Cohomology of Dowling Lattices and Lie (Super)algebras

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Abstract

We extend a well-known relationship between the representation of the symmetric group on the homology of the partition lattice and the free Lie algebra to Dowling lattices.

1 Introduction

There is a remarkable relationship between the homology of the partition lattice and the free Lie algebra. Namely, the representation of the symmetric group $\mathcal{S}_n$ on the homology of the partition lattice Π_n is isomorphic to the representation of the symmetric group on a multilinear component of the free Lie algebra on $[n] = \{1, 2, \ldots, n\}$ tensored with the sign representation. This relationship has been studied from various view points in Joyal [Jo], Barcelo [Ba], Hanlon-Wachs [Ha-Wa], and Wachs [Wa2]. Bergeron [Be] extended this relationship to the representation of the hyperoctahedral group on the homology of the signed partition lattice (more precisely on the Orlik-Solomon algebra of the signed partition lattice). In this paper we further extend the relationship to general Dowling lattices

Dowling lattices were introduced by Dowling [Do] as a generalization of the partition lattice. For any finite group G and positive integer n there is a Dowling lattice $Q_n(G)$. The partition lattice and the signed partition lattice are obtained when the

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order of G is 1 and 2 respectively. The wreath product of $\mathcal{S}_n$ and G, which we denote by $\mathcal{S}_n(G)$, acts naturally as an automorphism group of $Q_n(G)$, generalizing the action of the symmetric group S_n on the partition lattice Π_{n+1} and the action of the hyperoctahedral group on the signed partition lattice. The action of $\mathcal{S}_n(G)$ on $Q_n(G)$ induces dual representations of $\mathcal{S}_n(G)$ on the homology and cohomology of $Q_n(G)$, whose character was computed by Hanlon [Ha].

We show that as an $\mathcal{S}_n(G)$-module, the cohomology of $Q_n(G)$ is isomorphic to a multilinear component of the enveloping algebra of a certain Lie algebra tensored with a sign representation. The Lie algebra is obtained by taking the subalgebra of the free Lie algebra on $[n] \times G$ consisting of elements fixed by a natural action of G. We also show that the cohomology of $Q_n(G)$ is $\mathcal{S}_n(G)$-isomorphic to a multilinear component of the enveloping algebra of the fixed point subalgebra of the oddly generated free Lie superalgebra on $[n] \times G$. For the symmetric group acting on the partition lattice these results reduce to an observation of Stanley [St] that the representation of S_n on Π_{n+1} is the regular representation. For the hyperoctahedral group acting on the signed partition lattice, this yields the above mentioned result of Bergeron.

Bergeron produces an explicit isomorphism by giving a bijection between bases of the two modules and showing that the representation matrices with respect to these bases are the same. This extends the technique that Barcelo [Ba] used to explain the sign twisted isomorphism between the homology of the partition lattice and the multilinear component of the free Lie algebra. Our method of proof differs from Bergeron in that we construct a correspondence between natural generating sets rather than between bases. There are no representation matrices to compute. Instead we derive relations on the generating sets and show that relations for the cohomology module correspond to relations for the Lie module. This extends the technique that was used by Wachs [Wa2] to explain the sign twisted isomorphism between the homology of the partition lattice and the multilinear component of the free Lie algebra. A key ingredient of our proof is the use of labeled binary forests to encode generators for cohomology and for the Lie module.

We use the same technique to show that the Whitney cohomology of $Q_n(G)$ tensored with the sign representation is isomorphic to a multilinear component of the tensor product of the symmetric algebra of the fixed point Lie subalgebra with the exterior algebra of the fixed point Lie subalgebra. This extends a result of Lehrer and Solomon [Le-So] on the action of the symmetric group on the Whitney homology (equivalently Orlik-Solomon algebra) of the partition lattice, which was combinatorially explained in [Ba-Be] and [Wa2]. This result also reduces to Bergeron's result on the Whitney homology of the signed partition lattice.

Our result on Whitney cohomology can be applied to hyperplane arrangements. More precisely let $\mathcal{A}_{n,m}$ be the complex hyperplane arrangement in $\mathbf{C}^n$ consisting of hyperplanes $z_j = 0$ and $z_j - \omega^h z_k = 0$, where ω is a primitive mth root of unity,

$j, k \in [n]$ and $h \in [m]$. Let G be the cyclic group of order m. The group $\mathcal{S}_n(G)$ acts on $\mathcal{A}_{n,m}$ and this induces a representation of $\mathcal{S}_n(G)$ on the cohomology of the complement of $\mathcal{A}_{n,m}$. This representation is discussed in [Or-Te]. Our result implies, via the Orlik-Solomon formula, that the cohomology of the complement of $\mathcal{A}_{n,m}$ is $\mathcal{S}_n(G)$-isomorphic (up to sign) to the dual of the the multilinear component of the tensor product of the symmetric algebra of the fixed point Lie subalgebra with the exterior algebra of the fixed point Lie subalgebra.

In Section 2 we review the necessary poset homology notation and terminology. In Section 3 we recall the definition of the Dowling lattice $Q_n(G)$ and describe the action of the wreath product $\mathcal{S}_n(G)$ on $Q_n(G)$. In Section 4 we describe a natural generating set for the cohomology of $Q_n(G)$ in terms of labeled binary forests. We derive relations on the labeled forest generators from the coboundary relations in Section 5.

In Section 6 we describe a natural generating set for the Lie module in terms of labeled binary forests. More precisely, we use labeled binary forests to encode generators for the multilinear component of the enveloping algebra of the fixed point subalgebra of the free Lie (super)algebra on $[n] \times G$. We then derive relations on these generators. In Section 7 we give explicit isomorphisms between the cohomology module and the Lie module by giving a correspondence between the generating sets and showing that relations correspond to relations. Our results are extended to Whitney cohomology in Section 8 where the result on the cohomology of the complement of the complex hyperplane arrangement $A_{n,m}$ is derived.

An interesting basis for homology of $Q_n(G)$ is constructed in Section 9. This basis generalizes a basis for the homology of the partition lattice constructed in [Wa1] and a basis for the homology of the signed partition lattice constructed in [Bj-Wa].

2 Poset (co)homology

Let P be a finite poset of length $\ell \geq 1$ with maximum element $\hat{1}$ and minimum element $\hat{0}$. For $r = -1, \ldots, \ell - 2$, let $C_r(P)$ denote the complex vector space on the basis of chains $\hat{0} < x_1 < x_2 < \cdots < x_{r+1} < \hat{1}$ of P. ($C_{-1}(P)$ is the one dimensional space generated by the chain $\hat{0} < \hat{1}$.) Also let $C_{\ell-1}(P) = C_{-2}(P) = (0)$. For $r = -1, \ldots, \ell - 1$, the boundary map $\partial_r : C_r(P) \to C_{r-1}(P)$ is the linear map defined by

$$\partial_r(\hat{0} < x_1 < \cdots < x_{r+1} < \hat{1}) = \sum_{i=1}^{r+1} (-1)^i (\hat{0} < x_1 < \cdots < \hat{x}_i < \cdots < x_{r+1} < \hat{1}),$$

where $\hat{x}_i$ means that x_i is deleted. For $r = -2, \ldots, \ell - 2$, the coboundary map $\delta_r : C_r(P) \to C_{r+1}(P)$ is defined by

$$\langle \delta_r(\alpha), \beta \rangle = \langle \alpha, \partial_{r+1}(\beta) \rangle,$$

where $\alpha \in C_r(P)$, $\beta \in C_{r+1}(P)$ and $\langle \, , \, \rangle$ is the bilinear form on $\oplus_{r=-1}^{\ell-2} C_r(P)$ defined on chains by

$$\langle c_1, c_2 \rangle = \begin{cases} 1 & \text{if } c_1 = c_2 \\ 0 & \text{otherwise.} \end{cases}$$

It is equivalent to say that

$$\delta_r(\hat{0} < x_1 < \cdots < x_{r+1} < \hat{1}) =$$

$$\sum_{i=1}^{r+2} (-1)^i \sum_{x \in (x_{i-1}, x_i)} (\hat{0} < x_1 < \cdots < x_{i-1} < x < x_i < \cdots < x_{r+1} < \hat{1}),$$

where (a, b) denotes the open interval $\{x \in P : a < x < b\}$, $x_0 = \hat{0}$ and $x_{r+2} = \hat{1}$.

For $-1 \le r \le \ell - 2$, the rth homology of P is given by

$$H_r(P) = \ker \partial_r / \operatorname{im} \partial_{r+1},$$

and the rth cohomology of P is given by

$$H^r(P) = \ker \delta_r / \operatorname{im} \delta_{r-1}.$$

The elements of $\ker \partial_r$ and $\ker \delta_r$ are called *cycles* and *cocycles*, respectively; and the elements of $\operatorname{im} \partial_r$ and $\operatorname{im} \delta_r$ are called *boundaries* and *coboundaries*, respectively. Note that $H_r(P)$ and $H^r(P)$ are essentially the reduced simplicial homology and cohomology, respectively, of the order complex of P, i.e., the simplicial complex whose faces are the chains of $P \backslash \{\hat{0}, \hat{1}\}$. We say "essentially" because the chains that generate $C_r(P)$ contain $\hat{0}$ and $\hat{1}$ while the faces of the order complex do not.

The bilinear form $\langle , \rangle$ induces a pairing between $H_r(P)$ and $H^r(P)$ which allows one to view $H^r(P)$ as the dual space of $H_r(P)$. The following simple proposition is a useful device for identifying bases for homology and cohomology. It was used implicitly in [Wa1] to establish a splitting basis for the d-divisible partition lattice. A variation of it was also used in [Br], [Br-Wa] to establish bases for more general restricted block size partition lattices. We use it in Section 9 to establish bases for Dowling lattices.

Proposition 2.1 *Let P be a poset and suppose $H_r(P)$ has dimension d. If $\rho_1, \ldots, \rho_d$ are elements of $H_r(P)$ and $m_1, \ldots, m_d$ are elements of $H^r(P)$ such that the matrix $(\langle \rho_i, m_j \rangle)_{i,j \in [d]}$ is invertible then $\rho_1, \ldots, \rho_d$ is a basis for $H_r(P)$ and $m_1, \ldots, m_d$ is a basis for $H^r(P)$.*

Proof: We need only show that $\rho_1, \ldots, \rho_d$ are linearly independent since d is the dimension of $H_r(P)$. Suppose $\sum_{i=1}^d c_i \rho_i = 0$, where $c_i \in \mathbf{C}$. Then $\sum_{i=1}^d c_i \langle \rho_i, m_j \rangle = \langle \sum_{i=1}^d c_i \rho_i, m_j \rangle = 0$ for all j. It follows that $(c_1, c_2, \ldots, c_d)M = (0, \ldots, 0)$, where M is the matrix $(\langle \rho_i, m_j \rangle)_{i,j \in [d]}$. Since M is invertible $(c_1, \ldots, c_d) = (0, \ldots, 0)$. Hence $\rho_1, \ldots, \rho_d$ is a basis. The same argument shows that $m_1, \ldots, m_d$ is a basis. $\square$

If a group G acts as a group of automorphisms of P then P is said to be a G-poset. Since any automorphism of P commutes with the boundary and coboundary maps and respects the form $\langle , \rangle$, it induces a linear map on $H_r(P)$ and on $H^r(P)$ turning $H_r(P)$ and $H^r(P)$ into dual representations of G.

3 Dowling Lattices

We will use the notation $B_0/B_1/\ldots/B_k$ to denote the partition of the set $\{0, 1, \ldots, n\}$ whose blocks are $B_0, B_1, \ldots, B_k$ where B_0 is the block that contains 0.

Throughout this paper let G be a finite group with identity e. A *G-prepartition* is a pair (π, γ) where $\pi = B_0/B_1/\ldots/B_k$ is a partition of $\{0, 1, \ldots, n\}$ and γ is a map $B_1 \cup \ldots \cup B_k \to G$. We refer to B_0 as the *zero block*. For $i = 1, \ldots, k$, let $\gamma_i = \gamma|_{B_i}$. We refer to a pair $(B_i, \gamma_i), i = 1, \ldots, k$, as a *nonzero block* and we think of it as a subset $\{(a, \gamma_i(a)) : a \in B_i\}$ of $[n] \times G$. We view γ as a labeling of the the elements of the nonzero blocks with elements of G.

