ON THE PROPERTY M CONJECTURE FOR THE HEISENBERG LIE ALGEBRA

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ABSTRACT. We prove a fundamental case of a conjecture of the first author which expresses the homology of the extension of the Heisenberg Lie algebra by $\mathbb{C}[t]/(t^{k+1})$ in terms of the homology of the Heisenberg Lie algebra itself. More specifically, we show that both the 0^{th} and $k+1^{st}$ x-graded components of homology of this extension of the 3-dimensional Heisenberg Lie algebra have dimension 3^{k+1} by constructing a simple basis for cohomology.

1. Introduction

In a series of papers [4, 5, 6, 7] dating back to 1986, Hanlon conjectured that if L belongs to a certain class of complex Lie algebras, which includes all semisimple Lie algebras, the Heisenberg Lie algebra, and the Lie algebra of strictly upper triangular matrices, then the homology of the Lie algebra $L \otimes \mathbb{C}[t]/(t^{k+1})$ is related to the homology of L in a very natural way. More precisely, the conjectured relationship is as follows:

$$H_*(L \otimes \mathbb{C}[t]/(t^{k+1})) \cong H_*(L)^{\otimes(k+1)} \tag{1.1}$$

as graded vector spaces. A Lie algebra L that satisfies (1.1) is said to have *Property M*.

The Property M conjecture was originally stated for L semisimple. It is particularly important in this case, since it implies Macdonald's root-system conjecture [9] (see [4] and [7, Section 6]). The Property M conjecture for L semisimple is one of two conjectures known as the strong Macdonald conjectures. In 1990 Hanlon [5] proved that $sl_n(\mathbb{C})$ has Property M. Macdonald's original root-system conjecture was eventually proved by Cherednik [2] in 1995. More recently, Fishel, Grojnowski and Teleman [3] proved the strong Macdonald conjectures.

In a recent paper, Kumar [8] showed that the Property M conjecture is false in general for the Lie algebra $\mathcal{T}_n$ of strictly upper triangular $n \times n$

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matrices. More precisely he proved that (1.1) is false when $L = \mathcal{T}_4$ or $\mathcal{T}_5$ and $k = 1$. He showed, however, that (1.1) does hold for $L = \mathcal{T}_3$ and $k = 1$.

The Property M conjecture for the $(2n+1)$-dimensional Heisenberg Lie algebra, $\mathcal{H}_{2n+1}$, remains open. This is so even for $\mathcal{H}_3$ $(= \mathcal{T}_3)$. Partial results supporting the conjecture for $\mathcal{H}_3$ can be found in [6] and in [1]. Computational evidence is given in [5]. In this paper we provide further evidence by settling an important special case.

The Poincaré polynomial for $\mathcal{H}_3$ is easy to compute (see [7, Example 3.8]). It is given by

$$\sum_{r\geq 0} \dim H_r(\mathcal{H}_3)\ y^r = 1 + 2y + 2y^2 + y^3.$$

Hence Property M for $L = \mathcal{H}_3$ can be restated as

$$\sum_{r\geq 0} \dim H_r(L\otimes \mathbb{C}[t]/(t^{k+1}))\ y^r = (1 + 2y + 2y^2 + y^3)^{k+1},$$

which implies that

$$\sum_{r\geq 0} \dim H_r(L\otimes \mathbb{C}[t]/(t^{k+1})) = 6^{k+1}.$$

Even this simple statement is still open.

Hanlon conjectured that for $\mathcal{H}_3$, equation (1.1) holds for even finer gradings than the homological dimension grading. One such grading is an $\mathbb{N}^3$-grading by (e, f, x)-degree which is defined in the next section. For this grading,

$$\sum_{m,n,p\in\mathbb{N}} \dim H_{m,n,p}(\mathcal{H}_3)u^m v^n w^p = 1 + u + v + wu + wv + uvw$$

and the conjecture given in (1.1) becomes

$$\sum_{m,n,p\in\mathbb{N}} \dim H_{m,n,p}(\mathcal{H}_3\otimes \mathbb{C}[t]/(t^{k+1}))u^m v^n w^p = (1 + u + v + wu + wv + uvw)^{k+1}. \tag{1.2}$$

Setting $w = 0$ yields

$$\sum_{m,n\in\mathbb{N}} \dim H_{m,n,0}(\mathcal{H}_3\otimes \mathbb{C}[t]/(t^{k+1}))u^m v^n = (1 + u + v)^{k+1}, \tag{1.3}$$

which implies that the homology of the 0^{th} x-graded piece of $H_*(\mathcal{H}_3 \otimes \mathbb{C}[t])$ has dimension 3^{k+1}.

