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Action of the symmetric group on the free LAnKe: a CataLAnKe Theorem

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Abstract. We initiate a study of the representation of the symmetric group on the multilinear component of an n-ary generalization of the free Lie algebra, which we call a free LAnKe. Our central result is that the representation of the symmetric group S_{2n-1} on the multilinear component of the free LAnKe with $2n-1$ generators is given by an irreducible representation whose dimension is the nth Catalan number. This leads to a more general result on eigenspaces of a certain linear operator. A decomposition, into irreducibles, of the representation of S_{3n-2} on the multilinear component the free LAnKe with $3n-2$ generators is also presented. We also obtain a new presentation of Specht modules of shape λ, where λ has strictly decreasing column lengths, as a consequence of our eigenspace result.

Keywords: Free Lie algebra, Specht modules, Catalan numbers

1 Introduction

Lie algebras are defined as vector spaces equipped with an antisymmetric commutator and a Jacobi identity. They are a cornerstone of mathematics and have applications in a wide variety of areas of mathematics as well as physics. Also of fundamental importance is the free Lie algebra, a natural mathematical construction central in the field of algebraic combinatorics. The free Lie algebra has beautiful dimension formulas; elegant bases in terms of binary trees; and connections to the shuffle algebra, Lyndon words, necklaces, Witt vectors, the descent algebra of S_n, quasisymmetric functions, noncommutative symmetric functions, and the lattice of set partitions.

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It was the search for a tool to solve a problem in string theory that led the first author to define a generalization of Lie algebras, called a "LAnKe", or Lie Algebra of the n^{th} Kind **[Fr]**[4]. Specifically, the LAnKe arose as the algebraic object that plays a central role in generalizing a relation between codimension-4 singularities and ADE Lie algebras to a relation between higher-codimension singularities and algebraic objects. The singularities appear in the extra dimensions of string/M theory and affect the interactions seen in the 4-dimensional physical world.

In this extended abstract of **[FHSW1]**, we focus on generalizing the free Lie algebra to the free LAnKe and studying the representation of the symmetric group on its multilinear component. Our study generalizes the well-known representation $\mathrm{Lie}(k)$. The free LAnKe is based on an n-ary generalization of the Lie bracket. We define $\rho_{n,k}$ to be the representation of the symmetric group S_m on the multilinear component of the free LAnKe on m generators, where $m := n(k-1) - k + 2$. Our central result is that $\rho_{n,3}$ is isomorphic to the Specht module $S^{2^{n-1}1}$, whose dimension is the n^{th} Catalan number. We also present the result, whose proof will appear in **[FHSW2]**, that $\rho_{n,4}$ is isomorphic to a sum of two Specht modules.

An explicit S_m-isomorphism from $\rho_{n,3}$ to $S^{2^{n-1}1}$ can be obtained from presentations of free LAnKes and Specht modules following from results in **[DI]** and **[Fu]**, respectively. In this paper the relationship between $\rho_{n,3}$ and $S^{2^{n-1}1}$ is placed in a more general setting. The S_m-module $\rho_{n,k}$ has a presentation of the form $V_{n,k}/R_{n,k}$, where $V_{n,k}$ is generated by n-bracketed permutations involving an n-bracket that is antisymmetric only, and $R_{n,k}$ is the submodule of $V_{n,k}$ generated by the generalized Jacobi relations used in the definition of LAnKe. We consider a natural linear operator on $V_{n,3}$ whose 0th eigenspace is isomorphic to $\rho_{n,k}$. We show that all the eigenspaces are irreducible of the form $S^{2^i 1^{2n-1-2i}}$. Techniques from this proof also play a role in the proof of the above mentioned decomposition for $\rho_{n,4}$ obtained in **[FHSW2]**.

In Section 2 we introduce LAnKes and their relation to orbifold singularities (note that the rest of this extended abstract can be read independently of this section except for Definition 1). In Section 3 we introduce the free LAnKe and define the S_m-module $\rho_{n,k}$ which generalizes $\mathrm{Lie}(k)$. Our results for $\rho_{n,k}$ where $k = 3, 4$, are also included in Section 3. In Section 4 we present our result on eigenspaces of the above mentioned linear operator on $V_{n,3}$, which yields the $k = 3$ result. In Section 5 we discuss the presentations of LAnKes and Specht modules that yield the explicit isomorphism for $k = 3$. We also discuss a new presentation of Specht modules of shape λ, where λ has strictly decreasing column lengths. This is a consequence of a generalization of our eigenspace result.

