

From Poset Topology to q -Eulerian Polynomials to Stanley's Chromatic Symmetric Functions

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ABSTRACT. In recent years we have worked on a project involving poset topology, various analogues of Eulerian polynomials, and a refinement of Richard Stanley's chromatic symmetric function. Here we discuss how Stanley's ideas and results have influenced and inspired our own work.

1. A walk in the woods

It is a privilege and an honor to contribute an article to a volume celebrating Richard Stanley's 70th birthday. Over the years, Richard's work has had a tremendous influence on the research of so many of us working in algebraic and geometric combinatorics. It is our pleasure to discuss in this article the impact of Richard's work on our ongoing project involving Eulerian polynomials and symmetric functions. Working on this project has been very much like taking a walk in a beautiful forest. At every turn, we meet Richard Stanley, and each time we run into him our walk gets much more interesting.

The study of connections between poset topology and permutation enumeration has its roots in Stanley's pioneering work on edge labelings of posets, Cohen-Macaulay complexes, and group actions on posets. In Section 2 we describe how Stanley's work on this topic inspired our work on the topology of a q -analog of a certain poset introduced by Björner and Welker in connection with commutative algebra. We also describe how our work on this poset led to the discovery of a new q -analog, involving the major index and excedance number, of Euler's exponential generating function formula for the Eulerian polynomials.

We begin Section 3 by presenting a q -analog of Euler's exponential generating function formula due to Stanley. Stanley's formula involves the inversion number and the descent number. Next, we describe our own q -analog, which is obtained from a symmetric function identity using stable principal specialization. After discussing our symmetric function identity, we present in the remainder of Section 3 some consequences and refinements of our main results. One of these consequences is the q -unimodality of the q -Eulerian polynomials that appear in our q -analog. Here we find a close connection to earlier work of Stanley, in which the hard Lefschetz theorem is used to prove unimodality results in combinatorics.

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In Section 4, we explain how the symmetric function arising in the previous sections is an instance of a refinement of Stanley’s chromatic symmetric function for graphs. We were inspired to define this refinement by an observation of Stanley about our earlier work on q -Eulerian polynomials discussed in Section 3. The refinement is a polynomial in t , whose coefficients are in general not symmetric functions, but rather quasisymmetric functions. However, if the graph at hand is the incomparability graph G of a unit interval order (with an appropriate labeling of the vertices), the coefficients are symmetric.

We present in the remainder of Section 4 various results and conjectures on our chromatic quasisymmetric functions. Of particular interest is a conjecture that asserts that with G as above, the chromatic quasisymmetric function of G is e -positive and e -unimodal. This reduces to the Stanley-Stembridge e -positivity conjecture for unit interval orders when t is set equal to 1. We describe how our e -unimodality conjecture and an exercise in Stanley’s EC1 on generalized Eulerian polynomials led to a conjecture connecting the chromatic quasisymmetric functions to a certain class of subvarieties of flag varieties due to De Mari and Shayman, known as the regular semisimple Hessenberg varieties. Our conjecture¹ asserts that, with G as above, the chromatic quasisymmetric function of G is essentially the Frobenius characteristic of the representation of the symmetric group on the cohomology of the Hessenberg variety naturally associated to G . This conjecture implies Schur-positivity (which we prove by other means) of the chromatic quasisymmetric function of G . Schur-unimodality will follow from our conjecture and the hard Lefschetz theorem. We hope that our conjecture will lead eventually to a proof of the Stanley-Stembridge e -positivity conjecture.

We assume that the reader is familiar with various ideas from algebraic, enumerative and topological combinatorics. Luckily and unsurprisingly, all of these ideas are explained in the books [65, 70, 77].

2. Poset topology

Our project began while we were participating in a combinatorics program organized by Anders Björner and Richard Stanley, at the Mittag-Leffler Institute. We worked on a conjecture of Björner and Welker [9] on poset topology and permutation enumeration. Interest in this topic can be traced back to Stanley’s seminal work on edge labelings of posets in [52, 53, 54] and on Cohen-Macaulay complexes in [55, 57, 58, 4, 65]. The fact that Cohen-Macaulay posets in general, and such posets admitting a group action in particular, can be useful in enumeration problems is made clear in [64].

We give a brief description of some work of Stanley on the rank-selected Möbius invariant, which inspired the poset topology part of our project. For a ranked and bounded poset P of length n with rank function r_P , and a subset $S \subseteq [n - 1]$, we consider the rank selected subposet $P_S := \{x \in P : r_P(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. In [52] Stanley showed that if P is a distributive lattice then there is a way to label the edges of the Hasse diagram of P so that the rank-selected Möbius invariant $\mu_{P_S}(\hat{0}, \hat{1})$ is equal to $(-1)^n$ times the number maximal chains whose label sequences have descent set S . (Throughout this paper, the Möbius function on a poset P will be denoted by μ_P .) This result is generalized to the class of supersolvable lattices in [53] and further generalized to the class of admissible lattices in [54]. For the

¹See Section 5: *Further developments*.

Boolean algebra B_n on n elements, the label sequences in [52] correspond to the permutations in the symmetric group \mathfrak{S}_n . Thus for each $S \subseteq [n-1]$, the rank-selected Möbius invariant is given by

$$(2.1) \quad \mu_{(B_n)_S}(\hat{0}, \hat{1}) = (-1)^n |\{\sigma \in \mathfrak{S}_n : \text{DES}(\sigma) = S\}|,$$

where $\text{DES}(\sigma)$ is the descent set of σ .

The theory of lexicographic shellability, which relates edge labelings of posets to the topology of order complexes of the posets, has its roots in Stanley's theory of admissible labelings. Indeed, Stanley conjectured, and Björner proved in [3] that all admissible lattices are Cohen-Macaulay. This means that the homology of the order complex of each open interval in such a lattice vanishes below the top dimension. In proving this conjecture, Björner was led to introduce the theory of lexicographical shellability in [3]. This theory, which was further developed by Björner and Wachs in [5, 6, 7, 8], has proved to be an important tool for establishing Cohen-Macaulayness of posets and determining the homotopy type of order complexes; see also [84].

Through his work in [53, 54, 56, 64], Stanley demonstrates that one can obtain beautiful q -analogs of results involving the Möbius function of the Boolean algebra by replacing B_n with the lattice $B_n(q)$ of subspaces of the n -dimensional vector space over the finite field \mathbb{F}_q . For example, the following q -analog of (2.1) was obtained by Stanley in [56, Theorem 3.1]. For each $S \subseteq [n-1]$,

$$(2.2) \quad \mu_{(B_n(q))_S}(\hat{0}, \hat{1}) = (-1)^n \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{DES}(\sigma) = S}} q^{\text{inv}(\sigma)},$$

where inv is the number of inversions of σ .

By considering the natural action of the symmetric group \mathfrak{S}_n on B_n , Stanley obtains an equivariant version of (2.1) in [64, Theorem 4.3]. He proves that, in the representation of \mathfrak{S}_n on the top homology of the order complex $\Delta(\overline{(B_n)_S})$ of the rank-selected subposet $\overline{(B_n)_S} := (B_n)_S \setminus \{\hat{0}, \hat{1}\}$, the multiplicity of the irreducible representation of \mathfrak{S}_n indexed by the partition λ is equal to the number of standard Young tableaux of shape λ and descent set S .

In [9], Björner and Welker studied certain poset constructions arising in commutative algebra. They defined the *Rees product* $P * Q$ of ranked posets (P, \leq_P) and (Q, \leq_Q) with respective rank functions r_P and r_Q . The elements of $P * Q$ are those $(p, q) \in P \times Q$ satisfying $r_P(p) \geq r_Q(q)$. A partial order on $P * Q$ is defined by setting $(p, q) \leq (p', q')$ if

- $p \leq_P p'$,
- $q \leq_Q q'$, and
- $r_P(p') - r_P(p) \geq r_Q(q') - r_Q(q)$.

Björner and Welker [9] proved that $P * Q$ is Cohen-Macaulay if P is Cohen-Macaulay and Q is acyclic and Cohen-Macaulay. They applied this result to the poset

$$R_n := ((B_n \setminus \{\emptyset\}) * C_n) \uplus \{\hat{0}, \hat{1}\},$$

where C_n is the chain $\{0 < 1 < \dots < n-1\}$. Björner and Welker conjectured (and Jonsson proved in [36]) that

$$(2.3) \quad \mu_{R_n}(\hat{0}, \hat{1}) = (-1)^{n-1} d_n,$$

where d_n is the number of derangements in the symmetric group \mathfrak{S}_n . These results have been extended in several directions; see [48], [38] and [12]. Here we discuss just a few of these directions.

