

EULERIAN QUASISYMMETRIC FUNCTIONS AND POSET TOPOLOGY

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ABSTRACT. We introduce a family of quasisymmetric functions called *Eulerian quasisymmetric functions*, which have the property of specializing to enumerators for the joint distribution of the permutation statistics, major index and excedance number on permutations of fixed cycle type. This family is analogous to a family of quasisymmetric functions that Gessel and Reutenauer used to study the joint distribution of major index and descent number on permutations of fixed cycle type. Our central result is a formula for the generating function for the Eulerian quasisymmetric functions, which specializes to a new and surprising q -analog of a classical formula for the exponential generating function of the Eulerian polynomials. This q -analog computes the joint distribution of excedance number and major index, the only of the four important Euler-Mahonian distributions that had not yet been computed. Our study of the Eulerian quasisymmetric functions also yields results that include the descent statistic and refine results of Gessel and Reutenauer. We also obtain q -analogs, (q, p) -analogs and quasisymmetric function analogs of classical results on the symmetry and unimodality of the Eulerian polynomials. Our Eulerian quasisymmetric functions refine symmetric functions that have occurred in various representation theoretic and enumerative contexts such as in MacMahon's study of multiset derangements, in work of Procesi and Stanley on toric varieties of Coxeter complexes and in Stanley's work on symmetric chromatic polynomials. Here we present yet another occurrence in connection with the homology of a poset introduced by Björner and Welker.

Date: May 15, 2008.

2000 Mathematics Subject Classification. 05A30, 05E05, 05E25.

¹Supported in part by NSF Grants DMS 0300483 and DMS 0604233, and the Mittag-Leffler Institute.

²Supported in part by NSF Grants DMS 0302310 and DMS 0604562, and the Mittag-Leffler Institute.

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0.1. INTRODUCTION

Through our study of the homology of a certain partially ordered set introduced by Björner and Welker, we have discovered a remarkable q -analog of a classical formula for the Eulerian polynomials. The Eulerian polynomials enumerate permutations according to their number of descents or their number of excedances. Our q -Eulerian polynomials are the enumerators for the joint distribution of the excedance statistic and the major index. There is a vast literature on q -Eulerian polynomials that involves other combinations of Eulerian and Mahonian permutation statistics, but ours is the first result to address the combination of excedance number and major index. Although poset topology led us to conjecture our formula, it is symmetric function theory that provides the proof of our original formula, as well as of more refined versions involving additional permutation statistics.

Part 1 of this paper deals only with our permutation statistic and symmetric function theoretic results. In Part 2, results of Part 1 are used to compute the homology of a q -analog of the Björner and Welker poset and the character of the symmetric group acting on the homology of the Björner-Welker poset.

0.1.1. Permutation Statistics. The modern study of permutation statistics began with the work of Major Percy MacMahon [36, Vol. I, pp. 135, 186; Vol. II, p. viii], [37]. It deals with the enumeration of permutations according to natural statistics. A permutation statistic is simply a function from the union of all the symmetric groups \mathfrak{S}_n to the set of nonnegative integers. MacMahon studied four fundamental permutation statistics, the inversion index (inv), the major index (maj), the descent number (des), and the excedance number (exc), which are defined in Section 1.1.

MacMahon observed in [36, Vol. I, p. 186] the now well known result that the statistics des and exc are equidistributed, that is,

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}.$$

The coefficients of the polynomials $A_n(t)$ were studied by Euler, and are called *Eulerian numbers*. The polynomials are known as *Eulerian polynomials*. (Note that it is common in the literature to define the Eulerian polynomials to be $tA_n(t)$.) Any permutation statistic that is equidistributed with des and exc is called an *Eulerian statistic*. Eulerian numbers and polynomials have been extensively studied (see for example [23] and [35]). Euler proved (see [35, p. 39]) the generating function formula

$$(0.1.1) \quad 1 + \sum_{n \geq 1} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}.$$

For a positive integer n , the polynomials $[n]_q$ and $[n]_q!$ are defined as

$$[n]_q := 1 + q + \dots + q^{n-1}$$

and

$$[n]_q! := \prod_{j=1}^n [j]_q.$$

MacMahon proved in [37] the first equality in the equation

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q! = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)}$$

after the second equality had been obtained in [44] by Rodrigues. A permutation statistic that is equidistributed with maj and inv is called a *Mahonian* statistic.

Much effort has been put into the examination of joint distributions of pairs of permutation statistics, one Eulerian and one Mahonian (for a sample see, [2, 3, 13, 15, 21, 24, 25, 27, 28, 29, 30, 31, 41, 42, 48, 50, 51, 62]). One beautiful result on such joint distributions is found

in the paper [51] of Stanley. For permutation statistics f_1, \dots, f_k and a positive integer n , define the polynomial

$$A_n^{f_1, \dots, f_k}(t_1, \dots, t_k) := \sum_{\sigma \in \mathfrak{S}_n} t_1^{f_1(\sigma)} t_2^{f_2(\sigma)} \dots t_k^{f_k(\sigma)}.$$

Also, set

$$A_0^{f_1, \dots, f_k}(t_1, \dots, t_k) := 1.$$

Stanley showed that if we define

$$\text{Exp}_q(z) := \sum_{n \geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}$$

then we have the q -analogue

$$\sum_{n \geq 0} A_n^{\text{inv}, \text{des}}(q, t) \frac{z^n}{[n]_q!} = \frac{1 - t}{\text{Exp}_q(z(t - 1)) - t}$$

of (0.1.1).

Although there has been much work on the joint distributions of (maj, des), (inv, exc) and (inv, des) there are to our knowledge no results about $A_n^{\text{maj}, \text{exc}}(q, t)$ in the existing literature prior to [46], where our work was first announced. Our main result on these polynomials is as follows. Set

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}.$$

Theorem 0.1.1. *We have*

$$(0.1.2) \quad \sum_{n \geq 0} A_n^{\text{maj}, \text{exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1 - tq) \exp_q(z)}{\exp_q(z tq) - tq \exp_q(z)}.$$

It is well-known that $\exp_q(z)$ is a specialization (called the stable principal specialization) of the symmetric function

$$H(z) = H(\mathbf{x}, z) := \sum_{n \geq 0} h_n(\mathbf{x}) z^n,$$

where h_n is the complete homogeneous symmetric function of degree n (details are given in Section 1.1). Hence the right hand side of (0.1.2) is the stable principal specialization of the symmetric function

$$\frac{(1 - tq)H(z)}{H(z tq) - tqH(z)}.$$

We introduce a family of quasisymmetric functions $Q_{n,j,k}(\mathbf{x})$, called Eulerian quasisymmetric functions, whose generating function

$$\sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n$$

specializes to

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)} r^{\text{fix}(\sigma)} \frac{z^n}{[n]_q!},$$

where $\text{fix}(\sigma)$ is the number of fixed points of σ . The $Q_{n,j,k}$ are defined as sums of fundamental quasisymmetric functions that we associate (in a nonstandard way) with permutations $\sigma \in \mathfrak{S}_n$ satisfying $\text{exc}(\sigma) = j$ and $\text{fix}(\sigma) = k$. Our central result is the following theorem.

Theorem 0.1.2. *We have*

$$(0.1.3) \quad \sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

$$(0.1.4) \quad = \frac{H(rz)}{1 - \sum_{n \geq 2} t[n-1]_t h_n z^n}.$$

The proof of Theorem 0.1.2 appears in Section 1.2. It depends on combinatorial bijections and involves nontrivial extension of techniques introduced by Gessel and Reutenauer in [30], Désarménien and Wachs in [17] and Stembridge in [56]. Gessel and Reutenauer construct a bijection from multisets of primitive circular words of fixed content to permutations in order to enumerate permutations with a given descent set and cycle type. This bijection, which is related to Stanley's theory of P-partitions [50], has also proved useful in papers of Désarménien and Wachs [16, 17], Diaconis, McGrath and Pitman [18], Hanlon [32], and Wachs [61]. Reiner [43] introduced a type B analog. Here we introduce a bicolored version of the Gessel-Reutenauer bijection.

By specializing Theorem 0.1.2 we get the following strengthening of Theorem 0.1.1.

Corollary 0.1.3. *We have*

$$(0.1.5) \quad \sum_{n \geq 0} A_n^{\text{maj,exc,fix}}(q, t, r) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(rz)}{\exp_q(z tq) - tq \exp_q(z)}$$

By setting $t = 1$ in (0.1.5) one obtains a formula of Gessel and Reutenauer [30]. By setting $r = 0$, we obtain a new result on the (maj, exc)-enumerator of derangements.

We will show that a different specialization (the nonstable principal specialization) of Theorem 0.1.2 readily yields a further extension of Theorem 0.1.1 that Foata and Han obtained after seeing a preprint containing our work [46].

Corollary 0.1.4 (Foata and Han [22]). *We have*

$$\sum_{n \geq 0} A_n^{\text{maj,des,exc,fix}}(q, p, t, r) \frac{z^n}{(p; q)_{n+1}} = \sum_{m \geq 0} p^m \frac{(1 - qt)(z; q)_m (ztq; q)_m}{((z; q)_m - tq(ztq; q)_m)(zr; q)_{m+1}},$$

where

$$(a; q)_n := \begin{cases} 1 & \text{if } n = 0 \\ (1 - a)(1 - aq) \dots (1 - aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

Essential to our proof of Theorem 0.1.2 is a refinement of $Q_{n,j,k}$ which is quite interesting in its own right. Given a partition λ of n , we define $Q_{\lambda,j}$ to be a sum of fundamental quasisymmetric functions that we associate with permutations of cycle type λ that have j excedances. We prove that $Q_{\lambda,j}$ is, in fact, a symmetric function. It is well-known that the Eulerian polynomials are symmetric and unimodal. We prove q - and (q, p) -analogs of these results, as well as cycle type and quasisymmetric function analogs. These results appear in Section 1.4.

It follows from Theorem 0.1.2 that the Eulerian quasisymmetric functions $Q_{n,j,k}$ are h -positive. In Section 1.4 we show that this does not hold for the more refined $Q_{\lambda,j}$, but conjecture that they are Schur positive. Of particular interest are the $Q_{\lambda,j}$ when the partition λ consists of a single part. We observe in Section 1.5 that the $Q_{\lambda,j}$ for general λ can be expressed via plethysm in terms of these. We also present results and conjectures on the virtual representation of the symmetric group whose Frobenius characteristic is $Q_{\lambda,j}$ when λ consists of a single part.

The symmetric functions $Q_{\lambda,j}$ resemble the symmetric functions L_λ studied by Gessel and Reutenauer in [30] in their work on quasisymmetric functions and permutation statistics. However our $Q_{\lambda,j}$ are not refinements of the L_λ . Indeed, L_λ is the Frobenius characteristic of a representation induced from a linear character of the centralizer of a permutation of cycle type λ . On the other hand, we show that if $\lambda = (n)$, where $n \geq 3$, then $\sum_j Q_{\lambda,j}$ is the Frobenius characteristic of a virtual representation (conjecturally, an actual representation) whose character takes nonzero values on elements that do not commute with any element of cycle type λ (see Corollary 1.5.8).

The symmetric function on the right hand side of (0.1.3) and (0.1.4) refines symmetric functions that have been studied earlier in the literature. These symmetric functions include enumerators for multiset

derangements studied by MacMahon [36, Sec. III, Ch. III] and Askey and Ismail [1]; enumerators for words with no adjacent repeats studied by Carlitz, Scoville and Vaughan [14], Dollhopf, Goulden and Greene [20] and Stanley [52]; chromatic symmetric functions of Stanley [52]; and the Frobenius characteristic of the representation of the symmetric group on the degree $2j$ cohomology of the toric variety X_n associated to the Coxeter complex of the symmetric group \mathfrak{S}_n studied by Processi [40], Stanley [52], Stembridge [56, 57], and Dolgachev and Lunts [19]. It is a consequence of our work that these symmetric functions have nice interpretations as sums of fundamental quasisymmetric functions. These connections and others are discussed in Section 1.5.

0.1.2. Poset topology. Representations with Frobenius characteristic $Q_{n,j}$ also occur in poset topology. In fact, it was our study of the homology of a certain poset introduced by Björner and Welker [10] that led us to conjecture Theorems 0.1.1 and 0.1.2 in the first place. The poset we consider is the Rees product $(B_n \setminus \{\emptyset\}) * C_n$, where B_n is the Boolean algebra on $\{1, 2, \dots, n\}$ and C_n is an n -element chain. Rees products of posets were introduced by Björner and Welker in [10], where they study connections between poset topology and commutative algebra. (Rees products of affine semigroup posets arise from the ring-theoretic Rees construction.) The Rees product of two ranked posets is a subposet of the usual product poset (the precise definition is given in Section 2.1).

Björner and Welker [10] conjectured and Jonsson [33] proved that the dimension of the top homology of $(B_n \setminus \{\emptyset\}) * C_n$ is equal to the number of derangements in \mathfrak{S}_n . Here we prove a refinement of this result; namely that the dimension of the top homology of any open principal lower order ideal of $(B_n \setminus \{\emptyset\}) * C_n$ is an Eulerian number. Moreover, we use Theorems 0.1.1 and 0.1.2 to obtain a q -analog and an equivariant version of our refinement and of the Björner-Welker-Jonsson result.

The poset $(B_n \setminus \{\emptyset\}) * C_n$ has n maximal elements all of rank n . The usual action of \mathfrak{S}_n on B_n induces an action of \mathfrak{S}_n on each lower order ideal generated by a maximal element, which in turn induces a representation of \mathfrak{S}_n on the homology of the open lower order ideal. In Section 2.2 we prove the following equivariant version of our refinement.

Theorem 0.1.5. *Let x_1, \dots, x_n be the maximal elements of $(B_n \setminus \{\emptyset\}) * C_n$. For each $j = 1, \dots, n$, let $I_j(B_n)$ be the open lower order ideal*

generated by x_j . Then

$$\text{ch}(\tilde{H}_{n-2}(I_j(B_n))) = \omega \sum_{k=0}^n Q_{n,j-1,k},$$

where ch denotes the Frobenius characteristic and ω is the standard involution on the ring of symmetric functions

As a consequence we have the following equivariant version of the Björner-Welker-Jonsson result:

$$\text{ch}(\tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n)) = \omega \sum_{j=0}^{n-1} Q_{n,j,0}.$$

We also prove a q -analog of our refinement by considering the Rees product $(B_n(q) \setminus \{0\}) * C_n$, where $B_n(q)$ is the lattice of subspaces of the n -dimensional vector space \mathbb{F}_q^n over the q element field \mathbb{F}_q . Like $(B_n \setminus \{\emptyset\}) * C_n$, the q -analog $(B_n(q) \setminus \{0\}) * C_n$ has n maximal elements all of rank n .

Theorem 0.1.6. *Let x_1, \dots, x_n be the maximal elements of $(B_n(q) \setminus \{0\}) * C_n$. For each $j = 1, \dots, n$, let $I_{n,j}(q)$ be the lower order ideal generated by x_j . Then*

$$\dim \tilde{H}_{n-2}(I_{n,j}(q)) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j-1}} q^{\text{comaj}(\sigma) + j - 1},$$

where $\text{comaj}(\sigma) = \binom{n}{2} - \text{maj}(\sigma)$.

As a consequence, we have the following q -analog of the Björner-Welker-Jonsson result:

$$\dim \tilde{H}_{n-1}((B_n(q) \setminus \{0\}) * C_n) = \sum_{\sigma \in \mathcal{D}_n} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)},$$

where \mathcal{D}_n is the set of derangements in \mathfrak{S}_n .

It is also interesting to consider the Rees product of B_n (or $B_n(q)$) with a tree. We prove the following result in Section 2.2. For any poset P with a minimum element $\hat{0}$, let $P^- := P \setminus \{\hat{0}\}$.

Theorem 0.1.7. *For all $n, t \geq 1$, let $T_{t,n}$ be a poset whose Hasse diagram is a complete t -ary tree of height n with the root at the bottom.*

Then

$$\begin{aligned} \dim \tilde{H}_{n-2}((B_n * T_{t,n})^-) &= tA_n(t) \\ \dim \tilde{H}_{n-2}((B_n(q) * T_{t,n})^-) &= tq^{\binom{n}{2}} A_n^{\text{maj,exc}}(q^{-1}, qt) \\ \text{ch} \tilde{H}_{n-2}((B_n * T_{t,n})^-) &= \sum_{\substack{0 \leq k \leq n \\ 0 \leq j \leq n-1}} \omega Q_{n,j,k} t^{j+1}. \end{aligned}$$

We derive a general result relating the homology of lower order ideals of $P^- * C_n$ to the homology of the Rees product $P^* * T_{t,n}$, where P is any bounded, ranked poset of length n and P^* is the dual of P . This result enables us to show that Theorem 0.1.7 implies Theorems 0.1.5 and 0.1.6. We exploit the recursive nature of $B_n * T_{t,n}$ and $B_n(q) * T_{t,n}$ in our proof of Theorem 0.1.7.

Various authors have studied Mahonian (resp. Eulerian) partners to Eulerian (resp. Mahonian) statistics whose joint distribution is equal to a known Euler-Mahonian distribution. We mention, for example, Foata [21], Foata and Zeilberger [25], Clarke, Steingrímsson and Zeng [15], Haglund [31], Babson and Steingrímsson [2], Skandera [48] and Brändén [12]. In Section 2.3 we define a new Mahonian statistic to serve as a partner for des in the (maj, exc) distribution. We do not have a simple proof of the equidistribution. We have a highly nontrivial proof which uses Theorem 0.1.6 and poset topology techniques, such as an EL-labeling of $I_{n,j}(q)$, which is derived from an EL-labeling of Simion [47] for $B_n(q)$.

In the last section (Section 2.4) we use results from Section 2.1 to derive type BC analogs (in the sense of Coxeter groups) of the Björner-Welker-Jonsson derangement result and our q -analog.

Part 1. Permutation statistics

1.1. PERMUTATION STATISTICS AND QUASISYMMETRIC FUNCTIONS

For $n \geq 1$, let \mathfrak{S}_n be the symmetric group on the set $[n] := \{1, \dots, n\}$. A permutation $\sigma \in \mathfrak{S}_n$ will be represented here in two ways, either as a function that maps $i \in [n]$ to $\sigma(i)$, or in one line notation as $\sigma = \sigma_1 \dots \sigma_n$, where $\sigma_i = \sigma(i)$. If $n \leq 0$ then we set $[n] = \emptyset$ and $\mathfrak{S}_n = \{\theta\}$ where θ denotes the empty word.

The *descent set* of σ is

$$\text{DES}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\},$$

and the *excedance set* of σ is

$$\text{EXC}(\sigma) := \{i \in [n-1] : \sigma_i > i\}.$$

We now define the two basic Eulerian permutation statistics. The *descent number* and *excedance number* of σ are, respectively,

$$\text{des}(\sigma) := |\text{DES}(\sigma)|$$

and

$$\text{exc}(\sigma) := |\text{EXC}(\sigma)|.$$

For example, if $\sigma = 32541$, written in one line notation, then

$$\text{DES}(\sigma) = \{1, 3, 4\} \quad \text{and} \quad \text{EXC}(\sigma) = \{1, 3\};$$

hence $\text{des}(\sigma) = 3$ and $\text{exc}(\sigma) = 2$. If $i \in \text{DES}(\sigma)$ we say that σ has a descent at i . If $i \in \text{EXC}(\sigma)$ we say that $\sigma(i)$ is an excedance of σ and that i is an excedance position.

Next we define the two basic Mahonian permutation statistics. The *inversion index* of $\sigma \in \mathfrak{S}_n$ is

$$\text{inv}(\sigma) := |\{(i, j) : 1 \leq i < j \leq n \text{ and } \sigma_i > \sigma_j\}|,$$

and the *major index* of σ is

$$\text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i,$$

For example, if $\sigma = 32541$ then $\text{inv}(\sigma) = 6$ and $\text{maj}(\sigma) = 8$.

We review some basic facts of Gessel's theory of quasisymmetric functions; a good reference is [55, Chapter 7]. A *quasisymmetric function* is a formal power series $f(\mathbf{x}) = f(x_1, x_2, \dots)$ of finite degree with rational coefficients in the infinitely many variables x_1, x_2, \dots such that for any $a_1, \dots, a_k \in \mathbb{P}$, the coefficient of $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$ equals the coefficient of $x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$ whenever $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. Thus each symmetric function is quasisymmetric.

For a positive integer n and $S \subseteq [n-1]$, define

$$F_{S,n} = F_{S,n}(\mathbf{x}) := \sum_{\substack{i_1 \geq \dots \geq i_n \geq 1 \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \dots x_{i_n}$$

and let $F_{\emptyset,0} = 1$. Each $F_{S,n}$ is a quasisymmetric function. The set $\{F_{S,n} : S \subseteq [n-1], n \in \mathbb{N}\}$ is a basis for the ring \mathcal{Q} of quasisymmetric functions and $F_{S,n}$ is called a *fundamental* quasisymmetric function. If $S = \emptyset$ then $F_{S,n}$ is the complete homogeneous symmetric function h_n and if $S = [n-1]$ then $F_{S,n}$ is the elementary symmetric function e_n .

We review two important ways to specialize a quasisymmetric function. Let $\mathbb{Q}[q]$ denote the ring of polynomials in variable q with coefficients in \mathbb{Q} and let $\mathbb{Q}[[q]]$ denote the ring of formal power series in

variable q with coefficients in \mathbb{Q} . The *stable principal specialization* is the ring homomorphism $\Lambda : \mathcal{Q} \rightarrow \mathbb{Q}[[q]]$ defined by

$$\Lambda(x_i) = q^{i-1},$$

and the *principal specialization* of order m is the ring homomorphism $\Lambda_m : \mathcal{Q} \rightarrow \mathbb{Q}[q]$ defined by

$$\Lambda_m(x_i) = \begin{cases} q^{i-1} & \text{if } 1 \leq i \leq m \\ 0 & \text{if } i > m \end{cases}.$$

Sometimes we will need to apply Λ and Λ_m to quasisymmetric functions whose coefficient ring is \mathbb{Q} with indeterminants adjoined.

