TORSION IN THE MATCHING COMPLEX AND CHESSBOARD COMPLEX

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Abstract. Topological properties of the matching complex were first studied by Bouc in connection with Quillen complexes, and topological properties of the chessboard complex were first studied by Garst in connection with Tits coset complexes. Björner, Lovász, Vrécica and Živaljević established bounds on the connectivity of these complexes and conjectured that these bounds are sharp. In this paper we show that the conjecture is true by establishing the nonvanishing of integral homology in the degrees given by these bounds. Moreover, we show that for sufficiently large n, the bottom nonvanishing homology of the matching complex M_n is an elementary 3-group, improving a result of Bouc, and that the bottom nonvanishing homology of the chessboard complex $M_{n,n}$ is a 3-group of exponent at most 9. When $n \equiv 2 \bmod 3$, the bottom nonvanishing homology of $M_{n,n}$ is shown to be $\mathbb{Z}_3$. Our proofs rely on computer calculations, long exact sequences, representation theory, and tableau combinatorics.

Contents

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1. Introduction

A *matching* is a graph in which each vertex is contained in at most one edge. Given a graph $G = (V, E)$, the collection of all subgraphs (V, F) of G that are matchings forms an abstract simplicial complex $M(G)$. The vertices of $M(G)$ are the edges of G, and the k-dimensional faces of $M(G)$ are the edge sets F of size $k+1$ such that (V, F) is a matching. If G is the complete graph on vertex set $[n] := \{1, 2, \ldots, n\}$, then we write M_n for $M(G)$. Similarly, if G is the complete bipartite graph with parts $[m]$ and $[n]' := \{1', 2', \ldots, n'\}$ then we write $M_{m,n}$ for $M(G)$.

The complex M_n is called the *matching complex* and the complex $M_{m,n}$ is called the *chessboard complex*. A piece of M_7 (taken from [Bo]) is given in Figure 1.1 below. Here and throughout the paper, the vertex of $M(G)$ labelled ij represents the edge $\{i, j\}$ of the graph G. Each k-dimensional face of the chessboard complex $M_{m,n}$ corresponds to a placement of $k+1$ nontaking rooks on an $m \times n$ chessboard. Indeed, a rook in the ith row and jth column corresponds to the edge $\{i, j'\}$ in the bipartite graph, which corresponds to the vertex ij' in $M_{m,n}$. It is for this reason that the name "chessboard complex" is used.

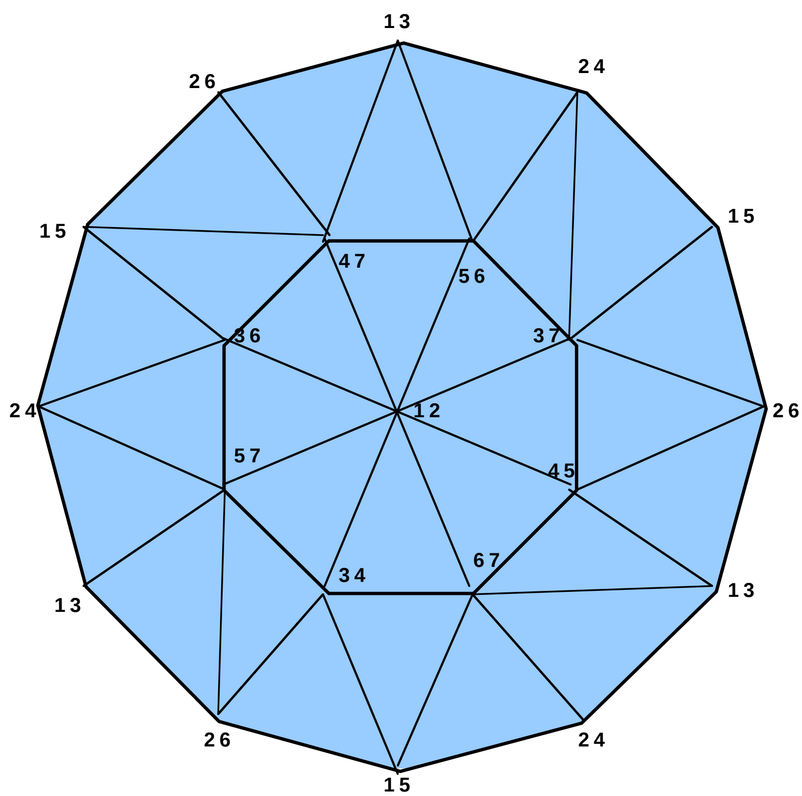

Figure 1.1: Piece of matching complex M_7

The matching complex, the chessboard complex and variations have arisen in a variety of fields such as group theory, representation theory, commutative algebra, Lie theory, computational geometry, and combinatorics; see the survey article [Wa] and its references. Topological properties of the matching complex were first studied by Bouc [Bo], in connection with the Quillen complex at the prime 2 for the symmetric group. Bouc obtains several beautiful results. He considers the representation of the symmetric group $\mathfrak{S}_n$ acting on the homology (over $\mathbb{C}$) of the matching complex M_n and obtains a decomposition into irreducibles. This yields a formula for the Betti numbers in terms of standard Young tableaux. Bouc also obtains results on torsion in integral homology, which we improve and extend to the chessboard complex in this paper.

Prior to Bouc's study of the matching complex, the chessboard complex was introduced in the 1979 thesis of Garst [Ga] dealing with Tits coset complexes. Garst shows that for $m \leq n$, $M_{m,n}$ is Cohen-Macaulay if and only if $2m - 1 \leq n$. Garst also obtains a decomposition of the representation of $\mathfrak{S}_n$ acting on the *top* homology (over $\mathbb{C}$) of $M_{m,n}$ into irreducibles, for $m \leq n$. This computation is a precursor of Friedman and Hanlon's [FrHa] decomposition of the representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on *each* homology of $M_{m,n}$ into irreducibles.

Questions on connectivity of the chessboard complex were raised by Živaljević and Vrécica [ZivVr] in connection with some problems in computational geometry. In response to these questions, Björner, Lovász, Vrécica, Živaljević [BLVZ] obtained bounds on connectivity of the chessboard complex and the matching complex which are given in the following theorem. The bound for the matching complex is also an immediate consequence of results in Bouc [Bo].

Theorem 1.1 (Björner, Lovász, Vrécica, Živaljević [BLVZ], Bouc [Bo])**.** *For positive integers m, n, let*

$$\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1 \quad \textit{and} \quad \nu_{m,n} = \min\{m, n, \lfloor \frac{m+n+1}{3} \rfloor\} - 1.$$

Then the matching complex M_n is $(\nu_n - 1)$-connected and the chessboard complex $M_{m,n}$ is $(\nu_{m,n} - 1)$-connected. Consequently, for all $t < \nu_n$,

$$\tilde{H}_t(M_n) = 0, \tag{1.1}$$

and for all $t < \nu_{m,n}$,

$$\tilde{H}_t(M_{m,n}) = 0. \tag{1.2}$$

Remark 1.2. Throughout this paper, by homology of a simplicial complex Δ, we mean reduced simplicial homology $\tilde{H}_*(\Delta)$ over the integers, unless otherwise stated.

It is conjectured in [BLVZ] that the connectivity bounds of Theorem 1.1 are sharp. The $n \equiv 0, 1 \bmod 3$ cases of the conjecture for the matching complex had already been established by Bouc [Bo] who proved the following result.

Theorem 1.3 (Bouc[Bo])**.**

(i) *Let $n \geq 3$. Then $\tilde{H}_{\nu_n}(M_n)$ is finite if and only if $n \geq 7$ and $n \notin \{8, 9, 11\}$.*
(ii) *If $n \equiv 1 \bmod 3$ and $n \geq 7$ then $\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3$.*
(iii) *If $n \equiv 0 \bmod 3$ and $n \geq 12$ then $\tilde{H}_{\nu_n}(M_n)$ is a nontrivial 3-group of exponent at most 9.*

Remark 1.4. Statement (i) is not explicitly stated in [Bo], but follows easily from the formula for the Betti numbers given in [Bo].

One can see the 3-torsion in $\tilde{H}_1(M_7)$ by looking at Figure 1.1. The union of the triangles shown is bounded by $3z$ where

$$z = (13, 24) + (24, 15) + (15, 26) + (26, 13).$$

Bouc shows that z is not a boundary; so z is a 3-torsion element.

Friedman and Hanlon [FrHa] derive the following analogue of Theorem 1.3 (i), which settles the chessboard complex version of the conjecture in the case that $n > 2m - 5$, but leaves the conjecture unresolved in the case that $m \leq n \leq 2m - 5$. Their result is a consequence of their formula for the Betti numbers of the chessboard complex derived in [FrHa] (see Theorem 6.1).

Theorem 1.5 (Friedman and Hanlon [FrHa])**.** *Let $1 \leq m \leq n$ and $m + n \geq 3$. Then the group $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is finite if and only if $n \leq 2m - 5$ and $(m, n) \notin \{(6, 6), (7, 7), (8, 9)\}$.*

In this paper we pick up where Bouc and Friedman-Hanlon left off. We prove the Björner-Lovász-Vrécica-Živaljević conjecture in the cases that were left unresolved in Bouc's work and Friedman-Hanlon's work (see Theorem 3.1). Moreover, we prove the following result which improves Theorem 1.3 by handling the remaining $n \equiv 2 \bmod 3$ case and making the exponent precise in all cases.

Theorem 1.6. *For $n = 7, 10$ or $n \geq 12$ (except possibly $n = 14$)*[1] *, $\tilde{H}_{\nu_n}(M_n)$ is a nontrivial elementary 3-group.*

[1] See New Developments Section at the end of the paper.

We also prove the following analogous result for the chessboard complex.

Theorem 1.7. *Let $m \leq n$.*

(i) *If $m + n \equiv 1 \bmod 3$ and $n \leq 2m - 5$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n}) \cong \mathbb{Z}_3$.*

(ii) *If $m + n \equiv 0 \bmod 3$ and $n \leq 2m - 9$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a nontrivial 3-group of exponent at most 9.*

(iii) *If $m + n \equiv 2 \bmod 3$ and $n \leq 2m - 13$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a nontrivial 3-group of exponent at most 9.*

Bouc proves the 1 mod 3 case of Theorem 1.3 using induction. His main tool is a long exact sequence which provides the induction step and also enables him to derive the 0 mod 3 case from the 1 mod 3 case. Bouc's "hand" calculation of $\tilde{H}_{\nu_7}(M_7)$ provides the base step of the induction. Here we further exploit Bouc's long exact sequence to derive the 2 mod 3 case from the 0 mod 3 case, and we use a computer calculation to provide another base case $\tilde{H}_{\nu_{12}}(M_{12})$, which enables us to bring the exponent down to 3 in Theorem 1.6.

The proof of Theorem 1.7, while patterned on the proof of the Theorem 1.6, is much more difficult. An essential ingredient is an interesting basis for the *top* homology of the chessboard complex. The construction of this basis has a surprising reliance on a result in tableau combinatorics, namely the classical Robinson-Schensted correspondence.

The computer program that we use for computing homology in the base steps, was first developed by Heckenbach and later improved by Dumas, Heckenbach, Saunders and Welker [DHSW]. With this software and Theorem 1.3 (ii), one can produce the following tables.

n	$\tilde{H}_{\nu_n}(M_n)$
2	0
3	$\mathbb{Z}^2$
4	$\mathbb{Z}^2$
5	$\mathbb{Z}^6$
6	$\mathbb{Z}^{16}$
7	$\mathbb{Z}_3$
8	$\mathbb{Z}^{132}$
9	$\mathbb{Z}^{42} \oplus \mathbb{Z}_3^8$
10	$\mathbb{Z}_3$
11	$\mathbb{Z}^{1188} \oplus Z_3^{45}$
12	$\mathbb{Z}_3^{56}$
13	$\mathbb{Z}_3$
14	?

Table 1.1: Bottom nonvanishing homology $\tilde{H}_{\nu_n}(M_n)$

$m\backslash n$	2	3	4	5	6	7	8
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{19}$	$\mathbb{Z}^{29}$	$\mathbb{Z}^{41}$
3		$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{47}$	$\mathbb{Z}^{104}$	$\mathbb{Z}^{191}$
4			$\mathbb{Z}^{15}$	$\mathbb{Z}^{20}$	$\mathbb{Z}^5$	$\mathbb{Z}^{225}$	$\mathbb{Z}^{641}$
5				$\mathbb{Z}_3$	$\mathbb{Z}^{152}$	$\mathbb{Z}^{98}$	$\mathbb{Z}^{14}$
6					$\mathbb{Z}^{25} \oplus \mathbb{Z}_3^{10}$	$\mathbb{Z}_3$	$\mathbb{Z}^{1316}$
7						$\mathbb{Z}^{588} \oplus \mathbb{Z}_3^{66}$	?

Table 1.2: Bottom nonvanishing homology $\tilde{H}_{\nu_{m,n}}(M_{m,n})$

Unfortunately we have not been able to get output for $n \geq 14$ nor for $m \geq 7$ and $n \geq 8$. This is what is responsible for the gap at $n = 14$ in Theorem 1.6[2] and the lack of precision with respect to the exponent in Theorem 1.7. Indeed, in Theorems 5.13 and 5.15, we show that if we could determine the exponent of the Sylow 3-subgroup of $\tilde{H}_{\nu_{7,8}}(M_{7,8})$ or the exponent of $\tilde{H}_{\nu_{9,9}}(M_{9,9})$ to be 3, then we could conclude that the exponent of $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is 3 for all m, n that satisfy the conditions of Theorem 1.7.

The paper is organized as follows. In Section 2, notation is established and the long exact sequences are derived. In Section 3, we prove the Björner-Lovász-Vrécica-Živaljević connectivity conjecture. The torsion result for the matching complex, Theorem 1.6, is proved in Section 4.

Sections 5, 6, 7 and 8 are devoted to the chessboard complex. The proof of Theorem 1.7 is given in Section 5. Partial results on torsion in the finite groups $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ not covered by Theorem 1.7 can also be found in Section 5. The basis for the top homology of the chessboard complex used in the proof of Theorem 1.7 is constructed in Section 6.

In Section 7, we deal with torsion in the case of infinite $\tilde{H}_{\nu_{m,n}}(M_{m,n})$. Here we use the results of previous sections and Friedman and Hanlon's representation theoretic result to show that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is torsion-free when $n = 2m - 2$. This leads to conjectures on higher dimensional homology.

In Section 8, we discuss the subcomplex of the square chessboard complex $M_{n,n}$ obtained by deleting a diagonal from the chessboard. This complex was shown to be $(\nu_{2n} - 1)$-connected by Björner and Welker [BjWe] as a consequence of a more general result of Ziegler [Zie] on nonrectangular boards. Here we show that the Björner-Welker-Ziegler bound is sharp.

[2]See New Developments Section at the end of the paper.

In Section 9, we answer another question of Björner, Lovász, Vrécica, and Živaljević [BLVZ]. Given the connectivity bounds on M_n and $M_{m,n}$, they ask whether the ν_n-skeleton of M_n and the $\nu_{m,n}$-skeleton of $M_{m,n}$ are shellable. Ziegler [Zie] answers this question affirmatively for the chessboard complex by establishing vertex decomposability. In Section 9, we answer the question affirmatively for the matching complex. We remark that in subsequent work, Athanasiadis [At] improves this result by establishing vertex decomposability.

In Section 10, bounds on the ranks of the finite 3-groups $\tilde{H}_{\nu_n}(M_n)$ and $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ are derived. This extends bounds given by Bouc for the $n \equiv 0, 1 \bmod 3$ cases of the matching complex.

2. Bouc's long exact sequence

In [Bo], Bouc produces a long exact sequence which enables him to prove that $\tilde{H}_t(M_n) = 0$ for $t < \nu(n)$ and to obtain Theorem 1.3. This sequence is a modification of the long exact sequence of the pair (M_n, X_n), where X_n is the subcomplex of M_n consisting of matchings in which either the vertices 1 and 2 form an edge or at least one of these vertices is isolated. As we will see in Section 3, it is easy to use Bouc's sequence to show that $\tilde{H}_{\nu_n}(M_n) \neq 0$ when $n \equiv 2 \bmod 3$, thereby establishing the matching complex case of the Björner-Lovász-Vrécica-Živaljević conjecture. This sequence will also play a role in the proof of Theorem 1.6 given in Section 4. In this section, we present Bouc's long exact sequence and an analogous sequence for the chessboard complex. The analogous sequence will be used to prove the chessboard complex version of the Björner-Lovász-Vrécica-Živaljević conjecture in Section 3, and to prove Theorem 1.7 in Section 5.

We use standard notation, $(C_*(\Delta), \partial)$ and $Z_*(\Delta)$, for the chain complex and the cycle group, respectively, of a simplicial complex Δ. For $z \in Z_*(\Delta)$, we let $\bar{z}$ denote the homology class of z in $\tilde{H}_*(\Delta)$.

2.1. **The long exact sequence for M_n.** In order to state Bouc's result in a manner that will be useful to us, we must introduce some additional notation. For finite set A, let M_A be the matching complex on the complete graph with vertex set A.

