ON THE (CO)HOMOLOGY OF THE PARTITION LATTICE AND THE FREE LIE ALGEBRA

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Dedicated to Adriano Garsia's 40 years of mathematics

Introduction.

There is a remarkable relationship between the top homology $H(\Pi_n)$ of Π_n, the lattice of partitions of the set $\{1, 2, \ldots, n\}$ and the free Lie algebra on $\{1, 2, \ldots, n\}$. Namely, the representation of the symmetric group $\mathcal{S}_n$ on $H(\Pi_n)$ is isomorphic to the representation of $\mathcal{S}_n$ on Lie_n, the 1^n homogeneous component of the free Lie algebra on $\{1, 2, \ldots, n\}$, tensored with the sign representation.

The character of the action of $\mathcal{S}_n$ on Lie_n has been known for quite some time. The first explicit formula is due to Brandt [Br] (1944) although it is already implicit in the work of Witt [Wi] (1937). The character of the action of $\mathcal{S}_n$ on $H(\Pi_n)$ was first calculated by Stanley [St] (1980) and Hanlon [H] (1981). The fact that one is a sign-twisted version of the other was brought to the attention of the combinatorial audience by Joyal in [J] (1985) as a byproduct of his theory of species. A beautiful combinatorial explanation of this phenomenon was given by Barcelo in her UCSD thesis (1988) where she succeeded in constructing an explicit $\mathcal{S}_n$-equivariant isomorphism between the two modules [Ba]. She accomplishes this by constructing an elegant bijection between a basis for $H(\Pi_n)$, due to Björner [Bj1], and the Lyndon basis for Lie_n and then computing the representation matrices.

More recently Hanlon and Wachs [HW] (1992) defined a k-analogue of the free Lie algebra whose 1^n homogeneous component is isomorphic as an $\mathcal{S}_n$ module to the top cohomology of the $1 \mod k$ partition poset with no sign twist. This isomorphism reduces to an isomorphism between the 1^n component of the free Lie superalgebra and the top cohomology of Π_n. The method of proof used in [HW] differs from that of [Ba] in that a correspondence between natural generating sets is constructed rather than a correspondence between bases. There are no representation matrices to compute since the correspondence reflects the fact that the action of $\mathcal{S}_n$ is preserved. Instead, it is shown that the relations on the generators

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of one module map to the relations on the other. Subsequent related work appears in [MR].

In this paper we further examine the relationship between the two $\mathcal{S}_n$ modules $H(\Pi_n)$ and Lie_n. In particular, we give two new combinatorial proofs of the sign twisted isomorphism between $H(\Pi_n)$ and Lie_n. The first proof (Section 4) is based on the method of [Ba], but involves a simpler bijection between simpler bases. The second proof (Section 5) is based on the method of [HW] but is more complex because it involves sign twists. The second proof is particularly interesting for several reasons. It constructs a general correspondence between natural generating sets for the two modules which makes the isomorphism transparent. This general correspondence reduces to the Barcelo bijection and to the bijection of the first proof when restricted to the appropriate basis. In other words, all three correspondences determine the identical isomorphism.

There are three key ingredients of our general correspondence. Just as the natural generating set for Lie_n can be expressed in terms of leaf labeled binary trees, we show that a natural generating set for $H^*(\Pi_n)$, the top cohomology of Π_n, can be expressed in terms of leaf labeled binary trees. The correspondence between the generating sets is then immediate except for the sign twists, whose determination is the second ingredient. The third ingredient is the idea, which originated in [HW], of mapping the cohomology relations to the Lie algebra relations.

In Section 2, we consider several pairs of dual bases for homology and cohomology of Π_n. In Section 3 we construct the representation matrix for the simplest of these bases, called the splitting basis, by exploiting the duality between homology and cohomology. We also derive nice identities for expanding the other bases in terms of the splitting basis.

In Section 4, we describe the Barcelo correspondence and construct the new correspondence, called the comb correspondence, which bijectively maps the dual splitting basis for $H^*(\Pi_n)$ to the comb basis for Lie_n. By examining the representation matrices, we prove that the comb correspondence determines an $\mathcal{S}_n$-isomorphism between $H^*(\Pi_n)$ and $\text{Lie}_n \otimes \text{sgn}$.

The general correspondence that unifies the Barcelo correspondence and the comb correspondence is given in Section 5. Here we show that every maximal chain of Π_n can be represented by a leaf labeled binary tree together with a linear extension of its internal nodes. Cohomology classes of maximal chains are shown to have a simpler representation in which the linear extension of internal nodes is always taken to be postorder. This gives a natural correspondence between cohomology classes and generators of Lie_n. The sign twist coefficient is the product of the sign of the permutation obtained by reading the leaf labels from left to right and the sign of the binary tree which is defined in Section 5. The comb correspondence is immediately seen to be a restriction of the general correspondence. The fact that the Barcelo correspondence is also a restriction of the general correspondence requires proof. The general correspondence can be used to transfer any tree basis for $H^*(\Pi_n)$ to a corresponding basis for Lie_n and vice versa. This is demonstrated by transferring a basis given in Section 2 to a "new" basis for Lie_n.

In Section 6, we use the isomorphism constructed in Section 5 and one of the expansion identities of Section 3 to obtain a nice identity for expanding the comb basis in terms of the Lyndon basis for Lie_n. This suggests a general identity for expressing any of the natural Lie generators in terms of the Lyndon basis. We prove the general identity by induction.

In Section 7, we construct a more general correspondence which generalizes both the correspondence of Section 5 and a correspondence of Barcelo and Bergeron [BB] which relates the Whitney homology (or equivalently the Orlik-Solomon algebra) of Π_n to the exterior algebra of the free Lie algebra. Instead of working with binary trees, we work, more generally, with ordered forests of binary trees.

The combinatorial significance of the free Lie algebra was demonstrated in a beautiful 1990 paper of Garsia [G]. I would like to thank Adriano Garsia and Hélène Barcelo for initiating my interest in this topic during a visit to UQAM in 1988. I am also grateful to Sheila Sundaram for rekindling my interest and for numerous valuable discussions. Parts of this paper were worked out and presented at the Mittag-Leffler Institute during the Combinatorics year in 1992. Other parts were presented at the Jerusalem Combinatorics Conference at the Hebrew University in 1993.

1. Preliminaries.

Let P be a finite poset of length $d+2 \geq 1$ with maximum element $\hat{1}$ and minimum element $\hat{0}$. The *order complex* of P, denoted by $\Delta(P)$, is defined to be the d-dimensional simplicial complex whose vertices are the elements of $P - \{\hat{0}, \hat{1}\}$ and whose faces are the chains of $P - \{\hat{0}, \hat{1}\}$.

Let $\mathbf{k}$ be any field. For $r = -2, \dots, d+1$, let $C_r(P)$ denote the vector space over $\mathbf{k}$ on the basis of chains $x_0 < x_1 < \cdots < x_r$ of $P - \{\hat{0}, \hat{1}\}$. ($C_{-1}(P)$ is the one dimensional space generated by the empty chain and $C_{d+1}(P) = C_{-2}(P) = (0)$.) For $r = -1, \dots, d+1$, the boundary map $\partial_r : C_r(P) \to C_{r-1}(P)$ is the linear map defined by

$$\partial_r(x_0 < x_1 < \cdots < x_r) = \sum_{i=0}^{r} (-1)^i (x_0 < \cdots < \hat{x}_i < \cdots < x_r),$$

where $\hat{x}_i$ means the x_i is deleted. For $r = -2, \dots, d$, the coboundary map $\delta_r : C_r(P) \to C_{r+1}(P)$ is defined by

$$\langle \delta_r(\alpha), \beta \rangle = \langle \alpha, \partial_{r+1}(\beta) \rangle,$$

where $\alpha \in C_r(P)$, $\beta \in C_{r+1}(P)$ and $\langle\ ,\ \rangle$ is the bilinear form on $\oplus_{r=-1}^{d} C_r(P)$ defined on chains by

$$\langle c_1, c_2 \rangle = \begin{cases} 1, & \text{if } c_1 = c_2 \\ 0, & \text{otherwise.} \end{cases}$$

It is equivalent to say that
(1-1)

$$\delta_r(x_0 < x_1 < \cdots < x_r) = \sum_{i=0}^{r+1} (-1)^i \sum_{x \in (x_{i-1}, x_i)} (x_0 < \cdots < x_{i-1} < x < x_i < \cdots < x_r),$$

where (x, y) denotes the open interval $\{z \in P \mid x < z < y\}$, $x_{-1} = \hat{0}$ and $x_{r+1} = \hat{1}$.

For $-1 \le r \le d$, the rth homology of P is given by

$$H_r(P) = \ker \partial_r / \operatorname{im} \partial_{r+1},$$

and the rth cohomology of P is given by

$$H^r(P) = \ker \delta_r / \operatorname{im} \delta_{r-1}.$$

The elements of $\ker \partial_r$ and $\ker \delta_r$ are called *cycles* and *cocycles*, respectively; and the elements of $\operatorname{im} \partial_r$ and $\operatorname{im} \delta_r$ are called *boundaries* and *coboundaries*, respectively. Note that $H_r(P)$ and $H^r(P)$ are just the reduced simplicial homology and cohomology, respectively, of the order complex $\Delta(P)$.

The bilinear form $\langle , \rangle$ induces a pairing between $H_r(P)$ and $H^r(P)$ which allows one to view $H^r(P)$ as the dual space of $H_r(P)$. The following simple proposition turns out to be a useful device for identifying bases for homology and cohomology.

Proposition 1.1. *Suppose $H_r(P)$ has dimension m. If $\rho_1, \rho_2, \dots, \rho_m$ are elements of $H_r(P)$ and $\gamma_1, \gamma_2, \dots, \gamma_m$ are elements of $H^r(P)$ such that*

$$\langle \rho_i, \gamma_j \rangle = \delta_{ij}, \quad \textit{for all } i, j = 1, \dots, m,$$

then $\{\rho_1, \rho_2, \dots, \rho_m\}$ is a basis for $H_r(P)$ with dual basis $\{\gamma_1, \gamma_2, \dots, \gamma_m\}$ for $H^r(P)$. □

Suppose a group G acts as a group of automorphisms of P. Since any automorphism of P commutes with the boundary and coboundary maps and respects the form $\langle , \rangle$, an automorphism of P induces a linear map on $H_r(P)$ and on $H^r(P)$. This turns $H_r(P)$ and $H^r(P)$ into dual representations of G. This means that if $M(g)$ is the representation matrix for g acting on $H_r(P)$ with respect to a basis $\{\rho_1, \rho_2, \dots, \rho_m\}$ then $M^t(g^{-1})$ is the representation matrix for g acting on $H^r(P)$ with respect to the dual basis $\{\gamma_1, \gamma_2, \dots, \gamma_m\}$. The representation matrices are easy to describe in terms of the form $\langle , \rangle$.

Proposition 1.2. *Suppose $\{\rho_1, \rho_2, \dots, \rho_m\}$ is a basis for $H_r(P)$ with dual basis $\{\gamma_1, \gamma_2, \dots, \gamma_m\}$ for $H^r(P)$. The representation matrix $M(g)$ for g acting on $H_r(P)$ with respect to the basis $\{\rho_1, \rho_2, \dots, \rho_m\}$ has i, j component given by*

$$M_{i,j}(g) = \langle g\rho_j, \gamma_i \rangle.$$

Equivalently, the representation matrix $M^*(g)$ for g acting on $H^r(P)$ with respect to the basis $\{\gamma_1, \gamma_2, \dots, \gamma_m\}$ has i, j component given by

$$M^*_{i,j}(g) = \langle \rho_i, g\gamma_j \rangle.$$

Proof. Clearly for each $i, j = 1, 2, \dots, m$,

$$\begin{aligned} \langle g\rho_j, \gamma_i \rangle &= \langle \sum_{k=1}^m M_{k,j}(g)\rho_k, \gamma_i \rangle \\ &= \sum_{k=1}^m M_{k,j}(g) \langle \rho_k, \gamma_i \rangle \\ &= M_{i,j}(g). \qquad \square \end{aligned}$$

Since Π_n is a geometric lattice of length $n-1$, it follows from [F] that $H_r(\Pi_n)$ and $H^r(\Pi_n)$ vanish in all dimensions except the top dimension $n-3$, when $n \geq 2$. (Unless stated otherwise we will assume that $n \geq 2$ when dealing with the partition lattice.) Let

$$H(\Pi_n) = H_{n-3}(\Pi_n) \qquad \text{and} \qquad H^*(\Pi_n) = H^{n-3}(\Pi_n).$$

Since $\operatorname{im} \partial_{n-2} = (0)$, $H(\Pi_n)$ is simply the subspace of cycles in the space spanned by $\mathcal{M}(\Pi_n)$, the set of maximal chains of $\Pi_n - \{\hat{0}, \hat{1}\}$. Since $\ker \delta_{n-3} = C_{n-3}(\Pi_n)$, $H^*(\Pi_n)$ is simply the space spanned by $\mathcal{M}(\Pi_n)$ modulo the coboundary relations. If m is a maximal chain of Π_n, we let $\bar{m}$ denote the chain $m - \{\hat{0}, \hat{1}\}$ in $\mathcal{M}(\Pi_n)$, as well its cohomology class depending on the context.