Lemma 1.1.1 ([30, Lemma 5.2]). *For all $n \geq 0$ and $S \in [n - 1]$, we have*

$$(1.1.1) \quad \Lambda(F_{S,n}) = \frac{q^{\sum_{i \in S} i}}{(q; q)_n}$$

and

$$(1.1.2) \quad \sum_{m \geq 0} \Lambda_m(F_{S,n}) p^m = \frac{p^{|S|+1} q^{\sum_{i \in S} i}}{(p; q)_{n+1}}.$$

It follows from Lemma 1.1.1 that

$$(1.1.3) \quad \Lambda(h_n) = \frac{1}{(1-q)(1-q^2) \dots (1-q^n)}$$

for all n and therefore

$$(1.1.4) \quad \Lambda(H(z(1-q))) = \exp_q(z).$$

The standard way to connect quasisymmetric functions with permutation statistics is by associating the fundamental quasisymmetric function $F_{\text{DES}(\sigma),n}$ with $\sigma \in \mathfrak{S}_n$. By Lemma 1.1.1, we have for any subset $A \in \mathfrak{S}_n$

$$\Lambda \left(\sum_{\sigma \in A} F_{\text{DES}(\sigma),n} \right) = \frac{1}{(q; q)_n} \sum_{\sigma \in A} q^{\text{maj}(\sigma)},$$

and

$$\sum_{m \geq 0} \Lambda_m \left(\sum_{\sigma \in A} F_{\text{DES}(\sigma),n} \right) p^m = \frac{1}{(p; q)_{n+1}} \sum_{\sigma \in A} p^{\text{des}(\sigma)+1} q^{\text{maj}(\sigma)}$$

Gessel and Reutenauer [30] used this to study the (maj, des)-enumerator for permutations of a fixed cycle type.

Here we introduce a new way to associate fundamental quasisymmetric functions with permutations. For $n \in \mathbb{N}$, we set

$$[\bar{n}] := \{\bar{1}, \dots, \bar{n}\}$$

and totally order the alphabet $[n] \cup [\bar{n}]$ by

$$(1.1.5) \quad \bar{1} < \dots < \bar{n} < 1 < \dots < n.$$

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we define $\bar{\sigma}$ to be the word over alphabet $[n] \cup [\bar{n}]$ obtained from σ by replacing σ_i with $\bar{\sigma}_i$ whenever $i \in \text{EXC}(\sigma)$. For example, if $\sigma = 531462$ then $\bar{\sigma} = \bar{5}\bar{3}14\bar{6}2$. We define a descent in a word $w = w_1 \dots w_n$ over any totally ordered alphabet to be any $i \in [n-1]$ such that $w_i > w_{i+1}$ and let $\text{DES}(w)$ be the set of all descents of w . Now, for $\sigma \in \mathfrak{S}_n$, we define

$$\text{DEX}(\sigma) := \text{DES}(\bar{\sigma}).$$

For example, $\text{DEX}(531462) = \text{DES}(\bar{5}\bar{3}14\bar{6}2) = \{1, 4\}$.

Lemma 1.1.2. *For all $\sigma \in \mathfrak{S}_n$,*

$$(1.1.6) \quad \sum_{i \in \text{DEX}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma),$$

and

$$(1.1.7) \quad |\text{DEX}(\sigma)| = \begin{cases} \text{des}(\sigma) & \text{if } \sigma(1) = 1 \\ \text{des}(\sigma) - 1 & \text{if } \sigma(1) \neq 1 \end{cases}$$

Proof. Let

$$J(\sigma) = \{i \in [n-1] : i \notin \text{EXC}(\sigma) \ \& \ i+1 \in \text{EXC}(\sigma)\},$$

and let

$$K(\sigma) = \{i \in [n-1] : i \in \text{EXC}(\sigma) \ \& \ i+1 \notin \text{EXC}(\sigma)\}.$$

If $i \in J(\sigma)$ then $\sigma(i) \leq i < i+1 < \sigma(i+1)$. Hence $i \notin \text{DES}(\sigma)$, but $i \in \text{DEX}(\sigma)$. If $i \in K(\sigma)$ then $\sigma(i) \geq i+1 \geq \sigma(i+1)$. Hence $i \in \text{DES}(\sigma)$, but $i \notin \text{DEX}(\sigma)$. It follows that $K(\sigma) \subseteq \text{DES}(\sigma)$ and

$$(1.1.8) \quad \text{DEX}(\sigma) = \text{DES}(\sigma) \uplus J(\sigma) - K(\sigma).$$

Hence

$$(1.1.9) \quad |\text{DEX}(\sigma)| = \text{des}(\sigma) + |J(\sigma)| - |K(\sigma)|.$$

Let $J(\sigma) = \{j_1 < j_2 < \dots < j_t\}$ and $K(\sigma) = \{k_1 < k_2 < \dots < k_s\}$. It is easy to see that if $\sigma(1) = 1$ then $t = s$ and

$$(1.1.10) \quad j_1 < k_1 < j_2 < k_2 < \dots < j_t < k_t,$$

since neither 1 nor n are excedance positions. On the other hand if $\sigma(1) \neq 1$ then $s = t + 1$ and

$$(1.1.11) \quad k_1 < j_1 < k_2 < j_2 < \cdots < j_t < k_{t+1}.$$

It now follows from (1.1.9) that (1.1.7) holds.

To prove (1.1.6), we again handle the cases $\sigma(1) = 1$ and $\sigma(1) \neq 1$ separately.

Case 1: Suppose $\sigma(1) = 1$. Then (1.1.10) holds. By (1.1.8),

$$\sum_{i \in \text{DEX}(\sigma)} i = \sum_{i \in \text{des}(\sigma)} i - \sum_{i=1}^t (k_i - j_i).$$

Clearly

$$\text{EXC}(\sigma) = \biguplus_{i=1}^t \{j_i + 1, j_i + 2, \dots, k_i\}.$$

Hence

$$\text{exc}(\sigma) = \sum_{i=1}^t (k_i - j_i)$$

and so (1.1.6) holds in this case.

Case 2: Suppose $\sigma(1) \neq 1$. Now (1.1.11) holds. By (1.1.8),

$$\sum_{i \in \text{DEX}(\sigma)} i = \sum_{i \in \text{des}(\sigma)} i - k_1 - \sum_{i=1}^t (k_{i+1} - j_i).$$

Now

$$\text{EXC}(\sigma) = \{1, 2, \dots, k_1\} \uplus \biguplus_{i=1}^t \{j_i + 1, j_i + 2, \dots, k_{i+1}\},$$

which implies that

$$\text{exc}(\sigma) = k_1 + \sum_{i=1}^t (k_{i+1} - j_i).$$

Hence (1.1.6) holds in this case too. \square

We now define the quasisymmetric functions that play a central role in this paper. Let $n \geq 1$, $j, k \geq 0$ and $\lambda \vdash n$. The *Eulerian quasisymmetric functions* are defined by

$$Q_{n,j} = Q_{n,j}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} F_{\text{DEX}(\sigma), n}(\mathbf{x}).$$

The *fixed point Eulerian quasisymmetric functions* are defined by

$$Q_{n,j,k} = Q_{n,j,k}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \text{fix}(\sigma) = k}} F_{\text{DEX}(\sigma),n}(\mathbf{x}).$$

The *cycle type Eulerian quasisymmetric functions* are defined by

$$Q_{\lambda,j} = Q_{\lambda,j}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\text{DEX}(\sigma),n}(\mathbf{x}),$$

where $\lambda(\sigma)$ denotes the cycle type of σ . For convenience we let $\text{DEX}(\theta) = \text{DES}(\theta) = \text{EXC}(\theta) = \emptyset$ and $\text{exc}(\theta) = \text{fix}(\theta) = \text{des}(\theta) = \text{maj}(\theta) = 0$. So $Q_{0,0,0} = 1$. It is a consequence of Theorem 0.1.2 that the Eulerian quasisymmetric functions $Q_{n,j}$ and $Q_{n,j,k}$ are symmetric functions. We prove that the refinement $Q_{\lambda,j}$ is symmetric, as well, in Section 1.4.2.

The (q, p) -Eulerian numbers are defined by

$$a_{n,j}(p, q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} q^{\text{maj}(\sigma)} p^{\text{des}(\sigma)}.$$

The *fixed point (q, p) -Eulerian numbers* are defined by

$$a_{n,j,k}(p, q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \text{fix}(\sigma) = k}} q^{\text{maj}(\sigma)} p^{\text{des}(\sigma)}.$$

The *cycle type (q, p) -Eulerian numbers* are defined by

$$a_{\lambda,j}(p, q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} q^{\text{maj}(\sigma)} p^{\text{des}(\sigma)}.$$

It follows from (1.1.1) and (1.1.6) that for $\sigma \in \mathfrak{S}_n$ we have

$$\Lambda(F_{\text{DEX}(\sigma),n}) = (q; q)_n^{-1} q^{\text{maj}(\sigma) - \text{exc}(\sigma)}.$$

Hence

$$(1.1.12) \quad a_{n,j,k}(q, 1) = q^j (q; q)_n \Lambda(Q_{n,j,k}),$$

$$(1.1.13) \quad a_{\lambda,j}(q, 1) = q^j (q; q)_n \Lambda(Q_{\lambda,j}).$$

From the first of these equations and (1.1.4), we see that Corollary 0.1.3 is obtained from Theorem 0.1.2 by applying the stable principal specialization.

The (nonstable) principle specialization is a bit more complicated. Given two partitions $\lambda \vdash n$ and $\mu \vdash m$, let (λ, μ) denote the partition of $n + m$ obtained by concatenating λ and μ and then appropriately rearranging the parts.

Lemma 1.1.3. *For $n, j, k \geq 0$ and $\lambda \vdash n - k$, where λ has no parts of size 1,*

$$a_{(\lambda, 1^k), j}(q, p) = (p; q)_{n+1} \sum_{m \geq 0} p^m \sum_{i=0}^k q^{im+j} \Lambda_m(Q_{(\lambda, 1^{k-i}), j}).$$

Consequently, for $n, j, k \geq 0$,

$$a_{n, j, k}(q, p) = (p; q)_{n+1} \sum_{m \geq 0} p^m \sum_{i=0}^k q^{im+j} \Lambda_m(Q_{n-i, j, k-i}).$$

Proof. For all $\sigma \in \mathfrak{S}_n$ we have, by (1.1.2) and Lemma 1.1.2,

$$\sum_{m \geq 0} \Lambda_m(F_{\text{DEX}(\sigma), n}) p^m = \frac{1}{(p; q)_{n+1}} p^{|\text{DEX}(\sigma)|+1} q^{\text{maj}(\sigma) - \text{exc}(\sigma)}.$$

It follows that

$$(1.1.14) \quad \sum_{m \geq 0} \Lambda_m(Q_{(\lambda, 1^k), j}) p^m = X_{\lambda, j}^k(q, p),$$

where

$$X_{\lambda, j}^k(q, p) := \frac{1}{(p; q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = (\lambda, 1^k)}} q^{\text{maj}(\sigma) - j} p^{|\text{DEX}(\sigma)|+1}.$$

By (1.1.7) we have

$$\begin{aligned} X_{\lambda, j}^k(q, p) := & \frac{1}{(p; q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = (\lambda, 1^k) \\ \sigma(1) = 1}} q^{\text{maj}(\sigma) - j} p^{\text{des}(\sigma)+1} \\ & + \frac{1}{(p; q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = (\lambda, 1^k) \\ \sigma(1) \neq 1}} q^{\text{maj}(\sigma) - j} p^{\text{des}(\sigma)}. \end{aligned}$$

Let

$$Y_{\lambda,j}^k(q,p) := \frac{1}{(p;q)_{n+1}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = (\lambda, 1^k) \\ \sigma(1) = 1}} q^{\text{maj}(\sigma) - j} p^{\text{des}(\sigma)}.$$

Write $a_{\lambda,j}^k(q,p)$ for $a_{(\lambda,1^k),j}(q,p)$. We have

$$\begin{aligned} \frac{a_{\lambda,j}^k(q,p)}{q^j(p;q)_{n+1}} &= Y_{\lambda,j}^k(q,p) + X_{\lambda,j}^k(q,p) - pY_{\lambda,j}^k(q,p) \\ (1.1.15) \qquad \qquad &= (1-p)Y_{\lambda,j}^k(q,p) + X_{\lambda,j}^k(q,p) \end{aligned}$$

Let $\varphi : \{\sigma \in \mathfrak{S}_n : \sigma(1) = 1\} \rightarrow \mathfrak{S}_{n-1}$ be the bijection defined by letting $\varphi(\sigma)$ be the permutation obtained by removing the 1 from the beginning of σ and subtracting 1 from each letter of the remaining word. It is clear that $\text{maj}(\sigma) = \text{maj}(\varphi(\sigma)) + \text{des}(\sigma)$, $\text{des}(\sigma) = \text{des}(\varphi(\sigma))$, $\text{exc}(\sigma) = \text{exc}(\varphi(\sigma))$ and $\text{fix}(\sigma) = \text{fix}(\varphi(\sigma)) + 1$. Hence

$$Y_{\lambda,j}^k(q,p) = \frac{a_{\lambda,j}^{k-1}(q,qp)}{q^j(p;q)_{n+1}}$$

Plugging this into (1.1.15) yields the recurrence

$$\frac{a_{\lambda,j}^k(q,p)}{q^j(p;q)_{n+1}} = \frac{a_{\lambda,j}^{k-1}(q,qp)}{q^j(qp;q)_n} + X_{\lambda,j}^k(q,p).$$

By iterating this recurrence we get

$$\frac{a_{\lambda,j}^k(q,p)}{q^j(p;q)_{n+1}} = \sum_{i=0}^k X_{\lambda,j}^{k-i}(q, q^i p).$$

The result now follows from this and (1.1.14). \square

Proof of Corollary 0.1.4. By Lemma 1.1.3 and Theorem 0.1.2, we have

$$\begin{aligned}
& \sum_{n \geq 0} A_n^{\text{maj,des,exc,fix}}(q, p, t, r) \frac{z^n}{(p; q)_{n+1}} \\
&= \sum_{n, j, k \geq 0} a_{n, j, k}(p, q) t^j r^k \frac{z^n}{(p; q)_{n+1}} \\
&= \sum_{n, j, k \geq 0} z^n t^j r^k \sum_{m \geq 0} p^m \sum_{i=0}^k q^{im+j} \Lambda_m(Q_{n-i, j, k-i}) \\
&= \sum_{m \geq 0} p^m \sum_{i \geq 0} (zr q^m)^i \sum_{\substack{n, k \geq i \\ j \geq 0}} \Lambda_m(Q_{n-i, j, k-i}) (qt)^j r^{k-i} z^{n-i} \\
&= \sum_{m \geq 0} \frac{p^m}{1 - zr q^m} \Lambda_m \frac{(1 - tq)H(zr)}{H(ztq) - tqH(z)} \\
&= \sum_{m \geq 0} p^m \frac{(1 - tq)(z; q)_m (ztq; q)_m}{((z; q)_m - tq(ztq; q)_m)(zr; q)_{m+1}},
\end{aligned}$$

with the last step following from

$$\Lambda_m(H(z)) = \Lambda_m \left(\prod_{i \geq 0} \frac{1}{1 - x_i z} \right) = \frac{1}{(z; q)_m}.$$

□

1.2. BICOLORED NECKLACES AND WORDS

This section is devoted to the proof of Theorem 0.1.2. There are three main steps. In the first step (Section 1.2.1) we modify a bijection that Gessel and Reutenauer [30] constructed in order to enumerate permutations with a fixed descent set and fixed cycle type. This yields an alternative characterization of the Eulerian quasisymmetric functions involving bicolored necklaces. In the second step (Section 1.2.2) we construct a bijection from multisets of bicolored necklaces to bicolored words, which involves Lyndon decompositions of words. This yields yet another characterization of the Eulerian quasisymmetric functions. In the third step (Section 1.2.3) we generalize a bijection that Stembridge constructed to study the representation of the symmetric group on the cohomology of the toric variety associated with the type A Coxeter complex. This enables us to derive a recurrence relation, which yields Theorem 0.1.2.

1.2.1. Step 1: bicolored version of the Gessel-Reutenauer bijection. The Gessel-Reutenauer bijection is a bijection between pairs (σ, s) , where σ is a permutation and s is a “compatible” weakly decreasing sequence, and multisets of primitive circular words over the alphabet of positive integers. This bijection enabled Gessel and Reutenauer to use the fundamental quasisymmetric functions $F_{\text{DES}(\sigma),n}$ to study properties of permutations with a fixed descent set and cycle type. Here we introduce a bicolored version of the Gessel-Reutenauer bijection.

We consider circular words over the alphabet of bicolored positive integers

$$\mathcal{A} := \{1, \bar{1}, 2, \bar{2}, 3, \bar{3}, \dots\}.$$

(We can think of “barred” and “unbarred” as colors assigned to each positive integer.) For each such circular word and each starting position, one gets a linear word by reading the circular word in a clockwise direction. If one gets a distinct linear word for each starting position then the circular word is said to be *primitive*. For example the circular word $(\bar{1}, 1, 1)$ is primitive while the circular word $(\bar{1}, 2, \bar{1}, 2)$ is not. (If w is a linear word then (w) denotes the circular word obtained by placing the letters of w around a circle in a clockwise direction.) The *absolute value* or just *value* of a letter a is the letter obtained by ignoring the bar if there is one. We denote this by $|a|$.

Definition 1.2.1. A *bicolored necklace* is a primitive circular word w over alphabet \mathcal{A} such that if the length of w is greater than 1 then

- (1) every barred letter is followed (clockwise) by a letter less than or equal to it in absolute value
- (2) every unbarred letter is followed by a letter greater than or equal to it in absolute value.

A circular word of length 1 is a bicolored necklace if its sole letter is unbarred. A *bicolored ornament* is a multiset of bicolored necklaces.

For example the following circular words are bicolored necklaces:

$$(\bar{3}, 1, 3, \bar{3}, 2, 2), (\bar{3}, 1, \bar{3}, \bar{3}, 2, 2), (\bar{3}, 1, 3, \bar{3}, \bar{2}, 2), (\bar{3}, 1, \bar{3}, \bar{3}, \bar{2}, 2), (2),$$

while $(\bar{3}, \bar{1}, 3, 3, 2, \bar{2})$ and $(\bar{3})$ are not.

From now on we will drop the word “bicolored”; so “necklace” will stand for “bicolored necklace” and “ornament” will stand for “bicolored ornament”. The type $\lambda(R)$ of an ornament R is the partition whose parts are the sizes of the necklaces in R . The weight of a letter a is the indeterminate $x_{|a|}$. The weight $\text{wt}(R)$ of an ornament R is the product of the weights of the letters of R . For example

$$\lambda((\bar{3}, 2, 2), (\bar{3}, \bar{2}, 1, 1, 2)) = (5, 3)$$

and

$$\text{wt}((\bar{3}, 2, 2), (\bar{3}, \bar{2}, 1, 1, 2)) = x_3^2 x_2^4 x_1^2.$$

For each partition λ and nonnegative integer j , let $\mathfrak{R}_{\lambda,j}$ be the set of ornaments of type λ with j bars.

Given a permutation $\sigma \in \mathfrak{S}_n$, we say that a weakly decreasing sequence $s_1 \geq s_2 \geq \dots \geq s_n$ of positive integers is σ -compatible if $s_i > s_{i+1}$ whenever $i \in \text{DEX}(\sigma)$. For example, $7, 7, 7, 5, 5, 4, 2, 2$ is σ -compatible, where $\sigma = 45162387$. Note that for all $\sigma \in \mathfrak{S}_n$,

$$F_{\text{DEX}(\sigma),n} = \sum_{s_1, \dots, s_n} x_{s_1} \dots x_{s_n}$$

where s_1, \dots, s_n ranges over all σ -compatible sequences.

For $\lambda \vdash n$ and $j = 0, \dots, n-1$, let $\text{Com}_{\lambda,j}$ be the set of pairs (σ, s) , where σ is a permutation of cycle type λ with j excedances and s is a σ -compatible sequence. Let $\phi : \text{Com}_{\lambda,j} \rightarrow \mathfrak{R}_{\lambda,j}$ be the map defined by letting $\phi(\sigma, s)$ be the ornament obtained by first writing σ in cycle form with bars above the letters that are followed (cyclicly) by larger letters (i.e., the excedances) and then replacing each i with s_i , keeping the bars in place. For example, let $\sigma = 45162387$ and $s = 7, 7, 7, 5, 5, 4, 2, 2$. First we write σ in cycle form,

$$\sigma = (1, 4, 6, 3)(2, 5)(7, 8).$$

Next we put bars above the letters that are followed by larger letters,

$$(\bar{1}, \bar{4}, 6, 3)(\bar{2}, 5)(\bar{7}, 8).$$

After replacing each i by s_i , we have the ornament

$$(\bar{7}, \bar{5}, 4, 7)(\bar{7}, 5)(\bar{2}, 2).$$

Theorem 1.2.2. *The map $\phi : \text{Com}_{\lambda,j} \rightarrow \mathfrak{R}_{\lambda,j}$ is a well-defined bijection.*

Proof. Let $(\sigma, s) \in \text{Com}_{\lambda,j}$. It is clear that the circular words in the multiset $\phi(\sigma, s)$ satisfy conditions (1) and (2) of Definition 1.2.1 and that the number of bars of $\phi(\sigma, s)$ is j . It is also clear that $\lambda(\phi(\sigma, s)) = \lambda$.

We must now show that the circular words in $\phi(\sigma, s)$ are primitive. Suppose there is a circular word $(a_{i_1}, \dots, a_{i_k})$ that is not primitive. Let this word come from the cycle (i_1, i_2, \dots, i_k) of σ , where i_1 is the smallest element of the cycle. Suppose $a_{i_1}, \dots, a_{i_k} = (a_{i_1}, \dots, a_{i_d})^{k/d}$, where $d < k$. Since $a_{i_1} = a_{i_{d+1}}$, $i_1 < i_{d+1}$, and the $|a_i|$'s form a weakly decreasing sequence, it follows that $|a_i| = |a_{i_1}|$ for all i such that $i_1 \leq i \leq i_{d+1}$. Hence since $(|a_1|, |a_2|, \dots, |a_n|)$ is σ -compatible, there can be no element of $\text{DEX}(\sigma)$ in the set $\{i_1, i_1 + 1, \dots, i_{d+1} - 1\}$.

Since a_{i_1} and $a_{i_{d+1}}$ are both barred or both unbarred, it follows that both i_1 and i_{d+1} are excedance positions or both are not. Hence

$$\sigma(i_1) < \sigma(i_1 + 1) < \cdots < \sigma(i_{d+1}).$$

Since $\sigma(i_1) = i_2$ and $\sigma(i_{d+1}) = i_{d+2}$ we have $i_2 < i_{d+2}$. Repeated use of the above argument yields $i_3 < i_{d+3}$ and eventually $i_{k+1-d} < i_1$, which is impossible since i_1 is the smallest element of its cycle. Hence all the circular words of $\phi(\sigma, s)$ are primitive and $\phi(\sigma, s)$ is an ornament of type λ with j bars, which means that ϕ is well-defined.

To show that ϕ is a bijection, we construct its inverse $\eta : \mathfrak{R}_{\lambda, j} \rightarrow \text{Com}_{\lambda, j}$, which takes the ornament R to the pair $(\sigma(R), s(R))$. Here $s(R)$ is simply the weakly decreasing rearrangement of the letters of R with bars removed. To construct the permutation $\sigma(R)$, we need to first order the alphabet \mathcal{A} by

$$(1.2.1) \quad 1 < \bar{1} < 2 < \bar{2} < \dots$$

Note that this ordering is different from the ordering (1.1.5) of the finite alphabet that was used to define DEX.

