ON AN n-ARY GENERALIZATION OF THE LIE REPRESENTATION AND TREE SPECHT MODULES

TAMAR FRIEDMANN, PHIL HANLON, AND MICHELLE L. WACHS[2]

Dedicated to the memory of Adriano Garsia.

ABSTRACT. We continue our study, initiated in our prior work with Richard Stanley, of the representation of the symmetric group on the multilinear component of an n-ary generalization of the free Lie algebra known as the free Filippov n-algebra with k brackets. Our ultimate aim is to determine the multiplicities of the irreducible representations in this representation. This had been done for the ordinary Lie representation ($n = 2$ case) by Kraskiewicz and Weyman. The $k = 2$ case was handled in our prior work, where the representation was shown to be isomorphic to $S^{2^{n-1}1}$. In this paper, for general n and k, we obtain decomposition results that enable us to determine the multiplicities in the $k = 3$ and $k = 4$ cases. In particular we prove that in the $k = 3$ case, the representation is isomorphic to $S^{3^{n-1}1} \oplus S^{3^{n-2}21^2}$. Our main result shows that the multiplicities stabilize in a certain sense when n exceeds k. As an important tool in proving this, we present two types of generalizations of the notion of Specht module that involve trees.

KEYWORDS. Lie algebra, Filippov algebra, representations of the symmetric group, Specht modules.

CONTENTS

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1. Introduction

There are various generalizations of the notion of Lie algebra that involve replacing the binary Lie bracket with an n-ary bracket. One of these was introduced by Filippov in [Fi] and another was introduced by Hanlon and Wachs in [HW]. Both generalizations are of interest in elementary particle physics, see e.g. [CZ, DI, Fr, Ji]. For both the Filippov generalization and the Hanlon-Wachs generalization, the representation of the symmetric group on the multilinear component of the free n-ary Lie algebra generalizes an extensively studied representation known as the Lie representation; see [Ba, BS, Ga, HS, Kl, KW, Re, Wa] for some earlier work on the Lie representation and [Su1, Su2] for some more recent work. The representation for the Hanlon-Wachs generalization was first studied in [HW]. More recently, a study of the representation for the Filippov generalization was initiated by Friedmann, Hanlon, Stanley, and Wachs in [FHSW1]. In this paper, we continue this study, and in particular, answer a question raised in [FHSW1, Question 4.1] in the affirmative.

Throughout this paper, all vector spaces are taken over a field of characteristic 0, say $\mathbb{C}$. Fix $n \geq 2$. A *Filippov algebra* is a vector space $\mathcal{L}$ equipped with an n-linear map (called a bracket)

$$[\cdot,\cdot, \quad ,\cdot] : \times^n \mathcal{L} \to \mathcal{L}$$

that satisfies the following antisymmetry relation for all σ in the symmetric group $\mathfrak{S}_n$:

$$[x_1, \dots, x_n] = \operatorname{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n)}] \tag{1.1}$$

and the following generalization of the Jacobi identity:

$$\begin{aligned}
&[[x_1, x_2, \dots, x_n], x_{n+1}, \dots, x_{2n-1}] \\
&\quad = \sum_{i=1}^{n} [x_1, x_2, \dots, x_{i-1}, [x_i, x_{n+1}, \dots, x_{2n-1}], x_{i+1}, \dots, x_n],
\end{aligned} \tag{1.2}$$

for $x_i \in \mathcal{L}$. When we need to specify n, we say Filippov n-algebra. Clearly Filippov 2-algebras are the same as ordinary Lie algebras. In [FHSW1], Filippov n-algebras are referred to as *Lie algebras of the nth kind* (LAnKe's for short, see [Fr]).

As defined in [FHSW1], the *free Filippov n-algebra* (or *free LAnKe*) on a set $X = \{x_1, x_2, \dots, x_m\}$ is a vector space generated by all possible n-ary bracketings of elements of X, where the only possible relations existing among these bracketings are consequences of n-linearity of the bracketing, the antisymmetry of the bracketing (1.1) and the generalized Jacobi identity (1.2).

The multilinear component of the free Filippov n-algebra on X is the subspace spanned by n-bracketed permutations of X. Every such n-bracketed permutation on X has the same number of brackets, which depends on m, the number of generators. Indeed, if the number of brackets is k then $m = (n-1)k + 1$.

The object we studied in [FHSW1] and continue to study in this paper is the representation of the symmetric group $\mathfrak{S}_{(n-1)k+1}$ on the multilinear component of the free Filippov n-algebra on $(n-1)k+1$ generators. We denote this representation by $\rho_{n,k}$ and note that $\rho_{2,k}$ is the classical Lie representation Lie_{k+1}. It follows from the antisymmetry of the bracket that when $k = 1$, $\rho_{n,k}$ is the sign representation sgn_n, which is the same as the Specht module S^{1^n}. A result of Kraskiewicz and Weyman [KW] on Lie_m gives the decomposition of $\rho_{n,k}$ into Specht modules (or irreducibles) when $n = 2$. Recall that the *major index* of a standard Young tableau T is the number of entries j of T for which $j + 1$ is in a lower row of T than j.

Theorem 1.1 (Kraskiewicz and Weyman [KW]). *For each partition $\lambda \vdash m \geq 2$, the multiplicity of the Specht module S^λ in Lie_m is equal to the number of standard Young tableaux of shape λ and major index congruent to $i \bmod m$, where i is any fixed positive integer relatively prime to m.*

Table 1 summarizes the decompositions of $\rho_{n,k}$ into irreducibles that were previously established or will be established in this paper. The partitions λ in the table stand for Specht modules S^λ.

In [FHSW1], the $k = 2$ column of the table was established with the following result.

Theorem 1.2 ([FHSW1]). *For all $n \geq 2$, the $\mathfrak{S}_{2n-1}$-module $\rho_{n,2}$ is isomorphic to the Specht module $S^{2^{n-1}1}$, whose dimension is the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$.*

In this paper we establish the $k = 3$ column of the table with the following result.

Theorem 1.3. *For all $n \geq 2$, as $\mathfrak{S}_{3n-2}$-modules,*

$$\rho_{n,3} \cong S^{3^{n-1}1} \oplus S^{3^{n-2}21^2}.$$

Consequently, by the hook length formula, $\dim \rho_{n,3} = \dfrac{4}{\prod_{i=1}^{3}(n+i)}\dbinom{3n}{n,n,n}$.

$n\backslash k$	1	2	3	4
2	1^2	21	$31 \oplus 21^2$	$41 \oplus 32 \oplus 31^2 \oplus 2^21 \oplus 21^3$
3	1^3	2^21	$3^21 \oplus 321^2$	$4^21 \oplus 432 \oplus 431^2 \oplus 42^21 \oplus 421^3$ $\oplus 3^21^3 \oplus 32^3$
4	1^4	2^31	$3^31 \oplus 3^221^2$	$4^31 \oplus 4^232 \oplus 4^231^2 \oplus 4^22^21 \oplus 4^221^3$ $\oplus 43^21^3 \oplus 432^3$
$\vdots$				
n	1^n	$2^{n-1}1$	$3^{n-1}1 \oplus 3^{n-2}21^2$	$4^{n-1}1 \oplus 4^{n-2}32 \oplus 4^{n-2}31^2 \oplus 4^{n-2}2^21 \oplus 4^{n-2}21^3$ $\oplus 4^{n-3}3^21^3 \oplus 4^{n-3}32^3$

TABLE 1. Decomposition of $\rho_{n,k}$ for $k \leq 4$.

We also establish the decomposition for $\rho_{n,4}$ given in the table. One can see from the table that for $k \in \{1, 2, 3, 4\}$ and $n \geq k$, increasing n by 1 adds a row of length k to the top of each Young diagram indexing the Specht module in the decomposition of $\rho_{n,k}$. More precisely, for $n \geq 2$ and $k \geq 1$, let $\beta_{n,k}$ be the $\mathfrak{S}_{k(n-1)+1}$-module whose decomposition into irreducibles is obtained from that of $\rho_{n-1,k}$ by adding a row of length k to the top of each indexing Young diagram. Also set $\beta_{1,k} = \rho_{1,k} = S^1$. For $k \in \{1, 2, 3, 4\}$, we have

$$\rho_{n,k} \cong \beta_{n,k} \iff n \geq k. \tag{1.3}$$

The question of whether (1.3) is true in general was raised in [FHSW1]. In this paper we obtain an affirmative answer to one direction with our main result, which is given in the following theorem, and we present a conjecture which refines the other direction. Our main result indicates that $\rho_{n,k}$ stabilizes in a certain sense when n exceeds k or is equal to k.

Theorem 1.4 (Stabilization Theorem)**.** *Let $n \geq k \geq 1$. Then as $\mathfrak{S}_{k(n-1)+1}$-modules,*

$$\rho_{n,k} \cong \beta_{n,k}. \tag{1.4}$$

We also prove the following theorem, which is used in the proof of the Stabilization Theorem and also gives some information on what happens above the $n = k$ diagonal.

Theorem 1.5. *Let $n, k \geq 1$. Then as $\mathfrak{S}_{k(n-1)+1}$-modules,*

$$\rho_{n,k} \cong \beta_{n,k} \oplus \gamma_{n,k}, \tag{1.5}$$

for some $\mathfrak{S}_{k(n-1)+1}$-module $\gamma_{n,k}$ all of whose irreducibles have Young diagrams with at most $k-1$ columns.

The following conjecture refines Conjecture 4.2 of [FHSW1], which addresses the other direction of (1.3).

Conjecture 1.6. *If $n \leq j < k$ then there is an irreducible in $\gamma_{n,k}$ that has exactly j columns.*

Note that the decompositions for $\rho_{n,2}$ and $\rho_{n,3}$, given in Theorems 1.2 and 1.3, respectively, are immediate consequences of the Stabilization Theorem. The decomposition for $\rho_{n,4}$ given in Table 1 also follows immediately from the Stabilization Theorem once the decomposition for the entries $\rho_{2,4}$ and $\rho_{3,4}$ are determined. The former can be computed by applying Theorem 1.1 and the latter is computed in this paper by using Theorem 1.5, the decompositions for $\rho_{2,4}$ and $\rho_{3,3}$, and a computer calculation for $\dim \rho_{3,4}$.

As tools for proving the Stabilization Theorem, we introduce generalizations of the notion of Specht module, called *tree* Specht modules. In the generalizations, we replace the ordinary partitions of Specht modules by Stanley's notion of P-partitions [St1, Sec. 3.15.1], where the poset P is a plane rooted tree. Our tree Specht modules are the same as ordinary Specht modules when P is a path.

Remark 1.7. *Table 1 suggests that it might be true that $\rho_{n,k}$ is multiplicity-free for all n, k. While true for $k \leq 4$, this fails for $k \geq 5$. Indeed, one can use Theorem 1.1 to see that the irreducible $S^{(k-2,2,1)}$ has multiplicity greater than 1 in $\rho_{2,k}$ for $k \geq 5$. Moreover, by Theorem 1.5, the same is true more generally for the irreducible $S^{(k^{n-2},k-2,2,1)}$ in $\rho_{n,k}$ for all $n \geq 2$ and $k \geq 5$. So $\rho_{n,k}$ is multiplicity-free if and only if $k \leq 4$.*

Remark 1.8. *Theorems 1.3 and 1.5 were announced in our extended abstract* [FHSW0] *and in our paper* [FHSW1]*, with a reference to our current paper (which was in preparation at the time of publication of* [FHSW1]*). Proofs of these theorems were posted in the first version of this paper (arXiv:2402.19174v1).*

The decomposition for $\rho_{n,4}$ given in Table 1 was proposed in [FHSW1, Proposition 4.7] *and the proof for $n = 3$ was added in the second version of this paper (arXiv:2402.19174v2) posted on March 19, 2024. The proof for general n was added as a consequence of the $n = 3$ case and Theorem 1.4 in the third version (arXiv:2402.19174v3) posted on October 13, 2024.*

Speculation on Theorem 1.4 (and Theorem 5.6) was raised in [FHSW1, Section 4] *and a proof was announced in a talk by one of the authors at the June 2024 conference "The Many Combinatorial Legacies of Richard P. Stanley"; the slides are available at*

https://www.math.harvard.edu/event/math-conference-honoring-richard-p-stanley/.
Proofs of these theorems, as well as the material on tree Specht modules, were added in the third version (arXiv:2402.19174v3) posted on October 13, 2024.

Motivated by [FHSW1] *and an earlier version of the current paper, Maliakas and Stergiopoulou* [MS] *have recently posted preprints (arXiv:2401.09405, arXiv:2410.06979) presenting proofs of Theorems 1.3 and 5.6, which appear to be very different from our approach.*

This paper is organized as follows. In Section 2, after recalling a presentation for Specht modules appearing in [Fu], we obtain a result on the kernel of a certain linear operator on the induction product of Specht modules. In addition to this result playing an important role in subsequent sections, a new presentation for Specht modules is obtained as a byproduct. In Section 3, we prove Theorem 1.5 and we obtain preliminary results on the submodules of $\rho_{n,k}$ spanned by "combs" and "non-combs". These results are used in a proof of Theorem 1.3 (that does not rely on the Stabilization Theorem), which is given in Section 4. The proof of the Stablilization Theorem is divided into two parts with Part 1 in Section 5 and Part 2 in Section 6. Our main tools in proving the Stabilization Theorem are the tree Specht modules of the first kind introduced in Section 5 and the tree Specht modules of the second kind introduced in Section 6. In Section 7, we establish the decomposition of $\rho_{n,4}$ given in Table 1 and we discuss Conjecture 1.6. A table of notation is provided in Section 8.

2. A linear operator on the induction product of Specht modules

In this section, we obtain a result on a linear operator that will play an important role in subsequent sections. A more immediate application of this result is a presentation for Specht modules that we have not seen in the literature before except for the special case of partitions with two columns [BF], where different techniques were used.

For ease of notation, throughout this paper, we will use the notation $A \bullet B$, for A an $\mathfrak{S}_n$-module and B an $\mathfrak{S}_m$-module, to denote the *induction product*

$$A \bullet B = (A \otimes B) \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}},$$

which is an associative operation that is distributive over direct sum.

2.1. **Preliminaries on Specht modules.** The Specht modules S^λ, where λ is a partition of n, give a complete set of irreducible representations of the symmetric group $\mathfrak{S}_n$ over a field of characteristic 0, say $\mathbb{C}$. They can be constructed as subspaces of the regular representation $\mathbb{C}\mathfrak{S}_n$ or as presentations given in terms of generators and relations, known as Garnir relations. We will need the latter type of construction in this paper.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ be a partition of n. A *Young tableau* of shape λ is a filling of the Young diagram associated with λ with distinct entries from the set $[n] := \{1, 2, \ldots, n\}$. Let $\mathcal{T}_\lambda$ be the set of Young tableaux of shape λ.