The symmetric group $\mathcal{S}_n$ acts as an automorphism group on Π_n by permuting the letters of the partitions. For example, the transposition $(3,4)$ acting on the partition 145/26/3 yields the partition 135/26/4. Hence $H(\Pi_n)$ and $H^*(\Pi_n)$ are dual $\mathcal{S}_n-$modules of dimension $(n-1)!$. In the next two sections we shall look at various bases and dual bases as well as representation matrices for these $\mathcal{S}_n-$modules.

2. Bases for (co)homology.

Björner [Bj1] [Bj2] has developed a general theory for constructing a basis for the top homology of a geometric lattice in terms of its NBC base . When this theory is applied to the partition lattice a very nice basis results whose precise formulation was given by Barcelo [Ba]. Here we shall give an alternative description of this basis. Then we will discuss two other bases for $H(\Pi_n)$ which do not come from Björner's general theory for geometric lattices. One of these is the splitting basis of [Wa] and the other is new and is dual to a basis for cohomology discussed in [BW] and [SW]. The Björner basis, the splitting basis and the new basis can all be described in terms of trees.

Let T be a tree (connected acyclic graph) on vertex set $\{1, \ldots, n\}$. Each subgraph of T on the full vertex set $\{1, \ldots, n\}$ is a forest which corresponds to a partition of $\{1, \ldots, n\}$ whose blocks are the sets of vertices in the connected components (trees) of the forest. For example, if F is the forest given in Figure 2.1 then the partition corresponding to F is 134/26/5.

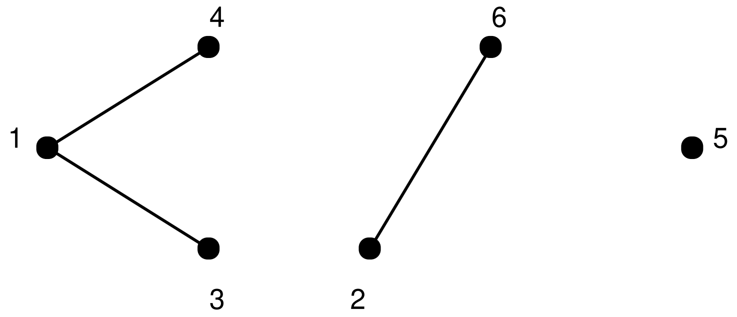

Figure 2.1

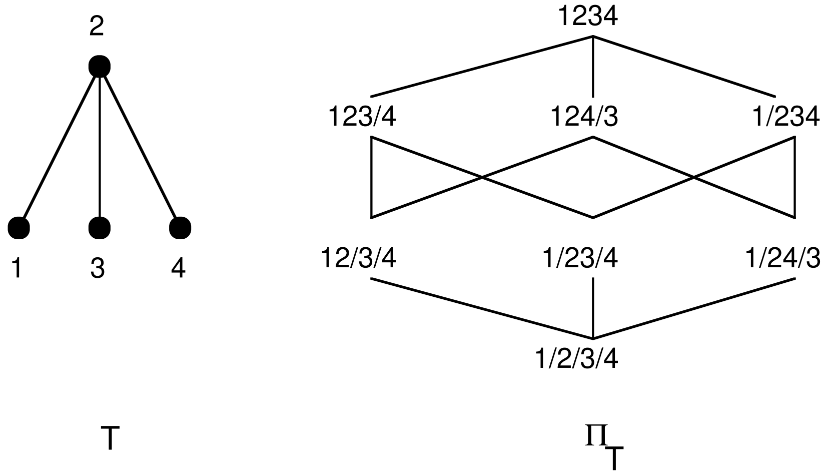

Figure 2.2

Let Π_T be the subposet of Π_n consisting of all partitions which correspond to subgraphs of T on the full vertex set. An example of Π_T is given in Figure 2.2.

Clearly Π_T is isomorphic to the lattice of subsets of $\{1, 2, \ldots, n-1\}$, denoted by B_{n-1}. Indeed, each subgraph of T on vertex set $\{1, \ldots, n\}$ is obtained by choosing some subset of the $n-1$ edges of T. A partition corresponding to one subgraph is finer than a partition corresponding to another subgraph if and only if the set of edges for the first subgraph is contained in the set of edges for the second subgraph. Since B_{n-1} is isomorphic to the lattice of faces of an $(n-2)$-simplex we have the following proposition.

Proposition 2.1. *For each tree T on vertex set $\{1, 2, \ldots, n\}$, $\Delta(\Pi_T)$ is the barycentric subdivision of the boundary of an $(n-2)$-simplex.* $\square$

We now define ρ_T to be a fundamental cycle of the spherical complex, $\Delta(\Pi_T)$. It turns out that the ρ_T span $H(\Pi_n)$. In fact, by restricting the index set of trees to certain collections of trees described below, we obtain various bases for $H(\Pi_n)$.

An *increasing* tree on $\{1, 2, \ldots, n\}$ is a tree in which every simple path ending at vertex n has increasing vertices. In other words, if we view our trees as rooted trees with root n then a tree is increasing if and only if every node except the root is less than its parent. There are $(n-1)!$ increasing trees on $\{1, 2, \ldots, n\}$. Indeed, an increasing tree T can be characterized as follows: for each vertex $i \in \{1, 2, \ldots, n-1\}$ there is a unique vertex $p(i) \in \{2, 3, \ldots, n\}$ such that $i < p(i)$ and $\{i, p(i)\}$ is an edge of T. (Note that $p(i)$ is simply the parent of i when we view our trees as rooted trees.) It follows that increasing trees can be represented by functions $p : \{1, \ldots, n-1\} \to \{2, \ldots, n\}$ where $i < p(i)$ for all i. Clearly there are $(n-1)!$ such functions.

For an increasing tree T, let m_T be the maximal chain of Π_n whose k block partition can be obtained from T by erasing edges $\{i, p(i)\}$, for $i = n-k+1, n-k+2, \ldots, n-1$. Since all maximal chains of $\Pi_T - \{\hat{0}, \hat{1}\}$ have coefficient ± 1 in a fundamental cycle of $\Delta(\Pi_T)$, we may choose ρ_T to be the fundamental cycle in which $\bar{m}_T$ has coefficient 1. (Recall that $\bar{m}_T$ denotes the chain $m_T - \{\hat{0}, \hat{1}\}$ or its cohomology class.)

Proposition 2.2. *(Björner basis) The set*

$$\{\rho_T \mid T \text{ is an increasing tree on } \{1, 2, \ldots, n\}\ \}$$

is a basis for $H(\Pi_n)$ with dual basis

$$\{\bar{m}_T \mid T \text{ is an increasing tree on } \{1, 2, \ldots, n\}\ \}$$

for $H^*(\Pi_n)$.

Proof. It is easy to see that m_T is a maximal chain of $\Pi_{T'}$ if and only if $T = T'$. Hence

$$\langle \rho_T, \bar{m}_{T'} \rangle = \delta_{T,T'}.$$

The result now follows from Proposition 1.1. □

The splitting basis of [Wa] is obtained by considering another set of trees as the index set. For each $\sigma \in \mathcal{S}_n$, let T_σ be the linear tree

Let $\Pi_\sigma = \Pi_{T_\sigma}$ and $\rho_\sigma = \rho_{T_\sigma}$. So Π_σ consists of all partitions obtained by "splitting" the word σ. Now let m_σ be the maximal chain of Π_n whose k block partition has $k-1$ singleton blocks $\{\sigma(1)\}, \{\sigma(2)\}, \ldots, \{\sigma(k-1)\}$ and one $n-k+1$ element block $\{\sigma(k), \ldots, \sigma(n)\}$. Clearly m_σ is a maximal chain of Π_σ and so now we choose ρ_σ to be the fundamental cycle of $\Delta(\Pi_\sigma)$ in which $\bar{m}_\sigma$ has coefficient 1.

Proposition 2.3. *(Splitting basis) The set*

$$\{\rho_\sigma \mid \sigma \in \mathcal{S}_n,\ \sigma(n) = n\}$$

is a basis for $H(\Pi_n)$ with dual basis

$$\{\bar{m}_\sigma \mid \sigma \in \mathcal{S}_n,\ \sigma(n) = n\}$$

for $H^*(\Pi_n)$.

Proof. Now observe that for $\sigma, \tau \in \mathcal{S}_n$ with $\sigma(n) = \tau(n) = n$, m_σ is a maximal chain of Π_τ if and only if $\sigma = \tau$. Again we apply Proposition 1.1 to obtain the result. □

We now present a third basis for $H(\Pi_n)$. Let T be a tree whose vertex set is some subset of $\{1, 2, \ldots, n\}$ of cardinality at least 2. By $\max(T)$ we mean the largest value of all the vertices of T. We shall say that T is a *star* if it has a distinguished vertex r, called the *root* of T, such that $r \neq \max(T)$ and all other vertices of T have degree 1. Suppose $T_1, T_2, \ldots, T_k$, is a sequence of stars whose vertex sets partition $\{1, 2, \ldots, n\}$. Suppose also that $n \in T_k$. By connecting the root of T_i with the root of T_{i+1} for each $i = 1, \ldots, k-1$, we form a tree on vertex set $\{1, 2, \ldots, n\}$ which we

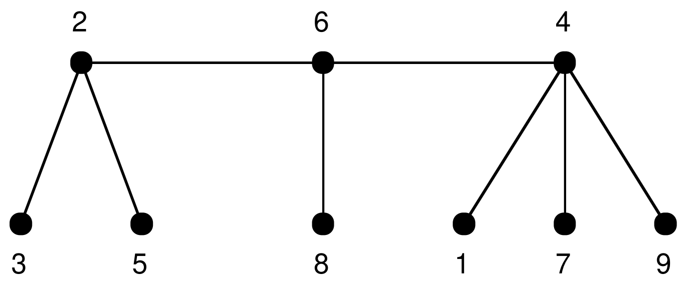

Figure 2.3

call a *caterpillar* and denote by $C(T_1, T_2, \ldots, T_k)$. The path connecting the roots will be called the *spine* of the caterpillar and the edges that are not on the spine will be called the *legs*. An example of a caterpillar is given in Figure 2.3.

Lemma 2.4. *The number of caterpillars c_n on vertex set $\{1, 2, \ldots, n\}$ is $(n-1)!$.*

Proof. We claim that the following recurrence relation holds:

$$c_n = n - 1 + \sum_{m=2}^{n-2} \binom{n-1}{m} (m-1) c_{n-m}. \tag{2-1}$$

Indeed, the $n-1$ before the summation counts the number of caterpillars whose spine consists of a single root, i.e., the caterpillar is a single star. The mth term of the summation counts caterpillars $C(T_1, T_2, \ldots, T_k)$ where $k \geq 2$ and the number of vertices of T_1 is m. Clearly $c_2 = 1$ and $c_3 = 2$. For $n \geq 4$, by induction, we can replace c_{n-m} by $(n-m-1)!$ in (2-1) to obtain

$$\begin{aligned}
c_n &= n - 1 + \sum_{m=2}^{n-2} \binom{n-1}{m} (m-1)(n-m-1)! \\
&= n - 1 + (n-1)! \sum_{m=2}^{n-2} \frac{m-1}{m!} \\
&= n - 1 + (n-1)! \left(\sum_{m=2}^{n-2} \frac{1}{(m-1)!} - \sum_{m=2}^{n-2} \frac{1}{m!} \right) \\
&= n - 1 + (n-1)! \left(1 - \frac{1}{(n-2)!} \right) \\
&= (n-1)!. \qquad \square
\end{aligned}$$

Remark. Gessel [Ge2] has come up with the following direct bijection from the set of caterpillars on $\{1, 2, \ldots, n\}$ to the set of permutations in S_{n-1} which is closely related to his "hook factorization" [Ge1]. Given a caterpillar $(T_1, \ldots, T_k)$, for each T_i, write down the children of the root in increaing order followed by the root. Write down these sequences in the same order as the trees and then remove n which must be in the next-to-last place.

Let T be the caterpillar $C(T_1, T_2, \ldots, T_k)$. For $i = 1, \ldots, k$, let r_i be the root of T_i. Also let

$$\{a_1 < a_2 < \cdots < a_{n-k}\} = \{1, 2, \ldots, n\} - \{r_1, r_2, \ldots, r_k\}.$$

We define μ_T to be the maximal chain of Π_T obtained from top to bottom by removing one edge at a time from T in the following manner: First remove edges $\{r_i, r_{i+1}\}$ on the spine of T in increasing order of $i = 1, 2, \ldots, k-1$ and then remove leg edges in increasing order of incident vertices $a_1 < a_2 < \cdots < a_{n-k}$. Note that after removing the spine edges from T, we have the partition $\{V(T_1), V(T_2), \ldots, V(T_k)\}$, where $V(T_i)$ is the set of vertices of T_i. We call this partition the pivot of T. After we reach the pivot of T we continue to go down μ_T by detaching singletons from their stars. For example, if T is the caterpillar given in Figure 2.3 then μ_T is the maximal chain

$$\begin{aligned}\hat{0} &\to 3/5/2/8/6/1/7/94 \to 3/5/2/86/1/7/94 \to 3/5/2/86/1/794 \\ &\to 3/52/86/1/794 \to 352/86/1/794 \to 352/86/1794 \to 352/861794 \to \hat{1}.\end{aligned}$$

with pivot 352/86/1794.

Now we choose ρ_T to be the fundamental cycle for $\Delta(\Pi_T)$ in which $\bar{\mu}_T$ has coefficient 1.