For each position x of each necklace of R , consider the infinite word w_x obtained by reading the necklace in a clockwise direction starting at position x . Let

$$w_x \leq_L w_y$$

mean that w_x is lexicographically less than or equal to w_y . We use the lexicographic order on words to order the positions: if $w_x <_L w_y$ then we say $x < y$. We break ties as follows: if $w_x = w_y$ and $x \neq y$ then x and y must be positions in distinct (but equal) necklaces since necklaces are primitive. If $w_x = w_y$ and x is a position in an earlier necklace than that of y under some fixed linear ordering of the necklaces then let $x < y$. If x is the i th *largest* position then replace the letter in position x by i . We now have a multiset of circular words in which each letter $1, 2, \dots, n$ appears exactly once. This is the cycle form of the permutation $\sigma(R)$.

For example, if

$$R = ((\bar{7}, \bar{3}, 3, 5), (\bar{7}, 3, \bar{5}, 3), (\bar{7}, 3, \bar{5}, 3), (5)),$$

then

$$\sigma(R) = (1, 8, 13, 6), (2, 11, 4, 9), (3, 12, 5, 10), (7)$$

and

$$s(R) = 7, 7, 7, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3.$$

It is not hard to see that the letter in position x of R is barred if and only if the letter that replaces it is an excedance position of $\sigma(R)$. This

implies that the number of bars of R equals the number of excedances of $\sigma(R)$.

We now show that $s(R)$ is $\sigma(R)$ -compatible. Suppose

$$s(R)_i = s(R)_{i+1}.$$

We must show $i \notin \text{DEX}(\sigma(R))$. Let x be the i th largest position of R and let y be the $(i+1)$ st largest. Then i is placed in position x and $i+1$ is placed in position y . Let $f(w)$ denote the first letter of a word w . Then one of the following must hold

- (1) $f(w_x) = s(R)_i = f(w_y)$
- (2) $f(w_x) = \overline{s(R)}_i = f(w_y)$
- (3) $f(w_x) = s(R)_i$ and $f(w_y) = s(R)_i$.

Cases (1) and (2): Either $w_x >_L w_y$ or $w_x = w_y$ and x is in an earlier necklace than y . Let u be the position that follows x clockwise and let v be the position that follows y . Since w_u is the word obtained from w_x by removing its first letter, w_v is obtained from w_y by removing its first letter, and the first letters are equal, we conclude that $u > v$. Hence the letter that gets placed in position u is smaller than the letter that gets placed in position v . Since the letter placed in position u is $\sigma(R)(i)$ and the letter placed in position v is $\sigma(R)(i+1)$, we have $\sigma(R)(i) < \sigma(R)(i+1)$. Since $i, i+1 \in \text{EXC}(\sigma(R))$ or $i, i+1 \notin \text{EXC}(\sigma(R))$, we conclude that $i \notin \text{DEX}(\sigma(R))$.

Case (3): Since the letter in position x is barred we have $i \in \text{EXC}(\sigma(R))$. Since the letter in position y is not barred, we have $i+1 \notin \text{EXC}(\sigma(R))$. Hence $i \notin \text{DEX}(\sigma(R))$.

In all three cases we have that $s(R)_i = s(R)_{i+1}$ implies $i \notin \text{DEX}(\sigma(R))$. Hence $s(R)$ is $\sigma(R)$ -compatible.

Now we show that the map η is the inverse of ϕ . It is easy to see that $\phi(\eta(R)) = R$. Establishing $\eta(\phi(\sigma, s)) = (\sigma, s)$ means establishing $\sigma(\phi(\sigma, s)) = \sigma$ and $s(\phi(\sigma, s)) = s$. The latter equation is obvious. To establish the former, let $R = \phi(\sigma, s)$. Recall that R is obtained by writing σ in cycle form, barring the excedances, and then replacing each i by s_i , keeping the bars intact. Let p_i be the position that i occupied before the replacement. By ordering the cycles of σ so that the minimum elements of the cycles increase, we get an ordering of the necklaces in R , which we use to break ties between the w_p . To show that $\sigma(R) = \sigma$, we need to show that

- (1) if $i < j$ then $w_{p_i} \geq_L w_{p_j}$
- (2) if $i < j$ and $w_{p_i} = w_{p_j}$ then i is in a cycle whose minimum element is less than that of j .

To prove (1) and (2), we will use the following implication:

$$(1.2.2) \quad i < j \text{ and } w_{p_i} \leq_L w_{p_j} \implies f(w_{p_i}) = f(w_{p_j}) \text{ and } \sigma(i) < \sigma(j),$$

(Recall $f(w)$ is the first letter of a word w .)

Proof of implication: Since $i < j$ and s is weakly decreasing, we have $s_i \geq s_j$. Since $w_{p_i} \leq_L w_{p_j}$, we have $f(w_{p_i}) \leq f(w_{p_j})$, which implies that $s_i \leq s_j$. Hence $s_i = s_j$, which implies

$$s_i = s_{i+1} = \cdots = s_j.$$

It follows from this and the fact that s is σ -compatible that $k \notin \text{DEX}(\sigma)$ for all $k = i, i+1, \dots, j-1$. This implies that either

$$i, i+1, \dots, j \in \text{EXC}(\sigma) \text{ and } \sigma(i) < \sigma(i+1) < \cdots < \sigma(j)$$

or

$$i, i+1, \dots, j \notin \text{EXC}(\sigma) \text{ and } \sigma(i) < \sigma(i+1) < \cdots < \sigma(j)$$

or

$$i \in \text{EXC}(\sigma) \text{ and } j \notin \text{EXC}(\sigma).$$

In the first case, $f(w_{p_i}) = \bar{s}_i = \bar{s}_j = f(w_{p_j})$. In the second case, $f(w_{p_i}) = s_i = s_j = f(w_{p_j})$. In the third case $f(w_{p_i}) = \bar{s}_i > s_i = s_j = f(w_{p_j})$, which is impossible. Hence the conclusion of the implication (1.2.2) holds.

Now we use (1.2.2) to prove (1). Suppose $i < j$ and $w_{p_i} <_L w_{p_j}$. Then by (1.2.2), we have $\sigma(i) < \sigma(j)$ and $f(w_{p_i}) = f(w_{p_j})$, which implies $w_{p_{\sigma(i)}} <_L w_{p_{\sigma(j)}}$. Hence we can apply (1.2.2) again with $\sigma(i)$ and $\sigma(j)$ playing the roles of i and j . This yields $f(w_{p_{\sigma^2(i)}}) = f(w_{p_{\sigma^2(j)}})$, $\sigma^2(i) < \sigma^2(j)$, and $w_{p_{\sigma^2(i)}} <_L w_{p_{\sigma^2(j)}}$. Repeated application of (1.2.2) yields $f(w_{p_{\sigma^m(i)}}) = f(w_{p_{\sigma^m(j)}})$ for all m , which implies $w_{p_i} = w_{p_j}$, a contradiction.

Next we prove (2). Repeated application of (1.2.2) yields $\sigma^m(i) < \sigma^m(j)$ for all m . Hence the cycle containing i has a smaller minimum than the cycle containing j . \square

Corollary 1.2.3. *For all $\lambda \vdash n$ and $j = 0, 1, \dots, n-1$,*

$$Q_{\lambda,j} = \sum_{R \in \mathfrak{R}_{\lambda,j}} w(R).$$

Corollary 1.2.3 has several interesting consequences. For one thing, it can be used to prove that the Eulerian quasisymmetric functions $Q_{\lambda,j}$ are actually symmetric (see Section 1.4). It also has the following useful consequence.

Corollary 1.2.4. *For all n, j, k ,*

$$Q_{n,j,k} = h_k Q_{n-k,j,0}.$$

It follows from Corollary 1.2.4 that Theorem 0.1.2 is equivalent to

$$\sum_{n,j \geq 0} Q_{n,j,0} t^j z^n = \frac{1-t}{H(zt) - tH(z)},$$

which in turn, is equivalent to the recurrence relation

$$(1.2.3) \quad Q_{n,j,0} = \sum_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} Q_{m,i,0} h_{n-m}.$$

1.2.2. Step 2: banners. In order to establish the recurrence relation (1.2.3), we introduce another type of configuration, closely related to ornaments.

Definition 1.2.5. A *banner* is a word B over alphabet \mathcal{A} such that for all $i = 1, \dots, \ell(B)$,

- (1) if $B(i)$ is barred then $|B(i)| \geq |B(i+1)|$
- (2) if $B(i)$ is unbarred then $|B(i)| \leq |B(i+1)|$ or $i = \ell(B)$,

where $B(i)$ denotes the i th letter of B and $\ell(B)$ denotes the length of B .

A *Lyndon word* over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements. A *Lyndon factorization* of a word over an ordered alphabet is a factorization into a weakly lexicographically increasing sequence of Lyndon words. It is a result of Lyndon (see [39, Theorem 5.1.5]) that every word has a unique Lyndon factorization. The *Lyndon type* $\lambda(w)$ of a word w is the partition whose parts are the lengths of the words in its Lyndon factorization.

To apply the theory of Lyndon words to banners, we use the ordering of \mathcal{A} given in (1.2.1). Using this order, the banner $B := \bar{2}2\bar{7}5\bar{7}\bar{5}47$ has Lyndon factorization

$$\bar{2}2 \cdot \bar{7}5 \cdot \bar{7}\bar{5}47.$$

So the Lyndon type of B is the partition $(4, 2, 2)$.

The weight $\text{wt}(B)$ of a banner B is the product of the weights of its letters, where as before the weight of a letter a is $x_{|a|}$. For each partition λ and nonnegative integer j , let $\mathfrak{B}_{\lambda,j}$ be the set of banners with j bars whose Lyndon type is λ .

Theorem 1.2.6. *For each partition λ and nonnegative integer j , there is a weight-preserving bijection*

$$\psi : \mathfrak{B}_{\lambda,j} \rightarrow \mathfrak{A}_{\lambda,j}.$$

Consequently,

$$Q_{\lambda,j} = \sum_{B \in \mathfrak{B}_{\lambda,j}} \text{wt}(B).$$

Proof. First note that there is a natural weight-preserving bijection from the set of Lyndon banners of length n to the set of necklaces of size n . To go from a Lyndon banner B to a necklace (B) simply attach the ends of B so that the left end follows the right end when read in a clockwise direction. To go from a necklace back to a banner simply find the lexicographically largest linear word obtained by reading the circular word in a clockwise direction. The number of bars of B clearly is the same as that of (B) .

Let $B \in \mathfrak{B}_{\lambda,j}$ and let

$$B = B_1 \cdot B_2 \cdots B_k$$

be the unique Lyndon factorization of B . Note that each B_i is a Lyndon banner. To see this we need only check that the last letter of each B_i is unbarred. The last letter of B_k is the last letter of B ; so it is clearly unbarred. For $i < k$, the last letter of B_i is strictly less than the first letter of B_{i+1} , which is less than or equal to the first letter of B_{i+1} . Since the last letter of B_i immediately precedes the first letter of B_{i+1} in the banner B , it must be unbarred.

Now define $\psi(B)$ to be the ornament whose necklaces are

$$(B_1), (B_2), \dots, (B_k).$$

This map is clearly weight preserving, type preserving, and bar preserving. To go from an ornament back to a banner simply arrange the Lyndon banners obtained from the necklaces in the ornament in weakly increasing order and then concatenate. \square

1.2.3. Step 3: the recurrence relation. Define a *marked sequence* (ω, i) to be a weakly increasing finite sequence ω of positive integers together with an integer i such that $1 \leq i \leq \text{length}(\omega) - 1$. (One can visualize this as a weakly increasing sequence with a mark above any of its elements except the last.) For $n \geq 2$, let \mathfrak{M}_n be the set of marked sequences of length n . For $n \geq 0$, let \mathfrak{B}_n^0 be the set of banners of length n whose Lyndon type has no parts of size 1. It will be convenient to consider the empty word to be a banner of length 0, weight 1, with no bars, and whose Lyndon type is the partition of 0 with no parts. So

\mathfrak{B}_0^0 consists of a single element, namely the empty word. Note that \mathfrak{B}_1^0 is the empty set.

Theorem 1.2.6 and Theorem 1.2.7 below are all that is needed to establish the recurrence relation (1.2.3), which we have shown is equivalent to Theorem 0.1.2.

Theorem 1.2.7. *For all $n \geq 2$, there is a bijection*

$$\gamma : \mathfrak{B}_n^0 \rightarrow \bigsqcup_{0 \leq m \leq n-2} \mathfrak{B}_m^0 \times \mathfrak{M}_{n-m},$$

such that if $\gamma(B) = (B', (\omega, b))$ then

$$(1.2.4) \quad \text{wt}(B) = \text{wt}(B')\text{wt}(\omega)$$

and

$$(1.2.5) \quad \text{bar}(B) = \text{bar}(B') + b,$$

where $\text{bar}(B)$ denotes the number of bars of B .

We will make use of another type of factorization of a word over an ordered alphabet used by Désarménien and Wachs [17]. For any alphabet A , let A^+ denote the set of words over A of positive length.

Definition 1.2.8 ([17]). An *increasing factorization* of a word w of positive length over a totally ordered alphabet A is a factorization $w = w_1 \cdot w_2 \cdots w_k$ such that

- (1) each w_i is of the form $a_i^{j_i} u_i$, where $a_i \in A$, $j_i > 0$ and

$$u_i \in \{x \in A : x < a_i\}^+$$

- (2) $a_1 \leq a_2 \leq \cdots \leq a_k$.

For example, $87 \cdot 8866 \cdot 995587 \cdot 95$ is an increasing factorization of the word $w := 87886699558795$ over the totally ordered alphabet of positive integers. Note that this factorization is different from the Lyndon factorization of w , which is $87 \cdot 8866 \cdot 99558795$

Proposition 1.2.9 ([17, Lemmas 3.1 and 4.3]). *A word over an ordered alphabet admits an increasing factorization if and only if its Lyndon type has no parts of size 1. Moreover, the increasing factorization is unique.*

Proof of Theorem 1.2.7. Given a banner B in \mathfrak{B}_n^0 , take its unique increasing factorization

$$B = B_1 \cdot B_2 \cdots B_k,$$

whose existence is guaranteed by Proposition 1.2.9. We will extract an increasing word from B_k . We have

$$B_k = a^p i_1 \cdots i_l,$$

where $p, l \geq 1$, and $a > i_1, i_2, \dots, i_l$. It follows from the definition of banner that a is a barred letter. Determine the unique index r such that $i_1 \geq \cdots \geq i_{r-1}$ are barred, while i_r is unbarred. Then let s be the unique index that satisfies $r \leq s \leq l$ and either

- (1) $i_r \leq i_{r+1} \leq \cdots \leq i_s$ are all unbarred and less than i_{r-1} (note that i_s can be equal to i_{r-1} in absolute value), while $i_{s+1} > i_{r-1}$ if $s < l$, or
- (2) $i_r \leq i_{r+1} \leq \cdots \leq i_{s-1}$ are all unbarred and less than i_{r-1} , and i_s is barred and less than or equal to i_{r-1} .

Case 1: $s = l$. In this case (1) must hold since the last letter of a banner is unbarred. Let ω be the weakly increasing rearrangement of B_k with bars removed and let

$$(1.2.6) \quad B' = B_1 \cdot B_2 \cdots B_{k-1}.$$

To see that B' is a banner, one need only note that the last letter of B_{k-1} is unbarred (as is the last letter of each B_i). Since (1.2.6) is an increasing factorization of B' , it follows from Proposition 1.2.9 that the Lyndon type of B' has no parts of size 1. Let

$$\gamma(B) = (B', (\omega, b)),$$

where b is the number of bars of B_k . Clearly $p \leq b < p + l$; so (ω, b) is a marked sequence for which (1.2.4) and (1.2.5) hold. For example, if

$$B = \bar{2}\bar{2}\bar{2}\bar{1} \cdot \bar{5}2\bar{2}\bar{4}2 \cdot \bar{8}\bar{8}\bar{7}\bar{5}2235.$$

then $a = \bar{8}$, $p = 2$, $r = 3$ and $s = 6 = l$. It follows that

$$(\omega, b) = (22355788, 4) \text{ and } B' = \bar{2}\bar{2}\bar{2}\bar{1} \cdot \bar{5}2\bar{2}\bar{4}2.$$

Case 2: $s < l$. In this case either (1) or (2) can hold. Let b be the number of bars of i_1, i_2, \dots, i_s . Clearly $b < s$. If (1) holds then $r > 1$ since $i_{s+1} > i_{r-1}$; so $b > 0$. If (2) holds clearly $b > 0$. Let ω be the increasing rearrangement of i_1, i_2, \dots, i_s with bars removed, let $B'_k = a^p i_{s+1} \cdots i_l$, let

$$(1.2.7) \quad B' = B_1 \cdot B_2 \cdots B_{k-1} \cdot B'_k,$$

and let

$$\gamma(B) = (B', (\omega, b)).$$

Clearly B' is a banner with increasing factorization given by (1.2.7) and (ω, b) is a marked sequence for which (1.2.4) and (1.2.5) hold. For example, if

$$B = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}2\bar{2}\bar{4}2 \cdot \bar{8}\bar{8}\bar{7}\bar{5}223\bar{5}\bar{6}24$$

then $a = \bar{8}$, $p = 2$, $r = 3$, and $s = 6 < l$. Hence

$$(\omega, b) = (223557, 2) \text{ and } B' = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}2\bar{2}\bar{4}2 \cdot \bar{8}\bar{8}\bar{6}24.$$

If

$$B = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}2\bar{2}\bar{4}2 \cdot \bar{8}\bar{8}\bar{7}\bar{5}223\bar{5}4\bar{6}24$$

then $a = \bar{8}$, $p = 2$, $r = 3$, and $s = 6 < l$. Hence

$$(\omega, b) = (223557, 3) \text{ and } B' = \bar{2}\bar{2}\bar{2}1 \cdot \bar{5}2\bar{2}\bar{4}2 \cdot \bar{8}\bar{8}4\bar{6}24.$$

In order to prove that the map γ is a bijection, we describe its inverse. Let $(B', (\omega, b)) \in \mathfrak{B}_m^0 \times \mathfrak{M}_{n-m}$, where $0 \leq m \leq n - 2$. Let

$$B' = B_1 \cdot B_2 \cdots B_{k-1}$$

be the unique increasing factorization of B' , whose existence is guaranteed by Proposition 1.2.9, and let a be the largest letter of B_{k-1} . We also let ω_i denote the i th letter of ω .

Case 1: $|a| \leq \omega_{n-m}$. Let

$$B_k = \bar{\omega}_{n-m} \cdots \bar{\omega}_{n-m-b+1} \omega_1 \cdots \omega_{n-m-b}.$$

Clearly B_k is a banner with exactly b bars that are placed on a rearrangement of ω . Now let

$$B = B_1 \cdot B_2 \cdots B_{k-1} \cdot B_k.$$

It is easy to see that this is an increasing decomposition of a banner and that equations (1.2.4) and (1.2.5) hold.

Case 2: $|a| > \omega_{n-m}$. In this case we expand the banner B_{k-1} by inserting the letters of ω in the following way. Suppose

$$B_{k-1} = a^p j_1 \cdots j_l,$$

where $p, l \geq 1$, and $a > j_i$ for all i . If $j_1 > \bar{\omega}_{n-m-b+1}$ let

$$\tilde{B}_{k-1} = a^p \bar{\omega}_{n-m} \cdots \bar{\omega}_{n-m-b+1} \omega_1 \cdots \omega_{n-m-b} j_1, \dots, j_l.$$

Otherwise if $j_1 \leq \bar{\omega}_{n-m-b+1}$ let

$$\tilde{B}_{k-1} = a^p \bar{\omega}_{n-m} \cdots \bar{\omega}_{n-m-b+2} \omega_1 \cdots \omega_{n-m-b} \bar{\omega}_{n-m-b+1} j_1, \dots, j_l.$$

In both cases \tilde{B}_{k-1} is a banner. Now let

$$B = B_1 \cdot B_2 \cdots B_{k-2} \cdot \tilde{B}_{k-1}.$$

It is easy to see that this is an increasing decomposition of a banner and that equations (1.2.4) and (1.2.5) hold. It is also easy to check that the map $(B', (\omega, b)) \mapsto B$ is the inverse of γ . \square

Remark 1.2.10. The bijection γ when restricted to banners with distinct letters (permutations) reduces to a bijection that Stembridge [56] constructed to study the representation of the symmetric group on the cohomology of the toric variety associated with the type A Coxeter complex (see Section 1.5). For words with distinct letters the notion of decreasing decomposition coincides with the notion of Lyndon decomposition. In Stembridge’s work the Lyndon decomposition corresponds to the cycle decomposition of a permutation. Although the term “marked sequence” is borrowed from Stembridge’s paper, he defines the term differently from the way in which we do. However there is a close connection between his marked sequences and ours.

Remark 1.2.11. In Section 1.5.2 we discuss a connection, pointed out to us by Richard Stanley, between banners and words with no adjacent repeats. This connection can be used to provide an alternative to Step 3 in our proof of Theorem 0.1.2.

1.3. ALTERNATIVE FORMULATIONS

In this section we present some equivalent formulations of Theorem 0.1.2 and some immediate consequences.

Corollary 1.3.1 (of Theorem 0.1.2). *Let $Q_n(t, r) = \sum_{j,k \geq 0} Q_{n,j,k} t^j r^k$. Then $Q_n(t, r)$ satisfies the following recurrence relation:*

$$(1.3.1) \quad Q_n(t, r) = r^n h_n + \sum_{k=0}^{n-2} Q_k(t, r) h_{n-k} t[n-k-1]_t.$$

Proof. The recurrence relation is equivalent to

$$\sum_{k=0}^n Q_k(t, r) h_{n-k} t[n-k-1]_t = -r^n h_n.$$

Taking the generating function we have

$$\left(\sum_{n \geq 0} Q_n(t, r) z^n \right) \left(\sum_{n \geq 0} h_n t[n-1]_t z^n \right) = -H(rz).$$

The result follows from this. \square

The right hand side of (0.1.3) is the Frobenius characteristic of a graded permutation representation that Stembridge [56] described in terms of \mathfrak{S}_n acting on “marked words”. From his work we were led to the following formula, which can easily be proved by showing that the right hand side satisfies the recurrence relation (1.3.1).

Corollary 1.3.2. *For all $n \geq 0$,*

$$(1.3.2) \quad Q_n(t, r) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_0 \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} r^{k_0} h_{k_0} \prod_{i=1}^m h_{k_i} t^{[k_i - 1]}_t.$$

Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad \text{and} \quad \begin{bmatrix} n \\ k_0, \dots, k_m \end{bmatrix}_q = \frac{[n]_q!}{[k_0]_q! [k_1]_q! \cdots [k_m]_q!}.$$

By taking the principal stable specialization of both sides of the recurrence relation (1.3.1) and the formula (1.3.2), we have the following result.

