connection with the doubly bracketed free Lie algebra

4.1. **The doubly bracketed free Lie algebra.** In this section $\mathbf{k}$ denotes an arbitrary field. Recall that a *Lie bracket* on a vector space V is a bilinear binary product $[\cdot,\cdot]:V\times V\to V$ such that for all $x,y,z\in V$,

$$[x,y]=-[y,x] \qquad \text{(Antisymmetry)} \tag{4.1}$$

$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0 \qquad \text{(Jacobi Identity).} \tag{4.2}$$

The *free Lie algebra on* $[n]$ (over the field $\mathbf{k}$) is the $\mathbf{k}$-vector space generated by the elements of $[n]$ and all the possible bracketings involving these elements subject only to the relations (4.1) and (4.2). Let $\mathcal{L}ie(n)$ denote the *multilinear* component of the free Lie algebra on $[n]$, i.e., the subspace generated by bracketings that contain each element of $[n]$ exactly once. For example $[[2,3],1]$ is an element of $\mathcal{L}ie(3)$, while $[[2,3],2]$ is not.

[3]See [7], [8] or [33] for the definition of recursive atom ordering. The property of admitting a recursive atom ordering is equivalent to that of being CL-shellable.

Now let V be a vector space equipped with two Lie brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$. The brackets are said to be *compatible* if any linear combination of them is a Lie bracket. As pointed out in [11, 20], compatibility is equivalent to the *mixed Jacobi* condition: for all $x, y, z \in V$,

$$[x,\langle y,z\rangle]+[z,\langle x,y\rangle]+[y,\langle z,x\rangle]+\langle x,[y,z]\rangle+\langle z,[x,y]\rangle+\langle y,[z,x]\rangle=0. \tag{4.3}$$

Let $\mathcal{L}ie_2(n)$ denote the multilinear component of the free Lie algebra on $[n]$ with two compatible brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$, that is, the multilinear component of the $\mathbf{k}$-vector space generated by (mixed) bracketings of elements of $[n]$ subject only to the five relations given by (4.1) and (4.2), for each bracket, and (4.3). We will call the bracketed words that generate $\mathcal{L}ie_2(n)$ *bracketed permutations*.

It will be convenient to refer to the bracket $[\cdot,\cdot]$ as the *blue* bracket and the bracket $\langle\cdot,\cdot\rangle$ as the *red* bracket. For each i, let $\mathcal{L}ie_2(n,i)$ be the subspace of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with exactly i red brackets and $n-1-i$ blue brackets.

A permutation $\tau \in \mathfrak{S}_n$ acts on the bracketed permutations by replacing each letter i by $\tau(i)$. For example $(1,2)\ \langle[\langle 3,5\rangle,[2,4]],1\rangle = \langle[\langle 3,5\rangle,[1,4]],2\rangle$. Since this action respects the five relations, it induces a representation of $\mathfrak{S}_n$ on $\mathcal{L}ie_2(n)$. Since this action also preserves the number of red and blue brackets, we have the following decomposition into $\mathfrak{S}_n$-submodules: $\mathcal{L}ie_2(n) = \oplus_{i=0}^{n-1}\mathcal{L}ie_2(n,i)$. Note that by replacing red brackets with blue brackets and vice verce, we get the $\mathfrak{S}_n$-module isomorphism,

$$\mathcal{L}ie_2(n,i) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1-i)$$

for all i. Also note that

$$\mathcal{L}ie_2(n,0) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(n).$$

A *bicolored binary tree* is a complete binary tree (i.e., every internal node has a left and a right child) for which each internal node has been colored red or blue. For a bicolored binary tree T with n leaves and $\sigma \in \mathfrak{S}_n$, define the *labeled bicolored binary tree* (T,σ) to be the tree T whose jth leaf from left to right has been labeled $\sigma(j)$. We denote by $\mathcal{BT}_n$ the set of labeled bicolored binary trees with n leaves and by $\mathcal{BT}_{n,i}$ the set of labeled bicolored binary trees with n nodes and i red internal nodes.

It will also be convenient to consider labeled bicolored trees whose label set is more general than $[n]$. For a finite set A, let $\mathcal{BT}_A$ be the set of bicolored binary trees whose leaves are labeled by a permutation of A and $\mathcal{BT}_{A,i}$ be the subset of $\mathcal{BT}_A$ consisting of trees with i red

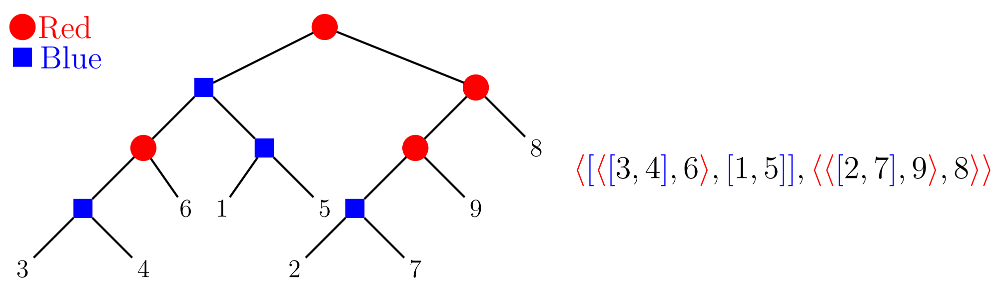


FIGURE 3. Example of a tree $(T, 346152798) \in \mathcal{BT}_{9,4}$ and $[T, 346152798] \in \mathcal{L}ie_2(9, 4)$

internal nodes. If $(S, \alpha) \in \mathcal{BT}_A$ and $(T, \beta) \in \mathcal{BT}_B$, where A and B are disjoint finite sets, and col $\in$ {red, blue} then $(S, \alpha)\overset{\mathrm{col}}{\wedge}(S, \beta)$ denotes the tree in $\mathcal{BT}_{A\cup B}$ whose left subtree is (S, α), right subtree is (T, β), and root color is col.

We can represent the bracketed permutations that generate $\mathcal{L}ie_2(n)$ with labeled bicolored binary trees. More precisely, let (T_1, σ_1) and (T_2, σ_2) be the left and right labeled subtrees of the root r of (T, σ). Then define recursively

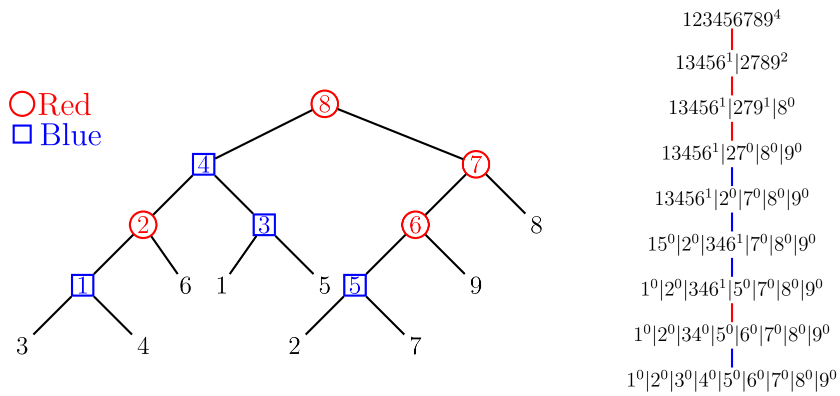
$$[T, \sigma] = \begin{cases} [[T_1, \sigma_1], [T_2, \sigma_2]] & \text{if } r \text{ is blue and } n > 1 \\ \langle [T_1, \sigma_1], [T_2, \sigma_2] \rangle & \text{if } r \text{ is red and } n > 1 \\ \sigma & \text{if } n = 1. \end{cases} \tag{4.4}$$

Clearly $(T, \sigma) \in \mathcal{BT}_{n,i}$ if and only if $[T, \sigma]$ is a bracketed permutation of $\mathcal{L}ie_2(n, i)$. See Figure 3.

4.2. **A generating set for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$.** In this section the ring of coefficients $\mathbf{k}$ for cohomology is either $\mathbb{Z}$ or an arbitrary field.

The top dimensional cohomology of a pure poset P, say of length ℓ, has a particularly simple description (see Appendix A). Let $\mathcal{M}(P)$ denote the set of maximal chains of P and let $\mathcal{M}'(P)$ denote the set of chains of length $\ell - 1$. We view the coboundary map δ as a map from the chain space of P to itself, which takes chains of length d to chains of length $d + 1$ for all d. Since the image of δ on the top chain space (i.e. the space spanned by $\mathcal{M}(P)$) is 0, the kernel is the entire top chain space. Hence top cohomology is the quotient of the space spanned by $\mathcal{M}(P)$ by the image of the space spanned by $\mathcal{M}'(P)$. The image of $\mathcal{M}'(P)$ is what we call the coboundary relations. We thus have the following presentation of the top cohomology

$$\tilde{H}^\ell(P) = \langle \mathcal{M}(P) | \text{ coboundary relations} \rangle.$$



(A) $(T,\sigma) \in \mathcal{BT}_9$ (B) $\mathrm{c}(T,\sigma)$

FIGURE 4. Example of postorder (internal nodes) of the binary tree T of Figure 3 and the chain $\mathrm{c}(T,\sigma)$

Recall that the *postorder listing* of the internal nodes of a binary tree T is defined recursively as follows: first list the internal nodes of the left subtree in postorder, then list the internal nodes of the right subtree in postorder, and finally list the root. The postorder listing of the internal nodes of the binary tree of Figure 3 is illustrated in Figure 4a.

Given k blocks $A_1^{w_1}, A_2^{w_2}, \dots, A_k^{w_k}$ in a weighted partition α and $u \in \{0,\dots,k-1\}$, by *u-merge* these blocks we mean remove them from α and replace them by the block $(\bigcup A_i)^{\sum w_i + u}$. Given col $\in \{\text{blue, red}\}$, let

$$u(\mathrm{col}) = \begin{cases} 0 & \text{if col} = \text{blue} \\ 1 & \text{if col} = \text{red.} \end{cases}$$

For $(T,\sigma) \in \mathcal{BT}_{A,i}$, let $\pi(T,\sigma) = A^i$.

Definition 4.1. For $(T,\sigma) \in \mathcal{BT}_n$ and $k \in [n-1]$, let $T_k = L_k \overset{\mathrm{col}_k}{\wedge} R_k$ be the subtree of (T,σ) rooted at the kth node listed in postorder. The chain $\mathrm{c}(T,\sigma) \in \mathcal{M}(\Pi_n^w)$ is the one whose rank k weighted partition is obtained from the rank $k-1$ weighted partition by $u(\mathrm{col}_k)$-merging the blocks $\pi(L_k)$ and $\pi(R_k)$. See Figure 4.

Not all maximal chains in $\mathcal{M}(\Pi_n^w)$ can be described as $\mathrm{c}(T,\sigma)$. For some maximal chains postordering of the internal nodes is not enough to describe the process of merging the blocks. We need a more flexible construction in terms of linear extensions (cf. [31]). Let $v_1,\dots,v_n$ be the postorder listing of the internal nodes of T. A listing $v_{\tau(1)}, v_{\tau(2)}, \dots, v_{\tau(n-1)}$ of the internal nodes such that each node precedes its parent is said to be a *linear extension* of T. We will say that the permutation τ induces the linear extension. In particular, the identity permutation ε induces postorder which is a linear extension. Denote by $\mathcal{E}(T)$ the set of permutations that induce linear extensions of the internal nodes of T. So we extend the construction of $\mathrm{c}(T,\sigma)$ by letting $\mathrm{c}(T,\sigma,\tau)$ be the chain in $\mathcal{M}(\Pi_n^w)$ whose rank k weighted partition is obtained from the rank $k-1$ weighted partition by $u(\mathrm{col}_{\tau(k)})$-merging the blocks $\pi(L_{\tau(k)})$ and $\pi(R_{\tau(k)})$, where $L_i \overset{\mathrm{col}_i}{\wedge} R_i$ is the subtree rooted at v_i. In particular, $\mathrm{c}(T,\sigma)=\mathrm{c}(T,\sigma,\varepsilon)$. From each maximal chain we can easily construct a binary tree and a linear extension that encodes the merging instructions along the chain. So it follows that any maximal chain can be obtained in this form.