In this paper we prove (1.3). This is accomplished by constructing the following basis for the 0^{th} x-graded piece of the cohomology of $\mathcal{H}_3 \otimes \mathbb{C}[t]/(t^{k+1})$:

$$\{e_{i_1}\wedge\cdots\wedge e_{i_m}\wedge f_{j_1}\wedge\cdots\wedge f_{j_n} | 0 \leq i_1 < \cdots < i_m \leq k, m \leq j_1 < \cdots < j_n \leq k\}$$

We show that this set is a basis by using the coboundary relations to show that the set spans and then applying a lower bound on the dimension which is established by considering a deformation of $\mathcal{H}_3 \otimes \mathbb{C}[t]/(t^{k+1})$. Poincaré duality enables us to construct a "complementary" basis for the $k+1^{st}$ x-graded component of homology. The paper ends with a new conjecture which presents similar looking bases for the 0^{th} and $k+1^{st}$ e- and f-graded pieces of (co)homology of $\mathcal{H}_3 \otimes \mathbb{C}[t]/(t^{k+1})$.

2. Preliminaries

In this section we recall some notation and background. The Heisenberg Lie algebra $\mathcal{H}_3$ is the subalgebra of $gl_3(\mathbb{C})$ spanned by the basis vectors z_{12}, z_{13}, z_{23}, where z_{ij} is the 3×3 matrix with 1 as the i, j entry and 0's elsewhere. We use the traditional notation for this basis which is obtained by letting $e = z_{12}$, $f = z_{23}$ and $x = z_{13}$. The only nonzero brackets on these basis elements are

$$[e, f] = -[f, e] = x.$$

Let L be a complex Lie algebra and let A be a commutative $\mathbb{C}$-algebra. The Lie algebra $L \otimes A$ is defined to be the vector space $L \otimes A$ with bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab,$$

where $[u, v]$ is the bracket of u and v in L, and ab is the product of a and b in A.

A basis for $L_k := \mathcal{H}_3 \otimes \mathbb{C}[t]/(t^{k+1})$ is given by

$$\mathcal{B} = \{e_0, e_1, \dots, e_k, f_0, f_1, \dots, f_k, x_0, x_1, \dots, x_k\},$$

where $e_i = e \otimes t^i$, $f_i = f \otimes t^i$ and $x_i = x \otimes t^i$ for all $i = 0, 1, \dots k$. Clearly, the only nonzero brackets on these basis elements are given by

$$[e_i, f_j] = -[f_j, e_i] = x_{i+j}$$

for all i, j such that $i + j \leq k$.

Let E, F and X denote the subspaces of L_k spanned by the e_i's, f_i's and x_i's, respectively. Then the exterior algebra of L_k has a $\mathbb{N}^3$-grading given by

$$\wedge L_k = \bigoplus_{m,n,p \in \mathbb{N}} \wedge^m(E) \otimes \wedge^n(F) \otimes \wedge^p(X),$$

where $\wedge^i$ denoted the ith exterior power. We will say that an element u in $\wedge^m(E) \otimes \wedge^n(F) \otimes \wedge^p(X)$ is (e,f,x)-homogeneous with (e,f,x)-degree equal to (m,n,p). We will also say that the e-degree of u is m, the f-degree is n and the x-degree is p.