Corollary 1.3.3. *For all $n \geq 0$,*

$$A_n^{\text{maj,exc,fix}}(q, t, r) = r^n + \sum_{k=0}^{n-2} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k^{\text{maj,exc,fix}}(q, t, r) tq[n-k-1]_{tq},$$

and

$$A_n^{\text{maj,exc,fix}}(q, t, r) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_0 \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} \begin{bmatrix} n \\ k_0, \dots, k_m \end{bmatrix}_q r^{k_0} \prod_{i=1}^m tq[k_i - 1]_{tq}.$$

Gessel and Reutenauer [30] and Wachs [60] derive a major index q -analog of the classical formula for the number of derangements in \mathfrak{S}_n (or more generally the number of permutations with a given number of fixed points). As an immediate consequence of [60, Corollary 3], one can obtain a (maj, exc) generalization. Since this generalization also follows from Corollary 0.1.3 and is not explicitly stated in [60], we state and prove it here. For all $n \in \mathbb{N}$, let \mathcal{D}_n be the set of derangements in \mathfrak{S}_n .

Corollary 1.3.4 (of Corollary 0.1.3). *For all $n \geq 0$, we have*

$$(1.3.3) \quad \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{fix}(\sigma) = k}} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\sigma \in \mathcal{D}_{n-k}} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}.$$

Consequently,

$$(1.3.4) \quad \sum_{\sigma \in \mathcal{D}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k}^{\text{maj,exc}}(q, t).$$

Proof. Since the left hand side of (1.3.3) equals the coefficient of $r^k \frac{z^n}{[n]_q!}$ on the left hand side of (0.1.5), we have that the left hand side of (1.3.3) is $\begin{bmatrix} n \\ k \end{bmatrix}_q$ times the coefficient of $\frac{z^{n-k}}{[n-k]_q!}$ in $(1-tq)/(\exp_q(z tq) - tq \exp_q(z))$. This coefficient is precisely $\sum_{\sigma \in \mathcal{D}_{n-k}} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$.

By summing (1.3.3) over all k and applying Gaussian inversion we obtain (1.3.4). \square

We point out that our results pertaining to major index have comajor index versions. Recall that the *comajor index* of $\sigma \in \mathfrak{S}_n$ is defined to be

$$\text{comaj}(\sigma) := \sum_{i \in [n-1] \setminus \text{DES}(\sigma)} i = \binom{n}{2} - \text{maj}(\sigma).$$

For example, we have the following comajor index version of Corollary 0.1.3.

Corollary 1.3.5 (of Corollary 0.1.3). *We have*

$$(1.3.5) \quad \sum_{n \geq 0} A_n^{\text{comaj,exc,fix}}(q, t, r) \frac{z^n}{[n]_q!} = \frac{(1-tq^{-1}) \text{Exp}_q(rz)}{\text{Exp}_q(z tq^{-1}) - (tq^{-1}) \text{Exp}_q(z)}$$

Proof. Use the facts that $[n]_{q^{-1}}! = q^{-\binom{n}{2}} [n]_q!$ and $\exp_{q^{-1}}(z) = \text{Exp}_q(z)$ to show that equations (0.1.5) and (1.3.5) are equivalent. \square

We also have the following comajor index version of Corollary 1.3.4.

Corollary 1.3.6. *For all $n \geq 0$, we have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{fix}(\sigma) = k}} q^{\text{comaj}(\sigma)} t^{\text{exc}(\sigma)} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\sigma \in \mathcal{D}_{n-k}} q^{\text{comaj}(\sigma)} t^{\text{exc}(\sigma)}.$$

Consequently,

$$\sum_{\sigma \in \mathcal{D}_n} q^{\text{comaj}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k}^{\text{comaj,exc}}(q, t).$$

1.4. SYMMETRY AND UNIMODALITY

1.4.1. Eulerian quasisymmetric functions. It is well known that the Eulerian numbers $a_{n,j}$ form a symmetric and unimodal sequence for each fixed n ; i.e., $a_{n,j} = a_{n,n-1-j}$ for all $j = 0, 1, \dots, n-1$ and

$$a_{n,0} \leq a_{n,1} \leq \dots \leq a_{n, \lfloor \frac{n-1}{2} \rfloor} = a_{n, \lfloor \frac{n}{2} \rfloor} \geq \dots \geq a_{n,n-2} \geq a_{n,n-1}.$$

In this subsection we discuss symmetry and unimodality of the coefficients of t^j in the Eulerian quasisymmetric functions and the q - and (q, p) -Eulerian polynomials.

Let $f(t) := f_r t^r + f_{r+1} t^{r+1} + \dots + f_s t^s$ be a nonzero polynomial in t whose coefficients come from a partially ordered ring R . We say that $f(t)$ is *t-symmetric* with center of symmetry $\frac{s+r}{2}$ if $f_{r+i} = f_{s-i}$ for all $i = 0, \dots, s - r$. We say that $f(t)$ is also *t-unimodal* if

$$(1.4.1) \quad f_r \leq_R f_{r+1} \leq_R \dots \leq_R f_{\lfloor \frac{s+r}{2} \rfloor} = f_{\lfloor \frac{s+r+1}{2} \rfloor} \geq_R \dots \geq_R f_{s-1} \geq_R f_s.$$

Let Par be the set of all partitions of all nonnegative integers. By choosing a \mathbb{Q} -basis $b = \{b_\lambda : \lambda \in \text{Par}\}$ for the space of symmetric functions, we obtain the partial order relation given by $f \leq_b g$ if $g - f$ is b -positive, where a symmetric function is said to be b -positive if it is a nonnegative linear combination of elements of the basis $\{b_\lambda\}$. Here we are concerned with the h -basis, $\{h_\lambda : \lambda \in \text{Par}\}$, where $h_\lambda = h_{\lambda_1} \dots h_{\lambda_k}$ for $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$, and the Schur basis $\{s_\lambda : \lambda \in \text{Par}\}$. Since h -positivity implies Schur-positivity, the following result also holds for the Schur basis.

Theorem 1.4.1. *Using the h -basis to partially order the ring of symmetric functions, we have for all n, j, k ,*

- (1) *the Eulerian quasisymmetric functions $Q_{n,j,k}$ and $Q_{n,j}$ are h -positive symmetric functions,*
- (2) *the polynomial $\sum_{j=0}^{n-1} Q_{n,j,k} t^j$ is t -symmetric and t -unimodal with center of symmetry $\frac{n-k}{2}$,*
- (3) *the polynomial $\sum_{j=0}^{n-1} Q_{n,j} t^j$ is t -symmetric and t -unimodal with center of symmetry $\frac{n-1}{2}$.*

Proof. Part (1) is a corollary of Theorem 0.1.2 (see also Corollary 1.3.2).

Parts (2) and (3) follow from Theorem 0.1.2 and results of Stembridge [56] on the symmetric function given on the right hand side of (0.1.3). For the sake of completeness, we include a proof of Parts (2) and (3) based on Stembridge's work.

It is well-known that the product of two symmetric unimodal polynomials in $\mathbb{N}[t]$ with respective centers of symmetry c_1 and c_2 , is symmetric and unimodal with center of symmetry $c_1 + c_2$. This result and the proof given in [52, Proposition 1.2] hold more generally for polynomials over an arbitrary partially ordered ring. By Corollary 1.3.2 we

have

$$\sum_{j=0}^{n-1} Q_{n,j,k} t^j = \sum_{m=0}^{\lfloor \frac{n-k}{2} \rfloor} \sum_{\substack{k_1, \dots, k_m \geq 2 \\ \sum k_i = n-k}} h_k \prod_{i=1}^m h_{k_i} t[k_i - 1]_t$$

Each term $h_k \prod_{i=1}^m h_{k_i} t[k_i - 1]_t$ is t -symmetric and t -unimodal with center of symmetry $\sum_{i \geq 0} \frac{k_i}{2} = \frac{n-k}{2}$. Hence the sum of these terms has the same property.

With a bit more effort we show that Part (3) also follows from Corollary 1.3.2. We have

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{j=0}^{n-1} Q_{n,j,1} t^j + \sum_{j=0}^{n-1} Q_{n,j,0} t^j + \sum_{k \geq 2} \sum_{j=0}^{n-1} Q_{n,j,k} t^j.$$

By Part (2) we need only show that

$$X(t) := \sum_{j=0}^{n-1} Q_{n,j,0} t^j + \sum_{k \geq 2} \sum_{j=0}^{n-1} Q_{n,j,k} t^j$$

is t -symmetric and t -unimodal with center of symmetry $\frac{n-1}{2}$. For any sequence of positive integers (k_1, \dots, k_m) , let

$$G_{k_1, \dots, k_m} := \prod_{i=1}^m h_{k_i} t[k_i - 1]_t.$$

We have by Corollary 1.3.2,

$$\sum_{j=0}^{n-1} Q_{n,j,0} t^j = \sum_{\substack{m \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} G_{k_1, \dots, k_m}$$

and

$$\sum_{k \geq 2} \sum_{j=0}^{n-1} Q_{n,j,k} t^j = \sum_{\substack{m \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} h_{k_1} G_{k_2, \dots, k_m}.$$

Hence

$$X(t) = \sum_{\substack{m \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} G_{k_1, \dots, k_m} + h_{k_1} G_{k_2, \dots, k_m}.$$

We claim that $G_{k_1, \dots, k_m} + h_{k_1} G_{k_2, \dots, k_m}$ is t -symmetric and t -unimodal with center of symmetry $\frac{n-1}{2}$. Indeed, we have

$$G_{k_1, \dots, k_m} + h_{k_1} G_{k_2, \dots, k_m} = h_{k_1} (t[k_1 - 1]_t + 1) G_{k_2, \dots, k_m}.$$

Clearly $t[k_1 - 1]_t + 1 = 1 + t + \dots + t^{k_1-1}$ is t -symmetric and t -unimodal with center of symmetry $\frac{k_1-1}{2}$, and G_{k_2, \dots, k_m} is t -symmetric and t -unimodal with center of symmetry $\frac{n-k_1}{2}$. Therefore our claim holds and $X(t)$ is t -symmetric and t -unimodal with center of symmetry $\frac{n-1}{2}$. \square

We now obtain analogous results for $A_n^{\text{maj,des,exc,fix}}$ and $A_n^{\text{maj,des,exc}}$ by applying the principal specializations. For the ring of polynomials $\mathbb{Q}[\mathbf{q}]$, where \mathbf{q} is a list of indeterminants, we use the partial order relation: for $f(\mathbf{q}), g(\mathbf{q}) \in \mathbb{Q}[(\mathbf{q})]$, define $f(\mathbf{q}) \leq_{\mathbf{q}} g(\mathbf{q})$ if $g(\mathbf{q}) - f(\mathbf{q})$ has nonnegative coefficients.

Lemma 1.4.2. *If f is a Schur positive homogeneous symmetric function of degree n then $(q; q)_n \Lambda(f)$ is a polynomial in q with nonnegative coefficients and $(p; q)_{n+1} \sum_{m \geq 0} \Lambda_m(f) p^m$ is a polynomial in q and p with nonnegative coefficients. Consequently if f and g are homogeneous symmetric functions of degree n and $f \leq_{\text{Schur}} g$ then*

$$(q; q)_n \Lambda(f) \leq_{(q)} (q; q)_n \Lambda(g)$$

and

$$(p; q)_{n+1} \sum_{m \geq 0} \Lambda_m(f) p^m \leq_{(q,p)} (p; q)_{n+1} \sum_{m \geq 0} \Lambda_m(g) p^m.$$

Proof. This follows from Lemma 1.1.1 and the fact that Schur functions are nonnegative linear combinations of fundamental quasisymmetric functions (cf. [55, pp. 360-361]). \square

Theorem 1.4.3. *For all n, k , let*

$$A_{n,k}^{\text{maj,des,exc}}(q, p, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{fix}(\sigma) = k}} q^{\text{maj}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)}.$$

Then

- (1) $A_{n,k}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is t -symmetric with center of symmetry $\frac{n-k}{2}$
- (2) $A_{n,0}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is t -symmetric and t -unimodal, with center of symmetry $\frac{n}{2}$
- (3) $A_{n,k}^{\text{maj,des,exc}}(q, 1, q^{-1}t)$ is t -symmetric and t -unimodal, with center of symmetry $\frac{n-k}{2}$
- (4) $A_n^{\text{maj,des,exc}}(q, 1, q^{-1}t)$ is t -symmetric and t -unimodal, with center of symmetry $\frac{n-1}{2}$.

Proof. Since h -positivity implies Schur positivity, we can use Lemma 1.4.2 to specialize Theorem 1.4.1. By Lemma 1.1.3, Parts (1) and (2) are obtained by specializing Part (2) of Theorem 1.4.1. By (1.1.12), Parts (3) and (4) are obtained by specializing Parts (2) and (3) of Theorem 1.4.1, respectively. \square

In Section 1.4.2 we conjecture that the t -symmetric polynomial $A_{n,k}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is t -unimodal even when $k \neq 0$ and $p \neq 1$. However, $A_n^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is not t -symmetric as can be seen in the computation,

$$\begin{aligned} A_4^{\text{maj,des,exc}}(q, p, t) = & 1 + (3p + 2pq + pq^2 + 2p^2q^2 + 2p^2q^3 + p^2q^4)t \\ & + (3p + pq + p^2q + 3p^2q^2 + 2p^2q^3 + p^3q^4)t^2 \\ & + pt^3. \end{aligned}$$

1.4.2. Cycle type Eulerian quasisymmetric functions. By means of ornaments we extend the symmetry results of Section 1.4.1 to the refinements $Q_{\lambda,j}$, and the *cycle type* (q, p) -Eulerian polynomials defined for each partition $\lambda \vdash n$ by

$$A_{\lambda}^{\text{maj,des,exc}}(q, p, t) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma) = \lambda}} q^{\text{maj}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)}.$$

Theorem 1.4.4. *For all $\lambda \vdash n$ and $j = 0, 1, \dots, n-1$, the Eulerian quasisymmetric function $Q_{\lambda,j}$ is a symmetric function.*

Proof. For each $k \in \mathbb{P}$, we will construct an involution ψ on necklaces that exchanges the number of occurrences of the letter k (barred or unbarred) and $k+1$ (barred or unbarred) in a necklace, but preserves the number of occurrences of all other letters and the total number of bars. The result will then follow from Corollary 1.2.3. We start with necklaces that contain only the letters $\{k, \bar{k}, k+1, \overline{k+1}\}$. Let R be such a necklace. We may assume without loss of generality that $k=1$. First replace all 1's with 2's and all 2's with 1's, leaving the bars in their original positions. The problem is that now each 2 that is followed by a 1 lacks a bar (call such a 2 *bad*) and each 1 that is followed by a 2 has a bar (call such a 1 *bad*). Since the number of bad 1's equals the number of bad 2's, we can move all the bars from the bad 1's to the bad 2's, thereby obtaining a necklace R' with the same number of bars as R but with the number of 1's and 2's exchanged. Let $\psi(R) = R'$. Clearly $\psi^2(R) = R$. For example if $R = (2\bar{2}1\bar{1}22\bar{2}\bar{2}11111)$ then we get $(1\bar{1}\bar{2}\bar{2}211\bar{1}\bar{1}\bar{2}2222)$ before the bars are adjusted. After the bars are moved we have $\psi(R) = (112\bar{2}\bar{2}11\bar{1}\bar{1}2222\bar{2})$.

Now we handle the case in which R has (barred and unbarred) k 's and $k+1$'s, and other letters which we will call intruders. The intruders enable us to form linear segments of R consisting only of (barred and unbarred) k 's and $k+1$'s. To obtain such a linear segment start with a letter of value k or $k+1$ that follows an intruder and read the letters of R in a clockwise direction until another intruder is encountered. For example if

$$(1.4.2) \quad R = (\bar{5}334\bar{4}\bar{3}\bar{3}\bar{3}\bar{3}6\bar{6}\bar{3}\bar{3}\bar{3}\bar{4}244)$$

and $k = 3$ then the segments are $334\bar{4}\bar{3}\bar{3}\bar{3}$, $\bar{3}\bar{3}\bar{3}\bar{4}$, and 44 .

There are two types of segments, even segments and odd segments. An even (odd) segment contains an even (odd) number of switches, where a switch is a letter of value k followed by one of value $k+1$ or a letter of value $k+1$ followed by one of value k . We handle the even and odd segments differently. In an even segment, we replace all k 's with $k+1$'s and all $k+1$'s with k 's, leaving the bars in their original positions. Again, at the switches, we have bad k 's and bad $k+1$'s, which are equinumerous. So we move all the bars from the bad k 's to the bad $k+1$'s to obtain a good segment. This preserves the number of bars and exchanges the number of k 's and $k+1$'s. For example, the even segment $334\bar{4}\bar{3}\bar{3}\bar{3}$ gets replaced by $443\bar{3}\bar{4}\bar{4}\bar{4}$ before the bars are adjusted. After the bars are moved we have $4\bar{4}\bar{3}\bar{3}\bar{4}\bar{4}\bar{4}$.

An odd segment either starts with a k and ends with a $k+1$ or vice-versa. Both cases are handled similarly. So suppose we have an odd segment of the form

$$k^{m_1}(k+1)^{m_2}k^{m_3}(k+1)^{m_4} \dots k^{m_{2r-1}}(k+1)^{m_{2r}},$$

where each $m_i > 0$ and the bars have been suppressed. The number of switches is $2r - 1$. We replace it with the odd segment

$$k^{m_2}(k+1)^{m_1}k^{m_4}(k+1)^{m_3} \dots k^{m_{2r}}(k+1)^{m_{2r-1}},$$

and put bars in their original positions. Again we may have created bad k 's (but not bad $k+1$'s); so we need to move some bars around. The positions of the bad k 's are in the set $\{N_1+m_2, N_2+m_4, \dots, N_r+m_{2r}\}$, where $N_i = \sum_{t=1}^{2i-2} m_t$. If there is a bar on the k in position N_i+m_{2i} we move it to position N_i+m_{2i-1} , which had been barless. This preserves the number of bars and exchanges the number of k 's and $k+1$'s. For example, the odd segment $\bar{3}\bar{3}\bar{3}\bar{4}$ gets replaced by $\bar{3}\bar{4}\bar{4}\bar{4}$ before the bars are adjusted. After the bars are moved we have $3\bar{4}\bar{4}\bar{4}$.

Let $\psi(R)$ be the necklace obtained by replacing all the segments in the way described above. For example if R is the necklace given in

(1.4.2) then

$$\psi(R) = (\bar{5}4\bar{4}33\bar{4}\bar{4}\bar{4}\bar{4}6\bar{6}3\bar{4}\bar{4}\bar{4}233).$$

It is easy to check that $\psi^2(R) = R$ for all necklaces R . Now extend the involution ψ to ornaments by applying ψ to each necklace of the ornament.

□

Theorem 1.4.5. *For all $\lambda \vdash n$ with exactly k parts equal to 1, and $j = 0, \dots, n - k$,*

$$Q_{\lambda,j} = Q_{\lambda,n-k-j}.$$

Proof. We construct a type-preserving involution γ on ornaments. Let R be an ornament of type λ . To obtain $\gamma(R)$, first we bar each unbarred letter of each nonsingleton necklace of R and unbar each barred letter. Next for each i , we replace each occurrence of the i th smallest value in R with the i th largest value leaving the bars intact. Clearly $\gamma(R)$ is an ornament with $n - k - j$ bars whenever R is an ornament with j bars. Also $\gamma^2(R) = R$. The result now follows from the fact that $Q_{\lambda,j}$ is symmetric (Theorem 1.4.4) and from Corollary 1.2.3. □

Remark 1.4.6. Although the equation

$$(1.4.3) \quad Q_{n,j} = Q_{n,n-1-j}$$

follows easily from Theorem 0.1.2, we can give a bijective proof by means of banners, using Theorem 1.2.6. We construct an involution τ on $\bigcup_{\lambda \vdash n} \mathfrak{B}_{\lambda,j}$ that is similar to the involution γ used in the proof of Theorem 1.4.5. Let B be a banner of length n with j bars. To obtain $\tau(B)$, first we bar each unbarred letter of B , except for the last letter, and unbar each barred letter. Next for each i , we replace each occurrence of the i th smallest value in B with the i th largest value, leaving the bars intact. Clearly $\tau(B)$ is a banner of length n , with $n - 1 - j$ bars.

Since $Q_{n,j,k}$ and $Q_{n,j}$ are h -positive, one might expect that the same is true for the refinement $Q_{\lambda,j}$. It turns out that this is not the case as the following computation using the Maple package ACE [59] shows,

$$(1.4.4) \quad Q_{(6),3} = 2h_{(4,2)} - h_{(4,1,1)} + h_{(3,2,1)} + h_{(5,1)}.$$

Therefore Theorem 1.4.1 cannot hold for $Q_{\lambda,n}$, as stated. However, by expanding in the Schur basis we obtain

$$Q_{(6),3} = 3s_{(6)} + 3s_{(5,1)} + 3s_{(4,2)} + s_{(3,3)} + s_{(3,2,1)},$$

which establishes Schur positivity of $Q_{(6),3}$. We have used ACE [59] to confirm the following conjecture for $\lambda = (n)$ up to $n = 9$.

Conjecture 1.4.7. *Let $\lambda \vdash n$. For all $j = 0, 1, \dots, n-1$, the Eulerian quasisymmetric function $Q_{\lambda,j}$ is Schur positive. Moreover, $Q_{\lambda,j} - Q_{\lambda,j-1}$ is Schur positive if $1 \leq j \leq \frac{n-k}{2}$, where k is the number of parts of λ that are equal to 1. Equivalently, the t -symmetric polynomial $\sum_{j=0}^{n-1} Q_{\lambda,j} t^j$ is t -unimodal under the partial order relation on the ring of symmetric functions induced by the Schur basis.*

From Stembridge's work [57] one obtains a nice combinatorial description of the coefficients in the Schur function expansion of $Q_{n,j,k}$. It would be interesting to do the same for the refinement $Q_{\lambda,j}$, at least when $\lambda = (n)$. In Section 1.5.1 we discuss the expansion in the power sum symmetric function basis.

By Lemmas 1.1.3 and 1.4.2, we have the following consequence of Theorem 1.4.5 (and refinement of Part (1) of Theorem 1.4.3).

Theorem 1.4.8. *Let λ be a partition of n with exactly k parts of size 1. Then $A_{\lambda}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is t -symmetric with center of symmetry $\frac{n-k}{2}$.*

Conjecture 1.4.9. *Let λ be a partition of n with exactly k parts of size 1. Then the t -symmetric polynomial $A_{\lambda}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is t -unimodal (with center of symmetry $\frac{n-k}{2}$). Consequently for all n, k , $A_{n,k}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is t -symmetric and t -unimodal with center of symmetry $\frac{n-k}{2}$.*

The next result is easy to prove and does not rely on the $Q_{\lambda,j}$.