[4]The $n = 3$ case of LAnKe is called a LATKe in **[Fr]**

2 The definition of the LAnKe

How was the definition of the LAnKe obtained, and what is the LAnKe? This section provides a roadmap that begins with the relation between singularities of codimension 4 and Lie algebras, and culminates in the definition of the LAnKe.

2.1 Orbifold singularities and Lie algebras

Consider an orbifold singularity of codimension 4, of the form $\mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $SU(2)$ acting naturally on the complex plane $\mathbb{C}^2$. For example, $\Gamma = \mathbb{Z}_n$ is generated by the diagonal 2×2 complex matrix $\mathrm{diag}\{e^{2\pi i/n}, e^{-2\pi i/n}\}$, and its action on $\mathbb{C}^2$ is given by $(x, y) \to (xe^{2\pi i/n}, ye^{-2\pi i/n})$. The action fixes the origin $(x, y) = (0, 0)$ while acting freely on the rest of the plane. The resolution of the singularity at the origin is a collection of intersecting 2-spheres known as the exceptional divisor of the singularity. When viewed as a graph, the intersecting 2-spheres form the Dynkin diagram of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, known as a Lie algebra of type A; further, the intersection matrix I_{ij} of the exceptional divisor is (the minus of) the Cartan matrix C_{ij} of the corresponding Lie algebra. Similarly, the other finite discrete subgroups of $SU(2)$ lead to singularities at the origin whose resolutions correspond to the Dynkin diagrams of Lie algebras of types D and E, with the same relation between the intersection numbers and the Cartan matrix. Due to their ADE classification, the codimension-4 orbifold singularities are known as ADE singularities.

The physical picture that involves codimension-4 singularities is as follows: the ADE singularities appear in the extra dimensions, and lead to a principal G_{ADE}-bundle over space-time crossed with the locus of the singularity. The group G_{ADE} is the Lie group of the corresponding ADE Lie algebra and it plays a central role in *gauge theories*, which are essentially a family of symmetries together with a physical model that can describe the interactions in the four-dimensional world.

What happens when the singularities have higher codimension? Can the relation between singularities and Lie algebras be generalized to codimension-$2n$ singularities? That is, given an orbifold singularity $\mathbb{C}^n/\Gamma$, where Γ is a finite, discrete subgroup of $SU(n)$, is there an algebraic object that can be associated with the singularity in a way analogous to that in which ADE Lie algebras are associated with ADE singularities? This question is illustrated in the following diagram.

$$\begin{tikzcd} \mathbb{C}^2/\Gamma_{ADE} \arrow[r, leftrightarrow] \arrow[d, dashed] & \text{ADE Lie algebra} \arrow[d, dashed] \\ \mathbb{C}^n/\Gamma \arrow[r, leftrightarrow, squiggly] & ? \end{tikzcd} \tag{2.1}$$

We discuss the answer in the next subsection. Here, we mention that just like the $n = 2$ case, the $n = 3$ case – with singularities of codimension 6 – has applications in string/M theory; for further details, see **[Fr]**.

2.2 The Commutator-Intersection-Relations and the LAnKe

To answer the question of the previous subsection, we look back at the traditional Lie algebra case. Recall that for any Lie algebra, the entries of the Cartan matrix, C_{ij}, play a central role in the Chevalley-Serre relations that contain the commutation relations of the generators of the Lie algebra. In our setting, said entries are equal to elements of the intersection matrix of the associated singularity (with a minus sign). Replacing C_{ij} by $-I_{ij}$ in the Chevalley-Serre relations leads to a fundamental reinterpretation. The relations become:

$$\begin{aligned} &[H_i, H_j] = 0 && [X_i, Y_j] = \delta_{ij} H_j \; ; \\ &[H_i, X_j] = -I_{ij} X_j && [H_i, Y_j] = I_{ij} Y_j \; ; \\ &\mathrm{ad}(X_i)^{1+I_{ij}}(X_j) = 0 && \mathrm{ad}(Y_i)^{1+I_{ij}}(Y_j) = 0 \; . \end{aligned} \tag{2.2}$$

Looking at these relations afresh reveals that actually, they provide all commutators of the Lie algebra *in terms of* the intersection numbers of the exceptional divisor of the corresponding singularity! These *Commutator-Intersection Relations* (CIRs) will now be generalized to the codimension 6 case, and then to codimension $2n$.