For disjoint subsets $A, B \subseteq [n]$, if z_1 and z_2 are oriented simplices of M_A and M_B, respectively, then $z_1 \wedge z_2$ will denote the oriented simplex of $M_{A\cup B}$ obtained by concatenating z_1 and z_2. We define a homomorphism

$$\bigwedge : C_{s-1}(M_A) \otimes C_{t-1}(M_B) \to C_{s+t-1}(M_{A\cup B})$$

by letting $z_1 \otimes z_2 \mapsto z_1 \wedge z_2$ for all oriented simplices z_1, z_2. This induces a homomorphism

$$\bigwedge : \tilde{H}_{s-1}(M_A) \otimes \tilde{H}_{t-1}(M_B) \to \tilde{H}_{s+t-1}(M_{A\cup B}),$$

defined by $\overline{z_1} \wedge \overline{z_2} = \overline{z_1 \wedge z_2}$ for all $z_1 \in Z_{s-1}(M_A)$ and $z_2 \in Z_{t-1}(M_B)$. (We write $z_1 \wedge z_2$ instead of $\bigwedge(z_1 \otimes z_2)$ and $\overline{z_1} \wedge \overline{z_2}$ instead of $\bigwedge(\overline{z_1} \otimes \overline{z_2})$ and note that $z_1 \wedge z_2$ is a cycle.)

For $a = 1, 2$ and $i = 3, \dots, n$, let

$$\phi_{a,i} : \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,i\}}) \to \tilde{H}_t(M_n)$$

be the homomorphism defined by

$$\phi_{a,i}(\overline{z}) = \overline{ai - 12} \ \wedge \ \overline{z}.$$

This determines the homomorphism

$$\phi : \bigoplus_{\substack{a \in \{1,2\} \\ i \in [n] \setminus \{1,2\}}} \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,i\}}) \ \to \ \tilde{H}_t(M_n),$$

defined by letting $\phi(\overline{z}) = \phi_{a,i}(\overline{z})$ for each $\overline{z}$ in each (a, i)-summand.

For $i \neq j \in \{3, \dots, n\}$, let

$$\psi_{i,j} : C_t(M_n) \to C_{t-2}(M_{[n]\setminus\{1,2,i,j\}})$$

be the map defined by letting

$$\psi_{i,j}(x) = \begin{cases} y & \text{if } \ x = 1i \wedge 2j \wedge y \text{ for some } y \in C_{t-2}(M_{[n]\setminus\{1,2,i,j\}}) \\ 0 & \text{otherwise,} \end{cases}$$

for each oriented simplex x. It is straightforward to show that the induced map

$$\psi_{i,j} : \tilde{H}_t(M_n) \to \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}})$$

given by $\psi_{i,j}(\overline{z}) = \overline{\psi_{i,j}(z)}$ is a well-defined homomorphism as is the map

$$\psi : \tilde{H}_t(M_n) \to \bigoplus_{i \neq j \in [n]\setminus\{1,2\}} \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}})$$

given by $\psi(\overline{z}) = (\psi_{i,j}(\overline{z}))$.

For $a = 1, 2,\quad h, i, j = 3, \dots, n$ and $i \neq j$, define

$$\delta^{i,j}_{a,h} : \tilde{H}_t(M_{[n]\setminus\{1,2,i,j\}}) \to \tilde{H}_t(M_{[n]\setminus\{1,2,h\}})$$

by

$$\delta^{i,j}_{a,h}(\overline{z}) = \begin{cases} \ \ \overline{z} & \text{if } \ a = 1 \ \text{ and } \ h = i \\ -\overline{z} & \text{if } \ a = 2 \ \text{ and } \ h = j \\ \ \ 0 & \text{otherwise,} \end{cases}$$

for $z \in Z_t(M_{[n]\setminus\{1,2,i,j\}})$. Again it is straightforward to show that $\delta^{i,j}_{a,h}$ is a well-defined homomorphism as is the homomorphism

$$\delta : \bigoplus_{i \neq j \in [n]\setminus\{1,2\}} \tilde{H}_t(M_{[n]\setminus\{1,2,i,j\}}) \rightarrow \bigoplus_{\substack{a \in \{1,2\} \\ h \in [n]\setminus\{1,2\}}} \tilde{H}_t(M_{[n]\setminus\{1,2,h\}})$$

defined by letting $\delta(\overline{z}) = (\delta^{i,j}_{a,h}(\overline{z}))$ for each $\overline{z}$ in each (i,j)-summand.

We can now state Bouc's result. For the sake of completeness, we will include a proof.

Lemma 2.1 ([Bo, Lemma 9]). *The sequence*

$$\cdots \xrightarrow{\delta} \bigoplus_{\substack{a \in \{1,2\} \\ h \in [n]\setminus\{1,2\}}} \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,h\}}) \xrightarrow{\phi} \tilde{H}_t(M_n) \xrightarrow{\psi}$$

$$\bigoplus_{i \neq j \in [n]\setminus\{1,2\}} \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}}) \xrightarrow{\delta} \bigoplus_{\substack{a \in \{1,2\} \\ h \in [n]\setminus\{1,2\}}} \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,h\}}) \xrightarrow{\phi} \cdots$$

is exact.

Proof. For any graph G on vertex set $[n]$, let $E(G)$ denote the edge set of G, and for $v \in [n]$, let $N_G(v)$ denote the set of neighbors of v, that is,

$$N_G(v) = \{u \in V : \{u,v\} \in E(G)\}.$$

Define

$$X_n := \{G \in M_n : |(N_G(1) \cup N_G(2)) \setminus \{1,2\}\,| \leq 1\}.$$

Then X_n is a subcomplex of M_n, and we examine the standard long exact sequence

$$\cdots \xrightarrow{\partial_*} \tilde{H}_t(X_n) \xrightarrow{i_*} \tilde{H}_t(M_n) \xrightarrow{\pi_*} \tilde{H}_t(M_n, X_n) \xrightarrow{\partial_*} \tilde{H}_{t-1}(X_n) \xrightarrow{i_*} \cdots$$

(see [Mu, Theorem 23.3]).

Let P_n be the subcomplex of X_n consisting of those $G \in X_n$ such that either $\{1,2\} \in E(G)$ or both 1 and 2 are isolated in G. Since P_n is a cone over $M_{[n]\setminus\{1,2\}}$, it is acyclic. Hence the natural projection of chain complexes induces an isomorphism

$$\tau : \tilde{H}_t(X_n) \to \tilde{H}_t(X_n, P_n).$$

For $a \in \{1,2\}$ and $h \in [n]\setminus\{1,2\}$, let

$$\alpha_{a,h} : C_t(X_n, P_n) \to C_{t-1}(M_{[n]\setminus\{1,2,h\}})$$

be the map defined by letting

$$\alpha_{a,h}(x) = \begin{cases} y & \text{if } x = ah \wedge y \text{ for some } y \in C_{t-2}(M_{[n]\setminus\{1,2,h\}}) \\ 0 & \text{otherwise,} \end{cases}$$

for each oriented simplex x. It is straightforward to show that the induced map

$$\alpha_{a,h} : \tilde{H}_t(X_n, P_n) \to \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,h\}})$$

given by $\alpha_{i,j}(\overline{z}) = \overline{\alpha_{i,j}(z)}$, is a well-defined homomorphism as is the map

$$\alpha : \tilde{H}_t(X_n, P_n) \; \to \bigoplus_{\substack{a \in \{1,2\} \\ h \in [n]\setminus\{1,2\}}} \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,h\}})$$

given by $\alpha(\overline{z}) = (\alpha_{a,h}(\overline{z}))$. If we define

$$\gamma_{a,h} : \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,h\}}) \to \tilde{H}_t(X_n, P_n)$$

by

$$\overline{w} \mapsto \overline{ah \wedge w}$$

then

$$\gamma := \bigoplus_{a,h} \gamma_{a,h}$$

is a well-defined inverse for α. We now have an isomorphism

$$\alpha\tau : \tilde{H}_t(X_n) \to \bigoplus_{\substack{a \in \{1,2\} \\ h \in [n]\setminus\{1,2\}}} \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,h\}}).$$

It is straightforward to show that the map

$$\beta_{i,j} : \tilde{H}_t(M_n, X_n) \to \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}})$$

induced by the restriction of $\psi_{i,j}$ to $C_t(M_n, X_n)$ is a well-defined homomorphism for all $i, j \in [n] \setminus \{1, 2\}$ with $i \neq j$. Define

$$\beta : \tilde{H}_t(M_n, X_n) \to \bigoplus_{\substack{i,j \in [n]\setminus\{1,2\} \\ i \neq j}} \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}})$$

by

$$\overline{z} \mapsto (\beta_{i,j}(\overline{z})).$$

If we define

$$\mu_{i,j} : \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}}) \to \tilde{H}_t(M_n, X_n)$$

by

$$\overline{w} \mapsto \overline{1i \wedge 2j \wedge w}$$

then

$$\mu := \bigoplus_{i,j} \mu_{i,j}$$

is an inverse for β. The result now follows from the fact that the diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_*} & \tilde{H}_t(X_n) & \xrightarrow{i_*} & \tilde{H}_t(M_n) & \xrightarrow{\pi_*} & \tilde{H}_t(M_n, X_n) & \xrightarrow{\partial_*} \cdots \\
& \downarrow \alpha\tau & & \downarrow \mathrm{id} & & \downarrow \beta & \\
\cdots \xrightarrow{\delta} & \bigoplus_{a,h} \tilde{H}_{t-1}(M_{[n]\setminus\{1,2,h\}}) & \xrightarrow{\phi} & \tilde{H}_t(M_n) & \xrightarrow{\psi} & \bigoplus_{i,j} \tilde{H}_{t-2}(M_{[n]\setminus\{1,2,i,j\}}) & \xrightarrow{\delta} \cdots
\end{array}$$

commutes. □

2.2. **The long exact sequence for $M_{m,n}$.** For any subset

$$Y = \{y_1, y_2, \ldots, y_k\} \subseteq [n],$$

let

$$Y' := \{y'_1, y'_2, \ldots, y'_k\} \subseteq [n]'.$$

For $X \subseteq [m]$ and $Y \subseteq [n]$, let $M_{X,Y}$ be the chessboard complex on X and Y'. In other words, $M_{X,Y}$ is the matching complex on the complete bipartite graph whose parts are X and Y'. Then $M_{X,Y}$ is a subcomplex of the matching complex $M_{X \uplus Y'}$, and the chain complex $C_*(M_{X,Y})$ is embedded in the complex $C_*(M_{X \uplus Y'})$.

After appropriate changes in notation, restrictions of the various functions defined in Section 2.1 will be used to produce a long exact sequence similar to the one described in Lemma 2.1. In particular, if $X = X_1 \biguplus X_2$ and $Y = Y_1 \biguplus Y_2$ then the restriction of the homomorphism $\bigwedge$ defined in Section 2.1 gives a homomorphism

$$\bigwedge : \tilde{H}_{s-1}(M_{X_1,Y_1}) \otimes \tilde{H}_{t-1}(M_{X_2,Y_2}) \to \tilde{H}_{s+t-1}(M_{X,Y}).$$

In Section 2.1, the graph vertices $1, 2$ were distinguished in order to produce the desired long exact sequence. Here, we distinguish the graph vertices $1, 1'$. For $i \in [m] \setminus \{1\}$, define

$$\phi_i : \tilde{H}_{t-1}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \to \tilde{H}_t(M_{m,n})$$

by

$$\overline{z} \mapsto (\overline{11' - i1'}) \wedge \overline{z},$$

and for $j \in [n] \setminus \{1\}$, define

$$\phi'_j : \tilde{H}_{t-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}}) \to \tilde{H}_t(M_{m,n})$$

by

$$\overline{z} \mapsto (\overline{11' - 1j'}) \wedge \overline{z}.$$

For ease of notation, we define

$$\tilde{H}_t(1) := \bigoplus_{j\in[n]\setminus\{1\}} \tilde{H}_t(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}}),$$
$$\tilde{H}_t(1') := \bigoplus_{i\in[m]\setminus\{1\}} \tilde{H}_t(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}).$$

The maps ϕ_i and ϕ'_j together determine a unique homomorphism

$$\phi : \tilde{H}_{t-1}(1') \oplus \tilde{H}_{t-1}(1) \to \tilde{H}_t(M_{m,n}).$$

For $i \in [m] \setminus \{1\}$ and $j \in [n] \setminus \{1\}$, define

$$\psi_{i,j} : C_t(M_{m,n}) \to C_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$$

by

$$x \mapsto \begin{cases} y & \text{if } x = 1j' \wedge 1'i \wedge y \text{ for some } y \in C_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}) \\ 0 & \text{otherwise.} \end{cases}$$

As in Section 2.1, $\psi_{i,j}$ induces a homomorphism, also called $\psi_{i,j}$, from $\tilde{H}_t(M_{m,n})$ to $\tilde{H}_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$. We define

$$\psi : \tilde{H}_t(M_{m,n}) \to \bigoplus_{\substack{i \in [m]\setminus\{1\} \\ j \in [n]\setminus\{1\}}} \tilde{H}_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}),$$

by

$$\overline{z} \mapsto (\psi_{i,j}(\overline{z})).$$

For $i \in [m] \setminus \{1\}$ and $j \in [n] \setminus \{1\}$ define

$$\delta^{i,j} : \tilde{H}_t(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}) \to \tilde{H}_t(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \oplus \tilde{H}_t(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}})$$

by

$$\overline{z} \mapsto (-\overline{z}, \overline{z}).$$

For ease of notation, we define

$$\tilde{H}_t(1,1') := \bigoplus_{\substack{i \in [m]\setminus\{1\} \\ j \in [n]\setminus\{1\}}} \tilde{H}_t(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}).$$

and let

$$\delta : \tilde{H}_t(1,1') \to \tilde{H}_t(1') \oplus \tilde{H}_t(1)$$

be the unique homomorphism whose restriction to $\tilde{H}_t(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$ is $\delta^{i,j}$ for each pair (i, j).

Lemma 2.2. *The sequence*

(2.1)

$$\cdots \xrightarrow{\delta} \bigoplus_{i\in[m]\setminus\{1\}} \tilde{H}_{t-1}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \oplus \bigoplus_{j\in[n]\setminus\{1\}} \tilde{H}_{t-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}})$$
$$\xrightarrow{\phi} \tilde{H}_t(M_{m,n}) \xrightarrow{\psi} \bigoplus_{\substack{i\in[m]\setminus\{1\}\\ j\in[n]\setminus\{1\}}} \tilde{H}_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}) \xrightarrow{\delta}$$
$$\bigoplus_{i\in[m]\setminus\{1\}} \tilde{H}_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \oplus \bigoplus_{j\in[n]\setminus\{1\}} \tilde{H}_{t-2}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}}) \xrightarrow{\phi} \cdots$$

is exact.

Proof. Define

$$X_{m,n} := \{G \in M_{m,n} : |(N_G(1) \cup N_G(1')) \setminus \{1,1'\}| \le 1\}.$$

Let $P_{m,n}$ be the subcomplex of $X_{m,n}$ consisting of those $G \in X_{m,n}$ such that either $\{1,1'\} \in E(G)$ or both 1 and $1'$ are isolated in G. As before, the natural projection of chain complexes induces an isomorphism

$$\tau : \tilde{H}_t(X_{m,n}) \to \tilde{H}_t(X_{m,n}, P_{m,n}).$$

For $i \in [m] \setminus \{1\}$ and $j \in [n] \setminus \{1\}$, let

$$\alpha_i : C_t(X_{m,n}, P_{m,n}) \to C_{t-1}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}})$$

and

$$\alpha'_j : C_t(X_{m,n}, P_{m,n}) \to C_{t-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}})$$

be the maps defined by letting

$$\alpha_i(x) = \begin{cases} y & \text{if } x = 1'i \wedge y \text{ for some } y \in C_{t-1}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\alpha'_j(x) = \begin{cases} y & \text{if } x = 1j' \wedge y \text{ for some } y \in C_{t-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}}) \\ 0 & \text{otherwise,} \end{cases}$$

for each oriented simplex x. It is straightforward to show that the induced maps,

$$\alpha_i : \tilde{H}_t(X_{m,n}, P_{m,n}) \to \tilde{H}_{t-1}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}})$$

and

$$\alpha'_j : \tilde{H}_t(X_{m,n}, P_{m,n}) \to \tilde{H}_{t-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}})$$

given by $\alpha_i(\bar{z}) = \overline{\alpha_i(z)}$ and $\alpha'_j(\bar{z}) = \overline{\alpha'_j(z)}$, are well-defined homomorphisms, as is the map

$$\alpha : \tilde{H}_t(X_{m,n}, P_{m,n}) \to \tilde{H}_{t-1}(1') \oplus \tilde{H}_{t-1}(1)$$

defined by

$$\overline{z} \mapsto ((\alpha_i(\bar{z})), (\alpha'_j(\bar{z}))).$$

The map α has an inverse analogous to the inverse γ defined in Section 2.1. Therefore, α is an isomorphism.

For $i \in [m] \setminus \{1\}$, and $j \in [n] \setminus \{1\}$, the map

$$\beta_{i,j} : \tilde{H}_t(M_{m,n}, X_{m,n}) \to \tilde{H}_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$$

induced by the restriction of $\psi_{i,j}$ to $C_t(M_{m,n}, X_{m,n})$ is a well-defined homomorphism. Define

$$\beta : \tilde{H}_t(M_{m,n}, X_{m,n}) \to \bigoplus_{\substack{i \in [m] \setminus \{1\} \\ j \in [n] \setminus \{1\}}} \tilde{H}_{t-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$$

by

$$\overline{z} \mapsto (\beta_{i,j}(\overline{z})).$$

As in Section 2.1, β is a well-defined isomorphism.

The diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_*} & \tilde{H}_t(X_{m,n}) & \xrightarrow{i_*} & \tilde{H}_t(M_{m,n}) & \xrightarrow{\pi_*} & \tilde{H}_t(M_{m,n}, X_{m,n}) & \xrightarrow{\partial_*} \cdots \\
& \downarrow \alpha\tau & & \downarrow \mathrm{id} & & \downarrow \beta & \\
\cdots \xrightarrow{\delta} & \tilde{H}_{t-1}(1) \oplus \tilde{H}_{t-1}(1') & \xrightarrow{\phi} & \tilde{H}_t(M_{m,n}) & \xrightarrow{\psi} & \tilde{H}_{t-2}(1, 1') & \xrightarrow{\delta} \cdots
\end{array}$$

commutes, which yields the result. □

2.3. **The tail end.**

For our purposes, we need only the tail end of each long exact sequence. Recall that

$$\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1.$$

Lemma 2.3. *Let ϕ and ψ be as in Lemma 2.1.*

(i) *If $n \equiv 0, 1 \bmod 3$ then the following is an exact sequence*

$$\bigoplus_{\substack{a \in \{1,2\} \\ i \in \{3,\dots,n\}}} \tilde{H}_{\nu_{n-3}}(M_{[n]\setminus\{1,2,i\}}) \xrightarrow{\phi} \tilde{H}_{\nu_n}(M_n) \to 0.$$

(ii) *If $n \equiv 2 \bmod 3$ then the following is an exact sequence*

$$\bigoplus_{\substack{a \in \{1,2\} \\ i \in \{3,\dots,n\}}} \tilde{H}_{\nu_{n-3}}(M_{[n]\setminus\{1,2,i\}}) \xrightarrow{\phi} \tilde{H}_{\nu_n}(M_n)$$

$$\xrightarrow{\psi} \bigoplus_{i \neq j \in \{3,\dots,n\}} \tilde{H}_{\nu_{n-4}}(M_{[n]\setminus\{1,2,i,j\}}) \to 0.$$

Proof. First note that $\nu_{n-3} = \nu_n - 1$ for all n. Hence the sequence of (i) is a piece of the long exact sequence of Lemma 2.1, provided that $\tilde{H}_{\nu_n - 2}(M_{n-4}) = 0$. This follows from (1.1), since $\nu_n - 2 < \nu_n - 1 = \nu_{n-4}$ when $n \equiv 0, 1 \bmod 3$.

If $n \equiv 2 \bmod 3$, we have that $\nu_{n-4} = \nu_n - 2$. Hence the sequence of (ii) is a piece of the long exact sequence of Lemma 2.1, by (1.1) and the fact that $\nu_n - 2 < \nu_{n-3}$. □

Now recall that,

$$\nu_{m,n} = \min\{m, n, \lfloor \frac{m+n+1}{3} \rfloor\} - 1.$$

Note that if $m \leq n$ then

$$\nu_{m,n} = \begin{cases} \lfloor \frac{m+n+1}{3} \rfloor - 1 & \text{if } n \leq 2m-1 \\ m-1 & \text{if } n \geq 2m-1, \end{cases} \tag{2.2}$$

and if $n < 2m - 1$ then

$$\nu_{m,n} < m - 1. \tag{2.3}$$

Lemma 2.4. *Suppose $m \leq n < 2m-1$. Let ϕ and ψ be as in Lemma 2.2.*

(i) *If $m + n \equiv 0, 1 \bmod 3$ then*

$$\bigoplus_{i \in [m]\setminus\{1\}} \tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \oplus \bigoplus_{j \in [n]\setminus\{1\}} \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}})$$

$$\xrightarrow{\phi} \tilde{H}_{\nu_{m,n}}(M_{m,n}) \to 0$$

is exact.

(ii) *If* $m+n \equiv 2 \bmod 3$ *then*

$$\bigoplus_{i\in[m]\setminus\{1\}} \tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{1,i\},[n]\setminus\{1\}}) \quad\oplus\quad \bigoplus_{j\in[n]\setminus\{1\}} \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{1\},[n]\setminus\{1,j\}})$$

$$\xrightarrow{\phi} \tilde{H}_{\nu_{m,n}}(M_{m,n}) \xrightarrow{\psi} \bigoplus_{\substack{i\in[m]\setminus\{1\}\\ j\in[n]\setminus\{1\}}} \tilde{H}_{\nu_{m-2,n-2}}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}) \to 0$$

is exact.

Proof. Note that for all m,n such that $m \le n < 2m-1$,

$$\nu_{m-2,n-1} = \nu_{m-1,n-2} = \nu_{m,n} - 1, \tag{2.4}$$

and

$$\nu_{m-2,n-2} = \lfloor \frac{m+n}{3} \rfloor - 2.$$

It follows that if $m+n \equiv 0,1 \bmod 3$ then

$$\begin{aligned} \nu_{m-2,n-2} &= \lfloor \frac{m+n+1}{3} \rfloor - 2 \\ &= \nu_{m,n} - 1. \end{aligned} \tag{2.5}$$

Hence by (1.2), we have $\tilde{H}_{\nu_{m,n}-2}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}}) = 0$, which together with (2.4) implies that the sequence in (i) is a piece of the long exact sequence of Lemma 2.2.

If $m+n \equiv 2 \bmod 3$ then

$$\begin{aligned} \nu_{m-2,n-2} &= \lfloor \frac{m+n+1}{3} \rfloor - 3 \\ &= \nu_{m,n} - 2. \end{aligned}$$

It follows from this, (1.2), and (2.4) that the sequence in (ii) is a piece of the long exact sequence of Lemma 2.2. □

Lemma 2.3 (resp., 2.4) will be used to decompose generators of $\tilde{H}_{\nu_n}(M_n)$ (resp., $\tilde{H}_{\nu_{m,n}}(M_{m,n})$) into wedge products of smaller cycles. An easy instance of this is given in the next lemma.

Lemma 2.5. *Suppose* $n \equiv 0,1 \bmod 3$*. Then* $\tilde{H}_{\nu_n}(M_n)$ *is generated by elements of the form*

$$\begin{aligned} (\sigma(1)\sigma(2) - \sigma(1)\sigma(3)) \quad &\wedge \quad (\sigma(4)\sigma(5) - \sigma(4)\sigma(6)) \ \wedge \ \ldots \\ \ldots \quad &\wedge \quad (\sigma(N-2)\sigma(N-1) - \sigma(N-2)\sigma(N)), \end{aligned}$$

where $\sigma \in \mathfrak{S}_n$ *and* $N = 3\lfloor \frac{n}{3} \rfloor$.

Proof. This follows from Lemma 2.3 (i) by induction on n. □

3. Proof of the BLVZ conjecture

The exact sequence given in Part (i) of Lemma 2.3 is one of the main tools of Bouc's proof of the $n \equiv 0, 1 \bmod 3$ cases of the conjecture of Björner, Lovász, Vrécica and Živaljević that $\tilde{H}_{\nu_n}(M_n)$ does not vanish. Bouc uses this exact sequence to establish nonvanishing homology in the most difficult case, the $n \equiv 1 \bmod 3$ case. He then observes that the tail end of another long exact sequence, which is given in (10.1), enables one to deduce the $n \equiv 0 \bmod 3$ case from the $n \equiv 1 \bmod 3$ case. Although not explicitly mentioned by Bouc, one can use Lemma 2.3 to deduce the remaining 2 mod 3 case from the 1 mod 3 case. Indeed, consider the surjective map ψ of Lemma 2.3 (ii). Since $n-4 \equiv 1 \bmod 3$, the range of ψ does not vanish. Hence neither does the domain $\tilde{H}_{\nu_n}(M_n)$.

We now prove the conjecture for the chessboard complex.

Theorem 3.1 (Björner-Lovász-Vrécica-Živaljević Conjecture)**.** *For $n \geq 3$,*

$$\tilde{H}_{\nu_n}(M_n) \neq 0, \tag{3.1}$$

and for $m+n \geq 3$,

$$\tilde{H}_{\nu_{m,n}}(M_{m,n}) \neq 0. \tag{3.2}$$

Proof of (3.2). If $n \geq 2m-1$, then the result follows from Theorem 1.5. So assume that $m \leq n < 2m-1$.

We will begin with the case that $m+n \equiv 0 \bmod 3$. The argument for $m+n \equiv 1 \bmod 3$ is similar and will be left to the reader. We will use the fact that $\tilde{H}_{\nu_{m+n}}(M_{m+n})$ does not vanish to prove that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ does not vanish. Since the chessboard complex $M_{m,n}$ is a subcomplex of the matching complex $M_{[m]\uplus[n]'}$, any cycle z of $M_{[m]\uplus[n]'}$ that is in the chain space of $M_{m,n}$ must be a cycle in $M_{m,n}$. Moreover, if z is a boundary in the subcomplex $M_{m,n}$ then it is also a boundary in $M_{[m]\uplus[n]'}$.

Let $k = \frac{2n-m}{3}$. It follows from $m+n \equiv 0 \bmod 3$, that k is an integer. The cycle

$$z := (1\,1' - 1\,2') \wedge (2\,3' - 2\,4') \wedge \;\ldots\; \wedge (k\,(2k-1)' - k\,(2k)') \wedge$$
$$((2k+1)'(k+1) - (2k+1)'(k+2)) \wedge \;\ldots\; \wedge (n'\,(m-1) - n'\,m)$$

of $M_{[m]\uplus[n]'}$ is not a boundary since it is one of the generators given by Lemma 2.5. Indeed, if any one of the cycles given by Lemma 2.5 is a boundary, they all are, which is impossible since $\tilde{H}_{\nu_{m+n}}(M_{[m]\uplus[n]'}) \neq 0$. The cycle z is clearly in the $(\frac{m+n}{3}-1)$-chain space of $M_{m,n}$. So it is a $(\frac{m+n}{3}-1)$-cycle of $M_{m,n}$ that is not a boundary. Since by (2.2), $\nu_{m,n}=\nu_{m+n}=\frac{m+n}{3}-1$, we have $\tilde H_{\nu_{m,n}}(M_{m,n})\neq 0$.

Now suppose $m+n\equiv 2 \bmod 3$. Just as for the matching complex, the 2 mod 3 case is a consequence of the 1 mod 3 case. We use Lemma 2.4 (ii). Since $m+n-4\equiv 1 \bmod 3$, we have that the range of the surjection ψ does not vanish, by the 1 mod 3 case. Hence, neither does the domain, $\tilde H_{\nu_{m,n}}(M_{m,n})$. ☐

4. Torsion in the matching complex

In this section we prove Theorem 1.6. We begin with the following lemma.

Lemma 4.1. *Suppose $n\equiv 2 \bmod 3$ and $n\geq 5$. Then $\tilde H_{\nu_n}(M_n)$ is generated by elements of the form $\gamma\wedge\rho$, where $\gamma\in\tilde H_1(M_S)$, $\rho\in\tilde H_{\nu_{n-5}}(M_{[n]-S})$, and $|S|=5$.*

Proof. The proof is by induction on n. The base step $n=5$ is trivial. Let $n\geq 8$. For distinct elements $i,j\in[n]$, recall the map

$$\psi_{i,j}:\tilde H_{\nu_n}(M_n)\to\tilde H_{\nu_{n-4}}(M_{[n]\setminus\{1,2,i,j\}})$$

defined in Section 2.1. Since $n-4\equiv 1 \bmod 3$, it follows from Lemma 2.5 that

$$\tilde H_{\nu_{n-4}}(M_{[n]\setminus\{1,2,i,j\}})=\langle\bar\rho:\rho\in Z_{\nu_{n-5}}(M_{[n]\setminus\{1,2,i,j,r\}}), r\in[n]\setminus\{1,2,i,j\}\rangle.$$

Therefore if $\zeta\in\tilde H_{\nu_n}(M_n)$ then

$$\psi_{i,j}(\zeta)=\sum_{r\in[n]\setminus\{1,2,i,j\}}\bar\rho_{i,j,r}, \tag{4.1}$$

for some $\rho_{i,j,r}\in Z_{\nu_{n-5}}(M_{[n]\setminus\{1,2,i,j,r\}})$.

For distinct elements $a,b,r\in[n]\setminus\{1,2\}$, let $\gamma_{a,b,r}$ be the cycle

$$(1a\wedge 2b)+(2b\wedge ra)+(ra\wedge 12)+(12\wedge rb)+(rb\wedge 1a)$$

in $Z_1(M_{\{1,2,a,b,r\}})$. Clearly $\gamma_{a,b,r}\wedge\rho_{a,b,r}\in Z_{\nu_n}(M_n)$ and

$$\psi_{i,j}(\gamma_{a,b,r}\wedge\rho_{a,b,r})=\begin{cases}\rho_{i,j,r} & \text{if } (i,j)=(a,b)\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}\psi_{i,j}(\zeta - \sum_{a\neq b\in[n]\setminus\{1,2\}} \sum_{r\in[n]\setminus\{1,2,a,b\}} \overline{\gamma_{a,b,r}\wedge\rho_{a,b,r}}) \\ &= \psi_{i,j}(\zeta) - \sum_{r\in[n]\setminus\{1,2,i,j\}} \bar{\rho}_{i,j,r} \\ &= 0.\end{aligned}$$

Hence, by Lemma 2.3 (ii), we have

$$\zeta - \sum_{a\neq b\in[n]\setminus\{1,2\}} \sum_{r\in[n]\setminus\{1,2,a,b\}} \overline{\gamma_{a,b,r}\wedge\rho_{a,b,r}} \in \ker(\psi) = \operatorname{im}(\phi). \tag{4.3}$$

Clearly $\operatorname{im}(\phi)$ is generated by elements of the form $\alpha\wedge\tau$, where $\alpha \in \tilde{H}_0(M_T)$, $\tau \in \tilde{H}_{\nu_{n-3}}(M_{[n]-T})$, and $|T| = 3$. By induction $\tilde{H}_{\nu_{n-3}}(M_{[n]-T})$ is generated by elements of the form $\gamma\wedge\omega$, where $\gamma \in \tilde{H}_1(M_S)$, $\omega \in \tilde{H}_{\nu_{n-8}}(M_{[n]-T-S})$, and $|S| = 5$. It follows that $\operatorname{im}(\phi)$ is generated by elements of the form $\alpha\wedge\gamma\wedge\omega$, where $\alpha \in \tilde{H}_0(M_T)$, $\gamma \in \tilde{H}_1(M_S)$, $\omega \in \tilde{H}_{\nu_{n-8}}(M_{[n]-T-S})$, $|T| = 3$, and $|S| = 5$. It now follows from (4.3) that ζ is an integral combination of elements of the form $\gamma\wedge\rho$, where $\gamma \in \tilde{H}_1(M_S)$, $\rho \in \tilde{H}_{\nu_{n-5}}(M_{[n]-S})$ and $|S| = 5$. Since ζ was arbitrary, $\tilde{H}_{\nu_n}(M_n)$ is generated by elements of this form. □

We are now ready to prove Theorem 1.6 which is restated here.

Theorem 1.6. *For $n \geq 12$ (except possibly $n = 14$)[3] ,$\tilde{H}_{\nu_n}(M_n)$ is a nontrivial elementary 3-group.*

Proof. By Theorem 1.3, we need only prove the result for $n \equiv 0, 2 \bmod 3$. We prove the $n \equiv 0 \bmod 3$ case by induction on n. Table 1.1 provides the base step,

$$\tilde{H}_{\nu_{12}}(M_{12}) = \mathbb{Z}_3^{56}.$$

The induction step follows from Lemma 2.3 (i) and Theorem 3.1, since the homomorphic image of a nontrivial elementary 3-group is either trivial or is a nontrivial elementary 3-group.

Now let $n \equiv 2 \bmod 3$ and $n \geq 17$. By Lemma 4.1, $\tilde{H}_{\nu_n}(M_n)$ is generated by elements of the form $\gamma\wedge\rho$ where $\gamma \in \tilde{H}_1(M_S)$, $\rho \in \tilde{H}_{\nu_{n-5}}(M_{[n]-S})$, and $|S| = 5$. Since $n - |S| \geq 12$ and $n - |S| \equiv 0 \bmod 3$, by the 0 mod 3 case,

$$3(\gamma\wedge\rho) = \gamma\wedge 3\rho = 0.$$

[3]See New Developments Section at the end of the paper.

Hence $\tilde{H}_{\nu_n}(M_n)$ has exponent at most 3. The result now follows from Theorem 3.1. $\square$

We conjecture that the result holds for $n = 14$ as well.[4] In principle, one need only check this on the computer. However, at the present time the computer, using the software of [DHSW], produces results only up to $n = 12$. We have the following partial result for $n = 14$.

Theorem 4.2. *$\tilde{H}_{\nu_{14}}(M_{14})$ is a finite group whose Sylow 3-subgroup is nontrivial.*

Proof. By Theorem 1.3(i), we have that $\tilde{H}_{\nu_{14}}(M_{14})$ is finite.

Let $n = 17$. It follows from Lemma 4.1, that $\tilde{H}_{\nu_n}(M_n)$ is generated by elements of the form $\gamma \wedge \rho$ where $\gamma \in \tilde{H}_1(M_S)$, $\rho \in \tilde{H}_{\nu_{n-5}}(M_{[n]-S})$, and $|S| = 5$. By Lemma 2.5, $\tilde{H}_{\nu_{n-5}}(M_{[n]-S})$ is generated by elements of the form $\alpha \wedge \omega$ where $\alpha \in \tilde{H}_0(M_T)$, $\rho \in \tilde{H}_{\nu_{n-8}}(M_{[n]-S-T})$, and $|T| = 3$. It follows that $\tilde{H}_{\nu_n}(M_n)$ is generated by elements of the form $\alpha \wedge \tau$ where $\alpha \in \tilde{H}_0(M_T)$, $\tau \in \tilde{H}_{\nu_{n-3}}(M_{[n]-T})$, and $|T| = 3$. By (3.1), at least one of these generators, say $\alpha \wedge \tau$, is nonzero.