There is a right action of G on $[n] \times G$, given by $(i, h) \cdot g = (i, hg)$. This action extends to the set of all subsets S of $[n] \times G$ by $S \cdot g = \{x \cdot g : x \in S\}$. We say that two subsets S_1 and S_2 of $[n] \times G$ are *equivalent* if there is a $g \in G$ so that $S_1 = S_2 \cdot g$. This is an equivalence relation on the power set of $[n] \times G$.

Two G-prepartitions are *equivalent* if their zero blocks are equal and there is a one-to-one correspondence between their nonzero blocks such that corresponding blocks are equivalent. This is an equivalence relation on G-prepartitions.

A *G-partition* is defined to be the equivalence class of a G-prepartition (π, γ) which we denote by $\overline{(\pi, \gamma)}$. The equivalence class of a nonzero block (B_i, γ_i) of a G-prepartition will be called a *nonzero G-block* of the G-partition and will be denoted by $\overline{(B_i, \gamma_i)}$. We will also refer to the zero block B_0 of the G-prepartition as the *zero G-block* of the G-partition and for convenience let $\overline{B}_0 = B_0$.

The Dowling lattice $Q_n(G)$ is the set of G-partitions with partial order determined by the covering relation described below. We first define a *merge* of two G-blocks. The merge of the zero G-block B_0 with a nonzero G-block $\overline{(B_i, \gamma_i)}$ is $B_0 \cup B_i$. There are $|G|$ merges of two nonzero G-blocks $\overline{(B_i, \gamma_i)}$ and $\overline{(B_j, \gamma_j)}$, each of which is of the form $\overline{(B_i, \gamma_i) \cup ((B_j, \gamma_j) \cdot g)}$ for some $g \in G$. Here the union is of two subsets of $[n] \times G$. Note that g determines the merge uniquely once equivalence class representatives are fixed for the G-blocks being merged. In $Q_n(G)$, the G-partition $\overline{(\pi,\gamma)}$ is covered by $\overline{(\pi',\gamma')}$ if $\overline{(\pi',\gamma')}$ can be obtained from $\overline{(\pi,\gamma)}$ by merging two G-blocks of $\overline{(\pi,\gamma)}$.

Let $\mathcal{S}_n$ be the symmetric group on $[n]$. Recall that the wreath product $\mathcal{S}_n(G)$ of $\mathcal{S}_n$ with G is defined to be the group $\mathcal{S}_n \times G^n$ with product defined by

$$(\sigma, f)\cdot(\tau, g) = \Big(\sigma\tau, (f_{\tau(1)}g_1, f_{\tau(2)}g_2, \ldots, f_{\tau(n)}g_n)\Big),$$

where $\sigma, \tau \in \mathcal{S}_n$, $f = (f_1, f_2, \ldots, f_n) \in G^n$ and $g = (g_1, g_2, \ldots, g_n) \in G^n$. For any subgroup H of $\mathcal{S}_n$ the wreath product $H(G)$ is defined similarly.

There is left action of $\mathcal{S}_n(G)$ on $[n]\times G$ defined by $(\sigma, f)\cdot(i,a) = (\sigma(i), f_i a)$, where $(\sigma, f) \in \mathcal{S}_n(G)$ and $(i,a) \in [n]\times G$. This action extends to a left action of $\mathcal{S}_n(G)$ on $([n]\times G)^*$, the set of all words over the alphabet $[n]\times G$, by letting (σ, f) act on each letter of the word. This action restricts to the left multiplication action of $\mathcal{S}_n(G)$ on itself by viewing an element (τ, g) of $\mathcal{S}_n(G)$ as the word $(\tau(1), g_1)(\tau(2), g_2)\cdots(\tau(n), g_n)$. We will often view the elements of $\mathcal{S}_n(G)$ in this way.

The action of $\mathcal{S}_n(G)$ on $[n]\times G$ also induces a left action on the power set of $[n]\times G$, which in turn induces a left action of $\mathcal{S}_n(G)$ on G-prepartitions by letting an element $(\sigma, f) \in \mathcal{S}_n(G)$ act on each nonzero block viewed as a subset of $[n]\times G$ and act on the zero block $B_0 = \{0, i_1, i_2, \ldots, i_j\}$ by $(\sigma, f)B_0 = \{0, \sigma(i_1), \sigma(i_2), \ldots, \sigma(i_j)\}$.

Since the left action of $\mathcal{S}_n(G)$ on nonzero blocks commutes with the right action of G, the left action of $\mathcal{S}_n(G)$ on G-prepartitions respects equivalence of G-prepartitions. Hence the left action of $\mathcal{S}_n(G)$ on G-prepartitions induces an action on G-partitions. It is easy to see that the action of $\mathcal{S}_n(G)$ on G-partitions respects the partial order relation. Hence the Dowling lattice $Q_n(G)$ is a $\mathcal{S}_n(G)$-poset.

The two most important examples of Dowling lattices are obtained by letting G be the trivial group (e) and the group $\mathbf{Z}/2\mathbf{Z}$. The Dowling lattice $Q_n(e)$ is the ordinary lattice of partitions of the set $\{0, 1, \ldots, n\}$ and $\mathcal{S}_n(e)$ is the symmetric group $\mathcal{S}_n$. Let Π_n denote the lattice of partitions of $[n]$. The symmetric group $\mathcal{S}_n$ acts as an automorphism group on Π_n by permuting the letters of the partitions. The action of $\mathcal{S}_n(e)$ on $Q_n(e)$ is isomorphic to the restriction of the action of $\mathcal{S}_{n+1}$ on Π_{n+1} to $\mathcal{S}_n$. The Dowling lattice $Q_n(\mathbf{Z}/2\mathbf{Z})$ is the signed partition lattice and $\mathcal{S}_n(\mathbf{Z}/2\mathbf{Z})$ is the hyperoctahedral group. These Dowling lattices are intersection lattices of Coxeter arrangements of types A_n and B_n, respectively.

4 Generators for Cohomology

Since $Q_n(G)$ is a geometric lattice of rank n, it follows from [Fo] that the cohomology of $Q_n(G)$ vanishes in all dimensions except the top dimension $n-2$. We write $H^*(Q_n(G)) = H^{n-2}(Q_n(G))$. Since $\mathrm{Ker}\delta_{n-2} = C_{n-2}(Q_n(G))$, $H^*(Q_n(G))$ is the vector space spanned by the maximal chains of $Q_n(G)$ modulo the coboundary relations.

Our goal in this section is to describe a subset of maximal chains which serve as generators for $H^*(Q_n(G))$. In the next section we derive relations on these generators from the coboundary relations.

We will generalize a construction given in [Wa2] for encoding maximal chains of the partition lattice with labeled binary trees. The binary trees considered here have the property that all internal nodes have a left child and a right child. By a *binary forest* we mean a sequence of binary trees. We denote the set of all binary trees with k leaves by $\mathcal{T}_k$ and the set of all binary forests with k leaves by $\mathcal{F}_k$.

For $F \in \mathcal{F}_k$ and word w of length k over some alphabet S, let $(F; w)$ denote the binary forest F whose ith leaf (from left to right) is labeled with the ith letter of w. Here we take our alphabet S to be $[n] \times G$ so that our leaves are labeled with pairs (i, g), where $i \in [n]$ and $g \in G$. We refer to i and g as the $[n]$-*label* and G-*label*, respectively. We also require that the $[n]$-labels of $(F; w)$ be distinct, i.e., no two letters of the word w share the same first coordinate. We refer to such an $(F; w)$ as a *labeled forest* . If F is a single binary tree then we refer to $(F; w)$ as a *labeled tree.* An example of a labeled forest is given in Figure 4.1, where $G = \mathbf{Z}/2\mathbf{Z} = \{e, g\}$. The leaf label (a, e) is denoted by a and the leaf label (a, g) is denoted by $\bar{a}$.

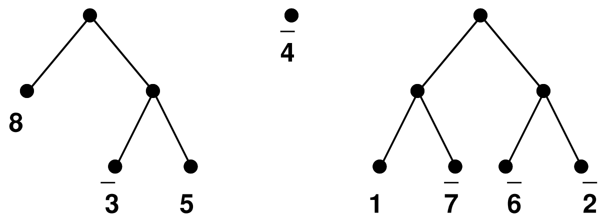

Figure 4.1

Of particular interest to us are the labeled forests $(F; w)$ with n leaves. That is, labeled forests $(F; w)$ for which $F \in \mathcal{F}_n$ and $w \in \mathcal{S}_n(G)$.

Suppose $T \in \mathcal{T}_{k+1}$ and w is a word in $([n] \times G)^k$ for which no two letters share the same first coordinate. Let $0w$ denote the concatenation of the letter 0 with the word w. We will refer to $(T; 0w)$, the binary tree T whose leaves are labeled by the word $0w$, as a 0-*labeled tree.* A 0-labeled forest is defined similarly.

We use the notation $(A_1 {}^\frown A_2)$ to denote a labeled tree whose labeled left subtree is A_1 and labeled right subtree is A_2. Similarly $(T_1 {}^\frown T_2)$ denotes an unlabeled binary tree whose left subtree is T_1 and right subtree is T_2.

For a forest $F = T_1, T_2, \ldots, T_k$ in $\mathcal{F}_n$, define $\hat{F}$ to be the binary tree in $\mathcal{T}_{n+1}$ given by

$$\hat{F} = (((\ldots(\bullet^\frown T_1)^\frown T_2)^\frown \ldots)^\frown T_k),$$

where $\bullet$ denotes the binary tree consisting of a single node. For example, suppose $(F; w)$ is the labeled forest of Figure 4.1. Then the 0-labeled tree $(\hat{F}; 0w)$ is given in Figure 4.2.

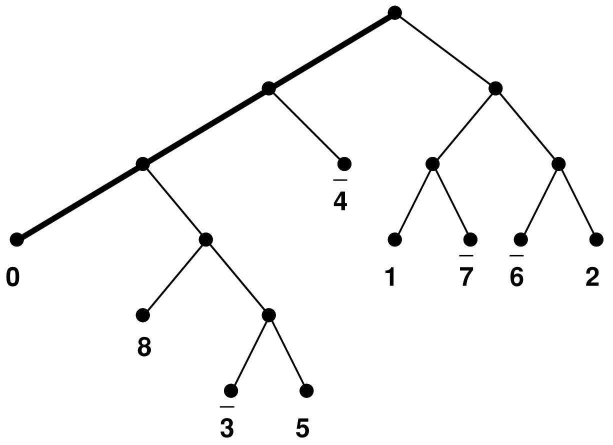

Figure 4.2

Recall that the *postorder* listing of the internal nodes of a binary tree T is defined recursively as follows: first list the internal nodes of the left subtree of T in postorder, then list the internal nodes of the right subtree of T in postorder, and finally list the root of T. Let $v_1, v_2, \ldots, v_n$ be the postorder listing of the internal nodes of T. A *linear extension* of the internal nodes of a binary tree T is a listing of the internal nodes $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}$ such that each node precedes its parent, i.e., if $v_{\sigma(j)}$ is the parent of $v_{\sigma(i)}$ then $i < j$. We shall say that the permutation σ induces the linear extension and let $E(T)$ be the set of all permutations that induce linear extensions of T. For example, if T is the binary tree in Figure 4.3 then $E(T) = \{1234, 1324, 3124\}$.

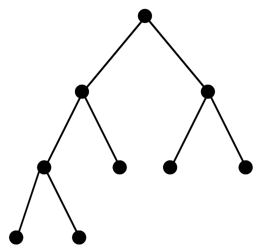

Figure 4.3

We need one final bit of notation before we can describe our construction. For a labeled tree A define $L(A)$ to be the set of leaf labels of A. For a 0-labeled tree A define $L(A)$ to be the set $\{0\} \cup \{i : (i,g) \text{ is a leaf label of } A\}$. If A is a labeled forest or 0-labeled tree and v is a node of A, let $L_v(A) = L(B)$ where B is the labeled subtree or 0-labeled subtree of A rooted at v. For an internal node v let $\ell(v)$ denote the left child of v and $r(v)$ denote the right child of v.

