WHITNEY HOMOLOGY OF SEMIPURE SHELLABLE POSETS

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Abstract. We generalize results of Calderbank, Hanlon and Robinson on the representation of the symmetric group on the homology of posets of partitions with restricted block size. Calderbank, Hanlon and Robinson consider the cases of block sizes that are congruent to $0 \mod d$ and $1 \mod d$ for fixed d. We derive a general formula for the representation of the symmetric group on the homology of posets of partitions whose block sizes are congruent to $k \mod d$ for any k and d. This formula reduces to the Calderbank-Hanlon-Robinson formulas when $k = 0, 1$ and to formulas of Sundaram for the virtual representation on the alternating sum of homology. Our results apply to restricted block size partition posets even more general than the k mod d partition posets. These posets include the lattice of partitions whose block sizes are bounded from below by some fixed k. Our main tools involve the new theory of nonpure shellability developed by Björner and Wachs and a generalization of a technique of Sundaram which uses Whitney homology to compute homology representations of Cohen-Macaulay posets. An application to subspace arrangements is also discussed.

Introduction.

Shellability is a well known notion in algebraic and topological combinatorics which until recently applied only to pure (i.e., all maximal chains have the same length) posets and simplicial complexes. In [BW2 & BW3] Björner and the author extend the theory of shellability to nonpure posets and complexes. The nonpure setting provides for a richer theory which enables one to analyze many important and natural classes of posets that are nonpure. One major distinction between pure shellability and nonpure shellability is that a pure shellable poset can have nonvanishing homology only in the top dimension while a nonpure shellable poset can have nonvanishing homology in various dimensions.

In this paper, we use the theory of nonpure shellability to generalize a powerful technique developed by Sundaram for computing the character of group actions

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on the top dimensional homology of Cohen-Macaulay posets. We then apply the generalized technique to computing the representation of the symmetric group on each homology for a general class of subposets of the partition lattice induced by restricting the block sizes of the partitions. This general class contains subposets which are nonpure shellable.

Homology of restricted block size partition posets was first considered by Calderbank, Hanlon and Robinson [CHR] who derived beautiful plethystic formulas yielding the character of the representation of the symmetric group on the top homology of the d-divisible partition lattice (all block sizes divisible by some fixed d) and the $1 \mod d$ partition lattice (all block sizes congruent to $1 \mod d$), both of which are pure. A key fact used in the Calderbank, Hanlon and Robinson proof is that the two posets are pure shellable which was proved respectively by Wachs and Björner (cf. [CHR] and [Sa]). For general k, the $k \mod d$ partition poset is not pure and that is where the difficulty lies in computing the homology representations of the general $k \mod d$ partition posets.

In [Su2], Sundaram gives a formula for the virtual representation of the symmetric group on the alternating sum of homology of the $k \mod d$ partition poset. When the poset is pure and shellable, as in the case $k = 0, 1$, Sundaram's formula gives the nonvirtual representation on the top dimensional homology since homology in every other dimension vanishes. However when the poset is not pure, as for general k, one cannot extract the nonvirtual representation on each homology from the virtual alternating sum representation, even though, as we establish here, the poset is shellable.

In this paper, we succeed in obtaining a formula which gives the representation of the symmetric group on each homology of the general $k \mod d$ partition poset. Our formula refines the alternating sum formula of Sundaram and reduces to the formulas of Calderbank, Hanlon and Robinson when $k = 0, 1$. More generally, we give two simple conditions on sets $T \subseteq \mathbb{P}$ and show that these conditions imply that Π_n^T, the poset of partitions of $[n]$ whose block sizes are in T, is shellable and its dual is semipure (i.e., all proper principal lower order ideals are pure). Then we derive a formula giving the representation of the symmetric group $\mathcal{S}_n$ on each homology of Π_n^T. The simple conditions are satisfied, for example, by the set $\{k + id \mid i \in \mathbb{N}\}$, for all $k, d \in \mathbb{P}$. When $k = d$ we have the d-divisible partition lattice and when $1 \leq k \leq d$ we have the $k \mod d$ partition poset. When $d = 1$, we have the at least k partition lattice, which was first proved to be shellable in [BW2]. Our formula, when $d = 1$, implies another formula of Sundaram for the virtual representation of the symmetric group on the the alternating sum of homology of the "at least k partition lattice".

Sundaram's technique for computing virtual representations on alternating sums of homology is based on her result equating alternating sums of homology representations with alternating sums of Whitney homology representations. When all but the top homology vanishes, this reduces to a formula expressing the top homology representation as an alternating sum of Whitney homology representations. We generalize this result to semipure posets that satisfy a certain homological condition implied by shellability, by introducing a new doubly indexed Whitney homology and expressing the homology representation of the poset in terms of the doubly indexed Whitney homology representations. As a consequence, we are also able to express the Betti numbers of the poset in terms of the Möbius function of certain intervals.

Sundaram expresses the Frobenius characteristic of each Whitney homology of the dual of a partition poset as a homogeneous component of the plethysm of certain symmetric functions. We refine and abstract her results by expressing a generating function for the Frobenius characteristic of the doubly indexed Whitney homology as a plethysm of a certain symmetric function with a certain generating function for the complete homogeneous symmetric functions.

In Section 1 we review Sundaram's result equating alternating homology representations with alternating Whitney homology representations. We also discuss some homological consequences of nonpure shellability, one of which is Stanley's recent notion of sequentially Cohen-Macaulay (the nonpure version of Cohen-Macaualay). We present a new characterization of sequentially Cohen-Macaulay which follows from one due to Duval [D].

In Section 2 we introduce the notion of doubly indexed Whitney homology and a variant of it. Doubly indexed Whitney homology is used to express homology representations of any semipure sequentially Cohen-Macaulay poset. The variant is used to do the same for sequentially Cohen-Macaulay posets whose dual is semipure.

In Section 3 we present the two simple conditions on a set $T \subseteq \mathbb{P}$ and show that these conditions imply that Π_n^T is shellable and its dual is semipure. We discuss examples of sets T that satisfy the conditions, one of which is the set $\{k+id \mid i \in \mathbb{N}\}$.

In Section 4 we derive the plethystic formulas for the generating function of the Frobenius characteristic of the doubly indexed Whitney homology of the dual of Π_n^T. By combining this with the results of Section 2, we obtain a generating function for the characteristic of homology of Π_n^T in each dimension. This result specializes to generating functions for the characteristic of homology of the $k \mod d$ and the at least k partition poset.

The doubly indexed Whitney homology representations of the dual of the partition posets decompose naturally into representations which are induced up from direct products of wreath products. In Section 5 we prove a general plethystic formula, stated and used in Section 4, for the generating function of the Frobenius characteristic of these induced representations.

In Section 6, we present some identities which can be derived from the formulas of Section 4. They can also be explained by considering the variant of doubly indexed Whitney homology on Π_n^T. We also touch upon connections with subspace arrangements. We use the formulas of Section 4 and an equivariant Goresky-MacPherson formula of Sundaram and Welker [SWe1] to derive a formula for the representation of the symmetric group on the cohomology of the complement of a complexified subspace arrangement whose intersection lattice is Π_n^T.

1. Preliminaries.

Let P be a finite bounded poset of length $\ell = \ell(P) \geq 0$, with minimum element $\hat{0}$ and maximum element $\hat{1}$. Let $\bar{P}$ denote the induced subposet $P \setminus \{\hat{0}, \hat{1}\}$. For $x \leq y$ in P, (x, y) denotes the open interval $\{z \in P \mid x < z < y\}$, $[x, y]$ denotes the closed interval $\{z \in P \mid x \leq z \leq y\}$ and $\ell(x, y)$ denotes the length $\ell([x, y])$ of the interval. P is said to be *pure* (also known as ranked or graded) if all its maximal chains have the same length.

Recall that if $\ell(P) \geq 1$, the *order complex* of P, denoted by $\Delta(P)$, is defined to be the $(\ell - 2)$-dimensional simplicial complex whose vertices are the elements of $\bar{P}$ and whose faces are the chains of $\bar{P}$. For $r \in \mathbb{Z}$ and $\ell(P) \geq 1$, let $\tilde{H}_r(P)$ denote the rth reduced simplicial homology $\tilde{H}_r(\Delta(P), \mathbb{C})$. For $\ell(P) = 0$ (i.e., $|P| = 1$), define $\tilde{H}_r(P)$ to be $\mathbb{C}$ if $r = -2$ and 0 otherwise. For $x \leq y$ in P, let $\tilde{H}_r(x, y)$ denote the rth homology $\tilde{H}_r([x, y])$ of the interval $[x, y]$. If $[x, y]$ has vanishing homology in all dimensions below the top dimension $\ell(x, y) - 2$ then we write $\tilde{H}(x, y)$ instead of $\tilde{H}_{\ell(x,y)-2}(x, y)$.

Whitney homology for geometric lattices was introduced by Baclawski [Ba]. A formulation due to Björner [Bj1] characterizes Whitney homology for any bounded poset P as the direct sum $\bigoplus_{x \in P} \tilde{H}_{r-2}(\hat{0}, x)$. Here, we slightly modify this formulation by defining the *rth Whitney homology* of P to be

$$WH_r(P) = \bigoplus_{x \in P \setminus \{\hat{1}\}} \tilde{H}_{r-2}(\hat{0}, x).$$

If a finite group G acts as a group of automorphisms of P then we say that P is a G-poset. For each $x \in P$, any element $g \in G$ acts as a map from chains in $(\hat{0}, x)$ to chains of the same length in $(\hat{0}, gx)$. This induces a linear map from the vector space whose basis is the set of length r chains of $(\hat{0}, x)$ to the vector space whose basis is the set of length r chains of $(\hat{0}, gx)$, for each r. Since this map commutes with the boundary maps on the corresponding chain vector spaces, it induces a linear map from $\tilde{H}_r(\hat{0}, x)$ to $\tilde{H}_r(\hat{0}, gx)$. Hence the action of G on P induces a representation of G on $\tilde{H}_r(P)$ and on $WH_r(P)$ for each r. The following important relationship between these G-modules was established by Sundaram [Su1] as a consequence of the Hopf trace formula.

Proposition 1.1. *(Sundaram* [Su1]*) Let P be a G-poset. Then the following isomorphism of sums of virtual G-modules holds:*

$$\bigoplus_{r=1}^{\ell}(-1)^r \tilde{H}_{r-2}(P) = \bigoplus_{r=0}^{\ell-1}(-1)^{r+1} WH_r(P). \tag{1-1}$$

Corollary 1.2. *(Sundaram* [Su1]*) Let P be a G-poset of length $\ell \geq 1$. If $\tilde{H}_i(P)$ vanishes for all $i \neq \ell - 2$ then the G-module $\tilde{H}_{\ell-2}(P)$ decomposes into a sum of* *virtual G-modules as follows*

$$\tilde{H}_{\ell-2}(P) = \bigoplus_{r=0}^{\ell-1}(-1)^{\ell+r+1}WH_r(P). \tag{1-2}$$

Consequently, if P is Cohen-Macaulay (i.e., homology of all intervals vanish below the top dimension) then (1-2) holds.

We assume familiarity with the theory of shellable simplicial complexes and posets (cf. [BW1]). Recall, in particular, that a pure shellable poset or simplicial complex is Cohen-Macaulay. In [Su1], Corollary 1.2 is applied to certain examples of pure shellable posets. Examples of nonpure posets have recently arisen which have lead to an extension of the theory of shellability from pure to nonpure complexes and posets [BW2 & BW3]. Since a nonpure shellable poset can have nonvanishing homology in more than one dimension, Corollary 1.2, as it stands, cannot be used to compute the homology of nonpure shellable posets. In the next section, we will generalize Corollary 1.2 to a class of posets which include posets that are nonpure and shellable.

Let Δ be a finite d-dimensional simplicial complex. For $r \in \mathbb{Z}$, let $C_r(\Delta)$ denote the rth simplicial chain space over $\mathbb{C}$ ($C_{-1}(\Delta)$ is the one dimensional vector space generated by the empty face) and $\tilde{H}_r(\Delta)$ denote the rth reduced simplicial homology of Δ over $\mathbb{C}$. For $-1 \le m \le d$, let $\Delta^{<m>}$ be the subcomplex of Δ generated by all facets of dimension at least m. We say that Δ has the *vanishing homology property* if $\tilde{H}_i(\Delta^{<m>}) = 0$ for all $i < m$.

Theorem 1.3. *Let Δ be a shellable simplicial complex. Then Δ has the vanishing homology property.*

Proof. By [BW2, Theorem 2.9], $\Delta^{<m>}$ is shellable for all $-1 \le m \le \dim \Delta$. Since the dimension of the ith homology of a shellable complex is equal to the number of i-dimensional facets with a certain property [BW2, Theorem 3.4 and Corollary 4.2], the ith homology vanishes if there are no i-dimensional facets. For $i < m$, there are no i dimensional facets in $\Delta^{<m>}$. Hence the ith homology of $\Delta^{<m>}$ vanishes when $i < m$. □

Recently Stanley [S] extended the connection between pure shellability and Cohen-Macaulayness to the nonpure case by finding a nonpure generalization of the notion of Cohen-Macaulay. Instead of giving Stanley's formulation we state the following characterization, established by Duval [D]. The *pure s-skeleton* $\Delta^{[s]}$ of a simplicial complex Δ is defined to be the subcomplex generated by all faces of dimension s.