We have
$$e(\alpha \wedge \tau) = \alpha \wedge e\tau = 0,$$
where e is the exponent of $\tilde{H}_{\nu_{14}}(M_{14})$. Since $\alpha \wedge \tau \neq 0$, it follows from Theorem 1.6 that 3 divides e, which implies that there is 3-torsion in $\tilde{H}_{\nu_{14}}(M_{14})$. $\square$

Corollary 4.3. *The Sylow 3-subgroup of $\tilde{H}_{\nu_n}(M_n)$ is nontrivial for all n such that $\tilde{H}_{\nu_n}(M_n)$ is finite.*

5. Torsion in the chessboard complex

In this section we prove Theorem 1.7. The general idea is patterned on the proof of the analogous result for the matching complex, given in the previous section. However there is a significant complication. Just as for the matching complex, the tail end of the long exact sequence will be used to decompose generators into smaller cycles, but this works only if n is sufficiently close to m. When n is not sufficiently close to m, it is necessary to understand the top homology of the chessboard complex in order to decompose the generators. A study of top homology is conducted in Section 6, where an essential decomposition result, Corollary 6.5, is obtained. This decomposition result and the tail end of

[4]See New Developments Section at the end of the paper.

the long exact sequence will enable us to prove the key decomposition result:

> For all $m \le n \le 2m-2$ except $(m,n) = (4,4)$, the group $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form
> $$(ij' - ik') \wedge \rho,$$
> where $i \in [m]$, $j,k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$.

We divide the proof of Theorem 1.7 into three cases which are handled in three separate subsections. An approach to determining torsion for all finite $\tilde{H}_{\nu_{m,n}}(M_{m,n})$, not covered by Theorem 1.7, is discussed in the final subsection.

5.1. **The** 1 mod 3 **case.** For $i,j \in [m]$ and $k,l \in [n]$, let

$$\alpha_{i,k',l'} := ik' - il' \ \in \ \tilde{H}_0(M_{\{i\},\{k,l\}}),$$

and

$$\beta_{i,j,k'} := ik' - jk' \ \in \ \tilde{H}_0(M_{\{i,j\},\{k\}}).$$

We refer to the fundamental cycle $\alpha_{i,k',l'}$ as an α-cycle and the fundamental cycle $\beta_{i,j,k'}$ as a β-cycle. We also need to view these fundamental cycles as elements of $\tilde{H}_0(M_{\{i,j\},\{k,l\}})$.

Lemma 5.1. *In $\tilde{H}_0(M_{\{i,j\},\{k,l\}})$ we have*

$$\alpha_{j,k',l'} = -\alpha_{i,k',l'} = -\beta_{i,j,k'} = \beta_{i,j,l'}.$$

Proof. The first equation follows from

$$\partial((ik' \wedge jl') - (il' \wedge jk')) = (ik' - il') + (jk' - jl').$$

The second equation follows from

$$\partial(il' \wedge jk') = (il' - ik') + (ik' - jk').$$

The third equation follows from

$$\partial((ik' \wedge jl') + (il' \wedge jk')) = (ik' - jk') + (il' - jl').$$

□

Lemma 5.2. *Suppose $m+n \equiv 1 \bmod 3$ and $m \le n \le 2m-2$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form*

$$\alpha_{i,j',k'} \wedge \rho, \tag{5.1}$$

where $i \in [m]$, $j,k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$.

Proof. First note that it follows from Lemma 2.4 (i) that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form given in (5.1) and elements of the form

$$\beta_{i,j,k'} \wedge \rho, \tag{5.2}$$

where $i, j \in [m]$, $k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$.

We will show by induction on m that the elements of the form given in (5.2) can be expressed as integral combinations of elements of the form given in (5.1). The base step, $m = n = 2$, follows from Lemma 5.1. Now suppose $m > 2$.

Case 1. Say $n < 2m - 2$. Then $n - 1 \leq 2(m-2) - 2$ and we apply the induction hypothesis to $\tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$. By replacing ρ in (5.2) by an integral combination of wedge products each of which contains an α-cycle, we are able to express $\beta_{i,j,k'} \wedge \rho$ as an integral combination of wedge products each of which contains an α-cycle.

Case 2. Say $n = 2m - 2$. Then $n - 1 > 2(m-2) - 2$ and it follows from (2.2) that

$$\nu_{m-2,n-1} = m - 3,$$

so we can apply Corollary 6.5 to $\tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$, which implies that the elements ρ of the generators $\beta_{i,j,k'} \wedge \rho$ given in (5.2) can be expressed as integral combinations of elements of the form

$$\rho_{U,V} \wedge \tau,$$

where $U \subseteq [m]\setminus\{i,j\}$, $V \subseteq [n]\setminus\{k\}$, $|U| = |V| - 1$, $\rho_{U,V} \in \tilde{H}_{|U|-1}(M_{U,V})$, and

$$\tau \in \tilde{H}_{m-3-|U|}(M_{[m]\setminus(\{i,j\}\cup U),[n]\setminus(\{k\}\cup V)}).$$

This implies that the generators of (5.2) can be expressed as integral combinations of elements of the form

$$\rho_{U,V} \wedge \gamma, \tag{5.3}$$

where $U \subseteq [m]$, $V \subseteq [n]$, $|U| = |V| - 1$, $\rho_{U,V} \in \tilde{H}_{|U|-1}(M_{U,V})$, and $\gamma \in \tilde{H}_{\nu_{m,n}-|U|}(M_{[m]\setminus U,[n]\setminus V})$.

We will show that if $|U| > 1$ then

$$\tilde{H}_{\nu_{m,n}-|U|}(M_{[m]\setminus U,[n]\setminus V}) = 0, \tag{5.4}$$

from which it follows that the wedge product in (5.3) is 0. From this it follows that the generators in given in (5.2) can be expressed as integral combinations of generators given in (5.1), since $\rho_{U,V}$ is an α-cycle when $|U| = 1$.

Since $n = 2m-2$ and $m > 2$, we have $n > m$. Thus $n-|V| \geq m-|U|$. It follows that

$$\nu_{m-|U|,n-|V|} = \min(m - |U|, \lfloor \frac{m - |U| + n - |V| + 1}{3} \rfloor) - 1.$$

Suppose $|U| > 1$. We will use (1.2) of Theorem 1.1 to prove (5.4). From (2.3) we have

$$\nu_{m,n} - |U| < m - |U| - 1. \tag{5.5}$$

We also need to check that

$$\nu_{m,n} - |U| < \lfloor \frac{m - |U| + n - |V| + 1}{3} \rfloor - 1. \tag{5.6}$$

By (2.2), we have

$$\nu_{m,n} - |U| = \frac{m + n - 1}{3} - 1 - |U|.$$

The right side of (5.6) equals

$$\lfloor \frac{m - |U| + n - |U| - 1 + 1}{3} \rfloor - 1 = \frac{m + n - 1}{3} + \lfloor \frac{-2|U| + 1}{3} \rfloor - 1.$$

So (5.6) is equivalent to

$$-|U| < \lfloor \frac{-2|U| + 1}{3} \rfloor,$$

which clearly holds when $|U| \geq 2$. Hence by (1.2), equation (5.4) holds. □

Lemma 5.3. *Suppose $m + n \equiv 1 \bmod 3$ and $m \leq n \leq 2m - 2$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form*

$$\alpha_{\sigma(1),\tau(1)',\tau(2)'} \wedge \alpha_{\sigma(2),\tau(3)',\tau(4)'} \wedge \cdots \wedge \alpha_{\sigma(t),\tau(2t-1)',\tau(2t)'} \wedge \beta_{\sigma(t+1),\sigma(t+2),\tau(2t+1)'} \wedge \beta_{\sigma(t+3),\sigma(t+4),\tau(2t+2)'} \wedge \cdots \wedge \beta_{\sigma(m-2),\sigma(m-1),\tau(n)'}, \tag{5.7}$$

where $\sigma \in \mathfrak{S}_m$, $\tau \in \mathfrak{S}_n$, and $t = \frac{2n-m+1}{3}$.

Proof. We use induction on m. When $m = 2$, the result is immediate from Lemma 5.2. When $2 < m < n$ the result follows from Lemma 5.2 and the induction hypothesis applied to $\tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$.

When $2 < m = n$, we also use Lemma 5.2 and apply the induction hypothesis to $\tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$. However there is an additional step. Since $m + n \equiv 1 \bmod 3$, we have $5 \leq m = n$. Hence

$n-2 \leq m-1 \leq 2(n-2)-2$. This allows us to apply the induction hypothesis with the role of the α-cycles and the β-cycles switched. Hence we have that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form

$$\alpha_{\sigma(1),\tau(1)',\tau(2)'} \wedge \alpha_{\sigma(2),\tau(3)',\tau(4)'} \wedge \cdots \wedge \alpha_{\sigma(t),\tau(2t-1)',\tau(2t)'} \quad \wedge$$
$$\beta_{\sigma(t+1),\sigma(t+2),\tau(2t+1)'} \wedge \beta_{\sigma(t+3),\sigma(t+4),\tau(2t+2)'} \wedge \cdots \wedge \beta_{\sigma(m-1),\sigma(m),\tau(n-1)'},$$

where $\sigma \in \mathfrak{S}_m$, $\tau \in \mathfrak{S}_n$, and $t = \frac{2n-m+1}{3} - 1$. To complete the proof, we use Lemma 5.1 to change one of the β-cycles to an α-cycle. $\square$

Theorem 5.4. *Suppose $m+n \equiv 1 \bmod 3$ and $m \leq n \leq 2m-5$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a cyclic group of order 3 generated by*

$$\alpha_{1,1',2'} \wedge \alpha_{2,3',4'} \wedge \cdots \wedge \alpha_{t,(2t-1)',(2t)'} \quad \wedge$$
$$\beta_{t+1,t+2,(2t+1)'} \wedge \beta_{t+3,t+4,(2t+2)'} \wedge \cdots \wedge \beta_{m-2,m-1,n'}, \tag{5.8}$$

where $t = \frac{2n-m+1}{3}$.

Proof. We use the relations of Lemma 5.1 to show that the generators of Lemma 5.3 are all equal up to sign. It suffices to show that

(5.9)
$$\begin{aligned}\alpha_{1,1',2'} \wedge \cdots \wedge \alpha_{t,(2t-1)',(2t)'} \wedge \beta_{t+1,t+2,(2t+1)'} \wedge \cdots \wedge \beta_{m-2,m-1,n'} \\ = \operatorname{sgn}(\sigma)\ \alpha_{\sigma(1),1',2'} \wedge \cdots \wedge \alpha_{\sigma(t),(2t-1)',(2t)'} \\ \wedge \beta_{\sigma(t+1),\sigma(t+2),(2t+1)'} \wedge \cdots \wedge \beta_{\sigma(m-2),\sigma(m-1),n'},\end{aligned}$$

and

(5.10)
$$\begin{aligned}\alpha_{1,1',2'} \wedge \cdots \wedge \alpha_{t,(2t-1)',(2t)'} \wedge \beta_{t+1,t+2,(2t+1)'} \wedge \cdots \wedge \beta_{m-2,m-1,n'} \\ = \operatorname{sgn}(\tau)\ \alpha_{1,\tau(1)',\tau(2)'} \wedge \cdots \wedge \alpha_{t,\tau(2t-1)',\tau(2t)'} \\ \wedge \beta_{t+1,t+2,\tau(2t+1)'} \wedge \cdots \wedge \beta_{m-2,m-1,\tau(n)'},\end{aligned}$$

for all $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$.

For the sake of ease of notation and getting to the heart of the argument, we prove (5.9) and (5.10) for $m = n = 5$. The general argument is essentially the same. To prove

(5.11)
$$\alpha_{1,1',2'} \wedge \alpha_{2,3',4'} \wedge \beta_{3,4,5'} = \operatorname{sgn}(\sigma)\ \alpha_{\sigma(1),1',2'} \wedge \alpha_{\sigma(2),3',4'} \wedge \beta_{\sigma(3),\sigma(4),5'}$$

for all σ, it suffices to prove this for σ in the set of transpositions $\{(1,5),(2,5),(1,3),(1,4)\}$, which generates $\mathfrak{S}_5$.

Case 1. $\sigma = (1,5)$. By Lemma 5.1, $\alpha_{1,1',2'} = -\alpha_{5,1',2'}$. Hence

$$\alpha_{1,1',2'} \wedge \alpha_{2,3',4'} \wedge \beta_{3,4,5'} = -\alpha_{5,1',2'} \wedge \alpha_{2,3',4'} \wedge \beta_{3,4,5'}.$$

Case 2. $\sigma = (2,5)$. This is similar to Case 1.

Case 3. $\sigma = (1,3)$. By repeated applications of Lemma 5.1, we have

$$\begin{aligned}
\alpha_{1,1',2'} \wedge \alpha_{2,3',4'} \wedge \beta_{3,4,5'} &= \alpha_{1,1',2'} \wedge \beta_{2,5,3'} \wedge \beta_{3,4,5'} \\
&= \alpha_{1,1',2'} \wedge \beta_{2,5,3'} \wedge \alpha_{4,4',5'} \\
&= -\alpha_{3,1',2'} \wedge \beta_{2,5,3'} \wedge \alpha_{4,4',5'} \\
&= -\alpha_{3,1',2'} \wedge \beta_{2,5,3'} \wedge \beta_{1,4,5'} \\
&= -\alpha_{3,1',2'} \wedge \alpha_{2,3',4'} \wedge \beta_{1,4,5'}
\end{aligned}$$

Case 4. $\sigma = (1,4)$. This is similar to Case 3.

To show

$$\alpha_{1,1',2'} \wedge \alpha_{2,3',4'} \wedge \beta_{3,4,5'} = \operatorname{sgn}(\tau)\ \alpha_{1,\tau(1)',\tau(2)'} \wedge \alpha_{2,\tau(3)',\tau(4)'} \wedge \beta_{3,4,\tau(5)'} \tag{5.12}$$

for all $\tau \in \mathfrak{S}_5$, we use Lemma 5.1 to exchange an α-cycle for a β-cycle. That is, by Lemma 5.1, equation (5.12) is equivalent to

$$\alpha_{1,1',2'} \wedge \beta_{2,5,3'} \wedge \beta_{3,4,5'} = \operatorname{sgn}(\tau)\ \alpha_{1,\tau(1)',\tau(2)'} \wedge \beta_{2,5,\tau(3)'} \wedge \beta_{3,4,\tau(5)'}.$$

This is equivalent to (5.11) with the role of the α-cycles and β-cycles switched.

It is straightforward to extend the argument for $m = n = 5$ to general $m \le n \le 2m-5$ since $\mathfrak{S}_m$ is generated by the set of transpositions $\{(1,m)\dots(t,m),(1,t+1),\dots,(1,m-1)\}$, and the expressions on each side of (5.9) and (5.10) contain at least two α-cycles and at least one β-cycle.

We now show that the order of the cyclic group $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is 3 by induction on m. The base step $\tilde{H}_{\nu_{5,5}}(M_{5,5}) = \mathbb{Z}_3$ is given in Table 1.2. Let $m \ge 6$. The generator given in (5.8) can be expressed as

$$\alpha_{1,1',2'} \wedge \rho$$

where $\rho \in \tilde{H}_{\nu_{m,n}-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,2\}})$. If $m < n$ then clearly $m-1 \le n-2 \le 2(m-1)-5$. If $m = n$ then $m = n \ge 8$ which implies $n-2 \le m-1 \le 2(n-2)-5$. In either case, $\nu_{m,n}-1 = \nu_{m-1,n-2}$, and we can apply the induction hypothesis to $\tilde{H}_{\nu_{m,n}-1}(M_{[m]\setminus\{1\},[n]\setminus\{1,2\}})$ to obtain

$$3(\alpha_{1,1',2'} \wedge \rho) = \alpha_{1,1',2'} \wedge 3\rho = 0.$$

Since, by Theorem 3.1, $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is nonvanishing, it has order 3. $\square$

5.2. **The** $0 \bmod 3$ **case.**

Lemma 5.5. *Suppose* $m+n \equiv 0 \bmod 3$ *and* $m \le n \le 2m-3$. *Then* $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ *is generated by elements of the form*

$$\alpha_{i,j',k'} \wedge \rho, \tag{5.13}$$

where $i \in [m]$, $j,k \in [n]$, *and* $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$.

Proof. The proof, although similar to the proof of Lemma 5.2, requires an additional step. By Lemma 2.4 (i), we have that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form given in (5.13) and elements of the form

$$\beta_{i,j,k'} \wedge \rho, \tag{5.14}$$

where $i,j \in [m]$, $k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$. It follows from this that $\tilde{H}_{\nu_{3,3}}(M_{3,3})$ is generated by elements of the form $\alpha_{i_1,j_1',j_2'} \wedge \beta_{i_2,i_3,j_3'}$, which takes care of the base step of an induction proof. Now assume $m > 3$.

Case 1. Say $n < 2m-3$. Then $n-1 \le 2(m-2)-3$. By applying the induction hypothesis to $\tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$, we have that the generators given in (5.14) can be expressed as integral combinations of generators given in (5.13).

Case 2. Say $n = 2m-3$. Then $n-1 > 2(m-2)-1$, so by (2.3), we have $\nu_{m-2,n-1} = m-3$. By applying Corollary 6.5 to $\tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$, we see that generators given in (5.14) can be expressed as integral combinations of elements of the form

$$\rho_{U,V} \wedge \gamma, \tag{5.15}$$

where $|U| = |V|-1$, $\rho_{U,V} \in \tilde{H}_{|U|-1}(M_{U,V})$, and $\gamma \in \tilde{H}_{\nu_{m,n}-|U|}(M_{[m]\setminus U,[n]\setminus V})$.

