SHELLABLE NONPURE COMPLEXES AND POSETS, II

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ABSTRACT. This is a direct continuation of *Shellable Nonpure Complexes and Posets, I*; which appeared in Transactions of the American Mathematical Society **348**(1996), 1299-1327.

8. INTERVAL-GENERATED LATTICES AND DOMINANCE ORDER

In this section and the following one we will continue exemplifying the applicability of lexicographic shellability to nonpure posets.

Let $\mathcal{F} = \{I_1, I_2, \ldots, I_n\}$ be a family of intervals of integers, by which is meant sets of the form $[a, b] = \{a, a+1, \ldots, b\}$, $a \le b$. We assume that there are no containments among these intervals, and that they are ordered so that their left and right endpoints are increasing. Let $L(\mathcal{F})$ be the lattice of all sets that are unions of subfamilies of $\mathcal{F}$, ordered by inclusion. Such *interval-generated lattices* $L(\mathcal{F})$ were introduced and studied by Greene [G].

Define an edge-labeling λ of $L(\mathcal{F})$ as follows. If $A \to B$ is a covering and $a = \max(B \setminus A)$, then

(8.1)

$$\lambda(A \to B) = \begin{cases} -a, & \text{if } (a+1) \in A \text{ and } a \text{ is the left endpoint of some } I \in \mathcal{F}, \\ a, & \text{otherwise.} \end{cases}$$

Figure 6 shows the labeling for $\mathcal{F} = \{[1,2],[2,3],[3,4],[4,5]\}$.

8.1. Definition. *A family of intervals $\mathcal{F}$ is said to satisfy the* left endpoint condition *if for every left endpoint a of an interval of $\mathcal{F}$ either $a-1$ is a left endpoint or $a-1$ is in at most one interval of $\mathcal{F}$.*

There are two natural classes of interval families that satisfy the left endpoint condition. The first class consists of those $\mathcal{F}$ for which the left endpoints of the intervals of $\mathcal{F}$ are consecutive integers. The second class consists of those $\mathcal{F}$ whose intervals overlap in at most one point, i.e., $|I \cap J| \le 1$ for all $I \ne J \in \mathcal{F}$.

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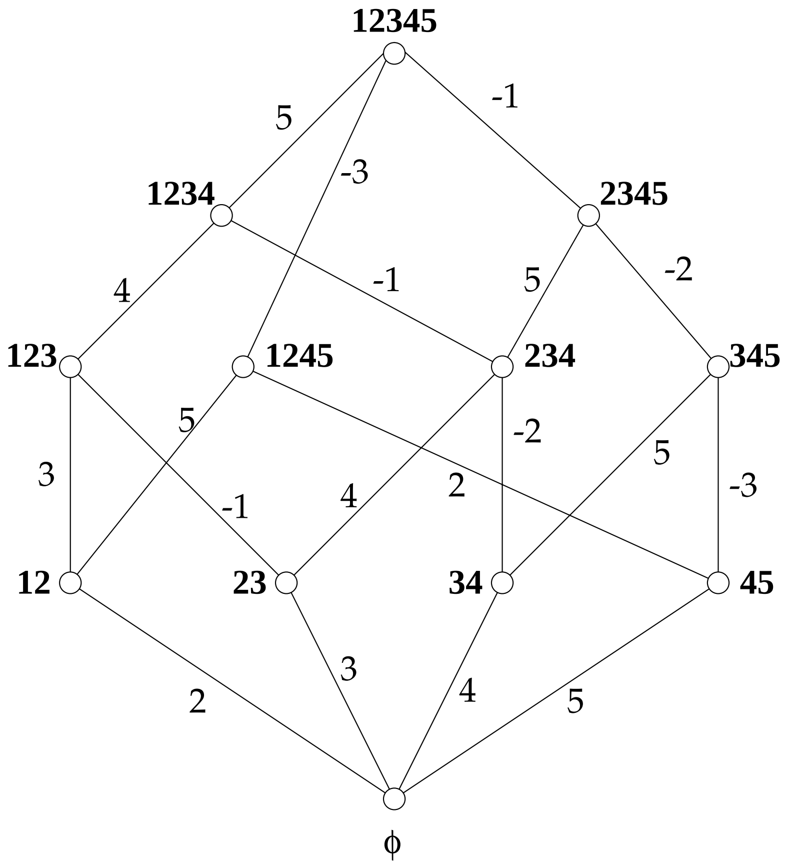

FIGURE 6.

8.2. Theorem. *Suppose that $\mathcal{F}$ satisfies the left endpoint condition. Then the rule (8.1) gives an EL-labeling of $L(\mathcal{F})$. Furthermore, with this labeling each interval has at most one falling chain.*

Proof. Consider an interval $[A, B]$ in $L(\mathcal{F})$. A rising maximal chain is constructed in two stages as follows.

(1) Suppose there exists a left endpoint $b \in B \setminus A$ such that $(b+1) \in A$. (If not put $k := 0$ and $A_k := A$ and go directly to stage 2.) Let a_1 be the greatest such b, and let $A_1 = A \cup \{a_1\}$. If there now is some left endpoint $b \in B \setminus A_1$ such that $(b+1) \in A_1$, then repeat: let a_2 be the greatest such b and let $A_2 = A_1 \cup \{a_2\}$. Continuing this way as long as possible we construct an unrefinable chain $A = A_0 \to A_1 \to \ldots \to A_k \leq B$ whose label $(-a_1, -a_2, \ldots, -a_k)$ is rising, since by construction $a_1 > a_2 > \ldots > a_k$.

(2) Let j be minimal such that $A_k \subsetneq A_k \cup I_j \subseteq B$ (unless $A_k = B$ in which case we are done). Then let $A_{k+1} = A_k \cup I_j$. It is easy to check that the left endpoint condition implies that this creates a covering with label $\lambda(A_k \to A_{k+1}) = c_1 = \max(I_j \setminus A_k)$. Note that there is still no left endpoint $b \in B \setminus A_{k+1}$ such that $(b+1) \in A_{k+1}$. If $A_{k+1} \neq B$ then repeat: let j' be minimal such that $A_{k+1} \subsetneq A_{k+1} \cup I_{j'} \subseteq B$, let $A_{k+2} = A_{k+1} \cup I_{j'}$, and let $c_2 = \max(I_{j'} \setminus A_{k+1})$. Repeating this we obtain in the end an unrefinable chain $A_k \to A_{k+1} \to \ldots \to A_{k+l} = B$ with

rising label $(c_1, c_2, \ldots, c_l)$.

Combining stages 1 and 2 we obtain a maximal chain $A = A_0 \to A_1 \to \ldots \to A_k \to A_{k+1} \to \ldots \to A_{k+l} = B$ with rising label $(-a_1, \ldots, -a_k,\ c_1, \ldots, c_l)$. We leave to the reader the easy verification of the following facts: If $A = B_0 \to B_1 \to \ldots \to B_p = B$ is some other maximal chain in $L(\mathcal{F})$ and $A_i = B_i$ for $i \leq j$, $A_{j+1} \neq B_{j+1}$, then

(i) $\lambda(A_j \to A_{j+1}) < \lambda(B_j \to B_{j+1})$
(ii) $\lambda(A_j \to A_{j+1}) \geq \lambda(B_k \to B_{k+1})$ for some $k > j$.

It follows that $A_0 \to A_1 \to \ldots$ is the unique rising chain and is lexicographically first among the maximal chains in $[A, B]$.

It remains to show that if there is a falling chain in $[A, B]$ then it is necessarily unique. This will follow from these two observations, whose proof we leave to the reader:

(iii) if $A_1, \ldots, A_e$ are the atoms of $[A, B]$ then $\lambda(A \to A_i) \neq \lambda(A \to A_j)$, $i \neq j$,
(iv) if $\lambda(A \to A_1) > \lambda(A \to A_j)$, for $j = 2, \ldots, e$, and $A = B_0 \to B_1 \to \ldots \to B_p = B$ has $B_1 \neq A_1$, then either $\lambda(A \to A_1) = \lambda(B_k \to B_{k+1})$ for some $k \geq 1$, or $\lambda(B_k \to B_{k+1}) < -\lambda(A \to A_1) = \lambda(B_l \to B_{l+1})$ for some $1 \leq k < l < p$.

Thus, if there is a falling chain in $[A, B]$ it is necessarily constructed by choosing the unique largest label among the coverings available at each step. □

8.3. Corollary. *Every interval in $L(\mathcal{F})$ has the homotopy type of a sphere or is contractible, if $\mathcal{F}$ satisfies the left endpoint condition.*

Proof. This follows from Theorem 5.9. □

For $\mathcal{F}$ that satisfy the left endpoint condition this sharpens the general result of Greene [G] that $\mu(A, B) \in \{+1, -1, 0\}$ for all $A \subseteq B$ in $L(\mathcal{F})$. Greene also studied the special case $L_{n,k} = L(\mathcal{F})$ for the family $\mathcal{F}$ of k-element subintervals $[a+1,\ a+k]$ of $[n]$, $a = 0, 1, \ldots, n-k$, for which more precise results can be stated. Figure 6 shows $L_{5,2}$.

8.4. Corollary. *The proper part of $L_{n,k}$ has the following homotopy type*

$$\overline{L}_{n,k} \simeq \begin{cases} S^{2n/(k+1)-2} & , \quad \textit{if} \quad n \equiv 0 \quad (\textit{mod } k+1) \\ S^{2(n+1)/(k+1)-3} & , \quad \textit{if} \quad n \equiv -1 \quad (\textit{mod } k+1) \\ \textit{point} & , \quad \textit{otherwise.} \end{cases}$$

Proof. We must determine in what cases there is a falling chain in $L_{n,k}$, and in such cases we need to know its length.

Assume that $n = d(k+1)$, and for $j = 1, \ldots, d$ let $I_j = [(j-1)(k+1)+2,\ j(k+1)]$. Then a maximal chain $\hat{0} = A_0 \to A_1 \to \ldots \to A_{2d} = \hat{1}$ with falling label

$$d(k+1),\ (d-1)(k+1), \ldots, k+1, -1, -(2(k+1)-k), \ldots, -(d(k+1)-k)$$

is constructed by successively enlarging $\emptyset$ by $I_d, I_{d-1}, \ldots, I_1, \{1\}, \{2(k+1)-k\}, \ldots, \{d(k+1)-k\}$.

Next, let $n = d(k+1)-1$. Then a maximal chain of length $2d-1$ is constructed in similar fashion with falling label

$$d(k+1)-1,\ (d-1)(k+1)-1,\ \ldots,k,-(k+1),-2(k+1),\ \ldots,-(d-1)(k+1).$$

Finally, suppose that $n \not\equiv 0,-1 \pmod{k+1}$. We know from the proof of Theorem 8.1 that if there were a falling chain in $L_{n,k}$ then it would be constructed by choosing the largest available label at each step. This greedy strategy leads however to the following non-falling label sequence if $n = d(k+1)+j$, $0 < j < k$:

$$n,\ n-k-1,\ \ldots,n-(d-1)(k+1),-(j+1),-j,\ \ldots,-1,-(j+k+2),\ \ldots. \quad \square$$

We find it likely that all lattices of the type $L(\mathcal{F})$ are lexicographically shellable, but the particular labeling rule (8.1) does not work in total generality. It fails for $\mathcal{F} = \{[1,2],[2,4],[3,5],[5,6]\}$, since the interval from $\{5,6\}$ to $\{2,3,4,5,6\}$ has no rising chain. Note that for $\mathcal{F} = \{[1,1],[2,2],\ldots,[n,n]\}$ rule (8.1) gives a nonstandard EL-labeling of the Boolean algebra.

The next class of posets that we will discuss are integer partitions ordered by dominance. Let λ and μ be two partitions of n, i.e., $\lambda = (\lambda_1,\ldots,\lambda_p)$, $\mu = (\mu_1,\ldots,\mu_q)$, $\lambda_1 \geq \ldots \geq \lambda_p > 0$, $\mu_1 \geq \ldots \geq \mu_q > 0$ and $\Sigma\lambda_i = \Sigma\mu_i = n$. We say that λ *dominates* (or majorizes) μ, if $\lambda_1+\ldots+\lambda_i \geq \mu_1+\ldots+\mu_i$ for all i. This relation, written $\lambda \geq \mu$, is a partial order. The poset P_n of all partitions of n ordered by dominance is in fact a lattice. It plays an important role in the representation theory of the symmetric groups. See [Br, G] for information concerning P_n.

Dominance order is closely related to the interval-generated lattices $L(\mathcal{F})$, as shown by the following result of Greene [G, Lemma 3.1].

8.5. Lemma. (Greene) *Let $\mu \leq \lambda$ in P_n. Then the $\wedge$-semilattice generated by the coatoms of $[\mu,\lambda]$ is isomorphic to the dual of a lattice $L(\mathcal{F})$ generated by intervals that overlap in at most one point.*

It has been shown by Bogart [Bo], Brylawski [Br] and Greene [G] that the Möbius function takes values $+1,-1$, or 0 on all intervals in P_n. This is strengthened by the following result.

8.6. Theorem. *Every interval in P_n has the homotopy type of a sphere or is contractible.*

Proof. The result follows from Corollary 8.3, Lemma 8.5 and Lemma 7.6. $\square$

Various algorithms for computing the Möbius function of P_n are given in Greene [G]. These make it possible to decide algorithmically whether a given interval $[\theta,\lambda]$ in P_n is contractible or spherical, according to whether $\mu(\theta,\lambda)=0$ or not. However, they do not give the dimension of the sphere in the latter case, only the parity of its dimension. This dimension can of course in principle be computed by passing from $[\theta,\lambda]$ to the $\wedge$-semilattice generated by its coatoms, and then tracing the steps of proof back to Corollary 8.3. However, a more direct route should be possible. It seems likely that dominance order P_n itself is lexicographically shellable. A good

CL-labeling would reprove Theorem 8.6 and hopefully give a reasonable formula for the dimension of spherical intervals.