Let $a_{n,j}$ be the number of permutations in \mathfrak{S}_n with j descents (or equivalently j excedances; see (3.1)). Recall that the numbers $a_{n,j}$ are called *Eulerian numbers*.

THEOREM 2.1 (Shareshian and Wachs [48, Theorem 1.2]). *If $S \in B_n$ has size $m > 0$, then*

$$\mu_{R_n}(\hat{0}, (S, j)) = (-1)^m a_{m,j}$$

for all $j \in \{0, 1, \dots, m-1\}$.

The conjecture of Björner and Welker (equation (2.3)) follows quickly from the standard inclusion-exclusion expression for d_n . There are various ways to prove Theorem 2.1. In [48] we apply the recursive definition of the Möbius function to a closely related Rees product poset to obtain the exponential generating function formula,

$$(2.4) \quad 1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} (-1)^n \mu_{R_n}(\hat{0}, ([n], j)) t^j \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}.$$

The result thus follows from Euler's exponential generating function formula for the Eulerian numbers, given below in (3.1), and the fact that each lower interval $[\hat{0}, (S, j)]$ of R_n is isomorphic to the interval $[\hat{0}, ([m], j)]$ in R_m . In our joint paper [38] with Linusson we gave another proof of Theorem 2.1, which involves counting the ascent-free chains in an EL-labeling. This enabled us to prove generalizations of both Theorem 2.1 and the Björner-Welker conjecture in which B_n is replaced with an arbitrary product of chains and \mathfrak{S}_n is replaced with a corresponding set of multiset permutations.

Inspired by (2.2), we decided to consider next the q -analogue of R_n obtained by replacing B_n by $B_n(q)$. Let

$$R_n(q) := ((B_n(q) \setminus \{0\}) * C_n) \uplus \{\hat{0}, \hat{1}\}.$$

Since $B_n(q)$ is Cohen-Macaulay and C_n is acyclic and Cohen-Macaulay, it follows from the general result of Björner and Welker that $R_n(q)$ is Cohen-Macaulay. From our joint paper with Linusson [38], we get the stronger result that $R_n(q)$ is EL-shellable. We have the following q -analog of Theorem 2.1.

THEOREM 2.2 (Shareshian and Wachs [48, Theorem 1.3]). *If $W \in B_n(q)$ has dimension $m > 0$ and $0 \leq j \leq m-1$, then*

$$\mu_{R_n(q)}(\hat{0}, (W, j)) = (-1)^m \sum_{\substack{\sigma \in \mathfrak{S}_m \\ \text{exc}(\sigma) = j}} q^{\binom{m}{2} - \text{maj}(\sigma) + j},$$

where $\text{exc}(\sigma)$ is the excedance number and $\text{maj}(\sigma)$ is the major index of σ .

The proof of Theorem 2.2 was substantially more difficult than any of the proofs of Theorem 2.1. We were originally led to conjecture the formula of Theorem 2.2 by inspection of data. To prove our conjectured formula, we first derived a q -analog of (2.4). This served to reduce our conjectured formula to the q -analog of Euler's exponential generating function formula given in (3.3). To our surprise this q -analog of Euler's exponential generating function formula was new and was not easy to prove.

As a consequence of Theorem 2.2 we obtain the following q -analog of (2.3).

COROLLARY 2.3 ([48, Corollary 1.4]). *Let \mathcal{D}_n be the set of derangements in \mathfrak{S}_n . Then*

$$\mu_{R_n(q)}(\hat{0}, \hat{1}) = (-1)^{n-1} \sum_{\sigma \in \mathcal{D}_n} q^{\binom{n}{2} - \text{maj}(\sigma) + \text{exc}(\sigma)}.$$

Combining the natural action of the symmetric group \mathfrak{S}_n on B_n with the trivial action of \mathfrak{S}_n on C_n , we obtain an action on R_n . This action yields a representation of \mathfrak{S}_n on the unique nontrivial homology group $\tilde{H}_{n-1}(\Delta(\overline{R}_n))$ of the order complex of $\overline{R}_n := R_n \setminus \{\hat{0}, \hat{1}\}$ and on the unique nontrivial homology group $\tilde{H}_{n-2}(\Delta I_j(B_n))$ of the order complex of $I_j(B_n) := \{x \in R_n : \hat{0} < x < ([n], j)\}$ for each j . We describe these representations (or \mathfrak{S}_n -modules) in [48]. The most efficient description involves the use of the Frobenius characteristic ch , which assigns to each \mathfrak{S}_n -module a homogeneous symmetric function of degree n . Recall that the elementary symmetric function e_n is the sum of all degree n squarefree monomials in infinitely many variables x_1, x_2, \dots . We define

$$E(z) := \sum_{n \geq 0} e_n z^n$$

and, for all $n \geq 1$,

$$[n]_t := \frac{t^n - 1}{t - 1} = 1 + t + \dots + t^{n-1}.$$

THEOREM 2.4 (Shareshian and Wachs [48, Theorem 1.5 and Corollary 1.6]). *The following equalities hold:*

$$(2.5) \quad 1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} \text{ch}(\tilde{H}_{n-2}(\Delta I_j(B_n))) t^j z^n = \frac{(1-t)E(z)}{E(tz) - tE(z)} \\ = \frac{\sum_{n \geq 0} e_n z^n}{1 - \sum_{n \geq 2} t[n-1]_t e_n z^n}$$

and

$$(2.6) \quad 1 + \sum_{n \geq 1} \text{ch}(\tilde{H}_{n-2}(\Delta \overline{R}_n)) z^n = \frac{1}{1 - \sum_{n \geq 2} (n-1) e_n z^n}.$$

Equation (2.5) specializes to (2.4) when one applies a variant of exponential specialization ex that takes e_n to $\frac{1}{n!}$ for all $n \geq 0$ (cf. [77, Sec. 7.8]). Indeed, it is not difficult to see that if V is an \mathfrak{S}_n -module then $\text{ex}(\text{ch}(V)) = \frac{1}{n!} \dim V$. Hence (2.5) specializes to

$$1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} \dim \tilde{H}_{n-2}(\Delta I_j(B_n)) t^j \frac{z^n}{n!} = \frac{(1-t)e^z}{etz - te^z}.$$

This is equivalent to (2.4) by the classical result of P. Hall relating Möbius functions to reduced Euler characteristics (see [70, Proposition 3.8.8]) and the Euler-Poincaré formula. Similarly (2.6) specializes to the Björner-Welker conjecture (equation (2.3)).

To prove Theorem 2.4 we use a technique of Sundaram [81], which can be viewed as an equivariant version of the technique of using the recursive definition

of Möbius function to compute the Euler characteristic of an order complex. An equivariant version of Theorem 2.1 is given in Corollary 3.4.

3. q -Eulerian polynomials

3.1. Euler's exponential generating function. It follows from work of Euler in [23], an equidistribution result of MacMahon in [39] and an observation of Riordan in [45] that

$$(3.1) \quad 1 + \sum_{n \geq 1} \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} \frac{z^n}{n!} = 1 + \sum_{n \geq 1} \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t},$$

where $\text{des}(\sigma)$ and $\text{exc}(\sigma)$ are the descent number and the excedance number of σ , respectively. As referred to above, (3.1) is *Euler's exponential generating function formula for the Eulerian polynomials*.

As mentioned in Section 2, in order to prove Theorem 2.2 we needed to prove a certain q -analog of (3.1). To obtain a q -analog of the Eulerian polynomials, one can combine a Mahonian statistic such as the major index maj or the inversion number inv with an Eulerian statistic such as exc or des . Such q -analogs have received considerable attention over the years. A q -analog, involving maj and des , of Euler's original definition of the Eulerian polynomials was first obtained by MacMahon in [39, Vol. 2, Sect. IX, Ch. IV, Art. 462] and rediscovered by Carlitz in [13]. It follows from (2.2) that a q -analog, involving inv and des , of the Eulerian polynomials can be obtained from the rank-selected Möbius invariant of $B_n(q)$. Using his theory of binomial posets, Stanley proved in [56], a beautiful q -analog of (3.1) also involving inv and des . For $n \in \mathbb{P}$, let $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$.

THEOREM 3.1 (Stanley [56, p. 351]). *Let $A_n^{\text{inv}, \text{des}}$ be the q -Eulerian polynomial defined by*

$$A_n^{\text{inv}, \text{des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}.$$

Then

$$(3.2) \quad 1 + \sum_{n \geq 1} A_n^{\text{inv}, \text{des}}(q, t) \frac{z^n}{[n]_q!} = \frac{1-t}{\text{Exp}_q(z(t-1)) - t},$$

where

$$\text{Exp}_q(z) := \sum_{n \geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

Our q -analog of (3.1) involves maj and exc .