One can show that if $|U| > 2$ then $\tilde{H}_{\nu_{m,n}-|U|}(M_{[m]\setminus U,[n]\setminus V}) = 0$ by using an argument similar to the one that was used to prove (5.4). We leave the straightforward details to the reader. This allows us to conclude that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements given in (5.13) and (5.15), where $2 = |U| = |V|-1$.

We now show that any generator of the form given in (5.15), where $(|U|,|V|) = (2,3)$, can be expressed as integral combination of generators given in (5.13), which will complete the proof. Since $m > 3$ and $n = 2m-3$, we have $m < n$. Thus

$$m-2 \le n-3 \le 2(m-2)-2.$$

By (2.2), we have $\nu_{m,n} - |U| = \nu_{m-|U|,n-|V|}$. It therefore follows from Lemma 5.2, that $\tilde{H}_{\nu_{m,n}-|U|}(M_{[m]\setminus U,[n]\setminus V})$ is generated by wedge products that contain an α-cycle. $\square$

The next result follows easily from Lemma 5.5 by induction.

Lemma 5.6. *Suppose $m+n \equiv 0 \bmod 3$ and $m \le n \le 2m-3$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form*

$$\alpha_{i_1,j_1',j_2'} \wedge \beta_{i_2,i_3,j_3'} \wedge \xi, \tag{5.16}$$

where $i_1, i_2, i_3 \in [m]$, $j_1, j_2, j_3 \in [n]$ and

$$\xi \in \tilde{H}_{\nu_{m,n}-2}(M_{[m]\setminus\{i_1,i_2,i_3\},[n]\setminus\{j_1,j_2,j_3\}}).$$

For distinct $i_1, i_2, i_3 \in [m]$ and distinct $j_1, j_2, j_3 \in [n]$, let

$$u_{i_1,i_2,j_1',j_2',j_3'} := i_1j_1' \wedge i_2j_2' + i_2j_2' \wedge i_1j_3' + i_1j_3' \wedge i_2j_1' + i_2j_1' \wedge i_1j_2' + i_1j_2' \wedge i_2j_3' + i_2j_3' \wedge i_1j_1'$$

and

$$v_{i_1,i_2,i_3,j_1',j_2'} := i_1j_1' \wedge i_2j_2' + i_2j_2' \wedge i_3j_1' + i_3j_1' \wedge i_1j_2' + i_1j_2' \wedge i_2j_1' + i_2j_1' \wedge i_3j_2' + i_3j_2' \wedge i_1j_1'.$$

When it suits our purposes, we shall view $u_{i_1,i_2,j_1',j_2',j_3'}$ and $v_{i_1,i_2,i_3,j_1',j_2'}$ as elements of $\tilde{H}_{\nu_{3,3}}(M_{\{i_1,i_2,i_3\},\{j_i,j_2,j_3\}})$ as well as of $\tilde{H}_{\nu_{2,3}}(M_{\{i_1,i_2\},\{j_i,j_2,j_3\}})$ and $\tilde{H}_{\nu_{3,2}}(M_{\{i_1,i_2,i_3\},\{j_i,j_2\}})$, respectively.

Lemma 5.7. *In $\tilde{H}_1(M_{3,3})$ we have,*

$$3(\alpha_{1,1',2'} \wedge \beta_{2,3,3'}) = -u_{2,3,1',2',3'} - v_{1,2,3,1',2'} - 2(v_{1,2,3,2',3'} + u_{1,2,1',2',3'}).$$

Proof. It is straightforward to verify that

$$\begin{aligned}&\partial(11' \wedge 22' \wedge 33' + 12' \wedge 23' \wedge 31' + 12' \wedge 21' \wedge 33' + 11' \wedge 32' \wedge 23') \\ &\quad = u_{2,3,1',2',3'} + v_{1,2,3,1',2'} - \alpha_{1,1',2'} \wedge \beta_{2,3,3'} - 2(\alpha_{3,2',3'} \wedge \beta_{1,2,1'}).\end{aligned}$$

Consequently, in $\tilde{H}_1(M_{3,3})$,

$$\alpha_{1,1',2'} \wedge \beta_{2,3,3'} = u_{2,3,1',2',3'} + v_{1,2,3,1',2'} - 2(\alpha_{3,2',3'} \wedge \beta_{1,2,1'}).$$

By symmetry (exchanging α with β, u with v, and i with i'),

$$\beta_{1,2,1'} \wedge \alpha_{3,2',3'} = v_{1,2,3,2',3'} + u_{1,2,1',2',3'} - 2(\beta_{2,3,3'} \wedge \alpha_{1,1',2'}).$$

By substituting the second equation into the first equation, we get

$$\begin{aligned}&\alpha_{1,1',2'} \wedge \beta_{2,3,3'} = \\ &u_{2,3,1',2',3'} + v_{1,2,3,1',2'} + 2(v_{1,2,3,2',3'} + u_{1,2,1',2',3'} - 2(\beta_{2,3,3'} \wedge \alpha_{1,1',2'})),\end{aligned}$$

which implies that

$$3(\alpha_{1,1',2'} \wedge \beta_{2,3,3'}) = -u_{2,3,1',2',3'} - v_{1,2,3,1',2'} - 2(v_{1,2,3,2',3'} + u_{1,2,1',2',3'}).$$

□

Theorem 5.8. *Suppose $m+n \equiv 0 \bmod 3$ and $m \le n \le 2m-9$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a nontrivial 3-group of exponent at most 9.*

Proof. It follows from Lemmas 5.6 and 5.7, that $3\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form

$$\rho_{U,V} \wedge \omega,$$

with $\{|U|,|V|\} = \{2,3\}$, $\rho_{U,V} \in \tilde{H}_1(M_{U,V})$ and $\omega \in \tilde{H}_{\nu_{m,n-2}}(M_{[m]\setminus U,[n]\setminus V})$. We can show that

$$3(\rho_{U,V} \wedge \omega) = \rho_{U,V} \wedge 3\omega = 0 \tag{5.17}$$

by applying Theorem 5.4, if we first check that $m-|U|$ and $n-|V|$ satisfy the hypothesis of the theorem. Clearly $m-|U|+n-|V| = m+n-5 \equiv 1 \bmod 3$. We leave it to the reader to check the inequalities in each of the three cases:

(1) $m < n$ and $(|U|,|V|) = (2,3)$
(2) $m = n$ and $(|U|,|V|) = (2,3)$
(3) $m \le n$ and $(|U|,|V|) = (3,2)$.

It follows from (5.17) that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ has exponent at most 9, and from Theorem 3.1 that the group is nontrivial. □

5.3. **The $2 \bmod 3$ case.**

Lemma 5.9. *Suppose $m+n \equiv 2 \bmod 3$ and $4 \le m \le n \le 2m-4$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form*

$$\alpha_{i,j',k'} \wedge \rho, \tag{5.18}$$

where $i \in [m]$, $j,k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$, and elements of the form

$$\beta_{i,j,k'} \wedge \rho, \tag{5.19}$$

where $i,j \in [m]$, $k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$.

Proof. We claim that $\bigoplus_{i,j} \tilde{H}_{\nu_{m-2,n-2}}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$ is generated by elements of the form $\psi(\alpha_{r,s',t'} \wedge \rho)$, where ψ is the surjection of Lemma 2.4 (ii), and

- $r \in [m] \setminus \{1\}$
- $s,t \in [n] \setminus \{1\}$
- $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{r\},[n]\setminus\{s,t\}})$.

We prove this claim by first using Lemma 5.2 to observe that

$$\tilde{H}_{\nu_{m-2,n-2}}(M_{[m]\setminus\{1,i\},[n]\setminus\{1,j\}})$$

is generated by elements of the form $\alpha_{r,s',t'} \wedge \tau$, where

- $r \in [m] \setminus \{1, i\}$
- $s, t \in [n] \setminus \{1, j\}$
- $\tau \in \tilde{H}_{\nu_{m-3,n-4}}(M_{[m]\setminus\{1,i,r\},[n]\setminus\{1,j,s,t\}})$.

The map

$$\psi : \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{r\},[n]\setminus\{s,t\}}) \to \bigoplus_{i,j} \tilde{H}_{\nu_{m-3,n-4}}(M_{[m]\setminus\{1,i,r\},[n]\setminus\{1,j,s,t\}})$$

is surjective by Lemma 2.4 (ii). Hence for

$$\tau \in \tilde{H}_{\nu_{m-3,n-4}}(M_{[m]\setminus\{1,i,r\},[n]\setminus\{1,j,s,t\}}),$$

we can let $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{r\},[n]\setminus\{s,t\}})$ be such that $\psi(\rho) = \tau$. It follows directly from the definition of ψ that

$$\psi(\alpha_{r,s',t'} \wedge \rho) = \alpha_{r,s',t'} \wedge \tau,$$

which proves our claim.

Let $\gamma \in \tilde{H}_{\nu_{m,n}}(M_{m,n})$. We express $\psi(\gamma)$ as an integral combination of generators:

$$\psi(\gamma) = \sum_{r,s,t,\rho} c_{r,s,t,\rho} \psi(\alpha_{r,s',t'} \wedge \rho) = \psi\left(\sum_{r,s,t,\rho} c_{r,s,t,\rho}(\alpha_{r,s',t'} \wedge \rho)\right),$$

for some $c_{r,s,t,\rho} \in \mathbb{Z}$. It follows from Lemma 2.4 (ii) that

$$\gamma - \sum_{r,s,t,\rho} c_{r,s,t,\rho}(\alpha_{r,s',t'} \wedge \rho) \in \operatorname{im}\phi.$$

Hence γ can be expressed as an integral combination of elements of the form given in the statement of the lemma. $\square$

Next we show that the elements given in (5.19) can be removed from the generating set.

Lemma 5.10. *Suppose $m + n \equiv 2 \bmod 3$ and $5 \le m \le n \le 2m - 4$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form*

$$\alpha_{i,j',k'} \wedge \rho, \tag{5.20}$$

where $i \in [m]$, $j, k \in [n]$, and $\rho \in \tilde{H}_{\nu_{m-1,n-2}}(M_{[m]\setminus\{i\},[n]\setminus\{j,k\}})$.

Proof. The proof is similar to the proofs of Lemmas 5.2 and 5.5. We use induction on m. The base step, $(m, n) = (5, 6)$, is part of Case 2 below, which does not require the induction hypothesis.

We will show that generators given in (5.19) can be expressed as integral combinations of generators given in (5.20).

Case 1. Say $n < 2m - 4$. Then $5 < m$ and $n - 1 \le 2(m - 2) - 4$. Moreover, $m \ne 6$ because otherwise $n - 1 \le 4$. Hence, $5 \le m - 2 \le$

$n-1 \leq 2(m-2)-4$, which enables us to apply the induction hypothesis to $\tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$.

Case 2. Say $n = 2m-4$. Since $m \geq 5$, it follows that $n > m$.

By Corollary 6.5 applied to $\tilde{H}_{\nu_{m-2,n-1}}(M_{[m]\setminus\{i,j\},[n]\setminus\{k\}})$, the generators given in (5.19) can be expressed as integral combinations of elements of the form

$$\rho_{U,V} \wedge \gamma, \tag{5.21}$$

with $|U| = |V|-1$, $\rho_{U,V} \in \tilde{H}_{|U|-1}(M_{U,V})$ and $\gamma \in \tilde{H}_{\nu_{m,n-|U|}}(M_{[m]\setminus U,[n]\setminus V})$.

An argument similar to the one used in the proof of Lemma 5.2 shows that if $|U| > 4$, then the wedge product in (5.21) is 0. From this it follows that the generators given in (5.19) can be expressed as integral combinations of generators of the form given in (5.21) where

$$(|U|,|V|) = (1,2), (2,3) \text{ or } (3,4).$$

As in the proof of Lemma 5.5, we will show that each of these generators $\rho_{U,V} \wedge \gamma$ can be written as an integral combination of generators given in (5.20), which will complete the proof.

If $(|U|,|V|) = (1,2)$ then we are done. If $(|U|,|V|) = (2,3)$ then we apply Lemma 5.6 since $m-2+n-3 \equiv 0 \bmod 3$. Since $m < n$, we have $m-2 \leq n-3 \leq 2(m-2)-3$. Hence by Lemma 5.6, we have that $\tilde{H}_{\nu_{m,n-|U|}}(M_{[m]\setminus U,[n]\setminus V})$ is generated by wedge products containing α-cycles. It follows that γ, and hence $\rho_{U,V} \wedge \gamma$, is an integral combination of wedge products containing α-cycles.

Now suppose $(|U|,|V|) = (3,4)$. Since $m < n$, we have $m-3 \leq n-4 \leq 2(m-3)-2$. We can therefore apply Lemma 5.2 since $m-3+n-4 \equiv 1 \bmod 3$. Hence, $\tilde{H}_{\nu_{m,n-|U|}}(M_{[m]\setminus U,[n]\setminus V})$ is generated by wedge products which contain α-cycles. It follows that γ, and hence $\rho_{U,V} \wedge \gamma$, is an integral combination of wedge products containing α-cycles. $\square$

The next result follows readily from Lemma 5.10 by induction.

Lemma 5.11. *Suppose $m+n \equiv 2 \bmod 3$ and $m \leq n \leq 2m-4$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is generated by elements of the form*

$$\omega \wedge \gamma,$$

where

$$\omega \in \tilde{H}_{\nu_{4,4}}(M_{U,V}), \qquad \gamma \in \tilde{H}_{\nu_{m-4,n-4}}(M_{[m]\setminus U,[n]\setminus V}),$$

and

$$4 = |U| = |V|.$$

Theorem 5.12. *Suppose $m+n \equiv 2 \bmod 3$ and $m \leq n \leq 2m-13$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a nontrivial 3-group of exponent at most 9.*

Proof. Since $m-4+n-4 \equiv 0 \bmod 3$ and $m-4 \le n-4 \le 2(m-4)-9$, the result follows from Lemma 5.11 and Theorem 5.8. $\square$

This completes the proof of Theorem 1.7. We conjecture that the exponent in Theorem 1.7 is 3. The following result shows that this conjecture need only be verified for $m = n = 9$.

Theorem 5.13. *For all m, n that satisfy the hypothesis of Theorem 1.7, the exponent of $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ divides the exponent of $\tilde{H}_{\nu_{9,9}}(M_{9,9})$. Consequently if $\tilde{H}_{\nu_{9,9}}(M_{9,9})$ is an elementary 3-group then so is $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ for all m, n that satisfy the hypothesis of Theorem 1.7.*

Proof. The proof is similar to that of Theorem 5.12. It follows from Lemmas 5.11 and 5.5. $\square$

5.4. **Finite homology.** This subsection contains some partial results on the finite $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ not covered by Theorem 1.7. We start with an analog of Corollary 4.3.

Theorem 5.14. *The Sylow 3-subgroup of $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is nontrivial for all m, n such that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is finite.*

Proof. The proof is similar to that of (3.2). Assume $m \le n$ and $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is finite with exponent e.

Case 1. $m + n \equiv 1 \bmod 3$. It follows from Theorem 1.5, that this case is covered by Theorem 1.7 (i).

Case 2. $m + n \equiv 0 \bmod 3$. Consider the cycle z in the proof of (3.2). Recall that z cannot be a boundary in $M_{[m]\uplus[n]'}$. Since ez is a boundary in $M_{m,n}$, it is also a boundary in $M_{[m]\uplus[n]'}$. Since by Theorem 1.5, $7 \le m \le n \le 2m-6$, we have that $m+n \ge 15$. Therefore Theorem 1.6 implies that 3 divides e, which means that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ has 3-torsion.

Case 3. $m + n \equiv 2 \bmod 3$. By Theorem 1.5, we have $9 \le m \le n \le 2m - 7$. Consider the surjection ψ of Lemma 2.4 (ii). Since $m+n-4 \equiv 1 \bmod 3$ and $5 \le m-2 \le n-2 \le 2(m-2)-5$, the range of ψ has 3-torsion by Theorem 1.7 (i). Since the domain is finite, the domain must also have 3-torsion. $\square$

We have not yet been able to eliminate p-torsion in finite $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ for primes $p \ne 3$ except in the cases covered by Theorem 1.7. However, the lemmas of the previous subsections provide an approach to doing so as well as to reducing the exponent in Theorem 1.7 to 3. This approach, which depends only on anticipated improvements in computer efficiency, is demonstrated by the following result.

Theorem 5.15.

(i) *If* $m+n \equiv 0 \bmod 3$ *and* $7 \leq m \leq n \leq 2m-6$ *then* $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ *is finite and its exponent divides the exponent of* $\tilde{H}_{\nu_{7,8}}(M_{7,8})$.

(ii) *If* $m+n \equiv 2 \bmod 3$ *and* $11 \leq m \leq n \leq 2m-10$ *then* $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ *is finite and its exponent divides the exponent of* $\tilde{H}_{\nu_{7,8}}(M_{7,8})$.

(iii) *If* $m+n \equiv 2 \bmod 3$ *and* $9 \leq m \leq n \leq 2m-7$ *and* $(m,n) \neq (10,10)$ *then* $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ *is finite and its exponent divides the exponent of* $\tilde{H}_{\nu_{9,11}}(M_{9,11})$.

Consequently if the Sylow 3*-subgroup of* $\tilde{H}_{\nu_{7,8}}(M_{7,8})$ *is elementary then* $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ *is an elementary* 3*-group for all* m,n *that satisfies the hypothesis of Theorem 1.7.*

Proof. Finiteness of the homology groups follow from Theorem 1.5.