Recall that the differential or boundary operator of the Koszul complex for Lie algebra homology is the map $\partial : \wedge L \to \wedge L$ defined by

$$\partial(u_1 \wedge \cdots \wedge u_r) = \sum_{i<j} (-1)^{i+j+1} [u_i, u_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_r,$$

where $\hat{}$ denotes deletion. The homology of L is

$$H_*(L) = \ker \partial / \operatorname{im} \partial.$$

When $L = L_k$, the differential ∂ reduces the e-degree and f-degree of an (e,f,x)-homogeneous element by 1 and increases the x-degree by 1. We can therefore let

$$\partial_{m,n,p} : \wedge^m(E) \otimes \wedge^n(F) \otimes \wedge^p(X) \to \wedge^{m-1}(E) \otimes \wedge^{n-1}(F) \otimes \wedge^{p+1}(X)$$

be the restriction of ∂. Now define

$$H_{m,n,p}(L_k) = \ker \partial_{m,n,p} / \operatorname{im} \partial_{m+1,n+1,p-1}.$$

So $H_*(L_k)$ has an $\mathbb{N}^3$-grading given by

$$H_*(L_k) = \bigoplus_{m,n,p \in \mathbb{N}} H_{m,n,p}(L_k).$$

3. A lower bound on the dimension

In this section we derive a lower bound on the dimension of $H_{m,n,0}(L_k)$ by considering a deformation of the Lie algebra L_k whose homology is relatively easy to compute.

Let L'_k denote the Lie algebra

$$L'_k = \mathcal{H}_3 \otimes \mathbb{C}[t]/(t^{k+1} - 1).$$

So the non-zero brackets in L'_k look like:

$$[e_i, f_j] = \begin{cases} x_{i+j}, & \text{if } i + j \le k \\ x_{i+j-(k+1)}, & \text{otherwise.} \end{cases}$$

Just as for L_k, we can restrict the Lie algebra boundary ∂' for L'_k to $\wedge^m(E) \otimes \wedge^n(F) \otimes \wedge^p(X)$ to obtain $\partial'_{m,n,p}$. This induces an (e,f,x)-grading of $H_*(L'_k)$ whose (m,n,p)-component $H_{m,n,p}(L'_k)$ is $\ker \partial'_{m,n,p} / \operatorname{im} \partial'_{m+1,n+1,p-1}$.

Lemma 3.1. *For all $k, m, n, p \in \mathbb{N}$,*

$$\dim H_{m,n,p}(L_k) \ge \dim H_{m,n,p}(L'_k).$$

Proof. As a basis for the two complexes, take the set $\wedge\mathcal{B}$ of wedges of distinct elements of $\mathcal{B} = \{e_0, e_1, \ldots, e_k, f_0, f_1 \ldots, f_k, x_0, x_1 \ldots, x_k\}$. The *weight* of a wedge of elements from $\mathcal{B}$ is the sum of their subscripts. Order the basis elements of $\wedge\mathcal{B}$ so that this weight is weakly increasing. With respect to this ordered basis, the boundary ∂ for L_k is block diagonal since that map preserves weight. The boundary ∂' has the form:

$$\partial' = \partial + U, \tag{3.1}$$

where U is strictly block upper triangular.

It is a simple fact from linear algebra that if A is block diagonal and B is strictly block upper triangular, then $\mathrm{rank}(A+B) \geq \mathrm{rank}(A)$. Applying that to equation (3.1) we see that

$$\mathrm{rank}(\partial) \leq \mathrm{rank}(\partial').$$

So, we see that the nullspace of ∂' has dimension that is no bigger than the dimension of the nullspace of ∂, whereas the image of ∂' has dimension that is no smaller than the dimension of the image of ∂. Since (3.1) holds for the restriction to each (e, f, x)-graded piece, it follows that

$$\dim H_{m,n,p}(L_k) \geq \dim H_{m,n,p}(L'_k)$$

for all $m, n, p \in \mathbb{N}$. □

The $(k+1)$-fold tensor power of $H_*(\mathcal{H}_3)$ has a natural (e, f, x)-grading. We denote the (m, n, p)-component of $H_*(\mathcal{H}_3)^{\otimes(k+1)}$ under this grading by $(H_*(\mathcal{H}_3)^{\otimes(k+1)})_{m,n,p}$.