Theorem 2.5. *(Caterpillar basis) The set*

$$\{\rho_T \mid T \text{ is a caterpillar on } \{1, 2, \ldots, n\}\}$$

is a basis for $H(\Pi_n)$ with dual basis

$$\{\bar{\mu}_T \mid T \text{ is a caterpillar on } \{1, 2, \ldots, n\}\}$$

for $H^*(\Pi_n)$.

Proof. By Proposition 1.1 and Lemma 2.4, we need only show that for all caterpillars T and T', μ_T is a maximal chain of $\Pi_{T'}$ if and only if $T = T'$. Let $T = C(T_1, T_2, \ldots, T_k)$ and $T' = C(T'_1, T'_2, \ldots, T'_j)$. If μ_T is a maximal chain of $\Pi_{T'}$ then the pivot of T is in $\Pi_{T'}$, which implies that the pivot of T corresponds to a subgraph of T' obtained by removing some spine edges from T'. Hence, each $V(T_i)$ is the vertex set of a segment of adjacent stars of T'. For i=1,

$$V(T_1) = V(T'_m) \cup V(T'_{m+1}) \cup \cdots \cup V(T'_t),$$

where $m \le t$.

We will show that $m = t$ and that $T'_m = T_1$. Suppose $m < t$. Let r_1 be the root of T_1. Since $\{r_1, \max(T_1)\}$ is a block of a partition in the chain μ_T, $\{r_1, \max(T_1)\}$ is an edge of T'. This implies that either r_1 and $\max(T_1)$ are adjacent roots of T' or they are both in T'_i, where $m \le i \le t$, with one of the two vertices being the root of T'_i. Since the root of a star cannot be the maximum element of the star, $\max(T_1)$ is not the root of any star in T', eliminating the first possibility. Hence r_1 is the root of T'_i and $\max(T_1)$ is in T'_i.

Since $m < t$, either $i+1$ or $i-1$ is in the interval from m to t. Suppose $m \le i+1 \le t$. Let r'_{i+1} be the root of T'_{i+1}. Then r'_{i+1} is adjacent to r_1 in T'. In traveling down the portion of μ_T below the pivot of T, viewed as a chain of $\Pi_{T'}$, edges are removed from T' which detach singleton vertices in increasing order. When the edge $\{r_1, r'_{i+1}\}$ is removed from T' either r_1 becomes detached or r'_{i+1} becomes detached. In the former case, the edge $\{r_1, \max(T_1)\}$ of T'_i would have already had to have been removed. But this is impossible since $\max(T_1)$ has to be the last vertex of $V(T_1)$ that is detached. This means that r'_{i+1} becomes detached when the edge is removed. Hence $\max(T'_{i+1})$ was already detached from the root r'_{i+1} which implies that $\max(T'_{i+1})$ was detached from T_1 prior to r'_{i+1}. This contradicts the fact that vertices must be detached in increasing order. Hence $m = t$ and $V(T_1) = V(T'_m)$. Also, as we have just seen, the root of T_1 must be the same as the root of T'_m. So $T_1 = T'_m$. The same argument shows that each $T_i = T'_{\sigma(i)}$ for some permutation $\sigma \in \mathcal{S}_k$.

We now claim that the permutation σ must be the identity. Indeed, this follows from the fact that all partitions of the form

$$\{V(T_1), V(T_2), \ldots, V(T_i), \bigcup_{j=i+1}^{k} V(T_j)\}, \quad i = 1, 2, \ldots, k$$

are in μ_T, and hence also in $\Pi_{T'}$. Since σ is the identity, we have

$$T = C(T_1, T_2, \ldots, T_k) = C(T'_1, T'_2, \ldots, T'_k) = T'. \qquad \square$$

Remark. The maximal chains m_T, m_σ and μ_T all arise as decreasing chains of respective lexicographical shellings of Π_n, cf. [Wa], [BW], and [SW]. The decreasing chains of a lexicographical shelling of a poset form a basis for the top cohomology of the poset. This follows from [BG, Theorem 5.1] and [G, Theorem 4.2] (or more directly from [Bj2, Theorem 7.7.2] or [BW, Theorem 5.9]).

3. Representation matrices for splitting basis.

In [Ba] Barcelo constructs the representation matrices for the action of $\mathcal{S}_n$ with respect to the Björner basis. Here we do the same for the splitting basis whose representation matrices are in fact easier to describe.

Define a *minmax* permutation to be a permutation $\sigma \in \mathcal{S}_n$ such that for all $i = 1, 2, \ldots, n-1$, either $\sigma(i) < \sigma(i+1), \sigma(i+2), \ldots, \sigma(n)$ or $\sigma(i) > \sigma(i+1), \sigma(i+2), \ldots, \sigma(n)$. In other words σ is a minmax permutation if σ avoids both the patterns 213 and 231. Let Γ_n be the set of all minmax permutations in $\mathcal{S}_n$.

For $\sigma \in \mathcal{S}_n$ define

$$\mathrm{sgn}^*(\sigma) = (-1)^{\mathrm{inv}(\sigma(1)\sigma(2)\ldots\sigma(n-1))},$$

where $\mathrm{inv}(w)$ denotes the number of inversions of the word w.

Lemma 3.1. *If $\alpha, \beta \in \mathcal{S}_n$, then*

$$\langle \rho_\alpha, \bar{m}_\beta \rangle = \begin{cases} \mathrm{sgn}^*(\alpha^{-1}\beta), & \text{if } \alpha^{-1}\beta \in \Gamma_n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First we compute $\langle \rho_\epsilon, \bar{m}_\tau \rangle$ where ϵ is the identity permutation $12\ldots n$ and τ is an arbitrary permutation in $\mathcal{S}_n$.

Recall that Π_ϵ is isomorphic to B_{n-1} the lattice of subsets of $\{1, 2, \ldots, n\}$. Here we use the isomorphism ϕ from the dual of B_{n-1}, to Π_ϵ defined by letting $\pi(S)$ be the partition obtained by splitting ϵ at all positions in the set S.

We also use the standard bijection between the maximal chains of B_{n-1} and permutations in S_{n-1} defined as follows: For $\sigma \in S_{n-1}$, let c_σ be the maximal chain

$$\emptyset \subset \{\sigma(1)\} \subset \{\sigma(1), \sigma(2)\} \subset \cdots \subset \{\sigma(1), \sigma(2), \ldots, \sigma(n-1)\}.$$

The fundamental cycles of the spherical complex $\Delta(B_{n-1})$ are $\pm \sum_{\sigma \in \mathcal{S}_{n-1}} \operatorname{sgn}(\sigma)\bar{c}_\sigma$. It follows that $\rho_\epsilon = \pm \sum_{\sigma \in \mathcal{S}_{n-1}} \operatorname{sgn}(\sigma)\pi(\bar{c}_\sigma)$. Since $\pi(c_{1,2,\ldots,n-1}) = m_\epsilon$ and ρ_σ was chosen so that $\langle \rho_\epsilon, \bar{m}_\epsilon \rangle = 1$, we conclude that

$$\rho_\epsilon = \sum_{\sigma \in \mathcal{S}_{n-1}} \operatorname{sgn}(\sigma)\pi(\bar{c}_\sigma). \tag{3-1}$$

Note that for $\sigma \in \mathcal{S}_{n-1}$, the maximal chain $\pi(c_\sigma)$ is obtained from top to bottom by successively splitting ϵ at positions $\sigma(1), \sigma(2), \ldots, \sigma(n-1)$. It follows that for $\tau \in \mathcal{S}_n$, $m_\tau = \pi(c_\sigma)$ if and only if $\tau \in \Gamma_n$ and $\sigma \in \Gamma_{n-1}$. Indeed, each successive split must create a singleton block and hence the split must occur at either end of the word $12\ldots n$. Moreover, for $i = 1, 2, \ldots, n-1$, if $\tau(i) < \tau(n)$ then $\sigma(i) = \tau(i)$ and if $\tau(i) > \tau(n)$ then $\sigma(i) = \tau(i) - 1$. It follows that

$$\operatorname{inv}(\tau(1)\tau(2)\ldots\tau(n-1)) = \operatorname{inv}(\sigma).$$

From these observations and (3-1) we have that

$$\langle \rho_\epsilon, \bar{m}_\tau \rangle = \begin{cases} \operatorname{sgn}^*(\tau), & \text{if } \tau \in \Gamma_n \\ 0, & \text{otherwise.} \end{cases}$$

To compute $\langle \rho_\alpha, \bar{m}_\beta \rangle$ for general $\alpha, \beta \in \mathcal{S}_n$, we use the fact that $\alpha\rho_\epsilon = \rho_\alpha$ and $\alpha m_\epsilon = m_\alpha$. We now have,

$$\begin{aligned} \langle \rho_\alpha, \bar{m}_\beta \rangle &= \langle \alpha\rho_\epsilon, \bar{m}_\beta \rangle \\ &= \langle \rho_\epsilon, \bar{m}_{\alpha^{-1}\beta} \rangle \\ &= \begin{cases} \operatorname{sgn}^*(\alpha^{-1}\beta) & \text{if } \alpha^{-1}\beta \in \Gamma_n \\ 0 & \text{otherwise.} \end{cases} \quad \square \end{aligned}$$

For $i = 1, 2, \ldots, (n-1)!$, let σ_i be the ith permutation of $\{\sigma \in \mathcal{S}_n \mid \sigma(n) = n\}$ under lexicographical order. Now for any permutation $\sigma \in \mathcal{S}_n$, we define $C(\sigma)$ and $C^*(\sigma)$ to be the $(n-1)! \times (n-1)!$ matrices whose respective i,j-entries are given by

$$C_{i,j}(\sigma) = \begin{cases} \operatorname{sgn}^*(\sigma_j^{-1}\sigma^{-1}\sigma_i) & \text{if } \sigma_j^{-1}\sigma^{-1}\sigma_i \in \Gamma_n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C^*_{i,j}(\sigma) = \begin{cases} \operatorname{sgn}^*(\sigma_i^{-1}\sigma\sigma_j) & \text{if } \sigma_i^{-1}\sigma\sigma_j \in \Gamma_n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2. *For all $\sigma \in \mathcal{S}_n$, and $j = 1, 2, \dots, (n-1)!$,*

$$\sigma \rho_{\sigma_j} = \sum_{i=1}^{(n-1)!} C_{i,j}(\sigma) \rho_{\sigma_i},$$

and

$$\sigma \bar{m}_{\sigma_j} = \sum_{i=1}^{(n-1)!} C^*_{i,j}(\sigma) \bar{m}_{\sigma_i},$$

In other words, $C(\sigma)$ is the matrix representing the action of $\sigma \in \mathcal{S}_n$ on $H(\Pi_n)$ with respect to the splitting basis and $C^*(\sigma)$ is the matrix representing the action of $\sigma \in \mathcal{S}_n$ on $H^*(\Pi_n)$ with respect to the dual splitting basis.

Proof. By Proposition 1.2 and Lemma 3.1, the i, j component of the representation matrix is

$$\begin{aligned} \langle \sigma \rho_{\sigma_j}, \bar{m}_{\sigma_i} \rangle &= \langle \rho_{\sigma\sigma_j}, \bar{m}_{\sigma_i} \rangle \\ &= C_{i,j}(\sigma). \qquad \square \end{aligned}$$

There is a simple way to express the Björner basis in terms of the splitting basis. Let T be a tree on $\{1, 2, \dots, n\}$. We shall say that $\sigma \in \mathcal{S}_n$ is a *pruning sequence* of T if $\sigma(n) = n$ and $\sigma(1)$ is a leaf of T, $\sigma(2)$ is a leaf of $T - \{\sigma(1)\}$, $\sigma(3)$ is a leaf of $T - \{\sigma(1), \sigma(2)\}$, etc.. If we view T as rooted at n and draw T with its root on top then T is the Hasse diagram of a poset. A pruning sequence for T is the same as a linear extension of the poset with Hasse diagram T. Let $\mathcal{P}(T)$ be the collection of pruning sequences for T.

Theorem 3.3. *Let T be a tree on $\{1, 2, \dots, n\}$. Then*

$$\rho_T = \pm \sum_{\sigma \in \mathcal{P}(T)} \operatorname{sgn}(\sigma) \rho_\sigma. \tag{3-2}$$

If T is increasing then

$$\rho_T = (-1)^{\binom{n-1}{2}} \sum_{\sigma \in \mathcal{P}(T)} \operatorname{sgn}(\sigma) \rho_\sigma. \tag{3-3}$$

If T is a caterpillar with spine $r_1, r_2, \dots, r_k$ then

$$\rho_T = (-1)^{r_1 + \cdots + r_k + \operatorname{inv}(r_k r_{k-1} \cdots r_1) + n} \sum_{\sigma \in \mathcal{P}(T)} \operatorname{sgn}(\sigma) \rho_\sigma. \tag{3-4}$$

Proof. The proof is similar to the proof of Lemma 3.1. First view T as rooted at n and let $p(i)$ be the parent of i for each $i = 1, 2, \dots, n-1$. Define an isomorphism π from the dual of B_{n-1} to Π_T by letting $\pi(S)$ be the partition obtained from T by erasing edges $\{i, p(i)\}$, for all $i \in S$. As in the proof of Lemma 3.1, the fundamental cycles of $\Delta(B_{n-1})$ map to fundamental cycles of $\Delta(\Pi_T)$ which are hence given by

$$\rho_T = \pm \sum_{\sigma \in \mathcal{S}_{n-1}} \operatorname{sgn}(\sigma)\pi(\bar{c}_\sigma). \tag{3-5}$$

Let

$$\nu_T = \sum_{\sigma \in \mathcal{S}_{n-1}} \operatorname{sgn}(\sigma)\pi(\bar{c}_\sigma).$$

The maximal chain $\pi(\bar{c}_\sigma)$ can be characterized as the chain whose partitions are obtained from top to bottom by first erasing edge $\{\sigma(1), p(\sigma(1))\}$ from T; then erasing edge $\{\sigma(2), p(\sigma(2))\}$, etc. Clearly, for $\tau \in \mathcal{S}_n$ with $\tau(n) = n$, $\bar{m}_\tau = \pi(\bar{c}_\sigma)$ for some $\sigma \in S_{n-1}$ if and only if $\tau(1)\tau(2)\ldots\tau(n-1) = \sigma$ and τ is a pruning sequence. It follows that

$$\langle \nu_T, \bar{m}_\tau \rangle = \operatorname{sgn}(\tau)\chi(\tau \text{ is a pruning sequence}), \tag{3-6}$$

where $\chi(s)$ is the function whose value is 1 if the statement s is true and 0 otherwise.