To construct the Specht module as a presentation, one can use column tabloids and Garnir relations. Let M^λ be the vector space (over $\mathbb{C}$) generated by $\mathcal{T}_\lambda$ subject only to column relations, which are of the form $t + s$, where $s \in \mathcal{T}_\lambda$ is obtained from $t \in \mathcal{T}_\lambda$ by switching two entries in the same column. Given $t \in \mathcal{T}_\lambda$, let $\bar{t}$ denote the coset of t in M^λ. These cosets, which are called *column tabloids*, generate M^λ. A Young tableau is *column strict* if the entries of each of its columns increase from top to bottom. Clearly, $\{\bar{t} : t \text{ is a column strict Young tableau of shape } \lambda\}$ is a basis for M^λ.

We will use a presentation of S^λ discussed in Fulton [Fu, page 102 (after Ex. 15)]. Suppose λ has m columns, labeled from left to right. The generators are the column tabloids $\bar{t}$, where $t \in \mathcal{T}_\lambda$. The Garnir relations are of the form[1]

$$g_c^t := \bar{t} - \sum_s \bar{s}, \tag{2.1}$$

where $c \in [m-1]$, $t \in \mathcal{T}_\lambda$, and the sum is over all $s \in \mathcal{T}_\lambda$ obtained from $t \in \mathcal{T}_\lambda$ by exchanging any entry of column c with the top entry of the next column.

Example: For

$$t = \begin{array}{|c|c|c|} \hline 1 & 5 & 7 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

[1]In our previous papers [FHSW0, FHSW1, FHW], we denote g_c^t by $g_{c,1}^t$.

we have

$$g_1^t = \overline{\begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 5&1&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1&2&7\\\hline 5&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1&3&7\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1} 4\\\cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 1&4&7\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 5\\\cline{1-1}\end{array}}.$$

Permutations σ in $\mathfrak{S}_n$ act on a tableau t by replacing each entry of t with its image under σ. Note that this action respects the column relations and the Garnir relations.

Proposition 2.1 ([Fu]). *For all $\lambda \vdash n$, the Specht module S^λ is generated by the column tabloids subject only to the Garnir relations g_c^t, where $t \in \mathcal{T}_\lambda$ and $c \in [m-1]$, where m is the number of columns of λ.*

We say that a Specht module S^λ, $\lambda \vdash n$, is contained in an $\mathfrak{S}_n$-module V if S^λ is isomorphic to a submodule of V, and we say that its multiplicity in V is c_λ if c_λ is the largest integer such that the direct sum of c_λ copies of S^λ is isomorphic to a submodule of V. If for each $\lambda \vdash n$, the multiplicity of S^λ in V is c_λ, we write

$$V \cong_{S_n} \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda.$$

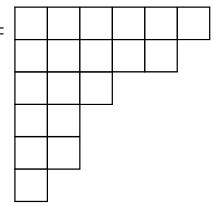
2.2. **The linear operator.** Given partitions λ_1 and λ_2 viewed as Young diagrams, we will say that λ_1 and λ_2 are *compatible* if the length of the last column of λ_1 is greater than or equal to the length of the first column of λ_2. In that case, we let $\lambda_1 +\!\!\!+ \lambda_2$ denote the *concatenation* of λ_1 and λ_2, i.e., the Young diagram obtained by placing the Young diagram λ_2 just to the right of the Young diagram λ_1 and aligning the first rows of both. Note that $\lambda_1 +\!\!\!+ \lambda_2$ is the partition obtained by componentwise addition of λ_1 and λ_2, and that $\lambda_1 +\!\!\!+ \lambda_2 \vdash |\lambda_1| + |\lambda_2|$. Note also that $(\lambda_1 +\!\!\!+ \lambda_2)/\lambda_1 = \lambda_2$.

Example: Let

$$\lambda_1 = \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ & \ \\\hline \ & \ & \ \\\hline \ & \ \\\cline{1-2} \ & \ \\\cline{1-2} \ \\\cline{1-1}\end{array}. \qquad \lambda_2 = \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ \\\cline{1-2}\end{array} \qquad \lambda_3 = \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline \ & \ \\\cline{1-2} \ & \ \\\cline{1-2} \ & \ \\\cline{1-2}\end{array}.$$

Then λ_1 and λ_2 are compatible but λ_1 and λ_3 are not, and so λ_1 and λ_2 can be concatenated but λ_1 and λ_3 cannot be concatenated. The concatenation is

$$\lambda_1 \mathbin{+\!\!\!+} \lambda_2 =$$



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Lemma 2.2. *Let λ_1 and λ_2 be partitions of n_1 and n_2, respectively, and let d be any integer that is greater than or equal to the number of columns of λ_1 viewed as a Young diagram. Define the linear operator $\varphi_d = \varphi_d^{\lambda_1,\lambda_2}$ on the induction product*

$$X := S^{\lambda_1} \bullet S^{\lambda_2}$$

by

$$\varphi_d = dn_2 I - \sum_{\substack{i\in[n_1]\\ j\in[n_1+n_2]\setminus[n_1]}} (i,j),$$

where the action of (i,j) is on the right. Then the eigenvalues of φ_d are nonnegative and

$$\ker(\varphi_d) = \begin{cases} S^{\lambda_1 \mathbin{+\!\!\!+} \lambda_2} & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are compatible and } \lambda_1 \text{ has } d \text{ columns} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Rewrite φ_d as

$$\varphi_d = dn_2 I - \sum_{(i,j)\in\mathfrak{S}_{n_1+n_2}} (i,j) + \sum_{(i,j)\in\mathfrak{S}_{n_1}} (i,j) + \sum_{(i,j)\in\mathfrak{S}_{[n_1+n_2]\setminus[n_1]}} (i,j). \tag{2.2}$$

For any partition λ, let

$$C(\lambda) = \sum_{(r,s)\in\lambda} c_\lambda(r,s),$$

where $c_\lambda(r,s) = s - r$ for the cell in row r and column s of λ. A well-known result from the representation theory of the symmetric groups (see [Ma, Section I7, Example 7]) states that the operator $\sum_{(a,b)\in\mathfrak{S}_n}(a,b)$ acts as multiplication by the scalar $C(\lambda)$ on the irreducible S^λ. Applying this result to the summands of (2.2), we have that φ_d acts as the scalar

$$a_\lambda := dn_2 - C(\lambda) + C(\lambda_1) + C(\lambda_2)$$

on every instance of the irreducible S^λ, $\lambda \vdash n_1 + n_2$, contained in X.

The irreducibles S^λ contained in X and their multiplicities are given by the Littlewood-Richardson rule, see [Fu, Ma, St2]. In particular, any such λ must contain λ_1 and the multiplicity of S^λ in X is the number of Littlewood-Richardson fillings of the skew shape λ/λ_1 of type λ_2. It follows from the containment of λ_1 in λ that

$$a_\lambda = dn_2 + C(\lambda_2) - \sum_{(r,s)\in\lambda/\lambda_1} c_\lambda(r,s).$$

Let $l(\lambda_2)$ be the length of λ_2 and consider the partition $d^{l(\lambda_2)}$, which is compatible with λ_2, and let $\nu = d^{l(\lambda_2)} + \!\!\!+ \, \lambda_2$. Then

$$\sum_{(r,s)\in\nu/d^{l(\lambda_2)}} c_\nu(r,s) = \sum_{(r,s)\in\lambda_2} (c_{\lambda_2}(r,s) + d) = dn_2 + C(\lambda_2).$$

Hence

$$a_\lambda = \sum_{(r,s)\in\nu/d^{l(\lambda_2)}} c_\nu(r,s) - \sum_{(r,s)\in\lambda/\lambda_1} c_\lambda(r,s). \tag{2.3}$$

We will show that the eigenvalue a_λ is nonnegative by constructing a bijection ψ from the set of cells of λ/λ_1 to the set of cells of $\nu/d^{l(\lambda_2)}$ that satisfies

$$c_\lambda(r,s) \le c_\nu(\psi(r,s)) \tag{2.4}$$

for all $(r,s) \in \lambda/\lambda_1$. First fix a Littlewood-Richardson filling of λ/λ_1 of type λ_2. Since the filling is column strict, for each $i \in [l(\lambda_2)]$, the entry i of the filling occurs in $(\lambda_2)_i$ distinct columns of λ/λ_1, where $(\lambda)_i$ denotes the ith part of a partition λ. Let (r,s) be a cell of λ/λ_1. Suppose that (r,s) contains the jth occurrence (from the left) of entry i of the filling and let $\psi(r,s) = (i, d+j)$. It is not difficult to see that ψ is a well-defined bijection from the set of cells of λ/λ_1 to the set of cells of $\nu/d^{l(\lambda_2)}$. Next we show that (2.4) holds.

We use the property of Littlewood-Richardson fillings that the rows contain no entries larger than the row index. This yields $r \ge i$. We claim that $s \le d+j$. Indeed, if $s \le \lambda_1$ then $s \le d < d+j$. On the other hand, let $s > \lambda_1$. It follows from column strictness of Littlewood-Richardson fillings and the property above that $r = i$ and i is the entry of each cell (r,m), $m = \lambda_1 + 1, \ldots, s$. This implies that $j \ge s - \lambda_1$, which implies that $d + j \ge d + s - \lambda_1 \ge s$. Therefore the claim holds in both cases, which implies

$$c_\lambda(r,s) = s - r \le d + j - i = c_\nu(i, d+j) = c_\nu(\psi(r,s)).$$

Thus the inequality (2.4) holds for all (r,s) in λ/λ_1. Looking back at the proof of the inequality, observe that equality in (2.4) holds if and only if $s > \lambda_1$, $j = s - \lambda_1$, and $(\lambda_1)_i = d$, for all $i = 1, \ldots, l(\lambda_2)$, which means that equality holds if and only if $\lambda = \lambda_1 \mathbin{+\!\!\!+} \lambda_2$ and λ_1 has d columns.

Now by (2.3) and (2.4),

$$a_\lambda = \sum_{(r,s)\in\lambda/\lambda_1} (c_\nu(\psi(r,s)) - c_\lambda(r,s)) \geq 0.$$

The above conditions on equality holding in (2.4) allows us to conclude that the eigenvalue a_λ is 0 if and only if $\lambda = \lambda_1 \mathbin{+\!\!\!+} \lambda_2$ and λ_1 has d columns. □

Lemma 2.2 can be used to give a presentation for Specht modules different from Fulton's presentation given in Proposition 2.1. Indeed, let λ be a partition of n with m columns. For $t \in \mathcal{T}_\lambda$ and $c \in \{2, \ldots, m\}$, define the *new* Garnir relation

$$\tilde{g}^t_c := (c-1)l_c\,\bar{t} - \sum_s \bar{s}, \tag{2.5}$$

where l_c is the length of column c and the sum is over all tableaux s that can be obtained from t by switching any element of column c with any element of any previous column. Now let

$$\tilde{S}^\lambda := M^\lambda / \langle \tilde{g}^t_c : t \in \mathcal{T}_\lambda,\, 2 \leq c \leq m\rangle. \tag{2.6}$$

Theorem 2.3. *For all $\lambda \vdash n$,*

$$\tilde{S}^\lambda \cong S^\lambda.$$

Proof. The proof is by induction on the number m of columns of λ. The result is obvious in the base case $m = 1$. Assume that $m \geq 2$ and that the result is true for all partitions with fewer columns. Let λ' be the partition obtained from λ by removing its last column. Since $M^\lambda = \mathrm{sgn}_{l_{c_1}} \bullet \cdots \bullet \mathrm{sgn}_{l_{c_m}} = M^{\lambda'} \bullet \mathrm{sgn}_{l_{c_m}}$, we have

$$\tilde{S}^\lambda \cong (\tilde{S}^{\lambda'} \bullet \mathrm{sgn}_{l_m}) / \langle \tilde{g}^t_m : t \in \mathcal{T}_\lambda\rangle.$$

By the induction hypothesis, $\tilde{S}^{\lambda'} \cong S^{\lambda'}$. Since λ' is clearly compatible with the partition 1^{l_m} and $\lambda = \lambda' \mathbin{+\!\!\!+} 1^{l_m}$, by Lemma 2.2,

$$\tilde{S}^\lambda \cong (S^{\lambda'} \bullet \mathrm{sgn}_{l_m}) / \operatorname{im}\varphi_{m-1} \cong S^\lambda,$$

since $\langle \tilde{g}^t_m : t \in \mathcal{T}_\lambda\rangle = \operatorname{im}\varphi_{m-1}$. □

Remark 2.4. Theorem 2.3 in the special case that λ has just two columns was obtained in [BF] using different techniques.

The following generalization of Lemma 2.2 is an easy consequence of Lemma 2.2.

Lemma 2.5. *For $i = 1, 2$, let Y_i be an $\mathfrak{S}_{n_i}$-module. Let d be an integer that is greater than or equal to the number of columns of every irreducible in Y_1. Define the linear operator $\varphi_d = \varphi_d^{Y_1,Y_2}$ on the induction product*

$$X := Y_1 \bullet Y_2$$

by

$$\varphi_d := dn_2 I - \sum_{\substack{i \in [n_1] \\ j \in [n_1+n_2] \setminus [n_1]}} (i, j),$$

where the action of (i, j) is on the right. Then S^λ is an irreducible in $\ker \varphi_d$ *if and only if λ has the form $\lambda_1 \mathbin{+\!\!\!+} \lambda_2$, where*

(1) S^{λ_i} is an irreducible of Y_i, $i = 1, 2$

(2) λ_1 has d columns and is compatible with λ_2.

Proof. Let S^λ be an irreducible of X. Then S^λ is in the kernel of φ_d if and only if S^λ is in the kernel of the restriction of φ_d to the induction product $S^{\lambda_1} \bullet S^{\lambda_2}$ for some irreducible S^{λ_1} of Y_1, and S^{λ_2} of Y_2. By Lemma 2.2, the kernel of this restriction is $S^{\lambda_1 \mathbin{+\!\!\!+} \lambda_2}$ if λ_1 has d columns and is compatible with λ_2. Otherwise the kernel is 0. Hence λ has the desired form if and only if S^λ is in the kernel of φ_d. $\square$

3. Decomposition results

3.1. **Preliminary results on free Filippov n-algebras.** In Appendix 1 of [DI], a proof that the generalized Jacobi relations (1.2) are equivalent to the relations

$$\begin{aligned}
&[[x_1, x_2, \dots, x_n], y_1, \dots, y_{n-1}] \\
&= \sum_{i=1}^{n} [[x_1, x_2, \dots, x_{i-1}, y_1, x_{i+1}, \dots, x_n], x_i, y_2, \dots, y_{n-1}]
\end{aligned} \tag{3.1}$$

is given. This gives an alternative useful presentation of $\rho_{n,k}$ for all n, k.