Proposition 1.4. *(Duval [D]) A simplicial complex Δ is sequentially Cohen-Macaulay if and only if its pure s-skeleton $\Delta^{[s]}$ is Cohen-Macaulay for all $-1 \le s \le d$.*

Recall that the *link* of a face F in Δ is defined to be the subcomplex

$$\mathrm{lk}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta, \quad F \cap G = \emptyset\}.$$

We will use the following characterization of sequential Cohen-Macaulay which follows readily from Duval's characterization.

Theorem 1.5. *A simplicial complex is sequentially Cohen-Macaulay if and only if the link of each of its faces has the vanishing homology property.*

Proof. Let Δ be a d-dimensional simplicial complex and let F be any face. We claim that for all m and r such that $\dim F \leq m \leq d$ and $-1 \leq r < m - \dim F$,

$$\tilde{H}_r(\mathrm{lk}_{\Delta^{[m]}} F) = \tilde{H}_r((\mathrm{lk}_\Delta F)^{<m'>}), \tag{1-3}$$

where $m' = m - \dim F$. To prove this claim we first observe that $C_r(\Delta^{[m]}) = C_r(\Delta^{<m>})$ if $r \leq m$. It follows that

$$\tilde{H}_r(\Delta^{[m]}) = \tilde{H}_r(\Delta^{<m>})$$

if $r < m$. Next we observe that

$$\mathrm{lk}_{\Delta^{[m]}} F = (\mathrm{lk}_\Delta F)^{[m']}.$$

It now follows that

$$\begin{aligned}\tilde{H}_r(\mathrm{lk}_{\Delta^{[m]}} F) &= \tilde{H}_r((\mathrm{lk}_\Delta F)^{[m']}) \\ &= \tilde{H}_r((\mathrm{lk}_\Delta F)^{<m'>}),\end{aligned}$$

for $r < m'$ as claimed.

Duval's characterization (Proposition 1.4), says that the condition that Δ is sequentially Cohen-Macaulay is equivalent to the condition that the left side of (1-3) equals 0. Since, setting the right side of (1-3) equal to 0 is the same as requiring that all links have the vanishing homology property, the result holds. □

Corollary 1.6. *(Stanley* [S]*) Let Δ be a shellable simplicial complex. Then Δ is sequentially Cohen-Macaulay.*

Proof. Since the link of any face of a shellable complex is shellable [BW3, Proposition 10.14], it follows from Proposition 1.3 that all the links have the vanishing homology property. Hence, by Theorem 1.5, Δ is sequentially Cohen-Macaulay. □

Lemma 1.7. *Let Δ be sequentially Cohen-Macaulay. Then $\Delta^{<m>}$ is sequentially Cohen-Macaulay for all m such that $-1 \leq m \leq d$.*

Proof. First note that if $-1 \leq m', m \leq d$, then $(\Delta^{<m>})^{<m'>} = \Delta^{<\max(m,m')>}$. It follows that if Δ has the vanishing homology property then so does $\Delta^{<m>}$. Also note that if F is a face of $\Delta^{<m>}$ then $\mathrm{lk}_{\Delta^{<m>}} F = (\mathrm{lk}_\Delta F)^{<m-\dim F>}$. It follows that if the link in Δ of each face of Δ has the vanishing homology property then the link in $\Delta^{<m>}$ of each face of $\Delta^{<m>}$ has the vanishing homology property. Hence, the result follows from Proposition 1.5. $\square$

Lemma 1.8. *Let Δ have the vanishing homology property. Then*

$$\tilde{H}_i(\Delta^{<m>}) = \begin{cases} \tilde{H}_i(\Delta) & \text{if } i \geq m \\ 0 & \text{otherwise.} \end{cases} \tag{1-4}$$

Proof. It is easy to see that the first case of (1-4) holds for any simplicial complex Δ. Indeed, this follows from the facts that $C_i(\Delta^{<m>}) = C_i(\Delta)$ and the boundary map ∂_i for $\Delta^{<m>}$ and Δ are the same. Since the second case of (1-4) is simply the vanishing homology property, the result is valid. $\square$

All of the above results pertaining to sequentially Cohen-Macaulay complexes and the vanishing homology property are valid for homology taken over any coefficient ring. We are grateful to an anonymous referee for the following important observation.

Proposition 1.9. *Let Δ be a simplicial complex with the vanishing homology property over the ring of integers. Then $\tilde{H}_i(\Delta, \mathbb{Z})$ is free for all i and vanishes whenever there is no facet of dimension i.*

Proof. Let i be such that $\tilde{H}_i(\Delta, \mathbb{Z}) \neq 0$ and let σ be any cycle of dimension i that is not a boundary. Since $\tilde{H}_i(\Delta^{<i+1>}, \mathbb{Z}) = 0$, the cycle σ must involve a facet of dimension i. Since any multiple of σ must also involve this facet, no multiple of σ can be a boundary either. Thus $\tilde{H}_i(\Delta, \mathbb{Z})$ is free. $\square$

We say that a bounded poset P has the vanishing homology property (resp., is shellable, sequentially Cohen-Macaulay) if its order complex $\Delta(P)$ has the vanishing homology property (resp., is shellable, sequentially Cohen-Macaulay). Note that each closed interval of P has order complex equal to the link of some face of $\Delta(P)$. Hence, by Proposition 1.5, all intervals of a sequentially Cohen-Macaulay poset are sequentially Cohen-Macaulay.

2. Refinement of Whitney homology.

In this section we refine the notion of Whitney homology and use the refined notion to extend Corollary 1.2 to nonpure posets. We give two different refinements which coincide for the class of semipure sequentially Cohen-Macaulay posets.

Definition 2.1. *Let P be a bounded poset of length $\ell \geq 1$. For each $x \in P$ let $m(x)$ be the length of the longest chain containing x (i.e., $m(x) = \ell(\hat{0}, x) + \ell(x, \hat{1})$). For $r, m \in \mathbb{Z}$, the (r, m)-Whitney homology of P is defined to be*

$$WH_{r,m}(P) = \bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ m(x)=m}} \tilde{H}_{r-2}(\hat{0}, x).$$

Remark. Baclawski's definition of Whitney homology involves a differential which makes Whitney homology an actual homology theory. One could formulate our refinement by using the same differential. However such a formulation is not needed in this work since we are merely using the refinement of Whitney homology as a tool in computing poset homology representations.

Note that if G is an automorphism group of P, then each (r, m)-Whitney homology is a $G-$invariant subspace of Whitney homology and each Whitney homology G-module decomposes into a direct sum of G-modules as follows,

$$WH_r(P) = \bigoplus_{m=r+1}^{\ell} WH_{r,m}(P).$$

If P is pure then

$$WH_r(P) = WH_{r,\ell}(P). \tag{2-1}$$

Note that $WH_{0,\ell}(P)$ is the trivial G-module and $WH_{0,m}(P)$ is (0) for $m \neq \ell$.

As in [BW3], we say that P is *semipure* if all proper lower intervals $[\hat{0}, x]$, $x < \hat{1}$ are pure. The following generalization of Corollary 1.2 is the main result of this section and will be used to compute homology representations of the semipure shellable posets considered in the next section.

Theorem 2.2. *Let P be a semipure G-poset of length $\ell \geq 1$ with the vanishing homology property. Then for each m, the G-module $\tilde{H}_{m-2}(P)$ decomposes into a sum of virtual G-modules as follows*

$$\tilde{H}_{m-2}(P) = \bigoplus_{r=0}^{m-1} (-1)^{m+r+1} WH_{r,m}(P). \tag{2-2}$$

Consequently, if P is semipure and shellable, or more generally semipure and sequentially Cohen-Macaulay, then (2-2) holds.

Proof. For each m such that $1 \leq m \leq \ell$, let $P^{<m>}$ be the induced subposet of P on the set $\{x \in P \mid m(x) \geq m\}$. Since P is semipure, $P^{<m>}$ is simply $\hat{1}$ together with the order ideal generated by coatoms of rank at least $m-1$. Hence all maximal chains in $P^{<m>}$ have length at least m. Therefore,

$$\Delta(P^{<m>}) = \Delta(P)^{<m-2>}. \tag{2-3}$$

By (2-3) and Lemma 1.8 we have that

$$\tilde{H}_{r-2}(P^{<m>}) = \begin{cases} \tilde{H}_{r-2}(P) & \text{if } r \geq m \\ 0 & \text{otherwise.} \end{cases} \tag{2-4}$$

We shall first establish (2-2) for $m = \ell$. Since $P^{<\ell>}$ has vanishing homology for all dimensions but $\ell - 2$, we can apply Corollary 1.2 to $P^{<\ell>}$ to obtain,

$$\tilde{H}_{\ell-2}(P^{<\ell>}) = \bigoplus_{r=0}^{\ell-1} (-1)^{\ell+r+1} WH_r(P^{<\ell>}). \tag{2-5}$$

Since $P^{<\ell>}$ is pure, by (2-1) we have

$$WH_r(P^{<\ell>}) = WH_{r,\ell}(P^{<\ell>}) = WH_{r,\ell}(P).$$

Equation (2-2) for $m = \ell$ now follows by substituting this and (2-4) into (2-5).

Now suppose that $m < \ell$. By Proposition 1.1 applied to $P^{<m>}$ and $P^{<m+1>}$, we have

$$\begin{aligned}
\bigoplus_{r=1}^{\ell} (-1)^r \tilde{H}_{r-2}(P^{<m>}) &- \bigoplus_{r=1}^{\ell} (-1)^r \tilde{H}_{r-2}(P^{<m+1>}) \\
&= \bigoplus_{r=0}^{\ell-1} (-1)^{r+1} WH_r(P^{<m>}) - \bigoplus_{r=0}^{\ell-1} (-1)^{r+1} WH_r(P^{<m+1>}) \\
&= \bigoplus_{r=0}^{\ell-1} (-1)^{r+1} \Big(\bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ m(x) \geq m}} \tilde{H}_{r-2}(\hat{0}, x) - \bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ m(x) \geq m+1}} \tilde{H}_{r-2}(\hat{0}, x) \Big) \\
&= \bigoplus_{r=0}^{\ell-1} (-1)^{r+1} \Big(\bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ m(x) = m}} \tilde{H}_{r-2}(\hat{0}, x) \Big) \\
&= \bigoplus_{r=0}^{m-1} (-1)^{r+1} WH_{r,m}(P).
\end{aligned}$$

By (2-4), we thus have

$$\begin{aligned}
\bigoplus_{r=0}^{m-1} (-1)^{r+1} WH_{r,m}(P) &= \bigoplus_{r=m}^{\ell} (-1)^r \tilde{H}_{r-2}(P) - \bigoplus_{r=m+1}^{\ell} (-1)^r \tilde{H}_{r-2}(P) \\
&= (-1)^m \tilde{H}_{m-2}(P). \quad \square
\end{aligned}$$

As a consequence of Theorem 2.2, we obtain a formula for the Betti numbers of P in terms of the Möbius function of P. Let $\beta_i(P)$ denote the ith Betti number, $\dim \tilde{H}_i(P)$, and let μ denote the Möbius function of P.

Corollary 2.3. *If P is semipure of length $\ell \geq 1$ and has the vanishing homology property then*

$$\beta_{m-2}(P) = (-1)^{m+1} \sum_{\substack{x \in P \setminus \{\hat{1}\} \\ m(x)=m}} \mu(\hat{0}, x),$$

for all m.

Proof. By taking dimensions of the representations in (2-2) we obtain

$$\beta_{m-2}(P) = (-1)^{m+1} \Big(\sum_{\substack{x \in P \setminus \{\hat{1}\} \\ m(x)=m}} \sum_{r=0}^{m-1} (-1)^r \beta_{r-2}([\hat{0}, x]) \Big).$$

By the Euler-Poincaré formula, the inner sum is the reduced Euler characteristic of $\Delta([\hat{0}, x])$ which is simply $\mu(\hat{0}, x)$. □

We can weaken the condition that P is semipure in the hypothesis of Theorem 2.2 provided that P is sequentially Cohen-Macaulay. To do this we define a variant of (r, m)-Whitney homology which coincides with the previous notion of (r, m)-Whitney homology when P is semipure and sequentially Cohen-Macaulay.

Definition 2.4. *Let P be a bounded poset of length $\ell \geq 1$. For $r, m \in \mathbb{Z}$, the variant (r, m)-Whitney homology of P is defined to be*

$$WH^{\#}_{r,m}(P) = \bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ \ell(x, \hat{1}) = m-r}} \tilde{H}_{r-2}(\hat{0}, x). \tag{2-6}$$

Each $WH^{\#}_{r,m}(P)$ is a G-invariant subspace of Whitney homology. Note that $WH^{\#}_{0,\ell}(P)$ is the trivial G-module and $WH^{\#}_{0,m}(P)$ is (0) for $m < \ell$.

Proposition 2.5. *If P is semipure and sequentially Cohen-Macaulay then*

$$WH^{\#}_{r,m}(P) = WH_{r,m}(P)$$

for all r, m.