Proposition 1.4.10. *Let λ be a partition of n with exactly k parts of size 1. Then for all $j = 0, 1, \dots, n-1$,*

$$a_{\lambda,j}(q, p) = a_{\lambda, n-k-j}(1/q, q^n p).$$

Proof. This follows from the type preserving involution $\phi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ defined by letting $\phi(\sigma)$ be obtained from σ by writing σ in cycle form and replacing each i by $n-i+1$. Clearly ϕ sends permutations with j excedences to permutations with $n-k-j$ excedences. Also $i \in \text{DES}(\sigma)$ if and only if $n-i \in \text{DES}(\phi(\sigma))$. It follows that ϕ preserves des and

$$\text{maj}(\phi(\sigma)) = \text{des}(\sigma)n - \text{maj}(\sigma).$$

□

By combining Proposition 1.4.10 and Theorem 1.4.8, we obtain the following result.

Corollary 1.4.11. *Let λ be a partition of n with exactly k parts of size 1. Then for all $j = 0, 1, \dots, n-1$,*

$$a_{\lambda,j}(q, p) = q^{2j+k-n} a_{\lambda,j}(1/q, q^n p).$$

Equivalently, the coefficient of p^{dt^j} in $A_\lambda^{\text{maj,des,exc}}(q, p, t)$ is q -symmetric with center of symmetry $j + \frac{k+nd-n}{2}$.

Conjecture 1.4.12. *The coefficient of p^{dt^j} in $A_\lambda^{\text{maj,des,exc}}(q, p, t)$ is q -unimodal.*

1.5. FURTHER PROPERTIES OF THE EULERIAN QUASISYMMETRIC FUNCTIONS

1.5.1. Partitions with a single part. The ornament characterization of $Q_{\lambda,j}$ yields a plethystic formula expressing $Q_{\lambda,j}$ in terms of $Q_{(i),k}$. (Note (i) stands for the partition with a single part i .) Let f be a symmetric function in variables x_1, x_2, \dots and let g be a formal power series with positive integer coefficients. Choose any ordering of the monomials of g , where a monomial appears in the ordering m times if its coefficient is m . The *plethysm* of f and g , denoted $f[g]$ is defined to be the formal power series obtained from f by replacing x_i by the i th monomial of g , for each i . The following result is an immediate consequence of Corollary 1.2.3.

Corollary 1.5.1. *Let λ be a partition with m_i parts of size i for all i . Then*

$$(1.5.1) \quad \sum_{j=0}^{|\lambda|-1} Q_{\lambda,j} t^j = \prod_{i \geq 1} h_{m_i} \left[\sum_{j=0}^{i-1} Q_{(i),j} t^j \right].$$

Consequently,

$$\sum_{n,j \geq 0} Q_{n,j} t^j z^n = \sum_{n \geq 0} h_n \left[\sum_{i,j \geq 0} Q_{(i),j} t^j z^i \right].$$

By specializing (1.5.1) we obtain the following result. Recall that (λ, μ) denotes the partition of $m+n$ obtained by concatenating λ and μ and then appropriately rearranging the parts.

Corollary 1.5.2. *Let λ and μ be partitions of m and n , respectively. If λ and μ have no common parts then*

$$A_{(\lambda,\mu)}^{\text{maj,exc}}(q, t) = \left[\begin{matrix} m+n \\ m \end{matrix} \right]_q A_\lambda^{\text{maj,exc}}(q, t) A_\mu^{\text{maj,exc}}(q, t).$$

Consequently, if λ has no parts equal to 1 then for all j ,

$$a_{(\lambda,1^m),j}(q, 1) = \left[\begin{matrix} m+n \\ m \end{matrix} \right]_q a_{\lambda,j}(q, 1).$$

Corollary 1.5.3. *For all j and k we have*

$$Q_{(2^j, 1^k), j} = h_j[h_2]h_k.$$

Moreover,

$$(1.5.2) \quad \sum_{\lambda, j} Q_{\lambda, j} t^j z^{|\lambda|} = \prod_{i \geq 1} (1 - x_i z)^{-1} \prod_{1 \leq i \leq j} (1 - x_i x_j t z^2)^{-1},$$

where λ ranges over all partitions with no parts greater than 2.

By specializing (1.5.2) one can obtain formulas for the generating functions of the (maj, des, exc) and the (maj, exc) enumerators of involutions. Since exc and fix determine each other for involutions, these formulas can be immediately obtained from the formulas that Gessel and Reutenauer derived for maj, des, fix in [30, Theorem 7.1].

It follows from Corollary 1.5.1 that in order to prove the conjecture on Schur positivity of $Q_{\lambda, j}$ (Conjecture 1.4.7), it suffices to prove it for all λ with a single part because the plethysm of Schur positive symmetric functions is a Schur positive symmetric function. Note that if λ has no parts greater than 2, then by Corollary 1.5.3, we conclude that $Q_{\lambda, j}$ is Schur positive.

The Frobenius characteristic is a fundamental isomorphism from the ring of virtual representations of the symmetric groups to the ring of homogeneous symmetric functions over the integers. We recall the definition here. For a virtual representation V of \mathfrak{S}_n , let χ_λ^V denote the value of the character of V on the conjugacy class of type λ . If λ has m_i parts of size i for each i define

$$z_\lambda := 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!.$$

Let

$$p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k},$$

for $\lambda = (\lambda_1, \dots, \lambda_k)$, where p_n is the power symmetric function $\sum_{i \geq 0} x_i^n$. The Frobenius characteristic of a virtual representation V of \mathfrak{S}_n is defined as follows:

$$\text{ch}V := \frac{1}{n!} \sum_{\lambda \vdash n} z_\lambda \chi_\lambda^V p_\lambda.$$

Recall that the Frobenius characteristic of a virtual representation of \mathfrak{S}_n is h -positive if and only if it is a permutation representation induced from Young subgroups. Hence by part (1) of Theorem 1.4.1, the Eulerian quasisymmetric function $Q_{n, j, k}$ is the Frobenius characteristic of a permutation representation induced from Young subgroups. However (1.4.4) shows that this is not the case in general for the refined Eulerian quasisymmetric functions $Q_{\lambda, j}$. (It can be shown that $Q_{(6), 3}$ is not the Frobenius characteristic of any permutation representation at all.)

Recall also that the Frobenius characteristic of a virtual representation is Schur positive if and only if it is an actual representation. Hence if $V_{\lambda,j}$ is the virtual representation whose Frobenius characteristic is $Q_{\lambda,j}$ then Conjecture 1.4.7 says that $V_{\lambda,j}$ is an actual representation.

Proposition 1.5.4. *Let $\lambda \vdash n$. Then the dimension of $V_{\lambda,j}$ equals the number of permutations of cycle type λ with j excedances. Moreover, the dimension of $V_{(n),j}$ is the Eulerian number $a_{n-1,j-1}$.*

Proof. The dimension of $V_{\lambda,j}$ is the coefficient of $x_1 x_2 \cdots x_n$ in $Q_{\lambda,j}$. Using the definition of $Q_{\lambda,j}$, this is the number of permutations of cycle type λ with j excedances.

The set of n -cycles with j excedances maps bijectively to the set $\{\sigma \in \mathfrak{S}_{n-1} : \text{des}(\sigma) = j - 1\}$. Indeed, the bijection is obtained by writing the cycle in the form (c_1, \dots, c_{n-1}, n) and then extracting the word $c_{n-1} \cdots c_1$. Hence the number of n -cycles with j excedances is the Eulerian number $a_{n-1,j-1}$. \square

We have a conjecture for the character of $V_{(n),j}$, which generalizes Proposition 1.5.4. We have confirmed our conjecture up to $n = 8$ using the Maple package ACE [59]. Equivalently, our conjecture gives the coefficients in the expansion of $Q_{(n),j}$ in the basis of power symmetric functions and implies that $Q_{(n),j}$ is p -positive. For a polynomial $F(t) = a_0 + a_1 t + \dots + a_k t^k$ and a positive integer m , define $F(t)_m$ to be the polynomial obtained from $F(t)$ by erasing all terms $a_i t^i$ such that $\gcd(m, i) \neq 1$. For example, if $F(t) = 1 + t + 2t^2 + 3t^3$ then $F(t)_2 = t + 3t^3$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, define

$$g(\lambda) := \gcd(\lambda_1, \dots, \lambda_k)$$

Conjecture 1.5.5. *For $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, let*

$$G_\lambda(t) := \left(t A_{k-1}(t) \prod_{i=1}^k [\lambda_i]_t \right)_{g(\lambda)}.$$

Then

$$\sum_{j=0}^{n-1} Q_{(n),j} t^j = \frac{1}{n!} \sum_{\lambda \vdash n} z_\lambda G_\lambda(t) p_\lambda.$$

Equivalently, the character of $V_{(n),j}$ evaluated on conjugacy class λ is the coefficient of t^j in $G_\lambda(t)$.

Conjecture 1.5.5 holds whenever $\lambda_k = 1$, see Corollary 1.5.8. It follows from this that the character of $V_{(n),j}$ evaluated at $\lambda = 1^n$ is the

Eulerian number $a_{n-1,j-1}$. This special case of the conjecture is also a consequence of Proposition 1.5.4 because any character evaluated at 1^n is the dimension of the representation. In the tables below we give the character values for $V_{(n),j}$ at conjugacy class λ , which were computed using ACE [59] and confirm the conjecture up to $n = 8$. The columns are indexed by n, j and the rows by the partitions λ .

	4, 1	4, 2
4	1	0
31	1	1
2 ²	1	0
21 ²	1	2
1 ⁴	1	4

	5, 1	5, 2
5	1	1
41	1	1
32	1	2
31 ²	1	2
2 ² 1	1	3
21 ³	1	5
1 ⁵	1	11

	6, 1	6, 2	6, 3
6	1	0	0
51	1	1	1
42	1	0	2
3 ²	1	2	0
41 ²	1	2	2
321	1	3	4
2 ³	1	0	6
31 ³	1	5	6
2 ² 1 ²	1	6	10
21 ⁴	1	12	22
1 ⁶	1	26	66

	7, 1	7, 2	7, 3
7	1	1	1
61	1	1	1
52	1	2	2
43	1	2	3
51 ²	1	2	2
421	1	3	4
3 ² 1	1	3	5
32 ²	1	4	7
41 ³	1	5	6
321 ²	1	6	11
2 ³ 1	1	7	16
31 ⁴	1	12	23
2 ² 1 ³	1	13	34
21 ⁵	1	27	92
1 ⁷	1	57	302

	8, 1	8, 2	8, 3	8, 4
8	1	0	1	0
71	1	1	1	1
62	1	0	2	0
53	1	2	3	3
4 ²	1	0	3	0
61 ²	1	2	2	2
521	1	3	4	4
431	1	3	5	6
42 ²	1	0	7	0
3 ² 2	1	4	8	10
51 ³	1	5	6	6
421 ²	1	6	11	12
3 ² 1 ²	1	6	12	16
32 ² 1	1	7	17	22
2 ⁴	1	0	23	0
41 ⁴	1	12	23	24
321 ³	1	13	35	46
2 ³ 1 ²	1	14	47	68
31 ⁵	1	27	93	118
2 ² 1 ⁴	1	28	119	184
21 ⁶	1	58	359	604
1 ⁸	1	120	1191	2416

Conjecture 1.5.5 resembles the following immediate consequence of Theorem 0.1.2 and Stembridge's computation [56, Proposition 3.3] of the coefficients in the expansion of the $r = 1$ case of the right hand side of (0.1.3), in the basis of power symmetric functions.

Proposition 1.5.6. *We have*

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \frac{1}{n!} \sum_{\lambda \vdash n} z_\lambda \left(A_{\ell(\lambda)}(t) \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_t \right) p_\lambda,$$

where $\ell(\lambda)$ denotes the length of the partition λ and λ_i denotes its i th part.

By using banners we are able to prove the following curious fact about the representation $V_{(n),j}$.

Theorem 1.5.7. *For all $j = 0, \dots, n-1$, the restriction of $V_{(n),j}$ to \mathfrak{S}_{n-1} is the permutation representation whose Frobenius characteristic is $Q_{n-1,j-1}$.*

Proof. Given a homogeneous symmetric function f of degree n , let \tilde{f} be the polynomial obtained from f by setting $x_i = 0$ for all $i > n$. Since f is symmetric, \tilde{f} determines f .

We use the fact that for any virtual representation V of \mathfrak{S}_n ,

$$(1.5.3) \quad \widetilde{\text{ch } V \downarrow_{\mathfrak{S}_{n-1}}} = \frac{\partial}{\partial x_n} \widetilde{\text{ch } V} \Big|_{x_n=0}.$$

By Theorem 1.2.6, $\widetilde{Q_{(n),j}}$ is the sum of weights of Lyndon banners of length n , with j bars, whose letters have value at most n . The partial derivative with respect to x_n of the weight of a such a Lyndon banner B is 0 unless n or \bar{n} appears in B . Since B is Lyndon, \bar{n} must be its first letter. If the partial derivative is not 0 after setting $x_n = 0$ then all the other letters of B must be less in absolute value than n . In this case, the partial derivative is the weight of the banner B' obtained from B by removing its first letter \bar{n} . We thus have

$$(1.5.4) \quad \frac{\partial}{\partial x_n} \widetilde{\text{ch } V_{(n),j}} \Big|_{x_n=0} = \sum_{B'} \text{wt}(B'),$$

where B' ranges over the set of all banners obtained by removing the first letter from a Lyndon banner of length n with j bars, whose first letter is \bar{n} and whose other letters have value strictly less than n . Clearly this is the set of all banners of length $n-1$, with $j-1$ bars, whose

letters have value at most $n - 1$. Thus the sum on the right hand side of (1.5.4) is precisely $\widetilde{Q_{n-1,j-1}}$. It therefore follows from (1.5.3) that

$$\text{ch } \widetilde{V_{(n),j} \downarrow_{\mathfrak{S}_{n-1}}} = \widetilde{Q_{n-1,j-1}},$$

which implies

$$\text{ch } V_{(n),j} \downarrow_{\mathfrak{S}_{n-1}} = Q_{n-1,j-1},$$

□

Theorem 1.5.7, along with Proposition 1.5.6, allows us to prove that Conjecture 1.5.5 holds when λ has a part of size one.

Corollary 1.5.8. *Let $\lambda = (\lambda_1, \dots, \lambda_k = 1)$ be a partition of n and let $\sigma \in \mathfrak{S}_n$ have cycle type λ . Then the character of $V_{(n),j}$ evaluated at σ is the coefficient of t^j in $tA_{k-1}(t) \prod_{i=1}^k [\lambda_i]_t$.*

Proof. We may assume that σ fixes n , which allows us to think of σ as an element of \mathfrak{S}_{n-1} . Let $V_{n-1,j-1}$ be the representation of \mathfrak{S}_{n-1} whose Frobenius characteristic is $Q_{n-1,j-1}$. By Theorem 1.5.7, the character value in question is equal to the character value of σ on $V_{n-1,j-1}$. Corollary 1.5.8 now follows from Proposition 1.5.6. □

1.5.2. Other occurrences. The Eulerian quasisymmetric functions refine symmetric functions that have appeared earlier in the literature. A multiset derangement of order n is a $2 \times n$ array of positive integers whose top row is weakly increasing, whose bottom row is a rearrangement of the top row, and whose columns contain distinct entries. An excedance of a multiset derangement $D = (d_{i,j})$ is a column j such that $d_{1,j} < d_{2,j}$. Given a multiset derangement $D = (d_{i,j})$, let $x^D := \prod_{i=1}^n x_{d_{1,i}}$. For all $j < n$, let $\mathcal{D}_{n,j}$ be the set of all derangements in \mathfrak{S}_n with j excedances and let $\mathcal{MD}_{n,j}$ be the set of all multiset derangements of order n with j excedances. Set

$$d_{n,j}(\mathbf{x}) := \sum_{D \in \mathcal{MD}_{n,j}} x^D.$$

Askey and Ismail [1] (see also [34]) proved the following t -analog of MacMahon's [36, Sec. III, Ch. III] result on multiset derangements

$$(1.5.5) \quad \sum_{j,n \geq 0} d_{n,j}(\mathbf{x}) t^j z^n = \frac{1}{1 - \sum_{i \geq 2} t[i-1]_t e_i z^i}.$$

Corollary 1.5.9 (to Theorem 0.1.2). *For all $n, j \geq 0$ we have*

$$(1.5.6) \quad d_{n,j}(\mathbf{x}) = \omega Q_{n,j,0} = \sum_{\sigma \in \mathcal{D}_{n,j}} F_{[n-1] \setminus \text{DEX}(\sigma), n},$$

where ω is the standard involution on the ring of symmetric functions, which takes h_n to e_n . Consequently,

$$(1.5.7) \quad d_{n,j}(1, q, q^2, \dots) = (q; q)_n^{-1} \sum_{\sigma \in \mathcal{D}_{n,j}} q^{\text{comaj}(\sigma)+j},$$

and

$$(1.5.8) \quad \sum_{m \geq 0} p^m \Lambda_m d_{n,j}(\mathbf{x}) = (p; q)_{n+1}^{-1} \sum_{\sigma \in \mathcal{D}_{n,j}} q^{\text{comaj}(\sigma)+j} p^{n-\text{des}(\sigma)+1}.$$

Proof. The right hand side of (1.5.5) can be obtained by applying ω to the right hand side of (0.1.4) and setting $r = 0$. Hence the first equation of (1.5.6) holds.

The involution on the ring \mathcal{Q} of quasisymmetric functions defined by $F_{S,n} \mapsto F_{[n-1] \setminus S, n}$ restricts to ω on the ring of symmetric functions (cf. [55, Exercise 7.94.a]). Hence the second equation of (1.5.6) follows from the definition of $Q_{n,j,0}$. Equations (1.5.7) and (1.5.8) now follow from Lemmas 1.1.1 and 1.1.2. \square

Let W_n be the set of all words of length n over alphabet \mathbb{P} with no adjacent repeats, i.e.,

$$W_n := \{w \in \mathbb{P}^n : w(i) \neq w(i+1) \forall i = 1, 2, \dots, n-1\}.$$

Define the enumerator

$$Y_n(x_1, x_2, \dots) := \sum_{w \in W_n} \mathbf{x}^w,$$

where $\mathbf{x}^w := x_{w(1)} \cdots x_{w(n)}$. In [14] Carlitz, Scoville and Vaughan prove the identity

$$(1.5.9) \quad \sum_{n \geq 0} Y_n(\mathbf{x}) z^n = \frac{\sum_{i \geq 0} e_i z^i}{1 - \sum_{i \geq 2} (i-1) e_i z^i}.$$

(See Dollhopf, Goulden and Greene [20] and Stanley [53] for alternative proofs.) It was observed by Stanley that there is a straightforward generalization of (1.5.9).

Theorem 1.5.10 (Stanley (personal communication)). *For all $j < n$, define*

$$Y_{n,j}(x_1, x_2, \dots) := \sum_{\substack{w \in W_n \\ \text{des}(w) = j}} \mathbf{x}^w.$$

Then

$$(1.5.10) \quad \sum_{n \geq 0} Y_{n,j}(\mathbf{x}) t^j z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where

$$E(z) = \sum_{n \geq 0} e_n z^n.$$

By combining Theorems 0.1.2 and 1.5.10 we conclude that

$$Q_{n,j} = \omega Y_{n,j}.$$

Stanley (personal communication) observed that there is a combinatorial interpretation of this identity in terms of P-partition reciprocity [54, Section 4.5]. Indeed, words in W_n with fixed descent set $S \subseteq [n-1]$ can be identified with strict P -partitions where P is the poset on $\{p_1, \dots, p_n\}$ generated by cover relations $p_i < p_{i+1}$ if $i \in S$ and $p_{i+1} < p_i$ if $i \notin S$. Banners of length n in which the set of positions of barred letters equals S can be identified with P -partitions for the same poset P . It therefore follows from P-partition reciprocity that

$$(1.5.11) \quad Y_{n,j}(\mathbf{x}) = \omega \sum_B \text{wt}(B),$$

summed over all banners of length n with j bars.

In [53] Stanley views words in W_n as proper colorings of a path P_n with n vertices and Y_n as the chromatic symmetric function of P_n . The chromatic symmetric function of a graph $G = (V, E)$ is a symmetric function analog of the chromatic polynomial χ_G of G . Stanley [53, Theorem 4.2] also defines a symmetric function analog of $(-1^{|V|})\chi_G(-m)$ which enumerates all pairs (η, c) where η is an acyclic orientation of G and $c : V \rightarrow [m]$ is a coloring satisfying $c(u) \leq c(v)$ if (u, v) is an edge of η . For $G = P_n$, one can see that these pairs can be identified with banners of length n . Hence Stanley's reciprocity theorem for chromatic symmetric functions [53, Theorem 4.2] reduces to an identity that is refined by (1.5.11) when $G = P_n$.

Another interesting combinatorial interpretation of the Eulerian quasisymmetric functions comes from Gessel (personal communication). He considers the set U_n of words of length n over the alphabet \mathbb{P} with no double (i.e., adjacent) descents and no descent in the last position $n-1$; and proves

$$(1.5.12) \quad \sum_{n \geq 0} z^n \sum_{w \in U_n} \mathbf{x}^w t^{\text{des}(w)} (1+t)^{n-1-2\text{des}(w)} = \frac{(1-t)H(z)}{H(zt) - tH(z)}.$$

The symmetric function on the right hand side of (1.5.12) has also occurred in the work of Processi [40], Stanley [52], Stembridge [56, 57], and Dolgachev and Lunts [19]. They studied a representation of the symmetric group on the cohomology of the toric variety X_n associated with the Coxeter complex of \mathfrak{S}_n . (See, for example [3], for a discussion

of Coxeter complexes and [26] for an explanation of how toric varieties are associated to polytopes.) The action of \mathfrak{S}_n on the Coxeter complex determines an action on X_n and thus a linear representation on the cohomology groups of X_n . As X_n is a complex manifold (of dimension $n - 1$), $H^d(X_n) = 0$ whenever d is odd. The action of \mathfrak{S}_n on X_n induces a representation of \mathfrak{S}_n on the cohomology $H^{2j}(X_n)$ for each $j = 0, \dots, n - 1$. Stanley [52], using a formula of Procesi [40], proves that

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \text{ch} H^{2j}(X_n) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}.$$

Combining this with Theorem 0.1.2 yields the following conclusion.

Theorem 1.5.11. *For all $j = 0, 1, \dots, n - 1$,*

$$\text{ch} H^{2j}(X_n) = Q_{n,j}.$$

In the next section we study another occurrence of the representation whose Frobenius characteristic is $Q_{n,j}$.