We are finally ready to describe the encoding of maximal chains of $Q_n(G)$ with labeled forests. We will construct maximal chains of $Q_n(G)$ by considering the labeled subtrees of $(\hat{F}; 0w)$, where $F \in \mathcal{F}_n$ and $w \in \mathcal{S}_n(G)$. The label sets of labeled subtrees and 0-labeled subtrees will form the blocks of the G-prepartitions. To each labeled forest $(F; w)$, where $F \in \mathcal{F}_n$ and $w \in \mathcal{S}_n(G)$, we associate a maximal chain $c(F; w)$ of $Q_n(G)$ as follows. For each $i = 1, 2, \ldots, n$, the G-partition of rank i in $c(F; w)$ is obtained from the G-partition of rank $i-1$ by merging the G-blocks $\overline{L_{\ell(v_i)}(\hat{F}; 0w)}$ and $\overline{L_{r(v_i)}(\hat{F}; 0w)}$, where v_i is the ith internal node of $\hat{F}$ in the postorder listing, to get the G-block $\overline{L_{v_i}(\hat{F}; 0w)}$. For the labeled forest $(F; w)$ of Figure 4.1, $c(F; w)$ is the maximal chain

$$0/8/\bar{3}/5/\bar{4}/1/\bar{7}/\bar{6}/2 < 0/8/\bar{3}5/\bar{4}/1/\bar{7}/\bar{6}/2 < 0/8\bar{3}5/\bar{4}/1/\bar{7}/\bar{6}/2 < 0835/\bar{4}/1/\bar{7}/\bar{6}/2$$

$$< 08354/1/\bar{7}/\bar{6}/2 < 08354/1\bar{7}/\bar{6}/2 < 08354/1\bar{7}/\bar{6}2 < 08354/1\bar{7}\bar{6}2 < 083541762,$$

where for the sake of simplicity of notation we have just given equivalence class representatives of the G-blocks here.

Not every maximal chain of $Q_n(G)$ is of the form $c(F; w)$. However a more general construction does encode every maximal chain. To each $F \in \mathcal{F}_n, w \in \mathcal{S}_n(G)$ and $\sigma \in E(\hat{F})$ we associate a maximal chain $c(F; w; \sigma)$ of $Q_n(G)$ as follows. For each $i = 1, 2, \ldots, n$, the G-partition of rank i in $c(F; w; \sigma)$ is obtained from the G-partition of rank $i-1$ by merging the G-blocks $\overline{L_{\ell(v_{\sigma(i)})}(\hat{F}; 0w)}$ and $\overline{L_{r(v_{\sigma(i)})}(\hat{F}; 0w)}$ to get the

G-block $\overline{L_{v_{\sigma(i)}}(\hat{F};0w)}$. Note that $c(F;w;12\cdots n)$ is simply $c(F;w)$ defined above. If $\sigma = 51236748$ and $(F;w)$ is the labeled forest of Figure 4.1 then $c(F;w;\sigma)$ is the maximal chain

$$0/8/\bar{3}/5/\bar{4}/1/\bar{7}/\bar{6}/2 < 0/8/\bar{3}/5/\bar{4}/1\bar{7}/\bar{6}/2 < 0/8/\bar{3}5/\bar{4}/1\bar{7}/\bar{6}/2 < 0/8\bar{3}5/\bar{4}/1\bar{7}/\bar{6}/2$$

$$< 0835/\bar{4}/1\bar{7}/\bar{6}/2 < 0835/\bar{4}/1\bar{7}/\bar{6}2 < 0835/\bar{4}/1\bar{7}\bar{6}2 < 08354/1\bar{7}\bar{6}2 < 083541762.$$

It is easy to see that every maximal chain in $Q_n(G)$ is of the form $c(F;w;\sigma)$ for some $F \in \mathcal{F}_n$, $w \in \mathcal{S}_n(G)$, and $\sigma \in E(\hat{F})$. Although not every maximal chain of $Q_n(G)$ is of the form $c(F;w)$, the next lemma shows that every maximal chain is in the same cohomology class (up to sign) as some maximal chain of this form. We shall let $c(F;w;\sigma)$ and $c(F;w)$ denote cohomology classes of maximal chains as well as maximal chains.

Lemma 4.1 *For any $F \in \mathcal{F}_n$, $w \in \mathcal{S}_n(G)$ and $\sigma \in E(\hat{F})$ we have in cohomology*

$$c(F;w;\sigma) = \operatorname{sgn}(\sigma)\ c(F;w).$$

Proof: The proof is by induction on $\operatorname{inv}(\sigma)$, the number of inversions of σ. If $\operatorname{inv}(\sigma) = 0$ then the result is trivial. If $\operatorname{inv}(\sigma) \geq 1$ then there is some i such that $\sigma(i) > \sigma(i+1)$. Note that $\sigma \cdot (i, i+1) \in E(\hat{F})$ (cf. [Wa2, Lemma 5.1]). Since $\operatorname{inv}(\sigma \cdot (i, i+1)) = \operatorname{inv}(\sigma) - 1$, induction gives

$$c(F;w;\sigma \cdot (i,i+1)) = \operatorname{sgn}(\sigma \cdot (i,i+1))\ c(F;w) = -\operatorname{sgn}(\sigma)\ c(F;w).$$

We must show that $c(F;w;\sigma) = -c(F;w;\sigma \cdot (i,i+1))$ in cohomology.

We shall say that a chain c is *submaximal* if it can be obtained from a maximal chain by removing an element other than $\hat{0}$ or $\hat{1}$. The missing element divides c into two chains; we call them the *upper* and *lower pieces* of c. We say that the missing element *completes* the submaximal chain.

Consider the submaximal chain c in $Q_n(G)$ obtained by removing the rank i element of $c(F;w;\sigma)$. The largest element of the lower piece of c contains the G-blocks $\overline{L_{\ell(v_{\sigma(i)})}(\hat{F};0w)}$, $\overline{L_{r(v_{\sigma(i)})}(\hat{F};0w)}$, $\overline{L_{\ell(v_{\sigma(i+1)})}(\hat{F};0w)}$, and $\overline{L_{r(v_{\sigma(i+1)})}(\hat{F};0w)}$. In the smallest element of the upper piece of c, the G-blocks $\overline{L_{\ell(v_{\sigma(i)})}(\hat{F};0w)}$ and $\overline{L_{r(v_{\sigma(i)})}(\hat{F};0w)}$ are merged and the G-blocks $\overline{L_{\ell(v_{\sigma(i+1)})}(\hat{F};0w)}$ and $\overline{L_{r(v_{\sigma(i+1)})}(\hat{F};0w)}$ are merged.

There are only two G-partitions that can complete c. They are identical to the largest element of the lower piece of c except that in one, the G-blocks $\overline{L_{\ell(v_{\sigma(i)})}(\hat{F};0w)}$ and $\overline{L_{r(v_{\sigma(i)})}(\hat{F};0w)}$ are merged, while in the other, the G-blocks $\overline{L_{\ell(v_{\sigma(i+1)})}(\hat{F};0w)}$ and

$\overline{L_{r(v_{\sigma(i+1)})}}(\hat{F}; 0w)$ are merged. The maximal chains formed by completing c with these G-partitions are $c(F; w; \sigma)$ and $c(F; w; \sigma \cdot (i, i+1))$ respectively. This shows that

$$\delta(c) = (-1)^i \ \ (c(F; w; \sigma) + c(F; w; \sigma \cdot (i, i+1)))$$

so that in cohomology

$$0 = c(F; w; \sigma) + c(F; w; \sigma \cdot (i, i+1))$$

as desired. $\square$

We have the following as an immediate consequence.

Theorem 4.2 *The set* $\{c(F; w) : F \in \mathcal{F}_n, w \in \mathcal{S}_n(G)\}$ *is a generating set for* $H^*(Q_n(G))$*. Moreover the action of* $\mathcal{S}_n(G)$ *on* $H^*(Q_n(G))$ *is given by*

$$u \cdot c(F; w) = c(F; u \cdot w) \tag{1}$$

where $u, w \in \mathcal{S}_n(G)$*.* $\square$

We refer to the generators of the form $c(F; w)$, where $F \in \mathcal{F}_n, w \in \mathcal{S}_n(G)$, as *labeled forest generators.*

5 Relations on Labeled Forest Generators

We now derive four relations on labeled forest generators which we refer to as *elementary relations.* We will show that the elementary relations generate all the cohomology relations on labeled forest generators.

The right action of G on the set $[n] \times G$ induces a right action of G on $([n] \times G)^*$ by letting G act on each letter of the word. This induces a right action of G on the set of labeled trees $(T; w)$ by letting $(T; w) \cdot g = (T; w \cdot g)$, where $g \in G$.

For any labeled tree A (or unlabeled binary tree T) let $i(A)$ (resp. $i(T)$) denote the number of internal nodes of A (resp. T).

Lemma 5.1 (Relations of Type 1) *Let* $A_1, \ldots, A_k$ *be a labeled forest with* n *leaves. Then for all* $g \in G$ *and* $j = 1, 2, \ldots, k$*, we have*

$$c(A_1, \ldots, A_j, \ldots, A_k) = c(A_1, \ldots, (A_j \cdot g), \ldots, A_k). \tag{2}$$

Proof: The expressions $c(A_1, \ldots, A_j, \ldots, A_k)$ and $c(A_1, \ldots, A_j \cdot g, \ldots, A_k)$ represent the same maximal chains of $Q_n(G)$ by the definition of equivalence of prepartitions. $\square$

Lemma 5.2 (Relations of Type 2) *Let $(A^\frown B)$ be a labeled subtree of one of the trees in a labeled forest with n leaves which we denote by $\cdots(A^\frown B)\cdots$. Then in cohomology,*

$$c(\cdots(A^\frown B)\cdots) + (-1)^{i(A)i(B)+1} c(\cdots(B^\frown A)\cdots) = 0. \tag{3}$$

(Our notation indicates that the forest of the second term is identical to the forest of the first term except that we are replacing the labeled subtree $(A^\frown B)$ with the labeled subtree $(B^\frown A)$.)

Proof: Consider the chain $c(\cdots(B^\frown A)\cdots;\sigma)$ where σ is the same as postorder except that the internal nodes of A are read before the internal nodes of B. By Lemma 4.1,

$$\begin{aligned} c(\cdots(A^\frown B)\cdots) &= c(\cdots(B^\frown A)\cdots;\sigma) \\ &= \operatorname{sgn}(\sigma)\ c(\cdots(B^\frown A)\cdots) \\ &= (-1)^{i(A)i(B)} c(\cdots(B^\frown A)\cdots). \quad \square \end{aligned}$$

Lemma 5.3 (Relations of Type 3) *Let $(A^\frown(B^\frown C))$ be a labeled subtree of one of the trees in a labeled forest with n leaves which we denote by $\cdots(A^\frown(B^\frown C))\cdots$. Then in cohomology,*

$$\begin{aligned} (-1)^{i(A)i(C)}\ c(\cdots(A^\frown(B^\frown C))\cdots) + (-1)^{i(B)i(C)}\ c(\cdots(C^\frown(A^\frown B))\cdots) \\ +(-1)^{i(A)i(B)}\ c(\cdots(B^\frown(C^\frown A))\cdots) = 0. \end{aligned} \tag{4}$$