8.7. Remark. After completing this paper we became aware that Kahn [Ka] has proved Corollary 8.3 for *all* interval-generated lattices. Furthermore, the same has been done by Linusson [L2], who also has given a dimension formula for the spheres appearing in Theorem 8.6. His method does not use lexicographic shellability. We are grateful to S. Linusson for pointing out an error in an earlier version of Theorem 8.2 and for several helpful discussions.

9. The Tamari lattices

The last class of posets that we will analyze using lexicographic shellability are the so called Tamari lattices. These are orderings of parenthesizations of words, or equivalently of binary trees, that were introduced by D. Tamari in 1951 and later shown to be lattices. They were independently rediscovered by Pallo [P1], who also proves the lattice property. See Huang and Tamari [HT], Pallo [P1, P2, P3] and Knuth [K] for more about the basic properties of these lattices.

The Tamari lattices T_n can be described in many ways via the known bijections between families of Catalan objects. The following approach is based on Pallo [P1] and Knuth [K].

9.1. Definition. *The elements of* T_n *are integer* n*-tuples* $(r_1, r_2, \ldots, r_n)$ *such that*

(i) $0 \le r_i \le n-i$, *for* $1 \le i \le n$,
(ii) $r_{k+i} \le r_k - i$, *for* $1 \le k \le n-2$, $1 \le i \le r_k$.

The order in T_n *is the product order:*

$$(r_1, \ldots, r_n) \le (s_1, \ldots, s_n) \iff r_i \le s_i, \quad \text{for } 1 \le i \le n.$$

It is clear from this definition that T_n is a bounded poset with top element $(n-1, n-2, \ldots, 0)$ and bottom element $(0, 0, \ldots, 0)$, and that meets exist and are given by

$$(r_1, \ldots, r_n) \wedge (s_1, \ldots, s_n) = (m_1, \ldots, m_n), \text{ where } m_i = \min\{r_i, s_i\}.$$

Hence, T_n is a lattice. See Figure 7 for a picture of T_4.

The connection with binary trees is important and interesting. By a binary tree we mean a rooted tree where each interior node has either a left child, or a right child, or both. Let $\mathcal{B}_n$ denote the set of all binary trees on n nodes. We will always consider the nodes of such a tree listed $v_1, v_2, \ldots, v_n$ in *inorder*, which is the linear order recursively defined by the requirement that if v_i is in the left subtree of v_j then $i < j$ and if v_i is in the right subtree of v_j then $i > j$. For $T \in \mathcal{B}_n$ let r_i be the size of the right subtree of v_i, and let $r(T) = (r_1, \ldots, r_n)$. Then $r(T)$ satisfies conditions (i) and (ii) of Definition 9.1, and conversely every $(r_1, \ldots, r_n) \in T_n$ is the right-subtree-size vector $r(T)$ of a unique binary tree $T \in \mathcal{B}_n$. Hence the elements of T_n should be seen as encodings of binary trees on n nodes. There are also explicit bijections between T_n and parenthesizations of words with $n+1$ letters, or with triangulations of a convex $(n+2)$-gon and

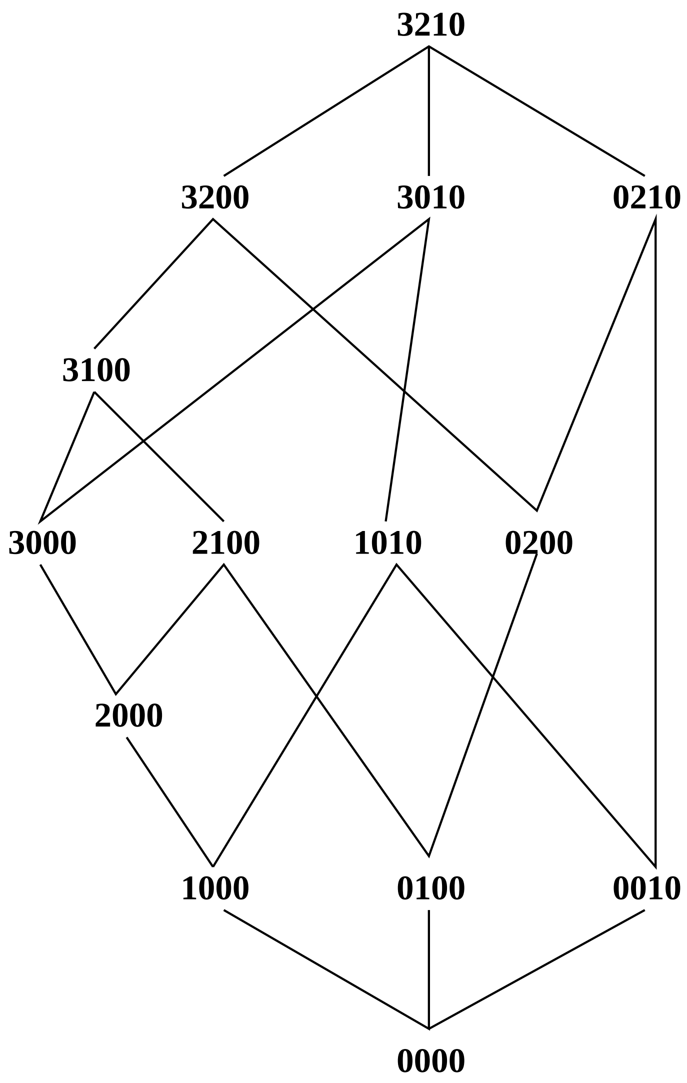

FIGURE 7.

many other classes of objects that are enumerated by the Catalan numbers. The coverings in T_n correspond to elementary mutations in each of these models, e.g. to rotations of binary trees, to reparenthesizations $\ldots((xy)z)\ldots \rightarrow \ldots(x(yz))\ldots$, and to diagonal flips in triangulations.

If $\mathbf{r} = (r_1, \ldots, r_n)$, $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{r} \rightarrow \mathbf{s}$ is a covering in T_n then there is a unique j such that $r_j \neq s_j$. Because if j is minimal such that $r_j \neq s_j$ and also $r_k \neq s_k$, $j < k$, then $\mathbf{r} < \mathbf{r}' < \mathbf{s}$ for $\mathbf{r}' = (r_1, \ldots, r_{j-1}, s_j, r_{j+1}, \ldots, r_n) \in T_n$. Define an edge-labeling $\lambda : \mathcal{E}(T_n) \rightarrow \mathbb{Z}^2$ as follows:

$$\lambda(\mathbf{r} \rightarrow \mathbf{s}) = (j, r_j) \quad , \quad \text{if } r_j \neq s_j, \tag{9.1}$$

with labels ordered lexicographically.

9.2. Theorem. *The rule (9.1) gives an EL-labeling of T_n. Furthermore, with this labeling each interval has at most one falling chain.*

Proof. Let $\mathbf{r} < \mathbf{s}$, $D = \{j | r_j \neq s_j\} = \{j_1, \ldots, j_d\}$, $j_1 < \ldots < j_d$. Define elements $\mathbf{r}_k \in T_n$, $k = 0, \ldots, d$, by replacing $r_{j_1}, \ldots, r_{j_k}$ in $\mathbf{r}$ by $s_{i_1}, \ldots, s_{i_k}$, respectively. Then we get a chain $\mathbf{r} = \mathbf{r}_0 < \mathbf{r}_1 < \ldots < \mathbf{r}_d = \mathbf{s}$, and since $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$ differ in only one coordinate j_i each interval $[\mathbf{r}_{i-1}, \mathbf{r}_i]$ is just a chain with rising label. Concatenating these chains we get a maximal chain m in $[\mathbf{r}, \mathbf{s}]$ with rising label.

The maximal chain m is constructed by choosing the least available label (j, t) at each step. Any other choice would produce a label (j', t') with $j' > j$, and would force us to later use a covering with label (j, t''). Hence, m is the only rising maximal chain and it is lexicographically first.

A falling chain in $[\mathbf{r}, \mathbf{s}]$, if one exists, must have label

$$(j_d, r_{j_d}), (j_{d-1}, r_{j_{d-1}}), \ldots, (j_1, r_{j_1}) \tag{9.2}$$

and is therefore unique. □

It follows via Theorem 5.9 that every open interval $(\mathbf{r}, \mathbf{s})$ in T_n is either contractible or has the homotopy type of a sphere, and consequently (or via Proposition 5.7) that the Möbius function $\mu(\mathbf{r}, \mathbf{s})$ takes values in $\{+1, -1, 0\}$. This result about the Möbius function was earlier obtained by Pallo [P3]. We are grateful to D. Knuth for bringing his work to our attention. Pallo proves that the coatoms of any interval $[\mathbf{r}, \mathbf{s}]$ in T_n $\wedge$-generate a Boolean lattice, which actually implies not only the result about the Möbius function but also via Lemma 7.6 the result about the homotopy type of $(\mathbf{r}, \mathbf{s})$. Pallo's method leads to formulas for $\mu(\mathbf{r}, \mathbf{s})$ based on computing the coatoms of the interval $[\mathbf{r}, \mathbf{s}]$. Our approach via falling chains leads to formulas with a somewhat different appearance. The following result characterizes the spherical intervals.

9.3. Theorem. *Let $\mathbf{r} < \mathbf{s}$ in T_n, and let $D = \{j | r_j \neq s_j\}$. Then $(\mathbf{r}, \mathbf{s}) \simeq S^{|D|-2}$ if*

(i) *$1 \leq k \leq n-2$, $1 \leq i \leq r_k$ and $(k+i) \in D$ implies that $s_{k+i} \leq r_k - i$, and*
(ii) *$s_j = r_j + 1$ or $s_j = r_j + 1 + s_{j+r_j+1}$, for all $j \in D$.*

Otherwise $(\mathbf{r}, \mathbf{s})$ is contractible.

Proof. Let $D = \{j_1, \ldots, j_d\}$, $j_1 < \ldots < j_d$. We must show that a maximal chain with label (9.2) exists if and only if conditions (i) and (ii) hold. Define n-tuples $\mathbf{r}'_k$, $k = 1, 2, \ldots, d+1$ by replacing $r_{j_k}, r_{j_{k+1}}, \ldots, r_{j_d}$ in $\mathbf{r}$ by $s_{j_k}, s_{j_{k+1}}, \ldots, s_{j_d}$, respectively. It is clear from the construction that the unique falling chain, if it exists must be $\mathbf{r} = \mathbf{r}'_{d+1} < \mathbf{r}'_d < \ldots < \mathbf{r}'_1 = \mathbf{s}$.

We leave to the reader the verification that condition (i) is equivalent to $\mathbf{r}'_k \in T_n$ for all $k = 1, \ldots, d+1$, and if (i) is satisfied that (ii) means precisely that every $\mathbf{r}'_{k+1} < \mathbf{r}'_k$ is a covering, $k = 1, \ldots, d$. The result then follows. □

9.4. Corollary.

$$\mu(\mathbf{r}, \mathbf{s}) = \begin{cases} (-1)^{|D|} & , \quad \textit{if (i) and (ii) are satisfied} \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

For example, $\mu(1000,\ 3010) = 1$, as can be checked also from Figure 7. Simplified statements are possible for the cases of lower and upper intervals.

9.5. Corollary.

$$\mu(\hat{0},\mathbf{s}) = \begin{cases} (-1)^{|D|} & , \quad \text{if } s_j \in \{0, 1+s_{j+1}\} \text{ for all } j < n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

$$\mu(\mathbf{r},\hat{1}) = \begin{cases} (-1)^{|D|} & , \quad \text{if } r_j \in \{0, n-j\} \text{ for all } j, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Proof. This is most easily seen using Pallo's atom/coatom method [P3]. Otherwise, here is how it follows from Corollary 9.4. If $\mathbf{r} = \hat{0} = (0, 0, \ldots, 0)$, then condition (i) is redundant and (ii) specializes directly to this form. If $\mathbf{s} = \hat{1} = (n-1, n-2, \ldots, 0)$, then $s_j = r_j + 1 + s_{j+r_j+1}$ is true for all $\mathbf{r}$ and all j, so condition (ii) is redundant. For (i) we first observe that if $0 < r_k < n-k$ for some k, then $r_{k+1} \le r_k - 1$ implies that $(k+1) \in D$. Hence, condition (i) would imply that $s_{k+1} = n-(k+1) \le r_k - 1$, which forces $r_k = n-k$, a contradiction. Therefore condition (i) is satisfied if and only if $r_k \in \{0, n-k\}$ for all k. □

For the remainder of this section we will depart from the topic of shellability in order to discuss a surprisingly close connection that exists between Tamari lattices and weak order on the symmetric groups. We will show that the lattice T_n is induced by weak order on a certain class of permutations, and it is also obtained as a quotient of weak order on S_n under a certain mapping.

Weak order on S_n (also called right weak order and permutohedron order) is the partial order on permutations whose cover relations $\sigma \to \pi$ are $\pi = \sigma \cdot (i, i+1)$ for some adjacent transposition $(i, i+1)$ such that $\text{inv}(\sigma) < \text{inv}(\pi)$. Here $\sigma \cdot (i, i+1)$ is the permutation obtained by transposing the letters in positions i and $i+1$ in σ, e.g. $51342 \to 51432$. It is a basic fact that weak order is a lattice. See Berge [Be] or Yanagimoto and Okamoto [YO] for more about weak order.

A permutation $\pi = \pi_1\pi_2\ldots\pi_n$ (where $\pi_i = \pi(i)$) is called *312-free* if there is no triple $i < j < k$ such that $\pi_i > \pi_k > \pi_j$. Let S_n^{312} denote the set of 312-free permutations.

9.6. Theorem. (i) *S_n^{312} is a sublattice of S_n.*

(ii) *There exists an order-preserving surjection $t : S_n \to T_n$ whose restriction to S_n^{312} gives a lattice isomorphism $S_n^{312} \cong T_n$.*

(iii) *$\mathbf{r} \le \mathbf{s}$ in T_n if and only if for every $\sigma \in t^{-1}(\mathbf{s})$ there is a $\pi \in t^{-1}(\mathbf{r})$ such that $\pi \le \sigma$ in S_n.*

The proof of this theorem will be given in bits and pieces as we develop the necessary side material. First of all we need the "inversion graph" characterization of weak order. For $\pi \in S_n$ let $I(\pi) = \{(\pi_i, \pi_j) \mid 1 \le i < j \le n,\ \pi_i > \pi_j\}$.