THEOREM 3.2 (Shareshian and Wachs [47, Theorem 1.1], [49, Corollary 1.3]). *Let $A_n^{\text{maj}, \text{exc}}$ be the q -Eulerian polynomial defined by*

$$A_n^{\text{maj}, \text{exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}.$$

Then

$$(3.3) \quad 1 + \sum_{n \geq 1} A_n^{\text{maj}, \text{exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1-tq) \text{exp}_q(z)}{\text{exp}_q(z tq) - tq \text{exp}_q(z)},$$

where

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}.$$

From here on, by q -Eulerian polynomial we mean

$$A_n(q, t) := A_n^{\text{maj, exc}}(q, tq^{-1}) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)},$$

and by q -Eulerian number we mean the coefficients of t^j in $A_n(q, t)$, that is

$$a_{n,j}(q) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} q^{\text{maj}(\sigma) - j}.$$

Now equation (3.3) can be rewritten as

$$(3.4) \quad 1 + \sum_{n \geq 1} A_n(q, t) \frac{z^n}{[n]_q!} = \frac{(1-t) \exp_q(z)}{\exp_q(z) - t \exp_q(z)}.$$

We prove (3.4) by proving an identity of (quasi)symmetric functions and applying stable principal specialization. (Recall that the stable principal specialization $\text{ps}(f)$ of a quasisymmetric function $f(x_1, x_2, \dots)$ is the power series in q obtained by replacing each x_i with q^{i-1} .) As usual, \mathfrak{h}_n will denote the complete homogeneous symmetric function of degree n , that is, the sum of all degree n monomials in x_1, x_2, \dots . Define

$$H(z) := \sum_{n \geq 0} \mathfrak{h}_n z^n.$$

For $S \subseteq [n-1]$, Gessel's fundamental quasisymmetric function $F_{n,S}$ is defined as

$$(3.5) \quad F_{n,S} := \sum_{\substack{i_1 \geq \dots \geq i_n \\ i_j > i_{j+1} \forall j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

It is known (see [77, Lemma 7.19.10]) that

$$(3.6) \quad \text{ps}(F_{n,S}) = \frac{q^{\sum_{i \in S} i}}{(q; q)_n},$$

where

$$(q; q)_n := \prod_{j=1}^n (1 - q^j).$$

As $F_{n, \emptyset} = \mathfrak{h}_n$, it follows that

$$(3.7) \quad \text{ps}(H(z(1-q))) = \exp_q(z).$$

We define the alphabet $A := [n] \cup [\bar{n}]$, where $[\bar{n}] := \{\bar{i} : i \in [n]\}$. From a permutation $\sigma = \sigma_1 \dots \sigma_n$, written in one-line notation, we obtain a word $\bar{\sigma} = \alpha_1 \dots \alpha_n$ in A by replacing σ_i with $\bar{\sigma}_i$ whenever σ has an excedance at position i . We order A by

$$\bar{1} < \dots < \bar{n} < 1 < \dots < n.$$

Now we define $\text{DEX}(\sigma)$ to be the descent set of $\bar{\sigma}$ in the given order on A . A key point is that

$$(3.8) \quad \sum_{i \in \text{DEX}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma).$$

Now, for positive integers n, j , define the *Eulerian quasisymmetric function*

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} F_{n, \text{DEX}(\sigma)}.$$

It turns out that the quasisymmetric functions $Q_{n,j}$ are in fact symmetric functions. It follows from (3.6) and (3.8) that

$$(3.9) \quad \text{ps}(Q_{n,j}) = \frac{a_{n,j}(q)}{(q; q)_n}.$$

Upon replacing z with $z(1-q)$, we obtain (3.4) from (3.7), (3.9) and stable principal specialization of both sides of (3.10) below.

THEOREM 3.3 (Shareshian and Wachs [49, Theorem 1.2]). *The following equalities hold:*

$$(3.10) \quad 1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} Q_{n,j} t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)} \\ = \frac{H(z)}{1 - \sum_{n \geq 2} t[n-1]_t h_n z^n}.$$

Let ω be the involution on the ring of symmetric functions that takes h_n to e_n . By comparing Theorems 2.4 and 3.3 we obtain the following equivariant version of Theorem 2.1.

COROLLARY 3.4. *For all $j = 0, 1, \dots, n-1$,*

$$\text{ch} \tilde{H}_{n-2}(\Delta I_j(B_n)) = \omega Q_{n,j}.$$

Equations (3.4) and (3.10) have been extended and applied in various ways; see e.g. [17, 24, 25, 26, 32, 33, 35, 37, 38, 46, 49, 50, 51]. In the remaining subsections we discuss just a few of these applications and related results.

3.2. Palindromicity and unimodality. Stanley proved many results about unimodality of sequences arising naturally in combinatorics. Some of his proofs involved striking applications of algebraic geometry ([60, 62]), commutative algebra ([73, 74]), convex geometry ([63]), and representations of groups, Lie algebras and Lie superalgebras ([59, 64, 67, 69, 72]). Some of this work is surveyed in [68, 71]. Motivated by Stanley's work on unimodality, we obtained in [49] a q -analog of the well-known fact that the Eulerian polynomials are palindromic and unimodal; see Theorem 3.5 below.

Let R be a \mathbb{Q} -algebra with basis \mathbf{b} . For instance, R can be the \mathbb{Q} -algebra of symmetric functions $\Lambda_{\mathbb{Q}}$ with basis $\mathbf{h} := \{h_{\lambda}\}$ of complete homogeneous symmetric functions or R can be the \mathbb{Q} -algebra of polynomials $\mathbb{Q}[q]$ with basis $\mathbf{q} := \{1, q, q^2, \dots\}$. For $f, g \in R$, we say that $f \geq_{\mathbf{b}} g$ if the expansion of $f - g$ in the basis \mathbf{b} has nonnegative coefficients. We say that $f \in R$ is \mathbf{b} -positive if $f \geq_{\mathbf{b}} 0$ and that a polynomial $f(t) = \sum_{i=0}^n a_i t^i$ in $R[t]$ is \mathbf{b} -positive, \mathbf{b} -unimodal and palindromic with center of symmetry $\frac{n}{2}$ if

$$0 \leq_{\mathbf{b}} a_0 \leq_{\mathbf{b}} a_1 \leq \dots \leq_{\mathbf{b}} a_{\lfloor \frac{n}{2} \rfloor} = a_{\lfloor \frac{n+1}{2} \rfloor} \geq_{\mathbf{b}} \dots \geq_{\mathbf{b}} a_{n-1} \geq_{\mathbf{b}} a_n \geq_{\mathbf{b}} 0,$$

and $a_i = a_{n-i}$ for all $i = 0, \dots, n$.

THEOREM 3.5 (Shareshian and Wachs [49]). *The polynomial $\sum_{j=0}^{n-1} Q_{n,j}t^j \in \Lambda_{\mathbb{Q}}[t]$ is \mathbf{h} -positive, \mathbf{h} -unimodal and palindromic with center of symmetry $\frac{n-1}{2}$, and the polynomial $A_n(q, t) := \sum_{j=0}^{n-1} a_{n,j}(q)t^j \in \mathbb{Q}[q][t]$ is \mathbf{q} -positive, \mathbf{q} -unimodal and palindromic with center of symmetry $\frac{n-1}{2}$.*

There are various ways to prove this theorem. In his survey paper [71], Stanley presents various tools for establishing unimodality of sequences. The most elementary tool [71, Proposition 1] states that the product of positive, unimodal, palindromic polynomials is positive, unimodal, and palindromic. Here we describe how this tool was used to prove Theorem 3.5 in [50, Appendix C]. In the next subsection we will see that the least elementary tool of [71] can also be used to prove the theorem.

Let $\left[\begin{matrix} n \\ k_1, k_2, \dots, k_m \end{matrix} \right]_q$ denote the q -multinomial coefficient $\frac{[n]_q!}{[k_1]_q! \cdots [k_m]_q!}$. In [50, Appendix C] we derive, by manipulating the right side of (3.4), the following formula, from which Theorem 3.5 is evident:

$$(3.11) \quad A_n(q, t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 2 \\ \sum k_i = n+1}} \left[\begin{matrix} n \\ k_1 - 1, k_2, \dots, k_m \end{matrix} \right]_q t^{m-1} \prod_{i=1}^m [k_i - 1]_t.$$

Indeed, each polynomial $t^{m-1} \prod_{i=1}^m [k_i - 1]_t$ is positive, unimodal and palindromic since each factor is. The center of symmetry of each product is the sum of the centers of symmetries its factors. This sum is $\frac{n-1}{2}$ in every case. By expressing $A_n(q, t)$ as a sum of \mathbf{q} -positive, \mathbf{q} -unimodal, palindromic polynomials all with the same center of symmetry we can conclude that $A_n(q, t)$ is \mathbf{q} -positive, \mathbf{q} -unimodal, and palindromic. The result for $Q_{n,j}$ is proved similarly in [50, Appendix C].