Proof: Consider the submaximal chain c of $c(\cdots(A^\frown(B^\frown C))\cdots)$ for which the largest element of the lower piece contains G-blocks $\overline{L(A)}, \overline{L(B)}, \overline{L(C)}$ and the smallest element of the upper piece contains G-block $\overline{L(A) \cup L(B) \cup L(C)}$. Note that all of these G-blocks are nonzero. There are exactly three completing G-partitions; one contains G-blocks $\overline{L(A)}$ and $\overline{L(B) \cup L(C)}$, another contains G-blocks $\overline{L(A) \cup L(B)}$ and $\overline{L(C)}$, and the last has G-blocks $\overline{L(A) \cup L(C)}$ and $\overline{L(B)}$. The completed chains are represented by

$$c(\cdots(A^\frown(B^\frown C))\cdots), \quad c(\cdots(C^\frown(A^\frown B))\cdots;\sigma) \text{ and } c(\cdots(B^\frown(C^\frown A))\cdots;\tau)$$

respectively, where σ and τ are the same as postorder except that σ reads the internal nodes of A and B before the internal nodes of C and τ reads the internal nodes of A before the internal nodes of B and C. Note that $\operatorname{sgn}(\sigma) = (-1)^{i(C)(i(A)+i(B))}$ and $\operatorname{sgn}(\tau) = (-1)^{i(A)(i(B)+i(C))}$. By setting the coboundary of c equal to 0 and applying Lemma 4.1, we have

$$\begin{aligned} c(\cdots(A^\frown(B^\frown C))\cdots) &+ (-1)^{i(C)(i(A)+i(B))}\ c(\cdots(C^\frown(A^\frown B))\cdots) \\ &+ (-1)^{i(A)(i(B)+i(C))}\ c(\cdots(B^\frown(C^\frown A))\cdots) = 0. \end{aligned}$$

Multiply through by $(-1)^{i(A)i(C)}$ to obtain the relation of type 3. $\square$

Lemma 5.4 (Relations of Type 4) *Let $A_1, \ldots, A_k$ be a labeled forest with n leaves. Then in cohomology for $j = 1, 2, \ldots, k-1$,*

$$c(A_1, \ldots, A_j, A_{j+1}, \ldots, A_k) - (-1)^{(i(A_j)+1)(i(A_{j+1})+1)}\, c(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k)$$
$$+(-1)^{i(A_{j+1})} \sum_{g \in G} c(A_1, \ldots, (A_j {}^\frown (A_{j+1} \cdot g)), \ldots, A_k) = 0. \qquad (5)$$

Proof: For any set $S \subseteq [n] \times G$ let $\tilde{S} = \{i : (i, g) \in S\}$. Let $X = \{0\} \cup \tilde{L}(A_1) \cup \ldots \cup \tilde{L}(A_{j-1})$. Consider the submaximal chain c of $c(A_1, \ldots, (A_j {}^\frown A_{j+1}), \ldots, A_k)$ for which the largest element of the lower piece contains G-blocks $X, \overline{L(A_j)}, \overline{L(A_{j+1})}$ and the smallest element of the upper piece contains the zero G-block $X \cup \tilde{L}(A_j) \cup \tilde{L}(A_{j+1})$.

There are exactly $|G| + 2$ completing G-partitions. Two of them are obtained by merging X with $\overline{L(A_j)}$ and with $\overline{L(A_{j+1})}$. The other $|G|$ completing G-partitions are obtained by merging the nonzero G-blocks $\overline{L(A_j)}$ and $\overline{L(A_{j+1})}$ in each of the $|G|$ possible ways. The completed chains are given by

$$c(A_1, \ldots, A_k; \sigma), \quad c(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k; \tau)$$

and, for each $g \in G$,

$$c(A_1, \ldots, (A_j {}^\frown (A_{j+1} \cdot g)), \ldots, A_k),$$

where σ and τ are just like postorder except that σ reads the internal nodes of A_{j+1} before the parent of the root of A_j and τ reads the internal nodes of A_j before the internal nodes of A_{j+1} and before the parent of the root of A_{j+1}. It is easy to see that $\mathrm{sgn}(\sigma) = (-1)^{i(A_{j+1})}$ and $\mathrm{sgn}(\tau) = (-1)^{i(A_j)(i(A_{j+1})+1)}$. By the setting the coboundary of c equal to 0 and applying Lemma 4.1, we have

$$(-1)^{i(A_{j+1})} c(A_1, \ldots, A_k) + (-1)^{i(A_j)(i(A_{j+1})+1)}\, \dot{c}(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k)$$
$$+ \sum_{g \in G} c(A_1, \ldots, (A_j {}^\frown (A_{j+1} \cdot g)), \ldots, A_k) = 0.$$

Multiply through by $(-1)^{i(A_{j+1})}$ to get the relation of type 4. $\square$

In order to show that the elementary relations generate all the cohomology relations we will consider a certain subset of the labeled forest generators whose cardinality equals the dimension of $H^*(Q_n(G))$. We will then show that any labeled forest generator can be straightened into a linear combination of elements in the subset using only the elementary relations. This will allow us to conclude that the subset is a basis and moreover that the elementary relations generate all the cohomology relations.

The basis that we construct here is a generalization of a basis for the cohomology of Π_n constructed in [Wa1] (see also [Wa2]) called the dual splitting basis. For any labeled tree A let $\min A$ be the smallest $[n]$-label in A. A labeled tree A is called a *comb* if every right child is a leaf and the leftmost leaf has label $(\min A, e)$. For a labeled forest $A_1, \ldots, A_k$ we define the *sequence of minima* to be the sequence $\min A_1, \ldots, \min A_k$. A labeled forest is called a *comb forest* if each tree in the forest is a comb and its sequence of minima is decreasing. An example of a comb forest for $G = \mathbf{Z}/2\mathbf{Z}$ is given in Figure 5.1.

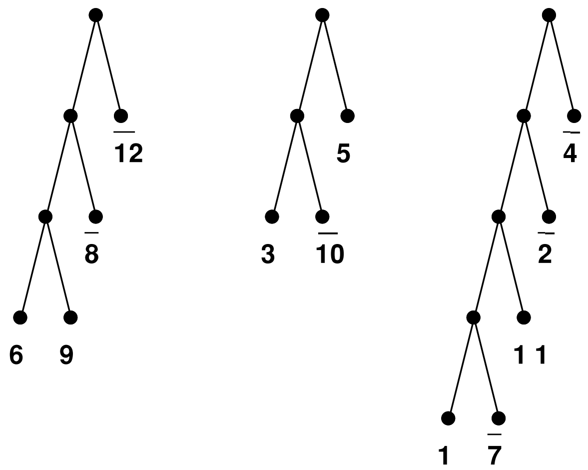

Figure 5.1

Lemma 5.5 *The number of comb forests with n leaves is $\prod_{j=1}^{n-1}(1 + j|G|)$ which is equal to the dimension of $H^*(Q_n(G))$.*

Proof: It was shown by Dowling [Do] that $\dim H^*(Q_n(G)) = \prod_{j=1}^{n-1}(1 + j|G|)$.

We can construct a comb forest as follows. Start with the single node labeled with $(1, e)$ and insert labeled leaves $(2, g_2), (3, g_3), \ldots, (n, g_n)$ one at a time. When we insert the leaf with label $(2, g_2)$ we can insert it either as a right sibling of $(1, e)$ forming the comb $((1, e)^\frown(2, g_2))$, or to the left of $(1, e)$ as a new comb forming the comb forest $(2, g_2), (1, e)$. In the former case we can choose the G-label g_2 to be any element of G while in the latter case g_2 has to be e. Hence there are $|G| + 1$ ways to insert the labeled leaf $(2, g_2)$ and choose g_2. Suppose we have already inserted the labeled leaves $(2, g_2), \ldots, (j, g_j)$. When we insert the labeled leaf $(j + 1, g_{j+1})$ we can insert it into an existing comb or insert it as the leaf label of a new single node comb. In the former case $(j+1, g_{j+1})$ can be inserted to the right of any leaf of an existing comb and g_{j+1} can be any element of G. In the latter case the new single node comb $(j+1, g_{j+1})$ must be the first comb in the sequence of combs and g_{j+1} must be e. Since there are j legal positions in which to insert the labeled leaf $(j+1, g_{j+1})$ in one of the existing combs and one legal position in which to insert into a new comb, the number of ways to insert the labeled leaf $(j+1, g_{j+1})$ and choose g_{j+1} is $j|G|+1$.□

Theorem 5.6 *The labeled forest generators $c(A)$, where A is a comb forest with n leaves, form a basis for $H^*(Q_n(G))$.*

Proof: By Lemma 5.5 it suffices to show that the $c(A)$, where A is a comb forest with n leaves, span $H^*(Q_n(G))$. We do this by describing a straightening algorithm. By Theorem 4.2, we need only straighten labeled forest generators. First we show how to express an arbitrary labeled forest generator $c(A)$ as a linear combination of labeled forest generators of the form $c(A')$, where A' is a labeled forest with a decreasing sequence of minima. Let A be the labeled forest $A_1, \ldots, A_k$. If the sequence of minima of A is not decreasing then it has an ascent, say at position j. We use the elementary relation of type 4 to write $c(A)$ as a linear combination of labeled forest generators of the form $c(A')$, where A' is a labeled forest whose sequence of minima read from right to left has fewer inversions than that of A. By repeatedly executing this step of the straightening algorithm, we keep reducing the number of inversions and eventually obtain a linear combination involving only labeled forests with decreasing sequence of minima.

Next we show how to straighten $c(A)$ if A is a labeled forest with a decreasing sequence of minima. First we use the elementary relation of type 1 to assume that each tree A_j has the leaf label $(\min A_j, e)$. Then we use repeated applications of the elementary relation of type 2 to assume that this leaf label is the leftmost leaf label of A_j.

We can now assume that A is a labeled forest $A_1, \ldots, A_k$ in which the leftmost leaf of each A_j has label $(\min A_j, e)$ and the sequence of minima is decreasing. Define $\chi(A)$ to be $\sum_{v \in I(A)} |L_{r(v)}(A)|$, where $I(A)$ is the set of internal nodes of A. In other words $\chi(A)$ is the sum of the number of leaves of the right subtree of each internal node of A. Note that $\chi(A) \geq n-k$ with equality if and only if A is a comb forest. When A is not a comb forest, we will use a relation derived from the elementary relations of types 2 and 3 to reduce the χ value of A. More precisely, we will use the relation to express $c(A)$ as a linear combination of forest generators of the form $c(A')$, where A' is a labeled forest for which (i) the leftmost leaf of each tree has minimum $[n]$-label and G-label e, (ii) A' has decreasing sequence of minima and (iii) $\chi(A') < \chi(A)$.

The relation derived from the elementary relations of types 2 and 3 is given by

$$c(\cdots(B_1{}^\frown(B_2{}^\frown B_3))\cdots) = (-1)^{i(B_3)+1}\; c(\cdots((B_1{}^\frown B_2){}^\frown B_3)\cdots)$$
$$+(-1)^{i(B_2)+i(B_2)i(B_3)+1}\; c(\cdots((B_1{}^\frown B_3){}^\frown B_2)\cdots). \qquad (6)$$

If A is not a comb forest, it has a labeled subtree of the form $(B_1{}^\frown(B_2{}^\frown B_3))$. That is $A = \cdots(B_1{}^\frown(B_2{}^\frown B_3))\cdots$. Set

$$A' = \cdots((B_1{}^\frown B_2){}^\frown B_3)\cdots \quad \text{and} \quad A'' = \cdots((B_1{}^\frown B_3){}^\frown B_2)\cdots.$$

Clearly the label set and leftmost leaf label of each tree of A' and A'' is the same as that of the corresponding tree of A. Also if we consider the sets of right subtrees of A, A' and A'' we see that the right subtree $(B_2{}^\frown B_3)$ in the set of right subtrees of A is replaced by the right subtree B_2 in both the sets of right subtrees of A' and of A''. It follows that $\chi(A'), \chi(A'') < \chi(A)$. Hence relation (6) expresses $c(A)$ as a linear combination of forest generators of the form $c(A')$ where (i),(ii), and (iii) hold. By repeatedly executing this step of the straightening algorithm, we keep reducing the χ values of the labeled forests and eventually obtain a linear combination involving only comb forests. □

Another proof of Theorem 5.6 which doesn't rely on the straightening algorithm is given in Section 9. The straightening algorithm is important nevertheless because it yields the following corollary.

Corollary 5.7 *The set $\{c(F;w) : F \in \mathcal{F}_n,\, w \in \mathcal{S}_n(G)\}$ is a generating set for $H^*(Q_n(G))$ subject only to the elementary relations of types 1, 2, 3 and 4.*

Proof: Since each step of the straightening algorithm relies only on the elementary relations, the elementary relations generate all the relations on labeled forest generators. □

6 The Lie Superalgebra

We begin this section by presenting the necessary notions and results from the theory of Lie superalgebras. Further information on Lie superalgebras can be found in [Sc].