Lemma 4.2 ([31, Lemma 5.1])**.** *Let T be a binary tree. Then*

(1) $\varepsilon \in \mathcal{E}(T)$
(2) *If $\tau \in \mathcal{E}(T)$ and $\tau(i) > \tau(i+1)$ then $\tau(i,i+1) \in \mathcal{E}(T)$,*

where $\tau(i,i+1)$ denotes the product of τ and the transposition $(i,i+1)$ in the symmetric group.

Proof. Postorder ε is a linear extension since in postorder we list children before parents. Now, $\tau(i) > \tau(i+1)$ means that $v_{\tau(i+1)}$ is listed in postorder before $v_{\tau(i)}$, and so $v_{\tau(i+1)}$ cannot be an ancestor of $v_{\tau(i)}$. This implies that $\tau(i,i+1)$ is also a linear extension. □

The number of inversions of a permutation $\tau \in \mathfrak{S}_n$ is defined by $\mathrm{inv}(\tau) := |\{(i,j) : 1 \le i < j \le n,\ \tau(i) > \tau(j)\}|$ and the sign of τ is defined by $\mathrm{sgn}(\tau) := (-1)^{\mathrm{inv}(\tau)}$. For $T \in \mathcal{BT}_{n,i}$, $\sigma \in \mathfrak{S}_n$, and $\tau \in \mathcal{E}(T)$, write $\bar{\mathrm{c}}(T,\sigma,\tau)$ for $\overline{c(T,\sigma,\tau)} := c(T,\sigma,\tau) \setminus \{\hat{0},[n]^i\}$ and $\bar{\mathrm{c}}(T,\sigma)$ for $\overline{c(T,\sigma)} := c(T,\sigma) \setminus \{\hat{0},[n]^i\}$.

Lemma 4.3 (cf. [31, Lemma 5.2])**.** *Let $T \in \mathcal{BT}_{n,i}$, $\sigma \in \mathfrak{S}_n$, $\tau \in \mathcal{E}(T)$. Then in $\tilde{H}^{n-3}((\hat{0},[n]^i))$*

$$\bar{\mathrm{c}}(T,\sigma,\tau) = \mathrm{sgn}(\tau)\bar{\mathrm{c}}(T,\sigma).$$

Proof. We proceed by induction on $\mathrm{inv}(\tau)$. If $\mathrm{inv}(\tau)=0$ then $\tau=\varepsilon$ and the result is trivial. If $\mathrm{inv}(\tau) \ge 1$, then there is some descent $\tau(i) > \tau(i+1)$ and by Lemma 4.2, $\tau(i,i+1) \in \mathcal{E}(T)$. Since $\mathrm{inv}(\tau(i,i+1)) = \mathrm{inv}(\tau)-1$, by induction we have,

$$\bar{\mathrm{c}}(T,\sigma,\tau(i,i+1)) = \mathrm{sgn}(\tau(i,i+1))\bar{\mathrm{c}}(T,\sigma) = -\,\mathrm{sgn}(\tau)\bar{\mathrm{c}}(T,\sigma).$$

We have to show then that

$$\bar{\mathrm{c}}(T,\sigma,\tau) = -\bar{\mathrm{c}}(T,\sigma,\tau(i,i+1)).$$

By the proof of Lemma 4.2 we know that the internal nodes $v_{\tau(i)}$ and $v_{\tau(i+1)}$ are unrelated in T and so $\pi(L_{\tau(i)})$, $\pi(R_{\tau(i)})$, $\pi(L_{\tau(i+1)})$ and $\pi(R_{\tau(i+1)})$ are pairwise disjoint sets which are all blocks of the rank $i-1$ partition in both $\bar{\mathrm{c}}(T,\sigma,\tau)$ and $\bar{\mathrm{c}}(T,\sigma,\tau(i,i+1))$. The blocks $\pi(L_{\tau(i)} \wedge R_{\tau(i)})$ and $\pi(L_{\tau(i+1)} \wedge R_{\tau(i+1)})$ are blocks of the rank $i+1$ partition in both $\bar{\mathrm{c}}(T,\sigma,\tau)$ and $\bar{\mathrm{c}}(T,\sigma,\tau(i,i+1))$. Hence the maximal chains $\bar{\mathrm{c}}(T,\sigma,\tau)$ and $\bar{\mathrm{c}}(T,\sigma,\tau(i,i+1))$ only differ at rank i. So if we denote by c either of these maximal chains with the rank i partition removed we get, using equation (A.1), a cohomology relation given by

$$\delta(c) = (-1)^i(\bar{\mathrm{c}}(T,\sigma,\tau) + \bar{\mathrm{c}}(T,\sigma,\tau(i,i+1)))$$

as desired. □

We conclude that in cohomology any maximal chain $c \in \mathcal{M}(\Pi_n^w)$ is cohomology equivalent to a chain of the form $\mathrm{c}(T,\sigma)$, more precisely, in cohomology $\bar{c} = \pm\bar{\mathrm{c}}(T,\sigma)$.

We will make further use of the elementary cohomology relations that are obtained by setting the coboundary (given in (A.1)) of a codimension 1 chain in $(\hat{0},[n]^i)$ equal to 0. There are three types of codimension 1 chains, which correspond to the three types of intervals of length 2 (see Figure 5). Indeed, if $\bar{c}$ is a codimension 1 chain of $(\hat{0},[n]^i)$ then $c = \bar{c} \cup \{\hat{0},[n]^i\}$ is unrefinable except between one pair of adjacent elements $x < y$, where $[x,y]$ is an interval of length 2. If the open interval $(x,y) = \{z_1,\ldots,z_k\}$ then it follows from (A.1) that

$$\delta(\bar{c}) = \pm(\bar{c} \cup \{z_1\} + \cdots + \bar{c} \cup \{z_k\}).$$

By setting $\delta(\bar{c}) = 0$ we obtain the elementary cohomology relation

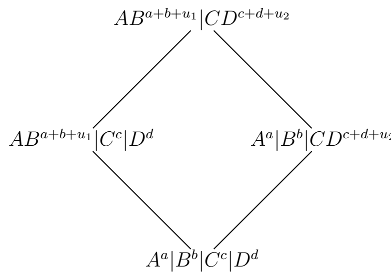
$$(\bar{c} \cup \{z_1\}) + \cdots + (\bar{c} \cup \{z_k\}) = 0.$$

Type I: Two pairs of distinct blocks of x are merged to get y. The open interval (x,y) equals $\{z_1,z_2\}$, where z_1 is obtained by u_1-merging the first pair of blocks and z_2 is obtained by u_2-merging the second pair of blocks for some $u_1,u_2 \in \{0,1\}$. Hence the Type I elementary cohomology relation is

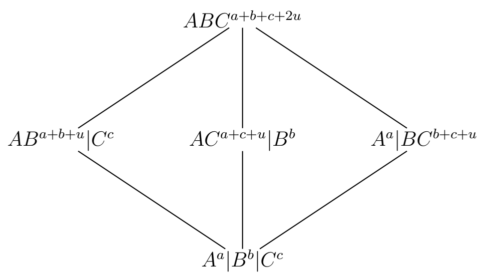
$$\bar{c} \cup \{z_1\} = -(\bar{c} \cup \{z_2\}).$$

Type II: Three distinct blocks of x are $2u$-merged to get y, where $u \in \{0,1\}$. The open interval (x,y) equals $\{z_1,z_2,z_3\}$, where each weighted partition z_i is obtained from x by u-merging two of the three blocks. Hence the Type II elementary cohomology relation is

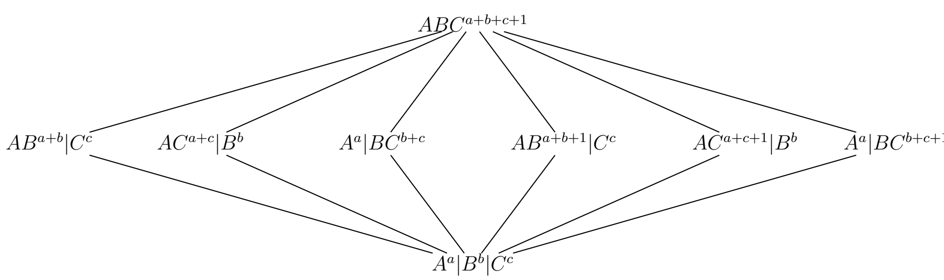
$$(\bar{c} \cup \{z_1\}) + (\bar{c} \cup \{z_2\}) + (\bar{c} \cup \{z_3\}) = 0.$$



(A) Type I



(B) Type II



(C) Type III

FIGURE 5. Intervals of length 2

Type III: Three distinct blocks of x are 1-merged to get y. The open interval (x, y) equals $\{z_1, z_2, z_3, z_4, z_5, z_6\}$, where each weighted partition z_i is obtained from x by either 0-merging or 1-merging two of the three blocks. Hence the Type III elementary cohomology relation is

$$(\bar{c}\cup\{z_1\})+(\bar{c}\cup\{z_2\})+(\bar{c}\cup\{z_3\})+(\bar{c}\cup\{z_4\})+(\bar{c}\cup\{z_5\})+(\bar{c}\cup\{z_6\}) = 0.$$

Let $I(\Upsilon)$ denote the set of internal nodes of the labeled bicolored binary tree Υ. Recall that $\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_2$ denotes the labeled bicolored binary tree whose left subtree is Υ_1, right subtree is Υ_2 and root color is col, where col $\in$ {blue, red}. If Υ is a labeled bicolored binary tree then $\alpha(\Upsilon)\beta$ denotes a labeled bicolored binary tree with Υ as a subtree. The following result generalizes [31, Theorem 5.3].

Theorem 4.4. *The set* $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathcal{BT}_{n,i}\}$ *is a generating set for* $\tilde{H}^{n-3}((\hat{0},[n]^i))$*, subject only to the relations*

$$\bar{c}(\alpha(\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_2)\beta) = (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\bar{c}(\alpha(\Upsilon_2 \overset{\text{col}}{\wedge} \Upsilon_1)\beta), \tag{4.5}$$

(4.6)
$$\begin{aligned} \bar{c}(\alpha(\Upsilon_1 \overset{\text{col}}{\wedge} (\Upsilon_2 \overset{\text{col}}{\wedge} \Upsilon_3))\beta) \quad &+ \quad (-1)^{|I(\Upsilon_3)|}\bar{c}(\alpha((\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_2) \overset{\text{col}}{\wedge} \Upsilon_3)\beta) \\ &+ \quad (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\bar{c}(\alpha(\Upsilon_2 \overset{\text{col}}{\wedge} (\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_3))\beta), \\ &= \quad 0, \end{aligned}$$

where col $\in \{\text{blue}, \text{red}\}$*, and*

$$\begin{aligned} &\bar{c}(\alpha(\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta) \quad + \quad \bar{c}(\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta) \\ + \quad (-1)^{|I(\Upsilon_3)|}\Big(&\bar{c}(\alpha((\Upsilon_1 \overset{\text{red}}{\wedge} \Upsilon_2) \overset{\text{blue}}{\wedge} \Upsilon_3)\beta) \quad + \quad \bar{c}(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_2) \overset{\text{red}}{\wedge} \Upsilon_3)\beta)\Big) \\ + \quad (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\Big(&\bar{c}(\alpha(\Upsilon_2 \overset{\text{red}}{\wedge} (\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta) \quad + \quad \bar{c}(\alpha(\Upsilon_2 \overset{\text{blue}}{\wedge} (\Upsilon_1 \overset{\text{red}}{\wedge} \Upsilon_3))\beta)\Big) \\ &\qquad\qquad\qquad\qquad\qquad = \quad 0. \end{aligned} \tag{4.7}$$

Proof. It is an immediate consequence of Lemma 4.3 that $\{\bar{c}(\Upsilon) | \Upsilon \in \mathcal{BT}_{n,i}\}$ generates $H^{n-3}((\hat{0},[n]^i))$.

