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Flagged Schur Functions, Schubert Polynomials, and Symmetrizing Operators

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0. INTRODUCTION

Flagged Schur functions are generalizations of Schur functions. They appear in the work of Lascoux and Schutzenberger [2] in their study of Schubert polynomials. Gessel [1] has shown that flagged Schur functions can be expressed both as a determinant in the complete homogeneous symmetric functions and in terms of column-strict tableaux just as can ordinary Schur functions (Jacobi–Trudi identity). For each row of these tableaux there is an upper bound (flag) on the entries.

The Schubert polynomials are obtained by applying certain symmetrizing operators to a monomial. In Section 1 we study the effect of applying these symmetrizing operators to flagged Schur functions. Although it is trivial to do this for the determinantal expression, we show, by direct means, how to apply the symmetrizing operators to the tableau expression (without the use of determinants). This produces another proof of Gessel's result and hence a new inductive proof of the Jacobi–Trudi identity.

Each Schubert polynomial is determined by some permutation. Lascoux and Schutzenberger [2] state a result which enables one to identify those permutations whose Schubert polynomial is a flagged Schur function. In Section 2 we present an explicit expression for the shape and flags (row bounds) in terms of the permutation. We do this by applying the symmetrizing operators to flagged Schur functions. We also show that any flagged Schur function can be obtained by applying a sequence of symmetrizing operators to some monomial.

In Section 4 we consider row (column) flagged skew Schur functions. Here, for each row (column) of the skew tableaux there is an upper and lower bound on the entries. In fact the above cited work of Gessel actually

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deals with row-flagged skew Schur functions. His proof involves a beautiful and clever combinatorial construction. We present an alternative proof which makes use of very natural recurrence relations on row-flagged skew Schur functions. We also prove an analogous result for column-flagged skew Schur functions by using recurrence relations.

1. Flagged Schur Functions

Let $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m>0)$ be a partition and let $b=(b_1\leqslant b_2\leqslant\cdots\leqslant b_m)$ be an increasing sequence of positive integers. Lascoux and Schutzenberger [2] define the *flagged Schur* function with shape λ and flags b to be

$$S_\lambda(b)=\det|h_{\lambda_i-i+j}(b_i)|_{i,j=1,2,\dots,m}, \tag{1.1}$$

where $h_d(k)$ is the complete homogeneous symmetric function of degree d in the variables $x_1, x_2,\dots, x_k$. Note that if $b_1=b_2=\cdots=b_m$ then the determinant in (1.1) is the well-known determinantal expression for ordinary Schur functions (cf. [3, 4]).

Just as ordinary Schur functions can be expressed determinantally and in terms of column-strict tableaux, so can flagged Schur functions. A *column-strict tableau* of shape λ and flags b is an array T of positive integers t_{ij}, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant\lambda_i$ such that $t_{ij}\leqslant t_{i,j+1}\leqslant b_i$ for all $1\leqslant i\leqslant m$, $1\leqslant j<\lambda_i$ and $t_{ij}<t_{i+1,j}$ for all $1\leqslant i<m$, $1\leqslant j\leqslant\lambda_{i+1}$. We define $\mathscr{T}(\lambda, b)$ to be the set of column-strict tableaux of shape λ and flags b. For each $T\in\mathscr{T}(\lambda, b)$ let $M(T)$ be the monomial $x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}$, where p_k is the number of entries t_{ij} that are equal to k. Now let

$$s_\lambda(b)=\sum_{T\in\mathscr{T}(\lambda,b)} M(T).$$

Gessel [1] has shown that $S_\lambda(b)$ and $s_\lambda(b)$ are the same. We will now show that this result can also be obtained by considering the following symmetrizing operators. Let f be a polynomial in the variables $x_1, x_2,\dots, x_n$. For $i=1, 2,\dots, n-1$ define the symmetrizing operator,

$$\partial_i(f)=\frac{f(x_1,\dots, x_n)-f(x_1,\dots, x_{i+1}, x_i,\dots, x_n)}{x_i-x_{i+1}}.$$

We will apply this operator to the determinantal expression and to the tableau expression.

LEMMA 1.1. *Let $\lambda_1 > \lambda_2$ and $b_1 < b_2$. Then*

(i) $\partial_{b_1} S_{\lambda_1\lambda_2\cdots\lambda_m}(b_1, b_2,..., b_m) = S_{\lambda_1-1,\lambda_2,...,\lambda_m}(b_1+1, b_2,..., b_m)$.

(ii) $\partial_{b_1} s_{\lambda_1\lambda_2\cdots\lambda_m}(b_1, b_2,..., b_m) = s_{\lambda_1-1,\lambda_2,...,\lambda_m}(b_1+1, b_2,..., b_m)$.

Proof. When $m=1$, (i) and (ii) become,

$$\partial_{b_1} h_d(b_1) = h_{d-1}(b_1+1). \tag{1.2}$$

This is easy to verify and is included in the proof of (ii).

We now prove (i) for $m>1$. Since $b_1 < b_2 \leqslant b_i$, $i=2,..., m$, every entry below the first row of the determinant in (1.1) is symmtric in x_{b_1} and x_{b_1+1}. Hence applying ∂_{b_1} to the determinant is the same as applying ∂_{b_1} to each of the elements in the first row. By (1.2) the resulting determinant is $S_{\lambda_1-1,\lambda_2,...,\lambda_m}(b_1+1, b_2,..., b_m)$.