We begin by generalizing the intersection matrix. For codimension-6 orbifolds, the intersection form of the exceptional divisor is no longer a matrix of size $r \times r$, where r is both the rank of the algebra and the number of irreducible components in the exceptional divisor. Rather, it is a hyper-matrix of dimension $r \times r \times r$. This follows from the fact that in order to obtain a zero-dimensional space from which we can define an intersection number, it is required that we take triple intersections. For general n, we need n-fold intersections.

Now consider equations (2.2), in particular $[H_i, X_j] = -I_{ij} X_j$. The number of indices in the intersection matrix (two: i and j) is the same as the number of entries in the commutator (two: H_i and X_j). For codimension 6, the number of indices in the intersection form is 3, and for codimension $2n$ this number is n. This leads to the concept of a commutator of 3, or n, entries. All we need now is antisymmetry and a generalized Jacobi identity, and we will have obtained a mathematical object generalizing Lie algebras and replacing the question mark in diagram (2.1).

Definition 1. *A **Lie algebra $\mathcal{L}$ of the n-th kind ("LAnKe")** is a vector space equipped with an n-linear bracket*

$$[\cdot, \cdot, \quad , \cdot] : \times^n \mathcal{L} \to \mathcal{L}$$

that satisfies the following antisymmetry relation for all σ in the symmetric group S_n:

$$[x_1, \ldots, x_n] = \mathrm{sgn}(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] \tag{2.3}$$

and the following generalization of the Jacobi identity:

$$\begin{aligned}&[[x_1, x_2, \ldots, x_n], x_{n+1}, \ldots, x_{2n-1}] \\ &= \sum_{i=1}^{n} [x_1, x_2, \ldots, x_{i-1}, [x_i, x_{n+1}, \ldots, x_{2n-1}], x_{i+1}, \ldots, x_n],\end{aligned} \tag{2.4}$$

for all $x_1, x_2, \ldots, x_{2n-1} \in \mathcal{L}$.

The definition of the LAnKe with the above-described motivation was given in **[Fr]**. The same algebraic structure arose before in other contexts as well; see **[Fi**, **Ta**, **DT**, **Ka**, **Li**, **BL**, **Gu]**. A different generalization of Lie algebras that also involves n-ary brackets appeared in the 1990's in work of Hanlon and Wachs **[HW]**.

We now proceed to use this definition in our generalization of the free Lie algebra.

3 The free LAnKe

There are beautiful results on the free Lie algebra involving the representation of the symmetric group S_k on the multilinear component of the free Lie algebra with k generators. This representation is known as $\mathrm{Lie}(k)$. The LAnKe provides an opportunity to generalize both the free Lie algebra itself and the representation $\mathrm{Lie}(k)$. To set the stage, we review the free Lie algebra and $\mathrm{Lie}(k)$ first (see **[Re]**).

3.1 The free Lie algebra and $\mathrm{Lie}(k)$

Let $X := \{x_1, x_2, \ldots, x_k\}$ be a set of generators. Then the multilinear component of the free Lie algebra on X is the subspace spanned by bracketed "words" where each generator in X appears exactly once. For example, $[[x_1, x_3], [[x_4, x_5], x_2]]$ is such a bracketed word when $k = 5$, while $[[x_1, x_3], [[x_1, x_5], x_3]]$ is not. A certain type of bracketed word in the multilinear component has the form

$$[\cdots[[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], x_{\sigma(4)}], \ldots, x_{\sigma(k)}], \qquad \sigma \in S_k. \tag{3.1}$$

Bracketed words that are not of this type, such as $[[x_1, x_3], [[x_4, x_5], x_2]]$, can be shown to be linear combinations of the bracketed words of the form in (3.1) using iterations of the Jacobi identity. Furthermore, if we restrict to the permutations that satisfy $\sigma(1) = 1$, then the words of the form in (3.1) form a basis for the vector space. This vector space admits a natural $(k-1)!$-dimensional representation of S_k denoted $\mathrm{Lie}(k)$.

The decomposition of $\mathrm{Lie}(k)$ into irreducibles is given by the following result.