Proof. It follows from the fact that P is semipure and sequentially Cohen-Macaulay, that each interval $[\hat{0}, x]$, $x \in P \setminus \{\hat{1}\}$, is pure and has homology equal to 0 in all dimensions other than $\ell(\hat{0}, x) - 2$. Hence $r = \ell(\hat{0}, x)$ for the nonvanishing terms of (2-6). The condition $\ell(x, \hat{1}) = m - r$ is thus equivalent to the condition $\ell(\hat{0}, x) + \ell(x, \hat{1}) = m$. Since $m(x) = \ell(\hat{0}, x) + \ell(x, \hat{1})$, the summation range for the nonvanishing terms of (2-6) is identical to that of Definition 2.1. . □

We shall say that a bounded poset P is *weakly semipure* if for each $x \in P$ either $[\hat{0}, x]$ or $[x, \hat{1}]$ is pure.

Lemma 2.6. *Let P be weakly semipure of length $\ell \geq 1$ and let $1 \leq m \leq \ell$. Then the relation $\leq_m$, defined by $x \leq_m y$ if $x \leq y$ in P and $x, y \in c$ for some chain c of length at least m, is a partial order relation on the set $\{x \in P \mid m(x) \geq m\}$. Moreover, if $P^{<m>}$ denotes the poset $(\{x \in P \mid m(x) \geq m\}, \leq_m)$ then*

$$\Delta(P^{<m>}) = \Delta(P)^{<m-2>}. \tag{2-7}$$

Remark. Note that when P is semipure, $P^{<m>}$ is the induced subposet of P given in the proof of Theorem 2.2.

Proof of Lemma 2.6. To show that $\leq_m$ is a partial order relation, one needs only to check transitivity. Suppose $x \leq y \leq z$ in P; $x, y \in c_1$ where c_1 has length at least m and $y, z \in c_2$ where c_2 has length at least m. Take c_1 and c_2 to be maximal chains. We must find a chain c of length at least m such that $x, z \in c$. We claim that the chain $c = (c_1 \cap [\hat{0}, y]) \cup (c_2 \cap [y, \hat{1}])$ is such a chain. Clearly $x, z \in c$. If $[\hat{0}, y]$ is pure then

$$\begin{aligned} \ell(c) &= \ell(c_1 \cap [\hat{0}, y]) + \ell(c_2 \cap [y, \hat{1}]) \\ &= \ell(c_2 \cap [\hat{0}, y]) + \ell(c_2 \cap [y, \hat{1}]) \\ &= \ell(c_2). \end{aligned}$$

Similarly, if $[y, \hat{1}]$ is pure $\ell(c) = \ell(c_1)$. In either case $\ell(c) \geq m$. Hence the relation defined on $P^{<m>}$ is indeed a partial order relation.

The construction of the chain c given in the verification of transitivity can be used to show that all maximal chains of $P^{<m>}$ have length at least m. Hence (2-7) holds. □

Lemma 2.7. *Let P be a weakly semipure, sequentially Cohen-Macaulay poset of length $\ell \geq 1$. Then for all $x \in P^{<m>}$, where $1 \leq m \leq \ell$, and $r \in \mathbb{Z}$,*

$$\tilde{H}_{r-2}([\hat{0}, x] \cap P^{<m>}) = \begin{cases} \tilde{H}_{r-2}(\hat{0}, x) & \text{if } r \geq m - \ell(x, \hat{1}) \\ 0 & \text{otherwise.} \end{cases} \tag{2-8}$$

Proof. Equation (2-8) is easy to check if $x = \hat{0}$; so assume $x > \hat{0}$. Observe that $[\hat{0}, x]$ is weakly semipure. Hence, (2-7) can be applied to $[\hat{0}, x]$. Since $[\hat{0}, x]$ also has the vanishing homology property, Lemma 1.8 applied to the order complex of $[\hat{0}, x]$ yields

$$\tilde{H}_{r-2}([\hat{0}, x]^{<m-\ell(x,\hat{1})>}) = \begin{cases} \tilde{H}_{r-2}(\hat{0}, x) & \text{if } r \geq m - \ell(x, \hat{1}) \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from the observation that $[\hat{0}, x] \cap P^{<m>} = [\hat{0}, x]^{<m-\ell(x,\hat{1})>}$. □

Theorem 2.8. *Let P be a weakly semipure and sequentially Cohen-Macaulay G-poset of length $\ell \geq 1$. Then for each m, the G-module $\tilde{H}_{m-2}(P)$ decomposes into a sum of virtual G-modules as follows*

$$\tilde{H}_{m-2}(P) = \bigoplus_{r=0}^{m-1} (-1)^{m+r+1} WH^{\#}_{r,m}(P).$$

Proof. By Lemmas 2.6 and 1.7, $P^{<m>}$ is sequentially Cohen-Macaulay for all $m \in [\ell]$.

Since $P^{<\ell>}$ is pure and sequentially Cohen-Macaulay, it is Cohen-Macaulay. Corollary 1.2 can therefore be applied to $P^{<\ell>}$ yielding (2-5) as in the proof of Theorem 2.2. Since $P^{<\ell>}$ is Cohen-Macaulay, $WH_r(P^{<\ell>}) = WH^{\#}_{r,\ell}(P^{<\ell>})$ for all r, ℓ. We have by Lemma 2.7

$$\begin{aligned} WH^{\#}_{r,\ell}(P^{<\ell>}) &= \bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ \ell(x,\hat{1}) = \ell - r \\ m(x) = \ell}} \tilde{H}_{r-2}([\hat{0}, x]) \cap P^{<\ell>}) \\ &= \bigoplus_{\substack{x \in P \setminus \{\hat{1}\} \\ \ell(x,\hat{1}) = \ell - r}} \tilde{H}_{r-2}(\hat{0}, x) \\ &= WH^{\#}_{r,\ell}(P). \end{aligned}$$

Hence, $WH_r(P^{<\ell>}) = WH^{\#}_{r,\ell}(P)$. Also by Lemma 2.7, $\tilde{H}_{\ell-2}(P^{<\ell>}) = \tilde{H}_{\ell-2}(P)$. Therefore, the result for $m = \ell$ again follows by substitution into (2-5).

Now suppose that $m < \ell$. By Proposition 1.1 applied to $P^{<m>}$ and $P^{<m+1>}$, we have

$$\begin{aligned} &\bigoplus_{r=1}^{\ell} (-1)^r \tilde{H}_{r-2}(P^{<m>}) - \bigoplus_{r=1}^{\ell} (-1)^r \tilde{H}_{r-2}(P^{<m+1>}) \\ &\qquad = \bigoplus_{r=0}^{\ell-1} (-1)^{r+1} \left(WH_r(P^{<m>}) - WH_r(P^{<m+1>}) \right). \end{aligned} \tag{2-9}$$

By Lemma 2.7, the left side of (2-9) is

$$\bigoplus_{r=m}^{\ell} (-1)^r \tilde{H}_{r-2}(P) - \bigoplus_{r=m+1}^{\ell} (-1)^r \tilde{H}_{r-2}(P) = (-1)^m \tilde{H}_{m-2}(P).$$

Lemma 2.7 is also used to evaluate the right side of (2-9). We have

$$\begin{aligned}
&WH_r(P^{<m>}) - WH_r(P^{<m+1>}) \\
&= \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)\geq m}} \tilde{H}_{r-2}([\hat{0},x]\cap P^{<m>}) \quad - \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)\geq m+1}} \tilde{H}_{r-2}([\hat{0},x]\cap P^{<m+1>}) \\
&= \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)\geq m\\ \ell(x,\hat{1})\geq m-r}} \tilde{H}_{r-2}(\hat{0},x) \quad - \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)\geq m+1\\ \ell(x,\hat{1})\geq m+1-r}} \tilde{H}_{r-2}(\hat{0},x) \\
&= \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)= m\\ \ell(x,\hat{1})\geq m-r}} \tilde{H}_{r-2}(\hat{0},x) \quad \oplus \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)\geq m+1\\ \ell(x,\hat{1})= m-r}} \tilde{H}_{r-2}(\hat{0},x).
\end{aligned}$$

Since $m(x) - \ell(x,\hat{1}) = \ell(\hat{0},x)$,

$$\tilde{H}_{r-2}(\hat{0},x) = 0 \qquad \text{if} \qquad r > m(x) - \ell(x,\hat{1}). \tag{2-10}$$

Hence we can replace $\ell(x,\hat{1}) \geq m-r$ with $\ell(x,\hat{1}) = m-r$ in the range of summation of the first sum. This yields

$$WH_r(P^{<m>}) - WH_r(P^{<m+1>}) = \bigoplus_{\substack{x\in P\setminus\{\hat{1}\}\\ m(x)\geq m\\ \ell(x,\hat{1})= m-r}} \tilde{H}_{r-2}(\hat{0},x).$$

Also by (2-10) we can eliminate $m(x) \geq m$ in the range of summation since if $m(x) < m$ then $m(x) - \ell(x,\hat{1}) < m - \ell(x,\hat{1}) = r$, forcing $\tilde{H}_{r-2}(\hat{0},x)$ to be 0. It follows that

$$WH_r(P^{<m>}) - WH_r(P^{<m+1>}) = WH^{\#}_{r,m}(P),$$

for $0 \leq r < \ell$. By plugging this into the right side of (2-9) we obtain the result. □

3. Shellability of posets of partitions with restricted block size.

In this section we establish shellability of certain subposets of the partition lattice obtained by restricting block sizes. The set of positive integers will be denoted by $\mathbb{P}$ and the set of nonnegative integers by $\mathbb{N}$. For any $n \in \mathbb{P}$, the lattice of partitions of $[n] = \{1, 2, \ldots, n\}$, ordered by refinement, is denoted by Π_n. Recall that Π_n is pure and bounded with minimum element $\hat{0} = 1/2/\ldots/n$ and maximum element $\hat{1} = 12\ldots n$.

Given any set $T \subseteq \mathbb{P}$, where $1 \notin T$, let Π_n^T be the induced subposet of Π_n consisting of $\hat{0}, \hat{1}$ and all partitions of $[n]$ whose block sizes are in T. If $1 \in T$, then in order for our results to hold we let Π_n^T be the induced subposet as above with a new minimum element $\hat{0}$ attached below $1/2/\ldots/n$. We give two conditions on T which together will be shown to guarantee the shellability of Π_n^T.

Definition 3.1. *Let $d \in \mathbb{P}$. We shall say that $T \subseteq \mathbb{P}$ is d-additive if for all $t_1, t_2, \dots, t_j \in T$, we have $t_1 + t_2 + \cdots + t_j \in T$ if and only if $j \equiv 1 \mod d$. We say that T is additive if it is d-additive for some d.*

Definition 3.2. *We shall say that a nonempty subset $T \subseteq \mathbb{P}$ is subtractive if for all $t_1, t_2 \in T$, $t_1 + t_2 - \min T \in T$.*

Example 3.3. Let $T = \mathbb{P}$. Then T is 1-additive and subtractive. The poset $\Pi_n^T \setminus \{\hat{0}\}$ is Π_n which is a well-known example of a pure shellable poset.

Example 3.4. For fixed integer $k \geq 1$, let $T = \{k, k+1, k+2, \dots\}$. Then T is 1-additive and subtractive. The poset Π_n^T is called the *at least k partition lattice* and is denoted by $\Pi_n^{\geq k}$. It is not pure in general and was shown to be shellable by Björner and Wachs [BW2, Section 7].

Example 3.5. For fixed integer $d \geq 1$, let $T = \{d, 2d, 3d, \dots\}$. Then T is 1-additive and subtractive. The poset Π_n^T is known as the *d-divisible partition lattice*. It is pure and was shown to be shellable by Wachs (cf. [Sa] and [W]).

Example 3.6. For fixed integers $d \geq 1$ and $k \geq 1$, let $T = \{kd, (k+1)d, (k+2)d, \dots\}$ then T is 1-additive and subtractive. By setting $d = 1$ we get Example 3.4 and by setting $k = 1$ we get Example 3.5.

Example 3.7. For fixed integer $d \geq 1$, let $T = \{1, d+1, 2d+1, 3d+1, \dots\}$. Then T is d-additive and subtractive. It is easy to see that this is the only d-additive T that contains 1. The poset $\Pi_n^T \setminus \{\hat{0}\}$ is known as the *$1 \mod d$ partition poset* and is denoted here by $\Pi_n^{1,d}$. It is pure and was shown to be shellable by Björner (see [CHR]).

The next example generalizes the previous examples.

Example 3.8. For fixed integers $k \geq 1$ and $d \geq 1$, let $T = \{k + id \mid i = 0, 1, 2, \dots\}$ and let $d_0 = \frac{d}{\gcd(d,k)}$. Then T is d_0-additive and subtractive. The poset Π_n^T which we shall denote by $\Pi_n^{(k,d)}$ is the main example of this paper. When $d = 1$, this example reduces to Example 3.4 and $\Pi_n^{(k,d)}$ is the at least k partition lattice. When $2 \leq k \leq d$, this example reduces to the *$k \mod d$ partition poset* which further reduces to Examples 3.5 by setting $k = d$. When $k = 1$, $\Pi_n^{(1,d)} \setminus \{\hat{0}\}$ is the $1 \mod d$ partition poset $\Pi_n^{1,d}$ (note the subtle difference in notation between $\Pi_n^{1,d}$ and $\Pi_n^{(1,d)}$). When d divides k, this example reduces to Example 3.6.