3.3. Geometric interpretation: the hard Lefschetz theorem. One of Stanley's many major contributions to combinatorics is the idea of using the hard Lefschetz theorem to solve combinatorial problems; see [60, 61, 62, 66, 68, 71]. In this section we will discuss how the hard Lefschetz theorem can be used to prove Theorem 3.5.

It is well-known that the Eulerian polynomial $A_n(t) := A_n(1, t)$ equals the h -polynomial of the polar dual Δ_n of the $(n-1)$ -dimensional permutohedron (or equivalently, the type A_{n-1} Coxeter complex).

Let Δ be a simplicial convex polytope of dimension d . The h -polynomial $h_{\Delta}(t)$ of Δ is defined by

$$h_{\Delta}(t) = \sum_{j=0}^d h_j t^j := \sum_{j=0}^d f_{d-1-j} (t-1)^j,$$

where f_i is the number of faces of Δ of dimension i . The coefficient sequence (h_0, h_1, \dots, h_d) is known as the h -vector of Δ . The Dehn-Sommerville equations state that $h_{\Delta}(t)$ is palindromic. Stanley's use of the hard Lefschetz theorem to prove the following theorem is a milestone in combinatorics.

THEOREM 3.6 (Stanley [62]). *If Δ be a simplicial convex polytope then $h_{\Delta}(t)$ is unimodal (as well as palindromic).*

Theorem 3.6 is an important consequence of the necessity part of the celebrated g -theorem. (The sufficiency part was proved by Billera and Lee in [2].) We describe the role of the hard Lefschetz theorem in the proof of Theorem 3.6. We may assume that the polytope Δ has rational vertices. There is a complex projective variety \mathcal{V}_Δ , of (complex) dimension d , naturally associated to the rational polytope Δ , namely, the toric variety associated with Δ . In [19] Danilov proved that

$$h_\Delta(t) = \sum_{i=0}^d \dim H^{2i}(\mathcal{V}_\Delta) t^i,$$

where H^{2i} denotes cohomology in degree $2i$. Stanley saw that the hard Lefschetz theorem can be applied in this situation. This theorem provides an injective map from $H^{2i-2}(\mathcal{V}_\Delta)$ to $H^{2i}(\mathcal{V}_\Delta)$ when $1 \leq i \leq \frac{d}{2}$ (as well as an isomorphism from $H^{2i}(\mathcal{V}_\Delta)$ to $H^{2d-2i}(\mathcal{V}_\Delta)$). Thus the polynomial $\sum_{i=0}^d \dim H^{2i}(\mathcal{V}_\Delta) t^i$ is palindromic and unimodal. By setting Δ equal to the dual permutohedron Δ_n , we obtain a geometric proof that the Eulerian polynomial $A_n(t) = h_{\Delta_n}(t)$ is palindromic and unimodal.

To obtain a geometric proof of \mathbf{q} -unimodality and palidromicity of the q -Eulerian polynomials $A_n(q, t)$, we use our Eulerian quasisymmetric functions $Q_{n,j}$ and a result of Procesi and Stanley. The reflection action of \mathfrak{S}_n on Δ_n determines a linear representation of \mathfrak{S}_n on each cohomology of \mathcal{V}_{Δ_n} . Stanley [71, Proposition 12] used a recurrence of Procesi [42] to obtain the following generating function formula for the Frobenius characteristic of the \mathfrak{S}_n -module $H^{2j}(\mathcal{V}_{\Delta_n})$:

$$(3.12) \quad 1 + \sum_{n \geq 1} \sum_{j=1}^{n-1} \text{ch}(H^{2j}(\mathcal{V}_{\Delta_n})) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}.$$

By comparing (3.12) with Theorem 3.3, we conclude that for $0 \leq j \leq n-1$,

$$(3.13) \quad \text{ch}(H^{2j}(\mathcal{V}_{\Delta_n})) = Q_{n,j}.$$

Now from (3.9) we obtain

$$(q; q)_n \text{ps}(\text{ch}(H^{2j}(\mathcal{V}_{\Delta_n}))) = a_{n,j}(q).$$

Next we use the observation of Stanley in [71, p. 528] that since the hard Lefschetz map commutes with the action of \mathfrak{S}_n , the polynomial $\sum_{i=0}^{n-1} \text{ch}(H^{2i}(\mathcal{V}_{\Delta_n})) t^i \in \Lambda_{\mathbb{Q}}[t]$ is palindromic and Schur-unimodal. Since $(q; q)_n \text{ps}(s_\lambda) \in \mathbb{N}[q]$ for every Schur function s_λ of degree n , we conclude that $A_n(q, t)$ is palindromic and \mathbf{q} -unimodal.

We remark that results analogous to Stanley's formula (3.12) for Coxeter complexes associated to arbitrary crystallographic root systems were obtained by Stembridge in [80] and by Dolgachev-Lunts in [22].

3.4. Refinements. The symmetric functions $Q_{n,j}$ and the q -Eulerian numbers $a_{n,j}(q)$ can be refined. For a partition λ of n and a nonnegative integer j , we define $\mathfrak{S}_{\lambda,j}$ to be the set of all $\sigma \in \mathfrak{S}_n$ with j excedances and cycle type λ . Now set

$$Q_{\lambda,j} := \sum_{\sigma \in \mathfrak{S}_{\lambda,j}} F_{n, \text{DEX}(\sigma)}$$

and

$$a_{\lambda,j}(q) := \sum_{\sigma \in \mathfrak{S}_{\lambda,j}} q^{\text{maj}(\sigma)-j}.$$

Note that by (3.6) and (3.8),

$$a_{\lambda,j}(q) = (q; q)_n \text{ps}(Q_{\lambda,j}).$$

It turns out that the quasisymmetric functions $Q_{\lambda,j}$ are symmetric and have some remarkable properties, see [49, 33, 46].

In [10], Brenti proved that $A_\lambda(t) := \sum_{j \geq 0} a_{\lambda,j}(1)t^j$ is unimodal and palindromic. We conjectured the following symmetric function analog and q -analog of this result in [49].

THEOREM 3.7 (Henderson and Wachs [33]). *The polynomial $\sum_{j \geq 0} Q_{\lambda,j}t^j$ is Schur-positive, Schur-unimodal, and palindromic. Consequently, the polynomial $A_\lambda(q, t) := \sum_{j \geq 0} a_{\lambda,j}(q)t^j$ is \mathbf{q} -unimodal, and palindromic.*

We do not have a geometric interpretation of this result like that for $\sum_{j=0}^{n-1} Q_{n,j}t^j$ and $A_n(q, t)$ given in Section 3.3. Schur-positivity is proved by deriving a plethystic formula for $Q_{\lambda,j}$ from one obtained by us in [49]. Schur-unimodality is established by using the plethystic formula to construct an \mathfrak{S}_n -module $V_{\lambda,j}$ whose Frobenius characteristic is $Q_{\lambda,j}$ and an \mathfrak{S}_n -module monomorphism from $V_{\lambda,j-1}$ to $V_{\lambda,j}$ when $1 \leq j \leq \frac{n-k}{2}$, where k is the number of parts of size 1 in λ . Many other related unimodality results and conjectures appear in [49, 33].

The $Q_{\lambda,j}$ also play a critical role in our joint paper [46] with Sagan, in which we show that the polynomials $a_{\lambda,j}(q)$ appear in an instance of the cyclic sieving phenomenon of Reiner, Stanton, and White (see [44]).

THEOREM 3.8 (Sagan, Shareshian, and Wachs [46, Theorem 1.2]). *Let $\gamma_n \in \mathfrak{S}_n$ be the n -cycle $(1, 2, \dots, n)$. Then, for each partition λ of n and each $j \in \{0, \dots, n-1\}$, the subgroup $\langle \gamma_n \rangle$ generated by γ_n acts by conjugation on $\mathfrak{S}_{\lambda,j}$. If $\tau \in \langle \gamma_n \rangle$ has order d , then the fixed point set of τ in this action has size $a_{\lambda,j}(e^{2\pi i/d})$.*

4. Chromatic quasisymmetric functions

4.1. An email message from Stanley. After distributing a preliminary version of our research announcement [47] on q -Eulerian polynomials, we received an email message from Stanley. Therein, he pointed out to us that a characterization of the Eulerian quasisymmetric functions $Q_{n,j}$ given in [47, Proposition 2.4] (see also [49, Theorem 3.6]) is equivalent to the characterization

$$(4.1) \quad \omega Q_{n,j} = \sum_{\substack{w \in W_n \\ \text{des}(w)=j}} x_w.$$

Here

- W_n , is the set of all words $w := w_1 \dots w_n$ over the alphabet of positive integers \mathbb{P} satisfying $w_i \neq w_{i+1}$ for all $i \in [n-1]$,
- $\text{des}(w)$ is the number of descents of w , and
- $x_w := x_{w_1} x_{w_2} \dots x_{w_n}$.