Theorem 1 (Kraskiewicz and Weyman [**KW**]). *Let $i \geq 1$ and $k \geq 2$ be relatively prime. For each $\lambda \vdash k$, the multiplicity of the Specht module S^λ in* $\mathrm{Lie}(k)$ *is equal to the number of standard Young tableaux of shape λ and of major index congruent to $i \mod k$.*

Interestingly, $\mathrm{Lie}(k)$ appears in a variety of other contexts, such as the top homology of the lattice of set partitions in work of Stanley [**St**], Hanlon [**Han**], Barcelo [**Ba**], and Wachs [**Wa**], in the homology of configuration spaces of k-tuples of distinct points in Euclidean space in work of Cohen [**Co**], and in scattering amplitudes in gauge theories in work of Kol and Shir [**KS**].

3.2 The free LAnKe and $\rho_{n,k}$

The following generalizes the standard definition of a free Lie algebra.

Definition 2. *Given a set X, the* **free LAnKe on** X *is a LAnKe $\mathcal{L}$ together with a mapping $i : X \to \mathcal{L}$ with the following universal property: for each LAnKe $\mathcal{K}$ and each mapping $f : X \to \mathcal{K}$, there is a unique LAnKe homomorphism $F : \mathcal{L} \to \mathcal{K}$ such that $f = F \circ i$.*

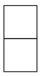
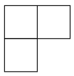
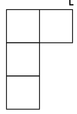
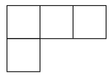
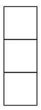
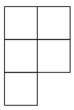
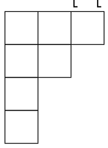
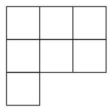
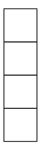
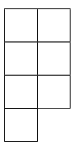
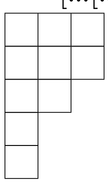
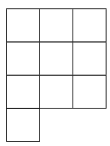
Similar to the free Lie algebra, a LAnKe is free on X if it is generated by all possible n-bracketings of elements of X, and if the only possible relations existing among these bracketings are consequences of n-linearity of the bracketing, the antisymmetry of the bracketing (2.3), and the generalized Jacobi identity (2.4).

The multilinear component of the free LAnKe on m generators is spanned by bracketed permutations; that is, bracketed words where each generator appears exactly once. We consider two variables: n, the number of entries in a given bracket (so $n = 2$ for Lie algebras), and k, the number of brackets plus 1. The number of generators for the multilinear component is then $kn - n - k + 2$, an expression symmetric in n and k. For example, an element of the form $[\cdot\cdot[\cdot\cdot\cdot]]$ has $n = 3$, $k = 3$, and $3 \cdot 3 - 3 - 3 + 2 = 5$ generators; $[\cdot\cdot[\cdot\cdot[\cdot\cdot\cdot]]]$ has $n = 3$, $k = 4$, and $4 \cdot 3 - 4 - 3 + 2 = 7$ generators.

The object we study here is the representation of S_m on the multilinear component of the free LAnKe on $m := kn - n - k + 2$ generators. We denote this representation by $\rho_{n,k}$, and view $(\rho_{n,k})$ as an array of representations, with $n, k \geq 2$.

Table 1 summarizes what we know about the decomposition of $\rho_{n,k}$ into irreducibles. The Young diagrams in the table stand for Specht modules of the indicated shape. The first row $\rho_{2,k}$, $k \geq 2$, is the Lie representation $\mathrm{Lie}(k)$. The sign representations that appear in the first column, $k = 2$, trivially follow from the antisymmetry of the bracket. The second column $\rho_{n,3}$, $n \geq 2$, follows from Theorem 2 below and the third column $\rho_{n,4}$, $n \geq 2$, follows from Theorem 3 below. The first three columns suggest that $\rho_{n,k}$ can be obtained from $\rho_{n-1,k}$ by adding a row of length $k-1$ to each irreducible of $\rho_{n-1,k}$. However the entry $\rho_{3,5}$, whose expansion follows from general results of the authors in [**FHSW2**], shows that this does not hold when $k = 5$. Relationships between $\rho_{n,k}$ and $\rho_{n-1,k}$ are explored in [**FHSW2**].