Proposition 3.2. *For all $k, n, m, p \in \mathbb{N}$,*

$$\dim H_{m,n,p}(L_k) \geq \dim(H_*(\mathcal{H}_3)^{\otimes(k+1)})_{m,n,p}.$$

Proof. By Lemma 3.1 it suffices to prove

$$\dim H_{m,n,p}(L'_k) = \dim(H_*(\mathcal{H}_3)^{\otimes(k+1)})_{m,n,p}. \tag{3.2}$$

It is straightforward to check that L'_k is isomorphic (as a Lie algebra) to the (Lie algebra) direct sum of $k+1$ copies of the Lie algebra $\mathcal{H}_3$. To see this isomorphism explicitly, let $\phi^{(j)}$ be the map from $\mathcal{H}_3$ to L'_k defined by

$$\phi^{(j)}(u) = u \otimes \frac{1}{k+1} \sum_{\ell=0}^{k} \omega^{j\ell}\, t^{\ell},$$

where ω is a primitive $(k+1)^{st}$ root of unity. Let $\mathcal{H}_3^{(j)}$ denote the image of $\phi^{(j)}$. It is easy to check that $\mathcal{H}_3^{(j)}$ is a subalgebra of L'_k isomorphic to $\mathcal{H}_3$ and that $[\mathcal{H}_3^{(i)}, \mathcal{H}_3^{(j)}] = 0$ for $i \neq j$. It follows that

$$L'_k \cong \bigoplus_{i=0}^{k} \mathcal{H}_3. \tag{3.3}$$

It is straightforward to show that $H_*(L \oplus M) \cong H_*(L) \otimes H_*(M)$ for Lie algebras L and M. Applying this to (3.3) yields

$$H_*\big(L'_k\big) \cong H_*(\mathcal{H}_3)^{\otimes(k+1)}.$$

One can easily check that this isomorphism carries the (e, f, x)-grading of $H_*(L'_k)$ to the natural (e, f, x)-grading of $H_*(\mathcal{H}_3)^{\otimes(k+1)}$. Hence (3.2) holds. □

Corollary 3.3. *For all $m, n \leq k + 1$,*

$$\dim H_{m,n,0}(L_k) \geq \binom{k+1}{m, n, k+1-(m+n)}.$$

Proof. We use the fact that

$$\sum_{m,n} \dim H_{m,n,0}(\mathcal{H}_3) u^m v^n = 1 + u + v.$$

□

Remark 3.4. The Lie algebras L_k and L'_k are part of a more general construction. For any complex Lie algebra L and complex number z define

$$L_k(z) := L \otimes \mathbb{C}[t]/(t^{k+1} - z).$$

For $z \neq 0$, an argument similar to the proof of (3.3) shows that the Lie algebra $L_k(z)$ is a Lie algebra direct sum of $k + 1$ copies of L. At the singular point, $z = 0$, the structure of $L_k(z)$ changes dramatically. However, the homology of $L_k(z)$ can remain constant at the singular point. This happens if and only if L has Property M.

4. A Basis for Cohomology

In this section we compute the dimension of $H_{m,n,0}(L_k)$ by switching to cohomology and constructing a spanning set for cohomology which turns out to be a basis.

Let the coboundary map $\delta : \wedge L_k \to \wedge L_k$ be the adjoint of ∂ with respect to the Hermitian form on $\wedge L_k$ that has $\wedge\mathcal{B}$ as an orthonormal basis. The cohomology of L_k, denoted $H^*(L_k)$, is defined to be $\ker \delta / \operatorname{im} \delta$. The key relationship between homology and cohomology is

$$H_*(L_k) \cong H^*(L_k). \tag{4.1}$$

By restricting δ we obtain a linear map

$$\delta_{m,n,p} : \wedge^m(E) \otimes \wedge^n(F) \otimes \wedge^p(X) \to \wedge^{m+1}(E) \otimes \wedge^{n+1}(F) \otimes \wedge^{p-1}(X).$$

Now define the (e, f, x)-graded cohomology component of degree (m, n, p) to be

$$H^{m,n,p}(L_k) = \ker \delta_{m,n,p} / \operatorname{im} \delta_{m-1,n-1,p+1}.$$