Relation (4.5): This is also a consequence of Lemma 4.3. Indeed, first note that

$$c(\alpha(\Upsilon_2 \overset{\text{col}}{\wedge} \Upsilon_1)\beta) = c(\alpha(\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_2)\beta, \tau),$$

where τ is the permutation that induces the linear extension that is just like postorder except that the internal nodes of Υ_2 are listed before those of Υ_1. Since $\text{inv}(\tau) = |I(\Upsilon_1)||I(\Upsilon_2)|$, relation (4.5) follows from Lemma 4.3. (Note that since Lemma 4.3 is a consequence only of the Type I cohomology relation, one can view (4.5) as a consequence only of the Type I cohomology relation.)

Relation (4.6): Note that the following relation is a Type II elementary cohomology relation:

$$\begin{aligned} \bar{c}(\alpha(\Upsilon_1 \overset{\text{col}}{\wedge} (\Upsilon_2 \overset{\text{col}}{\wedge} \Upsilon_3))\beta) \quad &+ \quad \bar{c}(\alpha((\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_2) \overset{\text{col}}{\wedge} \Upsilon_3)\beta, \tau_1) \\ &+ \quad \bar{c}(\alpha(\Upsilon_2 \overset{\text{col}}{\wedge} (\Upsilon_1 \overset{\text{col}}{\wedge} \Upsilon_3))\beta, \tau_2) = 0, \end{aligned}$$

where τ_1 is the permutation that induces the linear extension that is like postorder but that lists the internal nodes of Υ_3 before listing the root of $\Upsilon_1 \wedge \Upsilon_2$, and τ_2 is the permutation that induces the linear extension that is like postorder but lists the internal nodes of Υ_1 before listing the internal nodes of Υ_2. So then $\mathrm{inv}(\tau_1) = |I(\Upsilon_3)|$ and $\mathrm{inv}(\tau_2) = |I(\Upsilon_1)||I(\Upsilon_2)|$, and using Lemma 4.3 we obtain relation (4.6).

Relation (4.7): Note that the following relation is a Type III elementary cohomology relation:

$$\begin{aligned} &\bar{c}(\alpha(\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta) &+\quad& \bar{c}(\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta) \\ +\ &\bar{c}(\alpha((\Upsilon_1 \overset{\text{red}}{\wedge} \Upsilon_2) \overset{\text{blue}}{\wedge} \Upsilon_3)\beta, \tau_1) &+\quad& \bar{c}(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_2) \overset{\text{red}}{\wedge} \Upsilon_3)\beta, \tau_1) \\ +\ &\bar{c}(\alpha(\Upsilon_2 \overset{\text{red}}{\wedge} (\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta, \tau_2) &+\quad& \bar{c}(\alpha(\Upsilon_2 \overset{\text{blue}}{\wedge} (\Upsilon_1 \overset{\text{red}}{\wedge} \Upsilon_3))\beta, \tau_2) \\ & &=\quad& 0, \end{aligned}$$

where as in the previous case, τ_1 is the permutation that induces the linear extension that is like postorder but that lists the internal nodes of Υ_3 before listing the root of $\Upsilon_1 \wedge \Upsilon_2$, and τ_2 is the permutation that induces the linear extension that is like postorder but lists the internal nodes of Υ_1 before listing the internal nodes of Υ_2. So then $\mathrm{inv}(\tau_1) = |I(\Upsilon_3)|$ and $\mathrm{inv}(\tau_2) = |I(\Upsilon_1)||I(\Upsilon_2)|$, and using Lemma 4.3 we obtain relation (4.7).

To complete the proof, we need to show that these relations generate all the cohomology relations. In other words, we need to show that $\tilde{H}^{n-3}((\hat{0}, [n]^i)) = M/R$, where M is the free $\mathbf{k}$-module with basis $\{\bar{c}(T, \sigma) : (T, \sigma) \in \mathcal{BT}_{n,i}\}$ and R is the submodule spanned by elements given in the relations (4.5), (4.6), (4.7). We have already shown that $\operatorname{rank} \tilde{H}^{n-3}((\hat{0}, [n]^i)) \leq \operatorname{rank} M/R$. To complete the proof we need to establish the reverse inequality. This is postponed to Section 5.1. We will prove there, that a certain set S of maximal chains of $(\hat{0}, [n]^i)$ whose cardinality equals $\operatorname{rank} \tilde{H}^{n-3}((\hat{0}, [n]^i))$ generates M/R by showing that there is a straightening algorithm, which using only the relations (4.5),(4.6),(4.7), enables us to express every generator $\bar{c}(T, \sigma)$ as a linear combination of the elements of S. It follows that $\operatorname{rank} M/R \leq |S| = \operatorname{rank} \tilde{H}^{n-3}((\hat{0}, [n]^i))$. See Remark 5.4. $\square$

4.3. **The isomorphism.** In this section homology and cohomology are taken over an arbitrary field $\mathbf{k}$, as is $\mathcal{L}ie_2(n, i)$.

The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n^w. Indeed, let $\sigma \in \mathfrak{S}_n$ act on the weighted blocks of $\pi \in \Pi_n^w$ by replacing each element x of each weighted block of π with $\sigma(x)$. Since the maximal elements of Π_n^w are fixed by each $\sigma \in \mathfrak{S}_n$ and the order is preserved, each open interval $(\hat{0}, [n]^i)$ is a $\mathfrak{S}_n$-poset. Hence by (A.2) we have the $\mathfrak{S}_n$-module isomorphism,

$$\tilde{H}_{n-3}((\hat{0}, [n]^i)) \simeq_{\mathfrak{S}_n} \tilde{H}^{n-3}((\hat{0}, [n]^i)).$$

The symmetric group $\mathfrak{S}_n$ also acts naturally on $\mathcal{L}ie_2(n)$. Indeed, let $\sigma \in \mathfrak{S}_n$ act by replacing letter x of a bracketed permutation with $\sigma(x)$. Since this action preserves the number of brackets of each type, $\mathcal{L}ie_2(n,i)$ is an $\mathfrak{S}_n$-module for each i. In this section we obtain an explicit sign-twisted isomorphism between the $\mathfrak{S}_n$-modules $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and $\mathcal{L}ie_2(n,i)$.

Define the *sign* of a binary tree T recursively by

$$\operatorname{sgn}(T) = \begin{cases} 1 & \text{if } I(T) = \emptyset \\ (-1)^{|I(T_2)|} \operatorname{sgn}(T_1) \operatorname{sgn}(T_2) & \text{if } T = T_1 \wedge T_2 \end{cases}$$

where $I(T)$ is the set of internal nodes of the binary tree T. The sign of a bicolored binary tree is defined to be the sign of the binary tree obtained by removing the colors.

Theorem 4.5. *For each $i \in \{0, 1, \ldots, n-1\}$, there is an $\mathfrak{S}_n$-module isomorphism $\phi : \mathcal{L}ie_2(n,i) \to \tilde{H}^{n-3}((\hat{0}, [n]^i)) \otimes \operatorname{sgn}_n$ determined by*

$$\phi([T,\sigma]) = \operatorname{sgn}(\sigma) \operatorname{sgn}(T) \bar{\mathrm{c}}(T,\sigma),$$

for all $(T,\sigma) \in \mathcal{BT}_{n,i}$.

Before proving the theorem we make a few preliminary observations. The following lemma, which is implicit in [31, Proof of Theorem 5.4], is easy to prove. For a binary tree T, let $a(T)b$ denote a binary tree with T as a subtree.

Lemma 4.6. *For all binary trees T_1, T_2, T_3,*

(1) $\operatorname{sgn}(a(T_1 \wedge T_2)b) = (-1)^{|I(T_1)|+|I(T_2)|} \operatorname{sgn}(a(T_2 \wedge T_1)b)$

(2) $\operatorname{sgn}(a((T_1 \wedge T_2) \wedge T_3)b) = (-1)^{|I(T_3)|+1} \operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$

(3) $\operatorname{sgn}(a(T_2 \wedge (T_1 \wedge T_3))b) = (-1)^{|I(T_1)|+|I(T_2)|} \operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$.

For a word w denote by $l(w)$ the *length* or number of letters in w. We also have the following easy relation, which we state as a lemma.

Lemma 4.7. *For $uw_1w_2v \in \mathfrak{S}_n$, where u, w_1, w_2, v are subwords,*

$$\operatorname{sgn}(uw_1w_2v) = (-1)^{l(w_1)l(w_2)} \operatorname{sgn}(uw_2w_1v).$$

We give a presentation of $\mathcal{L}ie_2(n,i)$ in terms of labeled bicolored binary trees and a slightly modified, but clearly equivalent, form of the relations (4.1), (4.2) and (4.3) in the following proposition.

Proposition 4.8. *The set* $\{[T,\sigma] : (T,\sigma) \in \mathcal{BT}_{n,i}\}$ *is a generating set for* $\mathcal{L}ie_2(n,i)$*, subject only to the relations*

$$[\alpha(\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2)\beta] = -[\alpha(\Upsilon_2 \overset{\mathrm{col}}{\wedge} \Upsilon_1)\beta] \tag{4.8}$$

$$\begin{aligned}[\alpha(\Upsilon_1 \overset{\mathrm{col}}{\wedge} (\Upsilon_2 \overset{\mathrm{col}}{\wedge} \Upsilon_3))\beta] \;&-\; [\alpha((\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2) \overset{\mathrm{col}}{\wedge} \Upsilon_3)\beta] \\ &-\; [\alpha(\Upsilon_2 \overset{\mathrm{col}}{\wedge} (\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_3))\beta] \\ &=\; 0\end{aligned} \tag{4.9}$$

$$\begin{aligned}[\alpha(\Upsilon_1 \overset{\mathrm{red}}{\wedge} (\Upsilon_2 \overset{\mathrm{blue}}{\wedge} \Upsilon_3))\beta] \;&+\; [\alpha(\Upsilon_1 \overset{\mathrm{blue}}{\wedge} (\Upsilon_2 \overset{\mathrm{red}}{\wedge} \Upsilon_3))\beta] \\ -\; [\alpha((\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_2) \overset{\mathrm{blue}}{\wedge} \Upsilon_3)\beta] \;&-\; [\alpha((\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_2) \overset{\mathrm{red}}{\wedge} \Upsilon_3)\beta] \\ -\; [\alpha(\Upsilon_2 \overset{\mathrm{red}}{\wedge} (\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_3))\beta] \;&-\; [\alpha(\Upsilon_2 \overset{\mathrm{blue}}{\wedge} (\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_3))\beta] \\ &=\; 0.\end{aligned} \tag{4.10}$$

Proof of Theorem 4.5. The map ϕ maps generators onto generators and clearly respects the $\mathfrak{S}_n$ action. We will prove that the map ϕ extends to a well defined homomorphism by showing that the relations in $\mathcal{L}ie_2(n,i)$ of the generators in Proposition 4.8 map onto to the relations in Theorem 4.4. Since by Theorem 4.4 (whose proof will be completed in Section 5.1), the relations in Theorem 4.4 span all the relations in cohomology, this also implies that the map is an isomorphism.

For each Υ_j in the relations of Proposition 4.8, let w_j and T_j be such that $\Upsilon_j = (T_j, w_j)$. Let u be the permutation labeling the portion a of the tree corresponding to the preamble α, and let v be the permutation labeling the portion b of the tree corresponding to the tail β. Using Lemmas 4.6 and 4.7 we have the following.

Relation (4.8): Let $\wedge \in \{\overset{\mathrm{blue}}{\wedge}, \overset{\mathrm{red}}{\wedge}\}$. Then

$$\begin{aligned}\phi([\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta]) &= \mathrm{sgn}(uw_2w_1v)\,\mathrm{sgn}(a(T_2 \wedge T_1)b)\bar{c}(\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta) \\ \\ &= \mathrm{sgn}(uw_1w_2v)\,\mathrm{sgn}(a(T_1 \wedge T_2)b) \\ &\qquad \cdot (-1)^{l(w_1)l(w_2)+|I(T_1)|+|I(T_2)|}\bar{c}(\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta) \\ \\ &= \mathrm{sgn}(uw_1w_2v)\,\mathrm{sgn}(a(T_1 \wedge T_2)b) \\ &\qquad \cdot (-1)^{(|I(T_1)|+1)(|I(T_2)|+1)+|I(T_1)|+|I(T_2)|}\bar{c}(\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta) \\ \\ &= \mathrm{sgn}(uw_1w_2v)\,\mathrm{sgn}(a(T_1 \wedge T_2)b)\end{aligned}$$

$$\cdot (-1)^{|I(T_1)||I(T_2)|+1}\bar{c}(\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta).$$

Hence,

$$\begin{aligned}\phi([\alpha(\Upsilon_1 \wedge \Upsilon_2)\beta]) + \phi([\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta]) &= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(a(T_1 \wedge T_2)b) \\ &\cdot \big(\bar{c}(\alpha(\Upsilon_1 \wedge \Upsilon_2)\beta) - (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\bar{c}(\alpha(\Upsilon_2 \wedge \Upsilon_1)\beta)\big).\end{aligned}$$

We conclude that relation (4.8) maps to relation (4.5).