Table 1: What is known about the representations $\rho_{n,k}$

$n \backslash k$	2	3	4	5	k
2	S_2 [..]  1	S_3 [.[..]]  2	S_4 [.[.[..]]]  $\oplus$  6	S_5 [.[.[.[..]]]] $32 \oplus 41$ $\oplus 21^3 \oplus 31^2 \oplus 2^2 1$ 24	S_k $Lie(k)$ $(k-1)!$
3	S_3 [...]  1	S_5 [..[...]]  5	S_7 [..[..[...]]]  $\oplus$  56	S_9 [..[..[..[...]]]] $432 \oplus 4^2 1$ $\oplus 421^3 \oplus 431^2 \oplus 3^2 1^3$ $\oplus 42^2 1 \oplus 32^3$ 1077	S_{2k-1} $\rho_{3,k}$
4	S_4 [....]  1	S_7 [...[....]]  14	S_{10} [...[...[....]]]  $\oplus$  660	S_{13} [...[...[...[....]]]] $\rho_{4,5}$	S_{3k-2} $\rho_{4,k}$
n	S_n 1^n 1	S_{2n-1} $2^{n-1}1$ $\frac{1}{n+1}\binom{2n}{n}$	S_{3n-2} $3^{n-2}21^2 \oplus 3^{n-1}1$ $\frac{4}{\prod_{i=1}^{3}(n+i)}\binom{3n}{n,n,n}$	S_{4n-3} $\rho_{n,5}$	$S_{nk-n-k+2}$ $\rho_{n,k}$

Theorem 2. *For all $n \geq 2$, the representation $\rho_{n,3}$ is given by the Specht module $S^{2^{n-1}1}$, whose dimension is the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$.*

An explicit S_{2n-1}-module isomorphism between the CataLAnKe representation $\rho_{n,3}$ and the Specht module $S^{2^{n-1}1}$ can be obtained from an alternative generalized Jacobi relation in **[DI]** and a presentation for any Specht module S^{λ} in **[FU]**; see Section 5. In the next section we show that Theorem 2 can be viewed as a special case of a result on eigenspaces of a certain operator.

The following result is obtained by the authors in **[FHSW2]**. Techniques discussed in the next section play a role in the proof.

Theorem 3. *For all $n \geq 2$, the following S_{3n-2}-module isomorphism holds,*

$$\rho_{n,4} \cong S^{3^{n-2}21^2} \oplus S^{3^{n-1}1}.$$

4 The CataLAnKe representation and a linear operator

In this section we consider an operator whose null space is isomorphic to the CataLAnKe representation $\rho_{n,3}$. We show that all the eigenspaces of this operator are irreducible and use this to recover Theorem 2.

Let $V_{n,3}$ be the multilinear component of the vector space generated by all possible n-bracketed words on $[2n-1]$, subject only to antisymmetry of the brackets given in (2.3) (but not to the generalized Jacobi identity (2.4)). That is, $V_{n,3}$ is the subspace generated by

$$u_\tau := [[\tau_1, \dots, \tau_n], \tau_{n+1}, \dots, \tau_{2n-1}],$$

where $\tau \in S_{2n-1}$, $\tau_i = \tau(i)$ for each i, and $[\cdot, \dots, \cdot]$ is the antisymmetric n-linear bracket (that does not satisfy the generalized Jacobi relation).

The symmetric group S_{2n-1} acts on generators of $V_{n,3}$ by the following action: for $\sigma, \tau \in S_{2n-1}$,

$$\sigma u_\tau = u_{\sigma\tau}.$$

This induces a representation of S_{2n-1} on $V_{n,3}$ since the action respects the antisymmetry relation.

For each n-element subset $S := \{a_1, \dots, a_n\}$ of $[2n-1]$, let

$$v_S = [[a_1, \dots, a_n], b_1, \dots, b_{n-1}],$$

where $\{b_1, \cdots, b_{n-1}\} = [2n-1] \setminus S$, and the a_i's and b_i's are in increasing order. Clearly,

$$\left\{ v_S : S \in \binom{[2n-1]}{n} \right\} \tag{4.1}$$

is a basis for $V_{n,3}$. Thus $V_{n,3}$ has dimension $\binom{2n-1}{n}$.