Next we prove (ii). We will group together the terms of $s_\lambda(b)$ according to the configuration of tableau entries that are equal to b_1 or b_1+1. Let $t=b_1$. We define an equivalence relation $\sim$ on $\mathscr{T}(\lambda, b)$. For T_1, $T_2 \in \mathscr{T}(\lambda, b)$ let $T_1 \sim T_2$ if the collection of positions that contain either a t or a $t+1$ is the same for T_1 as it is for T_2. Clearly these positions form a skew diagram with one or more components. Let $\mathscr{A}$ be the equivalence class of $\mathscr{T}(\lambda, b)$ whose tableaux have the configuration of entries equal to t or $t+1$ shown in Fig. 1. Here each $*$ represents a t or a $t+1$ and any m_i or r_i can be 0. In Fig. 1, the skew diagram appears connected. However the argument to follow is unaffected by the occurrence of more than one component. We have,

$$\sum_{T\in\mathscr{A}} M(T) = x_t^{m_1}(x_t x_{t+1})^{r_1+r_2+\cdots+r_k}\left(\prod_{i=2}^{k}\sum_{v=0}^{m_i} x_t^v x_{t+1}^{m_i-v}\right) R(\mathscr{A}),$$

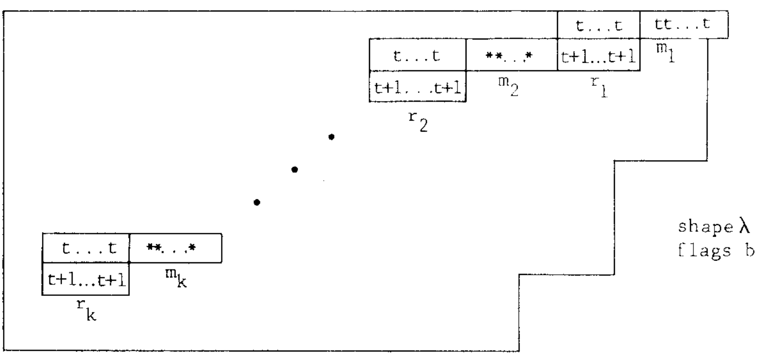

FIGURE 1

where $R(\mathscr{A})$ is the polynomial associated with entries other than t and $t+1$. Applying ∂_t gives,

$$\partial_t \sum_{T\in\mathscr{A}} M(T) = \frac{x_t^{m_1} - x_{t+1}^{m_1}}{x_t - x_{t+1}} (x_t x_{t+1})^{r_1+r_2+\cdots+r_k} \times \left(\prod_{i=2}^{k} \sum_{v=0}^{m_i} x_t^v x_{t+1}^{m_i-v}\right) R(\mathscr{A})$$

$$= \begin{cases} 0 & \text{if} \quad m_1 = 0, \\ (x_t x_{t+1})^{r_1+r_2+\cdots+r_k} \displaystyle\sum_{v=0}^{m_1-1} x_t^v x_{t+1}^{m_1-1-v} \\ \times \left(\displaystyle\prod_{i=2}^{k} \sum_{v=0}^{m_i} x_t^v x_{t+1}^{m_i-v}\right) R(\mathscr{A}) & \text{if} \quad m_1 > 0. \end{cases} \tag{1.3}$$

Now let $\lambda' = (\lambda_1 - 1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m)$ and $b' = (b_1 + 1 \leqslant b_2 \leqslant \cdots \leqslant b_m)$. If $m_1 > 0$ let $\mathscr{A}'$ be the equivalence class of $\mathscr{T}(\lambda', b')$ whose tableaux have the configuration of entries equal to t or $t+1$ shown in Fig. 2.

Clearly $\sum_{T\in\mathscr{A}'} M(T)$ is equal to (1.3). Hence we have

$$\partial_t \sum_{T\in\mathscr{A}} M(T) = \begin{cases} 0 & \text{if} \quad m_1 = 0, \\ \displaystyle\sum_{T\in\mathscr{A}'} M(T) & \text{if} \quad m_1 > 0. \end{cases} \tag{1.4}$$

Observe that there is a natural bijection $\mathscr{A} \leftrightarrow \mathscr{A}'$ between the equivalence classes of $\mathscr{T}(\lambda, b)$ with $m_1 > 0$ and the equivalence classes of $\mathscr{T}(\lambda', b')$. Therefore (ii) follows from (1.4), since $s_\lambda(b) = \sum_{\mathscr{A}} \sum_{T\in\mathscr{A}} M(T)$ and

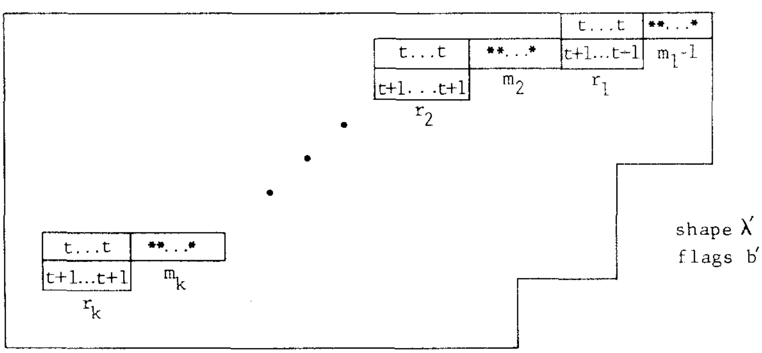

FIGURE 2

$s_{\lambda'}(b') = \sum_{\mathscr{A}'} \sum_{T \in \mathscr{A}'} M(T)$, where $\mathscr{A}$ and $\mathscr{A}'$ range over all equivalence classes of $\mathscr{T}(\lambda, b)$ and $\mathscr{T}(\lambda', b')$, respectively. ∎

By repeatedly applying the inverse of ∂_{b_1} to $S_\lambda(b)$ (or $s_\lambda(b)$) we can reduce b_1 to 1. When $b_1 = 1$ the next lemma shows that $S_\lambda(b)$ and $s_\lambda(b)$ can be further reduced.