An n-bracketed word on a set A is called an *n-comb* (or just *comb*) if it has the form

$$[[\cdots[[a_0, a_{1,1}, \dots, a_{1,n-1}], a_{2,1}, \dots, a_{2,n-1}], \dots], a_{k,1}, \dots, a_{k,n-1}], \tag{3.2}$$

where $a_0, a_{i,j} \in A$.

Proposition 3.1. *Let $Lie_{n,k}(X)$ be the component of the free Filippov n-algebra on a set X spanned by n-bracketed words with k brackets. Then $Lie_{n,k}(X)$ is spanned by its combs.*

Proof. Let w be a bracketed word in the free Filippov n-algebra. We will prove by induction on the number $b(w)$ of brackets of w that w is a linear combination of combs. If $b(w) = 0, 1$ then w is clearly a comb. Now let $b(w) \geq 2$ and $w = [w_1, w_2, \ldots, w_n]$. Define *degree* $d(w) := \max_{i\in[n]} b(w_i)$.

We claim that if w has the highest possible degree $b(w) - 1$ then w is a linear combination of combs. Indeed, in this case all but one w_i is a single letter. Using antisymmetry of the bracket, we can assume without loss of generality that $b(w_1) = b(w) - 1$ and $b(w_i) = 0$ for $i = 2, \ldots, n$. By induction w_1 is a linear combination of combs, which implies by multilinearity of the bracket that w is a linear combination of combs.

Now let $d(w) < b(w) - 1$. We only need to show that w is equal to a linear combination of bracketed words of higher degree. Clearly there are at least two w_i for which $b(w_i) > 0$. . By antisymmetry of the bracket, without loss of generality, we can assume that $0 < b(w_1) \leq b(w_2)$ and that $d(w) = b(w_2)$. Let $w_1 = [u_1, \ldots, u_n]$. Using the alternative generalized Jacobi relation given in (3.1), we have

$$w = \sum_{i=1}^{n} [[u_1, u_2, \ldots, u_{i-1}, w_2, u_{i+1}, \ldots, u_n], u_i, w_3, w_4, \ldots, w_n].$$

Note that the degree of each term is greater than or equal to $b(w_2) + 1 = d(w) + 1$. Thus w is a sum of bracketed words of degree greater than $d(w)$. □

Let $U_{n,k}$ be the multilinear component of the vector space generated by n-combs on $[k(n-1)+1]$ subject only to the anticommuting relations, but not the generalized Jacobi relations. Clearly $U_{n,k}$ is invariant under the action of $\mathfrak{S}_{k(n-1)+1}$ on the multilinear component of the full vector space of bracketed permutations on $[k(n-1)+1]$ subject only to the anticommuting relations. By Proposition 3.1, $\rho_{n,k}$ is a quotient (or submodule) of $U_{n,k}$. Note that as $\mathfrak{S}_{k(n-1)+1}$-modules,

$$U_{n,k} \cong \mathrm{sgn}_n \bullet \mathrm{sgn}_{n-1}^{\bullet(k-1)} . \tag{3.3}$$

As a consequence, by Pieri's rule, we have the following useful result.

Proposition 3.2. *Let λ be a partition of $k(n-1)+1$. If S^λ is in $\rho_{n,k}$ then the number of columns of λ, viewed as a Young diagram, is at most k.*

3.2. **Proof of Theorem 1.5.** Let $\hat{\rho}_{n,k}$ be the subspace of the free Filippov n-algebra on $[k(n-2)+2]$ spanned by bracketed words of type $1^{k(n-2)+1}k$, where the type of a bracketed word is the weak composition $(\mu_1, \mu_2, \dots, \mu_{k(n-2)+2})$ with μ_i equaling the number of i's in the bracketed word. This means that the letters $1, 2, \dots, k(n-2)+1$ each occur once in the bracketed words generating $\hat{\rho}_{n,k}$ and the letter $b := k(n-2)+2$ occurs k times. The symmetric group $\mathfrak{S}_{k(n-2)+1}$ acts on $\hat{\rho}_{n,k}$ by replacing each letter $1, \dots, k(n-2)+1$ in a bracketed word with its image under a permutation in $\mathfrak{S}_{k(n-2)+1}$, leaving b fixed.

We also need an extended definition of Specht modules that allows for repeated entries. We define a *tableau* to be a filling of a shape λ with positive integer entries. The *type* of a tableau t is the weak composition whose ith entry is the number of i's appearing in t. For each partition λ of $k(n-1)+1$, let $\hat{\mathcal{T}}_\lambda$ be the set of tableaux of shape λ and type $1^{k(n-2)+1}k$. The column relations, column tabloids, and Garnir relations on $\hat{\mathcal{T}}_\lambda$ are the same as on $\mathcal{T}_\lambda$ given in Section 2.1. Now let $\hat{S}^\lambda$ be the vector space generated by column tabloids $\bar{t}$, where $t \in \hat{\mathcal{T}}_\lambda$, subject to the Garnir relations g_c^t. The symmetric group $\mathfrak{S}_{k(n-2)+1}$ acts on $\bar{t}$ by replacing each entry $1, \dots, k(n-2)+1$ of t with its image under a permutation in $\mathfrak{S}_{k(n-2)+1}$, leaving b fixed.

In [DI, Proposition 43], the idea of obtaining Filippov $(n-1)$-algebras from Filippov n-algebras is discussed. We use this idea in the proof of the following result.

Lemma 3.3. *Let $n \geq 2$ and $k \geq 1$. Then*

$$\hat{\rho}_{n,k} \cong_{\mathfrak{S}_{k(n-2)+1}} \rho_{n-1,k}, \tag{3.4}$$

and for all $\lambda \vdash k(n-1)+1$ with k columns,

$$\hat{S}^\lambda \cong_{\mathfrak{S}_{k(n-2)+1}} S^{\lambda^-}, \tag{3.5}$$

where if λ is the partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ then λ^- is the partition $(\lambda_2 \geq \cdots \geq \lambda_\ell)$.

Proof of (3.4). The n-ary bracketed words of type $1^{k(n-2)+1}k$ have k occurrences of the letter $b := k(n-1)+2$. Since there are k brackets, by antisymmetry of the brackets, each bracket of a nonvanishing n-ary bracketed word contains exactly one b and the b can be placed at the right end of each bracket. For example,

$$[[[3,1,b],[5,2,b],b],[6,4,b],b]$$

is such a 3-bracketed word of type $1^{5(1)+1}5$. These bracketed words, which we refer to as b-*bracketings*, generate $\hat{\rho}_{n,k}$. The antisymmetry relations reduce to

$$[x_1, \dots, x_{n-1}, b] = \operatorname{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n-1)}, b],$$

where $\sigma \in \mathfrak{S}_{n-1}$, for the b-bracketings, and the generalized Jacobi relations reduce to

$$\begin{aligned}&[[x_1, x_2, \dots, x_{n-1}, b], x_{n+1}, \dots, x_{2n-2}, b]\\ &= \sum_{i=1}^{n-1} [x_1, x_2, \dots, x_{i-1}, [x_i, x_{n+1}, \dots, x_{2n-2}, b], x_{i+1}, \dots, x_{n-1}, b]\end{aligned}$$

for the b-bracketings, since the $i = n$ term of the summation in (1.2) is

$$[x_1, \dots, x_{n-1}, [b, x_{n+1}, \dots, x_{2n-2}, b]],$$

which by antisymmetry is 0. These relations imply all relations for $\hat{\rho}_{n,k}$. Indeed, one can check that any other relation on these generators that comes from (1.2) is trivial. For instance, by antisymmetry of the bracket, (1.2) with $x_{n+1} = x_{2n-1} = b$ yields the relation $0 = 0$ and (1.2) with $x_1 = x_n = b$ yields the relation $0 = [[x_{n+1}, \dots, x_{2n-1}, b], x_2, \dots, x_{n-1}, b] - [[x_{n+1}, \dots, x_{2n-1}, b], x_2, \dots, x_{n-1}, b]$.

Now let $\varphi : \hat{\rho}_{n,k} \to \rho_{n-1,k}$ be the linear map defined by letting $\varphi(w)$ be the $(n-1)$-ary bracketed word obtained from the b-bracketing w by removing all the b's. For example,

$$\varphi([[[3, 1, b], [5, 2, b], b], [6, 4, b], b]) = [[[3, 1], [5, 2]], [6, 4]].$$

We can see that φ respects the antisymmetry relations and the generalized Jacobi relations and is therefore a well-defined linear map. Indeed, φ takes

$$[x_1, \dots, x_{n-1}, b] - \operatorname{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n-1)}, b]$$

to

$$[x_1, \dots, x_{n-1}] - \operatorname{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n-1)}]$$

for all $\sigma \in \mathfrak{S}_{n-1}$ and φ takes

$$\begin{aligned}&[[x_1, x_2, \dots, x_{n-1}, b], x_{n+1}, \dots, x_{2n-2}, b]\\ &- \sum_{i=1}^{n-1} [x_1, x_2, \dots, x_{i-1}, [x_i, x_{n+1}, \dots, x_{2n-2}, b], x_{i+1}, \dots, x_{n-1}, b]\end{aligned}$$

to

$$[[x_1, x_2, \dots, x_{n-1}], x_{n+1}, \dots, x_{2n-2}]$$
$$- \sum_{i=1}^{n-1} [x_1, x_2, \dots, x_{i-1}, [x_i, x_{n+1}, \dots, x_{2n-2}], x_{i+1}, \dots, x_{n-1}].$$

Note that φ can easily be inverted. Hence φ is a vector space isomorphism. It is clear that φ respects the action of $\mathfrak{S}_{k(n-2)+1}$; so it is an $\mathfrak{S}_{k(n-2)+1}$-module isomorphism. □

Proof of (3.5). The proof is similar to the proof of (3.4). The column tabloids $\bar{t}$ of shape λ and type $1^{k(n-2)+1}k$ have k occurrences of $b := k(n-2)+2$. By antisymmetry of the columns, since λ has k columns, each column of t must have exactly one b and the b can be placed at the bottom of the column. We call such a t a b-tableau. Let $\mathcal{T}_\lambda^b$ be the subset of $\hat{\mathcal{T}}_\lambda$ consisting of b-tableaux of shape λ. The column relations on b-tableaux are of the form $t + s$, where s is obtained from t by exchanging two non-b entries in the same column. The column tabloids $\bar{t}$, where $t \in \mathcal{T}_\lambda^b$, generate $\hat{S}^\lambda$. The Garnir relations for these generators reduce to $g_c^t = \bar{t} - \sum_s \bar{s}$, where $t \in \mathcal{T}_\lambda^b$ and the sum is over all s obtained from t by exchanging any element of column c other than the bottom element b with the top element of the next column (provided the next column has length greater than 1).

Given a b-tableau t of shape λ, let t^- be the tableau obtained from t by removing the b from the bottom of each column. Clearly t^- is a tableau of shape λ^-. Now let $\psi : \hat{S}^\lambda \to S^{\lambda^-}$ be the linear map defined on the generating set $\{\bar{t} \mid t \in \mathcal{T}_\lambda^b\}$ by letting $\psi(\bar{t}) = \overline{t^-}$. To see that this is a well-defined $\mathfrak{S}_{k(n-2)+1}$-module homomorphism, one can easily verify that the column relations map to column relations, and the Garnir relation g_c^t maps to the Garnir relation $g_c^{t^-}$, for all $t \in \mathcal{T}_\lambda^b$. The inverse map is easily obtained by adding a b to the bottom of each column of a tableau of shape λ^-. Again since relations map to relations under this map, the inverse is well defined. Hence ψ is an isomorphism. Alternatively, one can see that ψ is an isomorphism, by noting that ψ maps basis elements of $\hat{S}^\lambda$ to basis elements of S^{λ^-}. Indeed, the column tabloids $\bar{t}$, where t is a semistandard b-tableau of shape λ, form a basis for $\hat{S}^\lambda$ and the column tabloids $\bar{t}$, where t is a standard tableau of shape λ^- form a basis for S^{λ^-} □

We need another lemma, which combined with the previous lemma yields Theorem 1.5.

Lemma 3.4. *Let c_λ be the multiplicity of the irreducible $\mathfrak{S}_{k(n-1)+1}$-module S^λ in $\rho_{n,k}$. Then as $\mathfrak{S}_{k(n-2)+1}$-modules*

$$\hat{\rho}_{n,k} \cong \bigoplus_{\substack{\lambda \vdash k(n-1)+1 \\ \lambda \text{ has } k \text{ columns}}} c_\lambda \hat{S}^\lambda.$$

(Here $c_\lambda \hat{S}^\lambda$ denotes the direct sum of c_λ copies of $\hat{S}^\lambda$.)