Indeed, this equivalence follows from P -partition reciprocity, which Stanley introduced in [52]; see [49, equation (7.7)].

Theorem 3.3 thus gives a generating function formula for $\sum_{w \in W_n} x_w t^{\text{des}(w)}$, which refines the following formula of Carlitz, Scoville and Vaughan (see [14]),

$$(4.2) \quad 1 + \sum_{n \geq 1} \sum_{w \in W_n} x_w z^n = \frac{\sum_{n \geq 0} e_n z^n}{1 - \sum_{n \geq 2} (n-1) e_n z^n}.$$

Stanley observes in [75, Proposition 5.3] that since the words in W_n can be viewed as proper colorings of the path graph $G_n := ([n], \{\{i, i+1\} : i \in [n-1]\})$, the symmetric function $\sum_{w \in W_n} x_w$ can be viewed as the chromatic symmetric function of G_n . These observations of Stanley brought us from the world of Eulerian polynomials to the world of chromatic symmetric functions. We consider in [50, 51] a refined notion of chromatic symmetric function, one for which $\sum_{w \in W_n} x_w t^{\text{des}(w)}$ is the refined chromatic symmetric function for G_n . Before describing our refinement we review chromatic symmetric functions.

4.2. Stanley's chromatic symmetric function. Let $G = (V, E)$ be a simple finite graph with no loops. A proper \mathbb{P} -coloring of G is a function $\kappa : V \rightarrow \mathbb{P}$ such that $\kappa(u) \neq \kappa(v)$ whenever $\{u, v\}$ lies in the edge set E . For such a coloring κ , set

$$x_\kappa := \prod_{v \in V} x_{\kappa(v)}.$$

In [75], Stanley defined the chromatic symmetric function

$$X_G(\mathbf{x}) := \sum_{\kappa} x_\kappa,$$

where the sum is taken over all proper \mathbb{P} -colorings κ of G . It is apparent that X_G is indeed a symmetric function and that by setting $x_i = 1$ for all $i \in [m]$ and $x_i = 0$ for all $i \geq m+1$, one gets the chromatic polynomial χ_G of G evaluated at m . Stanley studied various aspects of X_G in [75] and [76]. In particular, he examined expansions of X_G in various well-studied bases for the ring of symmetric functions.

Among the many interesting questions and theorems about X_G are ones that arise when the structure of G is restricted appropriately. Given a finite poset P , the incomparability graph $\text{inc}(P)$ has vertex set P . Its edge set consists of all $\{p, q\}$ such that neither $p < q$ nor $q < p$ holds in P . For positive integers a, b , we call P $(a+b)$ -free if there is no induced subposet of P that is the disjoint union of a chain of a elements and a chain of b elements. The next conjecture was first stated in the form given below in [75]. It is a generalization of a particular case of a conjecture of Stembridge on immanants, see [79]. The transformation of this case to a statement about chromatic symmetric functions is achieved implicitly by Stanley and Stembridge in [78].

CONJECTURE 4.1 (Stanley/Stembridge, see [75, Conjecture 5.1]). *If P is a $(3+1)$ -free poset then $X_{\text{inc}(P)}$ is \mathbf{e} -positive.*

Stanley and Stembridge showed in [78] that Conjecture 4.1 is true for all posets in some interesting infinite classes. For example, if P has no chain of length three then $X_{\text{inc}(P)}$ is \mathbf{e} -positive. Schur-positivity, a weaker property than \mathbf{e} -positivity, followed from results of Haiman [31]. Gasharov described in [27] the coefficients of the Schur functions in the expansion of $X_{\text{inc}(P)}$ in terms of tableaux. Guay-Paquet showed in [30] that if Conjecture 4.1 holds for all P that are both $(3+1)$ -free and

(2+2)-free, then it holds in general. Other work on chromatic symmetric functions can be found in [28, 15, 16, 85, 41, 40, 34].

4.3. A quasisymmetric refinement. Given a proper \mathbb{P} -coloring κ of a graph $G = ([n], E)$, we define the ascent number $\text{asc}(\kappa)$ to be the number of edges $\{i, j\} \in E$ such that $i < j$ and $\kappa(i) < \kappa(j)$. In [50, 51], we define the *chromatic quasisymmetric function*

$$X_G(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} x_{\kappa},$$

where the sum is taken over all proper \mathbb{P} -colorings κ of G . So, $X_G(\mathbf{x}, t)$ refines the chromatic symmetric function $X_G(\mathbf{x})$.

EXAMPLE 4.2. We show in [51, Example 3.2] that if G is the path $1 - 2 - 3$ then

$$X_G(\mathbf{x}, t) = \mathbf{e}_3 + (\mathbf{e}_3 + \mathbf{e}_{2,1})t + \mathbf{e}_3t^2$$

and if G is the path $1 - 3 - 2$ then

$$X_G(\mathbf{x}, t) = (\mathbf{e}_3 + F_{3,\{1\}}) + 2\mathbf{e}_3t + (\mathbf{e}_3 + F_{3,\{2\}})t^2,$$

where $F_{n,S}$ denotes Gessel's fundamental quasisymmetric function defined in (3.5).

Upon considering the paths with three vertices given in Example 4.2, one observes two clear differences between $X_G(\mathbf{x})$ and $X_G(\mathbf{x}, t)$. First, $X_G(\mathbf{x}, t)$ depends not only on the isomorphism type of G but also on the labeling of the vertices of G with the elements of $[n]$. Second, if we consider $X_G(\mathbf{x}, t)$ to lie in the polynomial ring in the variable t with coefficients in the ring of power series in x_1, x_2, \dots , then the coefficient of t^j in $X_G(\mathbf{x}, t)$ need not be a symmetric function. This coefficient is, however, a quasisymmetric function, hence the name.

In [50, 51] we consider a class of graphs on $[n]$ for which the coefficient of each t^j in $X_G(\mathbf{x}, t)$ is a symmetric function. Choose a finite set of closed intervals $[a_i, a_i + 1]$ ($1 \leq i \leq n$) of length one on the real line, with $a_i < a_{i+1}$ for $1 \leq i \leq n-1$. The associated natural unit interval order P is the poset on $[n]$ in which $i <_P j$ if $a_i + 1 < a_j$. A natural unit interval order is both (3+1)-free and (2+2)-free. Conversely, every finite poset that is both (3+1)-free and (2+2)-free is isomorphic with a unique natural unit interval order. We show that if G is the incomparability graph of a natural unit interval order then $X_G(\mathbf{x}, t)$ is symmetric in \mathbf{x} and palindromic as a polynomial in t .

CONJECTURE 4.3 (Shareshian and Wachs [51, Conjecture 5.1]). *Let G be the incomparability graph of a natural unit interval order. Then the palindromic polynomial $X_G(\mathbf{x}, t)$ is \mathbf{e} -positive and \mathbf{e} -unimodal.*

Conjecture 4.3 refines Conjecture 4.1 for posets that are not only (3+1)-free but also (2+2)-free. However, given the result of Guay-Paquet mentioned above, Conjecture 4.3 implies Conjecture 4.1.

EXAMPLE 4.4. The path $G_n := 1 - 2 - \dots - n$ is the incomparability graph of a natural unit interval order. In [51] we use (4.1) and (3.10) to derive

$$X_{G_n}(\mathbf{x}, t) = \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 2 \\ \sum k_i = n+1}} \mathbf{e}_{(k_1-1, k_2, \dots, k_m)} t^{m-1} \prod_{i=1}^m [k_i - 1]_t.$$

As discussed in [51, Appendix C], by expressing $X_{G_n}(\mathbf{x}, t)$ in the formula above as a sum of \mathbf{e} -positive, \mathbf{e} -unimodal, palindromic polynomials with the same center of symmetry, one can conclude that the conjecture holds for G_n .

Other classes of examples for which Conjecture 4.3 holds are given in [51, Section 8] and weaker versions of the conjecture are also discussed in [51]. For instance, in [51, Section 6] we establish Schur-positivity of $X_G(\mathbf{x}, t)$ when G is the incomparability graph of a natural unit interval order. In fact, we give a formula for the coefficient of each Schur function s_λ in the expansion of $X_G(\mathbf{x}, t)$, refining the unit interval order case of the result of Gasharov in [27] mentioned above. Given a poset P on $[n]$ and a partition λ of n , Gasharov defines a P -tableau of shape λ to be a filling of a Young diagram of shape λ (in English notation) with elements of P such that

- each element of P appears exactly once,
- if $y \in P$ appears immediately to the right of $x \in P$ then $y >_P x$,
- if $y \in P$ appears immediately below $x \in P$ then $y \not\prec_P x$.