Relations (4.9) and (4.10): Let $\wedge, \tilde{\wedge} \in \{\overset{\text{blue}}{\wedge}, \overset{\text{red}}{\wedge}\}$. Then

$$\begin{aligned}\phi([\alpha((\Upsilon_1 \wedge \Upsilon_2)\tilde{\wedge}\Upsilon_3)\beta]) &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a((T_1 \wedge T_2) \wedge T_3)b) \\ &\quad\cdot \bar{c}(\alpha((\Upsilon_1 \wedge \Upsilon_2)\tilde{\wedge}\Upsilon_3)\beta) \\ \\ &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b) \\ &\quad\cdot (-1)^{|I(T_3)|+1}\bar{c}(\alpha((\Upsilon_1 \wedge \Upsilon_2)\tilde{\wedge}\Upsilon_3)\beta).\end{aligned}$$

$$\begin{aligned}\phi([\alpha(\Upsilon_2 \wedge (\Upsilon_1\tilde{\wedge}\Upsilon_3))\beta]) &= \operatorname{sgn}(uw_2w_1w_3v)\operatorname{sgn}(a(T_2 \wedge (T_1 \wedge T_3))b) \\ &\quad\cdot\ \bar{c}(\alpha(\Upsilon_2 \wedge (\Upsilon_1\tilde{\wedge}\Upsilon_3))\beta) \\ \\ &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b) \\ &\quad\cdot (-1)^{l(w_1)l(w_2)+|I(T_1)|+|I(T_2)|}\bar{c}(\alpha(\Upsilon_2 \wedge (\Upsilon_1\tilde{\wedge}\Upsilon_3))\beta) \\ \\ &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b) \\ &\quad\cdot (-1)^{|I(T_1)||I(T_2)|+1}\bar{c}(\alpha(\Upsilon_2 \wedge (\Upsilon_1\tilde{\wedge}\Upsilon_3))\beta).\end{aligned}$$

Hence,

$$\begin{aligned}&\phi([\alpha(\Upsilon_1\wedge(\Upsilon_2\tilde{\wedge}\Upsilon_3))\beta]) - \phi([\alpha((\Upsilon_1 \wedge \Upsilon_2)\tilde{\wedge}\Upsilon_3)\beta]) - \phi([\alpha(\Upsilon_2 \wedge (\Upsilon_1\tilde{\wedge}\Upsilon_3))\beta]) \\ &\qquad= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b) \\ &\qquad\quad\cdot \Big(\bar{\mathrm{c}}(\alpha(\Upsilon_1 \wedge (\Upsilon_2\tilde{\wedge}\Upsilon_3))\beta) + (-1)^{|I(T_3)|}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \wedge \Upsilon_2)\tilde{\wedge}\Upsilon_3)\beta) \\ &\qquad\qquad + (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \wedge (\Upsilon_1\tilde{\wedge}\Upsilon_3))\beta)\Big).\end{aligned} \tag{4.11}$$

By setting $\wedge = \tilde{\wedge}$ in (4.11) we conclude that relation (4.9) maps to relation (4.6). By adding (4.11) with $\wedge = \overset{\text{blue}}{\wedge}$ and $\tilde{\wedge} = \overset{\text{red}}{\wedge}$ to (4.11) with $\wedge = \overset{\text{red}}{\wedge}$ and $\tilde{\wedge} = \overset{\text{blue}}{\wedge}$, we are also able to conclude that relation (4.10) maps to relation (4.7). $\square$

Theorem 4.5 and Corollary 3.7 yield the following result.

Corollary 4.9 (Liu [20], Dotsenko and Khoroshkin [12])**.** *For* $0 \leq i \leq n-1$*,* $\dim \mathcal{L}ie_2(n,i) = |\mathcal{T}_{n,i}|$.

5. Combinatorial bases

Throughout this section we take homology and cohomology over the integers or over an arbitrary field $\mathbf{k}$. We present three bases for cohomology and one for homology of each interval $(\hat{0}, [n]^i)$. Two of the three cohomology bases correspond to known bases for $\mathcal{L}ie_2(n,i)$ and one appears to be new. The homology basis also appears to be new. We also present two new bases for cohomology of the full weighted partition poset $\Pi_n^w \setminus \{\hat{0}\}$.

We say that a labeled binary tree is *normalized* if the leftmost leaf of each subtree has the smallest label in the subtree. Using cohomology relation (4.5), we see that $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ is generated by maximal chains of the form $\bar{c}(T,\sigma)$, where (T,σ) is a normalized binary tree in $\mathcal{BT}_{n,i}$. The first two bases for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ presented here are subsets of this set of maximal chains.

5.1. A bicolored comb basis for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and $\mathcal{L}ie_2(n,i)$.

In this section we present a generalization of a classical basis for $\tilde{H}^{n-3}(\overline{\Pi}_n)$ and a corresponding generalization of a classical basis for $\mathcal{L}ie(n)$; the classical bases are sometimes referred to as comb bases (see [31, Section 4]). The generalization for $\mathcal{L}ie_2(n)$ is due to Bershtein, Dotsenko and Khoroshkin (see [4] and [11, Theorem 4]).

A *bicolored comb* is a normalized bicolored binary tree that satisfies the following coloring restriction: for each internal node x whose right child y is not a leaf, x is colored red and y is colored blue. Let Comb_n^2 be the set of bicolored combs in $\mathcal{BT}_n$ and let $\mathsf{Comb}_{n,i}^2$ be the set of bicolored combs in $\mathcal{BT}_{n,i}$ The set of bicolored combs for $n = 3$ is depicted in Figure 6.

We refer to such trees as bicolored *combs* because the monochromatic ones are the usual left combs in the sense of [31]; indeed if a bicolored comb is monochromatic then the right child of every internal node is a leaf and the left-most leaf label of the tree is the smallest label. In this case we get the usual left comb, which has the form,

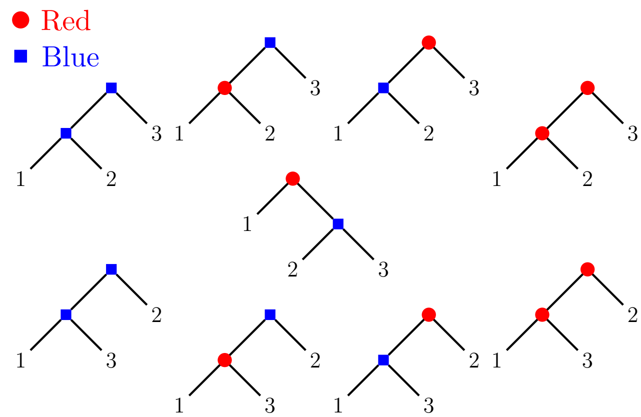
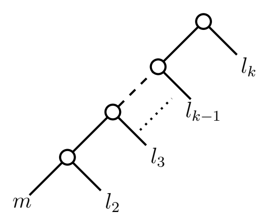


FIGURE 6. Set of bicolored combs for $n = 3$



where m and all the l_j are leaves, and m is the smallest label usually 1.

Bershtein, Dotsenko and Khoroshkin [4, Lemma 5.2] present the results that $\{[T,\sigma] : (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ spans $\mathcal{L}ie_2(n,i)$ and $|\mathsf{Comb}^2_n| = n^{n-1}$. Since it was already known from [20] and [11] that $\dim \mathcal{L}ie_2(n) = n^{n-1}$, they conclude that $\{[T,\sigma] : (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ is a basis for $\mathcal{L}ie_2(n,i)$. For the sake of completeness we give a detailed proof that the corresponding set $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ spans cohomology and we give an alternative proof of $|\mathsf{Comb}^2_n| = n^{n-1}$.

Proposition 5.1. *The set* $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}^2_{n,i}\}$ *spans* $\tilde{H}^{n-3}((\hat{0},[n]^i))$, *for all* $0 \leq i \leq n-1$.

Proof. We prove this result by "straightening" via the relations in Theorem 4.4. Define the *weight* $w(T)$ of a bicolored binary tree T to be

$$w(T) = \sum_{x \in I(T)} r(x),$$

where $I(T)$ is the set of internal nodes of T and $r(x)$ is the number of internal nodes in the right subtree of x. We say a node y of T is a *right descendent* of a node x if y can be reached from x along a path of right edges. Next we define an *inversion* of T to be a pair of internal nodes (x,y) of T such that x is blue and y is a red right descendent of x. Let $\mathrm{inv}(T)$ be the number of inversions of T. The *weight-inversion pair* of T is $(w(T),\mathrm{inv}(T))$. We order these pairs lexicographically, that is we say $(w(T),\mathrm{inv}(T)) < (w(T'),\mathrm{inv}(T'))$ if either $w(T) < w(T')$ or $w(T) = w(T')$ and $\mathrm{inv}(T) < \mathrm{inv}(T')$. For $\Upsilon = (T,\sigma) \in \mathcal{B}T_n$, let $w(\Upsilon) := w(T)$ and $\mathrm{inv}(\Upsilon) := \mathrm{inv}(T)$. Also define the weight-inversion pair of Υ to be that of T.

It follows from (4.5) that the chains of the form $\bar{c}(\Upsilon)$, where Υ is a normalized bicolored binary tree in $\mathcal{BT}_{n,i}$, span $\tilde{H}^{n-3}((\hat{0},[n]^i))$. Hence to prove the result we need only show that if $\Upsilon \in \mathcal{BT}_{n,i}$ is a normalized bicolored binary tree that is not a bicolored comb then $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\bar{c}(\Upsilon')$, where Υ' is a normalized bicolored binary tree in $\mathcal{BT}_{n,i}$ such that $(w(\Upsilon'),\mathrm{inv}(\Upsilon')) < (w(\Upsilon),\mathrm{inv}(\Upsilon))$ in lexicographic order. It will then follow by induction on the weight-inversion pair that $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\bar{c}(\Upsilon')$, where $\Upsilon' \in \mathsf{Comb}^2_{n,i}$.

Now let $\Upsilon \in \mathcal{BT}_{n,i}$ be a normalized bicolored binary tree that is not a bicolored comb. Then Υ must have a subtree of one of the following forms: $\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3)$, $\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3)$, or $\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3)$. We will show that in all three cases $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains with a smaller weight-inversion pair.

Case 1: Υ has a subtree of the form $\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3)$. We can therefore express Υ as $\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta$. Using relation (4.6) (and relation (4.5)) we have that

$$\bar{c}(\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta) = \pm\bar{c}(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_2) \overset{\text{blue}}{\wedge} \Upsilon_3)\beta) \pm \bar{c}(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_3) \overset{\text{blue}}{\wedge} \Upsilon_2)\beta).$$

(The signs in the relations of Theorem 4.4 are not relevant here and have therefore been suppressed.)

It is easy to see that

$$\begin{aligned} w(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_2) \overset{\text{blue}}{\wedge} \Upsilon_3)\beta) &= w(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_3) \overset{\text{blue}}{\wedge} \Upsilon_2)\beta) \\ &= w(\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta) - |I(\Upsilon_3)| - 1. \end{aligned}$$

Hence $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of smaller weight, and therefore of smaller weight-inversion pair.

Case 2: Υ has a subtree of the form $\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3)$. An argument analogous to that of Case 1 shows that $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of smaller weight-inversion pair.