For each $S \in \binom{[2n-1]}{n}$, use the generalized Jacobi identity (2.4) to define the relation

$$R_S = v_S - \sum_{i=1}^{n} [a_1, \dots, a_{i-1}, [a_i, b_1, \dots, b_{n-1}], a_{i+1}, \dots, a_n], \tag{4.2}$$

where $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_{n-1}$ are as in the previous paragraph. Let $R_{n,3}$ be the subspace of $V_{n,3}$ generated by the R_S. Then as S_{2n-1}-modules

$$V_{n,3}/R_{n,3} \cong \rho_{n,3}. \tag{4.3}$$

We consider the linear operator $\varphi : V_{n,3} \to V_{n,3}$ defined on basis elements by

$$\varphi(v_S) = R_S.$$

It is not difficult to see that this is an S_{2n-1}-module isomorphism. We show that as S_{2n-1}-modules,

$$\text{null } \varphi \cong \rho_{n,3}. \tag{4.4}$$

Hence Theorem 2 says that the null space is isomorphic to the Specht module $S^{2^{n-1}1}$. The next result generalizes this to all the eigenspaces of φ.

Theorem 4. *There are n distinct eigenvalues of φ, which are given by*

$$w_i := 1 + (n-i)(-1)^{n-i},$$

for $i = 0, 1, \ldots, n-1$. Moreover, if E_i is the eigenspace corresponding to w_i then as S_{2n-1}-modules

$$E_i \cong S^{2^i 1^{(2n-1)-2i}},$$

for each $i = 0, 1, \ldots, n-1$.

Note that we can recover Theorem 2 as a consequence of Theorem 4 and (4.4) since $w_i = 0$ when $i = n-1$.

Proof idea. [5] We use Young's rule to show

$$V_{n,3} \cong \bigoplus_{i=0}^{n-1} S^{2^i 1^{2n-1-2i}}.$$

Since there are no multiplicities, it follows from Schur's lemma that φ acts as a scalar on each irreducible submodule. It remains to determine said scalar for each irreducible $S^{2^i 1^{2n-1-2i}}$.

To compute the scalar, we start by letting t be the Young tableau of shape $2^i 1^{2n-1-2i}$ whose first column is $1, 2, \ldots, n, n+i+1, n+i+2, \cdots 2n-1$ and whose second column is $n+1, n+2, \ldots, n+i$. Let C_t be the column stabilizer of t and let R_t be the row stabilizer. Recall that the Young symmetrizer associated with t is defined by

$$e_t := \sum_{\alpha \in R_t} \alpha \sum_{\beta \in C_t} \operatorname{sgn}(\beta)\beta$$

and that the Specht module $S^{2^i 1^{2n-1-2i}}$ is the submodule of the regular representation $\mathbb{C}S_{2n-1}$ spanned by $\{\tau e_t : \tau \in S_{2n-1}\}$.

Now set $T := [n]$. We show that $e_t v_T \neq 0$. Let $\psi : \mathbb{C}S_{2n-1} \to V_{n,3}$ be the S_{2n-1}-module homomorphism defined by $\psi(\sigma) = \sigma v_T$. Now consider the restriction of ψ to the Specht module $S^{2^i 1^{2n-1-2i}}$. By the irreducibility of the Specht module and the fact that $e_t v_T \neq 0$, this restriction is an isomorphism from $S^{2^i 1^{2n-1-2i}}$ to the subspace of $V_{n,3}$ spanned by $\{\tau e_t v_T : \tau \in S_{2n-1}\}$. This subspace is therefore the unique subspace of $V_{n,3}$ isomorphic to $S^{2^i 1^{(2n-1)-2i}}$. Thus $\varphi(e_t v_T) = c e_t v_T$ for some scalar c. By computing the coefficient of v_T in $\varphi(e_t v_T)$ and in $e_t v_T$ we show that $c = w_i$. $\square$

[5] See **[FHSW1]** for a full-length manuscript with the complete proof of Theorem 4.

5 Alternative presentations

For each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ of m, let $\mathcal{T}_\lambda$ be the set of Young tableaux of shape λ. Let M^λ be the vector space generated by $\mathcal{T}_\lambda$ subject only to column relations, which are of the form $t + s$, where s is obtained from t by switching two entries in the same column. Given $t \in \mathcal{T}_\lambda$, let $\bar{t}$ denote the coset of t in M^λ. These cosets, which are called *column tabloids*, generate M^λ. The symmetric group S_m acts on $\mathcal{T}_\lambda$ by replacing each entry of a tableaux by its image under the permutation in S_m. This induces a representation of S_m on M^λ.