Since ν_T is a cycle let $\nu_T = \sum_{\substack{\alpha \in \mathcal{S}_n \\ \alpha(n)=n}} a_\alpha \rho_\alpha$. Then

$$\begin{aligned}\langle \nu_T, \bar{m}_\tau \rangle &= \sum_{\substack{\alpha \in \mathcal{S}_n \\ \alpha(n)=n}} a_\alpha \langle \rho_\alpha, \bar{m}_\tau \rangle \\ &= a_\tau.\end{aligned}$$

It follows from (3-6) that

$$\nu_T = \sum_{\tau \in \mathcal{P}(T)} \operatorname{sgn}(\tau)\rho_\tau. \tag{3-7}$$

Since $\rho_T = \pm\nu_T$, (3-2) holds.

If T is increasing then the orientation of ρ_T is determined by $\langle \rho_T, \bar{m}_T \rangle = 1$. Since $\pi(\bar{c}_{n-1,\ldots,1}) = \bar{m}_T$ and $\operatorname{sgn}(n-1,\ldots,1) = (-1)^{\binom{n-1}{2}}$, the orientation in (3-5) becomes

$$\begin{aligned}\rho_T &= (-1)^{\binom{n-1}{2}} \sum_{\sigma \in \mathcal{S}_{n-1}} \operatorname{sgn}(\sigma)\pi(\bar{c}_\sigma) \\ &= (-1)^{\binom{n-1}{2}} \nu_T.\end{aligned}$$

Hence (3-3) follows from (3-7).

If T is a caterpillar with spine $r_1, r_2, \ldots, r_k$ then the orientation of ρ_T is determined by $\langle \rho_T, \bar{\mu}_T \rangle = 1$. Since $\pi(c_{r_1,r_2,\ldots,r_{k-1},a_1,a_2,\ldots,a_{n-k},r_k}) = \mu_T$, the result follows from counting inversions of $\sigma = r_1, r_2, \ldots, r_{k-1}, a_1, a_2, \ldots, a_{n-k}, r_k$. $\square$

4. A Barcelo-like correspondence.

A Lie algebra L over field $\mathbf{k}$ is said to be the *free Lie algebra* on a set A if (1) A generates L and (2) given any map ϕ from A to a Lie algebra M over $\mathbf{k}$, there exists a unique Lie algebra homomorphism $\psi : L \to M$ extending ϕ. It is well known that L has the following realization as a subspace of $\mathbf{k}[A^*]$, the free associative $\mathbf{k}$-algebra generated by elements of A. For $f, g \in \mathbf{k}[A^*]$, define the bracket of f and g by

$$[f, g] = fg - gf. \tag{4-1}$$

The free Lie algebra on A is the smallest subspace of $\mathbf{k}[A^*]$ that is closed under $[,]$ and contains A. See [G] and [R] for comprehensive treatments of the free Lie algebra from a combinatorial point of view.

We now define the vector space Lie_n to be the intersection of the free Lie algebra on $A = \{1, 2, \dots, n\}$ and the subspace of $\mathbf{k}[A^*]$ spanned by all permutations in $\mathcal{S}_n$. The space Lie_n is known as the 1^n homogeneous component of the free Lie algebra on $\{1, 2, \dots, n\}$.

As in [G], we shall represent the bracketed permutations that generate Lie_n as binary trees with injective leaf labelings. More precisely, let $\mathcal{T}_n$ denote the set of binary trees with n leaves. The binary trees considered here have the property that all internal nodes have a left child and a right child. For $T \in \mathcal{T}_n$ and word w with n distinct letters in $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$, let (T, w) be the binary tree whose ith leaf from left to right is labeled with $w(i)$, the ith letter of w. We shall refer to (T, w) as a *labeled binary tree.* For $n > 1$, let (T_1, w_1) and (T_2, w_2) respectively denote the left and right labeled subtrees of (T, w). Now recursively set

$$[T, w] = \begin{cases} [[T_1, w_1], [T_2, w_2]] & \text{if } n > 1 \\ w(1) & \text{if } n = 1, \end{cases}$$

where the bracket on the right is the one defined in (4-1). For example, if (T, w) is the labeled binary tree given in Figure 4.1 then

$$[T, w] = [[[2, 3], 5], [4, [1, 6]]].$$

So Lie_n is the subspace of $\mathbf{k}[A^*]$ spanned by $\{[T, w] \mid T \in \mathcal{T}_n, \quad w \in S_n\}$. We shall refer to the elements $[T, w]$ as bracketed permutations.

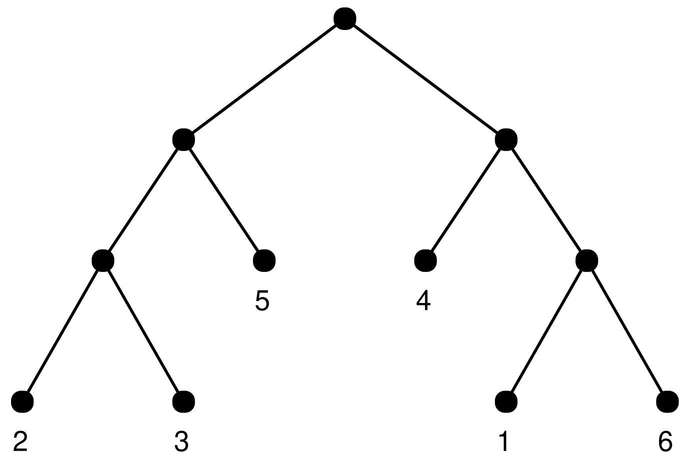

Figure 4.1

We now recall a well-known basis for Lie_n. A *Lyndon tree* is recursively defined to be a labeled binary tree (T, w) such that either T consists of only one node or if (T_1, w_1) and (T_2, w_2) are the nonempty left and right subtrees of (T, w) then

(1) (T_1, w_1) and (T_2, w_2) are Lyndon trees,
(2) the largest label of (T_1, w_1) is the second largest label of (T, w).

An example of a Lyndon tree is given in Figure 4.1. The set

$$\{[T, w] \mid (T, w) \text{ is a Lyndon tree and } w \in \mathcal{S}_n\}$$

is a basis for Lie_n called the *Lyndon basis.*

Remark. The Lyndon basis has a more general formulation as a basis for the full free Lie algebra on A. Here we have defined only the subset of the Lyndon basis that generates Lie_n. Also the formulation given here is nonstandard in that both left to right order and numerical order are reversed.

Let R_n denote the n-leaf binary tree for which the left child of every internal node is a leaf. For any word w consisting of n distinct letters in $\mathbf{Z}^+$, let $r_w = [R_n, w]$. Following [BS], we refer to r_w as a right comb. For example, the labeled binary tree in Figure 4.2 yields the right comb $r_{34125} = [3, [4, [1, [2, 5]]]]$. In [BS] it is observed that the set $\{r_w \mid w \in \mathcal{S}_n, \quad w(n) = n\}$ is a basis for Lie_n called the *right comb basis.*

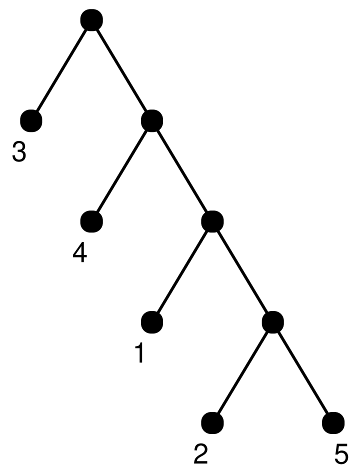

Figure 4.2

The action of the symmetric group $\mathcal{S}_n$ on Lie_n is given by $\sigma[T, w] = [T, \sigma w]$ for $T \in \mathcal{T}_n$ and $w, \sigma \in \mathcal{S}_n$. In [Ba], Barcelo constructs an elegant bijection between the Lyndon basis for Lie_n and the (dual) Björner basis for (co)homology in order to give a combinatorial explanation of the fact that Lie_n and $H^*(\Pi_n)$ tensored with the sign representation are isomorphic $\mathcal{S}_n$-modules.

We shall now give a slightly different description of the Barcelo correspondence from the one appearing in [Ba]. Let γ be the function from the set of Lyndon trees to the set of increasing trees defined recursively as follows: If A is a single node then $\gamma(A) = A$; if A has left subtree A_1 and right subtree A_2 then $\gamma(A)$ is the tree consisting of $\gamma(A_1), \gamma(A_2)$ and an edge from the root (i.e. maximum vertex) of $\gamma(A_1)$ to the root of $\gamma(A_2)$. For example, if A is the Lyndon tree of Figure 4.1 then $\gamma(A)$ is the increasing tree given in Figure 4.3.

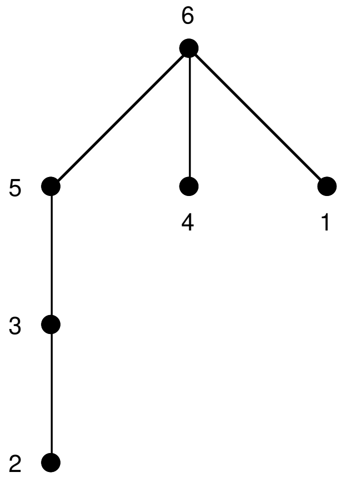

Figure 4.3

Note that the root of $\gamma(A_2)$ becomes the root of $\gamma(A)$ and the root of $\gamma(A_1)$ becomes the largest child of the root of $\gamma(A)$. It follows from this that γ has an inverse and is therefore a bijection. Note also that the leaf label set of A is the same as the node set of $\gamma(A)$. We remark that if we view increasing trees as ordered rooted trees in which the children of every node are decreasing from left to right as well as smaller than their parent then γ is the restriction of a classical bijection from the set of leaf labeled binary trees to the set of ordered rooted trees on subsets of $\mathbf{Z}^+$.

Theorem 4.1 (Barcelo correspondence). *The map*

$$[A] \mapsto \bar{m}_{\gamma(A)}$$

from the Lyndon basis for Lie_n *to the dual Björner basis for* $H^*(\Pi_n)$ *determines an* $\mathcal{S}_n$*-module isomorphism from* Lie_n *to* $H^*(\Pi_n) \otimes \mathrm{sgn}$.

Barcelo proves this result by computing the representation matrices with respect to the Lyndon basis and Björner basis. We now compute the representation matrices with respect to the right comb basis and use this to give a combinatorial derivation of the isomorphism between Lie_n and $H^*(\Pi_n) \otimes \mathrm{sgn}$ which is parallel to but somewhat simpler than Barcelo's derivation.

As in Section 3, let σ_i be the ith permutation of $\{\sigma \in \mathcal{S}_n \mid \sigma(n) = n\}$ in lexicographical order and let Γ_n be the set of minmax permutations in $\mathcal{S}_n$.

Theorem 4.2. *The representation matrix* $B(\sigma)$ *for the action of* $\sigma \in \mathcal{S}_n$ *on* Lie_n *with respect to the ordered right comb basis* $r_{\sigma_1}, r_{\sigma_2}, \dots, r_{\sigma_{(n-1)!}}$ *has* i, j*-entry,*

$$B_{i,j}(\sigma) = \begin{cases} (-1)^{\mathrm{des}(\sigma_i^{-1}\sigma\sigma_j)} & \textit{if } \sigma_i^{-1}\sigma\sigma_j \in \Gamma_n \\ 0 & \textit{otherwise,} \end{cases}$$

where $\operatorname{des}(\sigma)$ *is the number of descents of* σ.

Proof. Let $\langle , \rangle$ be the bilinear form on $\mathbf{k}[A^*]$ defined by $\langle u, v\rangle = \delta_{u,v}$ for $u, v \in A^*$. So if $f \in \mathrm{Lie}_n$ and $\sigma \in \mathcal{S}_n$ then $\langle f, \sigma\rangle$ is simply the coefficient of σ in f. Note that for all $i, k = 1, 2, \ldots, (n-1)!$,

$$\langle r_{\sigma_k}, \sigma_i\rangle = \delta_{i,k}, \tag{4-2}$$

since σ_k is the only permutation in the expansion of r_{σ_k} that ends with n.