9.7. Lemma. [Be] [YO] *For all $\pi, \sigma \in S_n$, $\pi \le \sigma$ if and only if $I(\pi) \subseteq I(\sigma)$.*

Let us say that a graph $G \subseteq \{(j, i) \mid 1 \le i < j \le n\}$ is *compressed* if $i < j < k$ and $(k, i) \in G$ implies that $(j, i) \in G$. The following lemma is immediate.

9.8. Lemma. *π is 312-free if and only if $I(\pi)$ is compressed.* □

Proof of Theorem 9.6, Part (i). Suppose $\pi, \sigma \in S_n^{312}$ and let $\pi \wedge \sigma$ and $\pi \vee \sigma$ be their meet and join in S_n. We must prove that $\pi \wedge \sigma$ and $\pi \vee \sigma$ are 312-free.

Suppose $m = \pi \wedge \sigma \notin S_n^{312}$. Then there exist $i < j < k$ such that $m_i > m_k > m_j$. By choosing i, j, k so that $k - j$ is minimal, we have $j + 1 = k$ or $m_{j+1} > m_k$. Either way $m_{j+1} > m_j$.

We have

$$I(m \cdot (j, j+1)) = I(m) \cup (m_{j+1}, m_j) \subseteq I(\pi) \cup (m_{j+1}, m_j). \tag{9.3}$$

We claim that

$$(m_{j+1}, m_j) \in I(\pi). \tag{9.4}$$

Since $(m_i, m_j) \in I(m) \subseteq I(\pi)$ and $I(\pi)$ is compressed, $(m_k, m_j) \in I(\pi)$. If $k = j+1$ then (9.4) is proved. For $k > j + 1$, we have $(m_{j+1}, m_k) \in I(m) \subseteq I(\pi)$. Now (9.4) follows by transitivity. It follows from (9.3) and (9.4) that $m \cdot (j, j+1) \leq \pi$. The same argument yields $m \cdot (j, j+1) \leq \sigma$. Hence $m \cdot (j, j+1) \leq \sigma \wedge \tau = m$ which is impossible.

The proof for join is similar. Suppose that $x = \pi \vee \sigma \notin S_n^{312}$. Then there exist $i < j < k$ such that $x_i > x_k > x_j$. Now choose i, j, k so that $j - i$ is minimal. This implies that $x_{j-1} > x_k$. We hence have

$$I(x \cdot (j-1, j)) = I(x) - (x_{j-1}, x_j). \tag{9.5}$$

Now we claim that

$$(x_{j-1}, x_j) \notin I(\pi). \tag{9.6}$$

If $(x_{j-1}, x_j) \in I(\pi)$ then since $I(\pi)$ is compressed, $(x_k, x_j) \in I(\pi) \subseteq I(x)$. But this is impossible since $j < k$. Hence (9.6) holds. It follows from (9.5) and (9.6) that $I(x \cdot (j-1, j)) \subseteq I(\pi)$ which implies that $x \cdot (j-1, j) \geq \pi$. The same argument yields $x \cdot (j-1, j) \geq \sigma$. Consequently $x \cdot (j-1, j) \geq \pi \vee \sigma = x$ which is impossible. □

9.9. Definition. (i) *For* $\pi = \pi_1 \pi_2 \dots \pi_n \in S_n$ *let*

$$R_k = max\{i \mid \pi_k > \pi_{k+1},\, \pi_{k+2},\, \dots,\, \pi_{k+i}\}$$

and let $R(\pi) = (R_1, R_2, \dots, R_n)$.

(ii) *Define* $t : S_n \to T_n$ *by* $t(\pi) = R\left(\pi^{-1}\right)$.

(iii) *Define* $\tau : S_n \to \mathcal{B}_n$ *recursively by* τ *(empty permutation) = empty tree, and for* $n > 0$ *and* $\pi \in S_n$, $\tau(\pi)$ *is the binary tree whose root is* π_n, *left subtree is* $\tau(\pi^-)$ *and right subtree is* $\tau(\pi^+)$, *where* π^- *and* π^+ *are the subwords of* π *consisting of all letters less than* π_n *and all letters greater than* π_n, *respectively.*

It is immediately clear that $t(\pi)$ satisfies the conditions of Definition 9.1, so $t(\pi) \in T_n$. For instance,

$$t(276938154) = 200541010.$$

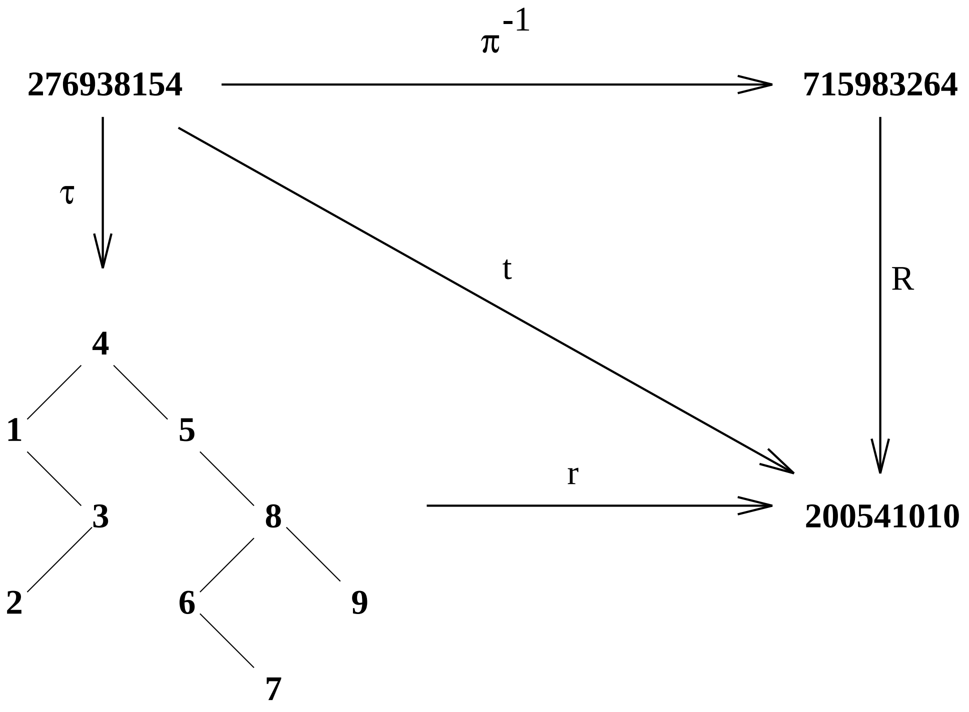

FIGURE 8.

In fact, we have that

$$t = r \circ \tau, \tag{9.7}$$

where $r : \mathcal{B}_n \to T_n$ is the right-subtree-size encoding of inorder labeled binary trees that was discussed at the beginning of this section. The relationship between these mappings is illustrated in Figure 8.

9.10. Proposition. (i) *The mapping $t : S_n \to T_n$ is surjective.*

(ii) *The preimage $t^{-1}(\mathbf{r})$ is a weak order interval in S_n, for all $\mathbf{r} \in T_n$.*

(iii) *A permutation is 312-free if and only if it is the minimal element of some $t^{-1}(\mathbf{r})$ interval.*

(iv) *A permutation is 132-free if and only if it is the maximal element of some $t^{-1}(\mathbf{r})$ interval.*

Proof. Corresponding properties of the mapping $\tau : S_n \to \mathcal{B}_n$ have been previously studied in [BW3, BW4]. Via (9.7) it is possible to fall back on Corollary 8.3 of [BW3] for parts (i) and (ii) and on Theorem 4.2 of [BW4] for parts (iii) and (iv).□

9.11. Lemma. *If $\pi \to \sigma$ in weak order, then either $t(\pi) = t(\sigma)$ or $t(\pi) \to t(\sigma)$ in T_n.*

Proof. Suppose that $\sigma = \pi \cdot (i, i+1)$, $\mathrm{inv}(\pi) < \mathrm{inv}(\sigma)$. Then $\pi^{-1} = \ldots i \ldots i + 1 \ldots$ and $\sigma^{-1} = \ldots i+1 \ldots i \ldots$, and letting $\pi_i = j$ and $R\left(\pi^{-1}\right) = (R_1, \ldots, R_n)$ one sees that either $R\left(\sigma^{-1}\right) = R\left(\pi^{-1}\right)$ or $R\left(\sigma^{-1}\right) = (R_1, \ldots, R_{j-1}, R_j{+}k, R_{j+1}, \ldots, R_n)$, for some $k \geq 1$. The second case happens according to whether $j + R_j + 1 = \pi_{i+1}$ or not, and one easily checks that it gives a covering in T_n. $\square$

Proof of Theorem 9.6, Parts (ii) *and* (iii). It follows from Lemma 9.11 that $t : S_n \to T_n$ is order-preserving.

Define a mapping $u : T_n \to S_n^{312}$ by sending $\mathbf{r} \in T_n$ to the least element of its preimage $t^{-1}(\mathbf{r})$. By Proposition 9.10 this is well-defined, and $u \circ t(\pi) = \pi$ for all $\pi \in S_n^{312}$. Thus, for proving part (ii) of the theorem it remains only to show that u is order-preserving.

For $\mathbf{r} \in T_n$ let $G_{\mathbf{r}} = \{(j, i) \mid 1 \leq i < j \leq i + r_i\}$. This graph is compressed in the sense used for Lemma 9.8. Since $u(\mathbf{r})$ is characterized as having minimal number of inversions among all π such that $R\left(\pi^{-1}\right) = \mathbf{r}$, we conclude that

$$u(\mathbf{r}) = \pi \iff I(\pi) = G_{\mathbf{r}}.$$

Hence,

$$\mathbf{r} \leq \mathbf{s} \Longrightarrow G_{\mathbf{r}} \subseteq G_{\mathbf{s}} \Longrightarrow u(\mathbf{r}) \leq u(\mathbf{s}).$$

This settles also part (iii), since $u(\mathbf{s}) = \min t^{-1}(\mathbf{s})$. $\square$

9.12. Remark. Weak order on S_n is not shellable for $n \geq 3$ (the order complex of the proper part of S_3 is the 1-dimensional nonshellable complex shown in Figure 1b). However every open interval is known to either be contractible or have the homotopy type of a sphere, just as in T_n (Theorem 9.3). It can be shown that the maps $t : \overline{S}_n \to \overline{T}_n$ and $u : \overline{T}_n \to \overline{S}_n$ defined above are homotopy inverses inducing this homotopy equivalence. Moreover, Proposition 9.10 and Corollary 9.5 yield the following characterization of open intervals that are spheres. For $u < v$ in S_n, $\Delta(u, v)$ has the homotopy type of a k-sphere if and only if $u^{-1}v$ is the maximal element of a Young subgroup of S_n generated by $k+2$ adjacent transpositions. This result is also known for general Coxeter groups via an argument using Lemma 7.6.

Let us sketch the way the stated homotopy type of $\Delta(u, v)$ can be deduced in the present context, since it is clearly not the case that t maps open intervals to open intervals. What is happening is that u and t form a Galois connection between the interval $(\hat{0}, x)$ in S_n and either the open interval $(\hat{0}, t(x))$ in T_n if x is 312 free or the half-closed interval $(\hat{0}, t(x)]$ if x is not 312 free. Hence one can indeed recover the homotopy type of lower open intervals in S_n from their images in T_n. In particular, since half closed intervals are contractible, $(\hat{0}, x)$ in S_n is contractible whenever x is not 312 free. Corollary 9.5 characterizes the homotopy type of the rest of the open lower intervals in S_n. Since any open interval (u, v) in S_n is isomorphic to the lower open interval $(\hat{0}, u^{-1}v)$, the homotopy type of all open intervals of S_n are characterized as stated.

9.13. Remark. The mapping $\tau : S_n \to \mathcal{B}_n$ was used in [BW4] for some results on permutation statistics. Namely, a mapping $s : S_n \to \mathbb{N}$ is called a *tree dependent statistic* if $s(\pi) = b(\tau(\pi))$ for some mapping $b : \mathcal{B}_n \to \mathbb{N}$. It was shown [BW4, Theorem 5.5] that

$$\sum_{\pi \in S_n} t_1^{s_1(\pi)} \dots t_p^{s_p(\pi)} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} t_1^{s_1(\pi)} \dots t_p^{s_p(\pi)} q^{\mathrm{inv}(\pi)} \tag{9.8}$$

for any collection of tree dependent statistics $s_1, \ldots, s_p$.

The mapping $t : S_n \to T_n$ is in a sense the "universal" tree dependent statistic, since the integer sequence $t(\pi)$ is an encoding of the tree $\tau(\pi)$. Thus, the tree dependent statistics $s(\pi)$, of interest for formulas of type (9.8), are precisely the functions $s : S_n \to \mathbb{N}$ that depend on the integers $R\left(\pi^{-1}\right) = (R_1, \ldots, R_n)$. This clarifies the general setting for Examples 5.1–5.4 of [BW4].

9.14. Remark. Billera and Sturmfels prove in [BiS] that the associahedron is a Minkowski summand of the permutohedron. This induces a many-to-one relation between the vertices of these polytopes, which are permutations and triangulations of an n-gon, respectively. Using a bijection with binary trees and some geometric arguments it can be shown that this relation is equivalent to the one induced by the mapping τ in Definition 9.9. We are grateful to L. Billera for an illuminating discussion which led to this insight.

10. Constructions that preserve shellability

In [B1,§4] and [BW2,§8] various operations on complexes and posets that preserve shellability or CL-shellability in the pure case are considered. These operations are rank selection, direct product, ordinal sum, cardinal power, and interval poset. The results concerning these operations, with the exception of rank selection, extend easily to the nonpure case, see Remark 10.22. The first part of this section deals with the difficulties of nonpure rank selection and the second part deals with variations of direct products.