A *Lie superalgebra* is a bigraded vector space $L = L_0 \oplus L_1$ equipped with a bilinear bracket operation $[\cdot,\cdot] : L \times L \to L$ such that

$$[x, y] \in L_{(\alpha+\beta) \bmod 2}$$

$$[x, y] + (-1)^{\alpha\beta}[y, x] = 0 \qquad (7)$$

$$(-1)^{\alpha\gamma}[x,[y,z]] + (-1)^{\beta\gamma}[z,[x,y]] + (-1)^{\alpha\beta}[y,[z,x]] = 0 \tag{8}$$

for each $x \in L_\alpha$, $y \in L_\beta$, $z \in L_\gamma$. The above equations are known as the *graded antisymmetry* and *graded Jacobi* relations, respectively. When $x \in L_i$ we say that x is homogeneous of *degree* i and write $i = \alpha(x)$. If $L_1 = (0)$ then L is an ordinary *Lie algebra*, and the relations are the *antisymmetry* and *Jacobi* relations.

Let $T(L)$ be the tensor algebra of L. The bigrading of L induces a bigrading of $T(L)$ making $T(L)$ an associative superalgebra. Let J be the two-sided ideal generated by elements of the form

$$[x,y] - x \otimes y + (-1)^{\alpha\beta} y \otimes x,$$

where $x \in L_\alpha$, $y \in L_\beta$. The *(universal) enveloping algebra*, denoted $\mathcal{E}(L)$, is defined to be the associative superalgebra $T(L)/J$. Let I be the two sided ideal generated elements of the form

$$x \otimes y - (-1)^{\alpha\beta} y \otimes x,$$

where $x \in L_\alpha$, $y \in L_\beta$. The *supersymmetric algebra* of L, denoted $\mathrm{Sym}_s(L)$, is the associative superalgebra $T(L)/I$. If L is an ordinary Lie algebra then $\mathrm{Sym}_s(L)$ is the ordinary symmetric algebra of L which we denote by $\mathrm{Sym}(L)$. The Poincaré-Birkhoff-Witt Theorem for Lie superalgebras (cf. [Sc]) states that a canonical homomorphism from the associated graded superalgebra of $\mathcal{E}(L)$ to $\mathrm{Sym}_s(L)$ is a superalgebra isomorphism. This yields a vector space isomorphism from $\mathcal{E}(L)$ to $\mathrm{Sym}_s(L)$.

Let L be a Lie superalgebra and let S be a set with a bigradation $S = S_0 \uplus S_1$ ($\uplus$ denotes disjoint union) such that $S_0 \subseteq L_0$ and $S_1 \subseteq L_1$. We say that L is *free on* S if S generates L by brackets and linear combinations and if, given a Lie superalgebra L' and a gradation respecting map $\rho : S \to L'$, there exists a unique Lie superalgebra homomorphism $\psi : L \to L'$ extending ρ.

It is a well-known consequence of the Poincaré-Birkhoff-Witt Theorem that the free Lie superalgebra on a bigraded set $S = S_0 \uplus S_1$ can be realized as a subset of a free associative superalgebra. To be precise let $\mathbf{C}[S^*]$ denote the free associative $\mathbf{C}$-superalgebra generated by S. The degree of a word $u = u_1 u_2 \cdots u_k$ in S^* with $u_i \in S_{\alpha_i}$ is given by

$$\alpha(u) = \sum_i \alpha_i \mod 2.$$

A Lie superalgebra structure on $\mathbf{C}[S^*]$ is determined by the following bracket operation on words. For words u and v over S, set

$$[u,v] = uv - (-1)^{\alpha(u)\alpha(v)} vu. \tag{9}$$

The free Lie superalgebra on bigraded set $S = S_0 \uplus S_1$, denoted by $\mathcal{L}(S_0, S_1)$, is the sub-Lie superalgebra of $\mathbf{C}[S^*]$ generated by S. Thus $\mathcal{L}(S_0, S_1)$ consists of linear combinations of bracketings of elements of S. We refer to such bracketings as *Lie monomials*. Any element of $\mathcal{L}(S_0, S_1)$ will be called a *Lie polynomial*. The bigradation is given by $\mathcal{L}(S_0, S_1) = \mathcal{L}_0 \oplus \mathcal{L}_1$, where $\mathcal{L}_0$ (resp. $\mathcal{L}_1$) is generated by Lie monomials that contain an even (resp. odd) number of elements of S_1. When every element of S is even (i.e., when $S_1 = \emptyset$) then $\mathcal{L}(S_0, S_1)$ is the ordinary free Lie algebra $\mathcal{L}(S)$. When every element of S is odd (i.e., when $S_0 = \emptyset$) then $\mathcal{L}(S_0, S_1)$ is said to be the oddly generated free Lie superalgebra on S.

Another well-known consequence of the Poincaré-Birkhoff-Witt Theorem is that the enveloping algebra $\mathcal{E}(\mathcal{L}(S_0, S_1))$ of the free Lie superalgebra $\mathcal{L}(S_0, S_1)$ can be realized as the free associative algebra $\mathbf{C}[S^*]$. Moreover, when L is a sub-Lie superalgebra of $\mathcal{L}(S_0, S_1)$ then the enveloping algebra $\mathcal{E}(L)$ has a realization as the subalgebra of $\mathbf{C}[S^*]$ generated by L. Thus $\mathcal{E}(L)$ consists of linear combinations of products of elements of L.

We take S to be the set $[n] \times G$ with grading $S_0 = N_0 \times G$ and $S_1 = N_1 \times G$ where N_0 and N_1 are disjoint sets whose union is $[n]$. Since the right action of G on $[n] \times G$ given by $(i, h) \cdot g = (i, hg)$ respects this grading, it extends to an action on $\mathbf{C}[S^*]$ viewed as a Lie superalgebra via the bracket defined in (9). This in turn restricts to an action of G on the free Lie superalgebra $\mathcal{L}(S_0, S_1)$. We are interested in the *fixed point subalgebra* $\mathcal{L}(S_0, S_1)^G$ consisting of Lie polynomials P that satisfy $P \cdot g = P$ for each $g \in G$. The fixed point subalgebra is clearly a sub-Lie superalgebra of $\mathcal{L}(S_0, S_1)$. We view its enveloping algebra $\mathcal{E}(\mathcal{L}(S_0, S_1)^G)$ as a sub-superalgebra of $\mathbf{C}[S^*]$.

Define the *type* $\nu(w)$ of a word w in $([n] \times G)^*$ to be the word $\nu_1(w)\nu_2(w) \cdots \nu_n(w)$, where $\nu_i(w)$ is the number of times i occurs as the first coordinate of a letter of w. For example, if $n = 5$, $w = (3, a)(5, b)(3, c)(2, d)$ and $a, b, c, d \in G$ then $\nu(w) = 01201$. We will say that an element x of $\mathbf{C}[([n] \times G)^*]$ is homogeneous of *type* ν if x is a linear combination of words in $([n] \times G)^*$ of type ν.

For any subspace X of $\mathbf{C}[([n] \times G)^*]$ and type $\nu \in \mathbf{N}^n$, the ν-component of X denoted $X|_\nu$ is the subspace of X spanned by homogeneous elements of type ν. Note that $X|_{1^n}$ consists of all linear combinations of elements of $\mathcal{S}_n(G)$ that are in X.

Now let

$$\mathcal{E}_{N_0,N_1}(G) = \mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)|_{1^n}.$$

We are particularly interested in two special cases; one in which $N_1 = \emptyset$ and the other in which $N_0 = \emptyset$. In these special cases the left action of $\mathcal{S}_n(G)$ on $[n] \times G$ extends to a left action on the superalgebra $\mathbf{C}[([n] \times G)^*]$ which restricts to a left action on $\mathcal{E}_{[n],\emptyset}(G)$ and on $\mathcal{E}_{\emptyset,[n]}(G)$. Note that $\mathcal{E}_{[n],\emptyset}(G)$ (resp. $\mathcal{E}_{\emptyset,[n]}(G)$) is a multilinear component of the enveloping algebra of the fixed point subalgebra of the ordinary free Lie algebra (resp. oddly generated free Lie superalgebra) on $[n] \times G$.

For the general case, define $\mathcal{S}_N$ to be the subgroup of $\mathcal{S}_n$ consisting of elements that permute only the elements of N where $N \subseteq [n]$. For general N_0 and N_1, $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$

(the wreath product of the Young subgroup $\mathcal{S}_{N_0} \times \mathcal{S}_{N_1}$ with G) acts on $\mathcal{E}_{N_0,N_1}(G)$.

We will now construct a spanning set for $\mathcal{E}_{N_0,N_1}(G)$ in terms of labeled forests. For any labeled tree A define the Lie monomial $b(A)$ by

$$b(A) = \begin{cases} (i,g) & \text{if } A \text{ is the single node labeled by } (i,g) \\ [b(A_1), b(A_2)] & \text{if } A = (A_1 \frown A_2). \end{cases}$$

Next define the Lie polynomial $\beta(A)$ by

$$\beta(A) = \sum_{g \in G} b(A) \cdot g.$$

If $A_1, A_2, \ldots, A_k$ is a labeled binary forest then define

$$\beta(A_1, A_2, \ldots, A_k) = \beta(A_1)\beta(A_2)\cdots\beta(A_k).$$

For any labeled tree A define $d(A)$ to be the number of leaves of A that have $[n]$-label in N_1. Clearly $\beta(A)$ is homogeneous of degree $d(A) \bmod 2$.

Theorem 6.1 *The set $\{\beta(F;w) : F \in \mathcal{F}_n,\ w \in \mathcal{S}_n(G)\}$ is a generating set for the vector space $\mathcal{E}_{N_0,N_1}(G)$ subject only to the following relations:*

- *(β-Fixed Point Relations) For any labeled forest $A_1, \ldots, A_k$, $g \in G$ and $j \in [k]$,*

$$\beta(A_1, \ldots, A_j, \ldots, A_k) = \beta(A_1, \ldots, (A_j \cdot g), \ldots, A_k).$$

- *(β-Anticommuting Relations) For any labeled subtree $(A \frown B)$ of a labeled forest $\cdots (A \frown B) \cdots$,*

$$\beta(\cdots (A \frown B) \cdots) + (-1)^{d(A)d(B)} \beta(\cdots (B \frown A) \cdots) = 0.$$

- *(β-Jacobi Relations) For any labeled subtree $(A \frown (B \frown C))$ of a labeled forest $\cdots (A \frown (B \frown C)) \cdots$,*

$$\begin{aligned} (-1)^{d(A)d(C)}\ \beta(\cdots (A \frown (B \frown C)) \cdots) \ &+\ (-1)^{d(B)d(C)}\ \beta(\cdots (C \frown (A \frown B)) \cdots) \\ &+\ (-1)^{d(A)d(B)}\ \beta(\cdots (B \frown (C \frown A)) \cdots) = 0. \end{aligned}$$

- *(β-Enveloping Relations) For any labeled forest $A_1, \ldots, A_k$ and $j \in [k-1]$,*

$$\beta(A_1, \ldots, A_j, A_{j+1}, \ldots, A_k) - (-1)^{d(A_j)d(A_{j+1})}\ \beta(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k)$$
$$- \sum_{g \in G}\ \beta(A_1, \ldots, (A_j \frown (A_{j+1} \cdot g)), \ldots, A_k) = 0.$$

Moreover, the action of $(\mathcal{S}_{N_0}\times\mathcal{S}_{N_1})(G)$ *on* $\mathcal{E}_{N_0,N_1}(G)$ *can be described on the generators as*

$$u\cdot\beta(F;w)=\beta(F;u\cdot w). \tag{10}$$

.