Part 2. Poset topology

2.1. A CONJECTURE OF BJÖRNER AND WELKER ON REES PRODUCTS

In this section we consider a partially ordered set introduced by Björner and Welker [10], which originally motivated the work in this paper. Björner and Welker conjectured and Jonsson [33] proved that the absolute value of the Möbius invariant of this poset is a derangement number. We refine this result by showing that the absolute value of the poset's Möbius function evaluated at any pair that includes the poset's bottom element, is an Eulerian number. We prove an equivariant version and a q -analog of both the Björner-Welker-Jonsson result and our refinement that involves the Eulerian quasisymmetric functions $Q_{n,j}$ and the q -Eulerian numbers $a_{n,j}(q, 1)$.

All posets in this paper are finite. Recall that for a poset P , the *order complex* ΔP is the abstract simplicial complex whose vertices are the elements of P and whose k -simplices are totally ordered subsets of size $k + 1$ from P . The (reduced) homology of P is given by $\tilde{H}_k(P) := \tilde{H}_k(\Delta P; \mathbb{C})$.

Given ranked posets P, Q with respective rank functions r_P, r_Q , the *Rees product* $P * Q$ is the poset whose underlying set is

$$\{(p, q) \in P \times Q : r_P(p) \geq r_Q(q)\},$$

with order relation given by $(p_1, q_1) \leq (p_2, q_2)$ if and only if all of the conditions

- $p_1 \leq_P p_2$,
- $q_1 \leq_Q q_2$, and
- $r_P(p_1) - r_P(p_2) \geq r_Q(q_1) - r_Q(q_2)$

hold. In other words, (p_2, q_2) covers (p_1, q_1) in $P * Q$ if and only if p_2 covers p_1 in P and either $q_2 = q_1$ or q_2 covers q_1 in Q .

Rees products were introduced by Björner and Welker in [10], where they study connections between poset topology and commutative algebra. (Rees products of affine semigroup posets arise from the ring-theoretic Rees construction on semigroup algebras.) We will need the following result from [10]. Recall that a poset is said to be Cohen-Macaulay if it, all its open intervals, and all its open principal (upper and lower) order ideals have vanishing homology below the top dimension. It is a well-known fact that Cohen-Macaulay posets are ranked.

Theorem 2.1.1 (Björner and Welker [10]). *The Rees product of two Cohen-Macaulay posets is a Cohen-Macaulay poset.*

Before proceeding we establish some poset notation and terminology. We say that a poset P is *bounded* if it has a minimum element $\hat{0}_P$ and a maximum element $\hat{1}_P$. For any poset P , let \hat{P} be the bounded poset obtained from P by adding a minimum element and a maximum element and let P^+ be the poset obtained from P by adding only a maximum element. For a poset P with minimum element $\hat{0}_P$, let $P^- = P \setminus \{\hat{0}_P\}$. For $x \leq y$ in P , let (x, y) denote the open interval $\{z \in P : x < z < y\}$ and $[x, y]$ denote the closed interval $\{z \in P : x \leq z \leq y\}$.

The Möbius invariant of any bounded poset P is given by

$$\mu(P) := \mu_P(\hat{0}_P, \hat{1}_P),$$

where μ_P is the Möbius function on P . It follows from a well known result of P. Hall (see [54, Proposition 3.8.5]) and the Euler-Poincaré formula that if poset P has length n then

$$(2.1.1) \quad \mu(\hat{P}) = \sum_{i=0}^n (-1)^i \dim \tilde{H}_i(P).$$

Hence if P is Cohen-Macaulay then for all $x \leq y$ in P

$$(2.1.2) \quad \mu_P(x, y) = (-1)^r \dim \tilde{H}_r((x, y)),$$

where $r = r_P(y) - r_P(x) - 2$, and if $y = x$ or y covers x we set $\tilde{H}_r((x, y)) = \mathbb{C}$.

Let B_n be the Boolean algebra on the set $[n]$ and let C_n be the chain $\{1 < 2 < \dots < n\}$. Jonsson [33] uses discrete Morse theory to prove the conjecture of Björner and Welker [10] that

$$(2.1.3) \quad \mu(\widehat{B_n^- * C_n}) = (-1)^n d_n,$$

where d_n is the number of derangements in \mathfrak{S}_n .

In Theorem 2.1.2 below we give a refinement of (2.1.3). Indeed, (2.1.3) follows immediately from Theorem 2.1.2 below, the well-known formula

$$(2.1.4) \quad d_n = \sum_{m=0}^n (-1)^m \binom{n}{m} (n-m)!$$

and the recursive definition of the Möbius function.

Theorem 2.1.2. *Let $S \subseteq [n]$ have size $m > 0$. Then for $1 \leq j \leq m$ we have*

$$\mu_{\widehat{B_n^- * C_n}}(\hat{0}, (S, j)) = (-1)^{m+1} a_{m, j-1}$$

We will present two different proofs of Theorem 2.1.2 both as consequences of general results on Rees products that we derive. The first proof, which is given in Section 2.2, is based on the recursive definition of Möbius function applied to the Rees product of B_n with a poset whose Hasse diagram is a tree and the second proof, which is given in Section 2.3, involves the theory of lexicographic shellability. The first proof yields an \mathfrak{S}_n -equivariant version (Theorem 2.1.3) and a q -analog (Theorem 2.1.6) of Theorem 2.1.2. By combining this q -analog with a q -analog of the second proof, we obtain a new Mahonian permutation statistic whose joint distribution with des is equal to the joint distribution of maj and exc .

Let P be any ranked and bounded poset of length n . Note that the minimal elements of $P^- * C_n$ are of the form $(a, 1)$ where a is an atom of P , and the maximal elements of $P^- * C_n$ are of the form $(\hat{1}_P, j)$ where $j \in [n]$. For $j \in [n]$, set

$$I_j(P) := \{x \in P^- * C_n : x < (1_P, j)\}.$$

Suppose a group G acts on a poset P by order preserving bijections (we say that P is a G -poset). The group G acts simplicially on ΔP and thus arises a linear representation of G on each homology group of P . Now suppose P is ranked of length n . The given action also determines an action of G on $P * X$ for any length n ranked poset X defined by $g(a, x) = (ga, x)$ for all $a \in P$, $x \in X$ and $g \in G$. For a bounded ranked G -poset P of length n , the action of G on P restricts

to an action on P^- , which gives an action of G on $P^- * C_n$. This action restricts to an action of G on each subposet $I_j(P)$.

Since B_n^- and C_n are Cohen-Macaulay, it follows from Theorem 2.1.1 that $B_n^- * C_n$ is a Cohen-Macaulay poset. Hence by (2.1.2), Theorem 2.1.2 is equivalent to

$$(2.1.5) \quad \dim \tilde{H}_{n-2}(I_j(B_n)) = a_{n,j-1},$$

for each $j \in [n]$. The symmetric group \mathfrak{S}_n acts on B_n in an obvious way and therefore on each $I_j(B_n)$. We prove the following result in Section 2.2.

Theorem 2.1.3. *For all $j = 1, 2, \dots, n$,*

$$(2.1.6) \quad \text{ch}(\tilde{H}_{n-2}(I_j(B_n)) \otimes \text{sgn}) = Q_{n,j-1}.$$

Combining this with Theorem 1.5.11 yields,

Corollary 2.1.4. *Let X_n be the toric variety associated with the type A Coxeter complex. For all $1 \leq j \leq n$,*

$$H^{2j}(X_n) \cong_{\mathfrak{S}_n} \tilde{H}_{n-2}(I_{j+1}(B_n)) \otimes \text{sgn},$$

where sgn is the one dimensional vector space on which \mathfrak{S}_n acts according to the sign character.

It would be interesting to find a topological explanation for this isomorphism, in particular one that extends the isomorphism to other Coxeter groups.

Our equivariant version of the Björner-Welker-Jonsson result involves the multiset derangement enumerator $d_{n,j}(\mathbf{x})$ discussed in Section 1.5.2.

Corollary 2.1.5. *For all $n \geq 1$,*

$$\text{ch}(\tilde{H}_{n-1}(B_n^- * C_n)) = \sum_{j=0}^{n-1} d_{n,j}(\mathbf{x}) = \omega \sum_{j=0}^{n-1} Q_{n,j,0}.$$

Proof. We use the following result of Sundaram [58] (see [63, Theorem 4.4.1]): If G acts on a bounded poset P of length n then we have the virtual G -module isomorphism,

$$(2.1.7) \quad \bigoplus_{r=0}^n (-1)^r \bigoplus_{x \in P/G} \tilde{H}_{r-2}((\hat{0}, x)) \uparrow_{G_x}^G \cong_G 0,$$

where P/G denotes a complete set of orbit representatives, G_x denotes the stabilizer of x , and $\uparrow_{G_x}^G$ denotes the induction of the G_x module from G_x to G . Here $H_{r-2}((\hat{0}, x))$ is the trivial representation of G_x

if the rank of x is $r = 0, 1$. Applying this to the Cohen-Macaulay \mathfrak{S}_n -poset $\widehat{B_n^- * C_n}$, we have

$$\tilde{H}_{n-1}(B_n^- * C_n) \cong_{\mathfrak{S}_n} \bigoplus_{m=0}^n (-1)^{n-m} \bigoplus_{j=1}^m \left(\tilde{H}_{m-2}(I_j(B_m)) \otimes 1_{\mathfrak{S}_{n-m}} \right) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n},$$

where 1_G denotes the trivial representation of a group G . From this we obtain

$$(2.1.8) \quad \text{ch} \tilde{H}_{n-1}(B_n^- * C_n) = \sum_{m=0}^n (-1)^{n-m} \sum_{j=1}^m \text{ch} \tilde{H}_{m-2}(I_j(B_m)) h_{n-m}$$

Hence

$$\begin{aligned} \sum_{n \geq 0} \text{ch} \tilde{H}_{n-1}(B_n^- * C_n) z^n &= H(-z) \sum_{n \geq 0} z^n \sum_{j=1}^n \text{ch} \tilde{H}_{n-2}(I_j(B_n)) \\ &= H(-z) \sum_{n \geq 0} z^n \sum_{j=0}^{n-1} \omega(Q_{n,j}). \end{aligned}$$

Recall that $E(z) = \sum_{n \geq 0} e_n z^n$. By the fact that $E(z)H(-z) = 1$, and (0.1.4), we thus have

$$\sum_{n \geq 0} \text{ch} \tilde{H}_{n-1}(B_n^- * C_n) z^n = \frac{1}{1 - \sum_{n \geq 2} (n-1) e_n z^n}.$$

The result now follows from (0.1.4) and (1.5.5). \square

Let $B_n(q)$ be the lattice of subspaces of the n -dimensional vector space \mathbb{F}_q^n over the finite field \mathbb{F}_q . Then $B_n(q)$ is bounded and ranked of length n , with the rank of a subspace equaling its dimension. Like B_n , the q -analog $B_n(q)$ is Cohen-Macaulay and therefore $B_n(q)^- * C_n$ is Cohen-Macaulay. Hence $I_j(B_n(q))$ has vanishing homology below its top dimension $n-2$. In Section 2.2 we prove the following result.

Theorem 2.1.6. *For all $j = 1, 2, \dots, n$,*

$$(2.1.9) \quad \dim \tilde{H}_{n-2}(I_j(B_n(q))) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j-1}} q^{\text{comaj}(\sigma) + j - 1}.$$

Corollary 2.1.7. *For all $n \geq 0$, let \mathcal{D}_n be the set of derangements in \mathfrak{S}_n . Then*

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{\sigma \in \mathcal{D}_n} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)}.$$

Proof. Since $B_n(q)^- * C_n$ is Cohen-Macaulay and the number of m -dimensional subspaces of \mathbb{F}_q^n is $\begin{bmatrix} n \\ m \end{bmatrix}_q$, the Möbius function recurrence for $(B_n(q)^- * C_n) \cup \{\hat{0}, \hat{1}\}$ is equivalent to

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^{n-m} \sum_{j=1}^m \dim \tilde{H}_{m-2}(I_j(B_m(q))).$$

It therefore follows from Theorem 2.1.6 that

$$\dim \tilde{H}_{n-1}(B_n(q)^- * C_n) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^{n-m} \sum_{\sigma \in \mathfrak{S}_m} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)}.$$

The result thus follows from Corollary 1.3.6 □

It follows from Theorems 2.1.3 and 2.1.6 that the symmetry properties of $Q_{n,j}$ and $a_{n,j}(q, 1)$ given in Theorems 1.4.3 and 1.4.8, respectively, are both consequences of the following general result. We suspect that there is a general Rees product result which implies the unimodality properties as well.

Proposition 2.1.8. *Let G be a group and let P be a ranked and bounded G -poset of length n . Then for all $j \in [n]$ we have the following isomorphism of G -posets,*

$$I_j(P) \cong_G I_{n-j+1}(P).$$

Proof. Let $f : I_j(P) \rightarrow I_{n-j+1}(P)$ be the map defined by $f(x, i) = (x, r_P(x) + 1 - i)$. It is straightforward to check that this is a well-defined poset isomorphism, which commutes with the action of G . □

2.2. REES PRODUCTS WITH TREES

For $n, t \in \mathbb{P}$, let $T_{t,n}$ be the poset whose Hasse diagram is a complete t -ary tree of height n , with the root at the bottom. By *complete* we mean that every nonleaf node has exactly t children and that all the leaves are distance n from the root. The following result, which is interesting in its own right, will be used to prove the results of Section 2.1.

Theorem 2.2.1. *For all $n, t \geq 1$ we have*

$$(2.2.1) \quad \dim \tilde{H}_{n-2}((B_n * T_{t,n})^-) = tA_n(t)$$

$$(2.2.2) \quad \dim \tilde{H}_{n-2}((B_n(q) * T_{t,n})^-) = tA_n^{\text{comaj,exc}}(q, qt)$$

$$(2.2.3) \quad \text{ch} \tilde{H}_{n-2}((B_n * T_{t,n})^-) = t \sum_{j=0}^{n-1} \omega Q_{n,j} t^j.$$

Corollary 2.2.2. *For all $n \geq 1$ we have*

$$\begin{aligned} \dim \tilde{H}_{n-2}((B_n * C_{n+1})^-) &= n! \\ \dim \tilde{H}_{n-2}((B_n(q) * C_{n+1})^-) &= \sum_{\sigma \in \mathfrak{S}_n} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)} \\ \text{ch} \tilde{H}_{n-2}((B_n * C_{n+1})^-) &= \omega \sum_{j=0}^{n-1} Q_{n,j}. \end{aligned}$$

2.2.1. Uniform posets. Our goal in this subsection is to prove Theorem 2.2.1.

A bounded ranked poset P is said to be *uniform* if $[x, \hat{1}_P] \cong [y, \hat{1}_P]$ whenever $r_P(x) = r_P(y)$ (see [54, Exercise 3.50]). We will say that a sequence of posets (P_0, P_1, \dots, P_n) is uniform if for each $k = 0, 1, \dots, n$, the poset P_k is uniform of length k and

$$P_k \cong [x, \hat{1}_{P_n}]$$

for each $x \in P_n$ of rank $n - k$. The sequences (B_0, \dots, B_n) and $(B_0(q), \dots, B_n(q))$ are examples of uniform sequences as are the sequences of set partition lattices (Π_0, \dots, Π_n) and the sequence of face lattices of cross polytopes $(\widehat{PCP}_0, \dots, \widehat{PCP}_n)$.

The following result is easy to verify.

Proposition 2.2.3. *Suppose P is a uniform poset of length n . Then for all $t \in \mathbb{P}$, the poset $R := (P * T_{t,n})^+$ is uniform of length $n + 1$. Moreover, if $x \in P$ and $y \in R$ with $r_P(x) = r_R(y) = k$ then*

$$[y, \hat{1}_R] \cong ([x, \hat{1}_P] * T_{t,n-k})^+.$$

Proposition 2.2.4. *Let (P_0, P_1, \dots, P_n) be a uniform sequence of posets. Then for all $t \in \mathbb{P}$,*

$$(2.2.4) \quad 1 + \sum_{k=0}^n W_k(P_n)[k+1]_t \mu((P_{n-k} * T_{t,n-k})^+) = 0,$$

where $W_k(P)$ is the number of elements of rank k in P .

Proof. Let $R := (P_n * T_{t,n})^+$ and let y have rank k in R . By Proposition 2.2.3,

$$\mu_R(y, \hat{1}_R) = \mu((P_{n-k} * T_{t,n-k})^+).$$

Clearly

$$W_k(R) = W_k(P_n)[k+1]_t$$

for all $0 \leq k \leq n$. Hence (2.2.4) is just the recursive definition of the Möbius function applied to the dual of R . \square

To prove (2.2.1) either take dimension in (2.2.3) or set $q = 1$ in the proof of (2.2.2) below.

Proof of (2.2.2). We apply Proposition 2.2.4 to the uniform sequence $(B_0(q), B_1(q), \dots, B_n(q))$. The number of k -dimensional subspaces of \mathbb{F}_q^n is given by

$$W_k(B_n(q)) = \left[\begin{matrix} n \\ k \end{matrix} \right]_q.$$

Write $\mu_n(q, t)$ for $\mu((B_n(q) * T_{t,n})^+)$. Hence by Proposition 2.2.4,

$$(2.2.5) \quad \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q [k+1]_t \mu_{n-k}(q, t) = -1.$$

Setting

$$F_{q,t}(z) := \sum_{j \geq 0} \mu_j(q, t) \frac{z^j}{[j]_q!}$$

and

$$G_{q,t}(z) := \sum_{k \geq 0} [k+1]_t \frac{z^k}{[k]_q!},$$

we derive from (2.2.5) that

$$(2.2.6) \quad F_{q,t}(z) = -\exp_q(z) G_{q,t}(z)^{-1}.$$

If we assume $t > 1$ we have

$$\begin{aligned} G_{q,t}(z) &= \frac{1}{1-t} \sum_{k \geq 0} (1-t^{k+1}) \frac{z^k}{[k]_q!} \\ &= \frac{\exp_q(z) - t \exp_q(tz)}{1-t}. \end{aligned}$$

We calculate that

$$F_{q,t}(-z) = -(1-t) - t \frac{(1-t) \exp_q(-tz)}{\exp_q(-z) - t \exp_q(-tz)}.$$

Using the fact that $\exp_q(-z) \text{Exp}_q(z) = 1$, we have

$$F_{q,t}(-z) = -(1-t) - t \frac{(1-t) \text{Exp}_q(z)}{\text{Exp}_q(tz) - t \text{Exp}_q(z)}.$$

It now follows from Corollary 1.3.5 that for all $n \geq 1$ and $t > 1$,

$$(2.2.7) \quad \mu_n(q, t) = (-1)^{n-1} t \sum_{\sigma \in \mathfrak{S}_n} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)} t^{\text{exc}(\sigma)}.$$

One can see from (2.2.5) and induction that $\mu_n(q, t)$ is a polynomial in t . Hence since (2.2.7) holds for infinitely many integers t , it holds as an identity of polynomials, which implies that it holds for $t = 1$.

By Theorem 2.1.1, the poset $(B_n(q) * T_{t,n})^-$ is Cohen-Macaulay. Hence (2.2.2) holds. \square

We say that a bounded ranked G -poset P is G -uniform if the following holds,

- P is uniform
- $G_x \cong G_y$ for all $x, y \in P$ such that $r_P(x) = r_P(y)$
- there is an isomorphism between $[x, \hat{1}_P]$ and $[y, \hat{1}_P]$ that intertwines the actions of G_x and G_y for all $x, y \in P$ such that $r_P(x) = r_P(y)$. We will write

$$[x, \hat{1}_P] \cong_{G_x, G_y} [y, \hat{1}_P].$$

Given a sequence of groups $G = (G_0, G_1, \dots, G_n)$. We say that a sequence of posets (P_0, P_1, \dots, P_n) is G -uniform if

- P_k is G_k -uniform of length k for each k
- $G_k \cong (G_n)_x$ and $P_k \cong_{G_k, (G_n)_x} [x, \hat{1}_{P_n}]$ whenever $r_{P_n}(x) = n - k$.

For example, the sequence (B_0, \dots, B_n) is $(\mathfrak{S}_0, \dots, \mathfrak{S}_n)$ -uniform.

The following proposition is easy to verify.

Proposition 2.2.5 (Equivariant version of Proposition 2.2.3). *Suppose P is a G -uniform poset of length n . Then for all $t \in \mathbb{P}$, $R := (P * T_{t,n})^+$ is G -uniform of length $n + 1$. Moreover, if $x \in P$ and $y \in R$ with $r_P(x) = r_R(y) = k$ then*

$$[y, \hat{1}_R] \cong_{G_y, G_x} ([x, \hat{1}_P] * T_{t, n-k})^+.$$

If (P_0, P_1, \dots, P_n) is a (G_0, G_1, \dots, G_n) -uniform sequence of posets, we can view G_k as a subgroup of G_n for each $k = 0, \dots, n$. For G -uniform poset P , let $W_k(P; G)$ be the number of G -orbits of the rank k elements of P . The Lefschetz character of a G -poset P of length $n \geq 0$ is defined to be the virtual representation

$$L(P; G) := \bigoplus_{j=0}^n (-1)^j \tilde{H}_j(P).$$

Note that by (2.1.1) the dimension of the Lefschetz character $L(P; G)$ is precisely $\mu(\hat{P})$.