First we claim that for all $\sigma \in \mathcal{S}_n$,

$$B_{i,j}(\sigma) = \langle r_{\sigma\sigma_j}, \sigma_i\rangle. \tag{4-3}$$

This follows from

$$\begin{aligned}\langle r_{\sigma\sigma_j}, \sigma_i\rangle &= \langle \sigma r_{\sigma_j}, \sigma_i\rangle \\ &= \langle \sum_{k=1}^{(n-1)!} B_{k,j}(\sigma) r_{\sigma_k}, \sigma_i\rangle \\ &= \sum_{k=1}^{(n-1)!} B_{k,j}(\sigma)\langle r_{\sigma_k}, \sigma_i\rangle,\end{aligned}$$

and from (4-2).

Next we claim that for any $\alpha \in \mathcal{S}_n$ and ϵ the identity permutation in $\mathcal{S}_n$,

$$\langle r_\alpha, \epsilon\rangle = (-1)^{\operatorname{des}(\alpha)}\chi(\alpha \in \Gamma_n). \tag{4-4}$$

To prove this we expand r_α as follows:

$$\begin{aligned} r_\alpha &= [\alpha(1), r_{\alpha(2)\alpha(3)\cdots\alpha(n)}] \\ &= \alpha(1) r_{\alpha(2)\alpha(3)\cdots\alpha(n)} - r_{\alpha(2)\alpha(3)\cdots\alpha(n)}\alpha(1).\end{aligned}$$

We have by induction,

$$\begin{aligned}\langle r_\alpha, \epsilon\rangle &= \langle \alpha(1) r_{\alpha(2)\alpha(3)\cdots\alpha(n)}, \epsilon\rangle - \langle r_{\alpha(2)\alpha(3)\cdots\alpha(n)}\alpha(1), \epsilon\rangle \\ &= \begin{cases} \langle r_{\alpha(2)\alpha(3)\cdots\alpha(n)}, 2\cdots n\rangle & \text{if } \alpha(1) = 1 \\ -\langle r_{\alpha(2)\alpha(3)\cdots\alpha(n)}, 1\cdots n-1\rangle & \text{if } \alpha(1) = n \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (-1)^{\operatorname{des}(\alpha(2)\alpha(3)\cdots\alpha(n))} & \text{if } \alpha(1) = 1 \text{ and } \alpha(2)-1, \alpha(3)-1, \ldots, \alpha(n)-1 \in \Gamma_{n-1} \\ -(-1)^{\operatorname{des}(\alpha(2)\alpha(3)\cdots\alpha(n))} & \text{if } \alpha(1) = n \text{ and } \alpha(2)\alpha(3)\cdots\alpha(n) \in \Gamma_{n-1} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Since $\alpha \in \Gamma_n$ if and only if either $\alpha(1)$ is 1 and $\alpha(2)-1, \alpha(3)-1, \ldots, \alpha(n)-1 \in \Gamma_{n-1}$ or $\alpha(1) = n$ and $\alpha(2)\alpha(3)\cdots\alpha(n) \in \Gamma_{n-1}$, (4-4) holds.

By (4-3) and (4-4), we have

$$\begin{aligned} B_{i,j}(\sigma) &= \langle r_{\sigma\sigma_j}, \sigma_i\rangle \\ &= \langle \sigma_i^{-1} r_{\sigma\sigma_j}, \epsilon\rangle \\ &= \langle r_{\sigma_i^{-1}\sigma\sigma_j}, \epsilon\rangle \\ &= (-1)^{\operatorname{des}(\sigma_i^{-1}\sigma\sigma_j)}\chi(\sigma_i^{-1}\sigma\sigma_j \in \Gamma_n). \qquad \square \end{aligned}$$

The representation matrices computed in Theorems 3.2 and 4.2 can now be used to derive the following result which was obtained jointly with S. Sundaram.

Theorem 4.3 (comb correspondence). *The map*

$$r_w \mapsto \operatorname{sgn}(w)\bar{m}_w$$

from the right comb basis for Lie_n *to the "signed" dual splitting basis for* $H^*(\Pi_n)$ *determines an* $\mathcal{S}_n$*-module isomorphism from* Lie_n *to* $H^*(\Pi_n) \otimes \operatorname{sgn}$.

Proof. It follows from Theorem 3.2 that the representation matrix $\hat{C}(\sigma)$ for the action of σ on $H^*(\Pi_n)$ with respect to the signed dual splitting basis

$$\{\operatorname{sgn}(\sigma_1)\bar{m}_{\sigma_1}, \operatorname{sgn}(\sigma_2)\bar{m}_{\sigma_2}, \dots, \operatorname{sgn}(\sigma_{(n-1)!})\bar{m}_{\sigma_{(n-1)!}}\}$$

has i,j-entry given by

$$\hat{C}_{i,j}(\sigma) = \operatorname{sgn}(\sigma_i\sigma_j)\operatorname{sgn}^*(\sigma_i^{-1}\sigma\sigma_j)\chi(\sigma_i^{-1}\sigma\sigma_j \in \Gamma_n). \tag{4-5}$$

We claim that for all $\alpha \in \Gamma_n$, $\operatorname{sgn}^*(\alpha) = (-1)^{\operatorname{des}(\alpha)}\operatorname{sgn}(\alpha)$. Indeed, this follows from the fact that $\operatorname{inv}(\alpha) = \operatorname{inv}(\alpha(1)\alpha(2)\cdots\alpha(n-1)) + \operatorname{des}(\alpha)$, where $\operatorname{inv}(\alpha)$ is the number of inversions of α. Plugging this into (4-5) yields

$$\operatorname{sgn}(\sigma)\hat{C}_{i,j}(\sigma) = (-1)^{\operatorname{des}(\sigma_i^{-1}\sigma\sigma_j)}\chi(\sigma_i^{-1}\sigma\sigma_j \in \Gamma_n),$$

which by Theorem 4.2 is the i,j-entry of the representation matrix $B(\sigma)$ for the action of σ on Lie_n. $\square$

5. The general correspondence.

The Barcelo correspondence and the comb correspondence map certain subsets of a natural generating set $\{[T,w] \mid T \in \mathcal{T}_n,\ \omega \in \mathcal{S}_n\}$ for Lie_n to certain subsets of a natural generating set $\mathcal{M}(\Pi_n)$ for $H^*(\Pi_n)$. In this section we describe a combinatorial correspondence between the two natural generating sets which restricts to the Barcelo correspondence and the comb correspondence. We show that this correspondence extends to a well-defined $\mathcal{S}_n$ isomorphism between Lie_n and $H^*(\Pi_n) \otimes \operatorname{sgn}$ by showing that the Lie algebra relations for Lie_n map to the cohomology relations for $H^*(\Pi_n)$.

Just as the natural generating set (of bracketed permutations) for Lie_n can be expressed in terms of labeled binary trees, we shall express the natural generating set (of maximal chains) for $H^*(\Pi_n)$ in terms of labeled binary trees.

First recall that the *postorder* listing of the nodes of a binary tree T is defined recursively as follows: first list the nodes of the left subtree of T in postorder, then list the nodes of the right subtree of T in postorder, and finally list the root of T. For any internal node v of (T,w), let $L(v)$ be the set of leaf labels in the left subtree of v and let $R(v)$ be the set of leaf labels in the right subtree of v. We define $c(T,w)$, where $T \in \mathcal{T}_n$, $w \in \mathcal{S}_n$, to be the maximal chain of Π_n whose rank i partition is obtained from its rank $i-1$ partition by merging blocks $L(v_i)$ and $R(v_i)$ where v_i is the ith internal node of T in the postorder listing. For example, if (T,w) is given in Figure 4.1 then $c(T,w)$ is the maximal chain

$$\hat{0} \to 23/5/4/1/6 \to 235/4/1/6 \to 235/4/16 \to 235/416 \to \hat{1}.$$

Note that the maximal chains m_w, $w \in \mathcal{S}_n$, $w(n) = n$ which give rise to the dual splitting basis have the form $c(R_n, w)$. (Recall that R_n is the n-leaf binary tree in which the left child of each internal node is a leaf.) Hence the comb correspondence can be expressed as

$$[R_n, w] \mapsto \mathrm{sgn}(w)\bar{c}(R_n, w) \quad w \in \mathcal{S}_n, w(n) = n.$$

Not all maximal chains are of the form $c(T, w)$ for some labeled binary tree (T, w). For example, the maximal chain

$$\hat{0} \to 12/3/4/5 \to 12/34/5 \to 125/34 \to \hat{1} \tag{5-1}$$

is not of this form. It does turn out (Lemma 5.2), however, that every maximal chain of $\Pi_n - \{\hat{0}, \hat{1}\}$ is in the same cohomology class as some $\bar{c}(T, w)$. So the set $\{\bar{c}(T, w) \mid T \in \mathcal{T}_n, w \in \mathcal{S}_n\}$ is a generating set for $H^*(\Pi_n)$.

We shall express all maximal chains in terms of binary trees by considering arbitrary linear extensions of the internal nodes of the tree. Let $v_1, v_2, \ldots, v_{n-1}$ be the postorder listing of the internal nodes of the binary tree T. A *linear extension* of the internal nodes of a binary tree T is a listing of the nodes $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n-1)}$ such that each node precedes its parent i.e., if $v_{\sigma(i)}$ is the parent of $v_{\sigma(j)}$ then $j < i$. We shall say that the permutation σ induces the linear extension $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n-1)}$ and let $E(T)$ be the set of all permutations that induce linear extensions of T. For example, if T is the binary tree in Figure 5.1, with postorder labeling of internal nodes indicated, then $E(T) = \{1234, 1324, 3124\}$.

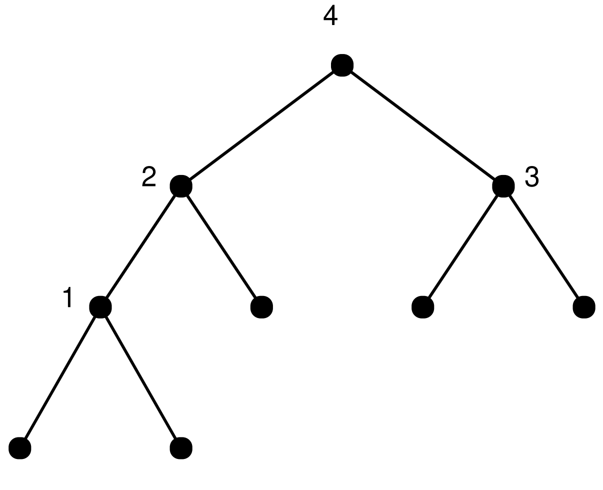

Figure 5.1

Lemma 5.1. *Let T be a binary tree. Then*

(1) $\epsilon \in E(T)$.
(2) *If $\sigma \in E(T)$ and $\sigma(i) > \sigma(i+1)$ then $\sigma \cdot (i, i+1) \in E(T)$.*

Proof. Since postorder is a linear extension of T, (1) holds. Since $\sigma(i) > \sigma(i+1)$, $v_{\sigma(i+1)}$ precedes $v_{\sigma(i)}$ in postorder. Hence $v_{\sigma(i+1)}$ is not an ancestor of $v_{\sigma(i)}$. It follows that $v_{\sigma(i)}$ and $v_{\sigma(i+1)}$ can be switched in the linear extension induced by σ to obtain a linear extension induced by $\sigma \cdot (i, i+1)$. $\square$

Now given $T \in \mathcal{T}_n$, $w \in \mathcal{S}_n$, $\sigma \in E(T)$, let $c(T, w, \sigma)$ be the maximal chain of Π_n whose rank i partition is obtained from its rank $i-1$ partition by merging blocks $L(v_{\sigma(i)})$ and $R(v_{\sigma(i)})$ where v_j is the jth internal node of T in the postorder listing. For example, if T is the binary tree of Figure 5.1, $w = 12534$, and $\sigma = 1324$ then $c(T, w, \sigma)$ is the maximum chain given by (5-1). Note that $c(T, w, \epsilon)$ is simply $c(T, w)$ defined above.

It is easy to see that every maximal chain of Π_n is of the form $c(T, w, \sigma)$ for some $T \in \mathcal{T}_n$, $w \in \mathcal{S}_n$, $\sigma \in E(T)$. It follows from this and from the next result that every maximal chain of $\Pi_n - \{\hat{0}, \hat{1}\}$ is in the same cohomology class as some $\bar{c}(T, w)$.