The next two examples of additive and subtractive T are not special cases of Example 3.8.

Example 3.9. For fixed integers $d \geq 1$ and $k \geq 1$, let $T = \{d, 2d, 3d, \dots\} \cup \{k, k+1, k+2, \dots\}$. Then T is 1-additive and subtractive.

Example 3.10. For fixed integers $k \geq 1$ and $d \geq 1$ let $T = \{jk + id \mid i \in \mathbb{N}, j \in \mathbb{P}\}$. Then T is 1-additive and subtractive. If $k = 3$ and $d = 2$ then $T = \{3, 5, 6, 7, 8, \dots\}$. If $k = 3$ and $d = 4$ then $T = \{3, 6, 7, 9, 10, 11, \dots\}$.

The following Lemma is obvious.

Lemma 3.11. *Let T be d-additive. Then for all $x \in \Pi_n^T \backslash \{\hat{0}\}$, the upper interval $[x, \hat{1}]$ of Π_n^T is isomorphic to $\Pi_{b(x)}^{1,d}$, where $b(x)$ is the number of blocks of x. Consequently, the dual of Π_n^T is semipure.* □

A bounded poset P is said to be *totally semimodular* if for all $x_1, x_2, y \in P$ such that $y > x_1, x_2$, whenever x_1 and x_2 cover a common element, there exists $z \in P$ such that $z \leq y$ and z covers both x_1 and x_2. Note that a totally semimodular poset is necessarily pure.

Lemma 3.12. *(Björner (see* [CHR]*)) For all $n, d \in \mathbb{P}$, $\Pi_n^{1,d}$ is totally semimodular.*

Recall that the notion of recursive atom ordering provides a combinatorial tool for establishing shellability of posets [BW2, Theorems 5.8 and 5.11]. In [BW2, Section 7], a recursive atom ordering for the at least k partition lattice is given. We shall generalize this recursive atom ordering to one for Π_n^T when T is d-additive and subtractive.

Lemma 3.13. *Suppose all proper upper intervals $[x, \hat{1}]$, $x \neq \hat{0}$ of a bounded poset P are totally semimodular. Then an ordering $a_1, a_2, \ldots, a_t$ of the atoms of P is a recursive atom ordering if and only if*

(*) *for all $i < j$, if $a_i, a_j < y$ then there is a $k < j$ and an element z covering a_j such that $a_k < z \leq y$.*

Proof. Since the intervals $[a_j, \hat{1}]$ are totally semimodular, every atom ordering of $[a_j, \hat{1}]$ is recursive by [BW1, Theorem 5.1]. Hence the first condition in the definition of recursive atom ordering given in [BW2, Definition 5.10] holds automatically. Since condition (*) is the second condition in [BW2, Definition 5.10], the result holds. □

Let $B_1, \ldots, B_p$ be the blocks of a partition π ordered by increasing order of minimum elements. Represent π by the p-tuple $w_\pi = (w_1, \ldots, w_p)$, where w_i is the word obtained by listing the elements of B_i increasingly. For example,

$$w_\pi = (135,\ 2679,\ 48)$$

represents the partition $\pi = 135/2679/48$ of Π_9.

The tuples w_π can be ordered lexicographically, that is, both individual words w_i and tuples of words w_π are compared in lexicographic order. This induces a total order on any subset of Π_n which we shall refer to as lexicographical order.

Theorem 3.14. *Let $T \subseteq \mathbb{P}$ be d-additive and subtractive. Then the lexicographical order of the atoms of Π_n^T is a recursive atom ordering. Consequently Π_n^T is shellable.*

Proof. We prove the result for $n \in T$. When $n \notin T$, the proof below requires slight modification which we leave to the reader.

By Lemmas 3.11, 3.12, and 3.13 it suffices to verify the following:

(**) if $w_{\pi'} < w_\pi$ and $\pi', \pi < \alpha$ for two atoms π' and π, then there exists an atom τ and an element β covering π such that $w_\tau < w_\pi$ and $\tau < \beta \leq \alpha$.

Let $\pi = B_1/B_2/\dots/B_p$ and $\pi' = B_1'/B_2'/\dots/B_q'$, with blocks ordered by increasing minimal elements. Assume that j is such that $B_j \neq B_j'$ and $B_i = B_i'$ for $i < j$. Since $w_{\pi'} < w_\pi$ we must have that $w_j' < w_j$, which leads to two cases.

Case 1. w_j' is a prefix in w_j. That is, w_j is the concatenation $w_j'u$, of words w_j' and $u \neq \emptyset$. Let $U = B_j \setminus B_j'$ and let $B_{f_1}', B_{f_2}', \dots, B_{f_k}'$ be the blocks of π' which intersect U. Then

$$U \subseteq \cup_{i=1}^k B_{f_i}' \subseteq C,$$

where C is the block of α that contains B_j. If $U = \cup_{i=1}^k B_{f_i}'$ then $B_j = B_j' \cup \cup_{i=1}^k B_{f_i}'$ which contradicts the fact that π is an atom. Therefore $U \subsetneq \cup_{i=1}^k B_{f_i}'$

Now choose an element $y \in \cup_{i=1}^k B_{f_i}' \setminus U$. Let B_g be the block of π that contains y. Clearly $g > j$ and $B_j \cup B_g \subseteq C$. Now let β be a partition obtained from π by merging B_j, B_g and any $d-1$ other blocks $B_{h_1}, B_{h_2}, \dots, B_{h_{d-1}}$ of π that are contained in C. Clearly β covers π in Π_n^T and $\beta \leq \alpha$. It remains to create another atom $\tau < \beta$ such that $w_\tau < w_\pi$. This is done by first partitioning the block $B_j \cup B_g \cup \cup_{i=1}^{d-1} B_{h_i}$ of β into $B, B_j \cup B_g \setminus B, B_{h_1}, \dots, B_{h_{d-1}}$; where B consists of the $\min T$ smallest elements of B_j. We have $|B_j \cup B_g \setminus B| = |B_j| + |B_g| - \min T$ which is in T since T is subtractive . Hence, the partition constructed is in Π_n^T. If this partition is not an atom of Π_n^T then further partitioning of $B_j \cup B_g \setminus B$ will yield an atom τ for which $w_\tau < w_\pi$.

Case 2. w_j' is not a prefix of w_j. Let $w_j' = x_1'x_2'\dots$ and $w_j = x_1x_2\dots$, and let t be minimal such that $x_t \neq x_t'$. Then $x_t' < x_t$ since $w_j' < w_j$, and $t \geq 2$ by construction since $B_i = B_i'$ for $i < j$. Let B_g, $g > j$, be the block of π that contains x_t'. Then $B_j \cup B_g \subseteq C$, where C is the block of α that contains B_j. Let β be a partition obtained from π by merging B_j and B_g and $d-1$ other blocks contained in C. Then create an atom $\tau < \beta$ by partitioning the new block of β into $B_j \setminus \{x_t\} \cup \{x_t'\}$, $B_g \setminus \{x_t'\} \cup \{x_t\}$ and the $d-1$ other blocks. The elements τ and β clearly satisfy condition $(**)$. $\square$

Corollary 3.15. *Let $T \subseteq \mathbb{P}$ be additive and subtractive. Then the dual of Π_n^T is semipure and sequentially Cohen-Macaulay.* $\square$

Example 3.16. For fixed $k \geq 1$, let $T = \{1, k, k+1, k+2, \dots\}$. Then T is subtractive, but not additive except when $k = 2$. Hence we cannot apply Theorem 3.14 to this T. The poset $\Pi_n^T \setminus \{\hat{0}\}$, known as the k-equal partition lattice, is not pure for $k \geq 3$ (it's not even weakly semipure), but was shown to be shellable by Björner and Wachs [BW2, Section 6]. Shellability is established by means of a lexicographical edge labeling, not a recursive atom ordering. The k-equal partition lattice first arose in the work of Björner Lovász and Yao [BLY] in connection with a computational complexity problem. Its homology was further studied in papers by Björner and Lovász [BL], Björner and Welker [BWe], Björner and Wachs [BW2], Sundaram and Wachs [SW] and Sundaram and Welker [SWe1].

Example 3.16 suggests that perhaps additivity can be dropped from the hypothesis of Theorem 3.14. This, however, is not the case as Example 3.17 below shows. Example 3.18 shows that subtractivity cannot be dropped either. Hence neither additivity nor subtractivity alone is sufficient for shellability. Example 3.19 below shows that they are not necessary either. These examples suggest that it may be possible to weaken the hypothesis of Theorem 3.14 so that it holds for Example 3.16. This would give a unified proof of shellability for two important classes of examples (Example 3.8 and Example 3.16).

Example 3.17. Let $T = \mathbb{P} - \{2,3,5,6,9,12\}$. It is easy to check that T is subtractive but not additive. We claim that Π_n^T is not shellable (or sequentially Cohen-Macaulay) when $n = 16$. Indeed, if Π_n^T were shellable (sequentially Cohen-Macaulay) then every interval would be shellable (sequentially Cohen-Macaulay). In particular the interval $[x, \hat{1}]$, where

$$x = 1,2,3,4/5,6,7,8/9,10,11,12/13,14,15,16,$$

would be shellable (sequentially Cohen-Macaulay). But $[x,\hat{1}]$ is isomorphic to $\Pi_4^{\{1,2,4\}} \setminus \{\hat{0}\}$ which is not shellable or even sequentially Cohen-Macaulay because its order complex consists of disconnected components which aren't points.

Example 3.18. Now let $T = \mathbb{P} \setminus \{1,2,4,7\}$ which is 1-additive but not subtractive since $5+5-3 \notin T$. This example is discussed in [BW2, Example 7.3] where it is shown that for $n = 15$, Π_n^T is not shellable.

Example 3.19. Let $T = \{2,3\}$ which is neither additive nor subtractive. The poset Π_n^T is shellable for all n because it has length 2 when $n \geq 4$ and length 1 when $n = 2,3$.

We shall now explore some further properties of additive and subtractive sets that we will need in the next section. First we set some standard notation. Let $A, B \subseteq \mathbb{N}$ and $j \in \mathbb{N}$. Then $A+B$ denotes the set $\{a+b \mid a \in A, b \in B\}$; jA denotes the set $\{ja \mid a \in A\}$; $A+j$ denotes the set $A+\{j\}$; for $j \geq 1$, A^{+j} denotes the set $\underbrace{A+\cdots+A}_{j \text{ times}}$; and A^+ denotes the set $\cup_{j\geq 1} A^{+j}$.

Lemma 3.20. *Let $T \subseteq \mathbb{P}$ have minimum element k. Then T is subtractive if and only if*

$$T^{+(j+1)} = T + jk, \tag{3-1}$$

for all $j \geq 0$.

Proof. Equation (3-1) with $j = 1$ is simply a reformulation of subtractivity. Hence, we need only prove that if T is subtractive then (3-1) holds for all $j \geq 0$. We use induction on j. The cases $j = 0,1$ are trivial; so assume that $j \geq 2$ and that (3-1) holds for $j-1$. We have

$$T^{+(j+1)} = T + T^{+j} = T + T + (j-1)k = T + k + (j-1)k = T + jk. \qquad \square$$

Corollary 3.21. *Let T be subtractive with minimum element k. Then T is d-additive if and only if (1) $T + jk \subseteq T$ for all $j \in \mathbb{N}$ such that $d \mid j$ and (2) $(T + jk) \cap T = \emptyset$ for all $j \in \mathbb{N}$ such that $d \nmid j$.* □

Lemma 3.22. *Let T be d-additive and subtractive. Then for all $i \in \mathbb{P}$ and $r \in \mathbb{N}$,*

$$T^{+(i+rd)} \subseteq T^{+i}. \tag{3-2}$$

Moreover the sets $T, T^{+2}, \dots, T^{+d}$ form a partition of T^+ into d distinct blocks.