Proof. Let $B := \{k(n-2)+2, k(n-2)+3, \ldots, k(n-1)+1\}$ and let $\alpha := \sum_{\sigma \in \mathfrak{S}_B} \sigma$. One can see that

$$\alpha(\rho_{n,k} \downarrow^{\mathfrak{S}_{k(n-1)+1}}_{\mathfrak{S}_{k(n-2)+1} \times \mathfrak{S}_B}) \cong \hat{\rho}_{n,k} \otimes 1_{\mathfrak{S}_B}$$

and

$$\alpha(S^\lambda \downarrow^{\mathfrak{S}_{k(n-1)+1}}_{\mathfrak{S}_{k(n-2)+1} \times \mathfrak{S}_B}) \cong \hat{S}^\lambda \otimes 1_{\mathfrak{S}_B},$$

where $1_{\mathfrak{S}_B}$ denotes the trivial representation of $\mathfrak{S}_B$. Now take the restriction of both sides of the decomposition

$$\rho_{n,k} \cong \bigoplus_{\lambda \vdash k(n-1)+1} c_\lambda S^\lambda,$$

from $\mathfrak{S}_{k(n-1)+1}$ to $\mathfrak{S}_{k(n-2)+1} \times \mathfrak{S}_B$ and then take the image under the action of α to get the $\mathfrak{S}_{k(n-2)+1}$-module isomorphism

$$\hat{\rho}_{n,k} \cong \bigoplus_{\lambda \vdash k(n-1)+1} c_\lambda \hat{S}^\lambda.$$

By Proposition 3.2, $c_\lambda = 0$ if λ has more than k columns. On the other hand $\hat{S}^\lambda = 0$ if λ has fewer than k columns. Indeed, if $t \in \hat{\mathcal{T}}_\lambda$ then entry $b := k(n-2)+2$ is repeated k times. If λ has fewer than k columns then entry b will appear more than once in some column of t. By antisymmetry of the columns, the column tabloid $\bar{t}$ is equal to 0. Thus we can limit the summation to λ with exactly k columns. □

We can now easily prove Theorem 1.5, which we restate here. First recall that for $n \geq 2$ and $k \geq 1$, we defined $\beta_{n,k}$ to be the $\mathfrak{S}_{k(n-1)+1}$-module whose decomposition into irreducibles is obtained from the $\mathfrak{S}_{k(n-2)+1}$-module $\rho_{n-1,k}$ by adding a row of length k to the top of each Young diagram in the decomposition of $\rho_{n-1,k}$. Also recall that for $k \geq 1$, we defined $\rho_{1,k}$ to be S^1. For example,

$$\rho_{1,2} = S^1 \text{ and } \beta_{2,2} = S^{21}$$

$$\rho_{2,3} = S^{31} \oplus S^{21^2} \text{ and } \beta_{3,3} = S^{3^21} \oplus S^{321^2}.$$

Theorem 3.5 (Theorem 1.5)**.** *Let $n, k \geq 1$. Then as $\mathfrak{S}_{k(n-1)+1}$-modules,*

$$\rho_{n,k} \cong \beta_{n,k} \oplus \gamma_{n,k}, \tag{3.6}$$

for some $\mathfrak{S}_{k(n-1)+1}$-module $\gamma_{n,k}$ all of whose irreducibles have Young diagrams with at most $k-1$ columns.

Proof. Lemmas 3.4 and 3.3 together imply that as $\mathfrak{S}_{k(n-2)+1}$-modules,

$$\rho_{n-1,k} \cong \bigoplus_{\substack{\lambda \vdash k(n-1)+1 \\ \lambda \text{ has } k \text{ columns}}} c_\lambda S^{\lambda^-},$$

where c_λ is the multiplicity of S^λ in $\rho_{n,k}$. If we add a part of size k to λ^-, we get λ. Thus

$$\beta_{n,k} \cong \bigoplus_{\substack{\lambda \vdash k(n-1)+1 \\ \lambda \text{ has } k \text{ columns}}} c_\lambda S^{\lambda}.$$

The result now follows from Proposition 3.2. □

3.3. **Another decomposition result.** We say that a bracketed permutation $[w_1, \dots, w_n]$ is a *non-comb* if either

- for some $i \neq j \in [n]$, the bracketed permutations w_i and w_j have more than one letter or
- for some $i \in [n]$, the bracketed permutation w_i is a non-comb.

These are the bracketed permutations that cannot be obtained from an n-comb (3.2) by just using the antisymmetry relations and not the generalized Jacobi relations. For $n, k \geq 1$, let $\tilde{\rho}_{n,k}$ be the submodule of $\rho_{n,k}$ spanned by the non-combs in $\rho_{n,k}$.

The next result shows that in order to obtain the decomposition of $\rho_{n,k}$ into irreducibles, we need only focus on the submodule $\tilde{\rho}_{n,k}$. This will play a critical role in the proof of Theorem 1.3 given in Section 4.

Theorem 3.6. *For all $k, n \geq 1$,*

(1) *the multiplicity of $S^{k^{n-1}1}$ in $\rho_{n,k}$ is 1*
(2) $\rho_{n,k} \cong S^{k^{n-1}1} \oplus \tilde{\rho}_{n,k}$
(3) $\tilde{\rho}_{n,k} = 0$ *if and only if* $k = 1, 2$.

Proof. (1) follows immediately from Theorem 3.5 and induction on n. It is clearly true when $n = 1$.

We prove (2) by constructing an $\mathfrak{S}_{k(n-1)+1}$-module isomorphism

$$\psi : S^{k^{n-1}1} \to \rho_{n,k}/\tilde{\rho}_{n,k}.$$

Let t be the tableau of shape $k^{n-1}1$ whose first column from top down is $a_0, a_{1,1}, \dots, a_{1,n-1}$ and whose ith column, where $2 \le i \le k$, from top down is $a_{i,1}, a_{i,2}, \dots, a_{i,n-1}$. We define ψ on generators of $S^{k^{n-1}1}$ by letting

$$\psi(\bar{t}) = [[\cdots[a_0, a_{1,1}, \dots, a_{1,n-1}], a_{2,1}, \dots, a_{2,n-1}], \dots], a_{k,1}, \dots, a_{k,n-1}].$$

We claim that this map induces a well-defined surjective $\mathfrak{S}_{k(n-1)+1}$-module homomorphism, $\psi : S^{k^{n-1}1} \to \rho_{n,k}/\tilde{\rho}_{n,k}$. To see this, first note that the column relations map to antisymmetry relations on brackets. The Garnir relations g_c^t map to relations on combs that are the same as the relations obtained by setting the non-combs equal to 0 in the alternative Jacobi relation (3.1). Indeed, let x_1 in (3.1) be a comb and let $x_2, \dots x_n, y_1, \dots, y_{n-1}$ be single letters, making the left hand side of (3.1) a comb. Then the summands of the right hand side of (3.1) are all combs except for the $i = 1$ term, which is set equal to 0. Hence ψ is well defined. Since, by Proposition 3.1, $\rho_{n,k}/\tilde{\rho}_{n,k}$ is spanned by combs, ψ is surjective.

We claim that $\ker\psi = 0$. Suppose not. Then since $S^{k^{n-1}1}$ is irreducible, $\ker\psi = S^{k^{n-1}1}$. It follows that $\rho_{n,k} = \tilde{\rho}_{n,k}$, which means that the non-combs span $\rho_{n,k}$. This implies that the non-combs of type $1^{k(n-2)+1}k$ span $\hat{\rho}_{n,k}$. Since the b-removing $\mathfrak{S}_{k(n-2)+1}$-module isomorphism $\varphi : \hat{\rho}_{n,k} \to \rho_{n-1,k}$ in the proof of (3.4) takes non-combs of type $1^{k(n-2)+1}k$ to non-combs of type $1^{k(n-2)+1}$, the non-combs span $\rho_{n-1,k}$. Continuing this argument leads to the conclusion that the non-combs span $\rho_{2,k} = \mathrm{Lie}_{k+1}$, which means that $\tilde{\rho}_{2,k} = \rho_{2,k}$.

However, it follows from Pieri's rule that $\tilde{\rho}_{2,k}$ and $\rho_{2,k}$ cannot be equal. Indeed, in any bracketed permutation w of $\rho_{2,k}$, there are three kinds of 2-brackets $[a, b]$, namely, those in which

(1) only one of a and b are single letters in $[k + 1]$
(2) both a and b are single letters in $[k + 1]$
(3) neither a nor b are single letters in $[k + 1]$.

For each $i = 1, 2, 3$, let $j_i(w)$ be the number of brackets in w of the ith kind. Then $\sum_{i=1}^{3} j_i(w) = k$. The cyclic submodule ρ_w of $\rho_{2,k}$ spanned by $\{\sigma w : \sigma \in \mathfrak{S}_{k+1}\}$ is isomorphic to a submodule of the induction product $\mathrm{sgn}_1^{\bullet j_1(w)} \bullet \mathrm{sgn}_2^{\bullet j_2(w)}$. Hence by Pieri's rule, all irreducibles in ρ_w have at most $j_1(w)+j_2(w)$ columns. The bracketed permutation w is a non-comb if and only if $j_3(w) \ge 1$. Thus if w is a non-comb,

$j_1(w) + j_2(w) \leq k-1$. Therefore, since $\tilde{\rho}_{n,k}$ is the sum of the cyclic submodules ρ_w, where w is a non-comb, all irreducibles in $\tilde{\rho}_{n,k}$ have at most $k-1$ columns. It follows that the multiplicity of $S^{k^1 1}$ in $\tilde{\rho}_{2,k}$ is 0, while by Part (1) of this theorem, the multiplicity of $S^{k^1 1}$ in $\rho_{2,k}$ is 1, which means that $\tilde{\rho}_{2,k}$ and $\rho_{2,k}$ cannot be equal. Thus the claim that $\ker \psi = 0$ holds, from which we conclude that ψ is indeed an isomorphism.

To prove (3) first note that when $k = 1$ or $k = 2$, there are no non-combs; so $\tilde{\rho}_{n,k} = 0$. For the converse, let $\tilde{\rho}_{n,k} = 0$. Then by part (2), $\rho_{n,k} = S^{k^{n-1}1}$. By Theorem 3.5, this implies $\rho_{2,k} = \mathrm{Lie}_{k+1} \cong S^{k^1 1}$. We use the Kraskiewicz-Weyman decomposition, Theorem 1.1, to show that $k = 1$ or $k = 2$. Indeed, $\rho_{2,3} = S^{31} \oplus S^{21^2}$. Thus $k \neq 3$. If $k \geq 4$, consider the standard Young tableau of shape $(k-1)2$ whose first row is $1, 2, 4, 5, \ldots, k$ and whose second row is $3, k+1$. Since its major index is $2 + k$, which is congruent to 1 mod $k+1$, we have that $S^{(k-1)^1 2}$ is in $\rho_{2,k}$. Thus $k < 4$. □

Note that Part (2) of the theorem generalizes Theorem 1.2 since there are no non-combs when $k = 2$.

4. The non-comb submodule for $k = 3$

In this section, we prove Theorem 1.3 by studying the submodule $\tilde{\rho}_{n,3}$ of $\rho_{n,3}$ spanned by the non-combs and then applying Theorem 3.6. This result can also be derived as an immediate consequence of results for general k, namely, the Kraskiewicz-Weyman decomposition (Theorem 1.1) and the Stabilization Theorem (Theorem 1.4), which is proved in the next two sections. We include the proof for $k = 3$ here because the proof of the Stabilization Theorem is substantially more difficult and because the $k = 3$ case serves as a warmup for what is to follow. Also this proof is what motivated our proof of the Stabilization Theorem.

Theorem 4.1. *For all $n \geq 2$,*

$$\tilde{\rho}_{n,3} \cong_{\mathfrak{S}_{3n-2}} S^{3^{n-2}21^2}.$$

Proof. Let $V_{n,k}$ be the vector space of n-ary bracketed permutations on $[k(n-1)+1]$ subject only to the anticommuting relations, but not the generalized Jacobi relations. Let $\tilde{V}_{n,k}$ be the subspace of $V_{n,k}$ generated by the bracketed permutations that are non-combs. Clearly $\tilde{V}_{n,k}$ is invariant under the action of $\mathfrak{S}_{k(n-1)+1}$ on the multilinear component of the full vector space $V_{n,k}$.

Now let $k=3$. For each $\alpha \in \mathfrak{S}_{3n-2}$, let v_α be the bracketed permutation (subject only to the anticommuting relations),

$$[[\alpha(1),\dots,\alpha(n)],[\alpha(n+1),\dots,\alpha(2n)],\alpha(2n+1),\dots,\alpha(3n-2)].$$

Clearly $\{v_\alpha : \alpha \in \mathfrak{S}_{3n-2}\}$ forms a of generating set for $\tilde{V}_{n,3}$ and $\sigma \in \mathfrak{S}_{3n-2}$ acts on the generators by $\sigma v_\alpha = v_{\sigma\alpha}$. For an $\mathfrak{S}_i$-module U and an $\mathfrak{S}_j$-module V, let $U[V]$ denote the $\mathfrak{S}_{ij}$-module obtained by inducing the $(\mathfrak{S}_j \wr \mathfrak{S}_i)$-module $U \otimes T^i(V)$, where $\wr$ denotes wreath product, and T^i denotes the ith tensor power. We will refer to $U[V]$ as the *composition product* of U and V. Now note that as $\mathfrak{S}_{3n-2}$-modules,

$$\tilde{V}_{n,3} \cong \mathrm{sgn}_2[\mathrm{sgn}_n] \bullet \mathrm{sgn}_{n-2}\,. \tag{4.1}$$

It is well known (see [Ma, Section I8, Example 9]) that the composition product $\mathrm{sgn}_2[\mathrm{sgn}_n]$ decomposes into irreducibles as follows

$$\mathrm{sgn}_2[\mathrm{sgn}_n] \cong \bigoplus_{\substack{i\in[n]\\ i \text{ odd}}} S^{2^{n-i}1^{2i}}.$$

Hence

$$\tilde{V}_{n,3} \cong \bigoplus_{\substack{i\in[n]\\ i \text{ odd}}} S^{2^{n-i}1^{2i}} \bullet \mathrm{sgn}_{n-2}\,. \tag{4.2}$$

For $\alpha \in \mathfrak{S}_{3n-2}$, we form the relation

$$R_\alpha := 2(n-2)v_\alpha - \sum_{\substack{i\in[2n]\\ j\in[3n-2]\setminus[2n]}} v_{\alpha(i,j)}.$$

Let R be the submodule of $\tilde{V}_{n,3}$ given by

$$R = \langle R_\alpha : \alpha \in \mathfrak{S}_{3n-2}\rangle.$$

Note that R_α is the sum of $2(n-2)$ relations, namely,

$$v_\alpha - \sum_{i=1}^{n} v_{\alpha(i,j)} \qquad \text{and} \qquad v_\alpha - \sum_{i=n+1}^{2n} v_{\alpha(i,j)}, \qquad \text{for each } j = 2n+1,\dots,3n-2.$$

By antisymmetry of the brackets these relations are alternative generalized Jacobi relations (3.1). Consequently,

$$\tilde{\rho}_{n,3} \subsetneq \tilde{V}_{n,3}/R, \tag{4.3}$$

where $U \subsetneq V$ means U is isomorphic to a submodule of V. We will show that the two modules are isomorphic to the irreducible $S^{3^{n-2}21^2}$ (and are therefore equal to each other).