First we present some general results on vertex induced subcomplexes. Let $A \subseteq V$ be a subset of the vertices of a complex Δ, and let $\Delta(A) = \{B \in \Delta \mid B \subseteq A\}$.

10.1. Theorem. *Suppose that Δ is shellable and that $F \cap A$ is a facet of $\Delta(A)$ for all facets F of Δ such that $\mathcal{R}(F) \subseteq A$. Then $\Delta(A)$ is also shellable. Furthermore,*

$$h_{s,j}(\Delta(A)) = \text{ number of facets } F \text{ of } \Delta \text{ such that } \mathcal{R}(F) \subseteq A, \\ |F \cap A| = s \text{ and } |\mathcal{R}(F)| = j.$$

Consequently, the Betti numbers for $\Delta(A)$ are given by

$$\tilde{\beta}_j = \text{ number of facets } F \text{ of } \Delta \text{ such that } \mathcal{R}(F) = F \cap A \text{ and } |\mathcal{R}(F)| = j+1.$$

Proof. Suppose that

$$\Delta = \dot{\bigcup_{i=1}^{t}} \, [\mathcal{R}(F_i), F_i] \tag{10.1}$$

is the Boolean interval partition induced by a shelling $F_1, \ldots, F_t$ with restriction map $\mathcal{R}$. If G is a facet of $\Delta(A)$, define $\mathcal{R}_A(G) = \mathcal{R}(F_i)$ for the unique i such that $\mathcal{R}(F_i) \subseteq G \subseteq F_i$. Note that by assumption $G = F_i \cap A$, hence $f(G) = F_i$ defines a bijective map from the facets of $\Delta(A)$ to the facets of Δ whose restriction is a subset of A. Label the facets of $\Delta(A)$ in the induced order: $G_1, G_2, \ldots, G_k$.

Intersecting with 2^A we obtain from (10.1)

$$\Delta(A) = \dot{\bigcup_{i=1}^{k}} \, [\mathcal{R}_A(G_i), G_i],$$

and "$\mathcal{R}_A(G_i) \subseteq G_j$ implies $i \leq j$" is inherited from the corresponding property of (10.1). Hence, by Proposition 2.5, $G_1, \ldots, G_k$ is a shelling of $\Delta(A)$ with restriction map $\mathcal{R}_A$. The statement about the h-triangle now follows from Theorem 3.4. $\square$

A similar result holds for posets. Now let Q be an induced subposet of a bounded poset P such that $\hat{0}, \hat{1} \in Q$.

10.2. Theorem. *Suppose that P is CL-shellable (resp. admits a CR-labeling) and that $m \cap Q$ is a maximal chain of Q for all maximal chains m of P such that $\mathcal{R}(m) \subseteq Q$. Then Q is also CL-shellable (resp. admits a CR-labeling). Furthermore, if P is CL-shellable then the Betti numbers for $\Delta(\bar{Q})$ are given by*

$\tilde{\beta}_j =$ *number of maximal chains m of P such that $\mathcal{R}(m) = m \cap \bar{Q}$ and $|\mathcal{R}(m)| = j+1$.*

If P admits a CR-labeling then the Möbius function for Q is given by

$$\mu_Q(\hat{0}, \hat{1}) = \sum_{\substack{m \in \mathcal{M}(P) \\ \mathcal{R}(m) = m \cap \bar{Q}}} (-1)^{|\mathcal{R}(m)|+1}.$$

Proof. We shall use the "weakly increasing" version of CR-labeling for the proof (cf. Remark 5.14). Suppose $\lambda : \mathcal{ME}(P) \to \Lambda$ is a CR-labeling of P. We will construct a CR-labeling of Q where the edges are labeled by words in Λ^* ordered lexicographically. First, define a map $f : \{m \in \mathcal{M}(P) \mid \mathcal{R}(m) \subseteq Q\} \to \mathcal{M}(Q)$ by $f(m) = m \cap Q$. We claim that f is a bijection. Indeed, the inverse of f is the map $g : \mathcal{M}(Q) \to \{m \in \mathcal{M}(P) \mid \mathcal{R}(m) \subseteq Q\}$ defined by letting $g(c)$ be the unique chain obtained by filling in the "gaps" of c with the unique rising chains of the respective rooted intervals of P from bottom to top.

For maximal chain $m : \hat{0} = x_0 \to x_1 \to \cdots \to x_k = \hat{1}$ with $\mathcal{R}(m) \subseteq Q$, we shall proceed to label the edges of $f(m)$. First let $0 \leq r_1 < s_1 \leq r_2 < s_2 \leq \cdots \leq r_j < s_j \leq r_{j+1} = k$ be such that $x_{r_i} < x_{s_i}$, $i = 1, 2, \ldots, j$, are the edges of $f(m)$ that are not edges of m, i.e. the "gaps" of $f(m)$. Since $\mathcal{R}(m) \subseteq Q$, m has rising label $(a_{i,1}, a_{i,2}, \ldots, a_{i,t_i})$, where $t_i = s_i - r_i$, on the segment from x_{r_i} to x_{s_i}, for all $i = 1, 2, \ldots, j$. For $0 \leq i < r_1$, we label edge $x_i \to x_{i+1}$ of $f(m)$ with the 1-tuple, $\lambda(m, x_i \to x_{i+1})$. Label edge $x_{r_1} \to x_{s_1}$ of $f(m)$ with the t_1-tuple $(a_{1,1}, a_{1,2}, \ldots, a_{1,t_1})$. For $s_1 \leq i < r_2$, label edge $x_i \to x_{i+1}$ with the t_1-tuple $(a_{1,1}, a_{1,2}, \ldots, a_{1,t_1-1}, \lambda(m, x_i \to x_{i+1}))$. Label edge $x_{r_2} \to x_{s_2}$ with the (t_1+t_2-1)-tuple $(a_{1,1}, a_{1,2}, \ldots, a_{1,t_1-1}, a_{2,1}, a_{2,2}, \ldots, a_{2,t_2})$. Continue labeling the edges in this way, so that for all $i = 1, 2, \ldots, j$, the edge $x_{r_i} \to x_{s_i}$ and all edges between x_{s_i} and $x_{r_{i+1}}$ are labeled with $(t_1 + t_2 + \cdots + t_i - (i-1))$-tuples.

It is not difficult to check that

$$\mathcal{R}(m) = \mathcal{R}(f(m)) \tag{10.2}$$

for all $m \in \mathcal{M}(P)$ such that $\mathcal{R}(m) \subseteq Q$. From this it follows that for $x < y$ in Q and maximal chain r of $[\hat{0}, x]$ in P, m is a rising chain of the rooted interval $[x, y]_r$ in P if and only if $m \cap Q$ is a rising chain of the rooted interval $[x, y]_{r \cap Q}$ in Q. Hence every rooted interval of Q has a unique rising chain and therefore the labeling constructed above is a CR-labeling. By applying Lemma 5.3, one can see that it is also a CL-labeling whenever λ is.

The statement about the Betti numbers follows from Theorem 10.1 or from (10.2), which implies that the set of falling chains of Q is $\{f(m) \mid m \in \mathcal{M}(P)$ and $\mathcal{R}(m) = m \cap \bar{Q}\}$. By counting these falling chains one also obtains the statement about the Möbius function. □

The intersection condition for maximal chains in Theorem 10.2 is clearly implied by the following:

(10.3) For any rooted interval $[x, y]_r$ of P with $x, y \in Q$, if the unique rising chain m in $[x, y]_r$ contains no elements of Q other than x and y, then the whole interval $[x, y]$ contains no element of Q other than x and y.

An application to Tamari lattices will be given now and another application will be given later in this section.

10.3. Example. Let $Y_n = [0, n-1] \times [0, n-2] \times \cdots \times [0, 1] \times [0, 0]$, where $[0, j] = \{0, 1, \ldots, j\}$ with the natural order. The poset Y_n, a direct product of chains, has an EL-labeling as in (9.1). Namely, if $\mathbf{r} \to \mathbf{s}$ is a covering and $r_j \neq s_j$ then $\lambda(\mathbf{r} \to \mathbf{s}) = (j, r_j)$, and these labels are ordered lexicographically.

By Definition 9.1 we have that the Tamari lattice T_n is embedded in Y_n as an induced subposet. We leave to the reader the easy verification that condition (10.3) is satisfied. In fact, one finds that for $[\mathbf{r}, \mathbf{s}]$ in Y_n with $\mathbf{r}, \mathbf{s} \in T_n$, if m is the rising chain in $[\mathbf{r}, \mathbf{s}]$ and $m \cap Q = \{\mathbf{r}, \mathbf{s}\}$ then there exists j such that $r_i = s_i$ for all $i \neq j$ and $[\mathbf{r}, \mathbf{s}] = m$. Therefore, by Theorem 10.2 we have that T_n is CL-shellable with a labeling that is induced by that of Y_n. This induced labeling is equivalent to the one used in Theorem 9.2.

Theorem 10.2 implies the result for pure posets that rank selection preserves CL-shellability (cf. [BW2]). There is a simplicial complex version of rank selection called type selection on a balanced pure simplicial complex (cf. [S4],[B3]). We shall now extend these notions from pure complexes to general complexes.

First we define a *balanced* $(d-1)$-complex (Δ, τ) to be a $(d-1)$-complex Δ together with a "coloring" function $\tau : V \to [d]$ such that $|\{v \in F \mid \tau(v) = i\}| \leq 1$ for all facets F and $i = 1, \ldots, d$. In other words, the restriction of τ to each facet is injective. We define the *type* of a face G to be $\tau(G)$. For $S \subseteq [d] = \{1, 2, \ldots, d\}$, the *type-selected* subcomplex of Δ, denoted by Δ_S is defined to be $\{G \in \Delta \mid \tau(G) \subseteq S\}$. For $G \in \Delta$ we let G_S denote the face $\{v \in G \mid \tau(v) \in S\}$ of Δ_S.

A *completely balanced* complex is defined to be a balanced complex (Δ, τ) such that for each facet F, $\tau(F) = \{1, 2, \ldots, |F|\}$. When Δ is pure these notions of balanced and completely balanced are identical and coincide with the notion of completely balanced complex given in [S4]. If P is a pure bounded poset with rank function ρ then $(\Delta(\bar{P}), \rho)$ is a completely balanced pure simplicial complex.

We shall use Theorem 10.1 to show that shellability is preserved by type selection on a completely balanced complex. To do this we need the following lemma.

10.4. Lemma. *Suppose (Δ, τ) is a completely balanced shellable $(d-1)$-complex and $S \subseteq [d]$. If F is a facet of Δ such that $\mathcal{R}(F) \subseteq F_S$ then F_S is a facet of the type-selected subcomplex Δ_S.*

Proof. By the Rearrangement Lemma (Lemma 2.6) we can assume that all facets of larger size precede those of smaller size in the shelling order. Suppose that F is a facet of Δ such that $\mathcal{R}(F) \subseteq F_S$ and F_S is not a facet of Δ_S. Then $F_S \subset G_S$ for some facet G of Δ that contains a vertex x such that $\tau(x) > |F|$ and $\tau(x) \in S$. It follows that $|G| > |F|$ which implies that G precedes F in the shelling order. But we have $\mathcal{R}(F) \subseteq F_S \subseteq G$ which contradicts (2β) of Proposition 2.5. $\square$

Suppose (Δ, τ) is a completely balanced $(d-1)$-complex. Then for all $S \subseteq [d]$, the type-selected subcomplex Δ_S is completely balanced with coloring function $\tau_S : \{v \in V : \tau(v) \in S\} \to |S|$ defined by $\tau_S(v) = i$ if $\tau(v)$ is the ith smallest element of S.

10.5. Theorem. *If (Δ, τ) is a completely balanced shellable $(d-1)$-complex then for all $S \subseteq [d]$, (Δ_S, τ_S) is a completely balanced shellable $(|S|-1)$-complex. Furthermore, the type-selected h-triangle is given by*

$$h_{s,j}(\Delta_S) = \textit{number of facets } F \textit{ of } \Delta \textit{ such that } \mathcal{R}(F) \subseteq F_S,$$
$$|F_S| = s \textit{ and } |\mathcal{R}(F)| = j.$$

Consequently, the type-selected Betti numbers are given by

$$\tilde{\beta}_j(\Delta_S) = \textit{ number of facets } F \textit{ of } \Delta \textit{ such that } \mathcal{R}(F) = F_S \textit{ and } |\mathcal{R}(F)| = j+1.$$

Proof. This is an immediate consequence of Theorem 10.1 and Lemma 10.4. $\square$

We will call a bounded poset P *semipure* if all proper lower intervals $[\hat{0}, x]$, $x < \hat{1}$, are pure. For semipure posets P we define a rank function $\rho : \bar{P} \to [\ell(P)-1]$ by

$$\rho(x) = \text{ common length of all maximal chains from } \hat{0} \text{ to } x.$$

We have already seen some examples of semipure posets that are not pure. The dual of the poset $\Pi'_{n,k}$ of Section 7 is one such example. The lattice of faces of a nonpure complex or more generally the augmented face poset of a nonpure regular cell complex (see Section 13) is another example.

Let P be semipure of length ℓ. For $S \subseteq [\ell-1]$, define the *rank-selected subposet* P_S of P to be the induced subposet on $\{x \in P \mid \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. Note that $(\Delta(\bar{P}), \rho)$ is a completely balanced $(\ell-2)$-complex and that $\Delta(\bar{P}_S)$ is the type-selected subcomplex $\Delta(\bar{P})_S$. Recall the notion of descent set $\mathcal{D}(m)$ from Definition 5.4.