LEMMA 1.2. *If $b_1 = 1$ then,*

(i) $S_{\lambda_1\lambda_2\cdots\lambda_m}(b_1, b_2,..., b_m) = x_1^{\lambda_1}(S_{\lambda_2\cdots\lambda_m}(b_2,..., b_m))^*$

(ii) $s_{\lambda_1\lambda_2\cdots\lambda_m}(b_1, b_2,..., b_m) = x_1^{\lambda_1}(s_{\lambda_2\cdots\lambda_m}(b_2,..., b_m))^*$,

*where the symbol * denotes the deletion of all terms that contain x_1.*

Proof. This time the proof of (ii) is trivial. It follows immediately from the definition of $s_\lambda(b)$.

To prove (i), observe that $(h_{d+1}(k))^* = h_{d+1}(k) - x_1 h_d(k)$. Hence if we subtract x_1 times column j from column $j+1$ of the determinant in (1.1) then we get

$$S_\lambda(b) = \det \begin{vmatrix} x_1^{\lambda_1} & 0 & \cdots & 0 \\ h_{\lambda_2-1}(b_2) & (h_{\lambda_2}(b_2))^* & \cdots & (h_{\lambda_2-2+m}(b_2))^* \\ \vdots & \vdots & & \vdots \\ h_{\lambda_m-m+1}(b_m) & (h_{\lambda_m-m+2}(b_m))^* & \cdots & (h_{\lambda_m}(b_m))^* \end{vmatrix}$$

$$= x_1^{\lambda_1}(S_{\lambda_2\cdots\lambda_m}(b_2\cdots b_m))^*. \quad ∎$$

THEOREM 1.3. $S_\lambda(b) = s_\lambda(b)$.

Proof. The result follows from Lemmas 1.1 and 1.2 and induction on the sum of the flags, $b_1 + b_2 + \cdots + b_m$. ∎

2. SCHUBERT POLYNOMIALS

We shall regard elements of the symmetric group $\mathscr{S}_n$ as words in the symbols 1, 2,..., n. We say that $w = w_1 w_2 \cdots w_n$ is an *increasing* permutation if $w_1 < w_2 < \cdots < w_n$. We say that w has a *descent* at i if $w_i > w_{i+1}$. Similarly w has an *ascent* at i if $w_i < w_{i+1}$.

For each $i = 1, 2,..., n-1$ let σ_i be the adjacent transposition $(i, i+1)$. Every $w \in \mathscr{S}_n$ can be expressed as a product of the σ_i. Here multiplication by σ_i on the right transposes the symbols in position i and $i+1$, i.e., if $w = w_1 w_2 \cdots w_n$ then $w\sigma_i = w_1 \cdots w_{i+1} w_i \cdots w_n$. The *length* of a permutation w denoted by $l(w)$, is the minimum number of σ_i (repetitions counted) needed to express w.

Let f be a polynomial in the variables $x_1, x_2, \ldots, x_n$. For $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$, $k = l(w)$, define the operator,

$$\partial_w(f) = \partial_{i_k}\partial_{i_{k-1}}\cdots\partial_{i_1}(f),$$

where the symmetrizing operators are applied from right to left. The fact that ∂_w is well defined follows the fact that the product depends only on w and not on the expression $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ (see [2]).

Let w_0 be the permutation $n, n-1, \ldots, 1$. For $w \in \mathcal{S}_n$ define the *Schubert polynomial*

$$F_w = \partial_{w_0 w}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}).$$

Hence $F_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ and if u covers w in weak Bruhat order (see [5] for the definition) then $F_w = \partial_i F_u$ for some $i = 1, 2, \ldots, n-1$.

We now review some fundamental properties of Schubert polynomials which appear in [2].

(1) F_w is a homogeneous polynomial with positive integral coefficients, of degree $l(w)$ in variables $x_1, x_2, \ldots, x_m$, where the right-most descent of w is at m.

(2) F_w is symmetric in the variables x_i and x_{i+1} if and only if w has an ascent at i.

(3) F_w is a Schur function in the variables $x_1, x_2, \ldots, x_m$ if and only if the only descent of w is at m.

Lascoux and Schutzenberger also characterize those permutations whose Schubert polynomials are flagged Schur functions. Before stating their result we need some additional terminology. For $w = w_1 w_2 \cdots w_n \in \mathcal{S}_n$ and $i = 1, 2, \ldots, n$ define the *inversion sets*,

$$I_i(w) = \{j = 1, 2, \ldots, n \mid i < j \text{ and } w_i > w_j\}.$$

Similarly the *inverse inversion sets* are

$$J_i(w) = \{j = 1, 2, \ldots, n \mid j < i \text{ and } w_j > w_i\}.$$

Let $\lambda(w)$ be the partition of $l(w)$ obtained by arranging the cardinalities of the non-empty inversion sets in decreasing order. Let $\mu(w)$ be the conjugate of the partition of $l(w)$ obtained by arranging the cardinalities of the non-empty inverse inversion sets in decreasing order.