Proof: Since $\mathcal{E}(\mathcal{L}(S_0,S_1)^G)$ is the subalgebra of the free associative algebra generated by $\mathcal{L}(S_0,S_1)^G$, it is spanned (as a vector space) by products of generators of $\mathcal{L}(S_0,S_1)^G$. We claim that $\mathcal{L}(S_0,S_1)^G$ is spanned by the Lie polynomials $\sum_{g\in G} b\cdot g$, where b is a Lie monomial. Clearly any Lie polynomial of the form $\sum_{g\in G} b\cdot g$ is a fixed point. Now let $P=\sum_{i=1}^k c_i b_i$ where $c_i\in\mathbf{C}$, b_i is a Lie monomial and $P\in\mathcal{L}(S_0,S_1)^G$. Then

$$P=\frac{1}{|G|}\sum_{g\in G}P\cdot g=\sum_{i=1}^k c_i\frac{1}{|G|}\sum_{g\in G}b_i\cdot g.$$

Hence our claim holds and therefore $\mathcal{E}(\mathcal{L}(S_0,S_1)^G)$ is spanned by products of Lie polynomials of the form $\sum_{g\in G} b\cdot g$ where b is a Lie monomial. Note that these Lie polynomials and their products are homogeneous with respect to type. Hence each ν-component of $\mathcal{E}(\mathcal{L}(S_0,S_1)^G)$ is spanned by type ν products of Lie polynomials $\sum_{g\in G} b\cdot g$. Since the set $\{\beta(F;w)\,:\,F\in\mathcal{F}_n,\ w\in\mathcal{S}_n(G)\}$ consists of all type 1^n products of Lie polynomials $\sum_{g\in G} b\cdot g$, it generates $\mathcal{E}_{N_0,N_1}(G)$.

The β-fixed point relations follow from the definition of $\beta(A_1,\ldots,A_k)$. The β-anticommuting relations and β-Jacobi relations are consequences of the graded antisymmetry relation (7) and the graded Jacobi relation (8), respectively.

To establish the β-enveloping relations we have

$$\begin{aligned}
[\beta(A_j),\beta(A_{j+1})] &= [\sum_{g\in G}\mathrm{b}(A_j)\cdot g,\sum_{h\in G}\mathrm{b}(A_{j+1})\cdot h]\\
&= \sum_{g,h\in G}[\mathrm{b}(A_j),\mathrm{b}(A_{j+1})\cdot hg^{-1}]\cdot g\\
&= \sum_{g,h\in G}[\mathrm{b}(A_j),\mathrm{b}(A_{j+1})\cdot h]\cdot g\\
&= \sum_{h\in G}\beta(A_j{}^\frown(A_{j+1}\cdot h)).
\end{aligned}$$

Hence the β-enveloping relations are the same as

$$\begin{aligned}
\beta(A_1)\cdots\beta(A_j)\beta(A_{j+1})\cdots\beta(A_k) &- (-1)^{d(A_j)d(A_{j+1})}\beta(A_1)\cdots\beta(A_{j+1})\beta(A_j)\cdots\beta(A_k)\\
&- \beta(A_1)\cdots[\beta(A_j),\beta(A_{j+1})]\cdots\beta(A_k)=0,
\end{aligned}$$

which follow from (9).

Since the β-relations are the same, up to sign, as the elementary relations for cohomology, the straightening algorithm that was used to prove Theorem 5.6 can be used to prove that the set $\{\beta(A) : A \,\text{is a comb forest with}\, n \,\text{leaves}\}$ spans $\mathcal{E}_{N_0,N_1}(G)$. Since each step of the straightening algorithm relies only on one of the four β-relations, we need only show that the Lie polynomials $\beta(A)$, where A is a comb forest with n leaves, are linearly independent in order to be able to conclude that the four β-relations generate all the relations on the generators $\beta(F; w)$. Independence of the comb forest Lie polynomials follows from the Poincaré-Birkhoff-Witt Theorem and the next lemma.

Equation (10) follows from the definition of multiplication in the wreath product. □

Lemma 6.2 *The Lie polynomials $\beta(A)$, where A is a comb with n leaves, are linearly independent in $\mathcal{L}(N_0 \times G, N_1 \times G)^G$.*

Proof: If A is a comb with n leaves then $A = (C_n; w)$ where

$$C_n = (((\ldots(\bullet\frown\bullet)\frown\bullet)\frown\ldots)\frown\bullet)$$

and $w \in \mathcal{S}_n(G)$ has first letter equal to $(1, e)$. Let

$$\mathcal{W} = \{w \in \mathcal{S}_n(G) : \text{first letter of } w \text{ is } (1, e)\}.$$

Now suppose $\sum_{w\in\mathcal{W}} c_w \beta(C_n; w) = 0$, where $c_w \in \mathbf{C}$. Note that the coefficient of a word $u \in \mathcal{W}$ in $\beta(C_n; w)$ is 1 if $u = w$ and is 0 otherwise. Hence c_u is the coefficient of u in $\sum_{w\in\mathcal{W}} c_w \beta(C_n; w)$. Consequently $c_u = 0$ for all $u \in \mathcal{W}$. □

Corollary 6.3 *The set $\{\beta(A) : A \,\text{is a comb forest with}\, n \,\text{leaves}\}$ is a basis for $\mathcal{E}_{N_0,N_1}(G)$.* □

7 The Isomorphisms

In this section we show that the three $\mathcal{S}_n(G)$-modules $H^*(Q_n(G))$, $\mathcal{E}_{[n],\emptyset}$ and $\mathcal{E}_{\emptyset,[n]}$ are isomorphic up to tensoring with a sign representation. More generally, we construct an explicit $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module isomorphism from $\mathcal{E}_{N_0,N_1}(G)$ to $H^*(Q_n(G))$ tensored with a sign representation.

Following [Wa2] the sign of a binary tree is defined as

$$\operatorname{sgn}(T) = \begin{cases} 1 & \text{if } T = \bullet \\ (-1)^{i(T_2)} \operatorname{sgn}(T_1) \operatorname{sgn}(T_2) & \text{if } T = (T_1\frown T_2). \end{cases} \tag{11}$$

We define the sign of a binary forest by

$$\mathrm{sgn}(T_1, \ldots, T_k) = \prod_{j=1}^{k} (-1)^{i(T_j)} \mathrm{sgn}(T_j).$$

We will need the following easy observations on the sign of a binary forest:

$$\begin{aligned}
\mathrm{sgn}(\cdots(T_2 {}^\frown T_1)\cdots) &= (-1)^{i(T_1)+i(T_2)}\ \mathrm{sgn}(\cdots(T_1 {}^\frown T_2)\cdots) && (12)\\
\mathrm{sgn}(\cdots(T_3 {}^\frown (T_1 {}^\frown T_2))\cdots) &= (-1)^{i(T_1)+i(T_2)}\ \mathrm{sgn}(\cdots(T_1 {}^\frown (T_2 {}^\frown T_3))\cdots) && (13)\\
\mathrm{sgn}(\cdots(T_2 {}^\frown (T_3 {}^\frown T_1))\cdots) &= (-1)^{i(T_2)+i(T_3)}\ \mathrm{sgn}(\cdots(T_1 {}^\frown (T_2 {}^\frown T_3))\cdots) && (14)\\
\mathrm{sgn}(T_1, \ldots, T_{j+1}, T_j, \ldots, T_k) &= \mathrm{sgn}(T_1, \ldots, T_j, T_{j+1}, \ldots, T_k) && (15)\\
\mathrm{sgn}(T_1, \ldots, (T_j {}^\frown T_{j+1}), \ldots, T_k) &= (-1)^{1+i(T_{j+1})}\ \mathrm{sgn}(T_1, \ldots, T_j, T_{j+1}, \ldots, T_k) && (16)
\end{aligned}$$

Theorem 7.1 *There is an $\mathcal{S}_n(G)$-module isomorphism $\varphi : \mathcal{E}_{\emptyset,[n]}(G) \to H^*(Q_n(G))$ determined by*

$$\varphi(\beta(F; w)) = \mathrm{sgn}(F)\ c(F; w), \tag{17}$$

where $F \in \mathcal{F}_n$, $w \in \mathcal{S}_n(G)$.

Proof: To show that (17) determines a well-defined vector space isomorphism it suffices to show that β-fixed point, β-antisymmetry, β-Jacobi, and β-enveloping relations are mapped to the elementary relations of types 1, 2, 3, and 4, respectively. It's clear from (1) and (10) that φ respects the action of $\mathcal{S}_n(G)$.

It is obvious that β-fixed point relations are mapped to elementary relations of type 1.

We show that β-antisymmetry relations are mapped to elementary relations of type 2. Define the sign of a labeled forest $(F; w)$ to be the sign of the underlying forest F, i.e.,

$$\mathrm{sgn}(F; w) = \mathrm{sgn}(F).$$

Let $(A {}^\frown B)$ be a labeled subtree of the labeled forest $\cdots(A {}^\frown B)\cdots$. We have

$$\varphi(\beta(\cdots(A {}^\frown B)\cdots)) + (-1)^{d(A)d(B)}\varphi(\beta(\cdots(B {}^\frown A)\cdots)) =$$

$$\mathrm{sgn}(\cdots(A {}^\frown B)\cdots)\ c(\cdots(A {}^\frown B)\cdots) + (-1)^{d(A)d(B)}\ \mathrm{sgn}(\cdots(B {}^\frown A)\cdots)\ c(\cdots(B {}^\frown A)\cdots).$$

Since $N_1 = [n]$, $d(X)$ is the number of leaves of X; that is $d(X) = i(X) + 1$. After substituting $(i(A)+1)(i(B)+1)$ for $d(A)d(B)$ on the right hand side of the equation and using (12) to factor out $\mathrm{sgn}(\cdots(A {}^\frown B)\cdots)$, we obtain

$$\varphi(\beta(\cdots(A {}^\frown B)\cdots)) + (-1)^{d(A)d(B)}\varphi(\beta(\cdots(B {}^\frown A)\cdots)) =$$

$$\operatorname{sgn}(\cdots(A^\frown B)\cdots)\ \left(c(\cdots(A^\frown B)\cdots)+(-1)^{i(A)i(B)+1}\ c(\cdots(B^\frown A)\cdots)\right).$$

Hence the β-antisymmetry relations are mapped to elementary relations of type 2.

Next we show that β-Jacobi relations are mapped to elementary relations of type 3. Suppose $(A^\frown(B^\frown C))$ is a labeled subtree of the labeled forest $\cdots(A^\frown(B^\frown C))\cdots$. We have

$$(-1)^{d(A)d(C)}\varphi(\beta(\cdots(A^\frown(B^\frown C))\cdots))+(-1)^{d(B)d(C)}\varphi(\beta(\cdots(C^\frown(A^\frown B))\ldots))$$
$$+(-1)^{d(A)d(B)}\varphi(\beta(\cdots(B^\frown(C^\frown A))\ldots))$$

$$\begin{aligned}
&=(-1)^{d(A)d(C)}\ \operatorname{sgn}(\cdots(A^\frown(B^\frown C))\cdots)\ c(\cdots(A^\frown(B^\frown C))\cdots)\\
&\quad+(-1)^{d(B)d(C)}\ \operatorname{sgn}(\cdots(C^\frown(A^\frown B))\cdots)\ c(\cdots(C^\frown(A^\frown B))\cdots)\\
&\quad+(-1)^{d(A)d(B)}\ \operatorname{sgn}(\cdots(B^\frown(C^\frown A))\cdots)\ c(\cdots(B^\frown(C^\frown A))\cdots).
\end{aligned}$$

Now if we replace $d(X)$ with $i(X)+1$ for $X=A,B,C$ and use (13) and (14) to factor out $\operatorname{sgn}(\cdots(A^\frown(B^\frown C))\cdots)$ from the right side of the equation, the right side of the equation becomes

$$(-1)^{i(A)+i(C)+1}\operatorname{sgn}(\cdots(A^\frown(B^\frown C))\cdots)\ \Big((-1)^{i(A)i(C)}\ c(\cdots(A^\frown(B^\frown C))\cdots)+$$
$$(-1)^{i(B)i(C)}c(\cdots(C^\frown(A^\frown B))\cdots)+(-1)^{i(A)i(B)}c(\cdots(B^\frown(C^\frown A))\cdots)\Big).$$

Hence the β-Jacobi relations map to elementary relations of type 3.