10.6. Theorem. *Let P be a semipure poset of length ℓ and let $S \subseteq [\ell-1]$. If P is CL-shellable (resp. dual CL-shellable) then so is the rank-selected subposet P_S. Furthermore, the rank selected Betti numbers are given by*

$$\tilde{\beta}_j(\bar{P}_S) = \textit{ number of maximal chains } m \textit{ of } P \textit{ such that}$$
$$\mathcal{D}(m) = S \cap [\ell(m)-1] \textit{ and } |\mathcal{D}(m)| = j+1.$$

Proof. Since $\Delta(\bar{P})$ is a completely balanced shellable $(\ell-2)$-complex, we can use Lemma 10.4 to conclude that for all maximal chains m of P such that $\mathcal{D}(m) \subseteq S$, the rank-selected chain m_S is a maximal chain of P_S. The respective conclusions follow by applying Theorem 10.2 to P and its dual. □

A weaker result can also be obtained for CR-labelings by applying Theorem 10.2.

10.7. Theorem. *Let P be a semipure poset of length ℓ and let $S = [r]$ for some $r \le \ell - 1$. If P admits a CR-labeling (resp. dual CR-labeling) then so does P_S. Furthermore, the rank-selected Möbius function is given by*

$$\mu_{P_S}(\hat{0}, \hat{1}) = \sum_{\substack{m \in \mathcal{M}(P) \\ \ell(m) \le r \\ \mathcal{D}(m) = [\ell(m)-1]}} (-1)^{\ell(m)} + \sum_{\substack{m \in \mathcal{M}(P) \\ \ell(m) > r \\ \mathcal{D}(m) = [r]}} (-1)^{r+1}.$$

□

The face lattice of the s-skeleton of a complex Δ can be obtained by rank-selection on the face lattice of Δ. Therefore, the $r = 0$ case of Theorem 2.9 is a consequence of Theorem 10.6. Namely, by Theorem 5.13 the face lattice $\hat{L}(\Delta)$ is dual CL-shellable. We have $\hat{L}(\Delta^{(0,s)}) = \hat{L}(\Delta)_{\{1,2,\ldots,s+1\}}$, which is dual CL-shellable by Theorem 10.6. Then by Theorem 5.13 once more, $\Delta^{(0,s)}$ is shellable.

The general (r,s)-skeleton of a complex can also be dealt with as a subposet of a semipure poset. Let P be a semipure poset of length ℓ. For $r = 1, 2, \ldots, \ell-1$, let P^r be the order ideal generated by coatoms of rank $\ge r$ with $\hat{1}$ attached. The following lemma is obvious.

10.8. Lemma. *Let Δ be a $(d-1)$-complex. Then for $0 \le r \le s \le d-1$,*

$$\hat{L}(\Delta^{(r,s)}) = ((\hat{L}(\Delta))^{r+1})_{\{1,2,\ldots,s+1\}}. \qquad \square$$

10.9. Theorem. *Let P be semipure of length ℓ. If P is CL-shellable (resp. dual CL-shellable) then so is P^j for all $j = 1, 2, \ldots, \ell-1$. Moreover, the falling chains of P^j are precisely the falling chains of P of length at least $j+1$.*

Proof. Since the maximal chains of P^j are maximal chains of P, P^j inherits a chain-edge labeling from P. We shall show that this inherited labeling is a CL-labeling (resp. dual CL-labeling). To do this, it suffices to show that if $x, y \in P^j$ then the lexicographically first maximal chain c of any rooted interval $[x,y]_r$ of P is in P^j. The only possible way that c is not in P^j is if $y = \hat{1}$, since $P^j - \{\hat{1}\}$ is an order ideal. Since $\bar{c}$ comes first in the induced shelling of $\Delta((x,\hat{1}))$, c has maximum length among all chains of $[x,\hat{1}]$ by Lemma 2.2. This implies that the maximum element of $\bar{c}$ is a coatom t of P with maximum rank among all coatoms above x. Since $x \in P^j$, $\rho(t) \ge j$. Hence $t \in P^j$, which implies that c is in P^j, since $P^j - \{\hat{1}\}$ is an order ideal. □

Now, in view of Lemma 10.8, Theorem 2.9 in its full generality is a consequence of Theorem 10.6 and Theorem 10.9. The proof given here shows that Theorem 2.9 is valid also for regular cell complexes (Theorem 13.4).

For general nonpure posets, there is still something we can say about rank selection.

10.10. Theorem. *If P is CL-shellable then so is the poset obtained from P by removing all the atoms or all the coatoms of P.*

Proof. We will again use Theorem 10.2. Let P be CL-shellable and let Q be the poset obtained from P by removing all atoms of P. Let $m : \hat{0} = x_0 \to x_1 \to \cdots \to x_k = \hat{1}$ be a maximal chain of P such that $\mathcal{R}(m) \subseteq Q$. This implies that $\lambda(m, \hat{0} \to x_1) < \lambda(m, x_1 \to x_2)$. Hence the chain $c : \hat{0} \to x_1 \to x_2$ is the unique rising chain of the rooted interval $[\hat{0}, x_2]_{\hat{0}}$, which means that it is lexicographically first. It follows that $\bar{c}$ comes first in the induced shelling of $\Delta((\hat{0}, x_2))$. By Lemma 2.2, $\bar{c}$ has maximal length in the interval $(\hat{0}, x_2)$. This means that $(\hat{0}, x_2)$ consists only of atoms of P. This implies that the chain $m \cap Q : x_2 \to \cdots \to x_k$ is maximal in Q. We can therefore apply Theorem 10.2 to conclude that Q is CL-shellable.

The proof for coatoms is similar. □

For each element x of a general bounded poset P, the *rank* of x, $\rho(x)$, is defined to be the maximal length of a chain from $\hat{0}$ to x in P. That is, $\rho(x) = \max \ell(c)$, where c ranges over all chains from $\hat{0}$ to x. The *corank* of x, $\rho^*(x)$ is defined to be the maximal length of a chain from x to $\hat{1}$ in P. So the corank of x is simply the rank of x in the dual poset. Note that when P is pure then the rank function is the usual rank function and the corank function is $\ell(P)$ minus the usual rank function. Also when P is semipure this definition of rank agrees with that previously used.

For $S, T \subseteq [\ell(P) - 1]$, define the *rank selected subposets*

$$\begin{aligned} P_S &= \{x \in P \mid \rho(x) \in S \cup \{0, \ell(P)\}\} \\ P^T &= \{x \in P \mid \rho^*(x) \in T \cup \{0, \ell(P)\}\} \\ P_S^T &= P_S \cap P^T. \end{aligned}$$

It turns out that rank selection in this general version does not preserve CL-shellability. For example, let P be the poset with EL-labeling given in Figure 9a. For $S = \{1, 2\}$, the rank selected poset P_S given in Figure 9b is clearly not shellable. However, a special type of rank selection, called truncation, does preserve CL-shellability as the next result shows.

10.11. Theorem. *Let P be CL-shellable of length ℓ. For $1 \le s, t < \ell$, let $S = \{s, s+1, \ldots, \ell - 1\}$ and $T = \{t, t+1, \ldots, \ell - 1\}$. Then the truncations P_S, P^T, and P_S^T are all CL-shellable.*

Proof. Since $P_S^T = (P_S)^T$ we need only prove that P_S and P^T are CL-shellable. We prove that P_S is CL-shellable by induction on s. If $s = 1$ then $P_S = P$. So assume $s > 1$. The poset P_S is obtained from $P_{\{s-1,s,\ldots,\ell-1\}}$ by removing the atoms of $P_{\{s-1,s,\ldots,\ell-1\}}$. Since by induction $P_{\{s-1,s,\ldots,\ell-1\}}$ is CL-shellable, it follows from Theorem 10.10 that P_S is CL-shellable.

The proof for P^T is the same with coatoms playing the role of atoms. □

Define the *k-coskeleton* of a simplicial complex Δ as the subcomplex obtained by removing all faces F such that $\dim lk_\Delta(F) < k$. For instance, the 0-coskeleton is obtained by removing all facets. Theorems 5.13 and 10.11 imply the following.

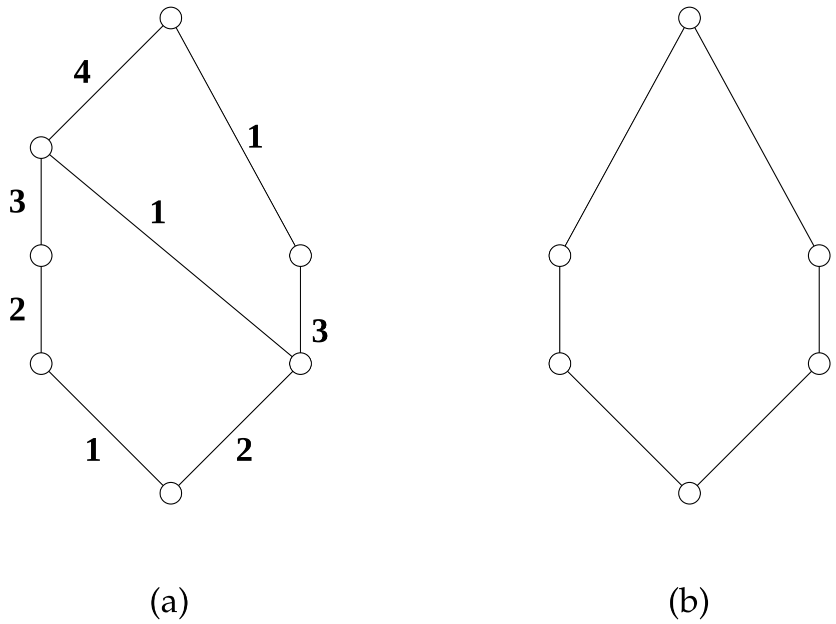

FIGURE 9.

10.12. Corollary. *Let Δ be a shellable $(d-1)$-complex and let $0 \le k \le d-2$. Then the k-coskeleton of Δ is shellable.* $\square$

10.13. Remark. All the preceding results of this section dealing with CL-shellability remain valid if we replace the notion of CL-shellability with the notion of shellability. It can also be shown that upper truncation $P^{\{t,t+1,\dots\}}$ preserves EL-shellability.

It was mentioned in Lemma 5.6 that CL-shellability is inherited by all intervals in a poset. The same is true for shellability, with a straightforward proof as in the pure case [B1, Proposition 4.2]. The same argument (taking the induced order of facets) actually shows that shellability is inherited by all links of faces in a complex. The *link* of a face $A \in \Delta$ is the subcomplex

$$lk_\Delta(A) = \{B \in \Delta \mid B \cap A = \emptyset,\, B \cup A \in \Delta\}.$$

10.14. Proposition. *If Δ is shellable then so is $lk_\Delta(A)$, for all faces $A \in \Delta$.* $\square$

As an alternative (but unnecessarily complicated) proof, Proposition 10.14 is directly implied by Lemma 5.6 and Theorem 5.13, since the face lattice of $lk_\Delta(A)$ is an upper interval in that of Δ.

For the rest of this section we will consider products of posets. In [B1,§4] and [BW2,§8] it is proved in the pure case that shellability, CL-shellability and EL-shellability are all preserved by taking direct products. The proof for lexicographic shellability goes through in the nonpure case with essentially no change, while the proof for shellability needs significant modification which we discuss below (Theorem 10.21). For pure or nonpure posets, there is a useful strengthening of the lexicographic-shellability-preserving result which we present next.

Let P_1 and P_2 be two CL-shellable posets with respective CL-labelings $\lambda_1 : \mathcal{ME}(P_1) \to \Lambda_1$ and $\lambda_2 : \mathcal{ME}(P_2) \to \Lambda_2$, where Λ_1 and Λ_2 are disjoint totally ordered label sets. Let $\lambda : \mathcal{ME}(P_1 \times P_2) \to \Lambda$ be the product chain-edge labeling defined by

$$\lambda(m, (a, b) \to (c, b)) = \lambda_1(m_1, a \to c)$$
$$\lambda(m, (a, b) \to (a, d)) = \lambda_2(m_2, b \to d),$$

where m_i is the maximal chain of P_i obtained by projecting m to P_i. In order for λ to be a CL-labeling of $P_1 \times P_2$, we need to impose an order on $\Lambda_1 \cup \Lambda_2$. In [BW2] the order is simply the ordinal sum $\Lambda_1 \oplus \Lambda_2$. It is just as easy to show that any shuffle of the two total orders will do.

10.15. Proposition. *Fix any shuffle of the total orders on Λ_1 and Λ_2 to get a total order of $\Lambda = \Lambda_1 \cup \Lambda_2$. Then the product chain-edge labeling λ defined above is a CL-labeling of $P_1 \times P_2$. Moreover, λ is an EL-labeling whenever λ_1 and λ_2 are.*□

In [BW2] the question of whether shellability is preserved by taking the direct product of pure posets which have bottom elements but no top elements, was left open. This is equivalent to asking whether shellability is preserved by taking the reduced product of two bounded posets (as defined in [Su]). The *upper reduced product* of bounded posets P_1 and P_2 is defined to be the poset obtained by attaching a top $\hat{1}$ to $(P_1 - \{\hat{1}_1\}) \times (P_2 - \{\hat{1}_2\})$. Similarly the *lower reduced product* is defined to be the poset obtained by attaching a bottom $\hat{0}$ to $(P_1 - \{\hat{0}_1\}) \times (P_2 - \{\hat{0}_2\})$.The upper reduced product of P_1 and P_2 is denoted $P_1 \hat{\times} P_2$ and the lower reduced product is denoted $P_1 \check{\times} P_2$. Note that the lower and upper reduced products $P_1 \check{\times} P_2$ and $P_1 \hat{\times} P_2$ are induced subposets of the direct product $P_1 \times P_2$.