Case 3: Υ has a subtree of the form $\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3)$. Using relation (4.7) (and relation (4.5)) we have that

$$\begin{aligned}\bar{c}(\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta) &= \pm\bar{c}(\alpha(\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta)\\ &\quad \pm\bar{c}(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_2) \overset{\text{red}}{\wedge} \Upsilon_3)\beta)\\ &\quad \pm\bar{c}(\alpha((\Upsilon_1 \overset{\text{red}}{\wedge} \Upsilon_2) \overset{\text{blue}}{\wedge} \Upsilon_3)\beta)\\ &\quad \pm\bar{c}(\alpha((\Upsilon_1 \overset{\text{blue}}{\wedge} \Upsilon_3) \overset{\text{red}}{\wedge} \Upsilon_2)\beta)\\ &\quad \pm\bar{c}(\alpha((\Upsilon_1 \overset{\text{red}}{\wedge} \Upsilon_3) \overset{\text{blue}}{\wedge} \Upsilon_2)\beta).\end{aligned}$$

Just as in Case 1, all the labeled bicolored trees on the right hand side of the equation, except for the first, have weight smaller than that of $\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta$. The first labeled bicolored tree $\alpha(\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta$ has the same weight as that of $\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta$. However the inversion number is reduced, that is

$$\operatorname{inv}(\alpha(\Upsilon_1 \overset{\text{red}}{\wedge} (\Upsilon_2 \overset{\text{blue}}{\wedge} \Upsilon_3))\beta) = \operatorname{inv}(\alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta) - 1.$$

Hence the weight-inversion pair for the first bicolored labeled tree is less than that of $\Upsilon := \alpha(\Upsilon_1 \overset{\text{blue}}{\wedge} (\Upsilon_2 \overset{\text{red}}{\wedge} \Upsilon_3))\beta$ just as it is for the other bicolored labeled trees on the right hand side of the equation. We conclude that $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of smaller weight-inversion pair. $\square$

Proposition 5.2 (Bershtein, Dotsenko and Khoroshkin [4])**.** *Let $n \geq 1$. Then $|\mathsf{Comb}^2_n| = n^{n-1}$.*

Proof. We present a different proof than that of [4]. Our proof is by induction on n. The cases $|\mathsf{Comb}^2_1| = 1$ and $|\mathsf{Comb}^2_2| = 2$ are trivially verified. For $n \geq 3$ assume that $|\mathsf{Comb}^2_k| = k^{k-1}$ for any $k < n$. We claim that

$$|\mathsf{Comb}^2_n| = (n-1)^{n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k} (n-k)^{n-k-1}(k-1)^{k-1}. \tag{5.1}$$

To prove the claim we show that the term that precedes the summation counts blue-rooted bicolored combs and the kth term of the sum counts red-rooted bicolored combs whose right subtree has k leaves. To construct a blue-rooted bicolored comb $T \in \mathsf{Comb}^2_n$, we can choose the right subtree, which is a leaf, in $n-1$ different ways, and the left subtree, which is a bicolored comb, in $(n-1)^{n-2}$ different ways, by induction. Hence there are $(n-1)^{n-1}$ blue-rooted bicolored combs. To construct a red-rooted bicolored comb $T \in \mathsf{Comb}^2_n$ whose right subtree has k leaves, first choose k labels for the right subtree in $\binom{n-1}{k}$ different ways. Then choose a right subtree that uses these labels. Since the right subtree must be a blue-rooted bicolored comb, there are $(k-1)^{k-1}$ ways to choose such a subtree by the previous case. Now choose the left subtree, which is a bicolored comb, in $(n-k)^{n-k-1}$ different ways by induction.

By setting $x, z := -1$, $y := n$ and $n := n-1$ in Abel's polynomial identity (2.4), we have

$$\begin{aligned} (n-1)^{n-1} &= -\sum_{k=0}^{n-1} \binom{n-1}{k} (n-k)^{n-k-1}(k-1)^{k-1} \\ &= n^{n-1} - \sum_{k=1}^{n-1} \binom{n-1}{k} (n-k)^{n-k-1}(k-1)^{k-1}. \end{aligned}$$

It therefore follows from (5.1) that $|\mathsf{Comb}_n^2| = n^{n-1}$. □

Theorem 5.3. *The set* $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}_{n,i}^2\}$ *is a basis for* $\tilde{H}^{n-3}((\hat{0},[n]^i))$.

Proof. It follows from Propositions 5.1 and 5.2 that $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}_n^2\}$ spans $\oplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0},[n]^i))$ and is of cardinality n^{n-1}. Since, by Corollary 3.7, rank $\oplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0},[n]^i)) = n^{n-1}$, the result holds. □

Remark 5.4. Since the only relations used in the straightening algorithm of Proposition 5.1 are the relations of the presentation given in Theorem 4.4, it follows from Theorem 5.3 that these relations are the *only* relations needed to present $\tilde{H}^{n-3}((\hat{0},[n]^i))$. Thus the final step of the proof of Theorem 4.4 is now complete.

Remark 5.5. Note that by switching left and right, small and large, blue and red, we get 8 different variations of bicolored comb bases.

5.2. **A bicolored Lyndon basis for $\tilde{H}^{n-3}((\hat{0},[n]^i))$ and $\mathcal{L}ie_2(n,i)$.** In this section, we describe the ascent-free chains of the EL-labeling of $[\hat{0},[n]^i]$ given in Theorem 3.2. Recall from Theorem 3.3 that these yield a basis for $H^{n-3}((\hat{0},[n]^i))$. By applying the isomorphism of Theorem 4.5, one gets a corresponding basis for $\mathcal{L}ie_2(n,i)$, which is the classical Lyndon basis for $\mathcal{L}ie(n)$ when $i = 0, n-1$.

We begin by recalling the Lyndon basis for $\mathcal{L}ie(n)$. A Lyndon tree is a labeled binary tree (T,σ) such that for each internal node x of T the smallest leaf label of the subtree T_x rooted at x is in the left subtree of T_x and the second smallest label is in the right subtree of T_x. Let Lyn_n be the set of Lyndon trees whose leaf labels form the set $[n]$. The set $\{[T,\sigma] : (T,\sigma) \in \mathsf{Lyn}_n\}$ is the classical Lyndon basis for $\mathcal{L}ie(n)$.

For each internal node x of a binary tree let $L(x)$ denote the left child of x and $R(x)$ denote the right child. For each node x of a bicolored labeled binary tree (T,σ) define its *valency* $v(x)$ to be the smallest leaf label of the subtree rooted at x. A Lyndon tree is depicted in

Figure 7 illustrating the valencies of the internal nodes. The following alternative characterization of Lyndon tree is easy to verify.

Proposition 5.6. *Let (T, σ) be a labeled binary tree. Then (T, σ) is a Lyndon tree if and only if it is normalized and for every internal node x of T we have*

$$v(R(L(x)) > v(R(x)). \tag{5.2}$$

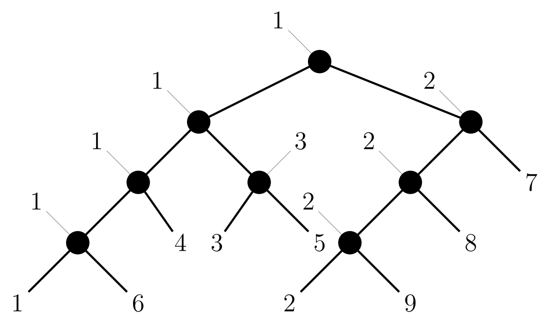


FIGURE 7. Example of a Lyndon tree. The numbers above the lines correspond to the valencies of the internal nodes

We will say that an internal node x of a labeled binary tree (T, σ) is a *Lyndon node* if (5.2) holds. Hence Proposition 5.6 says that (T, σ) is a Lyndon tree if and only if it is normalized and all its internal nodes are Lyndon nodes.

A *bicolored Lyndon tree* is a normalized bicolored binary tree that satisfies the following coloring restriction: for each internal node x that is not a Lyndon node, x is colored blue and its left child is colored red. The set of bicolored Lyndon trees for $n = 3$ is depicted in Figure 8.

Clearly if a bicolored Lyndon tree is monochromatic then all its nodes are Lyndon nodes. Hence the monochromatic ones are the classical Lyndon trees.

Let $\mathsf{Lyn}^2_{n,i}$ be the set of bicolored Lyndon trees in $\mathcal{BT}_{n,i}$. We will show that the ascent-free chains of the EL-labeling of $[\hat{0}, [n]^i]$ given in Theorem 3.2 are of the form $c(T, \sigma, \tau)$, where $(T, \sigma) \in \mathsf{Lyn}^2_{n,i}$ and τ is a certain linear extension of the internal nodes of T, which we now describe. It is easy to see that there is a unique linear extension of the internal notes of $(T, \sigma) \in \mathcal{BT}_{n,i}$ in which the valencies of the nodes weakly decrease. Let $\tau_{T,\sigma}$ denote the permutation that induces this linear extension.

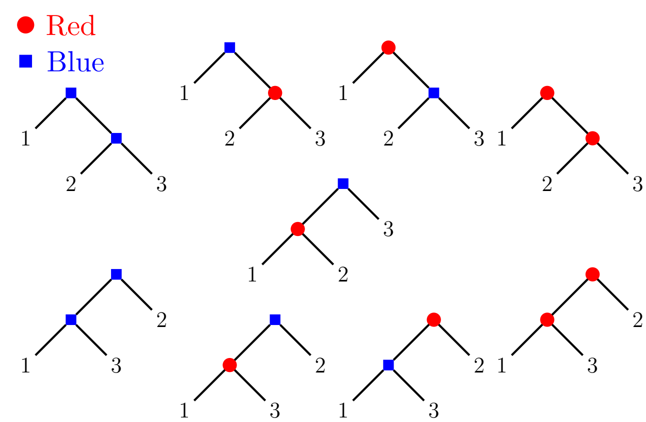


FIGURE 8. Set of bicolored Lyndon trees for $n = 3$

Theorem 5.7. *The set* $\{c(T, \sigma, \tau_{T,\sigma}) : (T, \sigma) \in \mathsf{Lyn}^2_{n,i}\}$ *is the set of ascent-free maximal chains of the EL-labeling of* $[\hat{0}, [n]^i]$ *given in Theorem 3.2.*

Proof. We begin by showing that $c := c(T, \sigma, \tau)$ is ascent-free whenever $(T, \sigma) \in \mathsf{Lyn}^2_{n,i}$ and $\tau = \tau_{T,\sigma}$. Let x_i be the ith internal node of T in postorder. Then by the definition of $\tau := \tau_{T,\sigma}$,

$$v(x_{\tau(1)}) \geq v(x_{\tau(2)}) \geq \cdots \geq v(x_{\tau(n-1)}), \tag{5.3}$$

where v is the valency. For each i, the ith letter of the label word $\lambda(c)$ is given by

$$\lambda_i(c) = (v(L(x_{\tau(i)})), v(R(x_{\tau(i)})))^{u_i} = (v(x_{\tau(i)}), v(R(x_{\tau(i)})))^{u_i},$$

where $u_i = 0$ if $x_{\tau(i)}$ is blue and is 1 if $x_{\tau(i)}$ is red. Note that since (T, σ) is normalized, $v(R(x_{\tau(i)})) \neq v(R(x_{\tau(i+1)}))$ for all $i \in [n-1]$. Now suppose the word $\lambda(c)$ has an ascent at i. Then it follows from (5.3) that

$$v(x_{\tau(i)}) = v(x_{\tau(i+1)}), \;\; v(R(x_{\tau(i)})) < v(R(x_{\tau(i+1)})), \;\text{ and }\; u_i \leq u_{i+1}. \tag{5.4}$$

The equality of valencies implies that $x_{\tau(i)} = L(x_{\tau(i+1)})$ since (T, σ) is normalized. Hence by (5.4),

$$v(R(L(x_{\tau(i+1)}))) < v(R(x_{\tau(i+1)})).$$

It follows that $x_{\tau(i+1)}$ is not a Lyndon node. So by the coloring restriction on bicolored Lyndon trees, $x_{\tau(i+1)}$ must be colored blue and its left child $x_{\tau(i)}$ must be colored red. This implies $u_i = 1$ and $u_{i+1} = 0$, which contradicts (5.4). Hence the chain c is ascent-free.