Proof. Let $k = \min T$. By Lemma 3.20 and Corollary 3.21,

$$T^{+(i+rd)} = T + (i + rd)k = T + rdk + ik \subseteq T + ik = T^{+i}.$$

It follows from (3-2) that $T^+ = \cup_{j=1}^{d} T^{+j}$. So we need only show that the sets T^{+j} are pairwise disjoint. Suppose $n \in T^{+i} \cap T^{+j}$, where $1 \le i \le j \le d$. Then by Lemma 3.20, $t + ik = n = t' + jk$, for some $t, t' \in T$. Consequently, $t = t' + (j - i)k$, which by d-additivity implies that $d \mid (j - i)$. Since $0 \le j - i \le d - 1$, we can conclude that $j = i$. □

Lemma 3.23. *Let T be d-additive and subtractive and let $n \in T^{+j}$, where $j \in [d]$. If $x \in \overline{\Pi_n^T}$ then $b(x)$, the number of blocks of x, satisfies*

$$b(x) \equiv j \mod d.$$

Proof. Since $x \in \overline{\Pi_n^T}$, we have $n \in T^{+b(x)}$. The result now follows from Lemma 3.22. □

Lemma 3.24. *Let T be d-additive and let $n \in T^+$. For all $x \in \Pi_n^T \setminus \{\hat{0}\}$,*

$$\ell(x, \hat{1}) = \ell(\Pi_{b(x)}^{1,d}) = \left\lceil \frac{b(x) - 1}{d} \right\rceil.$$

Proof. The first equation follows from Lemma 3.11 and the second is easy to see. □

Theorem 3.25. *Let T be d-additive and subtractive with minimum element k and let $n \in T^{+j}$, where $j \in [d]$. Then*

$$\ell(\Pi_n^T) = \max\{r \ge 0 \mid n - rdk \in T^{+j}\} + \begin{cases} 1 & \text{if } j = 1 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. For $\ell(\Pi_n^T) = 1$, the result is easy to see. So assume that $\ell(\Pi_n^T) > 1$. We shall use the obvious fact that

$$\ell(\Pi_n^T) = \max_{x \in \overline{\Pi_n^T}} \ell(x, \hat{1}) + 1.$$

For each $x \in \overline{\Pi_n^T}$, let $r(x) = (b(x) - j)/d$. By Lemma 3.23, $r(x) \in \mathbb{N}$. It follows from Lemma 3.24 that $\ell(x, \hat{1}) = r(x)$ for $j = 1$ and $\ell(x, \hat{1}) = r(x) + 1$ for $j \geq 2$. Hence

$$\ell(\Pi_n^T) = \max_{x \in \overline{\Pi_n^T}} r(x) + \begin{cases} 1 & \text{if } j = 1 \\ 2 & \text{otherwise.} \end{cases} \tag{3-3}$$

Since n is the sum of $b(x) = r(x)d + j$ elements of T, it follows that $n \in T^{+(r(x)d+j)}$, for any $x \in \overline{\Pi_n^T}$. Hence, by Lemma 3.20, $n - r(x)dk \in T^{+j}$ for all $x \in \overline{\Pi_n^T}$. Conversely, for any $r \geq 0$, such that $n - rdk \in T^{+j}$ there is an $x \in \overline{\Pi_n^T}$ such that $r(x) = r$. Namely, let x be the partition with rd blocks of size k and with remaining j blocks having sizes whose sum is equal to $n - rdk \in T^{+j}$. Consequently,

$$\max_{x \in \overline{\Pi_n^T}} r(x) = \max\{r \geq 0 \mid n - rdk \in T^{+j}\}.$$

Substituting this into (3-3) completes the proof. $\square$

For T d-additive and subtractive with minimum element k, define $\phi_T : T^+ \to \mathbb{N}$ by

$$\phi_T(n) = \max\{r \geq 0 \mid n - rdk \in T^{+j}\}, \tag{3-4}$$

where $j \in [d]$ is such that $n \in T^{+j}$. It follows from Lemma 3.22 that this map is well defined.

The *type* of a partition $x \in \Pi_n$, denoted $\lambda(x)$, is defined to be the integer partition of n whose parts are the block sizes of x. If λ is a partition of n, we say $|\lambda| = n$.

In the next result we express $m(x)$ (defined in Definition 2.1) in terms of the function ϕ_T and $\lambda(x)$.

Lemma 3.26. *Let T be d-additive and subtractive, let $n \in T^+$ and let $x \in \overline{\Pi_n^T}$ have type $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p)$. Then*

$$m(x) = \left\lceil \frac{p-1}{d} \right\rceil + 1 + \sum_{i=1}^{p} \phi_T(\lambda_i)$$

Proof. We use the fact that

$$m(x) = \ell(\hat{0}, x) + \ell(x, \hat{1}). \tag{3-5}$$

To compute the first length, note that the half open interval $(\hat{0}, x]$ is isomorphic to the direct product

$$\mathop{\times}_{i=1}^{p} (\Pi_{\lambda_i}^T \setminus \{\hat{0}\}).$$

It therefore follows from Theorem 3.25 that

$$\ell(\hat{0}, x) = 1 + \sum_{i=1}^{p} (\ell(\Pi_{\lambda_i}^T) - 1) = 1 + \sum_{i=1}^{p} \phi_T(\lambda_i),$$

since each $\lambda_i \in T$. We also have by Lemma 3.24, $\ell(x, \hat{1}) = \lceil \frac{p-1}{d} \rceil$. By substituting these expressions for the lengths into (3-5) we get the result. $\square$

For the sets T of Examples 3.8 and 3.6, we can give simple formulas for ϕ_T.

Lemma 3.27. *Let $k, d \in \mathbb{P}$ and $T = \{k + id \mid i \in \mathbb{N}\}$. Then for all $j \in \mathbb{P}$,*

$$T^{+j} = \{jk + id \mid i \in \mathbb{N}\}. \tag{3-6}$$

Moreover, each element $n \in T^+$ has a unique representation as $jk+id$, where $i \in \mathbb{N}$ and $j \in [d_0]$; and

$$\phi_T(n) = \phi_T(jk + id) = \left\lfloor \frac{i}{k_0} \right\rfloor,$$

where $d_0 = \frac{d}{\gcd(k,d)}$ and $k_0 = \frac{k}{\gcd(k,d)}$.

Proof. Equation (3-6) is an immediate consequence of Lemma 3.20.

Since T is d_0-additive and subtractive, by Lemmas 3.20 and 3.22, there is a unique $j \in [d_0]$ such that $n - (j-1)k \in T$. Clearly, there is a unique i such that $n - (j-1)k = k + id$. Hence each $n \in T^+$ has a unique representation as stated.

By the definition of ϕ_T given in (3-4), we have

$$\phi_T(jk + id) = \max\{r \mid jk + id - rd_0k \in T^{+j}\}.$$

By Lemma 3.20, $jk + id - rd_0k \in T^{+j}$ if and only if $jk + id - rd_0k - (j-1)k \in T$. But $jk + id - rd_0k - (j-1)k = k + id - rdk_0 = k + (i - rk_0)d$ is in T if and only if $i - rk_0 \in \mathbb{N}$. Hence $\phi_T(jk + id) = \max\{r \mid i - rk_0 \in \mathbb{N}\} = \left\lfloor \frac{i}{k_0} \right\rfloor$. □.

Corollary 3.28. *For $k, d \in \mathbb{P}$, let $T = \{kd, (k+1)d, (k+2)d, \dots\}$. Then for all $n \geq k$,*

$$\phi_T(nd) = \left\lfloor \frac{n}{k} \right\rfloor - 1. \quad \square$$

From any d-additive and subtractive set, where $d \geq 2$, one can construct other additive and subtractive sets as the following result indicates.

Proposition 3.29. *Let T be d-additive and subtractive. Then T^{+j} is $(\frac{d}{\gcd(d,j)})$-additive and subtractive for all $j \in \mathbb{P}$, and T^+ is 1-additive and subtractive.*

Proof. The proof is a straight forward application of Lemma 3.20 and Corollary 3.21. We leave the details to to the reader. □

Remark. One can see from (3-6) that if T is of the form given in Example 3.8, i.e., $T = \{k + id \mid i \in \mathbb{N}\}$, then taking T^{+j} gives no new examples. Indeed, the set T^{+j} is also of the form given in Example 3.8. However, taking T^+ does give a different example. It is precisely that of Example 3.10.

4. Representation of $\mathcal{S}_n$ on $WH_{r,m}((\Pi_n^T)^*)$.

A permutation σ in the symmetric group $\mathcal{S}_n$ acts on a partition $x \in \Pi_n$ by replacing the elements of each block of x by their images under σ. The induced subposet Π_n^T is invariant under this action. Hence Π_n^T is an $\mathcal{S}_n$-poset. It follows that homology and (r,m)-Whitney homology of Π_n^T are $\mathcal{S}_n$-modules. We compute the character of the $\mathcal{S}_n$-module $\tilde{H}_m(\Pi_n^T)$, when T is additive and subtractive, by first computing a generating function for the Frobenius characteristic of the $\mathcal{S}_n$-module $WH_{r,m}((\Pi_n^T)^*)$ and then applying Theorem 2.2.

Let $\left(\binom{T}{p}\right)$ denote the set of integer partitions with p parts all chosen from the set T. For $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p) \in \left(\binom{T}{p}\right)$, let

$$m(\lambda) = \left\lceil \frac{p-1}{d} \right\rceil + 1 + \sum_{i=1}^{p} \phi_T(\lambda_i), \tag{4-1}$$

where ϕ_T is defined in (3-4), and let

$$H_{\lambda,T} = \bigoplus_{x:\lambda(x)=\lambda} \tilde{H}(x,\hat{1}),$$

where $\tilde{H}(x,\hat{1})$ is the top homology of the interval $[x,\hat{1}]$ in $\Pi_{|\lambda|}^T$.

Proposition 4.1. *Suppose T is d-additive and $n \in T^{+j}$, where $j \in [d]$. Then for all $m \in \mathbb{P}$ and $r \in \mathbb{P}$ (and $r = 0$ when $j = 1$), $WH_{r,m}((\Pi_n^T)^*)$ decomposes into a direct sum of $\mathcal{S}_n$-modules as follows,*

$$WH_{r,m}((\Pi_n^T)^*) = \bigoplus_{\substack{\lambda \in \left(\binom{T}{p}\right) \\ |\lambda| = n \\ m(\lambda) = m}} H_{\lambda,T},$$

where

$$p = \begin{cases} rd+1 & \text{if } j = 1 \\ (r-1)d + j & \text{if } 2 \leq j \leq d. \end{cases}$$

Proof. For each $x \in \overline{\Pi_n^T}$, $[x,\hat{1}]$ has vanishing homology in all dimensions but dimension $\ell(x,\hat{1}) - 2$. Hence by Lemma 3.24, $\tilde{H}_{r-2}(x,\hat{1}) = 0$ if $r \neq \left\lceil \frac{b(x)-1}{d} \right\rceil$. Note that $r = \left\lceil \frac{b(x)-1}{d} \right\rceil$ if and only if $b(x) = rd+1$ or $b(x) = (r-1)d+i$ where $i = 2,3,\ldots,d$. Since by Lemma 3.23, $b(x) \equiv j \mod d$, we have $r = \left\lceil \frac{b(x)-1}{d} \right\rceil$ if and only $b(x) = p$. Hence $\tilde{H}_{r-2}(x,\hat{1}) = 0$ if $b(x) \neq p$. It follows that

$$WH_{r,m}((\Pi_n^T)^*) = \bigoplus_{\substack{x \in \overline{\Pi_n^T} \\ b(x) = p \\ m(x) = m}} \tilde{H}(x,\hat{1}).$$

The result now follows from Lemma 3.26. $\square$

We assume familiarity with the theory of symmetric functions and plethysm (cf. [M]). For any $\mathcal{S}_n$-module V, let $\operatorname{ch} V$ denote the Frobenius characteristic of V in the variables $x_1, x_2, \dots$. Let h_n denote the homogeneous symmetric function of degree n in the variables $x_1, x_2, \dots$. In [M], the plethysm of two symmetric functions f and g, where g has nonnegative integer coefficients, is computed by replacing the variables of f with the monomials of g. For this substitution to make sense g actually need not be symmetric. It can be any formal power series with nonnegative integer coefficients. Hence one can use the above substitution to define the plethysm of symmetric function f with any formal power series g having nonnegative integer coefficients and denote it by $f[g]$. (Plethysm can be defined for still more general g; see Section 6.)

We will need the following refinement and abstraction of results in [Su1]. For any partition $\lambda = (\lambda_1 \le \lambda_2 \le \dots \le \lambda_p)$ and variables $z_1, z_2, \dots$, let z_λ denote the product, $z_{\lambda_1} z_{\lambda_2} \cdots z_{\lambda_p}$.

Theorem 4.2. *Let T be d-additive and $p \in \mathbb{P}$. Then the generating function for the characteristic of $H_{\lambda,T}$ is given by*

$$\sum_{\lambda \in (\binom{T}{p})} \operatorname{ch} H_{\lambda,T}\ z_\lambda = \operatorname{ch} \tilde{H}(\Pi_p^{1,d}) \left[\sum_{i \in T} h_i z_i \right], \tag{4-2}$$

where the inner function of the plethysm, $\sum_{i \in T} h_i z_i$ is viewed as a formal power series in both $x_1, x_2, \dots$ and $z_1, z_2, \dots$.

The proof of Theorem 4.2 uses many of the ideas of the proofs of the specialized results in [Su1]. The proof appears in the next section. Now we apply the result to computing the Frobenius characteristic of the homology representations of Π_n^T.

Corollary 4.3. *Let T be d-additive and subtractive. Then for $j \in [d]$ and $r \in \mathbb{P}$ (or $j = 1$ and $r = 0$),*

$$\sum_{\substack{m > r \\ n \in T^{+j}}} \operatorname{ch} WH_{r,m}((\Pi_n^T)^*)\, u^n v^m = v^{r+1}\ \operatorname{ch} \tilde{H}(\Pi_p^{1,d}) \left[\sum_{i \in T} h_i\, u^i v^{\phi_T(i)} \right],$$

where $p = rd + 1$ if $j = 1$ and $p = (r-1)d + j$ if $2 \le j \le d$.

Proof. Set $z_i = u^i v^{\phi_T(i)}$ in (4-2). Then by (4-1), $z_\lambda = u^{|\lambda|} v^{m(\lambda) - r - 1}$. The result now follows from Proposition 4.1, Theorem 4.2 and the linearity of ch. □

We finally arrive at our main result.