Finally we show that the β-enveloping relations are mapped to the relations of type 4. Let $A_1,\ldots,A_k$ be a labeled forest. By (15) and (16) we have

$$\varphi(\beta(A_1,\ldots,A_j,A_{j+1},\ldots A_k))-(-1)^{d(A_j)d(A_{j+1})}\ \varphi(\beta(A_1,\ldots,A_{j+1},A_j,\ldots,A_k))$$
$$-\sum_{g\in G}\varphi(\beta(A_1,\ldots,(A_j{}^\frown(A_{j+1}\cdot g)),\ldots,A_k))$$

$$\begin{aligned}
&=\operatorname{sgn}(A_1,\ldots,A_j,A_{j+1},\ldots,A_k)\ c(A_1,\ldots,A_j,A_{j+1},\ldots,A_k)\\
&\quad-(-1)^{d(A_j)d(A_{j+1})}\ \operatorname{sgn}(A_1,\ldots,A_{j+1},A_j,\ldots,A_k)\ c(A_1,\ldots,A_{j+1},A_j,\ldots,A_k)\\
&\quad-\sum_{g\in G}\operatorname{sgn}(A_1,\ldots,(A_j{}^\frown A_{j+1}),\ldots,A_k)\ c(A_1,\ldots,(A_j{}^\frown(A_{j+1}\cdot g)),\ldots,A_k)
\end{aligned}$$

$$=\operatorname{sgn}(A_1,\ldots,A_k)\ \Big(c(A_1,\ldots,A_j,A_{j+1},\ldots,A_k)$$

$$-(-1)^{(1+i(A_j))(1+i(A_{j+1}))}\ c(A_1,\ldots,A_{j+1},A_j,\ldots,A_k)$$
$$-(-1)^{1+i(A_{j+1})}\ \sum_{g\in G} c(A_1,\ldots,(A_j\frown(A_{j+1}\cdot g)),\ldots,A_k)\Big). \qquad \square$$

We now generalize Theorem 7.1 to arbitrary N_0, N_1. For any $N \subseteq [n]$ and $\sigma \in \mathcal{S}_n$ let

$$\mathrm{inv}_N(\sigma) = |\{i<j : \sigma(i) > \sigma(j) \text{ and either } \sigma(i) \in N \text{ or } \sigma(j) \in N\}|$$

and

$$\mathrm{inv}^*_N(\sigma) = |\{i<j : \sigma(i) > \sigma(j),\ \sigma(i) \in N \text{ and } \sigma(j) \in N\}|.$$

Next define

$$\mathrm{sgn}_N(\sigma) = (-1)^{\mathrm{inv}_N(\sigma)} \quad \text{and} \quad \mathrm{sgn}^*_N(\sigma) = (-1)^{\mathrm{inv}^*_N(\sigma)}.$$

Note that $\mathrm{sgn}^*_N(\sigma)$ is simply the ordinary sign of the permutation obtained by taking the subword of σ consisting of letters of N.

For $w \in \mathcal{S}_n(G)$, let $\tilde{w} \in \mathcal{S}_n$ be such that $w = (\tilde{w}, f)$. Define

$$\mathrm{sgn}_N(w) = \mathrm{sgn}_N(\tilde{w}) \quad \text{and} \quad \mathrm{sgn}^*_N(w) = \mathrm{sgn}^*_N(\tilde{w}).$$

Clearly $\mathrm{sgn}_\emptyset(w) = \mathrm{sgn}^*_\emptyset(w) = 1$ for all $w \in \mathcal{S}_n(G)$ and $\mathrm{sgn}_{[n]}(w) = \mathrm{sgn}^*_{[n]}(w) = \mathrm{sgn}(w)$, where $\mathrm{sgn}(w)$ is the ordinary sign of the underlying permutation $\tilde{w}$. Observe also that for any $w \in \mathcal{S}_n(G)$

$$\mathrm{sgn}_{N_0}(w)\ \mathrm{sgn}^*_{N_1}(w) \ = \ \mathrm{sgn}^*_{N_0}(w)\ \mathrm{sgn}_{N_1}(w) \ = \ \mathrm{sgn}(w). \tag{18}$$

Lemma 7.2 *For $u \in (\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$, $w \in \mathcal{S}_n(G)$ and $\alpha = 0,1$,*

$$\mathrm{sgn}_{N_\alpha}(u\cdot w) = \mathrm{sgn}_{N_\alpha}(u)\ \mathrm{sgn}_{N_\alpha}(w) \tag{19}$$

and

$$\mathrm{sgn}^*_{N_\alpha}(u\cdot w) = \mathrm{sgn}^*_{N_\alpha}(u)\ \mathrm{sgn}^*_{N_\alpha}(w). \tag{20}$$

Consequently, sgn_{N_α} and $\mathrm{sgn}^*_{N_\alpha}$ are one dimensional representations of $(\mathcal{S}_{N_0}\times\mathcal{S}_{N_1})(G)$.

Proof: It follows from (18) that (19) and (20) are equivalent. In order to prove (20) note that $\widetilde{u\cdot w} = \tilde{u}\cdot\tilde{w}$. So it suffices to prove (20) for $u, w \in \mathcal{S}_n$. Suppose $u = (a,b)$ is an adjacent transposition in $\mathcal{S}_{N_0}$ or in $\mathcal{S}_{N_1}$. By adjacent we mean that a and b are adjacent in the natural linear ordering of N_0 or N_1. If $u \in \mathcal{S}_{N_\alpha}$ then $a, b \in N_\alpha$ which implies

$$\mathrm{sgn}^*_{N_\alpha}(u\cdot w) \ = \ -\mathrm{sgn}^*_{N_\alpha}(w) \ = \ \mathrm{sgn}^*_{N_\alpha}(u)\ \mathrm{sgn}^*_{N_\alpha}(w).$$

On the other hand, if $u \notin \mathcal{S}_{N_\alpha}$ then $a, b \notin N_\alpha$ which implies that

$$\mathrm{sgn}^*_{N_\alpha}(u \cdot w) \;=\; \mathrm{sgn}^*_{N_\alpha}(w) \;=\; \mathrm{sgn}^*_{N_\alpha}(u)\,\mathrm{sgn}^*_{N_\alpha}(w).$$

For general $u \in \mathcal{S}_{N_0} \times \mathcal{S}_{N_1}$ we let $u = s_1 \cdot s_2 \cdots s_k$ where s_i is an adjacent transposition in $\mathcal{S}_{N_0}$ or in $\mathcal{S}_{N_1}$. It follows by induction on k that

$$\begin{aligned} \mathrm{sgn}^*_{N_\alpha}(u \cdot w) &= \mathrm{sgn}^*_{N_\alpha}(s_1)\,\mathrm{sgn}^*_{N_\alpha}(s_2 \cdots s_k \cdot w) \\ &= \mathrm{sgn}^*_{N_\alpha}(s_1)\,\mathrm{sgn}^*_{N_\alpha}(s_2 \cdots s_k)\,\mathrm{sgn}^*_{N_\alpha}(w) \\ &= \mathrm{sgn}^*_{N_\alpha}(u)\,\mathrm{sgn}^*_{N_\alpha}(w). \qquad \square \end{aligned}$$

Remark: One can use Lemma 7.2 to prove that, although sgn and sgn* are different functions of $\mathcal{S}_n(G)$, they identical representations of $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$.

We will use Theorem 7.1 to prove the following generalization.

Theorem 7.3 *There is an $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module isomorphism $\varphi_{N_0,N_1} : \mathcal{E}_{N_0,N_1}(G) \to H^*(Q_n(G)) \otimes \mathrm{sgn}_{N_0}$ determined by*

$$\varphi_{N_0,N_1}(\beta(F; w)) = \mathrm{sgn}_{N_0}(w)\ \mathrm{sgn}(F)\ c(F; w), \tag{21}$$

where $F \in \mathcal{F}_n$, $w \in \mathcal{S}_n(G)$.

Proof: Let $\beta_0(F; w)$ be $\beta(F; w)$ in $\mathcal{E}_{[n],\emptyset}(G)$. We shall prove that there is an $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module isomorphism $\psi_{N_0,N_1} : \mathcal{E}_{N_0,N_1}(G) \to \mathcal{E}_{[n],\emptyset}(G) \otimes \mathrm{sgn}^*_{N_1}$ determined by

$$\psi_{N_0,N_1}(\beta(F; w)) = \mathrm{sgn}^*_{N_1}(w)\ \beta_0(F; w), \tag{22}$$

where $F \in \mathcal{F}_n$, $w \in \mathcal{S}_n(G)$. By taking φ_{N_0,N_1} to be the composition $\varphi \circ \psi^{-1}_{\emptyset,[n]} \circ \psi_{N_0,N_1}$ and applying (18), we have the result.

The proof that (22) determines a well-defined $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module isomorphism is similar to the proof of Theorem 7.1. In place of equations (12), (13), (14), (15) and (16), we use the following equations on labeled binary forests which are easy to verify. Let $\mathrm{sgn}^*_{N_1}(F; w) = \mathrm{sgn}^*_{N_1}(w)$. Then

$$\mathrm{sgn}^*_{N_1}(\cdots(A_2 \frown A_1)\cdots) = (-1)^{d(A_1)d(A_2)}\ \mathrm{sgn}^*_{N_1}(\cdots(A_1 \frown A_2)\cdots) \tag{23}$$

$$\mathrm{sgn}^*_{N_1}(\cdots(A_3 \frown (A_1 \frown A_2))\cdots) = (-1)^{d(A_3)(d(A_1)+d(A_2))}\ \mathrm{sgn}^*_{N_1}(\cdots(A_1 \frown (A_2 \frown A_3))\cdots) \tag{24}$$

$$\mathrm{sgn}^*_{N_1}(\cdots(A_2 \frown (A_3 \frown A_1))\cdots) = (-1)^{d(A_1)(d(A_2)+d(A_3))}\ \mathrm{sgn}^*_{N_1}(\cdots(A_1 \frown (A_2 \frown A_3))\cdots) \tag{25}$$

$$\mathrm{sgn}^*_{N_1}(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k) = (-1)^{d(A_j)d(A_{j+1})}\mathrm{sgn}^*_{N_1}(A_1, \ldots, A_j, A_{j+1}, \ldots, A_k) \tag{26}$$

$$\mathrm{sgn}^*_{N_1}(A_1,\ldots,(A_j{}^\frown A_{j+1}),\ldots,A_k)=\mathrm{sgn}^*_{N_1}(A_1,\ldots,A_j,A_{j+1},\ldots,A_k). \tag{27}$$

Equation (23) is used to show that the β-anticommuting relations for $\mathcal{E}_{N_0,N_1}(G)$ map to the β-anticommuting relations for $\mathcal{E}_{[n],\emptyset}(G)$. Equations (24) and (25) are used to do the same for the β-Jacobi relations. Equations (26) and (27) are used for the β-enveloping relations.

We give the proof for the β-anticommuting relations and leave the proofs for the other relations to the reader. Let $(A{}^\frown B)$ be a labeled subtree of the labeled forest $\cdots(A{}^\frown B)\cdots$. We have by (23)

$$\psi_{N_0,N_1}(\beta(\cdots(A{}^\frown B)\cdots))+(-1)^{d(A)d(B)}\psi_{N_0,N_1}(\beta(\cdots(B{}^\frown A)\cdots))$$

$$\begin{aligned}&=\mathrm{sgn}^*_{N_1}(\cdots(A{}^\frown B)\cdots)\ \beta_0(\cdots(A{}^\frown B)\cdots)\\&\qquad+(-1)^{d(A)d(B)}\ \mathrm{sgn}^*_{N_1}(\cdots(B{}^\frown A)\cdots)\ \beta_0(\cdots(B{}^\frown A)\cdots)\end{aligned}$$

$$=\mathrm{sgn}^*_{N_1}(\cdots(A{}^\frown B)\cdots)\ \Big(\beta_0(\cdots(A{}^\frown B)\cdots)+\beta_0(\cdots(B{}^\frown A)\cdots)\Big).$$

□

Corollary 7.4 *There is an $\mathcal{S}_n(G)$-module isomorphism $\varphi_{[n],\emptyset}:\mathcal{E}_{[n],\emptyset}(G)\to H^*(Q_n(G))\otimes$ sgn determined by*

$$\varphi_{[n],\emptyset}(\beta(F;w))=\mathrm{sgn}(w)\ \mathrm{sgn}(F)\ c(F;w),$$

where $F\in\mathcal{F}_n$, $w\in\mathcal{S}_n(G)$. □

Recall that the Poincaré-Birkhoff-Witt Theorem provides a vector space isomorphism between $\mathcal{E}(\mathcal{L}(N_0\times G,N_1\times G)^G)$ and $\mathrm{Sym}_s(\mathcal{L}(N_0\times G,N_1\times G)^G)$. We show that this isomorphism is in fact an $(\mathcal{S}_{N_0}\times\mathcal{S}_{N_1})(G)$-module isomorphism. The group $(\mathcal{S}_{N_0}\times\mathcal{S}_{N_1})(G)$ acts on $\mathrm{Sym}_s(\mathcal{L}(N_0\times G,N_1\times G)^G)$ by acting on each factor in a supersymmetric product.