Let \mathcal{T}_P be the set of all P -tableaux. For $T \in \mathcal{T}_P$ and graph $G = ([n], E)$, define a G -inversion of T to be an edge $\{i, j\} \in E$ such that $i < j$ and i appears below j in T (not necessarily in the same column). Let $\text{inv}_G(T)$ be the number of G -inversions of T and let $\lambda(T)$ be the shape of T .

EXAMPLE 4.5. Let $P := P_{n,r}$ be the poset on $[n]$ with $i <_P j$ if $j - i \geq r$ and let $G_{n,r}$ be the incomparability graph of $P_{n,r}$. That is, $G_{n,r} = ([n], \{\{i, j\} : i, j \in [n], 0 < |j - i| < r\})$. If $P := P_{9,3}$ and $G := G_{9,3}$ then

$$T = \begin{array}{|c|c|c|} \hline 2 & 6 & 9 \\ \hline 1 & 4 & 8 \\ \hline 3 & 7 & \\ \hline 5 & & \\ \hline \end{array}$$

is a P -tableau of shape $(3, 3, 2, 1)$ and

$$\text{inv}_G(T) = |\{\{1, 2\}, \{3, 4\}, \{4, 6\}, \{5, 6\}, \{5, 7\}, \{7, 8\}, \{7, 9\}, \{8, 9\}\}| = 8.$$

Upon setting $t = 1$ in (4.3) below, we obtain a formula that was shown to hold for all $(3 + 1)$ -free posets by Gasharov in [27].

THEOREM 4.6 (Shareshian and Wachs [51, Theorem 6.3]). *Let G be the incomparability graph of a natural unit interval order P . Then*

$$(4.3) \quad X_G(\mathbf{x}, t) = \sum_{T \in \mathcal{T}_P} t^{\text{inv}_G(T)} s_{\lambda(T)}.$$

Consequently, $X_G(\mathbf{x}, t)$ is Schur-positive.

We use Theorem 4.6 and the Murnaghan-Nakayama rule to prove in [51, Section 7] that for the incomparability graph $G := ([n], E)$ of a natural unit interval order, the coefficient of \mathbf{p}_n in the power-sum symmetric function basis expansion of $\omega X_G(\mathbf{x}, t)$ is

$$\frac{[n]_t}{n} \prod_{j=2}^n [b_j]_t,$$

where $b_j = |\{\{i, j\} \in E : i < j\}|$. Palindromicity and unimodality of the coefficient is a consequence of this formula. We also make two (equivalent) conjectures describing the coefficient of \mathbf{p}_λ for each partition λ of n , which can be shown to refine the corresponding result of Stanley for $X_G(\mathbf{x})$ in [75]. These conjectures have been proved by Athanasiadis in [1].

Chromatic quasisymmetric functions of incomparability graphs of natural unit interval orders are also related to representations of type A Hecke algebras, as explained in [18].

4.4. Generalized q -Eulerian polynomials and an exercise in Stanley's EC1. It follows from (3.9) and (4.1) that the stable principal specialization of the chromatic quasisymmetric function of the path G_n is given by

$$(4.4) \quad \text{ps}(\omega X_{G_n}(\mathbf{x}, t)) = \text{ps}\left(\sum_{j=0}^{n-1} Q_{n,j} t^j\right) = (q; q)_n^{-1} A_n(q, t).$$

To compute $\text{ps}(\omega X_G(\mathbf{x}, t))$ for an arbitrary incomparability graph $G = \text{inc}(P)$, we use a refinement of Chow's expansion of $X_G(\mathbf{x})$ in the fundamental quasisymmetric function basis (see [16]). In [51, Section 3], we show that

$$(4.5) \quad X_G(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{DES}_P(\sigma)} t^{\text{inv}_G(\sigma)},$$

where

$$\text{DES}_P(\sigma) := \{i \in [n-1] : \sigma(i) >_P \sigma(i+1)\}$$

and

$$\text{inv}_G(\sigma) := |\{\{\sigma(i), \sigma(j)\} \in E(G) : i < j \in [n], \sigma(i) > \sigma(j)\}|.$$

For a certain class of natural unit interval orders, the stable principal specialization of the corresponding chromatic quasisymmetric function has a particularly attractive description, which we present here.

For $\sigma \in \mathfrak{S}_n$, let

$$\begin{aligned} \text{inv}_{<r}(\sigma) &:= |\{(i, j) : i < j, 0 < \sigma(i) - \sigma(j) < r\}| \\ \text{maj}_{\geq r}(\sigma) &:= \sum_{i: \sigma(i) - \sigma(i+1) \geq r} i. \end{aligned}$$

The permutation statistic $\text{inv}_{<r} + \text{maj}_{\geq r}$ was introduced by Rawlings [43] who proved that it is Mahonian for all r , that is

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq r}(\sigma) + \text{inv}_{<r}(\sigma)} = [n]_q!.$$

Note that this Mahonian statistic interpolates between maj (when $r = 1$) and inv (when $r = n$).

Now consider the polynomial obtained by splitting the Rawlings statistic as follows

$$A_n^{(r)}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq r}(\sigma)} t^{\text{inv}_{<r}(\sigma)}.$$

As in Example 4.5, for each $r \in [n]$, let $P_{n,r}$ be the poset on $[n]$ with order relation given by $i <_{P_{n,r}} j$ if $j - i \geq r$. Note that when $r = 2$, $\text{inc}(P_{n,r})$ is the path G_n . By (4.5) and (3.6), we have

$$A_n^{(r)}(q, t) = (q; q)_n \text{ps}(\omega X_{\text{inc}(P_{n,r})}(x, t)).$$

It therefore follows from (4.4) that

$$(4.6) \quad A_n^{(2)}(q, t) = A_n(q, t).$$

This justifies calling $A_n^{(r)}(q, t)$ a generalized q -Eulerian polynomial. Since $\text{inv}_{<2}(\sigma) = \text{des}(\sigma^{-1})$, the $q = 1$ case of (4.6) is equivalent to the fact that des and exc are equidistributed on \mathfrak{S}_n , for which there is a well-known bijective proof. In [51, Problem 9.8] we pose the problem of finding a direct bijective proof of (4.6).

It is a consequence of the palindromicity of $X_{\text{inc}(P_{n,r})}(\mathbf{x}, t)$ that the generalized q -Eulerian polynomials are palindromic as polynomials in t . Conjecture 4.3 implies the following conjecture².

CONJECTURE 4.7 ([51, Conjecture 9.6]). *For all $r \in [n]$, the palindromic polynomial $A_n^{(r)}(q, t)$ is \mathbf{q} -unimodal.*

After formulating this conjecture we discovered that the $q = 1$ case of the conjecture appears as one of the 203 exercises of Chapter 1 of Stanley's EC1 (see [70, Exercise 1.50 f]). The solution in [70, page 157] is given by the following result, which involves a certain variety called a Hessenberg variety whose definition is given in the next subsection.

THEOREM 4.8 (De Mari and Shayman [21]). *Let $\mathcal{H}_{n,r}$ be the type A_{n-1} regular semisimple Hessenberg variety of degree r . Then*

$$A_n^{(r)}(1, t) = \sum_{j \geq 0} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

Consequently by the hard Lefschetz theorem, $A_n^{(r)}(1, t)$ is palindromic and unimodal.

Stanley asked for a more elementary proof of unimodality of $A_n^{(r)}(1, t)$ in [70, page 157]. (As far as we know this is still open.)

The result of De Mari and Shayman suggested to us an approach to proving Conjecture 4.7 along the lines of the geometric proof of \mathbf{q} -unimodality of $A_n(q, t)$ given in Section 3.3. What was needed was a representation of the symmetric group on the cohomology of the Hessenberg variety $\mathcal{H}_{n,r}$ whose Frobenius characteristic is $\omega X_{\text{inc}(P_{n,r})}(\mathbf{x}, t)$. In the next subsection we discuss a promising candidate for such a representation.

4.5. Hessenberg varieties. A weakly increasing sequence of integers $\mathbf{m} := (m_1, \dots, m_{n-1})$ is called a *Hessenberg vector* if $i \leq m_i \leq n$ for each i . Let \mathcal{F}_n be the variety of all flags of subspaces

$$F : V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n$$

with $\dim V_i = i$. Fix $s \in GL_n(\mathbb{C})$ such that s is diagonal with n distinct eigenvalues. The *regular semisimple Hessenberg variety of type A* associated to the Hessenberg vector \mathbf{m} is

$$\mathcal{H}(\mathbf{m}) := \{F \in \mathcal{F}_n \mid sV_i \subseteq V_{m_i} \text{ for all } i \in [n-1]\}.$$

The degree r Hessenberg variety $\mathcal{H}_{n,r} := \mathcal{H}(r, r+1, \dots, n, \dots, n)$ was studied initially by De Mari and Shayman in [21]. Further work on regular semisimple Hessenberg varieties associated to arbitrary root systems was done by De Mari, Procesi and Shayman in [20].