10.16. Theorem. *Let P_1 and P_2 be bounded posets. Then the following are equivalent.*

(1) *P_1 and P_2 are CL-shellable.*
(2) *$P_1 \times P_2$ is CL-shellable.*
(3) *$P_1 \hat{\times} P_2$ is CL-shellable.*
(4) *$P_1 \check{\times} P_2$ is CL-shellable.*

Proof. (1) implies (2) is a consequence of Proposition 10.15. (1) implies (3) and (4) is a consequence of Theorem 10.17 below. To prove that each of (2), (3) and (4) imply (1), we note that P_1 and P_2 are intervals of the product and reduced product. Indeed, for the upper reduced product, $P_1 \simeq [(\hat{0}_1, z_2), \hat{1}]$ and $P_2 \simeq [(z_1, \hat{0}_2), \hat{1}]$, where z_i is any maximal element of $P_i - \{\hat{1}_i\}$. Since intervals of CL-shellable posets are CL-shellable (Lemma 5.6), P_1 and P_2 are CL-shellable whenever the direct product or reduced product is. □

10.17. Theorem. *Let P_1 and P_2 admit CL-labelings. Then there are CL-labelings of $P_1 \times P_2$, $P_1 \check{\times} P_2$ and $P_1 \hat{\times} P_2$ whose respective falling chains are of the form*

$$(x_0, y_0) \to (x_0, y_1) \to \cdots \to (x_0, y_s) \to (x_1, y_s) \to \cdots \to (x_r, y_s), \tag{10.4}$$

$$\hat{0} \to (x_1, y_1) \to (x_1, y_2) \to \cdots \to (x_1, y_s) \to (x_2, y_s) \to \cdots \to (x_r, y_s), \tag{10.5}$$

and

$$(10.6) \qquad (x_0, y_0) \to \cdots \to (x_0, y_{s-1}) \to (x_1, y_{s-1}) \to \cdots \to (x_{r-1}, y_{s-1}) \to \hat{1},$$

where $\hat{0}_1 = x_0 \to x_1 \to \cdots \to x_r = \hat{1}_1$ *and* $\hat{0}_2 = y_0 \to y_1 \to \cdots \to y_s = \hat{1}_2$ *are falling chains of* P_1 *and* P_2 *respectively.*

Proof. The CL-labeling of $P_1 \times P_2$ whose falling chains have the form given in (10.4) is the product chain-edge labeling with label set equal to the ordinal sum of the label sets of P_1 and P_2.

Since the reduced products are induced subposets of $P_1 \times P_2$, we would like to be able to apply Theorem 10.2. To do this we must find an appropriate CL-labeling of $P_1 \times P_2$. The natural ordinal sum product chain-edge labeling given in the previous paragraph does not work. However, it turns out that there is a shuffle of label sets for P_1 and P_2 for which the product chain-edge labeling of Proposition 10.15 does work. Before describing this shuffle, we need some terminology and a technical lemma.

Given a poset P with CL-labeling $\lambda : \mathcal{ME}(P) \to \Lambda$, we shall refer to an element $s \in \Lambda$ as an *atomic* label if $s = \lambda(m, \hat{0} \to a)$ for some maximal chain m with atom a. We say that $s \in \Lambda$ is a *nonatomic* label if $s = \lambda(m, x \to y)$ for some $x \neq \hat{0}$. Note that s can be both atomic and nonatomic, but not if λ satisfies the following conditions, in which case we say that λ is *orderly*:

(i) Λ is totally ordered,
(ii) each nonatomic label is either less than every atomic label or greater than every atomic label.

A *coorderly* CL-labeling satisfies the above conditions with coatoms playing the role of atoms.

10.18. Lemma. *Suppose* P *admits a CL-labeling* λ. *Then* P *admits an orderly CL-labeling and a coorderly CL-labeling whose respective restriction maps are identical to that of* λ.

Proof. By following the "if" part of the proof of Theorem 3.2 of [BW2], we can construct a recursive atom ordering for P which is compatible with the CL-labeling λ. By then following the "only if" part of the proof we can construct an orderly CL-labeling of P from the recursive atom ordering. Indeed, labels from the real numbers $\mathbf{R}$ are first assigned to the bottom edges compatibly with the recursive atom ordering. Then the intervals above the atoms are assigned CL-labelings, as in the proof of [BW2,Theorem 3.2], with the additional requirement that respective label sets are disjoint from an interval of $\mathbf{R}$ containing the atomic labels. It is easy to check that the orderly CL-labeling constructed from the recursive atom ordering has the same restriction map as the original CL-labeling λ.

Although it is a bit trickier, a coorderly CL-labeling with the same restriction map can be constructed in a similar fashion. Now the labels are assigned in a top down manner with the requirement that every rooted interval $[x, \hat{1}]_r$ gets a coorderly CL-labeling in which each atomic label of $[x, \hat{1}]_r$, except for the smallest, is larger than all the coatomic labels of P. We leave the details to the reader. $\square$

Proof of Theorem 10.17 continued. Let $\lambda_1 : \mathcal{ME}(P_1) \to \Lambda_1$ and $\lambda_2 : \mathcal{ME}(P_2) \to \Lambda_2$ be CL-labelings. By Lemma 10.18 we can assume that λ_1 and λ_2 are orderly. Also assume that Λ_1 and Λ_2 are disjoint and that λ_i, $i = 1, 2$ is surjective. Let A_i be the set of atomic labels of Λ_i, S_i be the set of labels of Λ_i that are smaller than the atomic labels and let B_i be the set of labels that are bigger. Shuffle the totally ordered sets Λ_1 and Λ_2 so that S_1 precedes S_2 which precedes A_1 which precedes A_2 which precedes B_1 which precedes B_2.

By Proposition 10.15, the product labeling λ of $P_1 \times P_2$ with the above total order of $\Lambda_1 \cup \Lambda_2$, is a CL-labeling. We shall show that with this CL-labeling (10.3) holds for $P = P_1 \times P_2$ and $Q = P_1 \check{\times} P_2$. Let $[x, y]_r$ be a rooted interval of P with $x, y \in Q$. If a maximal chain m of $[x, y]_r$ contains no elements of Q other than x and y then $x = \hat{0}$ and $y = (u_1, u_2)$ where u_1 is an atom of P_1 or u_2 is an atom of P_2. The bottom label of m is an atomic label of either P_1 or P_2 and the top label of m is an atomic label of the other. Any label in between is nonatomic. Hence, the labels in between are either all less than the top and bottom labels or greater than the top and bottom labels. It follows that the only way that m can be increasing is if there are no labels in between. Consequently if m is increasing then both u_1 and u_2 are atoms. Hence, the interval $[x, y]$ consists only of $x = \hat{0}$, $(u_1, \hat{0}_2)$, $(\hat{0}_1, u_2)$ and $y = (u_1, u_2)$. Clearly the only elements of this interval in Q are x and y. So (10.3) holds.

By Theorem 10.2, $P_1 \check{\times} P_2$ is CL-shellable and its set of falling chains is

$$\{m \cap (P_1 \check{\times} P_2) \mid m \in \mathcal{M}(P_1 \times P_2) \text{ and } \mathcal{R}(m) = m \cap \overline{P_1 \check{\times} P_2}\}.$$

Let m be a maximum chain of $P_1 \times P_2$ such that $\mathcal{R}(m) = m \cap \overline{P_1 \check{\times} P_2}$. Then m starts at $(\hat{0}_1, \hat{0}_2)$, immediately leaves $P_1 \check{\times} P_2$ and later reenters and stays in $P_1 \check{\times} P_2$ all the way to the top. Up until the point of reentry m is rising and after reentry m is falling. The bottom label of the rising segment is an atomic label of P_1 and the top label is an atomic label of P_2. All the labels in between are nonatomic and less than the top label, which implies that they are less than the bottom label also. Hence the only way that this segment of m is rising is if there are no labels in between. This implies that the rising segment is of the form $(\hat{0}_1, \hat{0}_2) \to (x_1, \hat{0}_2) \to (x_1, y_1)$ where x_1 is an atom of P_1 and y_1 is an atom of P_2. The falling segment of m is of the form $(x_1, y_1) \to (x_1, y_2) \to \cdots \to (x_1, y_s) \to \cdots \to (x_r, y_s)$ where $\hat{0}_1 \to x_1 \to \cdots \to x_r = \hat{1}_1$ and $\hat{0}_2 \to y_1 \to \cdots \to y_s = \hat{1}_2$ are falling chains of P_1 and P_2, respectively. It follows that $m \cap \overline{P_1 \check{\times} P_2}$ is of the form given in (10.5).

The proof that $P_1 \hat{\times} P_2$ admits a CL-labeling with falling chains of the form given in (10.6) is essentially the same with coatoms playing the role of atoms. $\square$

We remark that the following immediate consequence of Theorems 5.9(i) and 10.17 is known to be valid for all bounded posets with torsion-free homology by means of the Künneth formula and results of Quillen [Q] and Walker [Wa] (see [Su]).

10.19. Corollary. *Suppose P_1 and P_2 are CL-shellable posets. Then*

$$\begin{aligned}\tilde{H}_i(\Delta(\overline{P_1 \times P_2})) &\cong \tilde{H}_{i-1}(\Delta(\overline{P_1 \hat{\times} P_2})) \\ &\cong \tilde{H}_{i-1}(\Delta(\overline{P_1 \check{\times} P_2})) \\ &\cong \bigoplus_{j+k=i-2} \tilde{H}_j(\Delta(\overline{P}_1)) \otimes \tilde{H}_k(\Delta(\overline{P}_2)). \qquad \square\end{aligned}$$

Products of CL-shellable posets can also be shown to be CL-shellable by using recursive atom orderings directly. We use the natural bijection between the set of atoms of $P_1 \times P_2$ (or $P_1 \hat{\times} P_2$) and the union of the sets of atoms of P_1 and P_2. Any ordering of the union induces an atom ordering of products. We leave the proof of the following result as an exercise.

10.20. Theorem. *Suppose Ω_i is a recursive atom ordering of P_i, i=1,2. Then*

(1) *Any shuffle of Ω_1 with Ω_2 induces a recursive atom ordering of $P_1 \times P_2$.*
(2) *Any shuffle of Ω_1 with Ω_2 in which the first atom is from Ω_1 and the second is from Ω_2 or vice versa, induces a recursive atom ordering of $P_1 \hat{\times} P_2$.*
(3) *Any linear extension of $\Omega_1 \times \Omega_2$ is a recursive atom ordering of $P_1 \check{\times} P_2$.* $\square$

CL-shellability is not preserved by taking *doubly reduced* products

$$((P_1 - \{\hat{0}_1, \hat{1}_1\}) \times (P_2 - \{\hat{0}_2, \hat{1}_2\})) \cup \{\hat{0}, \hat{1}\},$$

as is shown by the example in Figure 8.1 of [BW2].

The original question of whether shellability is preserved by taking reduced products of bounded posets remains open. (We don't know whether EL-shellability is preserved either.) It is shown in [BW2, Theorem 8.3] that direct products preserve shellability in the pure case. The proof for the nonpure case requires a modification which we now describe.

10.21. Theorem. *Let P and Q be bounded posets. Then $P \times Q$ is shellable if and only if P and Q are shellable.*

Proof. The modification that needs to be made to the proof of [BW2, Theorem 8.3] is as follows. In the original proof each maximal chain m of $P \times Q$ is represented by the triple $(\sigma(m), \pi_P(m), \pi_Q(m))$. Now, for the general case we need to represent m by the 4-tuple $(\ell(m), \sigma(m), \pi_P(m), \pi_Q(m))$, where $\ell(m)$ is the length of m and $\sigma(m), \pi_P(m), \pi_Q(m)$ are as in [BW2]. These 4-tuples are ordered lexicographically by using decreasing order on the first component, lexicographical order on the second component, and shelling orders on the third and fourth components. We leave it to the reader to check that this order on the 4-tuples induces a shelling order on the maximal chains of $P \times Q$. $\square$

10.22. Remark. All of the other shellability preserving results of [B1,§4] and [BW2,§8] go through in the nonpure case. There are minor changes that need to be made to some of the proofs. For example, in the proof of [B1, Theorem 4.6], the rank function ρ can be taken to be the general one defined here. Also one direction of the proof of [BW2, Theorem 8.6] dealing with ordinal sums requires that rank selection preserve CL-shellability. All that is really needed is the truncation result of Theorem 10.12. The result that the ordinal sum of two posets is shellable if and only if each of the posets is shellable ([B1, Theorem 4.4] in the pure case), is actually a special case of the result that the join of two simplicial complexes is shellable if and only if each of the simplicial complexes is shellable. The proof of the "if" direction of the general result is the same as that of [B1, Theorem 4.4]. The "only if" direction follows from Proposition 10.14, since each complex in the join is the link of any facet from the other complex.

11. Shifting, vertex-decomposability and CL-shellability

The property of being "vertex-decomposable" was defined for pure complexes by Provan and Billera [PB] in connection with their study of diameter problems for pure complexes. It implies shellability. We will extend this to nonpure complexes and prove that the order complex of a CL-shellable poset is vertex-decomposable. This also reveals some special combinatorial facts about the structure of CL-shellable posets.

Recall the definition of the *link* $lk_\Delta(x)$ of a complex Δ at a vertex x, and the *deletion*
$\Delta \setminus x$:

$$lk_\Delta(x) = \{A \in \Delta \mid x \notin A,\ A \cup x \in \Delta\} \quad , \quad \Delta \setminus x = \{A \in \Delta \mid x \notin A\}.$$

11.1. Definition. *A complex Δ is* vertex-decomposable *if*

(i) *Δ is a simplex or $\Delta = \{\emptyset\}$, or*

(ii) *there exists a vertex x such that*

  (α) *$\Delta \setminus x$ and $lk_\Delta(x)$ are vertex-decomposable*

  (β) *no facet of $lk_\Delta(x)$ is a facet of $\Delta \setminus x$.*

For pure Δ this specializes to the definition of Provan and Billera [PB]. The distinguished vertex x in (ii) is called a *shedding vertex*.

11.2. Definition. *A complex Δ on vertex set $[n] = \{1, 2, \dots, n\}$ is called* shifted *if $i < j$, $i \notin A$, $j \in A$ implies $A \setminus \{j\} \cup \{i\} \in \Delta$, for all $A \in \Delta$.*

Shifted complexes play an important role in extremal combinatorics and the theory of f-vectors, see [BK] for information and further references. The following implications were previously known in the pure case, the first one from [BK, Theorem 3] and the second one from [PB, Theorem 2.8]. Both implications are strict, already for pure complexes.