Proposition 2.2.6 (Equivariant version of Proposition 2.2.4). *Let (P_0, P_1, \dots, P_n) be a (G_0, G_1, \dots, G_n) -uniform sequence of posets. Then*

for all $t \in \mathbb{P}$,

(2.2.8)

$$1_{G_n} \oplus \bigoplus_{k=0}^n W_k(P_n; G_n)[k+1]_t L((P_{n-k} * T_{t,n-k})^-; G_{n-k}) \uparrow_{G_{n-k}}^{G_n} = 0.$$

Proof. Sundaram's equation (2.1.7) applied to the dual of a G -poset P is equivalent to the following equivariant version of the recursive definition of the Möbius function:

$$(2.2.9) \quad \bigoplus_{y \in P/G} L((y, \hat{1}_P); G_y) \uparrow_{G_y}^G = 0,$$

where $L((y, \hat{1}_P); G_y)$ is the trivial representation if $y = \hat{1}_P$ and is the negative of the trivial representation if y is covered by $\hat{1}_P$. We apply (2.2.9) to the G_n -uniform poset $R := (P_n * T_{t,n})^+$. Let y have rank k in R . It follows from Proposition 2.2.5 that

$$L((y, \hat{1}_R); (G_n)_y) \uparrow_{(G_n)_y}^{G_n} \cong L((P_{n-k} * T_{t,n-k})^-; G_{n-k}) \uparrow_{G_{n-k}}^{G_n}.$$

Clearly,

$$W_k(R; G_n) = W_k(P_n; G_n)[k+1]_t$$

for all k . Thus (2.2.8) follows from (2.2.9). \square

Proof of (2.2.3). Now we apply Proposition 2.2.6 to the $(\mathfrak{S}_0, \mathfrak{S}_1, \dots, \mathfrak{S}_n)$ -uniform sequence (B_0, B_1, \dots, B_n) . Let

$$L_n(t) := \text{ch } L((B_n * T_{t,n})^-; \mathfrak{S}_n).$$

Clearly $W_k(B_n; \mathfrak{S}_n) = 1$. Therefore by Proposition 2.2.6,

$$(2.2.10) \quad \sum_{k=0}^n [k+1]_t h_k L_{n-k}(t) = -h_n.$$

Setting

$$F_t(z) := \sum_{j \geq 0} L_j(t) z^j$$

and

$$G_t(z) := \sum_{k \geq 0} [k+1]_t h_k z^k$$

we derive from (2.2.10) that

$$(2.2.11) \quad F_t(z)G_t(z) = -H(z).$$

Now if $t > 1$,

$$\begin{aligned} G_t(z) &= \frac{1}{1-t} \sum_{k \geq 0} (1-t^{k+1}) h_k z^k \\ &= \frac{H(z) - tH(tz)}{1-t}, \end{aligned}$$

and we thus have

$$(2.2.12) \quad F_t(z) = -\frac{(1-t)H(z)}{H(z) - tH(tz)}.$$

We calculate that

$$(2.2.13) \quad F_t(-z) = -(1-t) - t \frac{(1-t)H(-tz)}{H(-z) - tH(-tz)}.$$

Using the fact that $H(-z)E(z) = 1$ we have

$$F_t(-z) = -(1-t) - t \frac{(1-t)E(z)}{E(tz) - tE(z)}.$$

By applying the standard symmetric function involution ω , we obtain

$$\omega F_t(-z) = -(1-t) - t \frac{(1-t)H(z)}{H(tz) - tH(z)}.$$

It follows from this and Theorem 0.1.2 that for all $n \geq 1$ and $t > 1$,

$$(2.2.14) \quad \omega L_n(t) = (-1)^{n-1} t \sum_{j=0}^{n-1} Q_{n,j} t^j$$

By (2.2.10) and induction, $L_n(t)$ is a polynomial in t . Hence (2.2.14) holds for $t = 1$ as well. Since $(B_n * T_{t,n})^-$ is Cohen-Macaulay we are done. \square

2.2.2. The tree lemma. By the following result, since B_n is self-dual and Cohen-Macaulay, Theorem 2.1.2 is equivalent to (2.2.1), and since $B_n(q)$ is self-dual and Cohen-Macaulay, Theorem 2.1.6 is equivalent to (2.2.2).

Theorem 2.2.7 (Tree Lemma). *Let P be a bounded, ranked poset of length n . Then for all $t \in \mathbb{P}$,*

$$(2.2.15) \quad \sum_{j=1}^n \mu(\widehat{I_j(P)}) t^j = -\mu((P^* * T_{t,n})^+),$$

where P^* is the dual of P .

Before we can prove Theorem 2.2.7, we need a few lemmas. Set

$$R(P) := P * \{x_0 < x_1 < \dots < x_n\}$$

and for $i \in [n]$ let $R_i(P)$ be the closed lower order ideal in $R(P)$ generated by $(\hat{1}_P, x_i)$. Set

$$R_i^+(P) := \{(a, x_j) \in R_i(P) : j > 0\}$$

and

$$R_i^-(P) := R_i(P) \setminus R_i^+(P).$$

Lemma 2.2.8. *The posets $R_i^+(P)$ and $I_i(P)^+$ are isomorphic.*

Proof. The map that sends (a, x_j) to (a, j) is an isomorphism. \square

An *antiisomorphism* from poset X to a poset Y is an isomorphism ψ from X to Y^* . In other words, ψ is an order reversing bijection from X to Y with order reversing inverse.

Lemma 2.2.9. *For $0 \leq i \leq n$, the map $\psi_i : R_i(P) \rightarrow R_i(P^*)$ given by $\psi_i((a, x_j)) = (a, x_{i-j})$ is an antiisomorphism.*

Proof. We show first that ψ_i is well-defined, that is, if $(a, x_j) \in R_i(P)$ then $(a, x_{i-j}) \in R_i(P^*)$. For $a \in P$ and $j \in \{0, \dots, n\}$ we have $(a, x_j) \in R_i(P)$ if and only if the three conditions

- (1) $0 \leq j \leq i$
- (2) $r_P(a) \geq j$
- (3) $n - r_P(a) \geq i - j$

hold. If (1), (2), (3) hold then so do all of

- (1') $0 \leq i - j \leq i$
- (2') $r_{P^*}(a) = n - r_P(a) \geq i - j$
- (3') $n - r_{P^*}(a) = r_P(a) \geq j = i - (i - j)$,

and (1'), (2'), (3') together imply that $(a, x_{i-j}) \in R_i(P^*)$. The map $\psi_i^* : R_i(P^*) \rightarrow R_i(P)$ given by $\psi_i^*((a, x_j)) = (a, x_{i-j})$ is also well-defined by the argument just given, and $\psi_i^* = \psi_i^{-1}$, so ψ_i is a bijection.

Now for (a, x_j) and (b, x_k) in $R_i(P)$, we have $(a, x_j) < (b, x_k)$ if and only if the three conditions

- (4) $a \leq_P b$
- (5) $j \leq k$
- (6) $r_P(b) - r_P(a) \geq k - j$

hold. If (4), (5), (6) hold then so do all of

- (4') $b \leq_{P^*} a$
- (5') $i - k \leq i - j$
- (6') $r_{P^*}(a) - r_{P^*}(b) = r_P(b) - r_P(a) \geq k - j = (i - j) - (i - k)$,

and (4'), (5'), (6') together imply that in $R_i(P^*)$ we have $(b, x_{i-k}) \leq (a, x_{i-j})$. Therefore, ψ_i is order reversing, and the same argument shows that ψ_i^* is order reversing. \square

Corollary 2.2.10. *For $1 \leq i \leq n$ we have*

$$(2.2.16) \quad \mu(\widehat{I_i(P)}) = \sum_{(a, x_i) \in R_i(P^*)} \mu_{R_i(P^*)}((\hat{1}_P, x_0), (a, x_i)).$$

In case the notation has confused the reader, we remark before proving Corollary 2.2.10 that the sum on the right side of equality (2.2.16) is taken over all pairs (a, x_i) such that $a \in P$ with $r_P(a) \leq n - i$ (so $r_{P^*}(a) \geq i$), and that $\hat{1}_P$, being the maximum element of P , is the minimum element of P^* (so $(\hat{1}_P, x_0)$ is the minimum element of $R_i(P^*)$).

Proof. We have

$$\begin{aligned} \mu(\widehat{I_i(P)}) &= - \sum_{\alpha \in I_i(P)^+} \mu_{\widehat{I_i(P)}}(\alpha, (\hat{1}_P, i)) \\ &= - \sum_{\beta \in R_i^+(P)} \mu_{R_i^+(P)}(\beta, (\hat{1}_P, x_i)) \\ &= \sum_{\gamma = (a, x_0) \in R_i^-(P)} \mu_{R_i(P)}(\gamma, (\hat{1}_P, x_i)) \\ &= \sum_{\gamma = (a, x_0) \in R_i^-(P)} \mu_{R_i(P^*)}(\psi_i((\hat{1}_P, x_i)), \psi_i(\gamma)) \\ &= \sum_{\gamma = (a, x_0) \in R_i^-(P)} \mu_{R_i(P^*)}((\hat{1}_P, x_0), (a, x_i)) \\ &= \sum_{(a, x_i) \in R_i(P^*)} \mu_{R_i(P^*)}((\hat{1}_P, x_0), (a, x_i)). \end{aligned}$$

Indeed, the first equality follows from the definition of the Möbius function; the second follows from Lemma 2.2.8; the third follows from the definition of the Möbius function and the fact that $\mu_{R_i^+(P)}$ is the restriction of $\mu_{R_i(P)}$ to $R_i^+(P) \times R_i^+(P)$ (as $R_i^+(P)$ is an upper order ideal in $R_i(P)$); the fourth follows from Lemma 2.2.9 and the last two follow from the definition of ψ_i . \square

Proof of Tree Lemma (Theorem 2.2.7). The poset $T_{t,n}$ has exactly t^j elements of rank j for each $j = 0, \dots, n$. Let r_T be the rank function of $T_{t,n}$ and let $\hat{0}_T$ be the minimum element of $T_{t,n}$.

We have

$$\begin{aligned} \mu((P^* * T_{t,n})^+) &= - \sum_{\alpha \in P^* * T_{t,n}} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha) \\ &= - \sum_{j=0}^n \sum_{\alpha \in P_{n,t,j}^*} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha), \end{aligned}$$

where

$$P_{n,t,j}^* := \{(a, w) \in P^* * T_{t,n} : r_T(w) = j\}.$$

We have

$$\begin{aligned} \sum_{\alpha \in P_{n,t,0}^*} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha) &= \sum_{a \in P^*} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), (a, \hat{0}_T)) \\ &= \sum_{a \in P^*} \mu_{P^*}(\hat{1}_P, a) \\ &= 0. \end{aligned}$$

Now fix $j \in [n]$. For any $w \in T_{t,n}$ with $r_T(w) = j$, the interval $[\hat{0}_T, w]$ in $T_{t,n}$ is a chain of length j . Therefore, for any $(a, w) \in P_{n,t,j}^*$, the interval $[(\hat{1}_P, \hat{0}_T), (a, w)]$ in $P^* * T_{t,n}$ is isomorphic with the interval $[(\hat{1}_P, x_0), (a, x_j)]$ in $R_j(P^*)$. For any $a \in P^*$, the four conditions

- $r_{P^*}(a) \geq j$,
- $(a, w) \in P_{n,t,j}^*$ for some $w \in T_{t,n}$,
- $(a, v) \in P_{n,t,j}^*$ for every $v \in T_{t,n}$ satisfying $r_T(v) = j$,
- $(a, x_j) \in R_j(P^*)$

are all equivalent. There are exactly t^j elements $v \in T_{t,n}$ of rank j . It follows that

$$\sum_{\alpha \in P_{n,t,j}^*} \mu_{P^* * T_{t,n}}((\hat{1}_P, \hat{0}_T), \alpha) = t^j \sum_{(a, x_j) \in R_j(P^*)} \mu_{R_j(P^*)}((\hat{1}_P, x_0), (a, x_j)),$$

and the Tree Lemma now follows from Corollary 2.2.10. \square

Since B_n is Cohen-Macaulay and self-dual, the following result shows that Theorem 2.1.3 is equivalent to (2.2.3).

Theorem 2.2.11 (Equivariant Tree Lemma). *Let P be a bounded, ranked G -poset of length n . Then for all $t \in \mathbb{P}$,*

$$(2.2.17) \quad \bigoplus_{j=1}^n t^j L(I_j(P); G) \cong_G -L((P^* * T_{t,n})^-; G).$$

Consequently, if P is Cohen-Macaulay then for all $t \in \mathbb{P}$,

$$\bigoplus_{j=1}^n t^j \tilde{H}_{n-2}(I_j(P)) \cong_G \tilde{H}_{n-1}((P^* * T_{t,n})^-).$$

Proof. The proof is an equivariant version of the proof of the Tree Lemma. In particular, the isomorphism of Lemma 2.2.8 is G -equivariant, as is the antiisomorphism of Lemma 2.2.9.

The equivariant version of (2.2.16) is

$$(2.2.18) \quad L(I_i(P); G) = \bigoplus_{(a, x_i) \in R_i(P^*)/G} L(((\hat{1}_P, x_0), (a, x_i)); G_a) \uparrow_{G_a}^G.$$

To prove (2.2.18) we let (2.2.9) play the role of the recursive definition of Möbius function in the proof of (2.2.16).

To prove (2.2.17) we follow the proof of the Tree Lemma again letting (2.2.9) play the role of the recursive definition of Möbius function, and in the last step applying (2.2.18) instead of (2.2.16). \square

2.3. LEXICOGRAPHICAL SHELLABILITY AND A NEW MAHONIAN STATISTIC

In this section we show that the posets $\widehat{I_j(B_n)}$ and $\widehat{I_j(B_n(q))}$ are EL-shellable and use the theory of lexicographical shellability to compute their Möbius invariants. This will yield a second proof of Theorem 2.1.2 and a new Mahonian permutation statistic to serve as a partner for the Eulerian statistic des in the joint (maj, exc)-distribution.

We recall some basic facts from the theory of lexicographic shellability (cf. [4, 7, 8, 9, 63]). Let P be a bounded poset and let $\text{Cov}(P)$ be the set of pairs $(x, y) \in P \times P$ such that y covers x in P . Let L be another poset and let W be the set of all finite sequences of elements of L . The given ordering of L induces a lexicographic ordering \preceq on W . An *edge labeling* of P by L is a function $\lambda : \text{Cov}(P) \rightarrow L$. Given such a function λ and a saturated chain $C = \{x_1 < \dots < x_m\}$ from P , we write $\lambda(C)$ for $(\lambda(x_1, x_2), \dots, \lambda(x_{m-1}, x_m)) \in W$. An *ascent* in C is any $i \in [m-1]$ satisfying $\lambda(x_i, x_{i+1}) < \lambda(x_{i+1}, x_{i+2})$. We say λ is *increasing* on C if each $i \in [m-1]$ is an ascent in C . The edge labeling λ is an *EL-labeling* of P if whenever $x < y$ in P there is a unique maximal chain C in the interval $[x, y]$ on which λ is increasing and for all other maximal chains D in $[x, y]$ we have $\lambda(C) \prec \lambda(D)$. A poset that admits an EL-labeling is said to be EL-shellable.

The notion of EL-shellability for *ranked* posets was introduced by Björner in [4]. A more general concept called CL-shellability, introduced by Björner and Wachs in [7], also associates label sequences

with maximal chains of a poset. We will not define CL-labelings here. (Both notions were subsequently extended to all bounded posets in [9].) Given an EL-labeling or a CL-labeling λ on P , we call a maximal chain C from P *ascent free* if its label sequence contains no ascent.

One of the main results in the theory of lexicographic shellability for ranked posets is the following result.

Theorem 2.3.1 (Björner [4], Björner and Wachs [7]). *If λ is an EL-labeling (or more generally a CL-labeling) of a bounded ranked poset P of length n then $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ is homotopy equivalent to a wedge of c spheres of dimension $(n - 2)$, where c is the number of ascent free maximal chains. Consequently*

$$\mu_P(\hat{0}, \hat{1}) = (-1)^n c.$$

To use Theorem 2.3.1 for our purposes, we need the following result.

Lemma 2.3.2. *Let P be a bounded ranked poset. Let $\lambda : \text{Cov}(P) \rightarrow L$ be an EL-labeling of P . Let $(\hat{0}_P, 1)$ denote the minimum element of $\widehat{I_j(P)}$ and let $(\hat{1}_P, j)$ denote the maximum element. Define the edge labeling*

$$\lambda^+ : \text{Cov}(\widehat{I_j(P)}) \rightarrow L \times \{0 < 1\}$$

by

$$\lambda^+((x, h), (y, i)) = (\lambda(x, y), i - h).$$

Then λ^+ is an EL-labeling of $\widehat{I_j(P)}$.

Proof. Let $(w, k) < (z, l)$ in $\widehat{I_j(P)}$. Then $w < z$ in P and there is a unique maximal chain $C = \{w = x_0 < \dots < x_m = z\}$ in $[w, z]$ on which λ is increasing. Thus if λ^+ is increasing on the maximal chain $D = \{(y_0, f_0) < \dots < (y_m, f_m)\}$ in the interval $I = [(w, k), (z, l)]$ then $y_j = x_j$ for $0 \leq j \leq m$. Moreover, if $\lambda^+(D) = ((a_1, d_1), \dots, (a_m, d_m))$ and λ^+ is increasing on D then we must have $d_i = 0$ for $1 \leq i \leq m - l + k$ and $d_i = 1$ for $m - l + k < i \leq m$. Since the sequence (d_1, \dots, d_m) , along with $f_0 = k$ determines f_i for $1 \leq i \leq m$, it follows that there is a unique maximal chain D in I on which λ^+ is increasing.

Now let $E = \{(v_0, e_0) < \dots < (v_m, e_m)\}$ be another maximal chain in I . Assume that $(v_i, e_i) = (y_i, f_i)$ for $1 \leq i < t$ but $(v_t, e_t) \neq (y_t, f_t)$. If $v_t = y_t$ then clearly $e_t \neq f_t$. If $v_t \neq y_t$ then we must have $\lambda(v_{t-1}, v_t) > a_t = \lambda(y_{t-1}, y_t)$ in L . Indeed, it is a basic property of EL-labelings that if P is a poset with EL-labeling λ then for each interval $[x, y]$, if a covers x in the unique increasing maximal chain of $[x, y]$ and b is an atom of $[x, y]$ other than a , then $\lambda(x, a) < \lambda(x, b)$ (cf. [4, Proposition

2.5], [9, Lemma 5.3]). In either case if $t \leq m - l + k$ then we have $f_t = f_{t-1}$, and if $t > m - l + k$ then $e_t = e_{t-1} + 1$. It follows that in all cases we have $\lambda^+((v_{t-1}, e_{t-1}), (v_t, e_t)) > \lambda^+((y_{t-1}, f_{t-1}), (y_t, f_t))$ in $L \times \{0 < 1\}$. Thus $\lambda^+(D) \prec \lambda^+(E)$ as required. \square

Remark 2.3.3. The EL-labeling Λ^+ given in Lemma 2.3.2 can be generalized in a straightforward way to the general case in which the chain C_n in the Rees product $(P \setminus \{0\}) * C_n$ is replaced by an arbitrary ranked EL-shellable poset. An analogous results holds for CL-labelings.

Given an EL-labeling λ^+ as in Lemma 2.3.2, we need to describe its ascent free chains. For $k = 0, \dots, n-1$, let $S_{n,k}$ be the set of sequences $(0 = d_1, \dots, d_n) \in \{0, 1\}^n$ such that the number of d_i equal to 1 is k . Given any maximal chain $D = \{(x_0, f_0) < \dots < (x_n, f_n)\}$ of $\widehat{I_j(P)}$, we have that $\{\hat{0}_P = x_0 < x_1 < \dots < x_n = \hat{1}_P\}$ is a maximal chain of P and $(f_1 - f_0, f_2 - f_1, \dots, f_n - f_{n-1}) \in S_{n,j-1}$. Conversely, given any maximal chain $C = \{\hat{0}_P = x_0 < x_1 < \dots < x_n = \hat{1}_P\}$ of P and any $d \in S_{n,j-1}$, there is a unique maximal chain $D = \{(x_0, f_0) < \dots < (x_n, f_n)\}$ of $\widehat{I_j(P)}$ such that $f_i - f_{i-1} = d_i$ for all $i \in [n]$. Hence the maximal chains of $\widehat{I_j(P)}$ can be identified with pairs (C, d) where C is a maximal chain of P and $d \in S_{n,j-1}$. We have

$$\lambda^+(C, d) = ((\lambda_1(C), d_1), \dots, (\lambda_n(C), d_n)),$$

where $\lambda_i(C)$ is the i th entry $\lambda(x_{i-1}, x_i)$ of $\lambda(C)$. The following result clearly holds.

Proposition 2.3.4. *The maximal chain (C, d) of $\widehat{I_j(P)}$ is ascent free if and only if*

$$(2.3.1) \quad \forall i \in [n], \quad \lambda_i(C) < \lambda_{i+1}(C) \implies d_i = 1 \text{ and } d_{i+1} = 0.$$

holds.

We turn now to the specific case where $P = B_n$. The labeling $\lambda : \text{Cov}(B_n) \rightarrow \{1 < 2 < \dots < n\}$ given by $\lambda(S, T) = x$ if $T \setminus S = \{x\}$ is an EL-labeling, and for any maximal chain C from B_n , each $i \in [n]$ appears exactly once in the sequence $\lambda(C)$. We can therefore view the sequence $\lambda(C)$ as a permutation in \mathfrak{S}_n (in one line notation). Hence by the discussion preceding Proposition 2.3.4, we can identify label sequences of maximal chains of $\widehat{I_j(B_n)}$ with pairs (σ, d) , where $\sigma \in \mathfrak{S}_n$ and $d \in S_{n,j-1}$. Since each permutation in \mathfrak{S}_n is the label sequence of a unique maximal chain of B_n , each pair (σ, d) , where $\sigma \in \mathfrak{S}_n$ and $d \in S_{n,j-1}$, is the label sequence of a unique maximal chain of $\widehat{I_j(B_n)}$. It will be convenient to view these pairs as barred permutations (that

is, type B permutations). Indeed, we identify (σ, d) with the barred permutation σ^B in which

$$\sigma^B(i) = \begin{cases} \sigma(i) & \text{if } d_i = 0 \\ \overline{\sigma(i)} & \text{if } d_i = 1 \end{cases}$$

for each $i \in [n]$. Let $|\sigma^B|$ be the permutation obtained from σ^B by removing the bars. Thus Proposition 2.3.4 in the case of $P = B_n$ asserts that σ^B is the label sequence of an ascent free maximal chain of $\widehat{I_j(B_n)}$ if and only if σ^B is a barred permutation of length n that satisfies

- (A) σ^B has exactly $j - 1$ bars
- (B) $\sigma^B(1)$ is not barred
- (C) $|\sigma^B|(i) < |\sigma^B|(i + 1) \implies \sigma^B(i)$ is barred and $\sigma^B(i + 1)$ is not, for each $i \in [n - 1]$.

Let $\mathcal{B}_{n,j-1}$ be the set of all such barred permutations. By Theorem 2.3.1 we have the following result.

Theorem 2.3.5. *For all $j \in [n]$,*

$$\dim \tilde{H}_{n-2}(I_j(B_n)) = |\mathcal{B}_{n,j-1}|.$$

Remark 2.3.6. One can use Theorem 2.1.3 and Gessel's formula (1.5.12) to obtain an alternative proof of Theorem 2.3.5, which does not involve lexicographic shellability.

Below we construct a bijection between $\mathcal{B}_{n,j}$ and $\{\sigma \in \mathfrak{S}_n : \text{des}(\sigma) = j\}$, resulting in an alternative proof that $\dim \tilde{H}_{n-2}(I_j(B_n))$ is the Eulerian number $a_{n,j-1}$. But first we use a result of Simion to derive a q -analog of Theorem 2.3.5.

Theorem 2.3.7 (Simion [47]). *There is an EL-labeling λ for $B_n(q)$ such that*

- (1) *for each maximal chain C of $B_n(q)$, we have $\lambda(C) \in \mathfrak{S}_n$*
- (2) *for each $\sigma \in \mathfrak{S}_n$, there are $q^{\text{inv}(\sigma)}$ maximal chains C of $B_n(q)$ with $\lambda(C) = \sigma$.*

It follows from (1) of Simion's result that for each maximal chain (C, d) of $\widehat{I_j(B_n(q))}$, the label sequence $\lambda^+(C, d)$ can be viewed as a barred permutation of length n . Hence by Proposition 2.3.4, a maximal chain (C, d) is ascent-free if and only if $\lambda^+(C, d) \in \mathcal{B}_{n,j-1}$. By (2) of Simion's result we have that for each $\sigma^B \in \mathcal{B}_{n,j-1}$, the number of maximal chains of $\widehat{I_j(B_n(q))}$ whose label sequence is σ^B is $q^{\text{inv}(|\sigma^B|)}$. Thus by Theorem 2.3.1, we have the following q -analog of Theorem 2.3.5.