²See Section 5: *Further developments*.

Given a Hessenberg vector \mathbf{m} , define $P(\mathbf{m})$ to be the poset on $[n]$ defined by $i <_{P(\mathbf{m})} j$ if $j > m_i$. In [51] we observe that $P(\mathbf{m})$ is a natural unit interval order for all Hessenberg vectors \mathbf{m} , and for every natural unit interval order P there is a unique Hessenberg vector \mathbf{m} satisfying $P = P(\mathbf{m})$.

As observed in [20], $\mathcal{H}_{n,2} := \mathcal{H}(2, 3, \dots, n)$ is the toric variety \mathcal{V}_{Δ_n} . This variety admits an \mathfrak{S}_n action as described in Section 3.3. We do not know whether for general Hessenberg vectors \mathbf{m} , each $\mathcal{H}(\mathbf{m})$ admits a faithful \mathfrak{S}_n action. However, there is a nice representation of \mathfrak{S}_n on $H^*(\mathcal{H}(\mathbf{m}))$, as described by Tymoczko in [83]. Let T be the group of diagonal matrices in $GL_n(\mathbb{C})$. It is straightforward to confirm that if $u \in T$ and the flag \mathcal{F} lies in $\mathcal{H}(\mathbf{m})$, then $u\mathcal{F} \in \mathcal{H}(\mathbf{m})$. This action of T on $\mathcal{H}(\mathbf{m})$ satisfies the technical conditions necessary to apply the theory of Goresky-Kottwitz-MacPherson on torus actions (see [29], and see [82] for a survey of GKM theory). The GKM theory says that the cohomology of $\mathcal{H}(\mathbf{m})$ is determined by the moment graph M associated to the T -action. It turns out that M admits an \mathfrak{S}_n action, which determines a representation of \mathfrak{S}_n on each cohomology of $\mathcal{H}(\mathbf{m})$.

When $\mathbf{m} = (2, 3, \dots, n)$, the representation of \mathfrak{S}_n on $H^{2j}(\mathcal{H}(\mathbf{m}))$, for each j , is isomorphic to the representation of \mathfrak{S}_n on $H^{2j}(\mathcal{V}_{\Delta_n})$ discussed in Section 3.3; see [83]. It therefore follows from (3.13) and (4.1) that the following conjecture³ holds when $\mathbf{m} = (2, 3, \dots, n)$.

CONJECTURE 4.9 (Shareshian and Wachs [51, Conjecture 10.1]). *For all Hessenberg vectors \mathbf{m} ,*

$$\omega X_{\text{inc}(P(\mathbf{m}))}(\mathbf{x}, t) = \sum_{j \geq 0} \text{ch}(H^{2j}(\mathcal{H}(\mathbf{m})))t^j.$$

There is considerable evidence in favor of Conjecture 4.9, see [50, Section 5] and [51, Section 10]. Since the hard Lefschetz map on $H^*(\mathcal{H}(\mathbf{m}))$ commutes with the action of the symmetric group, the conjecture implies that $X_{\text{inc}(P(\mathbf{m}))}(\mathbf{x}; t)$ is Schur-unimodal, which in turn implies that Conjecture 4.7 holds.

We aim to carry out the following two-step process, thereby repaying Richard Stanley in some small measure for all of the wonderful ideas he has shared with us and our colleagues.

- (1) Prove Conjecture 4.9.
- (2) Understand the action of \mathfrak{S}_n on $H^*(\mathcal{H}(\mathbf{m}))$ well enough to prove Conjecture 4.3, thereby proving the Stanley-Stembridge e-positivity conjecture (Conjecture 4.1).

5. Further developments

Brosnan and Chow have posted a proof of our Conjecture 4.9, see [11]. Thus step (1) in the process described at the end of Section 4.5 has been taken. Moreover, as was discussed above, Conjecture 4.7 follows as a consequence.

References

1. C. Athanasiadis, Power sum expansion of chromatic quasisymmetric functions, *Electronic Journal of Combinatorics* **22** (2015), Research Paper 2.7, 9pp (electronic).
2. L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, *J. Combin. Theory Ser. A* **31** (1981), no. 3, 237–255.

³See Section 5: *Further developments*.

3. A. Björner, Shellable and Cohen-Macaulay partially ordered sets, *Trans. AMS* **260** (1980), 159–183.
4. A. Björner, A. Garsia and R. P. Stanley, An introduction to the theory of Cohen-Macaulay partially ordered sets, in *Ordered Sets* (I. Rival, ed.), Reidel, Dordrecht/Boston/London, 1982, pp. 583–615.
5. A. Björner and M. L. Wachs, Bruhat order of Coxeter groups and shellability, *Advances in Math.* **43** (1982), 87–100.
6. A. Björner and M. L. Wachs, On lexicographically shellable posets, *Trans. AMS* **277** (1983), 323–341.
7. A. Björner and M. L. Wachs, Shellable nonpure complexes and posets I, *Trans. AMS* **348** (1996), 1299–1327.
8. A. Björner and M. L. Wachs, Shellable nonpure complexes and posets II, *Trans. AMS* **349** (1997), 3945–3975.
9. A. Björner and V. Welker, Segre and Rees products of posets, with ring-theoretic applications, *J. Pure Appl. Algebra* **198** (2005), 43–55.
10. F. Brenti, Unimodal polynomials arising from symmetric functions, *Proc. Amer. Math. Soc.* **108** (1990), 1133–1141.
11. P. Brosnan and T. Y. Chow, Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties, arXiv:1511.00773
12. P. M. Brown and M. Readdy, The Rees product of posets, *J. Comb.* **2** (2011), no. 2, 165–191.
13. L. Carlitz, A combinatorial property of q-Eulerian numbers, *Amer. Math. Monthly*, **82** (1975), 51–54.
14. L. Carlitz, R. Scoville, and T. Vaughan, Enumeration of pairs of sequences by rises, falls and levels, *Manuscripta Math.* **19** (1976), 211–243.
15. T. Chow, *Symmetric Function generalizations of graph polynomials*, MIT Ph.D. dissertation, 1995.
16. T. Chow, Descents, quasi-symmetric functions, Robinson-Schensted for posets, and the chromatic symmetric function, *J. Algebraic Combin.* **10** (1999), no. 3, 227–240.
17. F. Chung and R. Graham, Generalized Eulerian sums, *Journal of Combinatorics*, **3** (2012) no. 3, 299–316.
18. S. Clearman, M. Hyatt, B. Shelton, and M. Skandera, Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements, DMTCS proc. AS, 2013, 1119–1130.
19. V. I. Danilov, The geometry of toric varieties, *Russian Math. Surveys* **33** (1978), 97–154. Translated from *Uspekhi Mat. Nauk.* **33** (1978), 85–134.
20. F. De Mari, C. Procesi and M. Shayman, Hessenberg varieties, *Trans. Amer. Math. Soc.* **332** (1992), no. 2, 529–534.
21. F. De Mari and M. Shayman, Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix, *Acta Appl. Math.* **12** (1988), no. 3, 213–235.
22. I. Dolgachev and V. Lunts, A character formula for the representation of a Weyl group in the cohomology of the associated toric variety, *J. Algebra* **168** (1994), 741–772.
23. L. Euler, Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques, *Mémoires de l'Académie des Sciences de Berlin*, vol. **27**, (1768), 83–106. Also in *Opera Omnia*, Ser. I, *Commentationes analyticae ad theoriam serierum infinitarum pertinentes*, II, vol. 15, p. 70–90, Teubner, Leipzig, 1927.
24. D. Foata and G.-N. Han, Fix-Mahonian calculus. III. A quadruple distribution, *Monatsh. Math.* **154** (2008), no. 3, 177–197.
25. D. Foata and G.-N. Han, Signed words and permutations, V; a sextuple distribution, *Ramanujan J.* **19** (2009), 29–52.
26. D. Foata and G.-N. Han, The q-tangent and q-secant numbers via basic Eulerian polynomials, *Proc. AMS* **138** (2010), 385–393.
27. V. Gasharov, Incomparability graphs of $(3 + 1)$ -free posets are s -positive, Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994), *Discrete Math.* **157** (1996), no. 1-3, 193–197.
28. V. Gasharov, On Stanley's chromatic symmetric function and clawfree graphs, *Discrete Mathematics* **205** (1999), 229–234.
29. M. Goresky, R. Kottwitz and R. MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, *Invent. Math.* **131** (1998), no. 1, 25–83.