Lemma 5.2. *Let $T \in \mathcal{T}_n$, $w \in \mathcal{S}_n$, $\sigma \in E(T)$. Then in cohomology,*

$$\bar{c}(T, w, \sigma) = \operatorname{sgn}(\sigma)\bar{c}(T, w).$$

Proof. The proof is by induction on $\operatorname{inv}(\sigma)$. If $\operatorname{inv}(\sigma) = 0$ then the result is trivial. Suppose $\operatorname{inv}(\sigma) \geq 1$. Then there is some i such that $\sigma(i) > \sigma(i+1)$. By Lemma 5.1, $\sigma \cdot (i, i+1) \in E(T)$. Since $\operatorname{inv}(\sigma \cdot (i, i+1)) = \operatorname{inv}(\sigma) - 1$, by induction we have,

$$\begin{aligned} \bar{c}(T, w, \sigma \cdot (i, i+1)) &= \operatorname{sgn}(\sigma \cdot (i, i+1))\bar{c}(T, w) \\ &= -\operatorname{sgn}(\sigma)\bar{c}(T, w). \end{aligned}$$

To complete the proof, we must show

$$\bar{c}(T, w, \sigma) = -\bar{c}(T, w, \sigma \cdot (i, i+1)). \tag{5-2}$$

Recall in the proof of Lemma 5.1, we saw that $v_{\sigma(i)}$ and $v_{\sigma(i+1)}$ are unrelated in T. Hence $L(v_{\sigma(i)})$, $R(v_{\sigma(i)})$, $L(v_{\sigma(i+1)})$, and $R(v_{\sigma(i+1)})$ are pairwise disjoint sets. Moreover, $L(v_{\sigma(i)})$, $R(v_{\sigma(i)})$, $L(v_{\sigma(i+1)})$, and $R(v_{\sigma(i+1)})$ are blocks of the rank $i-1$ partition in $c(T, w, \sigma)$ and $L(v_{\sigma(i)}) \cup R(v_{\sigma(i)})$ and $L(v_{\sigma(i+1)}) \cup R(v_{\sigma(i+1)})$ are blocks of the rank $i+1$ partition in $c(T, w, \sigma)$. The maximal chains $c(T, w, \sigma)$ and $c(T, w, \sigma \cdot (i, i+1))$ are identical except at rank i. At rank i in $c(T, w, \sigma)$, blocks $L(v_{\sigma(i)})$, and $R(v_{\sigma(i)})$ are merged. While at rank i in $c(T, w, \sigma \cdot (i, i+1))$, blocks $L(v_{\sigma(i+1)})$, and $R(v_{\sigma(i+1)})$ are merged. Let c be the chain $\bar{c}(T, w, \sigma)$ with the rank i partition removed. Then clearly by (1-1), $\delta(c) = (-1)^{(i-1)}(\bar{c}(T, w, \sigma) + \bar{c}(T, w, \sigma \cdot (i, i+1)))$. This yields (5-2). $\square$

We need to use the following standard notation for a labeled binary tree (T, w). A labeled binary tree (T, w) is represented by a word over the alphabet $\{(,), \wedge\} \cup \mathbf{Z}^+$ as follows: If (T, w) consists of a single node labeled by a, then (T, w) is represented simply by the word a; otherwise (T, w) is represented by the word $(A \wedge B)$, where A is the word representing the left labeled subtree and B is the word representing the right labeled subtree. For example, the tree in Figure 4.1 is represented by the word $(((2 \wedge 3) \wedge 5) \wedge (4 \wedge (1 \wedge 6)))$. Unlabeled binary trees have a similar representation with a dot replacing each leaf label. For labeled (or unlabeled) binary trees A and X, we will write $A = \alpha X \beta$ to express the fact that X is a subtree of A, and α and β are the rest of the expression for A. For example, if X is the labeled tree $(1 \wedge 6)$ and A is the labeled tree of Figure 4.1 then we write $A = \alpha X \beta$, where α is the word $(((2 \wedge 3) \wedge 5) \wedge (4\wedge$ and β is the word $))$.

We also let $I(A)$ denote the number of internal nodes of labeled or unlabeled binary tree A.

The following result is equivalent to the $k = 1$ case of the first part of [HW, Theorem 3.11]. We reprove it within the framework of this paper.

Theorem 5.3. *The set $\{\bar{c}(T, w) \mid T \in \mathcal{T}_n,\ w \in \mathcal{S}_n\}$ is a generating set for $H^*(\Pi_n)$, subject only to the relations*

$$\bar{c}(\alpha(A \wedge B)\beta) = (-1)^{I(A)I(B)}\bar{c}(\alpha(B \wedge A)\beta), \tag{5-3}$$

(5-4)
$$\bar{c}(\alpha(A\wedge(B\wedge C))\beta)) + (-1)^{I(C)}\bar{c}(\alpha((A\wedge B)\wedge C)\beta) + (-1)^{I(A)I(B)}\bar{c}(\alpha(B\wedge(A\wedge C))\beta) = 0.$$

Proof. The fact that $\{\bar{c}(T, w) \mid T \in \mathcal{T}_n,\ w \in \mathcal{S}_n\}$ generates $H^*(\Pi_n)$ is an immediate consequence of Lemma 5.2.

To establish cohomology relation (5-3), note that

$$c(\alpha(B \wedge A)\beta) = c(\alpha(A \wedge B)\beta, \sigma),$$

where σ is the permutation that induces the linear extension which is just like postorder except that the internal nodes of B are listed before the internal nodes of A. Clearly $\mathrm{inv}(\sigma) = I(A)I(B)$. Hence (5-3) follows from Lemma 5.2.

We shall refer to a relation obtained by setting the coboundary of a codimension 1 chain of $\Pi_n - \{\hat{0}, \hat{1}\}$ equal to 0, as an *elementary cohomology relation.* There are two types of codimension 1 chains and they yield corresponding elementary cohomology relations. Indeed, if c is a codimension 1 chain of $\Pi_n - \{\hat{0}, \hat{1}\}$ then the chain $c \cup \{\hat{0}, \hat{1}\}$ is unrefinable except between one pair of adjacent elements, $x < y$, where y is obtained from x either by merging two pairs of distinct blocks (call this *type* 1) or by merging three distinct blocks (call this *type* 2).

By setting the coboundary of a type 1 codimension 1 chain equal to 0, we get an elementary cohomology relation of the form given in (5-2). The elementary cohomology relations (5-2) are equivalent to the relations given in Lemma 5.2 which, as we saw above, determine cohomology relation (5-3).

The elementary cohomology relations of type 2 are of the form

$$\bar{c}(\alpha(A \wedge (B \wedge C))\beta, \epsilon) + \bar{c}(\alpha((A \wedge B) \wedge C)\beta, \sigma) + \bar{c}(\alpha(B \wedge (A \wedge C))\beta, \tau) = 0, \tag{5-5}$$

where σ is the permutation that induces the linear extension which is just like postorder except that the internal nodes of C are listed before the root of $A \wedge B$; and τ is the permutation that induces the linear extension which is just like postorder except that the internal nodes of A are listed before the internal nodes of B. Clearly $\mathrm{inv}(\sigma) = I(C)$ and $\mathrm{inv}(\tau) = I(A)I(B)$. Hence, by Lemma 5.2, (5-4) is equivalent to the elementary cohomology relation of type 2 given in (5-5).

Since the relations (5-3) and (5-4) correspond to the elementary cohomology relations, it is reasonable that (5-3) and (5-4) should generate all the cohomology relations. This is indeed the case and it follows from the observation that any generator $\bar{c}(T,w)$ can be "straightened" by using relations (5-3) and (5-4), ultimately ending up with a linear combination of elements from the dual splitting basis. Alternatively, one can use the fact that the map ϕ of Theorem 5.4 is actually proved to be an isomorphism from the vector space Lie_n to the vector space generated by the chains $\bar{c}(T,w)$ subject to relations (5-3) and (5-4). Hence the dimensions of both vector spaces are the same, namely $(n-1)!$. Since this is also the dimension of $H^*(\Pi_n)$ we can conclude that (5-3) and (5-4) generate all the relations. □

Our general correspondence is given in the next result.

Theorem 5.4. *There is an $\mathcal{S}_n$-isomorphism $\phi : \mathrm{Lie}_n \to H^*(\Pi_n) \otimes \mathrm{sgn}$ determined by*

$$\phi([T,w]) = \mathrm{sgn}(w)\,\mathrm{sgn}(T)\bar{c}(T,w), \quad T \in \mathcal{T}_n, w \in \mathcal{S}_n,$$

where $\mathrm{sgn}(T)$ *is defined recursively by* $\mathrm{sgn}(T) = 1$ *if* $I(T) = 0$ *and*

$$\mathrm{sgn}(T_1 \wedge T_2) = (-1)^{I(T_2)}\,\mathrm{sgn}(T_1)\,\mathrm{sgn}(T_2)$$

otherwise.

Proof. Since Lie_n is a subspace of the free Lie algebra, the generating set $\{[T,w] \mid T \in \mathcal{T}_n, \quad w \in S_n\}$ is subject only to the anticommuting and Jacobi relations. In terms of labeled binary trees, the anticommuting relation is

$$[\alpha(A \wedge B)\beta] = -[\alpha(B \wedge A)\beta], \tag{5-6}$$

and the Jacobi relation is

$$[\alpha(A \wedge (B \wedge C))\beta] - [\alpha((A \wedge B) \wedge C)\beta] - [\alpha(B \wedge (A \wedge C))\beta] = 0. \tag{5-7}$$

To show that ϕ defined on the generators determines a well defined isomorphism, it suffices to show that ϕ maps the anticommuting and Jacobi relations to the cohomology relations (5-3) and (5-4), respectively. That ϕ respects the action of $\mathcal{S}_n$ is immediate.

We begin with the anticommuting relation (5-6). We will show that

$$\phi([\alpha(A\wedge B)\beta]) + \phi([\alpha(B\wedge A)\beta]) = \pm\left(\bar{c}(\alpha(A \wedge B)\beta) - (-1)^{I(A)I(B)}\bar{c}(\alpha(B \wedge A)\beta)\right). \tag{5-8}$$

First observe that if $uw_1w_2v \in \mathcal{S}_n$, where u, w_1, w_2, v are subwords, then

$$\mathrm{sgn}(uw_2w_1v) = (-1)^{\ell(w_1)\ell(w_2)}\,\mathrm{sgn}(uw_1w_2v), \tag{5-9}$$

where $\ell(w)$ is the length of the word w.

We also claim that if $\rho(T_1 \wedge T_2)\tau$ is a binary tree with subtree $(T_1 \wedge T_2)$, then

$$\mathrm{sgn}(\rho(T_2 \wedge T_1)\tau) = (-1)^{I(T_1)+I(T_2)}\,\mathrm{sgn}(\rho(T_1 \wedge T_2)\tau). \tag{5-10}$$

This claim is proved by induction on the depth of the root of $T_1 \wedge T_2$ in $\rho(T_1 \wedge T_2)\tau$. If the depth is 0, then $\rho(T_1 \wedge T_2)\tau = (T_1 \wedge T_2)$, $\rho(T_2 \wedge T_1)\tau = (T_2 \wedge T_1)$ and

$$\begin{aligned}\operatorname{sgn}(T_2 \wedge T_1) &= \operatorname{sgn}(T_2)\operatorname{sgn}(T_1)(-1)^{I(T_1)}\\ &= \operatorname{sgn}(T_1)\operatorname{sgn}(T_2)(-1)^{I(T_2)}(-1)^{I(T_1)+I(T_2)}\\ &= \operatorname{sgn}(T_1 \wedge T_2)(-1)^{I(T_1)+I(T_2)}.\end{aligned}$$

If the depth d is greater than 0 then $T_1 \wedge T_2$ is a subtree of depth $d-1$ in the left subtree or the right subtree of $\rho(T_2 \wedge T_1)\tau$. We can therefore apply the induction hypothesis to the appropriate subtree to establish (5-10).

In (5-8), let $A = (T_1, w_1)$, $B = (T_2, w_2)$ and $\alpha(A \wedge B)\beta = (\rho(T_1 \wedge T_2)\tau, uw_1w_2v)$. By (5-9) and (5-10) we have,

$$\begin{aligned}\phi([\alpha(B \wedge A)\beta]) &= \operatorname{sgn}(uw_2w_1v)\operatorname{sgn}(\rho(T_2 \wedge T_1)\tau)\bar{c}(\alpha(B \wedge A)\beta)\\ &= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(\rho(T_1 \wedge T_2)\tau)(-1)^{\ell(w_1)\ell(w_2)}(-1)^{I(T_1)+I(T_2)}\\ &\qquad \cdot \bar{c}(\alpha(B \wedge A)\beta)\\ &= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(\rho(T_1 \wedge T_2)\tau)(-1)^{(I(T_1)+1)(I(T_2)+1)}(-1)^{I(T_1)+I(T_2)}\\ &\qquad \cdot \bar{c}(\alpha(B \wedge A)\beta)\\ &= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(\rho(T_1 \wedge T_2)\tau)(-1)^{I(T_1)I(T_2)+1}\bar{c}(\alpha(B \wedge A)\beta)\end{aligned}$$

This yields

$$\begin{aligned}\phi([\alpha(A \wedge B)\beta]) + \phi([\alpha(B \wedge A)\beta]) &= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(\rho(T_1 \wedge T_2)\tau)\\ &\qquad \cdot \left(\bar{c}(\alpha(A \wedge B)\beta) - (-1)^{I(T_1)I(T_2)}\bar{c}(\alpha(B \wedge A)\beta)\right),\end{aligned}$$

which proves (5-8).