Theorem 4.4. *Let T be d-additive and subtractive. Then*

$$\sum_{\substack{m \in \mathbb{Z} \\ n \in T}} (-1)^m \operatorname{ch} \tilde{H}_{m-1}(\Pi_n^T)\, u^n v^m = \sum_{r \ge 0} (-v)^r\ \operatorname{ch} \tilde{H}(\Pi_{rd+1}^{1,d}) \left[\sum_{i \in T} h_i\, u^i v^{\phi_T(i)} \right], \tag{4-3}$$

and for $j = 2, 3, \ldots, d$,

(4-4)

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T^{+j}}} (-1)^m \operatorname{ch} \tilde{H}_m(\Pi_n^T)\, u^n v^m = \sum_{r\geq 0}(-v)^r \ \operatorname{ch} \tilde{H}(\Pi^{1,d}_{rd+j}) \left[\sum_{i\in T} h_i\, u^i v^{\phi_T(i)}\right] - \sum_{n\in T^{+j}} h_n\, u^n v^{\phi_T(n)}.$$

Proof. By Corollary 3.15, the dual of Π_n^T is semipure and has the vanishing homology property. It follows from Theorem 2.2 and the linearity of ch that

(4-5)

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T^{+j}}} (-1)^m \operatorname{ch} \tilde{H}_{m-2}(\Pi_n^T)\, u^n v^m = \sum_{\substack{m\in\mathbb{Z}\\ n\in T^{+j}}} \sum_{r=0}^{m-1} (-1)^{r+1} \operatorname{ch} WH_{r,m}((\Pi_n^T)^*)\, u^n v^m.$$

Equation (4-3) now follows from Corollary 4.3.

Now let $j = 2, \ldots, d$. Then (4-5) and Corollary 4.3 yield,

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T^{+j}}} (-1)^m \operatorname{ch} \tilde{H}_{m-2}(\Pi_n^T)\, u^n v^m = - \sum_{\substack{m\in\mathbb{Z}\\ n\in T^{+j}}} \operatorname{ch} WH_{0,m}((\Pi_n^T)^*) u^n v^m + \sum_{r\geq 1}(-v)^{r+1} \ \operatorname{ch} \tilde{H}(\Pi^{1,d}_{(r-1)d+j}) \left[\sum_{i\in T} h_i\, u^i v^{\phi_T(i)}\right].$$

Equation (4-4) now follows from the fact that $\operatorname{ch} WH_{0,m}(\Pi_n^T) = h_n$ if $m = \ell(\Pi_n^T) = \phi_T(n) + 2$ (by Lemma 3.25) and is 0 otherwise. $\square$

Corollary 4.5. *Fix* $k \geq 1$ *and* $d \geq 1$*. Let* $k_0 = \frac{k}{\gcd(k,d)}$ *and* $d_0 = \frac{d}{\gcd(k,d)}$*. Then*

(4-6)

$$\sum_{\substack{m\in\mathbb{Z}\\ i\geq 0}} (-1)^m \operatorname{ch} \tilde{H}_{m-1}(\Pi^{(k,d)}_{k+id})\, u^{k+id} v^m = \sum_{r\geq 0}(-v)^r \ \operatorname{ch} \tilde{H}(\Pi^{1,d_0}_{1+rd_0}) \left[\sum_{i\geq 0} h_{k+id}\, u^{k+id} v^{\lfloor \frac{i}{k_0}\rfloor}\right].$$

and for $j = 2, 3, \ldots, d_0$,

(4-7)

$$\sum_{\substack{m\in\mathbb{Z}\\ i\geq 0}} (-1)^m \operatorname{ch} \tilde{H}_m(\Pi^{(k,d)}_{jk+id})\, u^{jk+id} v^m = \sum_{r\geq 0}(-v)^r \ \operatorname{ch} \tilde{H}(\Pi^{1,d_0}_{j+rd_0}) \left[\sum_{i\geq 0} h_{k+id}\, u^{k+id} v^{\lfloor \frac{i}{k_0}\rfloor}\right] - \sum_{i\geq 0} h_{jk+id}\, u^{jk+id} v^{\lfloor \frac{i}{k_0}\rfloor}.$$

Proof. Use Lemma 3.27. $\square$

Corollary 4.6. *For $d \geq 1$,*

$$h_1 = \sum_{r\geq 0}(-v)^r \ \mathrm{ch}\,\tilde{H}(\Pi^{1,d}_{1+rd}) \left[\sum_{i\geq 0} h_{1+id}\ v^i\right]. \tag{4-8}$$

and for $j = 2, 3, \dots, d$,

$$\sum_{i\geq 0} h_{j+id}\ v^i = \sum_{r\geq 0}(-v)^r \ \mathrm{ch}\,\tilde{H}(\Pi^{1,d}_{j+rd}) \left[\sum_{i\geq 0} h_{1+id}\ v^i\right]. \tag{4-9}$$

.

Proof. To obtain (4-8) and (4-9) set $k = 1$ and $u = 1$ in (4-6) and (4-7), respectively, and observe that $\Pi^{(1,d)}_{j+id}$ is acyclic except when $j = 1$ and $i = 0$, in which case $|\Pi^{(1,d)}_1| = 2$. $\square$

Remark. By setting $v = 1$ in (4-8) and (4-9) we obtain the original formulas of Calderbank, Hanlon, and Robinson for the homology of the 1 mod d partition poset [CHR]. In particular, (4-8) says that $\sum_{i\geq 0}(-1)^i \ \mathrm{ch}\,\tilde{H}(\Pi^{1,d}_{1+id})$ is the plethystic inverse of $\sum_{i\geq 0} h_{1+id}$. At first glance, it may appear that (4-8) and (4-9) are more general than the Calderbank-Hanlon-Robinson formulas. However it is easy to derive (4-8) and (4-9) directly from these formulas (cf. Proposition 6.1).

Corollary 4.7. *For $k \geq 1$,*

$$\sum_{\substack{m\in\mathbb{Z}\\ n\geq k}}(-1)^m \ \mathrm{ch}\,\tilde{H}_{m-2}(\Pi^{\geq k}_n)\, u^n v^m = \sum_{r\geq 1}(-1)^r \ \mathrm{ch}\,\tilde{H}(\Pi_r) \left[\sum_{i\geq k} h_i\, u^i v^{\lfloor \frac{i}{k}\rfloor}\right]. \tag{4-10}$$

Proof. Set $d = 1$ in (4-6) or use Corollary 3.28. $\square$

Remark. By setting $u = v = 1$ in (4-6) (resp., (4-10)) we obtain Sundaram's formula [Su2] for the alternating sum of homology of the k mod d partition poset (resp., at least k partition lattice). By also setting $k = d$ in Corollary 4.5, we obtain the original formula of Calderbank, Hanlon and Robinson [CHR] for the homology of the d-divisible partition lattice.

One can obtain formulas for the Betti numbers of Π^T_n by extracting the square free terms in the coefficient of $u^n v^m$ in (4-3) and (4-4). In doing so, we obtain the following formula for the Betti numbers when T is 1-additive and subtractive. (A combinatorial description of these Betti numbers as well as a computation of the restriction of the action of $\mathcal{S}_n$ to $\mathcal{S}_{n-1}$ is given in [SaW].)

Corollary 4.8. *Let $S_T(n,r,m)$ be the number of r block partitions $x \in \Pi_n^T$ such that $m(x) = m$. If T is 1-additive and subtractive and $n \in T$ then*

$$\beta_{m-2}(\Pi_n^T) = \sum_{r \geq 1} (-1)^{m+r} (r-1)!\, S_T(n,r,m). \quad \square$$

5. Wreath product modules.

In this section we present a general result on $\mathcal{S}_n$-modules and plethysm from which Theorem 4.2 follows immediately. First we need to review some basic definitions and results from the representation theory of the symmetric group.

For partitions ν and μ such that $\mu \subseteq \nu$, $s_{\nu/\mu}$ denotes the Schur function of shape ν/μ and $S^{\nu/\mu}$ denotes the Specht module of shape ν/μ. Recall that $\operatorname{ch} S^{\nu/\mu} = s_{\nu/\mu}$.

Proposition 5.1. *(cf. [M]) Let ν be a nonempty integer partition and let f_i, $i \geq 1$, be a formal power series with nonnegative integer coefficients such that the sum $\sum_i f_i$ exists as a formal power series (e.g., if the monomial sets of the f_i are pairwise disjoint). Then*

$$s_\nu\left[\sum_{i\geq 1} f_i\right] = \sum_{\substack{\emptyset=\mu_0\subseteq\mu_1\subseteq\cdots\subseteq\mu_{j-1}\subsetneq\mu_j=\nu \\ j\geq 1}} \prod_{i=1}^{j} s_{\mu_i/\mu_{i-1}}\,[f_i]. \quad \square$$

Proposition 5.2. *(cf. [JK]) Let $\nu \vdash p$ and let $(m_1, m_2, \ldots, m_t)$ be a sequence of nonnegative integers whose sum is p. Then the restriction of the $\mathcal{S}_p$-module S^ν to the Young subgroup $\times_{i=1}^t \mathcal{S}_{m_i}$ decomposes into a direct sum of outer tensor products of $\mathcal{S}_{m_i}$-modules as follows,*

$$S^\nu\downarrow^{\mathcal{S}_r}_{\times \mathcal{S}_{m_i}} = \bigoplus_{\substack{\emptyset=\mu_0\subseteq\mu_1\subseteq\cdots\subseteq\mu_t=\nu \\ |\mu_i|-|\mu_{i-1}|=m_i}} \bigotimes_{i=1}^{t} S^{\mu_i/\mu_{i-1}}. \quad \square$$

It is well-known (cf. [M]) that the plethysm of the Frobenius characteristic of an $\mathcal{S}_m$-module with the Frobenius characteristic of an S_n-module is the Frobenius characteristic of the induction of a certain wreath product module. We shall closely follow the exposition given in [Su3] (see also [JK]) in describing this wreath product module.

Given a finite set $A = \{a_1 < a_2 < \cdots < a_n\}$, let $\mathcal{S}_A$ be the set of permutations of the set A. We shall view a permutation in $\mathcal{S}_A$ as a word whose letters come from A. If $\sigma \in S_n$, then let σ^A denote the word $a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}$. We shall view an element of the Young subgroup $\mathcal{S}_k \times \mathcal{S}_{n-k}$ of $\mathcal{S}_n$ as the concatenation $\alpha \star \beta$ of words $\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_{\{k+1,\ldots,n\}}$.

The *wreath product* of $\mathcal{S}_m$ and $\mathcal{S}_n$, denoted by $\mathcal{S}_m[\mathcal{S}_n]$, is defined to be the normalizer of the Young subgroup $\underbrace{\mathcal{S}_n \times \cdots \times \mathcal{S}_n}_{m \text{ times}}$ in $\mathcal{S}_{mn}$. Each $\sigma \in \mathcal{S}_m[\mathcal{S}_n]$, corresponds bijectively to an $(m+1)$-tuple $(\alpha_1, \alpha_2, \dots, \alpha_m; \tau)$ such that $\alpha_i \in \mathcal{S}_n$ and $\tau \in \mathcal{S}_m$. The correspondence is given by

$$(\alpha_1, \dots, \alpha_m; \tau) \quad \leftrightarrow \quad \sigma = \alpha_{\tau(1)}^{A_{\tau(1)}} \star \dots \star \alpha_{\tau(m)}^{A_{\tau(m)}},$$

where $A_i = [in] \backslash [(i-1)n]$. From now on we shall identify σ with $(\alpha_1, \alpha_2, \dots, \alpha_m; \tau)$. The following proposition is easy to check (cf. multiplication rule given in [Su3] or [JK]).

Proposition 5.3. *The map* $(\alpha_1, \alpha_2, \dots, \alpha_m; \tau) \mapsto \tau$ *is a homomorphism from* $\mathcal{S}_m[\mathcal{S}_n]$ *onto* $\mathcal{S}_m$. $\square$

Now let V be an $\mathcal{S}_m$-module and W be an $\mathcal{S}_n$-module. Then the wreath product of V with W, denoted $V[W]$, is the inner tensor product of two $\mathcal{S}_m[\mathcal{S}_n]$-modules:

$$V[W] = \widetilde{W^{\otimes m}} \otimes \hat{V},$$

where $\widetilde{W^{\otimes m}}$ is the vector space $W^{\otimes m}$ with $\mathcal{S}_m[\mathcal{S}_n]$ action given by

$$(\alpha_1, \dots, \alpha_m; \tau)(w_1 \otimes \dots \otimes w_m) = \alpha_1 w_{\tau^{-1}(1)} \otimes \dots \otimes \alpha_m w_{\tau^{-1}(m)}, \tag{5-1}$$

and $\hat{V}$ is the pullback of the representation of $\mathcal{S}_m$ on V to $\mathcal{S}_m[\mathcal{S}_n]$ through the homomorphism given in Proposition 5.3. That is, $\hat{V}$ is the representation of $\mathcal{S}_m[\mathcal{S}_n]$ on V defined by

$$(\alpha_1, \dots, \alpha_m; \tau)v = \tau v.$$

Note that we are using the same symbol $\otimes$ for both inner tensor product and outer tensor product.