Let τ be the linear operator on $\tilde{V}_{n,3} \cong \mathrm{sgn}_2[\mathrm{sgn}_n] \bullet \mathrm{sgn}_{n-2}$ that takes each generator v_α to R_α. It is not difficult to see that this is a well-defined linear operator that restricts to a linear operator on the direct summands of (4.2). Let τ_i be the restriction of τ to the direct summand $S^{2^{n-i}1^{2i}} \bullet \mathrm{sgn}_{n-2}$. Note that Lemma 2.2 can be applied since τ_i is equal to $\varphi_2^{\lambda_1,\lambda_2}$, where $\lambda_1 = 2^{n-i}1^{2i}$ and $\lambda_2 = 1^{n-2}$. For λ_1 and λ_2 to be compatible, i must equal $0, 1, 2$, or n. Since i is odd, we can eliminate $i = 0, 2$. We can also eliminate $i = n$, since in this case, the number of columns of λ_1 is 1, which is less than $d = 2$. This yields

$$\ker \tau_1 = S^{2^{n-1}1^2 + 1^{n-2}} = S^{3^{n-2}21^2}.$$

and $\ker \tau_i = (0)$ if $i \neq 1$. Hence

$$\ker \tau = \bigoplus_{\text{odd } i \in [n]} \ker \tau_i = \ker \tau_1 = S^{3^{n-2}21^2}. \tag{4.4}$$

We claim that

$$\operatorname{im} \tau = R. \tag{4.5}$$

Indeed, we have $\tau(v_\alpha) = R_\alpha$ for all $\alpha \in \mathfrak{S}_{3n-2}$. Since the v_α generate $\tilde{V}_{n,3}$ and the R_α generate R, the claim holds.

We conclude from this and (4.4) that

$$\tilde{V}_{n,3}/R \cong S^{3^{n-2}21^2}.$$

Now by (4.3) we have that $\tilde{\rho}_{n,3}$ is either trivial or is isomorphic to the irreducible $S^{3^{n-2}21^2}$. The former case is impossible by Part 3 of Theorem 3.6. Therefore $\tilde{\rho}_{n,3} \cong S^{3^{n-2}21^2}$, as desired. $\square$

Now by Theorem 3.6 we have,

Corollary 4.2 (Theorem 1.3). *For all $n \geq 2$, as $\mathfrak{S}_{3n-2}$-modules,*

$$\rho_{n,3} \cong S^{3^{n-1}1} \oplus S^{3^{n-2}21^2}.$$

Consequently, by the hook length formula, $\dim \rho_{n,3} = \dfrac{4}{\prod_{i=1}^{3}(n+i)} \dbinom{3n}{n, n, n}$.

5. Proof of the Stabilization Theorem, Part 1: trees and quotients

In this section we present the first part of the proof of the Stabilization Theorem discussed in the introduction, which we restate here. For $k \geq 1$, set $\beta_{1,k} = \rho_{1,k} = S^1$. For $n \geq 2$ and $k \geq 1$, let $\beta_{n,k}$ be the $\mathfrak{S}_{k(n-1)+1}$-module whose decomposition into irreducibles is obtained by adding a row of length k to the top of each Young diagram in the decomposition of $\rho_{n-1,k}$.

Theorem 5.1 (Theorem 1.4 (Stabilization Theorem))**.** *Let $n \geq k \geq 1$. Then as $\mathfrak{S}_{k(n-1)+1}$-modules,*

$$\rho_{n,k} \cong \beta_{n,k}. \tag{5.1}$$

In light of Theorem 3.5, equation (5.1) is equivalent to the assertion that all the irreducibles contained in $\rho_{n,k}$ have exactly k columns when $n \geq k$. In the previous section, we proved this for $3 = k \leq n$ by constructing the $\mathfrak{S}_{3(n-1)+1}$-module filtration

$$(0) \subset \tilde{\rho}_{n,3} \subset \rho_{n,3},$$

where $\tilde{\rho}_{n,3}$ is the submodule of $\rho_{n,3}$ spanned by the non-combs, and then showing that $\tilde{\rho}_{n,3}$ and $\rho_{n,3}/\tilde{\rho}_{n,3}$ are both irreducibles with exactly 3 columns; see Theorems 4.1 and 3.6. In this section, with considerably more effort, we again employ the idea of working with quotients of submodules generated by bracketed permutations of fixed "shapes".

It is natural to view n-ary bracketed permutations as leaf-labeled plane rooted n-ary trees. Let T be an (unlabeled) plane rooted n-ary tree with k internal nodes and let w be a labeling of the leaves of T with distinct labels from set A, where $|A| = k(n-1)+1$. Then the leaf-labeled tree (T, w) encodes the bracketed permutation $[(T, w)]$ defined recursively as follows. If $k = 0$ then $[(T, w)] = a$, where $A = \{a\}$, and if $k > 0$ then $[(T, w)] = [[(T_1, w_1)], [(T_2, w_2)], \ldots, [(T_n, w_n)]]$ where $(T_1, w_1), (T_2, w_2), \ldots, (T_n, w_n)$ are the subtrees of the root of (T, w) ordered from left to right. We say that T is the *shape* of the bracketed permutation $[(T, w)]$.

For example, in Figure 5.1 two leaf labeled plane rooted ternary trees (T, w) are given with their associated bracketed permutation $[(T, w)]$ beneath them.

For $n, k \geq 1$, let $\mathbb{T}_{n,k}$ be the set of plane rooted n-ary trees with k internal nodes. Note that for the leaf labeled trees (T, w) in Figure 5.1, the unlabeled tree T belongs to $\mathbb{T}_{3,3}$. We start with a simple observation.

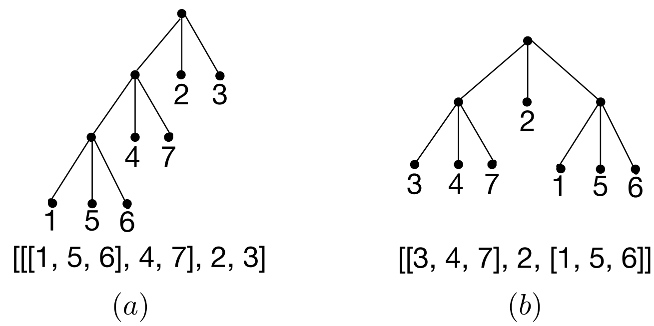


FIGURE 5.1. Leaf labeled ternary trees (T, w) and corresponding bracketed permutations [(T,w)]

Lemma 5.2. *Let $n \geq k \geq 1$ and $T \in \mathbb{T}_{n,k}$. Then every internal node of T has a child that is a leaf. (We call this a* leaf-child*.)*

Proof. Suppose T has an internal node a with no leaf-children. Then the subtree rooted at a has at least $n + 1$ internal nodes. But since k is the total number of internal nodes of T, we have $k \geq n + 1$ which contradicts the hypothesis. $\square$

Let $n, k \geq 1$. For each $T \in \mathbb{T}_{n,k}$, define the *depth vector* of T by

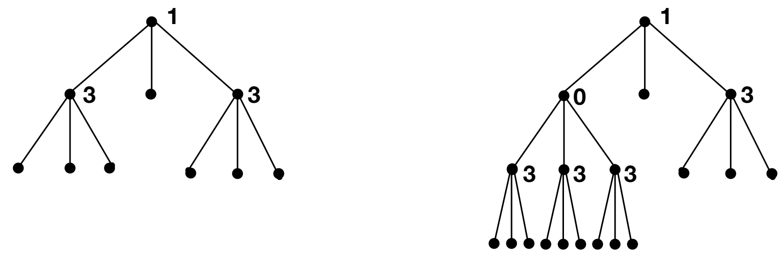
$$\delta(T) = (\delta_1(T), \delta_2(T), \ldots, \delta_{d_T}(T)), \tag{5.2}$$

where $\delta_i(T)$ is the number of leaves of T at depth i and d_T is the depth of T. (The depth of a node is the length of the path from the root to the node, and the depth of T is the maximum depth over all nodes of T). For the leaf labeled tree (T, w) in Figure 5.1 (a), the depth vector of the unlabeled tree T is $\delta(T) = (2, 2, 3)$, and for the tree (T, w) in Figure 5.1 (b), the depth vector of the unlabeled tree T is $\delta(T) = (1, 6)$.

Now for $T \in \mathbb{T}_{n,k}$ and internal node $a \in T$, let $\mu_T(a)$ denote the number of leaf-children of a. We say that $T \in \mathbb{T}_{n,k}$ is *increasing* if for all internal nodes a, b of T such that a is the parent of b,

$$\mu_T(a) \leq \mu_T(b).$$

In other words, when the internal nodes a of T are labeled with $\mu_T(a)$ then we say T is increasing if the labels increase as we read down the tree. Examples of increasing and non-increasing trees are given in Figure 5.2.



(a) increasing (b) non-increasing

FIGURE 5.2. Internal nodes a of T labeled with $\mu_T(a)$

Example 5.3. The *comb tree* $C_{n,k} \in \mathbb{T}_{n,k}$ is defined recursively as follows: if $k = 1$ then $C_{n,k}$ is a root with n children that are leaves and if $k > 1$ then the subtree rooted at the left most child of the root of $C_{n,k}$ is $C_{n,k-1}$ and the other children of the root are leaves. For example, the unlabeled tree T in Figure 5.1 (a) is a comb, while the one in Figure 5.1 (b) is not. Note that $C_{n,k}$ is the shape of the n-comb defined in Section 3.1. Also note that $C_{n,k}$ is increasing, its depth vector $\delta(C_{n,k})$ is $(n-1, n-1, \ldots, n-1, n)$, and $C_{n,k}$ has the lexicographically largest depth vector in $\mathbb{T}_{n,k}$.

Lemma 5.4. *Any bracketed permutation of non-increasing shape $T \in \mathbb{T}_{n,k}$ can be expressed as a sum (in $\rho_{n,k}$) of bracketed permutations whose shape S satisfies $\delta(S) <_{lex} \delta(T)$, where $<_{lex}$ denotes lexicographical order.*

Proof. Assume $[(T, w)]$ is a bracketed permutation such that $T \in \mathbb{T}_{n,k}$ is non-increasing. Then T has internal nodes a and b, where a is the parent of b and $\mu_T(a) > \mu_T(b)$. When we apply the generalized Jacobi relation (1.2) to the leaf-labeled subtree of (T, w) rooted at a, we express $[(T, w)]$ as a sum of bracketed permutations whose respective shapes S are obtained from T by exchanging all the subtrees rooted at the children of a other than b with the subtrees rooted at all but one child x of b. (See Figure 5.3 for an example.) This implies $\mu_S(a) = \mu_T(b) < \mu_T(a)$ if x is an internal node, and $\mu_S(a) = \mu_T(b) - 1 < \mu_T(a)$ if x is a leaf. In either case, $\mu_S(a) < \mu_T(a)$, which implies that $\delta_j(S) < \delta_j(T)$, where $j - 1$ is the depth of a. Since $\delta_i(S) = \delta_i(T)$ for all $i < j$, we have $\delta(S) <_{lex} \delta(T)$ as desired. □

The lemmas above will enable us to reduce the Stabilization Theorem (Theorem 5.1) to the following key lemma. For $T \in \mathbb{T}_{n,k}$, let ρ_T denote the submodule of

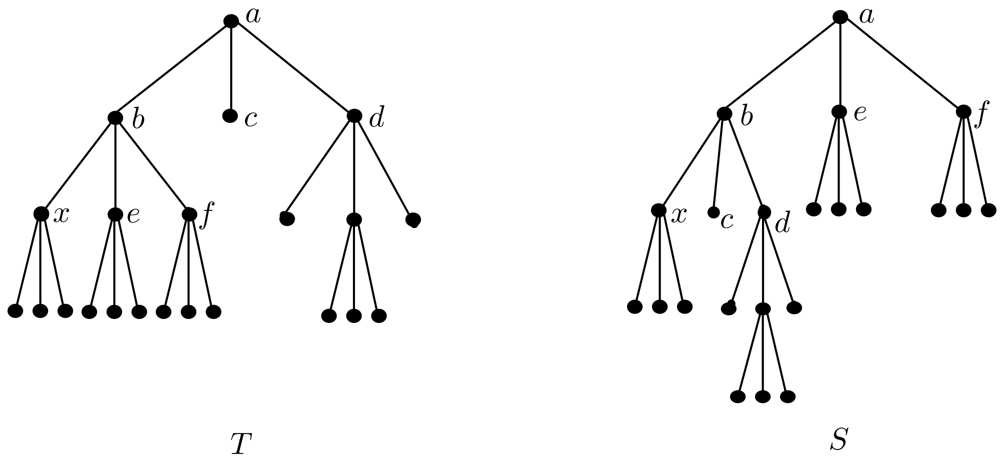


FIGURE 5.3. A tree S obtained from T by applying generalized Jacobi relation (1.2) to (T, w)

$\rho_{n,k}$ spanned by bracketed permutations of shape T. For any subset $\mathcal{S} \subseteq \mathbb{T}_{n,k}$, let $\rho_{\mathcal{S}}$ denote the submodule of $\rho_{n,k}$ spanned by bracketed permutations whose shapes are in $\mathcal{S}$. For $T \in \mathbb{T}_{n,k}$, let

$$D(T) = \{S \in \mathbb{T}_{n,k} : \delta(S) <_{lex} \delta(T)\}. \tag{5.3}$$

Lemma 5.5 (Quotient Lemma)**.** *Let $T \in \mathbb{T}_{n,k}$ be an increasing tree for which each internal node has a leaf-child. Then every irreducible contained in the quotient module $\rho_T/(\rho_T \cap \rho_{D(T)})$ has exactly k columns.*

Section 6 is devoted to the proof of the Quotient Lemma. We now use this lemma to prove the following result, which by Theorem 3.5 is equivalent to the Stabilization Theorem.

Theorem 5.6. *Let $n \geq k \geq 1$. Each irreducible in $\rho_{n,k}$ has exactly k columns.*

Proof. The result is equivalent to the following assertion: For all $T \in \mathbb{T}_{n,k}$, each irreducible in ρ_T has exactly k columns. We prove this assertion by induction on the depth vector $\delta(T)$, where the depth vectors are ordered lexicogaphically. Since by Lemma 5.2, every internal node of $T \in \mathbb{T}_{n,k}$ has a leaf-child, we will be able to apply the Quotient Lemma (Lemma 5.5) to any increasing $T \in \mathbb{T}_{n,k}$.

We start with the $\mathfrak{S}_{k(n-1)+1}$-module isomorphism

$$\rho_T \cong (\rho_T/(\rho_T \cap \rho_{D(T)})) \oplus (\rho_T \cap \rho_{D(T)}). \tag{5.4}$$

The base case handles the trees with minimum depth vector. Let T be such a tree. Then $D(T) = \emptyset$ and by Lemma 5.4, T is increasing. The second summand of (5.4) vanishes and by applying the Quotient Lemma to the first summand, we get the desired result that each irreducible in ρ_T has exactly k columns.

The induction step is handled with two cases. Suppose T does not have minimum depth vector. Then $D(T) \neq \emptyset$.