PROPOSITION 2.1. (*Lascoux and Schutzenberger* [2]). *Let* $w \in \mathcal{S}_n$. *Then the following are equivalent*:

(1) F_w *is a flagged Schur function of shape* λ.

(2) *The inversion sets of w ordered by inclusion form a chain.*

(3) $\lambda(w) = \mu(w)$.

Furthermore when the above conditions hold $\lambda = \lambda(w)$.

Lascoux and Schutzenberger also give bounds on the flags when F_w is a flagged Schur function. They do not, however, say what the flags are. In Theorem 2.3 we give an explicit description of the flags in terms of the descents of w.

A permutation w will be called a *single-shape permutation* if $\lambda(w) = \mu(w)$ and $\lambda(w)$ will be referred to as the *shape* of the permutation. Single-shape permutations are also significant in the enumeration of chains in weak Bruhat order (see [5]).

Proposition 2.1 (2) and (3) give two ways of describing single-shape permutations. We now present a third recursive way, which we will make use of in the sequel.

PROPOSITION 2.2. *Let the left-most descent of $w = w_1 w_2 \cdots w_n$ occur at d. (If w is an increasing permutation then $d = n$.) Then w is a single-shape permutation if and only if $n = 1$ or $n > 1$ and*

(1) *all symbols greater than w_d appear in ascending order to the right of w_d and,*

(2) *$w_1 w_2 \cdots \hat{w}_d \cdots w_n$ is a single-shape permutation ($\hat{}$ denotes deletion).*

Proof. Observe that (1) is equivalent to

$$I_j \subseteq I_d \qquad \text{for all} \quad j = 1, 2, ..., n, \tag{2.1}$$

and (2) is equivalent to $\{I_j \mid j \neq d\}$ forming a chain. Hence (1) and (2) together imply that $\{I_j \mid j = 1, 2, ..., n\}$ forms a chain which means that w is a single-shape permutation.

Conversely if w is a single-shape permutation then $\{I_j \mid j = 1, 2, ..., n\}$ forms a chain. This immediately implies (2). Since the left-most descent of w is at d, I_d is not properly contained in any I_j. Hence (2.1) must hold which means that (1) holds also. ∎

For each $i = 1, 2, ..., n$ such that the inversion set $I_i(w) \neq \phi$ let $c_i = \min I_i(w) - 1$. Let $b(w) = (b_1(w) \leqslant b_2(w) \leqslant \cdots \leqslant b_m(w))$ be the sequence obtained by arranging the c_i in increasing order. We will refer to $b(w)$ as the flag sequence of w. Note that the number of flags of w (repetitions counted) is the same as the number of parts of $\lambda(w)$, and that the set of flags of w is precisely the set of descents of w. We now present the main result of this section.

THEOREM 2.3. *If $w = w_1 w_2 \cdots w_n$ is a single-shape permutation then F_w is the flagged Schur function $S_{\lambda(w)}(b(w))$, where $\lambda(w)$ is the shape of w and $b(w)$ is the flag sequence of w.*

Proof. The proof is by induction on the parameters $(n, l(w_0) - l(w))$ ordered lexicographically. The result holds trivially for $w = w_0$ and for $n = 1$. Now assume that $l(w) < l(w_0)$. Suppose that the left-most descent of w is at d. There are three cases.

Case 1. Assume $d > 1$. Let $w' = w\sigma_{d-1}$. Then $w' = w_1, w_2, \ldots, w_d, w_{d-1}, \ldots, w_n$. It follows from Proposition 2.2 that w' is a single-shape permutation. Since $l(w') > l(w)$, by induction we have $F_{w'} = S_{\lambda(w')}(b(w'))$. Now since $F_w = \partial_{d-1} F_{w'}$ we have

$$F_w = \partial_{d-1} S_{\lambda(w')}(b(w')). \tag{2.2}$$

It is easy to see that

$$\lambda_1(w') = \lambda_1(w) + 1, \qquad \lambda_i(w') = \lambda_i(w), \qquad i = 2, \ldots, m,$$

and

$$b_1(w') = d - 1 = b_1(w) - 1, \qquad b_i(w') = b_i(w), \qquad i = 2, \ldots, m.$$

This implies, by Lemma 1.1, that $\partial_{d-1} S_{\lambda(w')}(b(w')) = S_{\lambda(w)}(b(w))$. Combining this with (2.2) completes the proof for Case 1.

Case 2. Assume $d = 1$ and $w_1 < n$. Let $w' = \sigma_{w_1} w = w_1 + 1, w_2, \ldots, w_1, \ldots, w_n$. We will show that $F_{w'} = x_1 F_w$. Let $u = w_1 + 1, w_1, w_1 + 2, \ldots, n, w_1 - 1, w_1 - 2, \ldots, 1$. Since all the symbols greater than w_1 appear in ascending order in w, it follows that $w'\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k} = u$ and $w\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\sigma_1 = u$, where $l(w') + k = l(u)$ and $i_j \neq 1$, $j = 1, \ldots, k$. Consequently,

$$F_w = \left(\prod_{j=1}^{k} \partial_{i_j}\right) \partial_1 F_u$$

and

$$F_{w'} = \left(\prod_{j=1}^{k} \partial_{i_j}\right) F_u.$$

It is easy to check that $F_u = x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}}$, where $a_1 = w_1$ and $a_2 = w_1 - 1$. Hence $\partial_1 F_u = x_1^{a_1 - 1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}} = F_u / x_1$. This implies that

$$\begin{aligned} F_w &= \left(\prod_{j=1}^{k} \partial_{i_j}\right) \frac{F_u}{x_1} \\ &= \frac{1}{x_1} \left(\prod_{j=1}^{k} \partial_{i_j}\right) F_u \qquad \text{since} \quad i_j \neq 1, \\ &= \frac{1}{x_1} F_{w'}. \end{aligned}$$