Theorem 2.3.8. *For all $j \in [n]$,*

$$\dim \tilde{H}_{n-2}(I_j(B_n(q))) = \sum_{\sigma^B \in \mathcal{B}_{n,j-1}} q^{\text{inv}(|\sigma^B|)}.$$

For our induction proofs, we need to extend the definition of $\mathcal{B}_{n,j}$ to barred permutations over arbitrary finite subsets X of \mathbb{P} . A barred permutation over X is a linear arrangement of the elements of X with bars over some (or none) of the elements. Let \mathcal{B}_X be the set of barred permutations σ^B of X that satisfy (B) and (C) of the definition of $\mathcal{B}_{n,j}$ given above. Let \mathfrak{S}_X be the set of ordinary permutations of X . If $X = \emptyset$ then $\mathfrak{S}_X = \mathcal{B}_X = \{\theta\}$. For $\sigma^B \in \mathcal{B}_X$, let $|\sigma^B|$ be the permutation in \mathfrak{S}_X obtained by removing the bars from σ^B .

Given barred permutations $\alpha \in \mathcal{B}_A$ and $\beta \in \mathcal{B}_B$, where A and B are disjoint, let $\alpha \cdot \beta$ denote the barred permutation in $\mathcal{B}_{A \uplus B}$ obtained by concatenating the words α and β . We define a map

$$\varphi : \bigsqcup_{\substack{X \subset \mathbb{P} \\ |X| < \infty}} \mathcal{B}_X \rightarrow \bigsqcup_{\substack{X \subset \mathbb{P} \\ |X| < \infty}} \mathfrak{S}_X,$$

recursively by

$$\varphi(\sigma^B) = \begin{cases} \theta & \text{if } \sigma^B = \theta \\ m \cdot \varphi(\alpha) & \text{if } \sigma^B = \alpha \cdot m \\ \varphi(\alpha) \cdot m & \text{if } \sigma^B = \alpha \cdot \bar{m} \text{ and } \alpha \neq \theta \\ \varphi(\beta) \cdot m \cdot \varphi(\alpha) & \text{if } \sigma^B = \alpha \cdot \bar{m} \cdot \beta \text{ and } \alpha, \beta \neq \theta, \end{cases}$$

where m is the minimum letter of $|\sigma^B|$.

Lemma 2.3.9. *The map φ is a well-defined bijection which satisfies*

- (1) $\varphi(\mathcal{B}_X) = \mathfrak{S}_X$
- (2) $\text{des}(\varphi(\sigma^B)) = \text{bar}(\sigma^B)$, where $\text{bar}(w)$ denotes the number of barred letters of a barred permutation w

for all finite subsets X of \mathbb{P} and $\sigma^B \in \mathcal{B}_X$.

Proof. By (C) of the definition of \mathcal{B}_X , if letter m is unbarred in the word $\sigma^B \in \mathcal{B}_X$ then it is the last letter of σ^B . By (B) if m is barred it cannot be the first letter. Hence the four cases of the definition of φ cover all possibilities. It is also clear from the definition of \mathcal{B}_X that if $\sigma^B \in \mathcal{B}_X$ and $|X| > 1$ then $\alpha \in \mathcal{B}_A$ and $\beta \in \mathcal{B}_{X \setminus (A \cup \{m\})}$ for some subset $A \subsetneq X$. Hence by induction on $|X|$ we have that φ is a well-defined map that takes elements of \mathcal{B}_X to \mathfrak{S}_X .

To show that φ is a bijection satisfying (1) we construct its inverse. Define

$$\psi : \bigsqcup_{\substack{X \subset \mathbb{P} \\ |X| < \infty}} \mathfrak{S}_X \rightarrow \bigsqcup_{\substack{X \subset \mathbb{P} \\ |X| < \infty}} \mathcal{B}_X,$$

recursively by

$$\psi(\sigma) = \begin{cases} \theta & \text{if } \sigma = \theta \\ \psi(\delta) \cdot m & \text{if } \sigma = m \cdot \delta \\ \psi(\gamma) \cdot \bar{m} & \text{if } \sigma = \gamma \cdot m \text{ and } \gamma \neq \theta \\ \psi(\delta) \cdot \bar{m} \cdot \psi(\gamma) & \text{if } \sigma = \gamma \cdot m \cdot \delta \text{ and } \gamma, \delta \neq \theta \end{cases},$$

where m is the minimum element of σ . One can see that conditions (B) and (C) of the definition of \mathcal{B}_X hold for $\psi(\sigma)$ whenever they hold for $\psi(\gamma)$ and $\psi(\delta)$. Hence by induction ψ is a well defined map. One can easily also show by induction that φ and ψ are inverses of each other.

We also prove (2) by induction on $|X|$, with the base case $|X| = 0$ being trivial. We do the fourth case and leave the others to the reader. Clearly

$$\text{bar}(\sigma^B) = \text{bar}(\alpha) + 1 + \text{bar}(\beta).$$

Since m is the smallest element of X and is not the first letter of $\varphi(\sigma^B)$, we have

$$\text{des}(\varphi(\sigma^B)) = \text{des}(\varphi(\beta)) + 1 + \text{des}(\varphi(\alpha)).$$

By induction we conclude that (2) holds in this case. \square

We now describe the permutation statistic that corresponds to $\text{inv}(|\cdot|)$ under the map φ . For a permutation $\sigma \in \mathfrak{S}_X$, an *admissible inversion* of σ is a pair $(\sigma(i), \sigma(j))$ such that

- $1 \leq i < j \leq n$
- $\sigma(i) > \sigma(j)$, and
- either
 - $j < n$ and $\sigma(j) < \sigma(j + 1)$ or
 - there is some k such that $i < k < j$ and $\sigma(k) < \sigma(j)$.

We write $\text{ai}(\sigma)$ for the number of admissible inversions of σ . For example, if $\sigma = 3167542$ then the admissible inversions are $(3, 1)$ and $(3, 2)$. So $\text{ai}(\sigma) = 2$.

Lemma 2.3.10. *For all $\sigma^B \in \mathcal{B}_X$,*

$$\text{inv}(|\sigma^B|) = \binom{|X|}{2} - \text{ai}(\varphi(\sigma^B))$$

Proof. Our proof proceeds by induction on $n = |X|$, the case $n = 0$ being trivial.

If $\sigma^B = \alpha \cdot m$ then

$$\begin{aligned}
\text{ai}(\varphi(\sigma^B)) &= \text{ai}(m \cdot \varphi(\alpha)) \\
&= \text{ai}(\varphi(\alpha)) \\
&= \binom{n-1}{2} - \text{inv}(|\alpha|) \\
&= \binom{n}{2} - (\text{inv}(|\alpha|) + n - 1) \\
&= \binom{n}{2} - \text{inv}(|\alpha \cdot m|).
\end{aligned}$$

Indeed, the first two equalities follow immediately from the definitions and the third follows from our inductive hypothesis.

If $\sigma^B = \alpha \cdot \bar{m}$ then we derive as in the case just above that

$$\begin{aligned}
\text{ai}(\varphi(\sigma^B)) &= \text{ai}(\varphi(\alpha) \cdot m) \\
&= \text{ai}(\varphi(\alpha)) \\
&= \binom{n-1}{2} - \text{inv}(|\alpha|) \\
&= \binom{n}{2} - \text{inv}(|\alpha \cdot \bar{m}|).
\end{aligned}$$

Finally, say $\sigma^B = \alpha \cdot \bar{m} \cdot \beta$ with $\alpha \in \mathcal{B}_A$ and $\beta \in \mathcal{B}_B$, where $|A| > 0$ and $|B| > 0$. Set $\text{inv}(A, B) := |\{(a, b) : a \in A, b \in B, a > b\}|$. It follows quickly from the inductive hypothesis and the definitions that

$$\begin{aligned}
\text{ai}(\varphi(\sigma^B)) &= \text{ai}(\varphi(\beta) \cdot m \cdot \varphi(\alpha)) \\
&= \text{ai}(\varphi(\beta)) + |B| + \text{ai}(\varphi(\alpha)) + \text{inv}(B, A) \\
&= \binom{|B|}{2} - \text{inv}(|\beta|) + n - 1 - |A| \\
&\quad + \binom{|A|}{2} - \text{inv}(|\alpha|) + |A||B| - \text{inv}(A, B).
\end{aligned}$$

Now

$$\text{inv}(|\alpha \cdot \bar{m} \cdot \beta|) = \text{inv}(|\alpha|) + |A| + \text{inv}(|\beta|) + \text{inv}(A, B)$$

and a straightforward calculation shows that

$$\binom{|B|}{2} + n - 1 + \binom{|A|}{2} + |A||B| = \binom{n}{2}.$$

□

By Theorem 2.3.8 and Lemmas 2.3.9 and 2.3.10 we obtain the following result.

Theorem 2.3.11. *For all $j \in [n]$,*

$$\dim \tilde{H}_{n-2}(I_j(B_n(q))) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{des}(\sigma) = j-1}} q^{\binom{n}{2} - \text{ai}(\sigma)}.$$

Now define the permutation statistic

$$\text{aid}(\sigma) := \text{ai}(\sigma) + \text{des}(\sigma)$$

for all $\sigma \in \mathfrak{S}_n$. By combining Theorem 2.3.11 with Theorem 2.1.6 we arrive at,

Theorem 2.3.12. *For all $n \geq 0$,*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{aid}(\sigma)} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}.$$

A considerable amount of work in symmetric function theory and poset topology has gone into proving Theorem 2.3.12. We pose the question of whether there is a nice direct bijective proof. Here we give a simple combinatorial proof that aid is Mahonian.

Proposition 2.3.13. *Let $F_n(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{aid}(\sigma)}$. Then $F_n(q)$ satisfies the following recurrence for all $n \geq 2$,*

$$F_n(q) := (1 + q)F_{n-1}(q) + \sum_{j=2}^{n-1} \left[\begin{matrix} n-1 \\ j-1 \end{matrix} \right]_q q^j F_{j-1}(q) F_{n-j}(q).$$

Consequently $F_n(q) = [n]_q!$.

Proof. The terms on the right side of the recurrence q -count permutations according to the position of 1 in the permutation. That is,

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(j) = 1}} q^{\text{aid}(\sigma)} = \begin{cases} \left[\begin{matrix} n-1 \\ j-1 \end{matrix} \right]_q q^j F_{j-1}(q) F_{n-j}(q) & \text{if } j = 2, \dots, n-1 \\ F_{n-1}(q) & \text{if } j = 1 \\ qF_{n-1}(q) & \text{if } j = n. \end{cases}$$

It is easy to see that $[n]_q!$ also satisfies the recurrence relation. \square

2.4. TYPE BC-ANALOGS

In this section we present type BC analogs (in the context of Coxeter groups) of both the Björner-Welker-Jonsson derangement result and its q -analog, Corollary 2.1.7.

A poset P with a $\hat{0}_P$ is said to be a *simplicial poset* if $[\hat{0}_P, x]$ is a Boolean algebra for all $x \in P$. The prototypical example of a simplicial poset is the poset of faces of a simplicial complex. In fact, every simplicial poset is isomorphic to the face poset of some regular CW complex (see [5]). The next result follows immediately from Theorem 2.1.2 and the definition of the Möbius function. For a ranked poset P of length n and $r \in \{0, 1, \dots, n\}$, let $W_r(P)$ be the r th Whitney number of the second kind of P , that is the number elements of rank r in P .

Corollary 2.4.1 (of Theorem 2.1.2). *Let P be a ranked simplicial poset of length n . Then*

$$\mu(\widehat{P^- * C_n}) = \sum_{r=0}^n (-1)^{r-1} W_r(P) r!.$$

We think of B_n as the poset of faces of a $(n-1)$ -simplex, whose barycentric subdivision is the Coxeter complex of type A_{n-1} . Then d_n is the number of derangements in the action of the associated Coxeter group \mathfrak{S}_n on the vertices of the simplex. Let PCP_n be the poset of *simplicial* (that is, proper) faces of the n -dimensional crosspolytope CP_n (see for example [6, Section 2.3]), whose barycentric subdivision is the Coxeter complex of type BC. The associated Weyl group, which is isomorphic to the wreath product $\mathfrak{S}_n[\mathbb{Z}_2]$, acts by reflections on CP_n and therefore on its vertex set. Let d_n^{BC} be the number of derangements in this action on vertices.

Theorem 2.4.2. *For all n , we have*

$$\dim \tilde{H}_{n-1}(PCP_n^- * C_n) = d_n^{BC}.$$

Proof. It is well known and straightforward to prove by induction on n that, for $0 \leq r \leq n$, the number of $(r-1)$ -dimensional faces of CP_n is $2^r \binom{n}{r}$. Corollary 2.4.1 gives

$$\mu(\widehat{PCP_n^- * C_n}) = \sum_{r=0}^n (-1)^{r-1} 2^r \binom{n}{r} r!.$$

Hence since PCP_n^- is Cohen-Macaulay, by Theorem 2.1.1 we have,

$$\dim \tilde{H}_{n-1}(PCP_n^- * C_n) = \sum_{r=0}^n (-1)^{n-r} 2^r \binom{n}{r} r!.$$

On the other hand, we may identify the vertices of CP_n with elements of $[n] \cup [\bar{n}]$, where $[\bar{n}] = \{\bar{1}, \dots, \bar{n}\}$, so that the action of the Weyl group $W \cong \mathfrak{S}_n[\mathbb{Z}_2]$ is determined by the following facts.

- Each element $w \in W$ can be written uniquely as $w = (\sigma, v)$ with $\sigma \in \mathfrak{S}_n$ and $v \in \mathbb{Z}_2^n$.
- Any element of the form $(\sigma, 0)$ maps $i \in [n]$ to $\sigma(i)$ and $\bar{i} \in [\bar{n}]$ to $\sigma(\bar{i})$.
- Any element of the form $(1, e_i)$, where e_i is the i^{th} standard basis vector in \mathbb{Z}_2^n , exchanges i and \bar{i} , and fixes all other vertices.

It follows that for each $S \subseteq [n]$, the pointwise stabilizer of S in W is exactly the pointwise stabilizer of $\bar{S} := \{\bar{i} : i \in S\}$ and is isomorphic to $\mathfrak{S}_{n-|S|}[\mathbb{Z}_2]$. Using inclusion-exclusion as is done to calculate d_n , we get

$$d_n^{BC} = \sum_{j=0}^n (-1)^j \binom{n}{j} 2^{n-j} (n-j)!.$$

□

Muldoon and Readdy [38] have recently obtained a dual version of Theorem 2.4.2 in which the Rees product of the dual of PCP_n with the chain is considered.

Next we consider a poset that can be viewed as both a q -analog of PCP_n and a type BC analog of $B_n(q)$. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate, alternating bilinear form on the vector space \mathbb{F}_q^{2n} . A subspace U of \mathbb{F}_q^{2n} is said to be *totally isotropic* if $\langle u, v \rangle = 0$ for all $u, v \in U$. Let $PCP_n(q)$ be the poset of totally isotropic subspaces of \mathbb{F}_q^{2n} . The order complex of $PCP_n(q)$ is the building of type C_n , naturally associated to a finite group of Lie type B_n or C_n (see for example [11, Chapter V], [45, Appendix 6]). Thus we have both a q -analog of PCP_n and a type BC analog of $B_n(q)$ (since the order complex of $B_n(q)$ is the building of type A_{n-1}).

Clearly $PCP_n(q)$ is a lower order ideal of $B_{2n}(q)$.

Proposition 2.4.3. *The maximal elements of $PCP_n(q)$ all have dimension n . For $r = 0, \dots, n$, the number of r -dimensional isotropic subspaces of \mathbb{F}_q^{2n} is given by*

$$W_r(PCP_n(q)) = \begin{bmatrix} n \\ r \end{bmatrix}_q (q^n + 1)(q^{n-1} + 1) \cdots (q^{n-r+1} + 1).$$

Proof. The first claim of the proposition is a well known fact (see for example [45, Chapter 1]). The second claim is also a known fact, we sketch a proof here. The number of ordered bases for any k -dimensional subspace of \mathbb{F}_q^{2n} is

$$\prod_{j=0}^{k-1} (q^k - q^j).$$

On the other hand, we can produce an ordered basis for a k -dimensional totally isotropic subspace of \mathbb{F}_q^{2n} in k steps, at each step i choosing $v_i \in \langle v_1, \dots, v_{i-1} \rangle^\perp \setminus \langle v_1, \dots, v_{i-1} \rangle$. The number of ways to do this is

$$\prod_{j=0}^{k-1} (q^{2n-j} - q^j),$$

and the proof is completed by division and manipulation. \square

It was shown by L. Solomon (see [49]) that $PCP_n(q)$ is Cohen-Macaulay. Hence by Theorem 2.1.1, so is $PCP_n(q)^- * C_n$. We will show that $\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n)$ is a polynomial in q with non-negative integral coefficients and give a combinatorial interpretation of the coefficients. We first need the following q -analog of Corollary 2.4.1. We say that a poset P with $\hat{0}_P$ is q -simplicial if each interval $[\hat{0}_P, x]$ is isomorphic to $B_j(q)$ for some j .

Corollary 2.4.4 (of Theorem 2.1.6). *Let P be a ranked q -simplicial poset of length n . Then*

$$\mu(\widehat{P^- * C_n}) = \sum_{r=0}^n (-1)^{r-1} W_r(P) \sum_{\sigma \in \mathfrak{S}_r} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)}.$$

Theorem 2.4.5. *For all $n \geq 0$, let $d_n(q) := \sum_{\sigma \in \mathcal{D}_n} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)}$. Then*

(2.4.1)

$$\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q q^{k^2} \prod_{i=k+1}^n (1 + q^i) d_{n-k}(q).$$

*Consequently, $\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n)$ is a polynomial in q with non-negative integer coefficients.*

Proof. We have by Proposition 2.4.3, Corollary 2.4.4, and the fact that $PCP_n(q)^- * C_n$ is Cohen-Macaulay,

$$\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n) = \sum_{j=0}^n (-1)^j \left[\begin{matrix} n \\ j \end{matrix} \right]_q \prod_{i=j+1}^n (1 + q^i) a_{n-j}(q),$$

where $a_n(q) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)}$. On the other hand by Corollary 1.3.6, the right hand side of (2.4.1) equals

$$\begin{aligned} & \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^n (1+q^i) \sum_{m=0}^{n-k} (-1)^m \begin{bmatrix} n-k \\ m \end{bmatrix}_q a_{n-k-m}(q) \\ &= \sum_{j \geq 0} a_{n-j}(q) \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^n (1+q^i) (-1)^{j-k} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_q \\ &= \sum_{j \geq 0} a_{n-j}(q) \begin{bmatrix} n \\ j \end{bmatrix}_q \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^n (1+q^i) (-1)^{j-k}. \end{aligned}$$

Thus to prove (2.4.1) we need only show that

$$\prod_{i=j+1}^n (1+q^i) = \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^n (1+q^i) (-1)^k,$$

holds for all n and j . By Gaussian inversion this is equivalent to,

$$q^{j^2} (-1)^j \prod_{i=j+1}^n (1+q^i) = \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{j-k} q^{\binom{j-k}{2}} \prod_{i=k+1}^n (1+q^i),$$

which is in turn equivalent to,

$$(2.4.2) \quad q^{j^2} (-1)^j = \sum_{k \geq 0} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{j-k} q^{\binom{j-k}{2}} \prod_{i=k+1}^j (1+q^i).$$

To prove (2.4.2) we use the q -binomial formula,

$$\prod_{i=0}^{n-1} (x + yq^i) = \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^{n-k} y^k.$$

Set $y = 1$ and use Gaussian inversion to obtain

$$x^n = \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} \prod_{i=0}^{k-1} (x + q^i)$$

Now set $x = q^n$ to obtain

$$\begin{aligned} q^{n^2} &= \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} \prod_{i=0}^{k-1} (q^n + q^i) \\ &= \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{k}{2}} \prod_{i=0}^{k-1} (q^{n-i} + 1). \end{aligned}$$

□

Using the standard identification of elements of $\mathfrak{S}_n[\mathbb{Z}_2]$ with barred permutations, the derangements of Theorem 2.4.2 are the barred permutations σ for which $\sigma(i) \neq i$ for all $i \in [n]$. Let \mathcal{D}_n^{BC} be the set of such barred permutations. For $\sigma \in \mathcal{D}_n^{BC}$, let $\tilde{\sigma}$ be the word obtained by rearranging the letters of σ so that the fixed points of $|\sigma|$, which are all barred in σ , come first in increasing order with bars intact, followed by subword of nonfixed points of $|\sigma|$ also with bars intact. Now let S be the set of positions in which bars appear in $\tilde{\sigma}$. Define the *bar index*, $\text{bnd}(\sigma)$ of σ to be $\sum_{i \in S} i$. For example if $\sigma = \overline{3}25\overline{4}\overline{6}1\overline{7}$ then $\tilde{\sigma} = \overline{2}\overline{4}\overline{7}\overline{3}5\overline{6}1$ and so $\text{bnd}(\sigma) = 1 + 2 + 3 + 4 + 6$.

Corollary 2.4.6.

$$\dim \tilde{H}_{n-1}(PCP_n(q)^- * C_n) = \sum_{\sigma \in \mathcal{D}_n^{BC}} q^{\text{comaj}(|\sigma|) + \text{exc}(|\sigma|) + \text{bnd}(\sigma)}$$

Proof. By Corollary 1.3.6 we have,

$$\begin{aligned} & \sum_{\sigma \in \mathcal{D}_n^{BC}} q^{\text{comaj}(|\sigma|) + \text{exc}(|\sigma|)} p^{\text{bnd}(\sigma)} \\ &= \sum_{k=0}^n \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{fix}(\sigma) = k}} q^{\text{comaj}(\sigma) + \text{exc}(\sigma)} p^{\binom{k+1}{2}} \prod_{i=k+1}^n (1 + p^i) \\ &= \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q q^{\binom{k}{2}} d_{n-k}(q) p^{\binom{k+1}{2}} \prod_{i=k+1}^n (1 + p^i). \end{aligned}$$

Now set $p = q$ and apply Theorem 2.4.5. □

ACKNOWLEDGEMENTS

The research presented here began while both authors were visiting the Mittag-Leffler Institute as participants in a combinatorics program organized by Anders Björner and Richard Stanley. We thank the Institute for its hospitality and support. We are also grateful to Ira Gessel for some very useful discussions.

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