(i) We prove this by induction on m. The base case, $(m,n) = (7,8)$, is trivial. Now assume $m > 7$. By Lemma 5.5, the exponent of $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ divides the exponent of $\tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$ if $\tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$ is finite. If $m < n$ then $7 \leq m-1 \leq n-2 \leq 2(m-1)-6$. Hence by induction, $\tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$ is finite and the exponent of $\tilde{H}_{\nu_{7,8}}(M_{7,8})$ is divisible by the exponent of $\tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$ which is divisible by the exponent of $\tilde{H}_{\nu_{m,n}}(M_{m,n})$. If $m = n$ then $7 \leq n-2 \leq m-1 \leq 2(n-2)-6$. So we can apply the induction hypothesis in this case as well.

(ii) By Lemma 5.11, $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ divides the exponent of $\tilde{H}_{\nu_{m-4,n-4}}(M_{m-4,n-4})$ if $\tilde{H}_{\nu_{m-4,n-4}}(M_{m-4,n-4})$ is finite. Since $7 \leq m-4 \leq n-4 \leq 2(m-4)-6$, we can apply (i).

(iii) This is similar to the proof of (i) and is left to the reader. $\square$

Remark 5.16. We conjecture that there is some m_0, such that if $n_0 = 2m_0 - 6$ or $n_0 = 2m_0 - 7$ then $\tilde{H}_{\nu_{m_0,n_0}}(M_{m_0,n_0})$ is an elementary 3-group. If this is so, then an argument like the one used in the proof of Theorem 5.15 would yield the conclusion that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is an elementary 3-group for all but a finite number of pairs (m,n) satisfying $m \leq n \leq 2m-5$. (Recall $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is infinite when $n > 2m-5$.)

6. Top homology of the chessboard complex

In this section we construct bases for the top homology and cohomology of the chessboard complex. The basis for homology yields the decomposition result used in proving the torsion results of Section 5.

Two important ingredients in the construction of our homology basis are the classical Robinson-Schensted correspondence of tableaux combinatorics and the fact that the complex $M_{n-1,n}$ is an orientable pseudomanifold. The basis elements are expressed as wedge products of fundamental cycles of copies of the orientable pseudomanifolds $M_{k-1,k}$ that result from applying the Robinson-Schensted correspondence to pairs of tableaux. These pairs of tableaux arise in Garst's [Ga] and Friedman and Hanlon's [FrHa] study of the representation of the symmetric group on the top homology of the chessboard complex.

We assume familiarity with the representation theory of the symmetric group $\mathfrak{S}_n$ and tableaux combinatorics, cf., [Sa], [St], [Fu]. The Specht module (or irreducible representation of $\mathfrak{S}_n$) over $\mathbb{C}$ indexed by the partition $\lambda \vdash n$, is denoted by S^λ. Recall that the dimension of S^λ is the number f^λ of standard Young tableaux of shape λ.

The direct product $\mathfrak{S}_m \times \mathfrak{S}_n$ acts on the chessboard complex $M_{m,n}$ by relabelling the graph vertices in $[m]$ and $[n]'$, and this induces a representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on $\tilde{H}_*(M_{m,n};\mathbb{C})$. The following result enables one to express the Betti numbers in terms of the number of pairs of standard Young tableaux of certain shapes.

Theorem 6.1 (Friedman and Hanlon [FrHa]). *For all $p, m, n \in \mathbb{Z}$, where $m, n \geq 1$, the following isomorphism of $(\mathfrak{S}_m \times \mathfrak{S}_n)$-modules holds:*

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong_{\mathfrak{S}_m \times \mathfrak{S}_n} \bigoplus_{(\lambda,\mu)\in\mathcal{R}(m,n,p)} S^{\lambda'} \otimes S^{\mu},$$

where $\mathcal{R}(m,n,p)$ is the set of all pairs of partitions $(\lambda \vdash m, \mu \vdash n)$ that can be obtained in the following way. Take a partition $\nu \vdash p$ that contains an $(m-p)\times(n-p)$ rectangle but contains no $(m-p+1)\times$ $\times(n-p+1)$ rectangle. Add a column of size $m-p$ to ν to obtain λ and add a row of size $n-p$ to ν to obtain μ. See Figure 6.1.

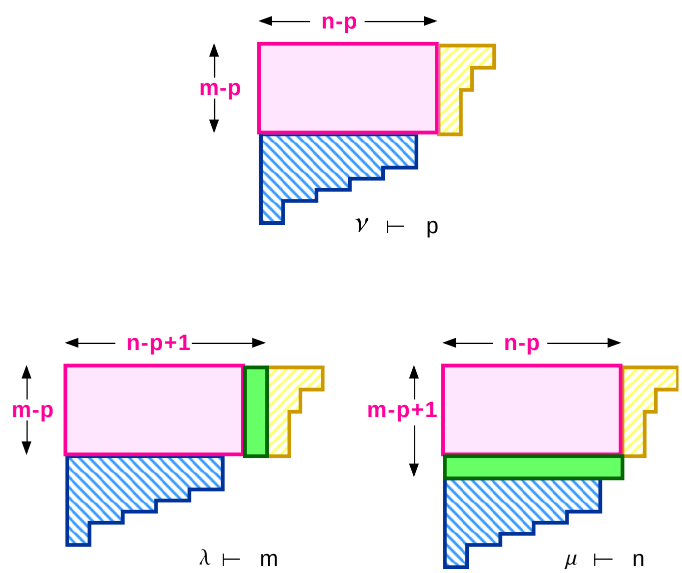

Figure 6.1

Corollary 6.2 (Garst[Ga]). *For all $m \leq n$, the following isomorphism of $\mathfrak{S}_n$-modules holds*

$$\tilde{H}_{m-1}(M_{m,n};\mathbb{C}) \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\lambda \vdash m \\ \lambda_1 \leq n-m}} f^\lambda\, S^{\lambda^*},$$

where λ^* is the partition obtained from λ by adding a part of size $n-m$.

It follows immediately from Corollary 6.2 that the rank of the top homology $\tilde{H}_{m-1}(M_{m,n})$ of the chessboard complex $M_{m,n}$ is the number of pairs of standard Young tableaux (S,T) such that S has m cells, T has n cells and the shape of S is the same as the shape of T minus the first row. Let $\mathcal{P}_{m,n}$ be the set of such pairs of standard tableaux. We construct for each $(S,T) \in \mathcal{P}_{m,n}$, a cycle $\eta(S,T) \in \tilde{H}_{m-1}(M_{m,n})$, and show that these cycles form a basis for homology.

In order to prove that the $\eta(S,T)$ form a basis for homology, we construct cocycles $\gamma(S,T)$ which form a basis for cohomology. Since our complex is finitely generated we can view the cohomology group as a subquotient of the chain group, just as is done for the homology group. Indeed, for any finite simplicial complex Δ on vertex set $\{x_1,\dots,x_r\}$, let $\langle\ ,\ \rangle$ be the bilinear form on $C_{k-1}(\Delta)$ for which the oriented simplices $(x_{i_1},\dots,x_{i_k})$, $i_1 < \cdots < i_k$, form an orthonormal basis. The coboundary map $\delta_k : C_k(\Delta) \to C_{k+1}(\Delta)$ is the adjoint of the boundary map. That is

$$\langle u, \delta_k(v)\rangle = \langle \partial_{k+1}(u), v\rangle,$$

for all $u \in C_{k+1}(\Delta)$ and $v \in C_k(\Delta)$. The kth cohomology group is defined to be the quotient of the cocycle group $Z^k(\Delta) := \ker \delta_k$ by the coboundary group $B^k(\Delta) := \operatorname{im} \delta_{k-1}$.

We construct the cycles and cocycles using the Robinson-Schensted correspondence. We begin with the cocycles. Let $(S,T) \in \mathcal{P}_{m,n}$. First add a cell with entry ∞ to the bottom of each of the first $n-m$ columns (some may be empty) of S to obtain a semistandard tableau S^* of the same shape as T. (Here ∞ represents a number larger than m.) See Figure 6.2. The inverse of the Robinson-Schensted bijection applied to (S^*, T) produces a permutation σ of the multiset $\{1, 2, \ldots, m, \infty^{n-m}\}$. The multiset permutation σ corresponds naturally to the oriented simplex of $M_{m,n}$ given by

$$\tau(\sigma) := (\sigma(i_1)i'_1, \sigma(i_2)i'_2, \ldots, \sigma(i_m)i'_m)\,, \tag{6.1}$$

where $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_m)$ is the subword of $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ obtained by removing the ∞'s. This oriented simplex is clearly a cocycle since it is in the top dimension. Let $\gamma(S,T)$ be the coset of the coboundary group $B^{m-1}(M_{m,n})$ that contains this oriented simplex.

We demonstrate the procedure for constructing $\gamma(S,T)$ by letting (S,T) be the pair of tableaux given in Figure 6.2. After applying the inverse of Robinson-Schensted to (S^*, T) we have the multiset permutation

$$\infty\,\infty\,2\,\infty\,4\,\infty\,3\,1.$$

The oriented simplex that corresponds to this multiset permutation is

$$(23', 45', 37', 18').$$

Hence, $\gamma(S,T)$ is the coset of $B^3(M_{4,8})$ that contains the oriented simplex $(23', 45', 37', 18')$.

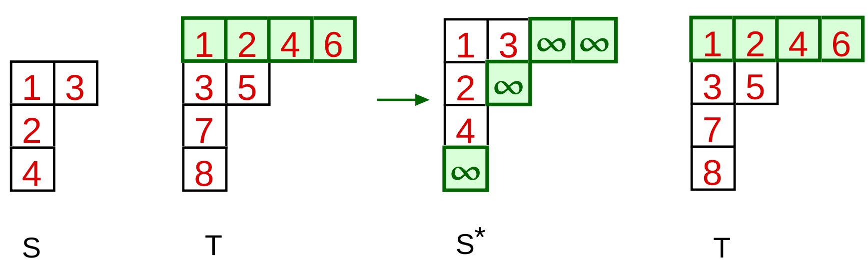

Figure 6.2

The construction of the cycles is a bit more involved. Recall that in the inverse Robinson-Schensted procedure, an entry "pops" from a cell in the top row of the left tableau when an entry is "crossed out" of the right tableau. For each top cell, we must keep track of the entries of

S^* that are popped and the corresponding entries of T that are crossed out. For each $i = 1, 2, \ldots, n - m$, let A_i^* be the multiset of entries that are popped from the ith cell of the top row of S^* and let B_i be the corresponding set of entries that are crossed out of T. One can easily see that A_i^* is actually a set and $\infty \in A_i^*$ for all i. Now let $A_i = A_i^* \setminus \{\infty\}$. So $|A_i| = |B_i| - 1$. It is easily observed that $M_{A,B}$ is an orientable pseudomanifold whenever $|A| = |B| - 1$, which implies that its top homology is cyclic. The fundamental cycle of $M_{A,B}$ (that is, generator of top homology, which is unique up to sign) is explicitly given by

$$\rho_{A,B} := \sum_{\sigma \in \mathfrak{S}_{A \cup \{\infty\}}} \operatorname{sgn}(\sigma)\tau(\sigma). \tag{6.2}$$

Now define

$$\eta(S,T) = \rho_{A_i,B_i} \wedge \cdots \wedge \rho_{A_{n-m},B_{n-m}}.$$

We demonstrate the procedure for constructing $\eta(S,T)$ on the tableaux S, T of Figure 6.2. Refer to Figure 6.3. First entry 8 is crossed out of T and entry 1 is popped from the first cell of the first row of S^*. So 1 is placed in A_1^* and 8 is placed in B_1. Next entry 7 is crossed out and entry 3 is popped from the second cell. So 3 is placed in A_2^* and 7 is placed in B_2. We eventually end up with

$$A_1^* = \{1, 2, \infty\},\ A_2^* = \{3, 4, \infty\},\ A_3^* = A_4^* = \{\infty\},$$

$$B_1 = \{1, 3, 8\},\ B_2 = \{2, 5, 7\},\ B_3 = \{4\},\ B_4 = \{6\}.$$

Hence

$$A_1 = \{1, 2\},\ A_2 = \{3, 4\},\ A_3 = A_4 = \emptyset.$$

Now

$$\eta(S,T) = \rho_{\{1,2\},\{1,3,8\}} \wedge \rho_{\{3,4\},\{2,5,7\}}.$$

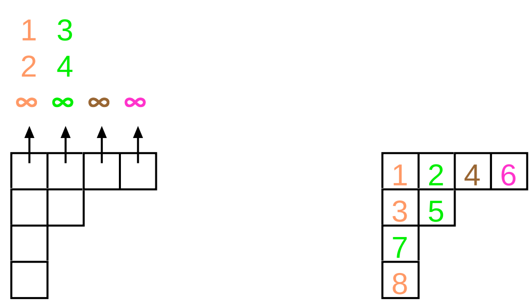

Figure 6.3

Theorem 6.3. *Let $m \leq n$. Then*

- $\{\eta(S,T) : (S,T) \in \mathcal{P}_{m,n}\}$ *is a basis for* $\tilde{H}_{m-1}(M_{m,n})$,
- $\{\gamma(S,T) : (S,T) \in \mathcal{P}_{m,n}\}$ *is a basis for a free subgroup of maximal rank in* $\tilde{H}^{m-1}(M_{m,n})$.

We need some general theory in order to prove this result. For any abelian group G, let G_{tor} denote the subgroup of G consisting of torsion elements of G

Proposition 6.4. *Let* Δ *be a simplicial complex. Suppose*

- $r = \operatorname{rank}(\tilde{H}_k(\Delta)/\tilde{H}_k(\Delta)_{\text{tor}})$,
- $u_1, \dots, u_r \in Z_k(\Delta)$,
- $v_1, \dots, v_r \in Z^k(\Delta)$,
- *the matrix* $(\langle u_i, v_j\rangle)_{i,j=1\dots,r}$ *is invertible over* $\mathbb{Z}$.

Then $\{\hat{u}_1, \dots, \hat{u}_r\}$ *is a basis for* $\tilde{H}_k(\Delta)/\tilde{H}_k(\Delta)_{\text{tor}}$ *and* $\{\hat{v}_1, \dots, \hat{v}_r\}$ *is a basis for* $\tilde{H}^k(\Delta)/\tilde{H}^k(\Delta)_{\text{tor}}$*, where* $\hat{x}$ *denotes the coset of* $\tilde{H}^k(\Delta)_{\text{tor}}$ *or* $\tilde{H}_k(\Delta)_{\text{tor}}$ *containing* $\bar{x}$.

Proof. The invertibility of the matrix $A := (\langle u_i, v_j\rangle)_{i,j=1\dots,r}$ implies that $\bar{u}_1, \dots, \bar{u}_r$ are independent in $\tilde{H}_k(\Delta, \mathbb{Q})$. Since $r = \dim \tilde{H}_k(\Delta, \mathbb{Q})$, we have that $\bar{u}_1, \dots, \bar{u}_r$ also spans $\tilde{H}_k(\Delta, \mathbb{Q})$.

Let $u \in Z_k(\Delta)$. Then $u \in Z_k(\Delta, \mathbb{Q})$. So

$$\bar{u} = \sum_{i=1}^{r} c_i\, \bar{u}_i, \quad c_i \in \mathbb{Q}$$

in $\tilde{H}_k(\Delta, \mathbb{Q})$. This means

$$u - \sum_{i=1}^{r} c_i\, u_i = \partial(y)$$

for some $y \in C_{k+1}(\Delta, \mathbb{Q})$. For each j, we have

$$\langle u, v_j\rangle - \sum_{i=1}^{r} c_i \langle u_i, v_j\rangle = \langle \partial(y), v_j\rangle = \langle y, \delta(v_j)\rangle = 0,$$

since v_j is a cocycle. It follows that

$$\begin{bmatrix} \langle u, v_1\rangle \\ \vdots \\ \langle u, v_r\rangle \end{bmatrix} = A \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix},$$

which implies

$$\begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = A^{-1} \begin{bmatrix} \langle u, v_1\rangle \\ \vdots \\ \langle u, v_r\rangle \end{bmatrix} \in \mathbb{Z}^r.$$

Let $t \in \mathbb{Z}^+$ be such that $ty \in C_{k+1}(\Delta)$. Since

$$t(u - \sum_{i=1}^{r} c_i u_i) = \partial(ty),$$

we have $\bar{u} - \sum_{i=1}^{r} c_i \bar{u}_i \in \tilde{H}_k(\Delta)_{\text{tor}}$. It follows that

$$\hat{u} = \sum_{i=1}^{r} c_i \hat{u}_i$$

in $\tilde{H}_k(\Delta)/\tilde{H}_k(\Delta)_{\text{tor}}$. Hence $\hat{u}_1, \dots, \hat{u}_r$ generates $\tilde{H}_k(\Delta)/\tilde{H}_k(\Delta)_{\text{tor}}$. Since $r = \text{rank}(\tilde{H}_k(\Delta)/\tilde{H}_k(\Delta)_{\text{tor}})$, these elements form a basis for $\tilde{H}_k(\Delta)/\tilde{H}_k(\Delta)_{\text{tor}}$. Similarly we have that $\hat{v}_1, \dots, \hat{v}_r$ forms a basis for $\tilde{H}^k(\Delta)/\tilde{H}^k(\Delta)_{\text{tor}}$. □

Proof of Theorem 6.3. For $(S, T) \in \mathcal{P}_{m,n}$, let

$$v(S,T) := \tau(\text{RS}^{-1}(S^*, T)) \in C_{m-1}(M_{m,n}),$$

where RS^{-1} denotes the inverse of the Robinson-Schensted map and τ is the map defined in (6.1). Let

$$u(S,T) := \sum_{\omega \in \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_{n-m}}} \text{sgn}(\omega)\ \tau(\text{RS}^{-1}(S^*, T)\omega) \in C_{m-1}(M_{m,n}),$$

where $B_1, \dots, B_{n-m}$ are the sets defined in the construction of $\eta(S, T)$.