Next we show that the Jacobi relation (5-7) maps to cohomology relation (5-4). That is we prove

(5-11) $$\phi([\alpha(A \wedge (B \wedge C))\beta]) - \phi([\alpha((A \wedge B) \wedge C)\beta]) - \phi([\alpha(B \wedge (A \wedge C))\beta])$$
$$= \pm\left(\bar{c}(\alpha(A \wedge (B \wedge C))\beta) + (-1)^{I(C)}\bar{c}(\alpha((A \wedge B) \wedge C)\beta) + (-1)^{I(A)I(B)}\bar{c}(\alpha(B \wedge (A \wedge C))\beta)\right).$$

First we claim that if $\rho(T_1 \wedge (T_2 \wedge T_3))\tau$ is a binary tree then

(5-12) $$\operatorname{sgn}(\rho((T_1 \wedge T_2) \wedge T_3)\tau) = (-1)^{I(T_3)+1}\operatorname{sgn}(\rho(T_1 \wedge (T_2 \wedge T_3))\tau)$$

and

(5-13) $$\operatorname{sgn}(\rho(T_2 \wedge (T_1 \wedge T_3))\tau) = (-1)^{I(T_1)+I(T_2)}\operatorname{sgn}(\rho(T_1 \wedge (T_2 \wedge T_3))\tau).$$

Again we use induction on the depth of the root of $(T_1 \wedge (T_2 \wedge T_3))$ in $\rho(T_1 \wedge (T_2 \wedge T_3))\tau$. If the depth is 0 then

$$\begin{aligned}\operatorname{sgn}((T_1 \wedge T_2) \wedge T_3) &= \operatorname{sgn}(T_1 \wedge T_2)\operatorname{sgn}(T_3)(-1)^{I(T_3)}\\ &= \operatorname{sgn}(T_1)\operatorname{sgn}(T_2)\operatorname{sgn}(T_3)(-1)^{I(T_2)+I(T_3)}\\ &= \operatorname{sgn}(T_1)\operatorname{sgn}(T_2 \wedge T_3)(-1)^{I(T_2)}\\ &= \operatorname{sgn}(T_1 \wedge (T_2 \wedge T_3))(-1)^{I(T_2)+I(T_2 \wedge T_3)}\\ &= \operatorname{sgn}(T_1 \wedge (T_2 \wedge T_3))(-1)^{I(T_2)+I(T_2)+I(T_3)+1}\\ &= \operatorname{sgn}(T_1 \wedge (T_2 \wedge T_3))(-1)^{I(T_3)+1},\end{aligned}$$

and

$$\begin{aligned}\operatorname{sgn}(T_2 \wedge (T_1 \wedge T_3)) &= \operatorname{sgn}(T_2)\operatorname{sgn}(T_1 \wedge T_3)(-1)^{I(T_1)+I(T_3)+1} \\ &= \operatorname{sgn}(T_2)\operatorname{sgn}(T_1)\operatorname{sgn}(T_3)(-1)^{I(T_1)+1} \\ &= \operatorname{sgn}(T_1)\operatorname{sgn}(T_2 \wedge T_3)(-1)^{I(T_1)+I(T_3)+1} \\ &= \operatorname{sgn}(T_1 \wedge (T_2 \wedge T_3))(-1)^{I(T_1)+I(T_2)}.\end{aligned}$$

The general equations (5-12) and (5-13) follow by induction.

In (5-11) let $A = (T_1, w_1)$, $B = (T_2, w_2)$, $C = (T_3, w_3)$ and $\alpha(A \wedge (B \wedge C))\beta = (\rho(T_1 \wedge (T_2 \wedge T_3))\tau, uw_1w_2w_3v)$. By (5-12), we have

$$\begin{aligned}\phi([\alpha((A \wedge B) \wedge C)\beta]) &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(\rho((T_1 \wedge T_2) \wedge T_3)\tau)\bar{c}(\alpha((A \wedge B) \wedge C)\beta) \\ &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(\rho(T_1 \wedge (T_2 \wedge T_3))\tau)(-1)^{I(T_3)+1}\bar{c}(\alpha((A \wedge B) \wedge C)\beta). \qquad (5\text{-}14)\end{aligned}$$

By (5-13) and (5-9), we have

$$\begin{aligned}\phi([\alpha(B \wedge (A \wedge C))\beta]) &= \operatorname{sgn}(uw_2w_1w_3v)\operatorname{sgn}(\rho(T_2 \wedge (T_1 \wedge T_3))\tau)\bar{c}(\alpha(B \wedge (A \wedge C))\beta) \\ &= \operatorname{sgn}(uw_1w_2w_3v)(-1)^{\ell(w_1)\ell(w_2)}\operatorname{sgn}(\rho(T_1 \wedge (T_2 \wedge T_3))\tau)(-1)^{I(T_1)+I(T_2)} \\ &\qquad \cdot \bar{c}(\alpha(B \wedge (A \wedge C))\beta) \\ &= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(\rho(T_1 \wedge (T_2 \wedge T_3))\tau)(-1)^{I(T_1)I(T_2)+1} \\ &\qquad \cdot \bar{c}(\alpha(B \wedge (A \wedge C))\beta). \qquad (5\text{-}15)\end{aligned}$$

Equations (5-14) and (5-15) yield

$$\begin{aligned}&\phi([\alpha(A \wedge (B \wedge C))\beta]) - \phi([\alpha((A \wedge B) \wedge C)\beta]) - \phi([\alpha(B \wedge (A \wedge C))\beta]) \\ &\quad = \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(\rho(T_1 \wedge (T_2 \wedge T_3))\tau) \\ &\qquad \cdot \big(\bar{c}(\alpha(A \wedge (B \wedge C))\beta) + (-1)^{I(T_3)}\bar{c}(\alpha((A \wedge B) \wedge C)\beta) \\ &\qquad\quad + (-1)^{I(T_1)I(T_2)}\bar{c}(\alpha(B \wedge (A \wedge C))\beta)\big),\end{aligned}$$

which proves (5-11) and completes the proof of Theorem 5.4. □

We now show that the map ϕ given in Theorem 5.4 restricts to the Barcelo correspondence on the Lyndon basis and to (essentially) the comb correspondence on the right comb basis.

Theorem 5.5. *For all Lyndon trees (T, w), with $w \in \mathcal{S}_n$ and $n \geq 2$,*

$$\phi([T, w]) = \operatorname{sgn}(w)\operatorname{sgn}(T)\bar{c}(T, w) = \bar{m}_{\gamma(T,w)}. \qquad (5\text{-}16)$$

For all $w \in \mathcal{S}_n$,

$$\phi([R_n, w]) = \operatorname{sgn}(w)\operatorname{sgn}(R_n)\bar{c}(R_n, w) = (-1)^{\binom{n-1}{2}}\operatorname{sgn}(w)\bar{m}_w \qquad (5\text{-}17)$$

Proof. Equation (5-17) is easy to prove. It suffices to verify $\operatorname{sgn}(R_n) = (-1)^{\binom{n-1}{2}}$. This follows from the recurrence

$$\begin{aligned}\operatorname{sgn}(R_n) &= \operatorname{sgn}(R_{n-1}) \cdot (-1)^{I(R_{n-1})} \\ &= \operatorname{sgn}(R_{n-1}) \cdot (-1)^{n-2},\end{aligned}$$

for $n > 1$.

Equation (5-16) is more difficult to prove. Let (T, w) be a Lyndon tree with internal nodes $v_1, v_2, \ldots, v_{n-1}$ listed in postorder. For $i = 1, 2, \ldots, n-1$, let

$$\sigma(i) = \max L(v_i). \tag{5-18}$$

It follows from the fact that (T, w) is a Lyndon tree that $\sigma \in \mathcal{S}_{n-1}$. Moreover, we claim that $\sigma^{-1} \in E(T)$. To verify this, it suffices to show that if v_i is a child of v_j then $\sigma(i) < \sigma(j)$. This is clearly true if v_i is a left child since then $L(v_i) \subseteq L(v_j)$. If v_i is a right child of v_j then since (T, w) is a Lyndon tree, the largest label of the subtree rooted at v_j is in $R(v_i)$ and the second largest label is in $L(v_j)$. This implies that $\max L(v_i) < \max L(v_j)$.

Now we claim that

$$m_{\gamma(T,w)} = c(T, w, \sigma^{-1}). \tag{5-19}$$

Note that σ^{-1} induces the linear extension of internal nodes v of T in increasing order of $\max L(v)$. Hence the merges of $L(v)$ and $R(v)$ in $c(T, w, \sigma^{-1})$ are being made in increasing order of $\max L(v)$. In $\gamma(T, w), \max L(v)$ is the largest child of $\max R(v)$ for all internal nodes v. Hence merging $L(v)$ and $R(v)$ corresponds to inserting the edge $(\max L(v), \max R(v))$ into the partially constructed subforest of $\gamma(T, w)$. Recall that by inserting the edges $(a, p(a))$ of an increasing tree A in increasing order of the a, we are forming the maximal chain m_A. This and the above correspondence imply (5-19).

Next we claim

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(T) \operatorname{sgn}(w). \tag{5-20}$$

We prove this by induction on n. For the induction to go through we need to work in the generality of allowing w to be a permutation of an arbitrary n element subset of $\mathbf{Z}^+$. We also need to allow $n = 1$ in which case σ is the empty word and $\operatorname{sgn}(\sigma) = 1$. The construction of σ given in (5-18) remains valid; but σ is now a permutation of the set $\{w(1), w(2), \ldots, w(n-1)\}$. When $n = 1$, the result is trivial. Suppose $n > 1$, $(T, w) = ((T_1, w_1) \wedge (T_2, w_2))$ and $I(T_1) = k$. Let σ_1 be the subword $\sigma(1)\sigma(2)\cdots\sigma(k)$ and let σ_2 be the subword $\sigma(k+1)\sigma(k+2)\cdots\sigma(n-1)$. Since (T_1, w_1) and (T_2, w_2) are Lyndon trees, by the induction hypotheses we have,

$$\operatorname{sgn}(\sigma_1) = \operatorname{sgn}(T_1) \operatorname{sgn}(w_1) \tag{5-21}$$

and

$$\operatorname{sgn}(\sigma_2) = \operatorname{sgn}(T_2) \operatorname{sgn}(w_2). \tag{5-22}$$

For permutations α and β on disjoint sets, define

$$\mathrm{INV}(\alpha;\beta) = \{(\alpha(i),\beta(j)) \mid \alpha(i) > \beta(j)\}$$

and

$$\mathrm{INV}(\alpha) = \{(\alpha(i),\alpha(j)) \mid i < j \text{ and } \alpha(i) > \alpha(j)\}.$$

Since (T,w) is a Lyndon tree, we have

$$\mathrm{INV}(\sigma) = \mathrm{INV}(\sigma_1) \uplus \mathrm{INV}(\sigma_2) \uplus \mathrm{INV}(\sigma_1;\sigma_2),$$

and

$$\mathrm{INV}(w) = \mathrm{INV}(w_1) \uplus \mathrm{INV}(w_2) \uplus \mathrm{INV}(w_1;w_2),$$

where $\uplus$ stands for the disjoint union. Note that

$$\mathrm{INV}(\sigma_1;\sigma_2) \uplus \{(w_1(k+1), w_2(i)) \mid i = 1,2,\dots,n-k-2\} = \mathrm{INV}(w_1;w_2),$$

since $w_1(k+1)$ is the second largest letter of w and $w_2(n-k-1)$ is the largest. It follows that

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma_1)\,\mathrm{sgn}(\sigma_2)(-1)^{|\mathrm{INV}(w_1;w_2)|-I(T_2)}.$$

By substituting (5-21) and (5-22) into this, we get

$$\begin{aligned}\mathrm{sgn}(\sigma) &= \mathrm{sgn}(T_1)\,\mathrm{sgn}(w_1)\,\mathrm{sgn}(T_2)\,\mathrm{sgn}(w_2)(-1)^{|\mathrm{INV}(w_1;w_2)|-I(T_2)}\\ &= \mathrm{sgn}(w)\,\mathrm{sgn}(T),\end{aligned}$$

which proves the assertion (5-20).

Equation (5-16) now follows from (5-19), (5-20), and Lemma 5.2. □

The following result is an immediate consequence of Theorem 5.4.

Corollary 5.6. *Let $\mathcal{B}$ be a set of labeled binary trees with n leaves. Then $\{[T,w] \mid (T,w) \in \mathcal{B}\}$ is a basis for* Lie_n *if and only if $\{\bar{c}(T,w) \mid (T,w) \in \mathcal{B}\}$ is a basis for $H^*(\Pi_n)$.*

We can use Corollary 5.6 to construct a basis for Lie_n which corresponds to the dual caterpillar basis for $H^*(\Pi_n)$ (cf. Theorem 2.5). We recursively define a *butterfly* to be a labeled binary tree of the form (R_k,α) or $(R_k,\alpha) \wedge (T,w)$, where $k \geq 2$, $\alpha(1) < \alpha(2) < \cdots < \alpha(k-1) > \alpha(k)$ and (T,w) is a butterfly whose maximum leaf label is greater than $\alpha(k-1)$.

Theorem 5.7. *Let $n \geq 2$. The set*

$$\{\bar{c}(T,w) \mid (T,w) \text{ is a butterfly and } w \in \mathcal{S}_n\}$$

is a basis for $H^*(\Pi_n)$ and the set

$$\{[T,w] \mid (T,w) \text{ is a butterfly and } w \in \mathcal{S}_n\}$$

is a basis for Lie_n.

Proof. We leave it to the reader to show that the maximal chains μ_A, where A is a caterpillar, have the form $c(T,w,\sigma)$ where (T,w) is a butterfly. Since $\bar{c}(T,w) = \pm\bar{c}(T,w,\sigma)$, it follows from Theorem 2.5 that the first set is a basis for $H^*(\Pi_n)$. It then follows from Corollary 5.6 that the second set is a basis for Lie_n. □

6. Lyndon basis expansion.

The identity (3-3) for expanding the Björner basis for $H(\Pi_n)$ in terms of the splitting basis can be transformed, via Theorems 5.4 and 5.5, into an identity for expanding the comb basis for Lie_n in terms of the Lyndon basis. The resulting identity has a nice combinatorial form which we give in Theorem 6.1 . It also suggests a more general identity for expanding any bracketed permutation in terms of the Lyndon basis, which turns out to be easy to prove by induction.