Case 1: T is increasing. By the Quotient Lemma, all the irreducibles in the first summand of (5.4) have exactly k columns. Since $D(T) \neq \emptyset$, we can apply the induction hypothesis to the trees in $D(T)$, and conclude that all the irreducibles in $\rho_{D(T)}$ have exactly k columns, which implies that all the irreducibles in the second summand of (5.4) have exactly k columns.

Case 2: T is not increasing. By Lemma 5.4, ρ_T is a submodule of a sum $\sum_S \rho_S$, where all the trees S have smaller depth vector. By the induction hypothesis, all the irreducibles of ρ_S have exactly k columns. Hence the same is true for the sum and therefore for ρ_T. $\square$

6. Proof of the Stabilization Theorem, Part 2: tree Specht modules

Recall that in the previous section we reduced the Stabilization Theorem to the Quotient Lemma. In this section we prove the Quotient Lemma by introducing two generalizations of the notion of Specht module. In addition to their application in the current work, we feel they might have some independent interest and be worthy of further study.

6.1. **Tree Specht modules of the first kind.**

Let T be an unlabeled plane rooted tree and let N be a positive integer. (Note that we are not requiring T to be an n-ary tree.) A *T-partition* μ of N is a labeling of the nodes of T with positive integers satisfying:

- $\sum_{a \in T} \mu(a) = N$
- If a is the parent of b then $\mu(a) \leq \mu(b)$.

Note that if we view T as the Hasse diagram of a poset P in which the root is the maximum element then the notion of T-partition of N is the same as Stanley's P-partition of N; see [St1, Sec. 3.15.1]. Note also that the T-partitions when T is a path (i.e., each internal node has exactly one child) are the same as ordinary partitions.

Definition 6.1. Let μ be a T-partition of N. A *μ-tableau* t is a filling of the nodes of T that satisfies

- each node $a \in T$ is filled with a column $Col(a)$ of distinct positive integers of length $\mu(a)$
- the sets of entries of the columns are disjoint and their union is $[N]$

Let $\mathcal{T}_{T,\mu}$ denote the set of μ-tableaux, where μ is a T-partition.

An example of a T-partition μ and a μ-tableau $t \in \mathcal{T}_{T,\mu}$ is given in Figure 6.1. We have oriented the tree from right to left with the root at the far right, instead of the usual top-down orientation with the root at the top. Note that top to bottom order matters in each column.

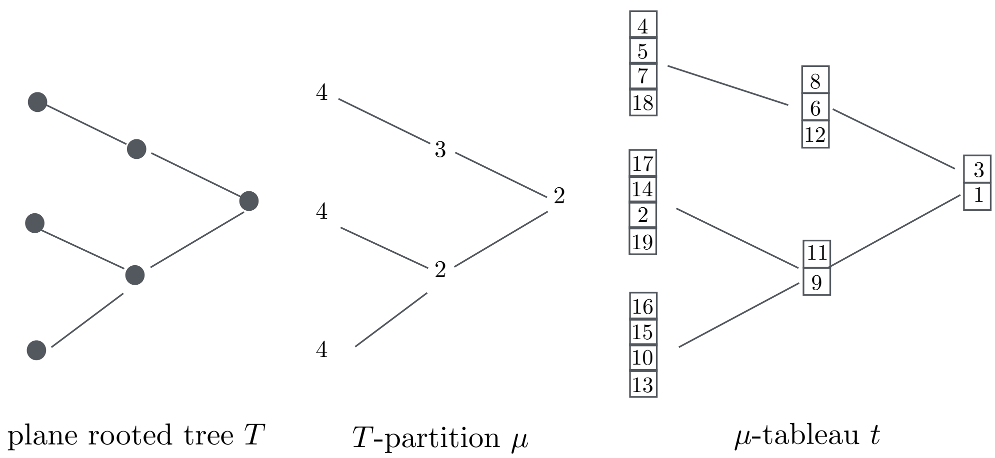


FIGURE 6.1. A tree partition and a tree tableau.

Note also that when T is a path, a T-partition μ is an ordinary partition and a μ-tableau is an ordinary Young tableau of conjugate shape μ^*. Indeed, we can visualize this by orienting the path T horizontally so that the root is the right endpoint. Then the columns of the μ-tableau are precisely the columns of the Young tableau in the same order. Since the sequence of column lengths of the Young tableau is given by μ, the sequence of row lengths (i.e., the shape of the Young tableau) is given by the conjugate μ^*. For example, the T-partition μ and μ-tableau t given by

$$3 \text{ —— } 3 \text{ —— } 1 \qquad\qquad \begin{array}{|c|}\hline 5\\ \hline 3\\ \hline 7\\ \hline\end{array} \text{ —— } \begin{array}{|c|}\hline 6\\ \hline 2\\ \hline 4\\ \hline\end{array} \text{ —— } \begin{array}{|c|}\hline 1\\ \hline\end{array}$$

T-partition μ $\qquad\qquad$ μ-tableau t

correspond, respectively, to the ordinary partition $(3,3,1)$ and the ordinary Young tableau $\begin{array}{|c|c|c|}\hline 5 & 6 & 1\\ \hline 3 & 2 \\ \cline{1-2} 7 & 4 \\ \cline{1-2}\end{array}$ of shape $(3,3,1)^* = (3,2,2)$.

Definition 6.2. Let $M^{T,\mu}$ be the vector space (over $\mathbb{C}$) generated by $\mathcal{T}_{T,\mu}$ subject to the *column relations*, which are of the form $t+s$, where $s \in \mathcal{T}_{T,\mu}$ is obtained from $t \in \mathcal{T}_{T,\mu}$ by switching two elements in the same column $Col(a)$ for any node $a \in T$. Given $t \in \mathcal{T}_{T,\mu}$, let $\bar{t}$ denote the coset of t in $M^{T,\mu}$. We call these cosets *μ-tabloids*.

A μ-tableau is *column strict* if each column is increasing. Clearly

$$\{\bar{t} : t \text{ is a column strict } \mu\text{-tableau}\}$$

is a basis for $M^{T,\mu}$. Note that $\mathfrak{S}_N$ permutes the entries of the μ-tableau $t \in \mathcal{T}_{T,\mu}$. So $M^{T,\mu}$ is an $\mathfrak{S}_N$-module. Note also that if we order the nodes $x_1, \dots, x_{|T|}$ of T in some canonical way, say in preorder, we can view μ as a sequence $(\mu(x_1), \dots, \mu(x_{|T|})$ and $M^{T,\mu}$ as the induction product

$$\operatorname{sgn}_{\mu(x_1)} \bullet \ \cdots \ \bullet \operatorname{sgn}_{\mu(x_{|T|})} .$$

This allows for a right action of $\mathfrak{S}_N$ as well as the left action described above.

Definition 6.3. Let μ be a T-partition of N. The *tree Garnir relations of the first kind* are defined for each $t \in \mathcal{T}_{T,\mu}$ and non-root node a of T by

$$g_a^t := \bar{t} - \sum_s \bar{s}, \tag{6.1}$$

where the sum is over all $s \in \mathcal{T}_{T,\mu}$ obtained from $t \in \mathcal{T}_{T,\mu}$ by exchanging any entry of $Col(a)$ with the top entry of $Col(p(a))$, where $p(a)$ denotes the parent of a in T. Let $G^{T,\mu}$ be the subspace of $M^{T,\mu}$ generated by the tree Garnir relations g_a^t. That is,

$$G^{T,\mu} := \langle g_a^t : a \in T \setminus \{r\},\, t \in \mathcal{T}_{T,\mu} \rangle,$$

where r denotes the root of T. The *tree Specht module of the first kind* $S^{T,\mu}$ is generated by the μ-tabloids subject only to the tree Garnir relations of the first kind, i.e.,

$$S^{T,\mu} = M^{T,\mu}/G^{T,\mu}.$$

Example 6.4. *Let T, μ, t be as in Figure 6.1 and let a be the child of the root of T that has two children. Then the entries of $Col(a)$ are* 11 *and* 9*, and the top of $Col(p(a))$ is* 3*. Thus there are two summands in the summation of (6.1); one is obtained obtained by exchanging* 11 *with* 3 *and the other is obtained by exchanging* 9 *with* 3*. The tree Garnir relation of the first kind g^t_a is given in Figure 6.2.*

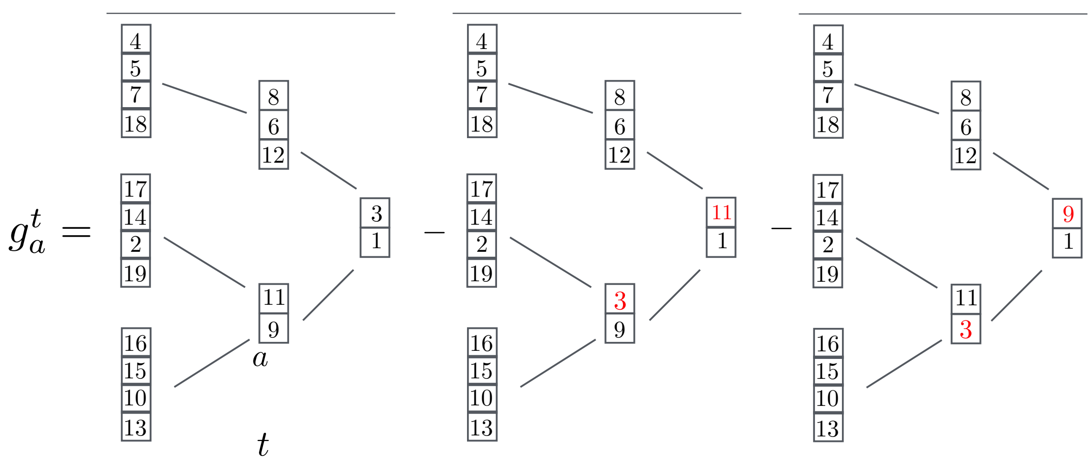


FIGURE 6.2. Tree Garnir relation of the first kind.

Note that when T is a path, the tree Garnir relations of the first kind given in (6.1) are precisely the ordinary Garnir relations given in (2.1). Indeed, as noted above, when T is a path oriented horizontally with the root at the right end, the columns of the tree tableau, ordered from left to right, are the same as the columns of the Young tableau. Thus the presentation defining the tree Specht module of the first kind is Fulton's presentation for the ordinary Specht module given in Proposition 2.1. Therefore if T is a path then for all T-partitions μ,

$$S^{T,\mu} \cong S^{\mu^*}, \tag{6.2}$$

where μ^* is the conjugate of μ viewed as an ordinary partition. Note that the irreducible S^{μ^*} has exactly $|T|$ columns, where $|T|$ is the number of nodes of T and the last column has length $\mu(r)$ where r is the root of T. We generalize this property in the following result, whose proof is delayed to the next subsection.

Theorem 6.5. *Let μ be a T-partition of N. Then every irreducible in the $\mathfrak{S}_N$-module $S^{T,\mu}$ has exactly $|T|$ columns and the last column has length $\mu(r)$ where r is the root of T.*

We now describe the relationship between tree Specht modules and the main subject of this paper, namely $\rho_{n,k}$. Let $n, k \geq 1$ and let T be an increasing tree in $\mathbb{T}_{n,k}$ for which each internal node has a leaf-child. We prune T by removing all its leaves to get the plane rooted tree $\widehat{T}$. The nodes of $\widehat{T}$ are the internal nodes of T. For each such node a, recall that $\mu_T(a)$ denotes the number of leaf-children of a in T. Since each internal node of T has a leaf-child, $\mu_T(a) \geq 1$ for all nodes a of $\widehat{T}$. Since T is increasing, μ_T is a $\widehat{T}$-partition of $k(n-1)+1$. We now associate with each μ_T-tableau $t \in \mathcal{T}_{\widehat{T},\mu_T}$, a leaf labeling w_t of T defined by labeling the ith leaf-child of a (from left to right), where a is an internal node of T and $i \in [\mu_T(a)]$, with the ith entry of $Col(a)$ of t (from top down); see Figure 6.3.

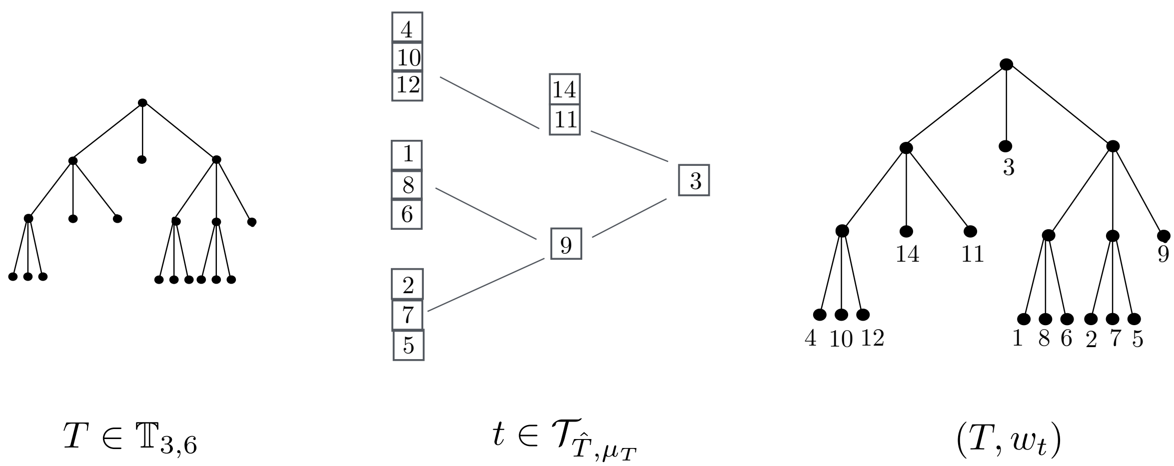


FIGURE 6.3. Leaf labeled tree (T, w_t) associated with μ_T-tableau t.

Lemma 6.6. *Let $T \in \mathbb{T}_{n,k}$ be an increasing tree for which each internal node has a leaf-child. Then the $\mathfrak{S}_{k(n-1)+1}$-module $\rho_T/(\rho_T \cap \rho_{D(T)})$ is isomorphic to a submodule of the tree Specht module $S^{\widehat{T},\mu_T}$.*

Proof. Define the linear map $\varphi : M^{\widehat{T},\mu_T} \to \rho_T$, which takes the μ_T-tabloid $\bar{t}$ to the bracketed permutation $[(T, w_t)]$. It follows from the antisymmetry of the bracket that this is a well-defined $\mathfrak{S}_{k(n-1)+1}$-module homomophism. Note also that the map is surjective.