11.3. Theorem. *Shifted $\Longrightarrow$ vertex-decomposable $\Longrightarrow$ shellable.*

Proof. Let Δ be shifted and not a simplex. Clearly, $\Delta \setminus n$ and $lk_\Delta(n)$ are shifted complexes on $[n-1]$. Also, no facet of $lk_\Delta(n)$ can be maximal in $\Delta \setminus n$, since n can be replaced in any $F \in \Delta$ containing n by some earlier element not in F. So n is a shedding vertex, and the proof of the first implication is completed by induction.

Now, suppose that Δ is vertex-decomposable and not a simplex, and let x be a shedding vertex. By induction we can let $F_1, \dots, F_t$ be a shelling order of the facets of $\Delta \setminus x$, and $E_1, \ldots, E_s$ a shelling order of the facets of $lk_\Delta(x)$. It follows directly from Proposition 2.5 that $F_1, \ldots, F_t, E_1 \cup \{x\}, \ldots, E_s \cup \{x\}$ is a shelling of Δ. $\square$

Putting the two parts of this proof together and using Proposition 2.5 for the restriction map, we obtain the following explicit description of how to shell a shifted complex.

11.4. Corollary. *For a shifted complex reverse lexicographic order of the facets is a shelling order, with restriction map*

$$\mathcal{R}(F) = F \setminus [j], \text{ if } j \geq 0 \text{ is minimal such that } (j+1) \notin F.$$

For instance, the shifted complex with facets 123, 124, 125 and 34 has the shelling 123, 124, 34, 125. It can be shown that also lexicographic order of the facets of a shifted complex is a shelling, and its restriction map is the same as that of reverse lexicographic order.

The rest of this section will concern CL-shellable posets and hinges on the following technical facts.

11.5. Lemma. *Suppose that $\lambda : \mathcal{ME}(P) \to \Lambda$ is a CL-labeling of a poset P that is not a chain. Let $\{x\}$ be the restriction of one of the lexicographically greatest maximal chains with exactly one descent. Then*

(i) *$[\hat{0}, x]$ is a chain.*

(ii) *If $m : \hat{0} = x_0 \to x_1 \to \ldots \to x_k = \hat{1}$ is any maximal chain through x, say $x_i = x$, then*

$$\lambda(m, x_{i-1} \to x) \not< \lambda(m, x \to x_{i+1}).$$

(iii) *If $a \to x \to b$ then there exists $y \neq x$ such that $a < y < b$.*

(iv) *λ restricts to a CL-labeling of $P \setminus x$.*

Proof. Let us first note that there exist maximal chains with one descent. In fact, any maximal chain whose only lexicographical predecessor is the rising chain has this property.

(i) Let $m_{\hat{0},x}$ be the rising chain in $[\hat{0}, x]$ and assume that $y \notin m_{\hat{0},x}$, $y \in [\hat{0}, x]$. If $m_{\hat{0},y}$ is the rising chain in $[\hat{0}, y]$ and $m_{y,\hat{1}}$ the rising chain in $[y, \hat{1}]_{m_{\hat{0},y}}$ then the concatenated chain $m_{\hat{0},y} m_{y,\hat{1}}$ has at most one descent (at y), so $m_{\hat{0},y} m_{y,\hat{1}} \not>_{lex} m_{\hat{0},x} m_{x,\hat{1}}$ by choice of x. On the other hand, $m_{\hat{0},x} m_{x,\hat{1}} <_{lex} m_{\hat{0},y} m_{y,\hat{1}}$ by Lemma 5.3.

(ii) Suppose that $\lambda(m, x_{i-1} \to x) < \lambda(m, x \to x_{i+1})$. It follows from part (i) that $\hat{0} = x_0 \to x_1 \to \ldots \to x_i = x$ is the rising chain $m_{\hat{0},x}$ in $[\hat{0}, x]$, so we now get that $x_0 \to x_1 \to \ldots \to x_{i+1}$ must be the rising chain $m_{\hat{0},x_{i+1}}$ in $[\hat{0}, x_{i+1}]$. Let $m_{x,\hat{1}}$ be the rising chain in $[x, \hat{1}]_{m_{\hat{0},x}}$ and similarly for $m_{x_{i+1},\hat{1}}$. If $x \to z$ is the first step of $m_{x,\hat{1}}$ then $z \neq x_{i+1}$, since the concatenation $m_{\hat{0},x} m_{x,\hat{1}}$ has a descent at x. It follows from Lemma 5.3 that $m_{\hat{0},x} m_{x,\hat{1}} <_{lex} m_{\hat{0},x_{i+1}} m_{x_{i+1},\hat{1}}$. This contradicts the choice of x, since the chain $m_{\hat{0},x_{i+1}} m_{x_{i+1},\hat{1}}$ has at most one descent (at x_{i+1}).

(iii) Take a maximal chain m containing $a \to x \to b$. In the notation of part (ii) we have that $a = x_{i-1}$, $b = x_{i+1}$ and $a \to x \to b$ was shown to be falling. The existence of a rising chain in the rooted interval $[a,b]_{\hat{0}\to\ldots\to a}$ forces the existence of y.

(iv) Part (iii) shows that the covering relations of $P\setminus x$ are precisely the coverings of P that don't involve x. Therefore the chain-edge labeling λ restricts to a chain-edge labeling of $P \setminus x$. If $[y,z]_r$ is a rooted interval in $P \setminus x$ then part (ii) shows that its rising chain is contained in $P \setminus x$. Hence, λ is a CL-labeling of $P \setminus x$. □

The following is a sharpening of Theorem 5.8.

11.6. Theorem. *If a bounded poset P is CL-shellable then $\Delta\left(\overline{P}\right)$ is vertex-decomposable.*

Proof. Suppose that P is not a chain (equivalently, $\Delta(P)$ is not a simplex). We will use induction on the size of P.

Take $x \in \overline{P}$ as described in Lemma 11.5. Then part (iv) shows that $P \setminus x$ is CL-shellable, hence $\Delta\left(\overline{P}\right) \setminus x = \Delta\left(\overline{P \setminus x}\right)$ is vertex-decomposable. Furthermore, the interval $[x, \hat{1}]$ is CL-shellable (Lemma 5.6), so $lk_{\Delta(\overline{P})}(x) = \Delta\left(\hat{0}, x\right) * \Delta\left(x, \hat{1}\right) =$ simplex $* \Delta\left(x, \hat{1}\right)$ is vertex-decomposable. Finally, part (iii) of Lemma 11.5 shows that no facet of $lk_{\Delta(\overline{P})}(x)$ is a facet of $\Delta\left(\overline{P}\right) \setminus x$. Thus, x is a shedding vertex. □

The following was shown for the pure case by Provan and Billera [PB, Theorem 3.3.1]. It is a consequence of Theorems 11.6 and 5.13.

11.7. Corollary. *If Δ is a shellable complex, then its barycentric subdivision* $\mathrm{sd}(\Delta) = \Delta(\overline{\hat{L}(\Delta)})$ *is vertex-decomposable.* □

In the context of Corollary 11.7 and Theorem 5.13, let us mention that Ziegler [Z2] has shown that if Δ is pure and vertex-decomposable then $\hat{L}(\Delta)$ is CL-shellable (has a recursive atom ordering). The result presumably extends to the nonpure case, but we have not checked this. Ziegler also conjectures the implication: $\hat{L}(\Delta)$ CL-shellable $\Longrightarrow \Delta$ shellable.

A pure $(d-1)$-dimensional complex Δ on n vertices is said to satisfy the *Hirsch bound* if one can walk in Δ from any facet to any other facet by a sequence of at most $n - d$ single-element exchanges. This notion comes from the theory of convex polytopes, where the validity of the Hirsch bound is an important open problem related to linear programming. Provan and Billera showed that pure vertex-decomposable complexes satisfy the Hirsch bound [PB, Corollary 2.11]. Thus, order complexes of distributive lattices (being vertex-decomposable) satisfy the Hirsch bound [PB, Example 3.4.2] and this was extended (without using vertex-decomposability) to all semimodular lattices in [B1, Theorem 6.4]. As a consequence of Theorem 11.6 and the result of Provan and Billera we can now extend this further.

11.8. Theorem. *If a pure bounded poset P is CL-shellable then its order complex $\Delta\left(\overline{P}\right)$ satisfies the Hirsch bound.*

The diameter of the adjacency graph of maximal chains in a CL-shellable poset is probably in most cases much smaller than the Hirsch bound. This is certainly the case for semimodular lattices [B1, Theorem 6.4], and it might be interesting to investigate some other cases such as Bruhat order of finite Coxeter groups [BW1].

We will end this section by pointing out some purely combinatorial consequences of Lemma 11.5. For that purpose it is convenient to introduce the following terminology.

11.9. Definition. *Let P be a poset with least element $\hat{0}$. Call $x \neq \hat{0}$* irreducible *if*

(i) *the interval $[\hat{0}, x]$ is a chain, and*

(ii) *if $a \to x \to b$ then there exists $y \neq x$ such that $a < y < b$.*

Furthermore, call P dismantlable *if P can be reduced to a single chain by successive removal of irreducibles.*

These definitions are similar but not equivalent to certain other definitions of "irreducible" elements and "dismantlable" posets in the literature. Note that if x is irreducible in P, then

$$l(P \setminus x) = l(P) \qquad \text{and} \qquad l^{-}(P \setminus x) \geq l^{-}(P), \tag{11.1}$$

where $l(P)$ denotes the length of the longest chain of P and $l^{-}(P)$ the length of the shortest maximal chain of P. Note also that if P is a lattice then $P \setminus x$ is a join-subsemilattice, since x is a join-irreducible element. Hence, if P is a dismantlable lattice there exists a chain m in P of maximal length and join-subsemilattices L_i such that $m = L_0 \subset \ldots \subset L_n = P$, $|L_i| = |m| + i$ and $l^{-}(L_0) \geq l^{-}(L_1) \geq \ldots \geq l^{-}(L_n)$. Note that if P is pure then all intermediate lattices L_i are also pure. An example of a non-dismantlable lattice is given by weak order on S_3 (the 6-element lattice shown in Figure 9b).

11.10. Theorem. *If P is CL-shellable then P is dismantlable.*

Proof. This follows from Lemma 11.5, which (as long as P is not a chain) shows the existence of an irreducible element x for which $P \setminus x$ is again CL-shellable. $\square$

12. Decompositions of the Stanley-Reisner ring

Here we will study some combinatorial properties of the Stanley-Reisner ring of a shellable complex. In particular, we will prove that shellability induces a certain direct sum decomposition. This was previously known in the pure case, for which it is equivalent to Cohen-Macaulayness.

Let Δ be a $(d-1)$-dimensional complex on vertex set $V = \{x_1, \ldots, x_n\}$ and let $\mathbf{k}$ be a field. Define the *Stanley-Reisner ring* $\mathbf{k}[\Delta]$ to be the polynomial ring $\mathbf{k}[x_1, \ldots, x_n]$ modulo the ideal generated by all squarefree monomials $x_{i_1} x_{i_2} \ldots x_{i_k}$ such that $\{x_{i_1}, \ldots, x_{i_k}\} \notin \Delta$. We will assume familiarity with the basic notions of commutative algebra, as they apply to Stanley-Reisner rings. See [S5] for all unexplained terminology and general background.

We begin by expressing the Hilbert series $F(\mathbf{k}[\Delta], t)$ in terms of the h-triangle $(h_{i,j})_{0 \leq j \leq i \leq d}$ and the polynomial $H(x, y)$ defined in Section 3.

12.1. Theorem. $F(\mathbf{k}[\Delta], t) = \sum_{0 \le j \le i} h_{i,j} \frac{t^j}{(1-t)^i} = H\left(\frac{t}{1-t}, \frac{1}{t}\right)$.

Proof. Using [S5, p. 63] for the first equality and (3.4) for the third we have

$$\begin{aligned} F(\mathbf{k}[\Delta], t) &= \sum_{0 \le j \le i} f_{i,j} \left(\frac{t}{1-t}\right)^j = F\left(\frac{t}{1-t}, \frac{1-t}{t}\right) = H\left(\frac{t}{1-t}, \frac{1}{t}\right) \\ &= \sum_{0 \le j \le i} h_{i,j} \frac{t^i}{(1-t)^i t^{i-j}}. \end{aligned}$$ □

Let $\theta_1, \ldots, \theta_d$ be linear forms in $\mathbf{k}[\Delta]$, and let $M = (m_{i,j})$ be the $d \times n$-matrix defined by $\theta_i = \sum_{j=1}^{n} m_{i,j} x_j$. So, the rows of M are indexed by the forms θ_i and the columns by the vertices x_j. Suppose that $F_1, \ldots, F_t$ are the facets of Δ and call $C : [t] \to 2^{[d]}$ a *nonsingular choice function* if $|C(j)| = |F_j|$ and the square submatrix with rows in $C(j)$ and columns in F_j is nonsingular, for all facets F_j.

12.2. Proposition. (Stanley [S4; Remark on p. 150]) *Let $\theta_1, \ldots, \theta_d$ be linear forms in $\mathbf{k}[\Delta]$. Then $(\theta_1, \ldots, \theta_d)$ is a linear system of parameters if and only if there exists a nonsingular choice function.*

If $m = a x_{i_1}^{e_1} \ldots x_{i_p}^{e_p}$, $a \neq 0$ and $e_j \ge 1$, is a monomial in $\mathbf{k}[x_1, \ldots, x_n]$ we will call the set $S_m = \{i_1, \ldots, i_p\} \subseteq [n]$ the *support* of m. This includes the case $S_a = \emptyset$. The monomials with support in Δ and coefficient one give a vector space basis of $\mathbf{k}[\Delta]$. We do not distinguish notationally between such monomials and their classes in $\mathbf{k}[\Delta]$. Faces of Δ and squarefree monomials are in one-to-one correspondence via

$$A = \{x_{i_1}, \ldots, x_{i_k}\} \in \Delta \leftrightarrow x_A = x_{i_1} x_{i_2} \ldots x_{i_k} \in \mathbf{k}[\Delta].$$

If $B = \{j_1, \ldots, j_b\} \subseteq [d]$ we will write "θ_B" as an abbreviation for $\{\theta_{j_1}, \ldots, \theta_{j_b}\}$, e.g. $p(\theta_B) = p(\theta_{j_1}, \ldots, \theta_{j_b})$ for polynomials p in b variables.