For all $(S, T) \in \mathcal{P}_{m,n}$, we have

$$\gamma(S,T) = \overline{v(S,T)}, \tag{6.3}$$

where $\overline{x}$ denotes the cohomology class of x in $\tilde{H}^{m-1}(M_{m,n})$. It is not hard to see that

$$\eta(S,T) = \text{sgn}(B_1, \dots, B_{n-m})\ \ u(S,T), \tag{6.4}$$

where $\text{sgn}(B_1, \dots, B_{n-m})$ is the sign of the permutation obtained by concatenating the words obtained by writing each B_i in decreasing order.

Next we claim that for all $(S_1, T_1), (S_2, T_2) \in \mathcal{P}_{m,n}$,

(6.5)

$$\langle u(S_1, T_1), v(S_2, T_2) \rangle \neq 0 \quad \Longrightarrow \quad \text{RS}^{-1}(S_2^*, T_2) \leq_{\text{lex}} \text{RS}^{-1}(S_1^*, T_1)$$

where $\leq_{\text{lex}}$ denotes lexicographical order. Note that the subword of $\text{RS}^{-1}(S^*, T)$ obtained by restricting to the positions in B_i, is decreasing for each $i = 1, \dots, n-m$. Hence any rearrangement of letters of $\mathrm{RS}^{-1}(S^*, T)$ occupying positions in B_i, produces a lexicographically smaller word. Hence for each $\omega \in \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_{n-m}} - \{e\}$,

$$\mathrm{RS}^{-1}(S^*, T)\omega \ <_{\text{lex}} \ \mathrm{RS}^{-1}(S^*, T).$$

The claim (6.5) follows from this. We also have that

$$\langle u(S,T), v(S,T)\rangle = 1 \tag{6.6}$$

for all $(S,T) \in \mathcal{P}_{m,n}$.

Now order the pairs of standard tableaux

$$(S_1, T_1), \dots, (S_r, T_r)$$

in $\mathcal{P}_{m,n}$ so that $\mathrm{RS}^{-1}(S_i^*, T_i) <_{\text{lex}} \mathrm{RS}^{-1}(S_j^*, T_j)$ if $i < j$. It follows from (6.5) and (6.6) that the matrix

$$(\langle u(S_i, T_i), v(S_j, T_j)\rangle)_{i,j=1,\dots,r}$$

is unitriangular. There is no torsion in the top homology, and by Corollary 6.2, $|\mathcal{P}_{m,n}| = \mathrm{rank}\tilde{H}_{m-1}(M_{m,n})$. Hence the result follows from (6.3), (6.4) and Proposition 6.4. $\square$

Corollary 6.5. *Let $m \le n$. Then $\tilde{H}_{m-1}(M_{m,n})$ is generated by cycles of the form*

$$\rho_{A,B} \wedge \tau,$$

where

- *$A \subseteq [m]$, $B \subseteq [n]$ and $1 \le |A| = |B| - 1$*
- *$\rho_{A,B}$ is a fundamental cycle of the pseudomanifold $M_{A,B}$*
- *$\tau \in \tilde{H}_{m-1-|A|}(M_{[m]-A,[n]-B})$.*

7. Infinite homology of the chessboard complex

In this section we study torsion in infinite $\tilde{H}_{\nu_{m,n}}(M_{m,n})$. Recall from Theorem 1.5 that for $m \le n$, the homology group $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is infinite if and only if $n \ge 2m-4$ or $(m,n) \in \{(6,6),(7,7),(8,9)\}$. From Table 1.2, we see that there is 3-torsion if $(m,n) = (6,6)$ or $(7,7)$. We expect that there is 3-torsion for $(m,n) = (8,9)$ as well, but have not yet been able to verify this by computer.

Conjecture 7.1. *Let $m \le n$. Then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is free if and only if $n \ge 2m-4$.*

The conjecture clearly holds in the case that $n \ge 2m-1$, since in this case $\nu_{m,n} = m-1$, which means that $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is top homology. The conjecture for $n = 2m-2$ is proved in the following result. The cases $n = 2m-3$ and $n = 2m-4$ are left open.

Theorem 7.2. *If $n = 2m-2$ then*

$$\tilde{H}_{\nu_{m,n}}(M_{m,n}) \cong \mathbb{Z}^{c_{m-1}},$$

where c_m is the Catalan number $\frac{1}{m+1}\binom{2m}{m}$.

Proof. Theorem 6.1 applied to $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2};\mathbb{C})$ yields a particularly nice formula. First note that $\nu_{m,2m-2} = m-2$. Next observe that the set $\mathcal{R}(m, 2m-2, m-1)$ consists of a single pair of partitions; namely the pair $((m), (m-1)^2)$. Hence Theorem 6.1 yields,

$$\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2};\mathbb{C}) \cong_{\mathfrak{S}_n} S^{(m-1)^2}.$$

It follows that the degree $\nu_{m,2m-2}$ Betti number of $M_{m,2m-2}$ is $f^{(m-1)^2}$, the number of standard Young tableaux of shape $(m-1)^2$. Hence

$$\text{rank } (\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})/\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})_{\text{tor}}\,) = f^{(m-1)^2}. \tag{7.1}$$

Since the number of standard Young tableaux of shape $(m-1)^2$ is the Catalan number c_{m-1}, we need only show that $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$ is free.

Given a partition λ, let $S^\lambda_{\mathbb{Z}}$ denote the Specht module indexed by λ with integer coefficients. It is well-known that $S^\lambda_{\mathbb{Z}}$ is a free group of rank f^λ, which is isomorphic to the group generated by the λ-tableaux subject to the column relations and the Garnir relations. For $\lambda = (m-1)^2$, these relations can be described as follows:

$$\begin{array}{|c|c|c|}\hline \cdots & a_j & \cdots \\ \hline \cdots & b_j & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \cdots & b_j & \cdots \\ \hline \cdots & a_j & \cdots \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|}\hline \cdots & a_j & a_{j-1} & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \cdots & a_{j-1} & \cdot & \cdots \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline \cdots & \cdot & a_j & \cdots \\ \hline \cdots & b_{j-1} & b_j & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|}\hline \cdots & \cdot & a_j & \cdots \\ \hline \cdots & b_j & b_{j-1} & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline \cdots & \cdot & b_j & \cdots \\ \hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \end{array}$$

Let $\phi : S^{(m-1)^2}_{\mathbb{Z}} \to \tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$ be the homomorphism defined on generators by

$$\phi\left(\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_{m-1} \\ \hline b_1 & b_2 & \cdots & b_{m-1} \\ \hline \end{array} \right) = \alpha_{1,a'_1,b'_1} \wedge \alpha_{2,a'_2,b'_2} \wedge \cdots \wedge \alpha_{m-1,a'_{m-1},b'_{m-1}}.$$

To verify that this map is well defined we need only check that the three relations for the Specht module given above are mapped to 0 in $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$. For the first relation we have

$$\phi\left(\begin{array}{|c|c|c|}\hline \cdots & a_j & \cdots \\ \hline \cdots & b_j & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \cdots & b_j & \cdots \\ \hline \cdots & a_j & \cdots \\ \hline \end{array} \right) = \cdots\wedge(\alpha_{j,a'_j,b'_j}+\alpha_{j,b'_j,a'_j})\wedge\ldots,$$

which is clearly 0.

For the second relation, we have

$$\phi\left(\begin{array}{|c|c|c|c|}\hline \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|}\hline \cdots & a_j & a_{j-1} & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \cdots & a_{j-1} & \cdot & \cdots \\ \hline \end{array} \right)$$

$$\begin{aligned} = \ \ldots \ \wedge \ &\big((\alpha_{j-1,a'_{j-1},b'_{j-1}} \wedge \alpha_{j,a'_j,b'_j}) - (\alpha_{j-1,a'_j,b'_{j-1}} \wedge \alpha_{j,a'_{j-1},b'_j}) \\ &+ (\alpha_{j-1,a'_j,a'_{j-1}} \wedge \alpha_{j,b'_{j-1},b'_j})\big) \ \wedge \ \ldots. \end{aligned}$$

We will show that this cycle, which we denote by ρ, is a boundary. After cancelling terms we get

$$\begin{aligned} \rho = \ \ldots \ \wedge \ &\big((\alpha_{j-1,a'_{j-1},b'_{j-1}} \wedge ja'_j) - (\alpha_{j-1,a'_j,b'_{j-1}} \wedge ja'_{j-1}) \\ &+ (\alpha_{j-1,a'_j,a'_{j-1}} \wedge jb'_{j-1})\big) \ \wedge \ \ldots, \end{aligned}$$

which is an element of the chain group $C_{m-2}(M_{[m-1],[2m-2]\setminus\{b_j\}})$. Hence $mb'_j \wedge \rho \in C_{m-1}(M_{m,2m-2})$. Since $\partial(mb'_j \wedge \rho) = \rho$, the second relation maps to 0. By symmetry the third relation maps to 0 as well. Hence ϕ is a well defined homomorphism.

We claim that ϕ is surjective. Indeed, by Lemma 5.3, $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$ is generated by elements of the form

$$\alpha_{\sigma(1),a'_1,b'_1} \wedge \alpha_{\sigma(2),a'_2,b'_2} \wedge \cdots \wedge \alpha_{\sigma(m-1),a'_{m-1},b'_{m-1}}.$$

It follows from Lemma 5.1 that σ can be taken to be the identity permutation, which means that $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$ is generated by the images of the $(m-1)^2$-tableaux.

Let

$$\pi : \tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2}) \to \tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})/\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})_{\text{tor}} \ ,$$

be the projection map. The composition

$$\pi \circ \phi : S_{\mathbb{Z}}^{(m-1)^2} \to \tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})/\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})_{\text{tor}} \ ,$$

is a surjective homomorphism between free groups. Since these groups have equal rank by (7.1), the composition $\pi \circ \phi$ is an isomorphism, which implies that the surjection ϕ is an isomorphism as well. We can now conclude that $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$ is free. $\square$

Corollary 7.3. *The set*

$$\{\phi(T) : T \text{ a standard tableau of shape } (m-1)^2\}$$

is a basis for $\tilde{H}_{\nu_{m,2m-2}}(M_{m,2m-2})$.

In [BBLSW, Section 9.1], it is observed that when $m = n$, the complex $M_{m,n}$ collapses to an $(n-2)$-dimensional complex. Hence for $m = n$, the homology group $\tilde{H}_i(M_{m,n})$ is free whenever $i \geq m-2$. The same is true for $n = m+1$, since in this case $M_{m,n}$ is an orientable psuedomanifold. Theorem 7.2 implies that the same is also true for $n = 2m-2$. This and the computer data suggest the following conjecture, which implies Conjecture 7.1.

Conjecture 7.4. *Let* $m \leq n$ *and* $i \geq \nu_{m,n}$. *Then* $\tilde{H}_i(M_{m,n})$ *is free if and only if* $i \geq m-2$.[5]

8. Subcomplexes of the chessboard complex

Our goal in this section is to establish sharpness of a connectivity bound for the simplicial complex of nontaking rooks on an $n \times n$ chessboard with a diagonal removed. This bound was obtained by Björner and Welker [BjWe] as a consequence of a more general result of Ziegler [Zie] on nonrectangular boards.

For any subset A of the set of positions on an $m \times n$ chessboard, let $M(A)$ be the simplicial complex of nontaking rooks on A. That is, for $A \subseteq [m] \times [n]$, the simplicial complex $M(A)$ has vertex set A and faces $\{(i_i, j_1), (i_2, j_2), \dots, (i_k, j_k)\} \subseteq A$ such that $i_s \neq i_t$ and $j_s \neq j_t$ for all $s \neq t$. Let

$$D_n = [n] \times [n] \setminus \{(1,1), (2,2), \dots, (n,n)\}.$$

Theorem 8.1 (Björner and Welker [BjWe])**.** *For all* $n \geq 2$, *the simplicial complex* $M(D_n)$ *is* $(\nu_{2n} - 1)$*-connected.*

Björner and Welker [BjWe] use computer calculations to obtain the following table which establishes sharpness of their connectivity bound for $3 \leq n \leq 7$. We will use results of the previous sections to establish sharpness for $n > 7$.

n	2	3	4	5	6	7
$\tilde{H}_{\nu_{2n}}(D_n)$	0	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}$	$\mathbb{Z}^{24} \oplus \mathbb{Z}_3^5$	$\mathbb{Z}^{415} \oplus \mathbb{Z}_3^{15}$

Table 8.1

[5]See New Developments Section at the end of the paper.

For $n \geq 3$ and $i = 0, \ldots, \lfloor \frac{n}{3} \rfloor - 1$, let

$$S_i = \{(3i+1, 3i+1), (3i+1, 3i+2), (3i+2, 3i+3), (3i+3, 3i+3)\},$$

and let

$$B_n = (\biguplus_{i=0}^{N} S_i) \ \uplus \ R_n,$$

where

$$N = \begin{cases} \frac{n-3}{3} & \text{if } n \equiv 0 \bmod 3 \\ \frac{n-5}{3} & \text{if } n \equiv 2 \bmod 3 \\ \frac{n-7}{3} & \text{if } n \equiv 1 \bmod 3 \end{cases}$$

and

$$R_n = \begin{cases} \emptyset & \text{if } n \equiv 0 \bmod 3 \\ \{(n-1, n-1), (n-1, n)\} & \text{if } n \equiv 2 \bmod 3 \\ \{n-3, n-2, n-1, n\} \times \{n-3, n-2, n-1, n\} & \text{if } n \equiv 1 \bmod 3. \end{cases}$$

Lemma 8.2. *For all $n \geq 3$, if A is a subset of $[n] \times [n]$ that contains B_n then $\tilde{H}_{\nu_{2n}}(M(A)) \neq 0$.*

Proof. For $n \equiv 0 \bmod 3$, let

$$\rho = \alpha_{1,1',2'} \wedge \beta_{2,3,3'} \wedge \alpha_{4,4',5'} \wedge \beta_{5,6,6'} \wedge \cdots \wedge \alpha_{n-2,(n-2)',(n-1)'} \wedge \beta_{n-1,n,n'},$$

and for $n \equiv 2 \bmod 3$, let

$$\rho = \alpha_{1,1',2'} \wedge \beta_{2,3,3'} \wedge \alpha_{4,4',5'} \wedge \beta_{5,6,6'} \wedge \cdots \wedge \alpha_{n-1,(n-1)',n'}.$$

In both cases ρ is a cycle in $C_{\nu_{2n}}(M(A))$, but not a boundary. Indeed, if ρ were a boundary in $C_{\nu_{2n}}(M(A))$ then it would be a boundary in $C_{\nu_{n,n}}(M_{n,n})$, which would imply that all the generators of $\tilde{H}_{\nu_{n,n}}(M_{n,n})$ given in Lemmas 5.6 and 5.3 are boundaries. This is impossible since by Theorem 3.1, $\tilde{H}_{\nu_{n,n}}(M_{n,n}) \neq 0$. Hence, $\tilde{H}_{\nu_{2n}}(M(A)) \neq 0$.

For $n \equiv 1 \bmod 3$, let

$$\rho = \alpha_{1,1',2'} \wedge \beta_{2,3,3'} \wedge \cdots \wedge \alpha_{n-6,(n-6)',(n-5)'} \wedge \beta_{n-5,n-4,(n-4)'}.$$

By Theorem 3.1 and Lemmas 5.11 and 5.6, there is a cycle ω in $C_2(M(R_n))$ such that the cycle $\rho \wedge \omega$ is not a boundary in $C_{\nu_{n,n}}(M_{n,n})$. So $\rho \wedge \omega$ is not a boundary in $C_{\nu_{2n}}(M(A))$. Hence $\tilde{H}_{\nu_{2n}}(M(A)) \neq 0$. $\square$

Theorem 8.3. *For $n \geq 3$, $\tilde{H}_{\nu_{2n}}(M(D_n)) \neq 0$.*

Proof. We claim that an isomorphic copy of D_n contains B_n for all $n \geq 3$ except for $n = 4, 7$. Indeed, if $n \equiv 0, 2 \bmod 3$ then the isomorphic copy of D_n is

$$[n] \times [n] \setminus (\{(i, i+2) : i = 1, \dots, n-2\} \cup \{(n-1, 1), (n, 2)\}).$$

If $n \equiv 1 \bmod 3$ and $n \geq 10$ then the isomorphic copy of D_n is

$$[n] \times [n] \setminus (\{(i, i+4) : i = 1, \dots, n-4\} \cup \{(i+n-4, i) : i = 1, 2, 3, 4\}).$$

The result now follows from Lemma 8.2 and Table 8.1. $\square$

Table 8.1 and the torsion results of Section 5 suggest the following conjecture.

Conjecture 8.4. *There exists an integer $n_0 \geq 8$ such that if $n \geq n_0$ then $\tilde{H}_{\nu_{2n}}(M(D_n))$ is an elementary 3-group. Moreover, if $n \geq n_0$ and $n \equiv 2 \bmod 3$ then $\tilde{H}_{\nu_{2n}}(M(D_n)) = \mathbb{Z}_3$.*

Björner and Welker's connectivity result is a consequence of a more general result of Ziegler. Indeed, Björner and Welker [BjWe] observe that an isomorphic copy of D_n contains the set $\Gamma(n, 2\nu_{2n} + 1 - n)$ described in the following theorem.