Conversely, assume c is an ascent-free maximal chain of $[\hat{0},[n]^i]$. Then $c = c(T,\sigma,\tau)$ for some bicolored labeled tree (T,σ) and some permutation $\tau \in \mathfrak{S}_{n-1}$. We can assume without loss of generality that (T,σ) is normalized. Since c is ascent-free, (5.3) holds. This implies that τ is the unique permutation that induces the valency-decreasing linear extension, namely $\tau_{T,\sigma}$.

If all internal nodes of (T,σ) are Lyndon nodes we are done. So let $i \in [n-1]$ be such that $x_{\tau(i)}$ is not a Lyndon node. That is

$$v(R(L(x_{\tau(i)}))) < v(R(x_{\tau(i)})).$$

Since (T,σ) is normalized and (5.3) holds, $L(x_{\tau(i)}) = x_{\tau(i-1)}$. Hence, $v(R(x_{\tau(i-1)})) < v(R(x_{\tau(i)}))$. Since (T,σ) is normalized we also have $v(L(x_{\tau(i-1)})) = v(L(x_{\tau(i)}))$. Hence to avoid an ascent at $i-1$ in c, we must color $x_{\tau(i-1)}$ red and $x_{\tau(i)}$ blue, which is precisely what we need to conclude that (T,σ) is a bicolored Lyndon tree. $\square$

From Theorem 3.3, Lemma 4.3 and Theorem 4.5 we have the following corollary.

Corollary 5.8. *The set* $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathsf{Lyn}^2_{n,i}\}$ *is a basis for* $\tilde{H}^{n-3}((\hat{0},[n]^i))$ *and the set* $\{[T,\sigma] : (T,\sigma) \in \mathsf{Lyn}^2_{n,i}\}$ *is a basis for* $\mathcal{L}ie_2(n,i)$.

Remark 5.9. Note that by switching left and right, small and large, blue and red, we get 8 different variations of bicolored Lyndon bases.

5.3. **Liu's bicolored Lyndon basis.** In this section we describe a different generalization of the Lyndon basis due to Liu [20]. The basis we present is actually a twisted version of the one in [20] and has an easier description. The two bases are related by a simple bijection. In Section 5.4 we will use this basis to prove that a certain naturally constructed set of fundamental cycles is a basis for homology of the interval $(\hat{0},[n]^i)$.

We need to define a different valency from that of the previous section. This valency is referred to in [20] as the *graphical root*. Recall that given an internal node x of a binary tree, $L(x)$ denotes the left child of x and $R(x)$ denotes the right child. For each node x of a bicolored labeled binary tree (T,σ), define its *valency* $v(x)$ recursively as follows:

$$v(x) = \begin{cases} \text{label of } x & \text{if } x \text{ is a leaf} \\ \min\{v(L(x)), v(R(x))\} & \text{if } x \text{ is a blue internal node} \\ \max\{v(L(x)), v(R(x))\} & \text{if } x \text{ is a red internal node.} \end{cases}$$

A *Liu-Lyndon tree* is a bicolored labeled binary tree (T,σ) such that for each internal node x of T,

(1) $v(L(x)) = v(x)$
(2) if x is blue and $L(x)$ is blue then
$$v(R(L(x))) > v(R(x))$$
(3) if x is red then $L(x)$ is red or is a leaf; in the former case,
$$v(R(L(x))) < v(R(x)).$$

Note that condition (1) is equivalent to the condition that $v(L(x)) < v(R(x))$ if x is blue and $v(L(x)) > v(R(x))$ if x is red. Note also that every subtree of a Liu-Lyndon tree is a Liu-Lyndon tree. The set of Liu-Lyndon trees for $n = 3$ is depicted in Figure 9.

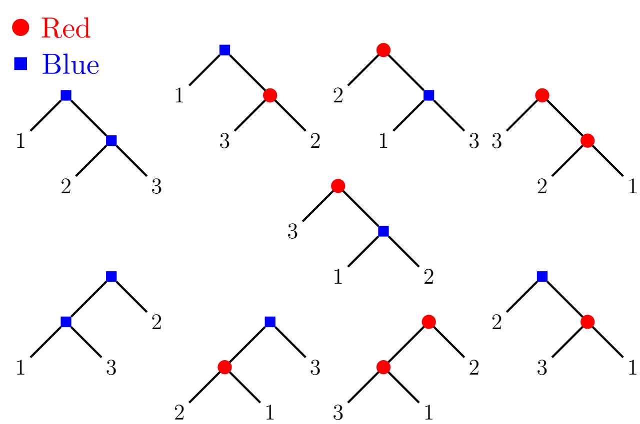


FIGURE 9. Set of Liu-Lyndon trees for $n = 3$

Let $\mathsf{Liu}^2_{n,i}$ be the set of Liu-Lyndon trees in $\mathcal{BT}_{n,i}$. When $i = 0$, all internal nodes are blue and it follows from the definition that $\mathsf{Liu}^2_{n,0}$ is the set of Lyndon trees on n leaves. When $i = n-1$, all internal nodes are red and it follows from the definition that $\mathsf{Liu}^2_{n,n-1}$ consists of labeled binary trees obtained from Lyndon trees by replacing each label j by label $n-j$.

In [20] Liu proves that $\{[T,\sigma] : (T,\sigma) \in \mathsf{Liu}^2_{n,i}\}$ is a basis for $\mathcal{L}ie_{n,i}$ by using a perfect pairing between $\mathcal{L}ie_{n,i}$ and another module that she constructs. In the next section, we will use the natural pairing between cohomology and homology of $(\hat{0}, [n]^i)$ to prove this result.

We will need a bijection of Liu [20]. Let A be a finite subset of the positive integers and let $0 \le i \le |A|-1$. Extend the definitions of $\mathcal{T}_{n,i}$ and $\mathsf{Liu}^2_{n,i}$ by letting $\mathcal{T}_{A,i}$ be the set of rooted trees on node set A with i descents and $\mathsf{Liu}^2_{A,i}$ be the set of Liu-Lyndon trees with leaf label set A and i red internal nodes. Define $\psi : \mathcal{T}_{A,i} \to \mathsf{Liu}^2_{A,i}$ recursively as follows: if $|A| = 1$, let $\psi(T)$ be the labeled binary tree whose single leaf is labeled with the sole element of A. Now suppose $|A| > 1$ and $r_T \in A$ is the root of T. Let x be the smallest child of r_T that is larger than r_T. If no such node exists let x be the largest child of r_T. Let T_x be the subtree of T rooted at x and let $T \setminus T_x$ be the subtree of T obtained by removing T_x from T. Now let

$$\psi(T) = \psi(T \setminus T_x) \overset{\text{col}}{\wedge} \psi(T_x),$$

where

$$\text{col} = \begin{cases} \text{blue} & \text{if } x > r_T \\ \text{red} & \text{if } x < r_T. \end{cases}$$

It will be convenient to refer to descent edges of T (i.e., edges $\{x, p_T(x)\}$, where $x < p_T(x)$) as red edges, and nondescent edges (i.e., edges $\{x, p_T(x)\}$, where $x > p_T(x)$) as blue edges. Hence ψ takes blue edges to blue internal nodes and red edges to red internal nodes. Consequently $\psi(T) \in \mathcal{BT}_{A,i}$ if $T \in \mathcal{T}_{A,i}$. By induction we see that the valuation of the root of $\psi(T)$ is equal to the root of T. It follows from this that $\psi(T) \in \mathsf{Liu}^2_{A,i}$. It is not difficult to describe the inverse of ψ and thereby prove the following result.

Proposition 5.10 ([20]). *For all finite sets A and $0 \le i \le |A|$, the map*

$$\psi : \mathcal{T}_{A,i} \to \mathsf{Liu}^2_{A,i}$$

is a well-defined bijection.

Remark 5.11. It follows from Corollary 3.7, Theorem 5.3, Corollary 5.8 and Proposition 5.10 that

$$|\mathcal{T}_{n,i}| = |\mathsf{Comb}^2_{n,i}| = |\mathsf{Lyn}^2_{n,i}| = |\mathsf{Liu}^2_{n,i}|.$$

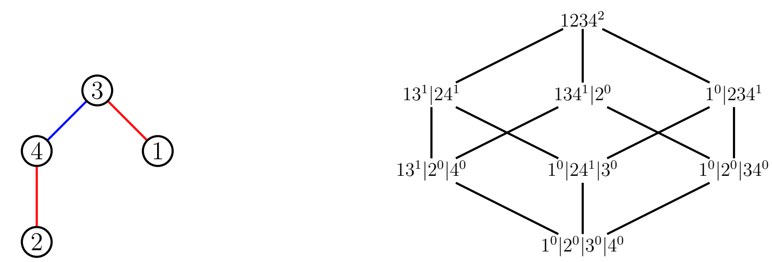
It would be desirable to find nice bijections between the given sets like that of Proposition 5.10. In [16] González D'León constructs such a bijection between $\mathsf{Comb}^2_{n,i}$ and $\mathsf{Lyn}^2_{n,i}$. We leave open the problem of finding a bijection between $\mathcal{T}_{n,i}$ and $\mathsf{Comb}^2_{n,i}$ or $\mathsf{Lyn}^2_{n,i}$.

5.4. **The tree basis for homology.** We now present a generalization of Björner's NBC basis for homology of $\overline{\Pi}_n$ (see [6, Proposition 2.2]). Recall that in Section 2.1, we associated a weighted partition $\alpha(F)$ with each forest $F = \{T_1, \dots, T_k\}$ on node set $[n]$, by letting

$$\alpha(F) = \{A_1^{w_1}, \dots, A_k^{w_k}\},$$

where A_i is the node set of T_i and w_i is the number of descents of T_i.

Let T be a rooted tree on node set $[n]$. For each subset E of the edge set $E(T)$ of T, let T_E be the subgraph of T with node set $[n]$ and edge set E. Clearly T_E is a forest on $[n]$. We define Π_T to be the induced subposet of Π_n^w on the set $\{\alpha(T_E) : E \in E(T)\}$. See Figure 10 for an example of Π_T. The poset Π_T is clearly isomorphic to the boolean algebra $\mathcal{B}_{n-1}$. Hence $\Delta(\overline{\Pi_T})$ is the barycentric subdivision of the boundary of the $(n-2)$-simplex. We let ρ_T denote a fundamental cycle of the spherical complex $\Delta(\overline{\Pi_T})$, that is, a generator of the unique nonvanishing integral simplicial homology of $\Delta(\overline{\Pi_T})$. Note that $\rho_T = \sum_{c\in\mathcal{M}(\Pi_T)} \pm\bar{c}$.



(A) T (B) Π_T

FIGURE 10. Example of a tree T with two descent edges (red edges) and the corresponding poset Π_T

The set $\{\rho_T : T \in \mathcal{T}_{n,0}\}$ is precisely the interpretation of the Björner NBC basis for homology of $\overline{\Pi}_n$ given in [31, Proposition 2.2], and the set $\{\rho_T : T \in \mathcal{T}_{n,n-1}\}$ is a variation of this basis. Björner's NBC basis is dual to the Lyndon basis $\{\bar{c}(\Upsilon) : \Upsilon \in \mathsf{Lyn}_n\}$ for cohomology of $\overline{\Pi}_n$ (using the natural pairing between homology and cohomology). While it is not true in general that $\{\rho_T : T \in \mathcal{T}_{n,i}\}$ is dual to any of the generalizations of the bases given in the previous sections, we are able to prove that it is a basis by pairing it with the Liu-Lyndon basis for cohomology.