30. M. Guay-Paquet, A modular law for the chromatic symmetric functions of $(3 + 1)$ -free posets, preprint, arXiv:1306.2400v1.
31. M. Haiman, Hecke algebra characters and immanant conjectures, *J. Amer. Math. Soc.*, **6** (1993), no. 3, 569–595.
32. G.-N. Han, Z. Lin, and J. Zeng, A symmetrical q -Eulerian identity, *Séminaire Lotharingien de Combinatoire*, **67** (2012), Article B67c.
33. A. Henderson and M. L. Wachs, Unimodality of Eulerian quasisymmetric functions, *J. Combin. Theory Ser. A* **119** (2012), no. 1, 135–145.
34. B. Humpert, A quasisymmetric function generalization of the chromatic symmetric function, *Electron. J. Combin.* **18** (2011), no. 1, P31.
35. M. Hyatt, Eulerian quasisymmetric functions for the type B Coxeter group and other wreath product groups, *Adv. in Appl. Math.* **48** (2012), 465–505.
36. J. Jonsson, The Rees product of a Boolean algebra and a chain, preprint, 2003.
37. Z. Lin, On some generalized q -Eulerian polynomials, *DMTCS proc.* **AS** (2013), 409–420.
38. S. Linusson, J. Shareshian and M. L. Wachs, Rees products and lexicographic shellability, *J. Comb.* **3** (2012), no. 3, 243–276.
39. P. A. MacMahon, *Combinatory Analysis*, 2 volumes, Cambridge University Press, London, 1915-1916. Reprinted by Chelsea, New York, 1960.
40. J. Martin, M. Morin and J. Wagner, On distinguishing trees by their chromatic symmetric functions, *J. Combin. Theory Ser. A* **115** (2008), no. 2, 237–253.
41. S. D. Noble and D. J. A. Welsh, A weighted graph polynomial from chromatic invariants of knots, *Annales de l'institut Fourier*, **49** (1999), no. 3, 1057–1087.
42. C. Procesi, The toric variety associated to Weyl chambers, in *Mots*, 153-161, Lang. Raison. Calc., Herms, Paris, 1990.
43. D. Rawlings, The r -major index, *J. Combin. Theory Ser. A* **31** (1981), no. 2, 175–183.
44. V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* **108** (2004), no. 1, 17–50.
45. J. Riordan, *An Introduction to Combinatorial Analysis*, J. Wiley, New York, 1958.
46. B. Sagan, J. Shareshian and M. L. Wachs, Eulerian quasisymmetric functions and cyclic sieving, *Adv. Applied Math* **46** (2011), 536–562.
47. J. Shareshian and M. L. Wachs, q -Eulerian polynomials: excedance number and major index, *Electron. Res. Announc. Amer. Math. Soc.* **13** (2007), 33–45 (electronic).
48. J. Shareshian and M. L. Wachs, Poset homology of Rees products, and q -Eulerian polynomials, *Electron. J. Combin.* **16** (2009), no. 2, Special volume in honor of Anders Björner, Research Paper 20, 29 pp.
49. J. Shareshian and M. L. Wachs, Eulerian quasisymmetric functions, *Adv. Math.* **225** (2010), no. 6, 2921–2966.
50. J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions and Hessenberg varieties, in *Configuration Spaces*, Proceedings, A. Björner, F. Cohen, C. De Concini, C. Procesi and M. Salvetti (eds.), Edizioni della Normale, Pisa, 2012, 433–460.
51. J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions, preprint, 2014.
52. R. P. Stanley, Ordered structures and partitions (revision of 1971 Harvard University thesis), *Memoirs of the Amer. Math. Soc.*, no. 119 (1972), iii + 104 pages.
53. R. P. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.
54. R. P. Stanley, Finite lattices and Jordan-Hölder sets, *Algebra Universalis* **4** (1974), 361–371.
55. R. P. Stanley, The Upper Bound Conjecture and Cohen-Macaulay rings, *Studies in Applied Math.* **54** (1975), 135–142.
56. R. P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, *J. Combinatorial Theory (A)* **20** (1976), 336–356.
57. R. P. Stanley, Cohen-Macaulay complexes, in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht/Boston, 1977, pp. 51–62.
58. R. P. Stanley, Balanced Cohen-Macaulay complexes, *Trans. Amer. Math. Soc.* **249** (1979), 139–157.
59. R. P. Stanley, Unimodal sequences arising from Lie algebras, in *Young Day Proceedings* (T. V. Narayana, R. M. Mathsen, and J. G. Williams, eds.), Dekker, New York/Basel, 1980, pp. 127–136.
60. R. P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Algebraic and Discrete Methods* **1** (1980), 168–184.

61. R. P. Stanley, An application of algebraic geometry to extremal set theory, in *Ring Theory and Algebra III* (B. R. McDonald, ed.), Dekker, New York/Basel, 1980, pp. 415–422.
62. R. P. Stanley, The number of faces of a simplicial convex polytope, *Advances in Math.* **35** (1980), 236–238.
63. R. P. Stanley, Two combinatorial applications of the Aleksandrov-Fenchel inequalities, *J. Combinatorial Theory (A)* **31** (1981), 56–65.
64. R. P. Stanley, Some aspects of groups acting on finite posets, *J. Combinatorial Theory (A)* **32** (1982), 132–161.
65. R. P. Stanley, *Combinatorics and Commutative Algebra*, Progress in Mathematics, vol. 41, Birkhuser, Boston/Basel/Stuttgart, 1983, viii + 88 pages; second edition, 1996, ix + 164 pages.
66. R. P. Stanley, The number of faces of simplicial polytopes and spheres, in *Discrete Geometry and Convexity* (J. E. Goodman, et al., eds.), Ann. New York Acad. Sci., vol. 440 (1985), pp. 212–223.
67. R. P. Stanley, $GL(n, \mathbb{C})$ for combinatorialists, in *Surveys in Combinatorics* (E. K. Lloyd, ed.), London Math. Soc. Lecture Note Series, vol. 82, Cambridge University Press, Cambridge, 1983, pp. 187–199.
68. R. P. Stanley, Combinatorial applications of the hard Lefschetz theorem, in *Proc. International Congress of Mathematicians* (Warsaw, 1983), North-Holland, Amsterdam (1984), pp. 447–453.
69. R. P. Stanley, Unimodality and Lie superalgebras, *Studies in Applied Math.* **72** (1985), 263–281.
70. R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986, xi + 306 pages; second printing, Cambridge University Press, Cambridge, 1996.
71. R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in *Graph Theory and Its Applications: East and West*, Ann. New York Acad. Sci., vol. 576, 1989, pp. 500–535.
72. R. P. Stanley, Some applications of algebra to combinatorics, *Discrete Applied Math.* **34** (1991), 241–277.
73. R. P. Stanley, A monotonicity property of h -vectors and h^* -vectors, *European J. Combinatorics* **14** (1993), 251–258.
74. R. P. Stanley, Subdivisions and local h -vectors, *J. Amer. Math. Soc.* **5** (1992), 805–851.
75. R. P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Advances in Math.* **111** (1995), 166–194.
76. R. P. Stanley, Graph colorings and related symmetric functions: ideas and applications, *Discrete Mathematics* **193** (1998), 267–286.
77. R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999, xii + 581 pages.
78. R. P. Stanley and J. Stembridge, On immanants of Jacobi-Trudi matrices and permutations with restricted position, *J. Combinatorial Theory (A)* **62** (1993), 261–279.
79. J. R. Stembridge, Some conjectures for immanants, *Can. J. Math.* **44** (1992), 1079–1099.
80. J. R. Stembridge, Some permutation representations of Weyl groups associated with the cohomology of toric varieties, *Adv. Math.* **106** (1994), 244–301.
81. S. Sundaram, *The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice*, *Advances in Math.* **104** (1994), 225–296.
82. J. Tymoczko, An introduction to equivariant cohomology and homology, following Goresky, Kottwitz, and MacPherson, *Snowbird lectures in algebraic geometry*, 169–188, *Contemp. Math.*, **388**, Amer. Math. Soc., Providence, RI, 2005.
83. J. Tymoczko, Permutation actions on equivariant cohomology, *Toric topology*, 365–384, *Contemp. Math.* **460**, Amer. Math. Soc., Providence, RI, 2008.
84. M. L. Wachs, *Poset topology: tools and applications*, Geometric Combinatorics, IAS/PCMI lecture notes series (E. Miller, V. Reiner, B. Sturmfels, eds.) **13** (2007), 497–615.
85. H. L. Wolfgang, *Two interactions between combinatorics and representation theory: monomial immanants and Hochschild cohomology*, MIT Ph.D. dissertation, 1997.

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