Let $a_1, a_2, \ldots, a_{n-1}$ be an integer sequence satisfying $i < a_i \leq n$ for all i. Define $\mathrm{Lyn}(a_1, a_2, \ldots, a_{n-1})$ to be the labeled binary tree obtained from the following insertion algorithm. Start with the singleton binary tree with leaf n. Then insert $n-1, n-2, \ldots, 1$ in the following way: First insert $n-1$ as the left sibling of $a_{n-1} = n$. In the ith insertion step we insert $n-i$ as the left sibling of a_{n-i} by replacing the leaf labeled by a_{n-i} with the subtree $((n-i) \wedge a_{n-i})$. Note that if γ is the bijection used in the Barcelo correspondence then $\mathrm{Lyn}(a_1, a_2, \ldots, a_{n-1}) = \gamma^{-1}(T)$ where T is the increasing tree in which a_i is the parent of i for each $i = 1, 2, \ldots, n-1$. This insertion algorithm is called grafting in [BB].

Theorem 6.1. *For each $w \in \mathcal{S}_n$,*

$$r_w = \sum_{\substack{i < a_i \leq n \\ w^{-1}(i) < w^{-1}(a_i)}} [\mathrm{Lyn}(a_1, a_2, \ldots, a_{n-1})].$$

Proof. By taking the dual of (3-3) we get

$$(-1)^{\binom{n-1}{2}} \operatorname{sgn}(w) \bar{m}_w = \sum_{T : w \in \mathcal{P}(T)} \bar{m}_T, \tag{6-1}$$

where the sum is over increasing trees T on $\{1, 2, \ldots, n\}$. Note that $w \in \mathcal{P}(T)$ if and only if $w^{-1}(i) < w^{-1}(p(i))$ for all $i = 1, 2, \ldots, n-1$. It follows that the sequences $a(1), a(2), \ldots, a(n-1)$, such that $i < a(i) \leq n$ and $w^{-1}(i) < w^{-1}(a(i))$, determine the increasing trees T for which $w \in \mathcal{P}(T)$. This implies that $w \in \mathcal{P}(\gamma(Lyn(a_1, a_2, \ldots, a_n)))$ if and only if $i < a(i) \leq n$ and $w^{-1}(i) < w^{-1}(a(i))$. The result now follows by taking ϕ^{-1} of both sides of (6-1) and applying Theorem 5.5. □

Corollary 6.2. *For ϵ, the identity in $\mathcal{S}_n$, the right comb r_ϵ equals the sum of the entire Lyndon basis for* Lie_n. □

We shall say that a labeled binary tree is *standard* if for all internal nodes v, $\max L(v) < \max R(v)$. Note that in a standard labeled binary tree the rightmost leaf of any subtree is the largest leaf of the subtree. Clearly all Lyndon trees and all right comb trees are standard. If (T, w) is any labeled binary tree then $[T, w] = (-1)^{\nu(T,w)}[T', w']$, where (T', w') is the standard labeled binary tree obtained from (T, w) by exchanging the left and right subtrees of every node v such that $\max L(v) > \max R(v)$ and $\nu(T, w)$ is the number of exchanges that are made.

Let (T, w) be a standard labeled binary tree with $w \in \mathcal{S}_n$. For each $i = 1, 2, \ldots, n-1$, let $b_i = \max R(v_i)$, where v_i is such that $i = \max L(v_i)$. Since (T, w) is standard, $i < b_i$ for all i. Now let

$$\mathcal{A}_{(T,w)}(i) = \{a \mid i < a \leq n,\ w^{-1}(i) < w^{-1}(a) \leq w^{-1}(b_i)\}.$$

In other words, $\mathcal{A}_{(T,w)}(i)$ is the set of all $a \in R(v_i)$ such that $a > i$. We also let

$$\mathcal{A}_{(T,w)} = \mathcal{A}_{(T,w)}(1) \times \mathcal{A}_{(T,w)}(2) \times \cdots \times \mathcal{A}_{(T,w)}(n-1).$$

The following is a generalization of Theorem 6.1.

Theorem 6.3. *Let (T, w) be a standard labeled binary tree with $w \in \mathcal{S}_n$. Then*

$$[T, w] = \sum_{a \in \mathcal{A}_{(T,w)}} [\mathrm{Lyn}(a)] \tag{6-2}$$

Proof. We prove this by induction on $|\mathcal{A}_{(T,w)}|$. Observe that $|\mathcal{A}_{(T,w)}|$ has minimum value 1 if and only if (T, w) is a Lyndon tree. In this case $\mathrm{Lyn}(a) = (T, w)$ and therefore (6-2) holds. Now suppose that (T, w) is not a Lyndon tree. Then

$$(T, w) = \alpha(A \wedge (B \wedge C))\beta$$

where A, B and C are labeled subtrees of (T, w) with $\max A < \max B < \max C$.

Let

$$(T', w') = \alpha((A \wedge B) \wedge C)\beta$$

and

$$(T'', w'') = \alpha(B \wedge (A \wedge C))\beta.$$

Then by the Jacobi relation

$$[T, w] = [T', w'] + [T'', w'']. \tag{6-3}$$

It is easy to see that if $i = \max A$ then

$$\mathcal{A}_{(T,w)}(i) = \mathcal{A}_{(T',w')}(i) \uplus \mathcal{A}_{(T'',w'')}(i)$$

and if $i \neq \max A$ then

$$\mathcal{A}_{(T,w)}(i) = \mathcal{A}_{(T',w')}(i) = \mathcal{A}_{(T'',w'')}(i).$$

Note that all of the sets in these two equations are nonempty. It follows that

$$\mathcal{A}_{(T,w)} = \mathcal{A}_{(T',w')} \uplus \mathcal{A}_{(T'',w'')}, \tag{6-4}$$

where each of the sets in the disjoint union are nonempty. This implies that $|\mathcal{A}_{(T',w')}| < |\mathcal{A}_{(T,w)}|$ and $|\mathcal{A}_{(T'',w'')}| < |\mathcal{A}_{(T,w)}|$. Hence by the induction hypothesis,

$$[T', w'] = \sum_{a \in \mathcal{A}_{(T',w')}} [\mathrm{Lyn}(a)] \quad \text{and} \quad [T'', w''] = \sum_{a \in \mathcal{A}_{(T'',w'')}} [\mathrm{Lyn}(a)].$$

The result now follows from (6-3) and (6-4). $\square$

7. Whitney cohomology.

Whitney homology for geometric lattices was introduced by Baclawski in [Bc]. Here we shall use a formulation due to Björner [Bj1] [Bj2] and Orlik and Solomon [OS]. The Whitney cohomology $WH^*(\Pi_n)$ of the partition lattice Π_n is the direct sum $\oplus_{x\in\Pi_n} H^*(\hat{0},x)$, where $H^*(\hat{0},x)$ is the unique nonvanishing cohomology of the interval $[\hat{0},x]$ for $x > \hat{0}$ and $H^*(\hat{0},\hat{0})$ is the 1-dimensional space spanned by the empty chain. From a character computation of Lehrer and Solomon [LS], it can be seen that $WH^*(\Pi_n)$ is isomorphic (as an $\mathcal{S}_n$-module) to $\wedge\mathrm{Lie}_n$, the 1^n component of the exterior algebra of the free Lie algebra, tensored with the sign representation.

The fact that $\wedge\mathrm{Lie}_n$ and $WH^*(\Pi_n)\otimes\mathrm{sgn}$ are isomorphic is first noted in Barcelo and Bergeron [BB], where the Barcelo correspondence is extended in order to construct an explicit isomorphism between the two modules. They actually work with the Orlik-Solomon algebra which is isomorphic to Whitney homology (cf. [OS], [Bj2]). Here we extend our general correspondence of Theorem 5.4 to a correspondence between natural generating sets of $\wedge\mathrm{Lie}_n$ and $WH^*(\Pi_n)\otimes\mathrm{sgn}$ which restricts (as in Theorem 5.5) to essentially the Barcelo-Bergeron correspondence.

Let $\mathcal{F}_n$ be the set of ordered forests consisting of binary trees with a total of n leaves. For $F\in\mathcal{F}_n$ and $w\in\mathcal{S}_n$, let (F,w) be the forest whose ith leaf from left to right is $w(i)$. If $(T_1,w_1),(T_2,w_2),\dots,(T_k,w_k)$ are the labeled binary trees of (F,w), then we set

$$[F,w]=[T_1,w_1]\wedge[T_2,w_2]\wedge\cdots\wedge[T_k,w_k],$$

where now $\wedge$ denotes the wedge product operation. The set $\{[F,w]\mid F\in\mathcal{F},\ w\in\mathcal{S}_n\}$ is a generating set for $\wedge\mathrm{Lie}_n$ subject only to the anticommuting relation (5-6) and Jacobi relation (5-7) within trees and the following wedge product relation between trees

$$\begin{aligned}&[T_1,w_1]\wedge[T_2,w_2]\wedge\cdots\wedge[T_i,w_i]\wedge[T_{i+1},w_{i+1}]\wedge\cdots\wedge[T_k,w_k]\\ &\qquad=-[T_1,w_1]\wedge[T_2,w_2]\wedge\cdots\wedge[T_{i+1},w_{i+1}]\wedge[T_i,w_i]\wedge\cdots\wedge[T_k,w_k].\end{aligned}\tag{7-1}$$

The labeled forests also provide a natural generating set for $WH^*(\Pi_n)$. For $F\in\mathcal{F}_n$ and $w\in\mathcal{S}_n$, let $c(F,w)$ be the unrefinable chain of Π_n whose rank i partition is obtained from its rank $i-1$ partition by merging the blocks $L(v_i)$ and $R(v_i)$, where v_i is the ith postorder internal node of F.

By a proof similar to that of Theorem 5.3 one can show:

Theorem 7.1. *The set $\{\bar{c}(F,w)\mid F\in\mathcal{F}_n,\ w\in\mathcal{S}_n\}$ is a generating set for $WH^*(\Pi_n)$ subject only to relations (5-3), (5-4), and*

$$\begin{aligned}&\bar{c}((T_1,w_1)\dots(T_i,w_i)(T_{i+1},w_{i+1})\dots(T_k,w_k))\\ &\qquad=(-1)^{I(T_i)I(T_{i+1})}\bar{c}((T_1,w_1)\dots(T_{i+1},w_{i+1})(T_i,w_i)\dots(T_k,w_k)),\end{aligned}\tag{7-2}$$

where $((T_1,w_1),\dots,(T_k,w_k))$ denotes the labeled forest whose ith tree is (T_i,w_i). □

Theorem 7.2. *There is an $\mathcal{S}_n$-isomorphism $\phi : \wedge\mathrm{Lie}_n \to WH^*(\Pi_n) \otimes \mathrm{sgn}$ determined by*

$$\phi([F, w]) = \mathrm{sgn}(w)\,\mathrm{sgn}(F)\bar{c}(F, w), \quad F \in \mathcal{F}, w \in \mathcal{S}_n,$$

where if F consists of binary trees $T_1, T_2, \ldots, T_k$, then $\mathrm{sgn}(F)$ is defined by

$$\mathrm{sgn}(F) = (-1)^{I(T_2)+I(T_4)+\cdots+I(T_{2\lfloor k/2\rfloor})}\,\mathrm{sgn}(T_1)\,\mathrm{sgn}(T_2)\cdots\mathrm{sgn}(T_k).$$

Proof. Most of the work of proving this has already been accomplished by the proof of Theorem 5.4. Indeed, equations (5-8) and (5-9) hold in this general setting. Hence we need only prove that relation (7-1) maps to relation (7-2). That is, we need to show

$$\begin{aligned}\phi([T_1, w_1] \wedge \cdots \wedge [T_i, w_i] \wedge [T_{i+1}, w_{i+1}] \wedge \cdots \wedge [T_k, w_k])&\\ + \phi([T_1, w_1] \wedge \cdots \wedge [T_{i+1}, w_{i+1}] \wedge [T_i, w_i] \wedge \cdots \wedge [T_k, w_k])&\\ = \pm(\bar{c}((T_1, w_1)\ldots(T_i, w_i)(T_{i+1}, w_{i+1})\ldots(T_k, w_k))&\\ - (-1)^{I(T_i)I(T_{i+1})}\bar{c}((T_1, w_1)\ldots(T_{i+1}, w_{i+1})(T_i, w_i)\ldots(T_k, w_k))).&\end{aligned}$$

The proof of this is similar to that of (5-8) and is left to the reader. $\square$

Note that ϕ restricted to Lie_n is precisely the correspondence of Theorem 5.4 since $\mathrm{sgn}(F)$ is the same as $\mathrm{sgn}(T)$ when F consists only of one binary tree T. It can also be shown, by modifying the proof of (5-14) that if ϕ is restricted to the "hedge row" basis for $\wedge\mathrm{Lie}_n$ of Barcelo and Bergeron [BB] then the resulting map is essentially the Barcelo-Bergeron correspondence.

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