Now by induction $F_{w'} = S_{\lambda(w')}(b(w'))$. Hence $F_w = (1/x_1)\ S_{\lambda(w')}(b(w'))$. But since

$$\lambda_1(w') = \lambda_1(w) + 1, \qquad \lambda_i(w') = \lambda_i(w), \qquad i = 2, ..., m,$$

and

$$b_1(w') = b_1(w) = 1, \qquad b_i(w') = b_i(w), \qquad i = 2, ..., m,$$

we also have

$$S_{\lambda(w)}(b(w)) = \frac{1}{x_1} S_{\lambda(w')}(b(w')).$$

Hence $F_w = S_{\lambda(w)}(b(w))$.

Case 3. Assume $d = 1$ and $w_1 = n$. Let $w_0 w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$, where $l(w) + k = l(w_0)$. Since $w_1 = n$, it follows that $i_j \neq 1$, $j = 1, ..., k$. Consequently,

$$\begin{aligned} F_w &= \partial_{i_k}\partial_{i_{k-1}}\cdots\partial_{i_1}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}) \\ &= x_1^{n-1}\partial_{i_k}\partial_{i_{k-1}}\cdots\partial_{i_1}(x_2^{n-2}\cdots x_{n-1}). \end{aligned}$$

Let $w' = w_2 w_3 \cdots w_m$. Then w' is a single-shape permutation in $\mathscr{S}_{n-1}$. If f is a polynomial in the variables $x_1, x_2, ..., x_n$ then let $f^\dagger$ be the polynomial obtained from f by replacing x_i by x_{i+1}, $i = 1, 2, ..., n$. We have

$$\partial_{i_k}\partial_{i_{k-1}}\cdots\partial_{i_1}(x_2^{n-2}x_3^{n-3}\cdots x_{n-1}) = (F_{w'})^\dagger.$$

Hence $F_w = x_1^{n-1}(F_{w'})^\dagger$. But since w' is a single shape permutation in $\mathscr{S}_{n-1}$, by induction we have $F_{w'} = S_{\lambda(w')}(b(w'))$. It follows that

$$\begin{aligned} F_w &= x_1^{n-1}(S_{\lambda(w')}(b(w')))^\dagger \\ &= x_1^{n-1}(S_{\lambda_2(w)\lambda_3(w)\cdots\lambda_m(w)}(b_2(w)-1, b_3(w)-1, ..., b_m(w)-1))^\dagger. \end{aligned}$$

By using the tableau expression for the flagged Schur function we can easily see that

$$\begin{aligned} &(S_{\lambda_2(w)\lambda_3(w)\cdots\lambda_m(w)}(b_2(w)-1, b_3(w)-1, ..., b_m(w)-1))^\dagger \\ &\quad = (S_{\lambda_2(w)\lambda_3(w)\cdots\lambda_m(w)}(b_2(w), b_3(w), ..., b_m(w)))^*. \end{aligned}$$

Applying Lemma 1.2 results in $F_w = S_{\lambda(w)}(b(w))$. ∎

Not all flagged Schur functions are Schubert polynomials. (Lascoux and Schutzenberger characterize those that are.) However all flagged Schur functions can be obtained in the same fashion in which the Schubert polynomials are obtained, that is, by applying the symmetrizing operators to some monomial.

THEOREM 2.4. *Every flagged Schur function* $S_\lambda(b)$ *is equal to* $\partial_w(x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m})$, *where* $a_i=\lambda_i+b_i-i$ *and* $w=\sigma_m\sigma_{m+1}\cdots\sigma_{b_m-1}\sigma_{m-1}\sigma_m\cdots\sigma_{b_{m-1}-1}\cdots\sigma_1\sigma_2\cdots\sigma_{b_1-1}$.

Proof. We use induction on the sum of the flags $b_1+b_2+\cdots+b_m$. If the sum of the flags is 1 then $b_1=1$ and $m=1$. Clearly, $S_\lambda(b)=x_1^{\lambda_1}=\partial_e(x_1^{a_1})$. Now suppose the sum of the flags is greater than 1. There are two cases.

Case 1. Assume $b_1>1$. By the induction hypothesis,

$$S_{\lambda_1+1,\lambda_2,\ldots,\lambda_m}(b_1-1,b_2,\ldots,b_m)=\partial_{w\sigma_{b_1-1}}(x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}).$$

By Lemma 1.1, applying ∂_{b_1-1} to both sides of this equation yields,

$$\begin{aligned}S_\lambda(b)&=\partial_{b_1-1}\partial_{w\sigma_{b_1-1}}(x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m})\\&=\partial_w(x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}).\end{aligned}$$

Case 2. Assume $b_1=1$. We have,

$$S_\lambda(b)=x_1^{\lambda_1}(S_{\lambda_2\cdots\lambda_m}(b_2-1,\ldots,b_m-1))^\dagger. \tag{2.3}$$