Now consider the composition

$$\pi \circ \varphi : M^{\widehat{T},\mu_T} \to \rho_T/(\rho_T \cap \rho_{D(T)}),$$

where $\pi : \rho_T \to \rho_T/(\rho_T \cap \rho_{D(T)})$ is the projection map. We will show that the tree Garnir relations g_a^t, where a is a nonroot node of $\widehat{T}$ and $t \in \mathcal{T}_{\widehat{T},\mu_T}$, satisfy

$$g_a^t \in \ker(\pi \circ \varphi). \tag{6.3}$$

Indeed, first note that

$$\varphi(g_a^t) = \varphi(\bar{t} - \sum_s \bar{s}) = \varphi(\bar{t}) - \sum_s \varphi(\bar{s}) = [(T, w_t)] - \sum_u [(T, u)],$$

where the summation index s is as in Definition 6.3 and the final sum is over all u obtained from w_t by switching the label of a leaf-child of a with the label of the left most leaf-child b of the parent $p(a)$ of a. We claim that the alternative generalized Jacobi relation (3.1) applied to the subtree of T rooted at $p(a)$ yields

$$[(T, w_t)] - \sum_u [(T, u)] \in \rho_{D(T)}. \tag{6.4}$$

Indeed, the bracketed permutations on the right hand side of (3.1) that are not one of the $[(T, u)]$'s have shapes S that can be obtained from T by switching the leaf b with the subtree rooted at any nonleaf-child c of a; see Figure 6.4. Clearly, the number of leaves at the depth of b (or a) in T is reduced by this switch, while nothing higher up in the tree changes. Hence $\delta(S) <_{lex} \delta(T)$, which implies (6.4).

Since $[(T, w_t)] - \sum_u [(T, u)]$ is clearly also in ρ_T, we have $\varphi(g_a^t) \in \rho_T \cap \rho_{D(T)}$, which means that (6.3) holds. Hence

$$G^{\widehat{T},\mu_T} \subseteq \ker(\pi \circ \varphi).$$

Since $\rho_T/(\rho_T \cap \rho_{D(T)})$ is the image of $\pi \circ \varphi$, it follows that $\rho_T/(\rho_T \cap \rho_{D(T)})$ is isomorphic to a submodule of $M^{\widehat{T},\mu_T}/G^{\widehat{T},\mu_T} = S^{\widehat{T},\mu_T}$. □

Remark 6.7. *Lemma 6.6 applied to the comb tree $C_{n,k}$ given in Example 5.3 asserts that*

$$\rho_{n,k}/\tilde{\rho}_{n,k} \precsim S^{k^{n-1}1}$$

since $\tilde{\rho}_{n,k} = \rho_{D(C_{n,k})}$ and by Proposition 3.1, $\rho_{n,k} = \rho_{C(n,k)}$. By Theorem 3.6(2), this containment is in fact an isomorphism.

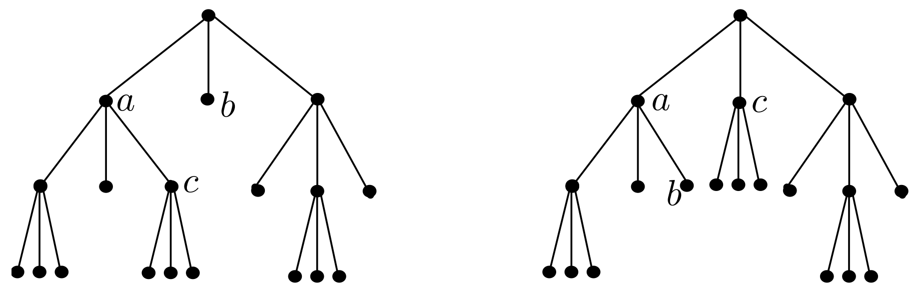


FIGURE 6.4. S obtained from T by switching leaf b with subtree rooted at nonleaf-child c of a.

We now use Theorem 6.5, which will be proved in the next subsection, to prove the Quotient Lemma (Lemma 5.5).

Proof of Lemma 5.5. It follows from Lemma 6.6 that each irreducible of $\rho_T/(\rho_T \cap \rho_{D(T)})$ is an irreducible of $S^{\widehat{T},\mu_T}$. By Theorem 6.5, which we have yet to prove, each irreducible of $S^{\widehat{T},\mu_T}$ has exactly $|\widehat{T}|$ columns. Since $|\widehat{T}|$ is the number of internal nodes of T, which is k, each irreducible of $\rho_T/(\rho_T \cap \rho_{D(T)})$ has exactly k columns. $\square$

6.2. **Tree Specht modules of the second kind.** Our goal in this subsection is to prove Theorem 6.5 which states, in particular, that all irreducibles in $S^{T,\mu}$ have exactly $|T|$ columns. We carry this out by introducing a second generalization $\tilde{S}^{T,\mu}$ of Specht module, and showing that $S^{T,\mu}$ is isomorphic to a submodule of $\tilde{S}^{T,\mu}$ and that all the irreducibles in $\tilde{S}^{T,\mu}$ have exactly $|T|$ columns.

Our second generalization of Specht module differs from the first only in the Garnir relations.

Definition 6.8. Let μ be a T-partition of N. The *tree Garnir relations of the second kind* are defined for each $t \in \mathcal{T}_{T,\mu}$ and internal node a of T by

$$\tilde{g}_a^t := |T_{<a}|\mu(a)\bar{t} - \sum_s \bar{s}, \tag{6.5}$$

where $T_{<a}$ denotes the set of descendants of a in T and the sum is over all $s \in \mathcal{T}_{T,\mu}$ obtained from $t \in \mathcal{T}_{T,\mu}$ by exchanging any entry of $Col(a)$ with any entry of $\bigcup_{b \in T_{<a}} Col(b)$. Let $\tilde{G}^{T,\mu}$ be the subspace of $M^{T,\mu}$ generated by the tree Garnir relations $\tilde{g}_a^t$. That is,

$$\tilde{G}^{T,\mu} := \langle \tilde{g}_a^t : a \text{ an internal node of } T,\ t \in \mathcal{T}_{T,\mu} \rangle. \tag{6.6}$$

The *tree Specht module of the second kind* $\tilde{S}^{T,\mu}$ is generated by the μ-tabloids subject only to the tree Garnir relations of the second kind , i.e.,

$$\tilde{S}^{T,\mu} = M^{T,\mu}/\tilde{G}^{T,\mu}. \tag{6.7}$$

Note that when T is a path, the tree Garnir relations of the second kind given by (6.5) are precisely the new Garnir relations given by (2.5). Indeed, recall the discussion that precedes Definition 6.2. For each node a of T, note that $T_{<a}$ is the set of nodes to the left of a in T oriented horizontally with the root as the right endpoint. Let $c = |T_{<a}| + 1$. Viewing the tree tableau t in $\mathcal{T}_{T,\mu}$ as an ordinary Young tableau, we have that $\mu(a)$ is equal to the length ℓ_c of column c in the Young tableau. Thus (6.5) reduces to (2.5) when T is a path. It follows that the presentation defining the tree Specht module of the second kind is the presentation for the ordinary Specht module given in Theorem 2.3. More precisely if T is a path, then for all T-partitions μ,

$$\tilde{S}^{T,\mu} \cong S^{\mu *}, \tag{6.8}$$

where μ^* is the conjugate of μ viewed as an ordinary partition.

Theorem 6.9. *Let μ be a T-partition of N. Then every irreducible in the $\mathfrak{S}_N$-module $\tilde{S}^{T,\mu}$ has exactly $|T|$ columns and the last column has length $\mu(r)$ where r is the root of T.*

Proof. The result is obvious in the case where $|T| = 1$. So assume $|T| > 1$. Let $T_1, T_2, \ldots, T_m$ be the subtrees rooted at the children of the root r indexed from left to right. The T-partition μ restricted to T_i defines a T_i-partition μ_i. Let $N_i = \sum_{a \in T_i} \mu_i(a)$. Clearly $\sum_{i=1}^m N_i = N - \mu(r)$.

Let Y be the $\mathfrak{S}_{N-\mu(r)}$-module

$$Y = \tilde{S}^{T_1,\mu_1} \bullet \cdots \bullet \tilde{S}^{T_m,\mu_m}.$$

Then

$$\tilde{S}^{T,\mu} \cong (Y \bullet \operatorname{sgn}_{\mu(r)})/\langle \tilde{g}_r^t : t \in \mathcal{T}_{T,\mu} \rangle.$$

Since $\tilde{S}^{T_i,\mu_i}$ is isomorphic to a submodule of M^{T_i,μ_i} for each i, by Pieri's rule, each irreducible in Y has at most $\sum_{i=1}^m |T_i| = |T| - 1$ columns. By the definition of $\varphi_d^{Y_1,Y_2}$

given in Lemma 2.5,

$$\langle \tilde{g}_r^t : t \in \mathcal{T}_{T,\mu} \rangle = \operatorname{im} \varphi_{|T|-1}^{Y,\mathrm{sgn}_{\mu(r)}}.$$

It follows that

$$\tilde{S}^{T,\mu} \cong \ker \varphi_{|T|-1}^{Y,\mathrm{sgn}_{\mu(r)}}.$$

Hence by Lemma 2.5, every irreducible in $\tilde{S}^{T,\mu}$ is of the form $S^{\lambda \# 1^{\mu(r)}}$, where S^λ is an irreducible of Y with exactly $|T|-1$ columns. Therefore the irreducibles of $\tilde{S}^{T,\mu}$ have exactly $|T|$ columns and the last column has length $\mu(r)$. □

Our final result of this section relates the two types of tree Specht modules and thereby completes the proof of the Stabilization Theorem. Recall that $U \lesssim V$ means that U that is isomorphic to a submodule of V.

Theorem 6.10. *Let μ be a T-partition of N. Then $S^{T,\mu} \lesssim \tilde{S}^{T,\mu}$ with isomorphism when T is a path.*

Proof. The isomorphism follows from (6.2) and (6.8). We prove the inclusion by induction on the number of nodes of T and split the argument into two cases.

Case 1: The root r of T has only one child b. Let T' be the tree obtained from T by removing r and let μ' be μ restricted to T'.

We have

$$\tilde{S}^{T,\mu} = (\tilde{S}^{T',\mu'} \bullet \mathrm{sgn}_{\mu(r)})/\langle \tilde{g}_r^t : t \in \mathcal{T}_{T,\mu} \rangle \tag{6.9}$$

By Theorem 6.9, the decomposition of $\tilde{S}^{T',\mu'}$ into irreducibles is of the form,

$$\tilde{S}^{T',\mu'} \cong \bigoplus_{\lambda \in D} \tilde{d}_\lambda S^\lambda, \tag{6.10}$$

where $\tilde{d}_\lambda$ is a nonnegative integer and D is the set all Young diagrams $\lambda \vdash N - \mu(r)$ whose number of columns is $|T|-1$ and last column length is $\mu(b)$. Therefore, since the virtual representations of the symmetric groups form a ring under sum and induction product, by (6.9)

$$\tilde{S}^{T,\mu} \cong \bigoplus_{\lambda \in D} \tilde{d}_\lambda ((S^\lambda \bullet \mathrm{sgn}_{\mu(r)})/\langle \tilde{g}_{|T|}^t : t \in \mathcal{T}_{\lambda \# 1^{\mu(r)}} \rangle),$$

where $\tilde{g}^t_{|T|}$ is as defined in (2.5). It now follows from Theorem 2.3 that

$$\begin{aligned}\tilde{S}^{T,\mu} &\cong \bigoplus_{\lambda\in D} \tilde{d}_\lambda((\tilde{S}^\lambda \bullet \mathrm{sgn}_{\mu(r)})/\langle \tilde{g}^t_{|T|} : t \in \mathcal{T}_{\lambda+1^{\mu(r)}}\rangle).\\ &\cong \bigoplus_{\lambda\in D} \tilde{d}_\lambda \tilde{S}^{\lambda+1^{\mu(r)}}\\ &\cong \bigoplus_{\lambda\in D} \tilde{d}_\lambda S^{\lambda+1^{\mu(r)}}.\end{aligned}$$

By the induction hypothesis and (6.10), we have that

$$S^{T',\mu'} \cong \bigoplus_{\lambda\in D} d_\lambda S^\lambda,$$

for some d_λ satisfying $0 \le d_\lambda \le \tilde{d}_\lambda$. It follows that

$$\begin{aligned}S^{T,\mu} &\cong S^{T',\mu'} \bullet \mathrm{sgn}_{\mu(r)} / \langle g^t_b : t \in \mathcal{T}_{T,\mu}\rangle\\ &\cong \bigoplus_{\lambda\in D} d_\lambda((S^\lambda \bullet \mathrm{sgn}_{\mu(r)})/\langle g^t_{|T|-1} : t \in \mathcal{T}_{\lambda+1^{\mu(r)}}\rangle),\end{aligned}$$

where $g^t_{|T|}$ is as defined in (2.1), since $\lambda \in D$ means λ has $|T|-1$ columns and its last column has length $\mu(b) \ge \mu(r)$. By Fulton's presentation (Proposition 2.1) for Specht modules, we now have

$$S^{T,\mu} \cong \bigoplus_{\lambda\in D} d_\lambda S^{\lambda+1^{\mu(r)}} \subseteq \bigoplus_{\lambda\in D} \tilde{d}_\lambda S^{\lambda+1^{\mu(r)}} \cong \tilde{S}^{T,\mu},$$

which completes the proof for Case 1.