The isomorphism in 4.1 restricts to

$$H_{m,n,p}(L_k) \cong H^{m,n,p}(L_k).$$

Now let us set $p = 0$. Note that $\ker \delta_{m,n,0} = \wedge^m(E) \otimes \wedge^n(F)$. So $H^{m,n,0}(L_k)$ is generated by elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_{j_1} \wedge \cdots \wedge f_{j_n}$ subject only to the cohomology relations. The cohomology relations are obtained by setting the coboundary of elements of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_{m-1}} \wedge f_{j_1} \wedge \cdots \wedge f_{j_{n-1}} \wedge x_t$$

equal to 0. This results in relations of the form:

$$\sum_{s=0}^{t} e_{i_1} \wedge \cdots \wedge e_{i_{m-1}} \wedge f_{j_1} \wedge \cdots \wedge f_{j_{n-1}} \wedge e_s \wedge f_{t-s} = 0, \tag{4.2}$$

for each $t = 0, \dots, k$. We allow $e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_{j_1} \wedge \cdots \wedge f_{j_n}$ to represent an element of cohomology of L_k as well as of the exterior algebra of L_k.

Theorem 4.1. *For all $k, m, n \in \mathbb{N}$, the set $S_{m,n} :=$*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_{j_1} \wedge \cdots \wedge f_{j_n} | 0 \le i_1 < \cdots < i_m \le k, m \le j_1 < \cdots < j_n \le k\}$$

is a basis for $H^{m,n,0}(L_k)$.

We will prove Theorem 4.1 by first showing that $S_{m,n}$ spans $H^{m,n,0}(L_k)$. To conclude that the spanning set is a basis we appeal to the lower bound given by Corollary 3.3.

The following ordering of symbols

$$\bar{0} < 0 < \bar{1} < 1 < \cdots < \bar{k} < k$$

induces a lexicographical ordering of sequences of these symbols which we denote by $<_L$. We assign to each wedge product

$$\omega = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \wedge f_j,$$

where $0 \le i_1, i_2, \dots, i_m, j \le k$, the sequence

$$\mu(\omega) := \begin{cases} (i_1, i_2, \dots, \bar{i}_{j+1}, \dots, i_m) & \text{if } j < m \\ (i_1, i_2, \dots, i_m) & \text{if } j \ge m. \end{cases}$$

Lemma 4.2. *Each wedge product*

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \wedge f_j,$$

where $0 \leq i_1, i_2, \ldots, i_m \leq k$ *and* $0 \leq j < m$*, can be expressed as a linear combination of wedge products with lexicographically smaller* μ*-value.*

Proof. By the cohomology relation given in (4.2)

(4.3)

$$\begin{aligned} e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_j &= -\sum_{r\geq 1} e_{i_1} \wedge \cdots \wedge e_{i_{j+1}+r} \wedge \cdots \wedge e_{i_m} \wedge f_{j-r} \\ &\quad -\sum_{r\geq 1} e_{i_1} \wedge \cdots \wedge e_{i_{j+1}-r} \wedge \cdots \wedge e_{i_m} \wedge f_{j+r}. \end{aligned}$$

Here we let $e_i = f_i = 0$ if $i < 0$ or $i > k$.

Consider the r^{th} term of the first sum and suppose this term is nonzero. We have

$$\begin{aligned} \mu(e_{i_1} \wedge \cdots \wedge e_{i_{j+1}+r} \wedge & \cdots \wedge e_{i_m} \wedge f_{j-r}) \\ &= (i_1, \ldots, \bar{i}_{j-r+1}, \ldots, i_{j+1}+r, \ldots, i_m) \\ &<_L (i_1, \ldots, \bar{i}_{j+1}, \ldots, i_m) \\ &= \mu(e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_j). \end{aligned}$$

The r^{th} term of the second sum is handled similarly. Assume the term is nonzero. If $j + r < m$ then

$$\begin{aligned} \mu(e_{i_1} \wedge \cdots \wedge e_{i_{j+1}-r} \wedge & \cdots \wedge e_{i_m} \wedge f_{j+r}) \\ &= (i_1, \ldots, i_{j+1}-r, \ldots, \bar{i}_{j+r+1}, \ldots, i_m) \\ &<_L (i_1, \ldots, \bar{i}_{j+1}, \ldots, i_m) \\ &= \mu(e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_j). \end{aligned}$$