Theorem 8.5 (Ziegler [Zie])**.** *For $0 \leq k \leq n-1$, let*

$$\Gamma(n, k) = \{(i, j) \in [n] \times [n] : |j - i| \leq k\}.$$

Let A be a subset of $[n] \times [n]$ that contains $\Gamma(n, 2\nu_{2n} + 1 - n)$. Then $M(A)$ is $(\nu_{2n} - 1)$-connected.

Note that $B_n \subseteq \Gamma(n, 2\nu_{2n} + 1 - n)$ if $n = 6$ or $n \geq 8$. It therefore follows from Lemma 8.2 that Ziegler's connectivity bound is sharp for $n = 6$ and $n \geq 8$. When $n = 3$ or 5, $M(\Gamma(n, 2\nu_{2n} + 1 - n))$ is a simplex, which is contractible. Hence Ziegler's bound is not sharp in these cases.

9. Shellability of the ν_n-skeleton of M_n

In this section we describe a shelling of the ν_n-skeleton of M_n along with a discrete Morse function on M_n that is closely related to our shelling. We assume that the reader is familiar with the basic definitions from shellability theory (see for example [BjWa]) and discrete Morse theory (see [Fo]). Before presenting our results, we remark that in [At], Athanasiadis has shown that the ν_n-skeleton of M_n is vertex decomposable, which implies that it is shellable. In light of this fact, we will not provide a proof that our ordering of the facets of the ν_n-skeleton is in fact a shelling.

Our shelling and Morse function are determined with use of the following recursive algorithm, which gives, for any graph $G \in M_n$, an ordered partition $\rho(G) = (G_1, \dots, G_r)$ of G into subgraphs $G_i = (V_i, E_i)$. We begin with $G_0 = (\emptyset, \emptyset)$. Having defined G_j for all $j < i$, we define G_i as follows.

- If $\bigcup_{j<i} V_j = [n]$, stop.
- If $\bigcup_{j<i} V_j = [n] \setminus \{t\}$, set $G_i = (\{t\}, \emptyset)$.
- If $|\bigcup_{j<i} V_j| < n - 1$, let a, b be the two smallest elements of $[n] \setminus \bigcup_{j<i} V_j$. Set $V_i = \{a, b\} \cup N_G(a) \cup N_G(b)$ and define E_i to be the set of all edges in G that have both vertices in V_i.

For example, if $n = 10$ and $E(G) = \{17, 38, 45\}$ then our algorithm will give $G_1 = (\{1, 2, 7\}, \{17\})$, $G_2 = (\{3, 4, 5, 8\}, \{38, 45\})$, $G_3 = (\{6, 9\}, \emptyset)$ and $G_4 = (\{10\}, \emptyset)$.

It follows immediately from the definition of our partition that $|V_i| \le 4$ for all $i \in [r]$ and that $|V_i| > 1$ if $i < r$. Moreover, we have $|E_i| = \lfloor \frac{|V_i|}{2} \rfloor$ whenever $|V_i| \neq 2$. We now partially order the set of all graphs $G = (V, E)$ such that $V \subseteq [n]$ by setting $(V, E) \preceq (V', E')$ if either $|V| < |V'|$ or we have $V = V' = \{i, j\}$ and $E = \{ij\}$ while $E' = \emptyset$. The partial order $\preceq$ gives rise to a lexicographic partial order $\preceq_l$ on M_n. That is, if $G, H \in M_n$ with $\rho(G) = (G_1, \dots, G_r)$ and $\rho(H) = (H_1, \dots, H_s)$, we set $G \preceq_l H$ if either $G_i = H_i$ for all $i \in [r]$ or, for some $i \le r$, we have $G_j = H_j$ for all $j < i$ and $G_i \prec H_i$.

Theorem 9.1. *Let $F_1 < F_2 < \ldots < F_t$ be any linear extension of the restriction of $\preceq_l$ to the set of ν_n-dimensional faces of M_n. Then $F_1, F_2, \dots, F_t$ is a shelling of the ν_n-skeleton of M_n.*

To a shelling $F_1, \dots, F_t$ of any complex Δ, one can associate a discrete Morse function (actually, many such functions) as follows. For each nonhomology facet F_i of the shelling, let $R_i \subset F_i$ be the restriction face of F_i, that is, the unique minimal new face obtained when F_i is added to the complex built from $\{F_j : j < i\}$. The interval $[R_i, F_i]$ in the face poset of Δ is isomorphic to the face poset of a simplex (of dimension at least one), and if we fix an isomorphism between these two posets then any simplicial collapse of the simplex to a point gives rise to a pairing $\mathcal{M}_i$ of the faces in $[R_i, F_i]$. The union of all such pairings $\mathcal{M}_i$ determines (the gradient flow of) a discrete Morse function on Δ whose critical cells are the homology facets of the given shelling.

A discrete Morse function associated to the shelling of Theorem 9.1 is quite easy to describe. For $G \in M_n$ with $\rho(G) = (G_1, \dots, G_r)$, define

$$\mu(G) := \begin{cases} \infty & \text{if no } V_i \text{ has size two,} \\ \min\{i : |V_i| = 2\} & \text{otherwise.} \end{cases}$$

Let X_n be the set of all $G \in M_n$ such that $\mu(G) \neq \infty$ and $E_{\mu(G)} \neq \emptyset$. For $G \in X_n$, let G^- be the graph obtained from G by removing the unique edge in $E_{\mu(G)}$. The next result is straightforward to prove using standard techniques from discrete Morse theory.

Theorem 9.2. *The set $\{(G, G^-) : G \in X_n\}$ determines the gradient flow of a discrete Morse function on M_n whose critical cells are those $G \in M_n$ such that $\mu(G) = \infty$.*

One can show that the shelling of Theorem 9.1 gives rise to the restriction of the Morse function of Theorem 9.2 to the ν_n-skeleton of M_n.

10. Bounds on the rank of $\widetilde{H}_\nu$

In this section we give upper and lower bounds on the rank (that is, smallest size of a generating set) of $\widetilde{H}_{\nu_n}(M_n)$ when $n \equiv 0, 2 \bmod 3$. (Note that the case $n \equiv 1 \bmod 3$ is settled by Theorem 1.3 and that our lower bound in the case $n \equiv 0 \bmod 3$ is given in [Bo].) We do the same for $\widetilde{H}_{\nu_{m,n}}(M_{m,n})$, although we need conditions on m, n similar to those found in Theorem 1.7 for the lower bounds.

Set

$$r_n := \operatorname{rank}(\widetilde{H}_{\nu_n}(M_n)).$$

We can get upper bounds on r_n using the Morse function of Section 9. If we let c_n be the size of the set $\mathcal{C}_n$ of graphs $G \in M_n$ with ν_n edges such that $\mu(G) = \infty$, then by [Fo, Corollary 3.7(i)], we have

$$r_n \leq c_n.$$

For $G \in M_n$ with $\rho(G) = (G_1, \dots, G_r)$, let $\lambda(G)$ be the partition of n such that the number of parts of size m in $\lambda(G)$ is the number of V_i of size m. Straightforward calculation shows that for $G \in M_n$ we have $G \in \mathcal{C}_n$ if and only if

$$\lambda(G) = \begin{cases} (3, \dots, 3) & n \equiv 0 \bmod 3, \\ (3, \dots, 3, 1) & n \equiv 1 \bmod 3, \\ (4, 3, \dots, 3, 1) & n \equiv 2 \bmod 3. \end{cases}$$

Now further calculation gives

$$c_n = \begin{cases} 2^{n/3} \prod_{j=1}^{n/3} (n - 3j + 1) & n \equiv 0 \bmod 3, \\ 2^{(n-1)/3} \prod_{j=1}^{(n-1)/3} (n - 3j + 1) & n \equiv 1 \bmod 3, \\ 2^{(n-5)/3} \sum_{k=1}^{(n-2)/3} \prod_{j=1}^{k} (n - 3j + 1) \prod_{j=k}^{(n-2)/3} (n - 3j) & n \equiv 2 \bmod 3. \end{cases}$$

Of course when $n \equiv 1 \bmod 3$ and $n \geq 7$, we know that $r_n = 1$ and our upper bound is both useless and horribly inaccurate. It turns out that one can improve the upper bound on r_n in the case $n \equiv 2 \bmod 3$ using the long exact sequence of Lemma 2.3. Indeed, if $n \equiv 0 \bmod 3$, the tail end

$$\bigoplus_{a,h} \widetilde{H}_{\nu_n - 1}(M_{[n]\setminus\{1,2,h\}}) \to \widetilde{H}_{\nu_n}(M_n) \to 0$$

of the sequence gives

$$r_n \leq 2(n-2)r_{n-3},$$

and one simply reobtains the bound $r_n \leq c_n$ using induction. However, if $n \equiv 2 \bmod 3$ and $n \geq 11$, the tail end of the sequence is

$$\bigoplus_{a,h} \widetilde{H}_{\nu_n - 1}(M_{[n]\setminus\{1,2,h\}}) \to \widetilde{H}_{\nu_n}(M_n) \to \bigoplus_{i,j} \mathbb{Z}_3 \to 0,$$

from which we obtain

$$r_n \leq 2(n-2)r_{n-3} + (n-2)(n-3).$$

This recursive formula leads to a somewhat better upper bound than that given by c_n. However, as we shall see momentarily, all the bounds we have found so far are so distant from the known lower bounds on r_n that differences between them are insignificant. Before going on to lower bounds, we examine upper bounds for chessboard complexes. Set

$$r_{m,n} := \operatorname{rank}(\widetilde{H}_{\nu_{m,n}}(M_{m,n})).$$

Using the long exact sequence of Lemma 2.4 as we used that of Lemma 2.3 for the matching complexes, we get

$$r_{m,n} \leq \begin{cases} (m-1)r_{m-2,n-1} + (n-1)r_{m-1,n-2} & m+n \equiv 0 \bmod 3, \\ (m-1)r_{m-2,n-1} + (n-1)r_{m-1,n-2} + (m-1)(n-1) & m+n \equiv 2 \bmod 3. \end{cases}$$

Now we examine lower bounds. In [Bo], Bouc gets a lower bound for r_n when $n \equiv 0 \bmod 3$ using the standard long exact sequence associated to the pair (M_n, M_{n-1}), where we consider M_{n-1} as the subcomplex of M_n consisting of all matchings in which vertex n is isolated. It is straightforward to show that the quotient complex M_n/M_{n-1} has the homotopy type of a wedge of $n-1$ complexes, each homotopy equivalent to the suspension of M_{n-2}, from which it follows that the sequence under discussion is

$$\ldots \longrightarrow \widetilde{H}_t(M_{n-1}) \longrightarrow \widetilde{H}_t(M_n) \longrightarrow \bigoplus_{i=1}^{n-1} \widetilde{H}_{t-1}(M_{n-2}) \longrightarrow \ldots.$$

When $n \equiv 0 \bmod 3$, the tail end of this sequence is

$$\widetilde{H}_{\nu_n}(M_n) \longrightarrow \bigoplus_{i=1}^{n-1} \widetilde{H}_{\nu_{n-2}}(M_{n-2}) \longrightarrow 0, \tag{10.1}$$

from which Bouc obtains

$$r_n \geq n - 1.$$

When $n \equiv 2 \bmod 3$ and $n \geq 8$, the tail end of the sequence is

$$\widetilde{H}_{\nu_n}(M_n) \longrightarrow \bigoplus_{i=1}^{n-1} \widetilde{H}_{\nu_{n-2}}(M_{n-2}) \longrightarrow \mathbb{Z}_3 \longrightarrow 0,$$

from which we obtain

$$r_n \geq (n-1)r_{n-2} - 1 \geq (n-1)(n-3) - 1.$$

We can obtain similar results for the chessboard complexes using the long exact sequence for the pair $(M_{m,n}, M_{m-1,n})$, where we consider $M_{m-1,n}$ to be the subcomplex of $M_{m,n}$ consisting of all matchings in which vertex m is isolated. We get

$$r_{m,n} \geq \begin{cases} n & m+n \equiv 0 \bmod 3 \text{ and } m \leq n \leq 2m-3, \\ n(n-1)-1 & m+n \equiv 2 \bmod 3 \text{ and } m \leq n \leq 2m-7. \end{cases}$$

Certainly the distance between the upper and lower bounds we have provided is unsatisfactory in all cases.

11. New Developements

In this section, we mention some important recent work of Jonsson, which was done after the first version of this paper was submitted and extends the results of this paper. In a surprising developement, Jonsson [J2] has shown that $\tilde{H}_{\nu_{14}}(M_{14})$ has 5-torsion. So 14 is the only value of n for which $\tilde{H}_{\nu_n}(M_n)$ has torsion other than 3-torsion. Also Jonsson [J4] gives a proof of $\tilde{H}_{\nu_{5,5}}(M_{5,5}) = \mathbb{Z}_3$ that doesn't involve the computer.

Jonsson [J1, J3, J4] is also able to use our results on 3-torsion in the bottom nonvanishing homology of the matching complex and the chessboard complex to establish 3-torsion in higher dimensional homology groups. He shows that if $\nu_n < i \leq \lfloor \frac{n-6}{2} \rfloor$ then $\tilde{H}_i(M_n)$ also has 3-torsion and conjectures that the same is true for $\nu_n < i = \lfloor \frac{n-5}{2} \rfloor$. (Apart from $\nu_n \leq i \leq \lfloor \frac{n-5}{2} \rfloor$, the only other value of i for which $\tilde{H}_i(M_n) \neq 0$ is given by $i = \lfloor \frac{n-3}{2} \rfloor$ when $n \geq 3$. This is a consequence of a representation theoretic result of Bouc [Bo] analogous to Theorem 6.1. It follows from elementary considerations that $\tilde{H}_i(M_n)$ is torsion-free for this value of i.)

In [J1, J4] Jonsson shows that $\tilde{H}_i(M_{m,n})$ also has 3-torsion for all i such that $\nu_{m,n} < i \leq m-3$ when $n \geq m+1$, and for all i such that

$\nu_{m,n} < i \leq m-4$ when $n = m$. Hence the only cases of Conjecture 7.4 that remain open are

- $n = m \geq 8$ and $i = m-3$ (conjectured to have 3-torsion).
- $(m, n) = (8, 9)$ and $i = m-3$ (conjectured to have 3-torsion).
- $9 \leq m+2 \leq n \leq 2m-3$ and $i = m-2$ (conjectured to be torsion-free).

Jonsson [J3, J4] also derives upper bounds on the rank of the 3-torsion in the homology groups of both the matching complex and the chessboard complex.

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References

[At] C. A. Athanasiadis, Decompositions and connectivity of matching and chessboard complexes, *Discrete Comput. Geom.* **31** (2004), 395-403.

[BBLSW] E. Babson, A. Björner, S. Linusson, J. Shareshian and V. Welker, Complexes of not i-connected graphs, *Topology* **38** (1999), 271-299.

[BLVZ] A. Björner, L. Lovász, S. T. Vrećica, and R. T. Živaljević, Chessboard complexes and matching complexes, *J. London Math. Soc. (2)* **49** (1994), 25-39.

[BjWa] A. Björner and M. L. Wachs, Shellable nonpure complexes and posets, I, *Trans. Amer.Math. Soc.* **348** (1996), 1299-1327.

[BjWe] A. Björner and V. Welker, Complexes of directed graphs, *SIAM J. Discrete Math.* **12** (1999), 413-424.

[Bo] S. Bouc, Homologie de certains ensembles de 2-sous-groupes des groupes symétriques, *J. Algebra* **150** (1992), 158-186.

[DHSW] J. G. Dumas, F. Heckenbach, D. Saunders and V. Welker, *Homology software package*, http://www.cis.udel.edu/heckenba.

[Fo] R. Forman, Morse theory for cell complexes, *Adv. Math.* **134** (1998), 90-145.

[FrHa] J. Friedman and P. Hanlon, On the Betti numbers of chessboard complexes, *J. Algebraic Combin.* **8** (1998), 193-203.

[Fu] W. Fulton, *Young tableaux: with applications to representation theory and geometry*, London Mathematical Society Student Texts, **35**, Cambridge University Press, Cambridge, 1997.

[Ga] P. F. Garst, *Cohen-Macaulay complexes and group actions*, Ph.D. Thesis, University of Wisconsin-Madison, 1979.

[J1] J. Jonsson, *Simplicial Complexes of Graphs*, Ph.D. Thesis, KTH, Stockholm, 2005.

[J2] J. Jonsson, *Five-torsion in the homology of the matching complex on 14 vertices*, preprint 2006.

[J3] J. Jonsson, *Exact sequences for the homology of the matching complex*, preprint 2006.

[J4] J. Jonsson, *On the 3-torsion part of the homology of the chessboard complex* , preprint 2006.

[Mu] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[Sa] B. E. Sagan, *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, Second edition, Graduate Texts in Mathematics, **203**, Springer-Verlag, New York, 2001.

[St] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, Cambridge Studies in Advanced Mathematics, **62**, Cambridge University Press, Cambridge, 1999.

[Wa] M. L. Wachs, Topology of matching, chessboard, and general bounded degree graph complexes, Dedicated to the memory of Gian-Carlo Rota, *Algebra Universalis* **49** (2003), 345-385.

[Zie] G. M. Ziegler, Shellability of chessboard complexes, *Israel J. Math.* **87** (1994), 97-110.

[ZivVr] R. T. Živaljević and S. T. Vrećica, The colored Tverberg's problem and complexes of injective functions, *J. Combin. Theory Ser. A* **61** (1992), 309-318.

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