Theorem 5.12. *The set $\{\rho_T : T \in \mathcal{T}_{n,i}\}$ is a basis for $\tilde{H}_{n-3}((\hat{0},[n]^i))$ and the set $\{\bar{c}(\Upsilon) : \Upsilon \in \mathsf{Liu}^2_{n,i}\}$ is a basis for $\tilde{H}^{n-3}((\hat{0},[n]^i))$.*

Our main tool in proving this theorem is Proposition A.2 (of the Appendix), which involves the bilinear form $\langle,\rangle$ defined in Appendix A. In order to apply Proposition A.2 we need total orderings of the sets $\mathcal{T}_{n,i}$ and $\mathsf{Liu}^2_{n,i}$. Recall Liu's bijection $\psi : \mathcal{T}_{n,i} \to \mathsf{Liu}^2_{n,i}$ given in Proposition 5.10. We will show that any linear extension $\{T_1, T_2, \dots, T_{|\mathcal{T}_{n,i}|}\}$ of a certain partial ordering on $\mathcal{T}_{n,i}$ provided by Liu [20] yields a matrix $\langle \rho_{T_j}, \bar{c}(\psi(T_k))\rangle_{1\le j,k\le|\mathcal{T}_{n,i}|}$ that is upper-triangular with diagonal entries equal to ± 1. Theorem 5.12 will then follow from Proposition A.2 and Theorem 3.6 (2).

We define Liu's partial ordering $\leq_{\mathrm{Liu}}$ of $\mathcal{T}_{A,i}$ recursively. For $|A| \leq 2$, the set $\mathcal{T}_{A,i}$ has only one element. So assume that $|A| \geq 3$ and that $\leq_{\mathrm{Liu}}$ has been defined for all $\mathcal{T}_{B,j}$ where $|B| < |A|$. Let $T, T' \in \mathcal{T}_{A,i}$. We say that $T \preceq T'$ if there exist edges e of T and e' of T' such that the following conditions hold

- e and e' have the same color,
- e' contains the root of T',
- $\alpha(T_{E(T)\setminus\{e\}}) = \alpha(T'_{E(T')\setminus\{e'\}})$
- $T_1 \leq_{\mathrm{Liu}} T'_1$,
- $T_2 \leq_{\mathrm{Liu}} T'_2$,

where T_1 and T_2 are the connected components (trees) of the forest obtained by removing e from T, and T'_1 and T'_2 are the corresponding connected components (trees) of the forest obtained by removing e' from T'.

Now define $\leq_{\mathrm{Liu}}$ to be the transitive closure of the relation $\preceq$ on $\mathcal{T}_{A,i}$. It follows from [20, Lemma 8.12] that this relation is the same as the relation $\leq_{\mathrm{op}}$ that was defined in [20, Definition 7.11] and was proved to be a partial order in [20, Lemma 7.13].

Lemma 5.13. *Let $T, T' \in \mathcal{T}_{n,i}$ and let $\psi : \mathcal{T}_{n,i} \to \mathsf{Liu}^2_{n,i}$ be the bijection of Proposition 5.10. If $c(\psi(T')) \in \mathcal{M}(\Pi_T)$ then $T \leq_{Liu} T'$.*

Proof. First note that if $\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2$ is a bicolored labeled binary tree such that $c(\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2)$ is a maximal chain in Π_T then there is an edge e of T whose color equals col and whose removal from T yields a forest whose connected components (trees) T_1 and T_2 satisfy: $c(\Upsilon_1)$ is a maximal chain in Π_{T_1} and $c(\Upsilon_2)$ is a maximal chain in Π_{T_2}.

Now recalling the definition of ψ, let x be the child of the root $r_{T'}$ of T', for which

$$\psi(T') = \psi(T' \setminus T'_x) \overset{\mathrm{col}}{\wedge} \psi(T'_x),$$

where col equals the color of the edge $\{x, r_{T'}\}$. Let e be the edge of T whose removal yields the subtrees T_1 and T_2 such that $c(\psi(T' \setminus T'_x)) \in \mathcal{M}(\Pi_{T_1})$ and $c(\psi(T'_x)) \in \mathcal{M}(\Pi_{T_2})$. Then the color of e is the same as that of the edge $\{x, r_{T'}\}$. By induction we can assume that

$$T_1 \leq_{\mathrm{Liu}} T' \setminus T'_x \quad \text{and} \quad T_2 \leq_{\mathrm{Liu}} T'_x.$$

Since e and $e' := \{x, r_{T'}\}$ satisfy the conditions of the definition of $\preceq$, we have $T \preceq T'$, which implies the result. $\square$

Proof of Theorem 5.12. Let $T_1, \dots, T_m$ be any linear extension of $\leq_{\mathrm{Liu}}$ on $\mathcal{T}_{n,i}$, where $m = |\mathcal{T}_{n,i}|$. It follows from Lemma 5.13 that the matrix

$$M := \langle \rho_{T_j}, \bar{c}(\psi(T_k)) \rangle_{1 \leq j,k \leq m}$$

is upper-triangular, where $\langle , \rangle$ is the bilinear form defined in Appendix A. Since $c(\psi(T))$ is a maximal chain of Π_T for all $T \in \mathcal{T}_{n,i}$, the diagonal entries of M are equal to ± 1. Hence M is invertible over $\mathbb{Z}$ or any field. The result now follows from Propositions 5.10 and A.2 and Theorem 3.6 (2). ☐

Remark 5.14. Theorems 4.5 and 5.12 yield an alternative proof of Liu's result that $\{[T,\sigma] : (T,\sigma) \in \mathsf{Liu}^2_{n,i}\}$ is a basis for $\mathcal{L}ie_{n,i}$.

5.5. **Bases for cohomology of the full weighted partition poset.** In this section we use bicolored combs and bicolored Lyndon trees to construct bases for $\tilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$.

For a chain c in Π_n^w, let

$$\breve{c} := c \setminus \{\hat{0}\}.$$

The codimension 1 chains of $\Pi_n^w \setminus \{\hat{0}\}$ are of the form $\breve{c}$, where c is either

(1) unrefinable in some maximal interval $[\hat{0}, [n]^i]$ except between one pair of adjacent elements $x < y$, where $[x, y]$ is an interval of length 2 in $[\hat{0}, [n]^i]$, or
(2) unrefinable in $[\hat{0}, x]$, where x is a weighted partition of $[n]$ consisting of exactly two blocks.

The former case yields the cohomology relations of Types I, II and III given in Section 4.2, with $\bar{c}$ replaced by $\breve{c}$. The latter case yields the additional cohomology relation:

Type IV: The two blocks of x are either 0-merged to get a single-block partition z_1 or 1-merged to get a single-block partition z_2. The open interval $(x, \hat{1})$ is equal to $\{z_1, z_2\}$, see Figure 11. Hence the Type IV elementary cohomology relation is

$$(\breve{c} \cup \{z_1\}) + (\breve{c} \cup \{z_2\}) = 0.$$

The reader can verify, using the cohomology relations of Type I (with $\bar{c}$ replaced by, $\breve{c}$), that the proof of Lemma 4.3 goes through for $\tilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$. Hence $\tilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$ is generated by chains of the form $\breve{c}(\Upsilon)$ where $\Upsilon \in \mathcal{BT}_n$. The reader can also check, using the relations of Types I, II, and III, that the relations in Theorem 4.4 hold (with $\bar{c}$ replaced by $\breve{c}$). It follows from the cohomology relation of Type IV that

$$\breve{c}(\Upsilon_1 \overset{\mathrm{red}}{\wedge} \Upsilon_2) = -\breve{c}(\Upsilon_1 \overset{\mathrm{blue}}{\wedge} \Upsilon_2), \tag{5.5}$$

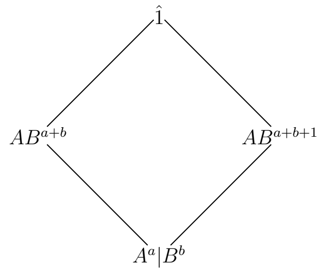


FIGURE 11. Type IV cohomology relation

for all $\Upsilon_1 {}^{\mathrm{red}}_{\wedge} \Upsilon_2 \in \mathcal{BT}_n$.

Recall $\mathsf{Comb}_n^2 = \bigcup_{i=0}^{n-1} \mathsf{Comb}_{n,i}^2$ and let $\mathsf{Lyn}_n^2 = \bigcup_{i=0}^{n-1} \mathsf{Lyn}_{n,i}^2$.

Theorem 5.15. *The sets*

$$\{\breve{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}_n^2,\ \mathrm{col}(\mathrm{root}(T)) = \mathrm{blue}\}$$

and

$$\{\breve{c}(T,\sigma) : (T,\sigma) \in \mathsf{Lyn}_n^2,\ \mathrm{col}(\mathrm{root}(T)) = \mathrm{red}\}$$

are bases for $\tilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$.

Proof. The Comb Basis: We prove, by induction on the size $r(\Upsilon)$ of the right subtree of Υ, that if Υ is a normalized tree in $\mathcal{BT}_n$ then $\breve{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\breve{c}(\Upsilon')$, where Υ' is a blue-rooted bicolored comb. Since the relations in Theorem 4.4 hold (with $\bar{c}$ replaced by $\breve{c}$), we can use the straightening algorithm in the proof of Proposition 5.1 to express $\breve{c}(\Upsilon)$ as a linear combination of chains of the form $\breve{c}(\Upsilon')$, where Υ' is a bicolored comb whose right subtree has size at most $r(\Upsilon)$. If Υ' is red-rooted we can use relation (5.5) to change the root color to blue. The only way that the modified blue-rooted Υ' will fail to be a bicolored comb is if the right child of its root is blue, in which case we can apply Case 1 of the straightening algorithm to Υ'. We thus have that $\breve{c}(\Upsilon')$ is a linear combination of two chains $\breve{c}(\Upsilon_1)$ and $\breve{c}(\Upsilon_2)$, where each $\Upsilon_i \in \mathcal{BT}_n$ and $r(\Upsilon_i) < r(\Upsilon') \leq r(\Upsilon)$. By induction, each $\breve{c}(\Upsilon_i)$ is a linear combination of chains associated with blue-rooted bicolored combs. The same is thus true for each $\breve{c}(\Upsilon')$ and for $\breve{c}(\Upsilon)$. Hence $\{\breve{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}_n^2,\ \mathrm{col}(\mathrm{root}(T)) = \mathrm{blue}\}$ spans. We conclude that this set is a basis by the step in the proof of Proposition 5.2 that shows that there are $(n-1)^{n-1}$ blue-rooted combs and Corollary 3.7.

The Lyndon Basis: From the EL-labeling of Theorem 3.2 we have that all the maximal chains of $\widehat{\Pi_n^w}$ have last label $(1, n+1)^0$. Then for a maximal chain to be ascent-free it must have a second to last label of the form $(1, a)^1$ for $a \in [n]$. By Theorem 5.7, we see that the ascent-free chains correspond to red-rooted bicolored Lyndon trees. It therefore follows from Theorem 3.3 and Lemma 4.3 (with $\bar{c}$ replaced by $\check{c}$) that the second set is a basis for $\tilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$. $\square$

Since the comb basis was shown to span $\tilde{H}^{n-3}(\Pi_n^w \setminus \{\hat{0}\})$ by using only the relations of Theorem 4.4 and relation (5.5) we can conclude that these are the only relations in a presentation of $\tilde{H}^{n-3}(\Pi_n^w \setminus \{\hat{0}\})$. We summarize with the following result.

Theorem 5.16. *The set $\{\check{c}(\Upsilon) : \Upsilon \in \mathcal{BT}_n\}$ is a generating set for $\tilde{H}^{n-3}(\Pi_n^w \setminus \{\hat{0}\})$, subject only to the relations of Theorem 4.4 (with $\bar{c}$ replaced by $\check{c}$) and relation (5.5).*

6. Whitney cohomology

Whitney cohomology (over the field $\mathbf{k}$) of a poset P with a minimum element $\hat{0}$ is defined for each integer r as follows

$$WH^r(P) := \oplus_{x \in P} \tilde{H}^{r-2}((\hat{0}, x); \mathbf{k}).$$

Whitney (co)homology was introduced in [1] and further studied in [29, 32]. It is shown in [21] that if P is a geometric lattice then there is a vector space isomorphism between $\oplus_r WH^r(P)$ and the Orlik-Solomon algebra of P that becomes a graded G-module isomorphism when G is a group acting on P. The symmetric group $\mathfrak{S}_n$ acts naturally on $WH^r(\Pi_n)$ and on the multilinear component $\wedge^r \mathcal{L}ie(n)$, of the rth exterior power of the free Lie algebra on $[n]$. In [3] Barcelo and Bergeron, working with the Orlik-Solomon algebra, establish the following $\mathfrak{S}_n$-module isomorphism

$$WH^{n-r}(\Pi_n) \simeq_{\mathfrak{S}_n} \wedge^r \mathcal{L}ie(n) \otimes \operatorname{sgn}_n .$$

In [31] Wachs shows that an extension of her correspondence between generating sets of $\tilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n) \otimes \operatorname{sgn}_n$ can be used to prove this result.