If $j + r \geq m$ then

$$\begin{aligned} \mu(e_{i_1} \wedge \cdots \wedge e_{i_{j+1}-r} \wedge & \cdots \wedge e_{i_m} \wedge f_{j+r}) \\ &= (i_1, \ldots, i_{j+1}-r, \ldots, i_m) \\ &<_L (i_1, \ldots, \bar{i}_{j+1}, \ldots, i_m) \\ &= \mu(e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_j). \end{aligned}$$

Hence each nonzero wedge product on the right side of (4.3) has lexicographically smaller μ value than that of $e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_j$. □

Proof of Theorem 4.1. It follows from Lemma 4.2, the anticommuting exterior algebra relations and induction that $S_{m,1}$ spans $H^{m,1,0}(L_k)$. We use this to show that $S_{m,n}$ spans $H^{m,n,0}(L_k)$ for all m and n. Consider an arbitrary generator of $H_{m,n,0}(L_k)$,

$$\omega = e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_{j_1} \wedge \cdots \wedge f_{j_n},$$

where $0 \leq i_1 < \cdots < i_m \leq k$ and $0 \leq j_1 < \cdots < j_n \leq k$. We show that ω is in the span of $S_{m,n}$ by induction on the minimum f-index, j_1.

If $j_1 \geq m$ then $\omega \in S_{m,n}$. Now suppose $j_1 < m$. We will show that ω can be expressed as a linear combination of wedge products in $H^{m,n,0}(L_k)$ with larger minimum f-index. Since $S_{m,1}$ spans $H_{m,1,0}(L_k)$, the wedge product $e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_{j_1}$ is a linear combination of elements of the form $e_{i'_1} \wedge \cdots \wedge e_{i'_m} \wedge f_{j'_1}$, where $0 \leq i'_1 < \cdots < i'_m \leq k$ and $m \leq j'_1 \leq k$. It follows that $\omega = e_{i_1} \wedge \cdots \wedge e_{i_m} \wedge f_{j_1} \wedge \cdots \wedge f_{j_n}$ is a linear combination of elements of the form $e_{i'_1} \wedge \cdots \wedge e_{i'_m} \wedge f_{j'_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_n}$, where $0 \leq i'_1 < \cdots < i'_m \leq k$ and $m \leq j'_1 \leq k$. Clearly the wedge products of this form have larger minimum j-index than that of ω. It follows by induction these wedge products are in the span of $S_{m,n}$. Hence ω is in the span of $S_{m,n}$ and we can conclude that $S_{m,n}$ spans $H^{m,n,0}(L_k)$.

We now have

$$\dim H^{m,n,0}(L_k) \leq |S_{m,n}| = \frac{(k+1)!}{m!n!(k+1-m-n)!}.$$

Hence by Corollary 3.3, $S_{m,n}$ is a basis for $H^{m,n,0}(L_k)$. □

Remark 4.3. The proofs of Lemma 4.2 and Theorem 4.1 show that (4.3) provides the basic step of a straightening algorithm for expressing wedge products in $H_{m,n,0}(L_k)$ in terms of elements of the basis $S_{m,n}$.

The conjecture given in (1.3), which is restated here, is an immediate consequence.

Theorem 4.4. *For all $k \geq 0$,*

$$\sum_{m,n} \dim H_{m,n,0}(L_k) u^m v^n = (1+u+v)^{k+1}.$$

Equivalently,

$$\dim H_{m,n,0}(L_k) = \binom{k+1}{m, n, k+1-m-n}. \tag{4.4}$$

Corollary 4.5. *The dimension of the 0^{th} x-graded component of $H_*(L_k)$ is 3^{k+1}.*

Remark 4.6. Adin and Athanasiadis [1] derive the special case of (4.4) obtained by setting $m = 1$.