Case 2: The root r of T has children $b_1,\dots,b_m$, where $m \ge 2$. For each i, let T_i be the subtree of T rooted at b_i and let μ_i be the restriction of μ to T_i. Let T_i^r be the subtree $T_i \cup \{r\}$ of T and let μ_i^r be the restriction of μ to T_i^r. Let $G_i^{T,\mu}$ be the submodule of $M^{T,\mu}$ generated by the Garnir relations of the first kind that involve b_i or any of its descendants, i.e.

$$G_i^{T,\mu} = \langle g^t_c : c \in T_i, t \in \mathcal{T}_{T,\mu}\rangle.$$

Clearly

$$\bigoplus_{i=1}^m G_i^{T,\mu} = G^{T,\mu}.$$

It is not difficult to see that

$$M^{T,\mu}/G_i^{T,\mu} \cong M^{T_1,\mu_1} \bullet \cdots \bullet M^{T_{i-1},\mu_{i-1}} \bullet S^{T_i^r,\mu_i^r} \bullet M^{T_{i+1},\mu_{i+1}} \bullet \cdots \bullet M^{T_m,\mu_m}. \tag{6.11}$$

Now for $t \in \mathcal{T}_{T,\mu}$ and internal node $c \in T_i^r$, define

$$\tilde{g}_{c,i}^t := \begin{cases} \tilde{g}_c^t & \text{if } c \neq r \\ |T_i|\mu(r)\bar{t} - \sum_{s\in A_i} \bar{s} & \text{if } c = r, \end{cases}$$

where A_i is the set of μ-tableaux that can be obtained from t by switching any entry of $Col(r)$ with any entry of $Col(a)$ for any node a in T_i other than the root r. Let

$$\tilde{G}_i^{T,\mu} = \langle \tilde{g}_{c,i}^t : c \text{ is an internal node of } T_i^r, t \in \mathcal{T}_{T,\mu}\rangle.$$

Note that

$$M^{T,\mu}/\tilde{G}_i^{T,\mu} \cong M^{T_1,\mu_1} \bullet \cdots \bullet M^{T_{i-1},\mu_{i-1}} \bullet \tilde{S}^{T_i^r,\mu_i^r} \bullet M^{T_{i+1},\mu_{i+1}} \bullet \cdots \bullet M^{T_m,\mu_m}. \tag{6.12}$$

It follows from Case 1 or the induction hypothesis that $S^{T_i^r,\mu_i^r} \precsim \tilde{S}^{T_i^r,\mu_i^r}$. Thus by (6.11) and (6.12),

$$M^{T,\mu}/G_i^{T,\mu} \precsim M^{T,\mu}/\tilde{G}_i^{T,\mu}.$$

This means that $\tilde{G}_i^{T,\mu} \precsim G_i^{T,\mu}$ for all i. We therefore have

$$\bigoplus_{i=1}^m \tilde{G}_i^{T,\mu} \precsim \bigoplus_{i=1}^m G_i^{T,\mu} = G^{T,\mu}. \tag{6.13}$$

We claim that for all $t \in \mathcal{T}_{T,\mu}$ and internal nodes $c \in T$,

$$\tilde{g}_c^t = \begin{cases} \tilde{g}_{c,i}^t & \text{if } \ c \in T_i \\ \sum_{i=1}^m \tilde{g}_{r,i}^t & \text{if } \ c = r. \end{cases}$$

Indeed, the first case follows immediately from the definitions. For the second case, we have

$$\begin{aligned} \sum_{i=1}^m \tilde{g}_{r,i}^t &= \sum_{i=1}^m \left(|T_i|\mu(r)\bar{t} - \sum_{s\in A_i} \bar{s}\right) \\ &= \sum_{i=1}^m |T_i|\mu(r)\bar{t} - \sum_{i=1}^m \sum_{s\in A_i} \bar{s} \\ &= (|T|-1)\mu(r)\bar{t} - \sum_{s\in\bigcup A_i} \bar{s} \\ &= \tilde{g}_r^t \end{aligned}$$

From this claim and (6.13) we can conclude that $\tilde{G}^{T,\mu} \precsim \bigoplus_{i=1}^m \tilde{G}_i^{T,\mu} \precsim G^{T,\mu}$, which implies that

$$S^{T,\mu} = M^{T,\mu}/G^{T,\mu} \precsim M^{T,\mu}/\tilde{G}^{T,\mu} = \tilde{S}^{T,\mu}.$$

Therefore the induction step is complete. □

Proof of Theorem 6.5. This is an immediate consequence of Theorems 6.10 and 6.9. □

6.3. **Recap of proof of Stabilization Theorem.** Let $n \geq k$ and let $T \in \mathbb{T}_{n,k}$ be an increasing tree. By Lemma 5.2 every internal node of T has a leaf-child. From Lemma 6.6 and Theorem 6.10, we have the inclusions

$$\rho_T/(\rho_T \cap \rho_{D(T)}) \subsetneq S^{\widehat{T},\mu_T} \subsetneq \tilde{S}^{\widehat{T},\mu_T},$$

where $\widehat{T}$ is obtained from T by removing its leaves. Since by Theorem 6.9, the irreducibles in $\tilde{S}^{\widehat{T},\mu_T}$ have exactly $|\widehat{T}|$ columns, the same is true for $S^{\widehat{T},\mu_T}$ and for the quotient $\rho_T/(\rho_T \cap \rho_{D(T)})$. Since $|\widehat{T}|$ is equal to the number of internal nodes of T, which is k, the Quotient Lemma (Lemma 5.5) holds. Recall that the only thing left to do in the induction proof of Theorem 5.6, was to prove the Quotient Lemma. Therefore Theorem 5.6 is now proved. This implies that $\gamma_{n,k}$ in Theorem 3.5 is (0), which means that $\rho_{n,k} = \beta_{n,k}$, as asserted by the Stabilization Theorem.

7. The $k = 4$ case and further considerations

7.1. **Decomposition for k=4.** With the Stabilization Theorem now proved and the decomposition for $k = 3$ established, we are also now able to fill in the $k = 4$ column of Table 1. Using a computer program written in C++, we found that $\dim \rho_{3,4} = 1077$. This is used in our proof of the following result.

Theorem 7.1. *For all $n \geq 3$, the $\mathfrak{S}_{4n-3}$-module $\rho_{n,4}$ decomposes as*

$$\rho_{n,4} \cong S^{4^{n-1}1} \oplus S^{4^{n-2}32} \oplus S^{4^{n-2}31^2} \oplus S^{4^{n-2}2^21} \oplus S^{4^{n-2}21^3} \oplus S^{4^{n-3}3^21^3} \oplus S^{4^{n-3}32^3}.$$

Proof. We begin by proving the result for $n = 3$, which is,

$$\rho_{3,4} \cong S^{4^21} \oplus S^{432} \oplus S^{431^2} \oplus S^{42^21} \oplus S^{421^3} \oplus S^{3^21^3} \oplus S^{32^3}. \tag{7.1}$$

From the Kraskiewicz-Weyman decomposition (Theorem 1.1) we obtain

$$\rho_{2,4} \cong S^{4^11} \oplus S^{32} \oplus S^{31^2} \oplus S^{2^21} \oplus S^{21^3}.$$

It follows that

$$\beta_{3,4} \cong S^{4^21} \oplus S^{432} \oplus S^{431^2} \oplus S^{42^21} \oplus S^{421^3}.$$

By the well-known hook length formula (c.f. [Fu, St2]) we have $\dim \beta_{3,4} = 873$. As was mentioned above, our computer calculation gave $\dim \rho_{3,4} = 1077$. Hence by Theorem 3.5,

$$\rho_{3,4} \cong S^{4^21} \oplus S^{432} \oplus S^{431^2} \oplus S^{42^21} \oplus S^{421^3} \oplus \gamma_{3,4},$$

where $\gamma_{3,4}$ is an $\mathfrak{S}_9$-module of dimension 204 that decomposes into irreducibles whose Young diagrams have fewer than 4 columns. We need only show that $\gamma_{3,4} \cong S^{3^21^3} \oplus S^{32^3}$, which by the hook length formula has dimension 204.

By Proposition 3.1, we have that $\rho_{3,4}$ is isomorphic to a submodule of the induction product $\rho_{3,3} \bullet \mathrm{sgn}_2$. Therefore by Theorem 1.3, $\rho_{3,4}$ is isomorphic to a submodule of

$$(S^{3^21} \oplus S^{321^2}) \bullet \mathrm{sgn}_2 \cong S^{3^21} \bullet \mathrm{sgn}_2 \oplus S^{321^2} \bullet \mathrm{sgn}_2\,.$$

By Pieri's rule, this module decomposes into

$$\nu \oplus (S^{3^221} \oplus S^{3^21^3}) \oplus (S^{3^221} \oplus S^{3^21^3} \oplus S^{32^3} \oplus S^{32^21^2} \oplus S^{321^4}),$$

where ν is an $\mathfrak{S}_9$ module whose irreducibles have 4 columns. It follows that $\gamma_{3,4}$ is a submodule of

$$2S^{3^221} \oplus 2S^{3^21^3} \oplus S^{32^3} \oplus S^{32^21^2} \oplus S^{321^4}.$$

By the hook length formula, the dimensions of these irreducibles are $168, 120, 84, 108, 105$, respectively. The only way to get a submodule of dimension 204 from these irreducibles is by taking $S^{3^21^3} \oplus S^{32^3}$. Therefore

$$\gamma_{3,4} = S^{3^21^3} \oplus S^{32^3},$$

which means that (7.1) holds.

The result for general n now follows from the Stabilization Theorem. $\square$

7.2. **Above the diagonal.** The Stabilization Theorem addresses what happens at or below the $n = k$ diagonal. We conclude this section with an observation on what happens above the diagonal and a conjectural converse, which generalizes Conjecture 4.2 of [FHSW1].

Theorem 7.2. *For all $n \leq k$, if $\rho_{n,k}$ has an irreducible with exactly j columns then $n \leq j \leq k$.*

Proof. We prove this by induction on k. Suppose $\rho_{n,k}$ has an irreducible S^λ with exactly j columns. Then by Proposition 3.2, $j \leq k$.

We must now show that $n \le j$. If $k = n$ then by Theorem 5.6, we have $j = k = n$ as desired. So assume $k > n$. Since the combs span $\rho_{n,k}$ (Proposition 3.1), we have that $\rho_{n,k}$ is isomorphic to a submodule of the induction product $\rho_{n,k-1} \bullet \mathrm{sgn}_{n-1}$. It follows that S^λ is in the induction product. Therefore by Pieri's rule, $\rho_{n,k-1}$ has an irreducible with h columns where $h \in \{j-1, j\}$. Since $k > n$, we have $k - 1 \ge n$, which means we can apply the induction hypothesis to $\rho_{n,k-1}$ and conclude that $n \le h \le j$, as desired. □

Conjecture 7.3. *If $n \le j \le k$ then there is an irreducible in $\rho_{n,k}$ that has exactly j columns.*

The conjecture is true when $n = 2$. This can be proved by using the result of Kraskiewicz and Weyman given in Theorem 1.1. We leave this as an exercise for the reader. The conjecture is also true when $1 \le k \le 4$ as one can see from our results displayed Table 1.

8. Table: Selected notation

Notation	**Meaning**	**Section**
$\rho_{n,k}$	The representation of $S_{k(n-1)+1}$ on the multilinear component of the free Filippov n-algebra on $k(n-1)+1$ generators.	1
$\beta_{n,k}$	The $S_{k(n-1)+1}$-module whose irreducible decomposition is obtained from that of $\rho_{n-1,k}$ by adding a row of length k to the top of each indexing Young diagram.	1
$\gamma_{n,k}$	A complementary $S_{k(n-1)+1}$-module in (1.5).	1
$\mathcal{T}_\lambda$	Set of Young tableaux of shape λ.	2.1
g_c^t	Fulton's Garnir relation given in (2.1).	2.1
$\lambda_1 +\!\!\!+ \lambda_2$	Concatenation of compatible Young diagrams.	2.2

Continued on the next page

Notation	**Meaning**	**Section**
$\phi_d^{Y_1,Y_2}$	A linear operator on the induction product $Y_1 \bullet Y_2$ given in Lemmas 2.2 and 2.5.	2.2
$\tilde{g}_c^t$	A new Garnir relation introduced in (2.5).	2.2
$\tilde{S}^\lambda$	The quotient given in (2.6) of the column-tabloid module M^λ by the submodule generated by the $\tilde{g}_c^t$.	2.2
$\hat{\rho}_{n,k}$	The subspace of the Fillipov n-algebra spanned by bracketed words of type $1^{k(n-2)+1}k$.	3.2
$\hat{\mathcal{T}}_\lambda$	The set of tableaux of shape λ and type $1^{k(n-2)+1}k$	3.2
$\hat{S}^\lambda$	The analogue of S^λ generated by tableaux of shape λ and type $1^{k(n-2)+1}k$.	3.2
$\tilde{\rho}_{n,k}$	The submodule of $\rho_{n,k}$ spanned by the non-combs.	3.3
$\tilde{V}_{n,k}$	The submodule of the anticommuting bracket space $V_{n,k}$ spanned by the non-combs.	4
$[(T,w)]$	The bracketed permutation associated with a tree $T \in \mathbb{T}_{n,k}$ and a bijective labeling w of its leaves.	5
$\mathbb{T}_{n,k}$	Set of plane rooted n-ary trees with k internal nodes	5
$\delta(T)$	The depth vector of $T \in \mathbb{T}_{n,k}$ defined in (5.2).	5
$\mu_T(a)$	Number of leaf-children of internal node a of $T \in \mathbb{T}_{n,k}$.	5
ρ_T	Submodule of $\rho_{n,k}$ spanned by bracketed permutations of shape $T \in \mathbb{T}_{n,k}$.	5
$\rho_{\mathcal{S}}$	For $\mathcal{S} \subseteq \mathbb{T}_{n,k}$, the submodule spanned by bracketed permutations whose shapes belong to $\mathcal{S}$.	5

Continued on the next page

Notation	**Meaning**	**Section**
$D(T)$	Set of trees in $\mathbb{T}_{n,k}$ whose depth vectors are lexicographically smaller than that of T, cf. (5.3).	5
$\mathcal{T}_{T,\mu}$	Set of μ-tableaux given in Definition 6.1.	6.1
$M^{T,\mu}$	The S_N-module generated by $\mathcal{T}_{T,\mu}$ subject only to the column relations, as given in Definition 6.2.	6.1
$G^{T,\mu}$	The submodule of $M^{T,\mu}$ generated by the tree Garnir relations g_a^t of the first kind, as given in Definition 6.3.	6.1
$S^{T,\mu}$	The tree Specht module of the first kind, $S^{T,\mu} = M^{T,\mu}/G^{T,\mu}$ as given in Definition 6.3	6.1
$\widehat{T}$	The plane rooted tree obtained from $T \in \mathbb{T}_{n,k}$ by pruning, that is, by removing all the leaves of T.	6.1
$\tilde{G}^{T,\mu}$	The submodule of $M^{T,\mu}$ generated by the tree Garnir relations $\tilde{g}_a^t$ of the second kind, cf. (6.5), 6.6).	6.2
$\tilde{S}^{T,\mu}$	The tree Specht module of the second kind, $\tilde{S}^{T,\mu} = M^{T,\mu}/\tilde{G}^{T,\mu}$, cf. (6.7).	6.2

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Department of Mathematics, Colby College
Email address: tfriedma@colby.edu

Department of Mathematics, Dartmouth College
Email address: philip.j.hanlon@dartmouth.edu

Department of Mathematics, University of Miami
Email address: wachs@math.miami.edu