There are various different presentations of S^λ in the literature, which involve the column relations and Garnir relations. Here we are interested in a presentation of S^λ discussed in Fulton **[Fu]**. The Garnir relations are of the form $\bar{t} - \sum \bar{s}$, where the sum is over all $s \in \mathcal{T}_\lambda$ obtained from $t \in \mathcal{T}_\lambda$ by exchanging any k entries of any column with the top k entries of the next column, while maintaining the vertical order of each of the exchanged sets. There is a Garnir relation $g^t_{c,k}$ for every $t \in \mathcal{T}_\lambda$, every column $c \in [\lambda_1 - 1]$, and every k from 1 to the length of the column $c+1$. Let G^λ be the subspace of M^λ generated by these Garnir relations. Clearly G^λ is invariant under the action of S_m. The presentation of S^λ obtained in Section 7.4 of **[Fu]** is given by

$$M^\lambda / G^\lambda \cong S^\lambda. \tag{5.1}$$

On page 102 (after Ex. 15) of **[Fu]**, a presentation of S^λ with fewer relations is given. The presentation is

$$M^\lambda / G^{\lambda,1} \cong S^\lambda, \tag{5.2}$$

where $G^{\lambda,1}$ is the subspace of G^λ generated by

$$\{g^t_{c,1} : c \in [\lambda_1 - 1], t \in \mathcal{T}_\lambda\}.$$

In Appendix 1 of **[DI]**, a proof that the generalized Jacobi relations (2.4) are equivalent to the relations

$$[[x_1, x_2, \ldots, x_n], y_1, \ldots, y_{n-1}] = \sum_{i=1}^{n} [[x_1, x_2, \ldots, x_{i-1}, y_1, x_{i+1}, \ldots, x_n], x_i, y_2, \ldots, y_{n-1}] \tag{5.3}$$

is given.[6] This gives an alternative presentation of $\rho_{n,k}$ for all n, k.

Using the natural correspondence between generators $[[a_1, \ldots, a_n], b_1, \ldots, b_{n-1}]$ of $V_{n,3}$ and column tabloids $\bar{t}$, where t is the tableau whose first column is $a_1, \ldots, a_n$ and whose second column is $b_1, \ldots, b_{n-1}$, we see that the alternative Jacobi relations (5.3) correspond to the Garnir relation $g^t_{1,1}$ for $\lambda = 2^{n-1}1$. Thus the natural correspondence between generators yields an isomorphism from $\rho_{n,3}$ to the realization of $S^{2^{n-1}1}$ given in (5.2).

[6]This equivalence can also be obtained as a consequence of Theorem 4 and (5.2).

The natural correspondence between generators of $V_{n,3}$ and generators of $M^{2^{n-1}1}$ also takes the generalized Jacobi relations (2.4) to the Garnir relations $g^t_{1,n-1}$. This enables us to give another presentation of $S^{2^{n-1}1}$ with fewer relations than that of (5.1). In fact, we can extend this to a wider class of Specht modules. Indeed, the natural correspondence between generators of $V_{n,3}$ and generators of $M^{2^{n-1}1}$ can be used to transfer the operator φ on $V_{n,3}$ of Theorem 4 to $M^{2^{n-1}1}$. For $d \in [n]$, a generalization $\varphi_{n,d}$ of φ can be defined on $M^{2^d 1^{n-d}}$. We obtain a generalization of Theorem 4 for $\varphi_{n,d}$ and use it to prove the following result, which gives a new presentation with fewer relations than that of (5.1) for Specht modules S^λ whose conjugate shape λ' has strictly decreasing parts.[7]

Theorem 5. *For $\lambda \vdash m$, let $\mathcal{T}^*_\lambda$ be the set of Young tableaux of shape λ in which each element of $[m]$ appears once and the columns increase, and let $\tilde{G}^\lambda$ be the subspace of M^λ generated by*

$$\{g^t_{c,\lambda'_{c+1}} : c \in [\lambda_1 - 1], t \in \mathcal{T}^*_\lambda\}.$$

If λ' has strictly decreasing parts then $S^\lambda \cong M^\lambda / \tilde{G}^\lambda$.

In [**BF**], Brauner and Friedmann obtain a result analogous to the generalization of Theorem 4 discussed above and use it to obtain an interesting new presentation of Specht modules of *all* shapes, in which the number of relations has been similarly reduced. The new presentation implies the presentation (5.2). Another proof of Theorem 2 is also discussed in [**BF**].

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