Since L_k is nilpotent, we can use Poincaré duality to conclude that the dimension of the $k+1^{st}$ x-graded component of (co)homology of L_k is also 3^{k+1}. In fact, we can explicitly transfer the basis for the 0^{th} x-graded component of cohomology to the $k+1^{st}$ x-graded component of homology. For $I = \{i_1 < i_2 < \cdots < i_n\} \subseteq \{0, 1, \ldots, k\}$, let

$$e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$$

and define f_I and x_I similarly. Let $\psi : \wedge L_k \to \wedge L_k$ be the isomorphism defined by

$$\psi(e_I \wedge f_J \wedge x_K) = e_{\bar{I}} \wedge f_{\bar{J}} \wedge x_{\bar{K}},$$

where $\bar{S}$ denotes the complement $\{0, 1, \ldots, k\} - S$. It is straightforward to check that $\psi \circ \partial = (-1)^k \delta \circ \psi$. Hence ψ determines a well-defined isomorphism $\psi : H^{m,n,p}(L_k) \to H_{k+1-m,k+1-n,k+1-p}(L_k)$.

Theorem 4.7. *For all $k, m, n \in \mathbb{N}$,*

$$\{e_I \wedge f_J \wedge x_{0,1,\ldots,k} \mid 0, 1, \ldots, k-n \in J, |I| = m, |J| = n\}$$

is a basis for $H_{m,n,k+1}$.

Proof. Apply ψ to the basis given in Theorem 4.1. □

Corollary 4.8. *For all $k \geq 0$,*

$$\sum_{m,n} \dim H_{m,n,k+1}(L_k) u^m v^n = (u + v + uv)^{k+1}.$$

Equivalently,

$$\dim H_{m,n,k+1}(L_k) = \binom{k+1}{k+1-m, k+1-n, m+n-k-1}.$$

Remark 4.9. Conjecture (1.2) is a special case of a conjecture involving the Laplacian of L_k (see [7, Conjecture 6E]). In this context Theorem 4.4 states that the dimension of the $(m, n, 0)$-graded piece of the kernel of the Laplacian of L_k has dimension $\binom{k+1}{m,n,k+1-m-n}$. and Corollary 4.8 states that the dimension of the $(m, n, k+1)$-graded piece of the kernel of the Laplacian of L_k has dimension $\binom{k+1}{k+1-m,k+1-n,m+n-k-1}$.

5. Conjectured bases for other components of (co)homology

We now consider the 0^{th} e- and f-graded pieces of homology. Since

$$H_{n,0,p}(L_k) \cong H_{0,n,p}(L_k)$$

we need only consider the 0^{th} e-graded piece. Analogous to the 0^{th} x-graded piece of cohomology, $H_{0,n,p}(L_k)$ is generated by all the elements of $\wedge^n(F) \otimes \wedge^p(X)$ subject to the boundary relations.

By setting $u = 0$ in (1.2) one has the following conjectured generating function

$$\sum_{n,p\in\mathbb{N}} \dim H_{0,n,p}(L_k) v^n w^p = (1 + v + vw)^{k+1}. \tag{5.1}$$

This implies that the total dimension of the 0^{th} e-graded piece of homology is 3^{k+1}. It also implies that if $p > n$ then $H_{0,n,p}(L_k)$ vanishes. The second consequence is not hard to prove.

The following conjecture implies (5.1).

Conjecture 5.1. *Let $0 \le p \le n$ and $\omega_p = f_0 \wedge \cdots \wedge f_{p-1}$. Then*

$$\{\omega_p \wedge f_{i_1} \wedge \cdots \wedge f_{i_{n-p}} \wedge x_{j_1} \wedge \cdots \wedge x_{j_p} \mid p \le i_1 < \cdots < i_{n-p} \le k, \quad 0 \le j_1 < \cdots < j_p \le k\}$$

is a basis for $H_{0,n,p}(L_k)$.

We can prove this conjecture for $p \le 2$ by using ideas similar to those of Section 4, and expect that these ideas will eventually lead to a proof for all p. By applying the Poincaré duality isomorphism ψ given in Section 4, one can formulate an equivalent conjecture for $H^{k+1,n,p}(L_k)$ (and $H^{m,k+1,p}(L_k)$).

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