Proposition 5.4. *(*[M]*) Let* V *be an* $\mathcal{S}_m$*-module and* W *an* $\mathcal{S}_n$*-module. Then*

$$\mathrm{ch}\,(V \otimes W)\big\uparrow_{\mathcal{S}_m \times \mathcal{S}_n}^{\mathcal{S}_{m+n}} = \mathrm{ch}\, V \ \mathrm{ch}\, W$$

and

$$\mathrm{ch}\, V[W]\big\uparrow_{\mathcal{S}_m[\mathcal{S}_n]}^{S_{mn}} = \mathrm{ch}\, V[\mathrm{ch}\, W]. \quad \square$$

Our aim now is to present an abstraction and refinement of results of Sundaram [Su1] dealing with certain specific representations of direct products of wreath products. Let V be an $\mathcal{S}_p$-module and let W_i be an $\mathcal{S}_i$-module for each $i = 1, 2, \dots$. Also let λ be a partition with p parts and let $m_i(\lambda)$ denote the multiplicity of i in λ. We form a $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$-module

$$\bigotimes_i \widetilde{W_i^{\otimes m_i(\lambda)}} \ \otimes \ V_\lambda,$$

where $\bigotimes_i \widetilde{W_i^{\otimes m_i(\lambda)}}$ is the outer tensor product of the $\mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$-modules $\widetilde{W_i^{\otimes m_i(\lambda)}}$, defined as in (5-1), and V_λ is the pullback of $V\big\downarrow_{\times_i \mathcal{S}_{m_i(\lambda)}}^{\mathcal{S}_p}$ to $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$ through the product of the canonical homomorphisms $\mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i] \to \mathcal{S}_{m_i(\lambda)}$ given in Proposition 5.3.

Theorem 5.5. *Let V be an $\mathcal{S}_p$-module and let W_i be an $\mathcal{S}_i$-module for each $i = 1, 2, \ldots$. Then for any $T \subseteq \mathbb{P}$,*

$$\sum_{\lambda \in \left(\binom{T}{p}\right)} \operatorname{ch}\left(\bigotimes_i \widetilde{W_i^{\otimes m_i(\lambda)}} \;\otimes\; V_\lambda\right)\Big\uparrow_{\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{|\lambda|}} z_\lambda = \operatorname{ch} V\left[\sum_{i \in T} \operatorname{ch} W_i \; z_i\right].$$

Proof. We can assume that V is irreducible since restrictions, pullbacks, inductions, and ch are linear, inner tensor products are bilinear, and plethysm is linear in the outer function. So assume that V is the irreducible $\mathcal{S}_p$-module S^ν where $\nu \vdash p$. Let t be the maximum part of λ. By Proposition 5.2, we have

$$\begin{aligned}
\bigotimes_{i=1}^{t} \widetilde{W_i^{\otimes m_i(\lambda)}} \;\otimes\; V_\lambda &= \bigotimes_{i=1}^{t} \widetilde{W_i^{\otimes m_i(\lambda)}} \;\otimes \bigoplus_{\substack{\emptyset = \mu_0 \subseteq \cdots \subseteq \mu_t = \nu \\ |\mu_i| - |\mu_{i-1}| = m_i(\lambda)}} \bigotimes_{i=1}^{t} \widehat{S^{\mu_i/\mu_{i-1}}} \\
&= \bigoplus_{\mu} \left(\bigotimes_{i=1}^{t} \widetilde{W_i^{\otimes m_i(\lambda)}} \;\otimes\; \bigotimes_{i=1}^{t} \widehat{S^{\mu_i/\mu_{i-1}}}\right) \\
&= \bigoplus_{\mu} \bigotimes_{i=1}^{t} \left(\widetilde{W_i^{\otimes m_i(\lambda)}} \;\otimes\; \widehat{S^{\mu_i/\mu_{i-1}}}\right) \\
&= \bigoplus_{\mu} \bigotimes_{i=1}^{t} S^{\mu_i/\mu_{i-1}}\left[W_i\right].
\end{aligned}$$

Taking Frobenius characteristic of the induced representation yields

$$\begin{aligned}
\operatorname{ch}\left(\bigotimes_{i=1}^{t} \widetilde{W_i^{\otimes m_i(\lambda)}} \;\otimes\; V_\lambda\right)\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{|\lambda|}} &= \operatorname{ch} \bigoplus_{\mu} \left(\bigotimes_i S^{\mu_i/\mu_{i-1}}\left[W_i\right]\right)\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{|\lambda|}} \\
&= \sum_{\mu} \operatorname{ch}\left(\bigotimes_i S^{\mu_i/\mu_{i-1}}\left[W_i\right]\right)\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{|\lambda|}}.
\end{aligned}$$

By transitivity of induction and Proposition 5.4, we have

$$\begin{aligned}
\operatorname{ch}\left(\bigotimes_i S^{\mu_i/\mu_{i-1}}\left[W_i\right]\right)\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{|\lambda|}} &= \operatorname{ch}\left(\bigotimes_i S^{\mu_i/\mu_{i-1}}\left[W_i\right]\right)\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\times \mathcal{S}_{m_i(\lambda)i}}\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)i}}^{\mathcal{S}_{|\lambda|}} \\
&= \operatorname{ch}\left(\bigotimes_i \left(S^{\mu_i/\mu_{i-1}}\left[W_i\right]\right)\Big\uparrow_{\mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{m_i(\lambda)i}}\right)\Bigg\uparrow_{\times \mathcal{S}_{m_i(\lambda)i}}^{\mathcal{S}_{|\lambda|}} \\
&= \prod_i \operatorname{ch}\left(S^{\mu_i/\mu_{i-1}}\left[W_i\right]\right)\Big\uparrow_{\mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{m_i(\lambda)i}} \\
&= \prod_i s_{\mu_i/\mu_{i-1}}\left[\operatorname{ch} W_i\right].
\end{aligned}$$

Since $s_{\mu_i/\mu_{i-1}}[\mathrm{ch}\, W_i z_i] = s_{\mu_i/\mu_{i-1}}[\mathrm{ch}\, W_i]\ \ z_i^{|\mu_i|-|\mu_{i-1}|}$, we now have

(5-2)
$$\sum_{\lambda\in\left(\binom{T}{p}\right)} \mathrm{ch}(\widetilde{\bigotimes_i W_i^{\otimes m_i(\lambda)}\ \otimes V_\lambda})\uparrow^{\mathcal{S}_{|\lambda|}}_{\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}\ \ z_\lambda$$
$$= \sum_{\lambda\in\left(\binom{T}{p}\right)} \sum_{\substack{\emptyset\subseteq\mu_0\subseteq\cdots\subseteq\mu_t=\nu\\ |\mu_i|-|\mu_{i-1}|=m_i(\lambda)}} \prod_i s_{\mu_i/\mu_{i-1}}[\mathrm{ch}\, W_i z_i]\,.$$

Since $s_{\mu_i/\mu_{i-1}} = s_\emptyset = 1$ if $i\notin T$, we can rewrite the right hand side of (5-2) as

$$\sum_{\lambda\in\left(\binom{T}{p}\right)} \sum_{\substack{\emptyset\subseteq\mu_0\subseteq\cdots\subseteq\mu_j=\nu\\ |\mu_i|-|\mu_{i-1}|=m_{t_i}(\lambda)}} \prod_i s_{\mu_i/\mu_{i-1}}[\mathrm{ch}\, W_{t_i} z_{t_i}]$$
$$= \sum_{\emptyset\subseteq\mu_0\subseteq\cdots\subseteq\mu_{j-1}\subsetneq\mu_j=\nu} \prod_i s_{\mu_i/\mu_{i-1}}[\mathrm{ch}\, W_{t_i} z_{t_i}]\,,$$

where $T=\{t_1<t_2<\dots\}$ and j is such that t_j is the maximum part of λ (i.e. $t_j=t$). By Proposition 5.1, the right hand side is precisely $s_\nu\left[\sum_{i\in T}\mathrm{ch}\, W_i z_i\right]$. □

Our final goal is to use Theorem 5.5 to prove Theorem 4.2. To accomplish this, we first need to choose a canonical set partition of type λ, for each integer partition $\lambda=(\lambda_1\le\cdots\le\lambda_p)\vdash n$, . The canonical set partition of type λ, denoted x_λ, is defined to be the partition with blocks $B_1,\dots,B_p$, where $B_i=[\sum_{j=1}^i\lambda_j]\backslash[\sum_{j=1}^{i-1}\lambda_j]$ for all $i=2,\dots,p$. Note that x_λ is the partition whose stabilizer is $\times_i\mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$.

The following result is extracted from [Su1, proof of Theorem 1.4].

Proposition 5.6. *Let T be d-additive and $\lambda\in\left(\binom{T}{p}\right)$. Then the interval $[x_\lambda,\hat{1}]$ in $\Pi^T_{|\lambda|}$ is an $\times_i\mathcal{S}_{m_i(\lambda)}[S_i]$-poset. Moreover, the following $\times_i\mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$-module isomorphism holds*

$$\tilde{H}(x_\lambda,\hat{1}) = \bigotimes_i \widetilde{(1_{\mathcal{S}_i})^{\otimes m_i(\lambda)}}\ \ \bigotimes\ \tilde{H}(\Pi_p^{1,d})_\lambda, \tag{5-3}$$

where $1_{\mathcal{S}_i}$ is the trivial $\mathcal{S}_i$-module.

Proof. Since $\times_i\mathcal{S}_{m_i(\lambda)}[S_i]$ is the stabilizer of x_λ, it follows that $[x_\lambda,\hat{1}]$ is a $\times_i\mathcal{S}_{m_i(\lambda)}[S_i]$-poset.

To verify (5-3), we need a simple observation. If P and Q are two G-posets and $f:P\to Q$ is a poset isomorphism that commutes with the action of G then we say that f is a G-poset isomorphism. A G-poset isomorphism from P to Q induces a G-module isomorphism from $\tilde{H}_r(P)$ to $\tilde{H}_r(Q)$ for each r.

Since $\widetilde{\bigotimes_i (1_{\mathcal{S}_i})^{\otimes m_i(\lambda)}}$ is the trivial $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$-module, (5-3) is equivalent to the $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$-module isomorphism

$$\tilde{H}(x_\lambda, \hat{1}) = \tilde{H}(\Pi_p^{1,d})_\lambda.$$

This isomorphism follows from the above observation and the fact that the $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$-poset $[x_\lambda, \hat{1}]$ is isomorphic to $\Pi_p^{1,d}$ under the pullback action of $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$. Indeed, the isomorphism that replaces each B_i with i in each partition $y \geq x_\lambda$ clearly commutes with the action of $\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]$. $\square$

Proof of Theorem 4.2. Since

$$H_{\lambda,T} = \tilde{H}(x_\lambda, \hat{1}) \Big\uparrow_{\times_i \mathcal{S}_{m_i(\lambda)}[\mathcal{S}_i]}^{\mathcal{S}_{|\lambda|}},$$

the result is an immediate consequence of Theorem 5.5, Proposition 5.6 and the fact that $h_i = \operatorname{ch} 1_{\mathcal{S}_i}$. $\square$

6. Related results.

In this section we derive some identities involving the characteristic of the homology $\mathcal{S}_n$-modules of Π_n^T as easy consequences of the formulas in Section 4. These identities turn out to be precisely the identities that one gets by applying Theorem 2.8 (which involves the variant form of (r, m)-Whitney homology) to Π_n^T. We also discuss the connection between the homology of Π_n^T and subspace arrangements. We use the formulas of Section 4 to compute the representation of $\mathcal{S}_n$ on the cohomology of the complement of complexified subspace arrangements whose intersection lattice is of the form Π_n^T.

Let g be a formal power series with integer coefficients and let g^+ and g^- be formal power series with nonnegative coefficients such that $g = g^+ - g^-$. If ν is any partition then the plethysm of the Schur function s_ν with g is given by

$$s_\nu[g] = \sum_{\emptyset \subseteq \mu \subseteq \nu} (-1)^{|\mu|} s_{\mu'}[g^-] s_{\nu/\mu}[g^+],$$

where μ' denotes conjugate shape of μ. Now if f is any symmetric function, the plethysm of f with g is obtained by expressing f in terms of Schur functions and extending linearly.