The following direct sum decomposition of $\mathbf{k}[\Delta]$ is the main result of this section.

12.3. Theorem. *Let $F_1, F_2, \ldots, F_t$ be a shelling of a $(d-1)$-complex Δ. Also, let $\theta_1, \theta_2, \ldots, \theta_d$ be linear forms in $\mathbf{k}[\Delta]$ and let $C : [t] \to 2^{[d]}$ be a nonsingular choice function. Then*

$$\mathbf{k}[\Delta] = \bigoplus_{j=1}^{t} x_{\mathcal{R}(F_j)} \cdot \mathbf{k}[\theta_{C(j)}].$$

The decomposition means that every element of $\mathbf{k}[\Delta]$ can be uniquely expressed as a sum of the form

$$\sum_{j=1}^{t} x_{\mathcal{R}(F_j)} \cdot p_j(\theta_{C(j)}),$$

where $p_j \in \mathbf{k}[y_1, \ldots, y_{|C(j)|}]$, see Example 12.8 below. The crucial points of the proof will be dealt with in two lemmas. We let $I_j = [\mathcal{R}(F_j), F_j]$, $j = 1, \ldots, t$, be the pieces of the Boolean interval decomposition of Δ, see Proposition 2.5.

12.4. Lemma. *Let* $P(x_1, \dots, x_n) = x_{\mathcal{R}(F_j)} p(\theta_{C(j)})$, *for* $0 \neq p \in \mathbf{k}[y_1, \dots, y_{|C(j)|}]$. *Then*

(i) *P contains some monomial m with support $S_m \in I_j$,*
(ii) *if $S_m \in I_k$ for some monomial m in P then $k \geq j$.*

Proof. Suppose for notational ease that $F_j = \{x_1, \dots, x_q\}$ and $C(j) = \{1, \dots, q\}$. Let $\tilde{\theta}_i = \sum_{j=1}^{q} m_{i,j} x_j$, for $i = 1, \dots, q$. Since the matrix $(m_{i,j})_{i,j=1}^{q}$ is by assumption nonsingular it follows that $\tilde{\theta}_1, \dots, \tilde{\theta}_q$ are algebraically independent over $\mathbf{k}$, and hence $p\left(\tilde{\theta}_1, \dots, \tilde{\theta}_q\right) \neq 0$. If m is a nonzero term of $p\left(\tilde{\theta}_1, \dots, \tilde{\theta}_q\right)$ then $S_m \subseteq F_j$. Such a term m cannot be canceled by terms involving $x_i \notin F_j$, hence $m \neq 0$ is also a term in the polynomial $p(\theta_1, \dots, \theta_q)$. But then $0 \neq m' = m \cdot x_{\mathcal{R}(F_j)}$ is a term in P and $\mathcal{R}(F_j) \subseteq S_{m'} \subseteq F_j$.

For part (ii) it suffices to observe that $\mathcal{R}(F_j) \subseteq S_m$, for all terms m occurring in P, and hence $k < j$ would contradict condition (β) of Proposition 2.5. $\square$

12.5 Lemma. *Let $m \in \mathbf{k}[\Delta]$ be a monomial such that $S_m \in I_j$. Then*

$$m = x_{\mathcal{R}(F_j)} \cdot p(\theta_{C(j)}) + \sum_{m'} a_{m'} m',$$

where $p \in \mathbf{k}[y_1, \dots, y_{|C(j)|}]$ and the sum runs over finitely many monomials m' for which $S_{m'} \in I_l$ with $l > j$.

Proof. Assume again for simplicity that $F_j = \{x_1, \dots, x_q\}$ and $C(j) = \{1, \dots, q\}$. Since the submatrix $(m_{i,j})_{i,j=1}^{q}$ of M is nonsingular we can solve $\theta = M\mathbf{x}$ for $x_1, \dots, x_q$ and obtain

$$x_i = h_i(\theta_1, \dots, \theta_q) + h'_i(x_{q+1}, \dots, x_n) \qquad , \qquad \text{for } i = 1, \dots, q,$$

where h_i and h'_i are linear forms over $\mathbf{k}$.

Since $\mathcal{R}(F_j) \subseteq S_m \subseteq F_j$ we obtain

$$m = x_{\mathcal{R}(F_j)} \cdot x_{i_1}^{e_1} \dots x_{i_c}^{e_c} = x_{\mathcal{R}(F_j)} \cdot h_{i_1}^{e_1}(\theta_{C(j)}) \dots h_{i_c}^{e_c}(\theta_{C(j)}) + A(x_1, \dots, x_n),$$

where $1 \leq i_1 < \dots < i_c \leq q$, $e_1, \dots, e_c \geq 1$, and every monomial m' occurring in A is divisible by $x_{\mathcal{R}(F_j)}$ and by at least one of $x_{q+1}, \dots, x_n$ (coming from powers of nontrivial $h'_i(x_{q+1}, \dots, x_n)$). It follows that $\mathcal{R}(F_j) \subseteq S_{m'} \not\subseteq F_j$, which by Proposition 2.5 forces that $S_{m'} \in I_l$ for some $l > j$. $\square$

Proof of Theorem 12.3. We must show that every monomial $m \in \mathbf{k}[\Delta]$ can be expressed on the form

$$(*) \qquad m = \sum_{j=1}^{t} x_{\mathcal{R}(F_j)} \cdot p_j(\theta_{C(j)}).$$

Since such monomials span $\mathbf{k}[\Delta]$ this will establish the existence part of the proof.

Suppose $S_m \in I_j$. If $j = t$ then the expression $(*)$ with $p_1 = \ldots = p_{t-1} = 0$ is already provided by Lemma 12.5. If $j < t$ then we again obtain an expression $(*)$ by recursively expanding the monomials m' in Lemma 12.5, whose supports belong to intervals I_l with $l > j$.

To prove uniqueness, assume that

$$g = \sum_{j=1}^{t} x_{\mathcal{R}(F_j)} \cdot p_j(\theta_{C(j)}) \qquad , \qquad p_1 = \ldots = p_{s-1} = 0 \qquad , \qquad p_s \neq 0.$$

Then by part (i) of Lemma 12.4 the polynomial $x_{\mathcal{R}(F_s)} p_s(\theta_{C(s)})$ contains a monomial m such that $S_m \in I_s$, and by part (ii) this term cannot be canceled by any of the later contributions to g. Hence, $g \neq 0$. □

We will make separate statements of two special cases of Theorem 12.3 that are of particular interest. Call a collection of linear forms $\theta_1, \ldots, \theta_d$ *generic* if all minors of the $d \times n$ matrix M are nonzero. This can be achieved for large enough fields e.g. by taking all entries $m_{i,j}$ algebraically independent over the prime field, or by choosing distinct field elements $z_1, \ldots, z_n$ and letting $m_{i,j} = z_j^{i-1}$.

12.6. Corollary. *If $\theta_1, \ldots, \theta_d$ is a generic collection of linear forms then*

$$\mathbf{k}[\Delta] = \bigoplus_{j=1}^{t} x_{\mathcal{R}(F_j)} \cdot \mathbf{k}[\theta_1, \theta_2, \ldots, \theta_{|F_j|}].$$

Proof. The choice function $C(j) = \{1, \ldots, |F_j|\}$ is nonsingular. □

12.7. Corollary. *Suppose $\tau : V \to [d]$ is such that (Δ, τ) is balanced (defined after Example 10.3) and let $\theta_i = \sum_{\tau(x)=i} x$, $i = 1, \ldots, d$. Then*

$$\mathbf{k}[\Delta] = \bigoplus_{j=1}^{t} x_{\mathcal{R}(F_j)} \cdot \mathbf{k}[\theta_i \mid i \in \tau(F_j)].$$

Proof. The function $C(j) = \tau(F_j)$ is nonsingular. □

In the pure case Corollary 12.6 specializes to a result of Kind & Kleinschmidt [KK], while Corollary 12.7 specializes to one of Garsia [Ga]. A short proof of Garsia's theorem appears in [B3].

It has been shown by Rees [R] and by Baclawski & Garsia [BaG] that for every finitely generated graded $\mathbf{k}$-algebra R there exists a homogeneous system of parameters $(\theta_1, \ldots, \theta_d)$, a finite sequence of homogeneous elements $(\eta_1, \ldots, \eta_t)$ and a function $q : [t] \to \{0, 1, \ldots, d\}$ such that

$$R = \bigoplus_{j=1}^{t} \eta_j \cdot \mathbf{k}[\theta_1, \ldots, \theta_{q(j)}].$$

Corollaries 12.6 and 12.7 provide combinatorial constructions of such a decomposition for Stanley-Reisner rings of shellable complexes. In particular, if P is any shellable bounded poset then $(\Delta(\overline{P}), \rho)$, where ρ is the restriction of the rank function of P (defined after Theorem 10.10) to $\overline{P}$, is a shellable balanced complex. Hence Corollary 12.7 can be applied to P.

12.8. Example. Let $\Delta = \Delta\left(\overline{P}\right)$ be the order complex of the proper part of the poset in Figure 10. Thus, $\mathbf{k}[\Delta] = \mathbf{k}[x_1, x_2, x_3]/(x_1x_3, x_2x_3)$. The given edge-labeling is an EL-labeling, so from Corollary 12.7 we obtain the decomposition

$$\mathbf{k}[\Delta] = \mathbf{k}[x_1 + x_3, x_2] \oplus x_3 \cdot \mathbf{k}[x_1 + x_3].$$

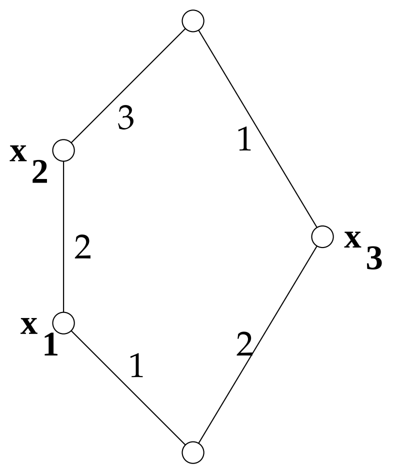

FIGURE 10.

13. Regular cell complexes

So far in this paper the word "complex" has meant simplicial complex. In this brief section we will outline how the concept of nonpure shellability and some of its main properties can be extended to regular cell complexes. This was previously done for the pure case in [BW2] and [B2], a more detailed account of which is given in Section 4.7 of [B+]. Since everything generalizes straightforwardly we will only state the main facts and refer to these sources for all details.

Let Γ be a regular cell complex (i.e. a regular finite CW complex) with face poset $\mathcal{F}(\Gamma)$ [B+, Definition 4.7.4]. Then the open interval $\left(\hat{0}, \sigma\right)$ in $\mathcal{F}(\Gamma)$ is homeomorphic to a sphere for all $\sigma \in \Gamma$, and this in fact characterizes face posets of regular cell complexes [B2, Proposition 3.1] [B+, Proposition 4.7.23]. The homeomorphism of Γ and its face poset

$$\Gamma \cong \Delta\left(\mathcal{F}(\Gamma)\right), \tag{13.1}$$

is important [B+, Proposition 4.7.8].

13.1. Definition. *Let* Γ *be a regular cell complex. For each cell* $\sigma \in \Gamma$ *let* $\delta\sigma$ *denote the subcomplex consisting of all proper faces of* σ. *A linear ordering* $\sigma_1, \sigma_2, \ldots, \sigma_t$ *of the maximal cells of* Γ *is called a* shelling *if either* $\dim \Gamma = 0$, *or if* $\dim \Gamma \geq 1$ *and the following conditions are satisfied:*

(i) $\delta\sigma_1$ *admits a shelling,*

(ii) $\delta\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \delta\sigma_i \right)$ *is pure and* $(\dim \sigma_j - 1)$*-dimensional, for* $2 \leq j \leq t$,

(iii) $\delta\sigma_j$ *admits a shelling in which the* $(\dim \sigma_j - 1)$*-cells of* $\delta\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \delta\sigma_i \right)$ *come first, for* $2 \leq j \leq t$.

A complex that admits a shelling is said to be shellable.

One concludes as in [B2, Proposition 4.3] that the successive intersections $\delta\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \delta\sigma_i \right)$ are topological balls or spheres. For simplicial complexes conditions (i) and (iii) are automatically true, so that the definition specializes to the nonrecursive Definition 2.1.

Just as in the pure case [BW2, Theorem 4.3] [B2, Proposition 4.2] and the simplicial case (Theorem 5.13) we have the following connection with lexicographic shellability. The proof proceeds via recursive coatom orderings (Theorem 5.11).

13.2. Theorem. *A regular cell complex* Γ *is shellable if and only if the dual of its augmented face poset* $\widehat{\mathcal{F}}(\Gamma) = \mathcal{F}(\Gamma) \cup \{\hat{0}, \hat{1}\}$ *is CL-shellable.*

This result and the homeomorphism (13.1) transfer most questions about shellable regular cell complexes to questions about a certain class of CL-shellable posets.

13.3. Corollary. *If* Γ *is shellable then it has the homotopy type of a wedge of spheres.*

Proof. Use Theorem 5.9 and (13.1). □

The two rearrangement lemmas 2.6 and 2.7 are true also for cell complexes. For 2.7 this is easy to see directly and also 2.6 can be done this way with a little more effort. Alternatively one applies the simplicial version of Lemma 2.6 to the face poset $\widehat{\mathcal{F}}(\Gamma)$ and uses the correspondence between shellings of Γ and CL-labelings of $\widehat{\mathcal{F}}(\Gamma)$ underlying Theorem 13.2. Define the subcomplexes $\Gamma^{(r,s)}$ as in Definition 2.8.

13.4. Theorem. *If* Γ *is shellable then so is* $\Gamma^{(r,s)}$, *for all* $r \leq s$.

Proof. Use Theorem 10.6, Lemma 10.8 and Theorem 10.9. □

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