By the induction hypothesis,

$$S_{\lambda_2\cdots\lambda_m}(b_2-1,\ldots,b_m-1)=\partial_{w'}(x_1^{a_1'}x_2^{a_2'}\cdots x_{m-1}^{a_{m-1}'}), \tag{2.4}$$

where $a_i'=\lambda_{i+1}+b_{i+1}-1-i=a_{i+1}$ and $w'=\sigma_{m-1}\sigma_m\cdots\sigma_{b_m-2}\sigma_{m-2}\sigma_{m-1}\cdots\sigma_{b_{m-1}-2}\cdots\sigma_1\sigma_2\cdots\sigma_{b_2-2}$. Observe that $(\partial_{w'}(x_1^{a_1'}x_2^{a_2'}\cdots x_{m-1}^{a_{m-1}'}))^\dagger=\partial_w(x_2^{a_2}x_3^{a_3}\cdots x_m^{a_m})$. Hence (2.3) and (2.4) combined become

$$S_\lambda(b)=x_1^{\lambda_1}\partial_w(x_2^{a_2}x_3^{a_3}\cdots x_m^{a_m}).$$

Since none of the transpositions involved in the expression for w is equal to σ_1, we have $S_\lambda(b)=\partial_w(x_1^{\lambda_1}x_2^{a_2}x_3^{a_3}\cdots x_m^{a_m})$. Since $\lambda_1=a_1$, we are done. ∎

3. FLAGGED SKEW SCHUR FUNCTIONS

Let $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m>0)$ and $\mu=(\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_m\geqslant 0)$ be two partitions such that $\lambda_i-\mu_i\geqslant 0$ for all $i=1,2,\ldots,m$. Let $a=(a_1,a_2,\ldots,a_m)$ and $b=(b_1,b_2,\ldots,b_m)$ be two sequences of positive integers.

A column-strict *skew* tableau of shape λ/μ and *row flags* a and b is an array T of positive integers t_{ij}, $1\leqslant i\leqslant m$, $\mu_i<j\leqslant\lambda_i$ such that

$a_i \leqslant t_{ij} \leqslant t_{i,j+1} \leqslant b_i$ for all $1 \leqslant i \leqslant m$, $\mu_i < j < \lambda_i$, and $t_{ij} \leqslant t_{i+1,j}$ for all $1 \leqslant i < m$, $\mu_i < j \leqslant \lambda_{i+1}$. We now let $\mathscr{T}(\lambda/\mu, a, b)$ denote the set of all column-strict tableaux of shape λ/μ and row flags a and b. The *row-flagged skew Schur function* $s_{\lambda/\mu}(a, b)$ is defined to be $\sum_{T \in \mathscr{T}(\lambda/\mu, a, b)} M(T)$. Note that this definition does not require $b_i \geqslant a_i$ for all $i = 1,..., n$. For example, the tableau

$$\begin{array}{cccc} & & \boxed{3} & \boxed{4} \\ & & & \\ \boxed{1} & \boxed{2} & & \end{array}$$

is in $\mathscr{T}(\lambda/\mu, a, b)$, where $\lambda = (4, 2, 2)$ $\mu = (2, 2, 0)$, $a = (3, 5, 1)$, $b = (4, 1, 2)$. Also if $\lambda_i = \mu_i$ for all $i = 1, 2,..., m$ then we will adopt the convention that $s_{\lambda/\mu}(a, b) = 1$.

For λ, μ, a, b as described above we define

$$S_{\lambda/\mu}(a, b) = \det|h_{\lambda_i - \mu_j + j - i}(a_j, b_i)|_{i,j, = 1,2,...,m}, \tag{3.1}$$

where $h_d(u, v)$ is the complete homogeneous symmetric function of degree d in the variables $x_u x_{u+1} \cdots x_v$ if $u \leqslant v$ and $d > 0$. If $d = 0$ then $h_d(u, v) = 1$ and if $d < 0$ then $h_d(u, v) = 0$. If $u > v$ and $d \neq 0$ then $h_d(u, v) = 0$.

That flagged Schur functions can be expressed determinantally and in terms of tableaux is a special case of Gessel's result. He actually proves that $S_{\lambda/\mu}(a, b) = s_{\lambda/\mu}(a, b)$ when the flags are increasing. We now present an alternative proof of this result by establishing natural recurrence relations given in the 3 lemmas to follow.

LEMMA 3.1. *Let k be such that $\mu_k \geqslant \lambda_{k+1}$. Then,*

(i) $S_{\lambda/\mu}(a, b) = S_{\hat{\lambda}/\hat{\mu}}(\hat{a}, \hat{b})\ S_{\check{\lambda}/\check{\mu}}(\check{a}, \check{b})$,

(ii) $s_{\lambda/\mu}(a, b) = s_{\hat{\lambda}/\hat{\mu}}(\hat{a}, \hat{b})\ s_{\check{\lambda}/\check{\mu}}(\check{a}, \check{b})$,

where the symbols $\hat{}$ and $\check{}$ *applied to a sequence $t_1, t_2,..., t_m$ denote the subsequences $t_1, t_2,..., t_k$ and $t_{k+1} t_{k+2},..., t_m$, respectively.*

Proof. (i) Since $\mu_k \geqslant \lambda_{k+1}$, it follows that $\mu_j \geqslant \lambda_i$ whenever $j \leqslant k < i$. This implies that $\lambda_i - \mu_j + j - i < 0$ whenever $j \leqslant k < i$. Hence the i, j-entry of the determinant in (3.1) is 0 for all $j \leqslant k < i$. Therefore the determinant can be expressed as a product of the determinants given by $S_{\hat{\lambda}/\hat{\mu}}(\hat{a}, \hat{b})$ and $S_{\check{\lambda}/\check{\mu}}(\check{a}, \check{b})$.