Proposition 6.1. *For $d \geq 1$,*

$$h_1 = \sum_{i \geq 0} v^i \; h_{1+id} \left[\sum_{r \geq 0} (-v)^r \; \operatorname{ch} \tilde{H}(\Pi_{1+rd}^{1,d}) \right]. \tag{6-1}$$

and for $j = 2, 3, \ldots, d$,

$$\sum_{r \geq 0} (-v)^r \; \operatorname{ch} \tilde{H}(\Pi_{j+rd}^{1,d}) = \sum_{i \geq 0} v^i \; h_{j+id} \left[\sum_{r \geq 0} (-v)^r \; \operatorname{ch} \tilde{H}(\Pi_{1+rd}^{1,d}) \right]. \tag{6-2}$$

Proof. We show that (6-1) and (6-2) are equivalent to the following formulas of Calderbank, Hanlon and Robinson,

$$h_1 = \sum_{i\geq 0} h_{1+id}\left[\sum_{r\geq 0}(-1)^r \operatorname{ch}\tilde{H}(\Pi^{1,d}_{1+rd})\right]. \tag{6-3}$$

and for $j = 2, 3, \ldots, d$,

$$\sum_{r\geq 0}(-1)^r \operatorname{ch}\tilde{H}(\Pi^{1,d}_{j+rd}) = \sum_{i\geq 0} h_{j+id}\left[\sum_{r\geq 0}(-1)^r \operatorname{ch}\tilde{H}(\Pi^{1,d}_{1+rd})\right]. \tag{6-4}$$

If we equate terms of like degree on both sides of (6-3) and (6-4) we get

$$h_1\, w = \sum_{i\geq 0} h_{1+id}\left[\sum_{r\geq 0}(-1)^r w^{1+rd} \operatorname{ch}\tilde{H}(\Pi^{1,d}_{1+rd})\right].$$

and for $j = 2, 3, \ldots, d$,

$$\sum_{r\geq 0}(-1)^r w^{j+rd} \operatorname{ch}\tilde{H}(\Pi^{1,d}_{j+rd}) = \sum_{i\geq 0} h_{j+id}\left[\sum_{r\geq 0}(-1)^r w^{1+rd} \operatorname{ch}\tilde{H}(\Pi^{1,d}_{1+rd})\right].$$

By pulling w^1 through the plethysm the right hand side of both equations becomes

$$\sum_{i\geq 0} w^{j+id}\, h_{j+id}\left[\sum_{r\geq 0}(-1)^r w^{rd} \operatorname{ch}\tilde{H}(\Pi^{1,d}_{1+rd})\right],$$

for $j = 1, 2, \ldots, d$. Now divide both sides by w^j and replace w^d by v to obtain (6-1) and (6-2). □

Theorem 6.2. *Let T be d-additive and subtractive. Then*

$$\sum_{i\geq 0} v^i\, h_{id+1}\left[\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \operatorname{ch}\tilde{H}_{m-1}(\Pi^T_n)\, u^n(-v)^m\right] = \sum_{n\in T} h_n\, u^n v^{\phi_T(n)} \tag{6-5}$$

and for $j = 2, 3, \ldots, d$,

(6-6)

$$\sum_{r\geq 0}(-v)^r \operatorname{ch}\tilde{H}(\Pi^{1,d}_{rd+j})\left[\sum_{i\in T} h_i\, u^i v^{\phi_T(i)}\right] = \sum_{i\geq 0} v^i\, h_{id+j}\left[\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \operatorname{ch}\tilde{H}_{m-1}(\Pi^T_n)\, u^n(-v)^m\right]$$

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T^{+j}}} \operatorname{ch}\tilde{H}_m(\Pi^T_n)\, u^n(-v)^m = \sum_{i\geq 0} v^i\, h_{id+j}\left[\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \operatorname{ch}\tilde{H}_{m-1}(\Pi^T_n)\, u^n(-v)^m\right] - \sum_{n\in T^{+j}} h_n\, u^n v^{\phi_T(n)}. \tag{6-7}$$

Proof. To prove (6-5) take the plethysm of $\sum_{i\geq 0} v^i\, h_{1+id}$ with both sides of (4-3). Equation (6-5) then follows from associativity and (6-1).

To prove (6-6) take the plethysm of $\sum_{i\geq 0} v^i\, h_{j+id}$ with both sides of (4-3). Equation (6-6) follows from associativity and (6-2).

To prove (6-7) substitute (6-6) into (4-4). □

Remark. By letting $T = \{k + id \mid i \in \mathbb{N}\}$ and setting $u = v = 1$, equations (6-5) and (6-7) reduce to equations (4.5) and (4.6) of [Su2].

Equations (6-5) and (6-7) are precisely what we get when we apply Theorem 2.8 to Π_n^T. To see this we need the following result.

Lemma 6.3. *For fixed $r \geq 1$ and $T \subseteq \mathbb{P}$,*

$$\text{(6-8)} \qquad \sum_{\substack{m\in\mathbb{Z}\\ n\in T^+\\ x\in\Pi_n^T\setminus\{\hat{0}\}\\ b(x)=r}} \mathrm{ch}\, \tilde{H}_{m-1}(\hat{0}, x)\, u^n(-v)^m = h_r \left[\sum_{\substack{m\in Z\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T) u^n(-v)^m\right],$$

where $b(x)$ is the number of blocks of x.

Sketch of Proof. Let $B_1, B_2, \ldots, B_r$ be the blocks of $x \in \Pi_n^T \setminus \{\hat{0}\}$ and let G_x be the stabilizer of x in $\mathcal{S}_n$. Then the G_x-poset $[\hat{0}, x]$ is G_x-isomorphic to the reduced product $\{\hat{0}\}\cup\times_{i=1}^{r}(\Pi_{B_i}^T\setminus\{\hat{0}\})$, where $\Pi_{B_i}^T$ is the poset of partitions of the set B_i with block sizes in T. In [SWe2, Theorem 1.1 (ii)], a description of the representation of a wreath product $\mathcal{S}_i[G]$ on the homology of a reduced product of i copies of a G-poset is given. By applying this reduced product result, one can derive (6-8) similarly to the way in which Corollary 4.3 was derived. □

Remark. By setting $u = v = 1$, equation (6-8) reduces to Theorem 4.1 of [Su 2].

We shall now describe the steps involved in deriving (6-5) from Theorem 2.8. First rewrite (6-5) as

(6-9)

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T)\, u^n(-v)^m = -\sum_{i\geq 1} v^i\, h_{id+1}\left[\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T)\, u^n(-v)^m\right] + \sum_{n\in T} h_n\, u^n v^{\phi_T(n)}.$$

By Lemma 6.3 we have for $i \geq 1$,
(6-10)

$$v^i\, h_{id+1}\left[\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T)\, u^n(-v)^m\right] = \sum_{\substack{m> i\\ n\in T}} (-1)^i W H^{\#}_{m-i,m}(\Pi_n^T) u^n(-v)^{m-1}.$$

Next observe that

$$\sum_{n\in T} h_n\, u^n v^{\phi_T(n)} = \sum_{\substack{m\in\mathbb{Z}\\ n\in T}} WH^{\#}_{0,m}(\Pi_n^T)u^n v^{m-1}. \tag{6-11}$$

From Theorem 2.8 we have

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T)u^n(-v)^m = \sum_{\substack{r,m\in\mathbb{Z}\\ n\in T}} (-1)^r\, \mathrm{ch}\, WH^{\#}_{r,m}(\Pi_n^T)u^n v^{m-1}.$$

By plugging (6-10) and (6-11) into this equation we obtain (6-9). Equation (6-7) is obtained in a similar manner using the fact that for $i\ge 0$,

$$v^i\, h_{id+j}\left[\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T)\, u^n(-v)^m\right] = \sum_{\substack{m>i+1\\ n\in T^{+j}}} (-1)^i WH^{\#}_{m-i-1,m}(\Pi_n^T)u^n(-v)^{m-2},$$

which also can be proved using Lemma 6.3.

When T is 1-additive, Π_n^T takes on the added significance of being the intersection lattice of a subspace arrangement. The connection between subspace arrangements and restricted block size partition posets was first considered in work of Björner, Lovász, and Yao [BLY] on a complexity theory problem. It has been further studied in [BL], [BWe], [L], [SWe1]. For a survey of recent developments in the theory of subspace arrangements see [Bj2].

For each $\pi\in\Pi_n$, let ℓ_π be the linear subspace of $\mathbb{C}^n$ consisting of all points $(x_1,x_2,\dots,x_n)$ such that $x_i=x_j$ whenever i and j are in the same block of π. Then any 1-additive set T, where $1\notin T$, determines the complexified subspace arrangement $\mathcal{A}^{\mathbb{C}}_{n,T}=\{\ell_\pi\mid \pi\in\Pi_n^T\setminus\hat{0}\}$ whose lattice of intersections is Π_n^T. Let $V^{\mathbb{C}}_{n,T}$ be the union $\cup_{\ell\in\mathcal{A}_{n,T}}\ell$ of the arrangement and let $M^{\mathbb{C}}_{n,T}$ be the complement $\mathbb{C}^n - V_{n,T}$. By the Goresky-MacPherson formula [GM], an immediate consequence of the shellability of Π_n^T, when T is 1-additive and subtractive, is that the manifold $M_{n,T}$ has free integral cohomology. Another consequence, which follows from the a result of Ziegler and Živaljević [ZZ] is that the link $V^{\mathbb{C}}_{n,T}\cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{C}^n$, has the homotopy type of a wedge of spheres. These two consequences are true for the real arrangements as well.

An equivariant version of the Goresky-MacPherson formula due to Sundaram and Welker [SWe1] enables one to compute the action of $\mathcal{S}_n$ on $M^{\mathbb{C}}_{n,T}$. For $T=\{d,2d,\dots\}$ and $T=\{1,k,k+1,\dots\}$, formulas for the S_n-module $M^{\mathbb{C}}_{n,T}$ are given in [SWe1]. For general 1-additive T we have the following result.

Theorem 6.4. *Let T be 1-additive with $1\notin T$. Then*

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\, \tilde{H}^m(M^{\mathbb{C}}_{n,T})\, u^n\, (-v)^{2n-m-1} = \sum_{r\ge 1} v^{2r}h_r\left[\sum_{\substack{m\in Z\\ n\in T}} \mathrm{ch}\, \tilde{H}_{m-1}(\Pi_n^T)u^n(-v)^m\right].$$

Proof. By the equivariant Goresky-MacPherson formula for complexified arrangements [SWe1, Corollary 2.8], we have the $\mathcal{S}_n$-module isomorphism,

$$\tilde{H}^{2n-m-1}(M_{n,T}^{\mathbb{C}}) = \bigoplus_{x\in\Pi_n^T\setminus\{\hat{0}\}} \tilde{H}_{m-2b(x)-1}(\hat{0},x).$$

By Lemma 6.3, we have

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T\\ x\in\Pi_n^T\setminus\{\hat{0}\}\\ b(x)=r}} \mathrm{ch}\,\tilde{H}_{m-2r-1}(\hat{0},x)\,u^n(-v)^m = v^{2r}h_r\left[\sum_{\substack{m\in Z\\ n\in T}} \mathrm{ch}\,\tilde{H}_{m-1}(\Pi_n^T)u^n(-v)^m\right].$$

Hence the result follows by summing over all r. $\square$

If we combine this result with Theorem 4.4 we get the following corollary.

Corollary 6.5. *Let T be 1-additive and subtractive with $1\notin T$. Then*

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\,\tilde{H}^m(M_{n,T}^{\mathbb{C}})\,u^n\,(-v)^{2n-m-1} = \sum_{r\geq 1} h_r\left[\sum_{j\geq 1}(-v)^{j+1}\,\mathrm{ch}\,\tilde{H}(\Pi_j)\left[\sum_{i\in T} h_i\,u^i v^{\phi_T(i)}\right]\right].$$

Corollary 6.6. *Let T be 1-additive and subtractive with $1\notin T$. Then*

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in T}} \mathrm{ch}\,\tilde{H}^m(M_{n,T}^{\mathbb{C}})\,u^n\,(-v)^{2n-m-1} = \sum_{\substack{r\geq 0\\ n\geq 1}}(-v)^{2n-r}\,\mathrm{ch}\,H^r(B_n^{\mathbb{C}})\left[\sum_{i\in T} h_i\,u^i v^{\phi_T(i)}\right],$$

where $B_n^{\mathbb{C}}$ is the complement of the complexified braid hyperplane arrangement. (Note that cohomology of $B_n^{\mathbb{C}}$ is not reduced.)

Proof. The result follows from Lehrer and Solomon's formula for the cohomology $\mathcal{S}_n$-module of the complement of the complexified braid arrangement [LS, Theorem 4.5] (see [Su1, Theorem 1.8] for the symmetric function formulation and a direct computation of the Whitney homology representation of the partition lattice), associativity of plethysm and Corollary 6.5. $\square$

In [SWe1] significant consequences of the computation of the cohomology $\mathcal{S}_n$-module of the complement of the d-divisible arrangement and the k-equal arrangement are given. We leave the task of generalizing such consequences, for general T, to a future paper.

A striking consequence of the plethystic formula for the d-divisible partition lattice is obtained by restricting the representation to $\mathcal{S}_{n-1}$. Namely, Calderbank, Hanlon and Robinson prove a conjecture of Stanley that the restricted representation is isomorphic to a skew representation of a certain skew hook shape. In [W] bases for homology and cohomology of the d-divisible partition lattice are constructed and used to give a combinatorial proof of this result. Sanders and Wachs [SaW] generalize this result to Π_n^T when T is 1-additive and subtractive also by constructing bases for homology and cohomology. They decompose the restriction to $\mathcal{S}_{n-1}$ into a direct sum of skew hook representations.

There are also important connections between problems in computational complexity and Betti numbers of Π_n^T when T is 1-additive. In particular it is shown in [BL] that the Betti numbers of Π_n^T determine lower bounds on the computational complexity of certain problems arising in computer science. These connections were initiated in [BLY] and further developed in [BL] and [L]. The results obtained in this paper could conceivably be useful in improving the lower bounds given in [L].

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