Let $\wedge^r \mathcal{L}ie_2(n)$ be the multilinear component of the exterior algebra of the free Lie algebra on $[n]$ with two compatible brackets. A *bicolored binary forest* is a sequence of bicolored binary trees. Given a bicolored binary forest F with n leaves and $\sigma \in \mathfrak{S}_n$, let (F, σ) denote the *labeled* bicolored binary forest whose ith leaf from left to right has label $\sigma(i)$. Let $\mathcal{BF}_{n,r}$ be the set of labeled bicolored binary forests with n leaves and r trees. If the jth labeled bicolored binary tree of (F, σ) is (T_j, σ_j) for each $j = 1, \ldots r$ then define

$$[F, \sigma] := [T_1, \sigma_1] \wedge \cdots \wedge [T_r, \sigma_r],$$

where now $\wedge$ denotes the wedge product operation in the exterior algebra. The set $\{[F,\sigma] : (F,\sigma) \in \mathcal{BF}_{n,r}\}$ is a generating set for $\wedge^r \mathcal{L}ie_2(n)$.

The set $\mathcal{BF}_{n,r}$ also provides a natural generating set for $WH^{n-r}(\Pi_n^w)$. For $(F,\sigma) \in \mathcal{BF}_{n,r}$, let $c(F,\sigma)$ be the unrefinable chain of Π_n^w whose rank i partition is obtained from its rank $i-1$ partition by col_i-merging the blocks L_i and R_i, where col_i is the color of the ith postorder internal node v_i of F, and L_i and R_i are the respective sets of leaf labels in the left and right subtrees of v_i.

The symmetric group $\mathfrak{S}_n$ acts naturally on $\wedge^r \mathcal{L}ie_2(n)$ and on $WH^r(\Pi_n^w)$ for each r. We have the following generalization of Theorem 4.5 and [31, Theorem 7.2]. The proof is similar to that of Theorem 4.5 and is left to the reader.

Theorem 6.1. *For each r, there is an $\mathfrak{S}_n$-module isomorphism*

$$\phi : \wedge^r \mathcal{L}ie_2(n) \to WH^{n-r}(\Pi_n^w) \otimes \mathrm{sgn}_n$$

determined by

$$\phi([F,\sigma]) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(F)\bar{c}(F,\sigma), \qquad (F,\sigma) \in \mathcal{BF}_{n,r},$$

where if F is the sequence $T_1, \dots, T_r$ of bicolored binary trees then

$$\mathrm{sgn}(F) := (-1)^{I(T_2)+I(T_4)+\cdots+I(T_{2\lfloor r/2\rfloor})}\,\mathrm{sgn}(T_1)\,\mathrm{sgn}(T_2)\dots\mathrm{sgn}(T_r).$$

Corollary 6.2. *For $0 \le r \le n-1$,*

$$\dim \wedge^{n-r} \mathcal{L}ie_2(n) = \dim WH^r(\Pi_n^w) = \binom{n-1}{r} n^r.$$

Moreover if $\wedge\mathcal{L}ie_2(n)$ is the multilinear component of the exterior algebra of the free Lie algebra on n generators and $WH(\Pi_n^w) = \oplus_{r\ge 0} WH^r(\Pi_n^w)$ then

$$\dim \wedge\mathcal{L}ie_2(n) = \dim WH(\Pi_n^w) = (n+1)^{n-1}.$$

Proof. Since $\dim WH^r(\Pi_n^w)$ equals the signless rth Whitney number of the first kind $|w_r(\Pi_n^w)|$, the result follows from Theorem 6.1, equation (2.6), and the binomial formula. $\square$

For a result that is closely related to Corollary 6.2, see [4, Theorem 2].

7. Related work

In [15] González D'León considers a more general version of Π_n^w and uses it to study $\mathcal{L}ie_k(n)$, the multilinear component of the free Lie algebra with k compatible brackets, where k is an arbitrary positive integer. In particular, he uses an EL-labeling of the generalized version of Π_n^w to obtain a combinatorial description of the dimension of $\mathcal{L}ie_k(n)$. This answers a question posed by Liu [20] on how to generalize $\mathcal{L}ie(n)$

further and to find the right combinatorial objects to compute the dimensions. The comb basis and the Lyndon basis are also further generalized in this paper to multicolored versions.

By Theorem 5.3 and Corollary 5.8 we conclude that the set of bicolored combs and bicolored Lyndon trees are equinumerous (cf. Remark 5.11). In [15] González D'León presents bijections between the multicolored combs, multicolored Lyndon trees and a certain class of permutations, which generalize the classical bijections between the sets of combs, Lyndon trees and permutations in $\mathfrak{S}_{n-1}$.

It can be concluded from equation (2.3) that the generating polynomial of rooted trees enumerated by number of descents $\sum_{i=0}^{n-1}|\mathcal{T}_{n,i}|t^i$ has only negative real roots. Since the polynomial is also palindromic (or symmetric), this implies it can be written using nonnegative coefficients in the basis $\{t^i(1+t)^{n-1-2i}\}_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}$, a property known as γ-positivity. In [17] the γ-positivity property is discussed further and generalized. In particular, formulas and combinatorial interpretations of the γ-coefficients in terms of sets of normalized labeled binary trees are provided.

In a forthcoming paper we will study a more general weighted partition poset obtained by associating weights to the bonds of an arbitrary graph on n-vertices.

Acknowledgement

The authors would like to thank the referee for valuable comments and for pointing out an error in an earlier proof of Proposition 5.1.

Appendix A. Homology and Cohomology of a Poset

We give a brief review of poset (co)homology with group actions. For further information see [33].

Let P be a finite poset of length ℓ. The reduced simplicial (co)homology of P is defined to be the reduced simplicial (co)homology of its order complex $\Delta(P)$, where $\Delta(P)$ is the simplicial complex whose faces are the chains of P. We will review the definition here by dealing directly with the chains of P, and not resorting to the order complex of P.

Let $\mathbf{k}$ be an arbitrary field or the ring of integers $\mathbb{Z}$. The (reduced) chain and cochain complexes

$$\cdots \underset{\delta_r}{\overset{\partial_{r+1}}{\rightleftarrows}} C_r(P) \underset{\delta_{r-1}}{\overset{\partial_r}{\rightleftarrows}} C_{r-1}(P) \underset{\delta_{r-2}}{\overset{\partial_{r-1}}{\rightleftarrows}} \cdots$$

are defined by letting $C_r(P)$ be the $\mathbf{k}$-module generated by the chains of length r in P, for each integer r, and letting the boundary maps

$\partial_r : C_r(P) \to C_{r-1}(P)$ be defined on chains by

$$\partial_r(\alpha_0 < \alpha_1 < \cdots < \alpha_r) = \sum_{i=0}^{r} (-1)^i (\alpha_0 < \cdots < \hat{\alpha}_i < \cdots < \alpha_r),$$

where $\hat{\alpha}_i$ means that the element α_i is omitted from the chain. Note that $C_{-1}(P)$ is generated by the empty chain and $C_r(P) = (0)$ if $r < -1$ or $r > \ell$.

Let $\langle , \rangle$ be the bilinear form on $\bigoplus_{r=-1}^{\ell} C_r(P)$ for which the chains of P form an orthonormal basis. This allows us to define the coboundary map $\delta_r : C_r(P) \to C_{r+1}(P)$ by

$$\langle \delta_r(c), c' \rangle = \langle c, \partial_{r+1}(c') \rangle.$$

Equivalently,

(A.1)

$$\delta_r(\alpha_0 < \cdots < \alpha_r) = \sum_{i=0}^{r+1} (-1)^i \sum_{\alpha \in (\alpha_{i-1}, \alpha_i)} (\alpha_0 < \cdots < \alpha_{i-1} < \alpha < \alpha_i < \cdots < \alpha_r),$$

for all chains $\alpha_0 < \cdots < \alpha_r$, where $\alpha_{-1} = \hat{0}$ and $\alpha_{r+1} = \hat{1}$ of the augmented poset $\hat{P}$ in which a minimum element $\hat{0}$ and a maximum element $\hat{1}$ have been adjoined to P.

Let $r \in \mathbb{Z}$. Define the cycle space $Z_r(P) := \ker \partial_r$ and the boundary space $B_r(P) := \operatorname{im} \partial_{r+1}$. Homology of the poset P in dimension r is defined by

$$\tilde{H}_r(P) := Z_r(P)/B_r(P).$$

Define the cocycle space $Z^r(P) := \ker \delta_r$ and the coboundary space $B^r(P) := \operatorname{im} \delta_{r-1}$. Cohomology of the poset P in dimension r is defined by

$$\tilde{H}_r(P) := Z^r(P)/B^r(P).$$

For $x \le y$ consider the open interval (x, y) of P. Note that if y covers x then (x, y) is the empty poset whose only chain is the empty chain. Therefore $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = 0$ unless $r = -1$, in which case $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = \mathbf{k}$. If $y = x$ then we adapt the convention that $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = 0$ unless $r = -2$, in which case $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = \mathbf{k}$.

Proposition A.1. *Let P be a finite poset of length ℓ whose order complex has the homotopy type of a wedge of m spheres of dimension $\ell - 2$. Then $\tilde{H}_{\ell-2}(P)$ and $\tilde{H}^{\ell-2}(P)$ are isomorphic free $\mathbf{k}$-modules of rank m.*

The following proposition gives a useful tool for identifying bases for top homology and top cohomology.

Proposition A.2 (see [33, Theorem 1.5.1], [23, Proposition 6.4])**.** *Let P be a finite poset of length ℓ whose order complex has the homotopy type of a wedge of m spheres of dimension $\ell-2$. Let $\{\rho_1, \rho_2, ..., \rho_m\} \subseteq Z_{\ell-2}(P)$ and $\{\gamma_1, \gamma_2, ..., \gamma_m\} \subseteq Z^{\ell-2}(P)$. If the matrix $(< \rho_i, \gamma_j >)_{i,j\in[m]}$ is invertible over $\mathbf{k}$ then the sets $\{\rho_1, \rho_2, ..., \rho_m\}$ and $\{\gamma_1, \gamma_2, ..., \gamma_m\}$ are bases for $\tilde{H}_{\ell-2}(P;\mathbf{k})$ and $\tilde{H}^{\ell-2}(P;\mathbf{k})$ respectively.*

Let G be a finite group. A G-poset is a poset P together with a G-action on its elements that preserves the partial order; i.e., $x < y \implies gx < gy$ in P.

Now assume that $\mathbf{k}$ is a field. Let P be a G-poset and let $0 \le r \le \ell$. Since $g \in G$ takes r-chains to r-chains, g acts as a linear map on the chain space $C_r(P)$ (over $\mathbf{k}$). It is easy to see that for all $g \in G$ and $c \in C_r(P)$,

$$g\partial_r(c) = \partial_r(gc) \quad \text{and} \quad g\delta_r(c) = \delta_r(gc).$$

Hence g acts as a linear map on the vector spaces $\tilde{H}_r(P)$ and on $\tilde{H}^r(P)$. This implies that whenever P is a G-poset, $\tilde{H}_r(P)$ and $\tilde{H}^r(P)$ are G-modules. The bilinear form $\langle,\rangle$, induces a pairing between $\tilde{H}_r(P)$ and $H^r(P)$, which allows one to view them as dual G-modules. For $G = \mathfrak{S}_n$ we have the $\mathfrak{S}_n$-module isomorphism

$$\tilde{H}_r(P) \simeq_{\mathfrak{S}_n} \tilde{H}^r(P) \tag{A.2}$$

since dual $\mathfrak{S}_n$-modules are isomorphic.

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