(ii) This follows immediately from the definition of $s_{\lambda/\mu}(a, b)$. ∎

LEMMA 3.2. *Let k be such that $\mu_k < \lambda_k$, $a_k \leqslant b_k$. If $a_k < a_{k+1}$ (or $k = m$) and $\mu_{k-1} > \mu_k$ (or $k = 1$) then,*

(i) $S_{\lambda/\mu}(a, b) = x_{a_k} S_{\lambda/\mu'}(a, b) + S_{\lambda/\mu}(a', b)$,

(ii) $s_{\lambda/\mu}(a, b) = x_{a_k} s_{\lambda/\mu'}(a, b) + s_{\lambda/\mu}(a', b)$,

where the symbol, ′, applied to a sequence denotes adding 1 *to the* kth *element of the sequence.*

Proof. (i) The determinants in the equation are identical except for the kth column. Hence the equation holds if and only if the corresponding equation for the kth column holds, i.e.,

$$\begin{aligned} h_{\lambda_i - \mu_k + k - i}(a_k, b_i) = x_{a_k} h_{\lambda_i - \mu_k - 1 + k - i}(a_k, b_i) \\ + h_{\lambda_i - \mu_k + k - i}(a_k + 1, b_i) \end{aligned} \tag{3.2}$$

holds for all $i = 1, 2, \ldots, m$. This is easily observed to hold. (It is also a special case of (ii).) Note that the conditions on μ_k, λ_k, a_k, b_k were not used in establishing (i).

(ii) $\mathscr{T}(\lambda/\mu, a, b)$ can be partitioned into two sets $\mathscr{T}_1$ which consists of those tableaux in which the first entry of row k is a_k and $\mathscr{T}_2$ which consists of those tableaux in which the first entry of row k is greater than or equal to $a_k + 1$.

By removing the first entry of row k from a tableau in $\mathscr{T}_1$ we obtain a tableau in $\mathscr{T}(\lambda/\mu', a, b)$. Conversely, given any tableau in $\mathscr{T}(\lambda/\mu', a, b)$ we can add a_k to the beginning of row k without violating the column strictness. Indeed since $a_{k+1} > a_k$, all elements of row $k+1$ are strictly greater than a_k. We can conclude that $\sum_{T \in \mathscr{T}_1} M(T) = x_{a_k} s_{\lambda/\mu'}(a, b)$. Since we also have that $\sum_{T \in \mathscr{T}_2} M(T) = s_{\lambda/\mu}(a', b)$, the proof is complete. ∎

LEMMA 3.3. *Let k be such that $a_{k-1} = a_k$, $\mu_{k-1} = \mu_k$. Then*

(i) $S_{\lambda/\mu}(a, b) = S_{\lambda/\mu}(a', b)$,

(ii) $s_{\lambda/\mu}(a, b) = s_{\lambda/\mu}(a', b)$,

where the symbol ′ is as in Lemma 3.2.

Proof. (i) In the determinant for $S_{\lambda/\mu}(a', b)$ we add x_{a_k} times column $k-1$ to column k. The i, k-entry of the new determinant is $h_{\lambda_i - \mu_k + k - i}(a_k + 1, b_i) + x_{a_k} h_{\lambda_i - \mu_{k-1} + k - 1 - i}(a_{k-1}, b_i)$. Since $\mu_{k-1} = \mu_k$ and $a_{k-1} = a_k$, the result follows from (3.2).

(ii) The result is a consequence of the column strictness of the tableaux. ∎

The next lemma establishes the "boundary" conditions that are satisfied by the row-flagged skew Schur functions.

LEMMA 3.4. *Let* $a_i \leqslant a_{i+1}$ *and* $b_i \leqslant b_{i+1}$ *whenever* $\mu_i < \lambda_{i+1}$. *If* $b_k < a_k$ *and* $\mu_k < \lambda_k$ *for some* k *then*

(i) $S_{\lambda/\mu}(a, b) = 0$

(ii) $s_{\lambda/\mu}(a, b) = 0$.

Proof. (i) The proof is by induction on m. If $m = 1$, $S_{\lambda/\mu}(a, b) = h_{\lambda_1 - \mu_1}(a_1, b_1) = 0$. Suppose $m > 1$. If $\mu_i \geqslant \lambda_{i+1}$ for some i then the result follows from Lemma 3.1 and the induction hypothesis. Now suppose $\mu_i < \lambda_{i+1}$ for all i. It follows that $a_i \leqslant a_{i+1}$ and $b_i \leqslant b_{i+1}$ for all i. Hence $b_k < a_k$ implies that $b_i < a_j$ whenever $i \leqslant k \leqslant j$. Similarly $\mu_k < \lambda_k$ implies that $\mu_j < \lambda_i$ whenever $i \leqslant k \leqslant j$. Therefore the i, j-entry of the determinant in (3.1) is 0 for all $i \leqslant k \leqslant j$. It follows that $S_{\lambda/\mu}(a, b) = 0$.

(ii) Since $\mathscr{T}(\lambda/\mu, a, b) = \phi$, the result holds. ∎

THEOREM 3.5. *If* $a_i \leqslant a_{i+1}$ *and* $b_i \leqslant b_{i+1}$, *whenever* $\mu_i < \lambda_{i+1}$, *then* $S_{\lambda/\mu}(a, b) = s_{\lambda/\mu}(a, b)$.

Proof. We prove this by induction on $(m, \lambda - \mu, b - a)$ ordered lexicographically. If $m = 1$ then $S_{\lambda/\mu}(a, b) = h_{\lambda_1 - \mu_1}(a_1, b_1) = s_{\lambda/\mu}(a, b)$.

Suppose $m > 1$. If $\lambda_i - \mu_i = 0$ for some i then $\lambda_{i+1} \leqslant \lambda_i = \mu_i$ and $\lambda_i = \mu_i \leqslant \mu_{i-1}$. Hence we can invoke Lemma 3.1 with $k = i$ or $k = i - 1$. The result now follows by induction.

Now suppose $m > 1$ and $\mu_i < \lambda_i$ for all $i = 1, 2, \ldots, m$. If $b_k - a_k < 0$ for some k then the result is a consequence of Lemma 3.4.

Finally suppose $m > 1$, $\lambda_i - \mu_i > 0$, $b_i - a_i \geqslant 0$ for all $i = 1, 2, \ldots, m$. Let k be such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k < a_{k+1}$ (or $k = m$). If $\mu_{k-1} > \mu_k$ (or $k = 1$) then the result follows from Lemma 3.2 and induction. If $\mu_{k-1} = \mu_k$ then $\lambda_k > \mu_{k-1}$ which means that $a_{k-1} \leqslant a_k$. Hence $a_{k-1} = a_k$. We now invoke Lemma 3.3 and the induction hypothesis to obtain the result. ∎

We now consider column-flagged skew Schur functions. Let $\mathscr{T}^*(\lambda/\mu, a, b)$ be the set of column-strict skew tableaux whose shape is the conjugate of λ/μ and whose column flags are a and b (i.e., the entries of column i are bounded below by a_i and above by b_i). The column-flagged skew Schur function $s^*_{\lambda/\mu}(a, b)$ is defined to be $\sum_{T \in \mathscr{T}^*(\lambda/\mu, a, b)} M(T)$. We also define,

$$S^*_{\lambda/\mu}(a, b) = \det|e_{\lambda_i - \mu_j + j - i}(a_j, b_i)|_{i,j = 1,2,\ldots,m},$$

where $e_d(u, v)$ is the elementary symmetric function of degree d in the variables $x_u, x_{u+1}, \ldots, x_v$ if $v - u + 1 \geqslant d > 0$. If $d = 0$ then $e_d(u, v) = 1$ and if $d < 0$ then $e_d(u, v) = 0$. If $v - u + 1 < d \neq 0$ then $e_d(u, v) = 0$.

We will present a result analogous to Theorem 3.5 for column-flagged skew Schur functions. This result is also obtainable from Gessel's combinatorial construction. There are four lemmas that are analogous to Lemmas 3.1–3.4, whose proofs we will omit.

LEMMA 3.1*. *Let k be such that $\mu_k \geqslant \lambda_{k+1}$. Then,*

(i) $S^*_{\lambda/\mu}(a, b) = S^*_{\hat{\lambda}/\hat{\mu}}(\hat{a}, \hat{b})\ S^*_{\check{\lambda}/\check{\mu}}(\check{a}, \check{b})$

(ii) $s^*_{\lambda/\mu}(a, b) = s^*_{\hat{\lambda}/\hat{\mu}}(\hat{a}, \hat{b})\ s^*_{\check{\lambda}/\check{\mu}}(\check{a}, \check{b})$

where $\hat{}$ *and* $\check{}$ *are as in Lemma* 3.1.

LEMMA 3.2*. *Let k be such that $\mu_k < \lambda_k$, $a_k - \mu_k \leqslant b_k - \lambda_k + 1$. If $a_k - \mu_k < a_{k+1} - \mu_{k+1} + 1$ (or $k = m$) and $\mu_{k-1} > \mu_k$ (or $k = 1$) then,*

(i) $S^*_{\lambda/\mu}(a, b) = x_{a_k} S^*_{\lambda/\mu'}(a', b) + S^*_{\lambda/\mu}(a', b)$,

(ii) $s^*_{\lambda/\mu}(a, b) = x_{a_k} s^*_{\lambda/\mu'}(a', b) + s^*_{\lambda/\mu}(a', b)$,

where $'$ is as in Lemma 3.2.

LEMMA 3.3*. *Let k be such that $a_{k-1} = a_k + 1$, $\mu_{k-1} = \mu_k$. Then*

(i) $S^*_{\lambda/\mu}(a, b) = S^*_{\lambda/\mu}(a', b)$,

(ii) $s^*_{\lambda/\mu}(a, b) = s^*_{\lambda/\mu}(a', b)$,

where $'$ is as in Lemma 3.2.

LEMMA 3.4*. *Let $a_i - \mu_i \leqslant a_{i+1} - \mu_{i+1} + 1$ and $b_i - \lambda_i \leqslant b_{i+1} - \lambda_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$. If $b_k - \lambda_k + 1 < a_k - \mu_k$ and $\mu_k < \lambda_k$ for some k then*

(i) $S^*_{\lambda/\mu}(a, b) = 0$,

(ii) $s^*_{\lambda/\mu}(a, b) = 0$.

THEOREM 3.5*. *If $a_i - \mu_i \leqslant a_{i+1} - \mu_{i+1} + 1$ and $b_i - \lambda_i \leqslant b_{i+1} - \lambda_{i+1} + 1$ whenever $\mu_i < \lambda_{i+1}$ then $S^*_{\lambda/\mu}(a, b) = s^*_{\lambda/\mu}(a, b)$.*

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