Proposition 7.5 *As $(\mathcal{S}_{N_0}\times\mathcal{S}_{N_1})(G)$-modules*

$$\mathrm{Sym}_s(\mathcal{L}(N_0\times G,N_1\times G)^G)=\mathcal{E}(\mathcal{L}(N_0\times G,N_1\times G)^G).$$

Proof: For $k = 0, \ldots, n$, let V_k be the subspace of $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ spanned by all products of k or fewer Lie polynomials in $\mathcal{L}(N_0 \times G, N_1 \times G)^G$. Since each V_k is an invariant subspace of $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ under the action of $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$, the sequence $V_0 \subseteq V_1 \subseteq \ldots$ is an $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module filtration of $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$. It follows that $\oplus_{k \geq 1} V_k / V_{k-1}$ and $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ are isomorphic $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-modules. The space $\oplus_{k \geq 0} V_k / V_{k-1}$ is the associated graded superalgebra of $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$. It is easy to check that the canonical isomorphism (of the Poincaré-Birkhoff-Witt Theorem) from the associated graded superalgebra to the supersymmetric algebra respects the action of $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$. Hence $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ and $\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ are isomorphic $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-modules. $\square$

We extend the notion of type to $\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ by defining the type of a supersymmetric product of homogeneous elements of $\mathcal{L}(N_0 \times G, N_1 \times G)^G$ to be the componentwise sum of the types of the factors, i.e. the ith entry of the type of the product is the sum of the ith entries of the types of the factors. Clearly $\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)|_{1^n}$ is an invariant subspace of $\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$ under the action of $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$. The image of $\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)|_{1^n}$ under the isomorphism described in the proof of Proposition 7.5 is $\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G)|_{1^n}$. Thus the results of this section can be rephrased by substituting "supersymmetric algebra" for "enveloping algebra". In particular, we have

Corollary 7.6 *As $\mathcal{S}_n(G)$-modules*

$$H^*(Q_n(G)) \;=\; \mathrm{Sym}_s(\mathcal{L}(\emptyset, [n] \times G)^G)|_{1^n} \;=\; \mathrm{Sym}(\mathcal{L}([n] \times G)^G)|_{1^n} \otimes \mathrm{sgn}. \qquad \square$$

Theorem 7.1 and Corollary 7.4 reduce to the following result of Stanley [St] when G is the trivial group (e).

Corollary 7.7 *(Stanley) The $\mathcal{S}_n$-module $H^*(\Pi_{n+1})$ is the regular representation.*

Proof: Since the $\mathcal{S}_n$-poset Π_{n+1} is isomorphic to the $\mathcal{S}_n$-poset $Q_n(e)$, we need to consider the enveloping algebra of the free Lie superalgebra on $N_0 \uplus N_1$. Recall that this is the free associative superalgebra $\mathbf{C}[[n]^*]$. Since the 1^n-component of $\mathbf{C}[[n]^*]$ is just $\mathbf{C}[\mathcal{S}_n]$, the span of permutations in $\mathcal{S}_n$, the action of $\mathcal{S}_n$ on $\mathcal{E}_{\emptyset,[n]}(e)$ (and on $\mathcal{E}_{[n],\emptyset}(e) \otimes$ sgn) is the regular representation. $\square$

8 Whitney Cohomology

Whitney homology for geometric lattices was introduced by Baclawski [Bc]. We use the following formulation due to Björner [Bj1], [Bj2]. For each integer r, the rth *Whitney homology* of a geometric lattice P is defined to be

$$WH_r(P) = \bigoplus_{r(x)=r} H_{r-2}([\hat{0}, x]),$$

and the rth *Whitney cohomology* of P is defined to be

$$WH^r(P) = \bigoplus_{r(x)=r} H^{r-2}([\hat{0}, x]),$$

where $r(x)$ is the rank of x. Any automorphism of P induces linear maps on $WH_r(P)$ and $WH^r(P)$. If P is a G-poset then $WH_r(P)$ and $WH^r(P)$ are dual G-modules. We will work with the Whitney cohomology G-module.

Each $H^{r-2}([\hat{0}, x])$ in the direct sum is generated by maximal chains of the interval $[\hat{0}, x]$ modulo the coboundary relations. We say that a chain is *lower maximal* if there is an $x \in P$ so that the chain is maximal in $[\hat{0}, x]$.

We will encode lower maximal chains of $Q_n(G)$ with labeled forest pairs. First we need to extend the notion of postorder from trees to forests. The postorder listing of the internal nodes of a forest F is obtained by listing the internal nodes of the first tree in postorder followed by the internal nodes of the second tree in postorder, etc.

A *labeled forest pair* is a pair of labeled forests $(F_1; w_1), (F_2; w_2)$, whose $[n]$-labels are distinct. We denote a labeled forest pair by $((F_1; w_1); (F_2; w_2))$ or by $(F_1; F_2; w_1 w_2)$. Given two forests F_1 and F_2 let $F_1 F_2$ denote the concatenation of the forests, i.e., the forest consisting of the trees of F_1 followed by the trees of F_2.

For each labeled forest pair $(F_1; F_2; w)$ where $F_1 F_2 \in \mathcal{F}_n$ and $w \in \mathcal{S}_n(G)$, we associate the lower maximal chain $c(F_1; F_2; w)$ of $Q_n(G)$ whose rank i G-partition is obtained from its rank $i-1$ G-partition by merging the G-blocks $\overline{L_{\ell(v_i)}(\hat{F}_1 F_2; 0w)}$ and $\overline{L_{r(v_i)}(\hat{F}_1 F_2; 0w)}$ to get the G-block $\overline{L_{v_i}(\hat{F}_1 F_2; 0w)}$, where v_i is the ith internal node of the forest $\hat{F}_1 F_2$ in the postorder listing. If $\sigma \in E(\hat{F}_1 F_2)$ then $c(F_1; F_2; w; \sigma)$ is the lower maximal chain of $Q_n(G)$ whose rank i G-partition is obtained from its rank $i-1$ G-partition by merging the G-blocks $\overline{L_{\ell(\sigma(v_i))}(\hat{F}_1 F_2; 0w)}$ and $\overline{L_{r(\sigma(v_i))}(\hat{F}_1 F_2; 0w)}$ to get the G-block $\overline{L_{\sigma(v_i)}(\hat{F}_1 F_2; 0w)}$.

Suppose $(\hat{F}_1 F_2; 0w) = A_0, A_1, \ldots, A_k$. Then $c(F_1; F_2; w)$ and $c(F_1; F_2; w; \sigma)$ are maximum chains of the interval $[\hat{0}, x]$ where x is the G-partition consisting of G-blocks $\overline{L(A_0)}, \overline{L(A_1)}, \ldots, \overline{L(A_k)}$. Clearly $r(x) = n - k$.

An easy modification of the proof of Lemma 4.1 yields the following lemma.

Lemma 8.1 *Let $(F_1; F_2; w)$ be a labeled forest pair with $F_1F_2 \in \mathcal{F}_n$ and $w \in \mathcal{S}_n(G)$. If $\sigma \in E(\hat{F}_1 F_2)$ then we have in Whitney cohomology*

$$c(F_1; F_2; w; \sigma) = \operatorname{sgn}(\sigma)\, c(F_1; F_2; w). \qquad \square$$

By using Lemma 8.1 and arguments similar to those used to prove Corollary 5.7, one can prove the following result. Let $t(F)$ be the number of trees of the forest F.

Theorem 8.2 *Fix $r = 0, 1, \ldots, n$. The set*

$$\{c(F_1; F_2; w) : F_1F_2 \in \mathcal{F}_n,\ t(F_2) = r,\ w \in \mathcal{S}_n(G)\}$$

is a generating set for $WH^{n-r}(Q_n(G))$ subject only to the following relations:

- *(Relations of Type 1) Let $(A_1, \ldots, A_k; A_{k+1}, \ldots, A_{k+r})$ be a labeled forest pair with n leaves. Then for $g \in G$ and $j \in [k+r]$ we have*

  $$c(A_1, \ldots, A_j, \ldots, A_{k+r}) = c(A_1, \ldots, (A_j \cdot g), \ldots, A_{k+r})$$

  regardless of whether A_j occurs in the first or second forest.

- *(Relations of Type 2) Let $(\cdots(A \frown B)\cdots)$ be a labeled forest pair with n leaves. Then*

  $$c(\cdots(A \frown B)\cdots) + (-1)^{i(A)i(B)+1}\, c(\cdots(B \frown A)\cdots) = 0$$

  regardless of whether the subtree occurs in a tree of the first or second forest.

- *(Relations of Type 3) Let $(\cdots(A \frown (B \frown C))\cdots)$ be a labeled forest pair with n leaves. Then*

  $$(-1)^{i(A)i(C)}\, c(\cdots(A \frown (B \frown C))\cdots) + (-1)^{i(B)i(C)}\, c(\cdots(C \frown (A \frown B))\cdots)$$
  $$+(-1)^{i(A)i(B)}\, c(\cdots(B \frown (C \frown A))\cdots) = 0.$$

  regardless of whether the subtree occurs in a tree of the first or second forest.

- *(Relations of Type 4) Let $(A_1, \ldots, A_k; B)$ be a labeled forest pair with n leaves. Then for $j \in [k-1]$*

  $$c(A_1, \ldots, A_j, A_{j+1}, \ldots, A_k; B)$$
  $$-(-1)^{(i(A_j)+1)(i(A_{j+1})+1)}\, c(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k; B)$$
  $$+(-1)^{i(A_{j+1})} \sum_{g \in G} c(A_1, \ldots, (A_j \frown (A_{j+1}) \cdot g), \ldots, A_k; B) = 0.$$

- *(Relations of Type 5) Let $(A; B_1, \ldots, B_r)$ be a labeled forest pair with n leaves. Then for $j \in [r-1]$*

$$c(A; B_1, \ldots, B_j, B_{j+1}, \ldots, B_r) = (-1)^{i(B_j)i(B_{j+1})}\; c(A; B_1, \ldots, B_{j+1}, B_j, \ldots, B_r).$$

Moreover, the action of $\mathcal{S}_n(G)$ on $WH^r(Q_n(G))$ can be described on the generators as

$$u \cdot c(F_1; F_2; w) = c(F_1; F_2; u \cdot w). \qquad \square$$

The *superexterior algebra* of a bigraded vector space $L = L_0 \oplus L_1$ is denoted by $\wedge_s(L)$ and is defined by $\wedge_s(L) = T(L)/K$ where K is the two sided ideal generated by elements of the form

$$x \otimes y + (-1)^{\alpha\beta} y \otimes x, \quad x \in L_\alpha, y \in L_\beta.$$

The superexterior algebra $\wedge_s(L)$ is an associative $\mathbf{Z}$-graded superalgebra with product denoted by $\wedge$ and $\mathbf{Z}$-grading given by

$$\wedge_s(L) = \bigoplus_{r \in \mathbf{Z}} \wedge_s^r(L),$$

where $\wedge_s^r(L) = \{0\}$ if $r < 0$ and $\wedge_s^r(L)$ is the vector space generated by products of the form $x_1 \wedge \ldots \wedge x_r$, $x_i \in L$, if $r \geq 0$. If $L = L_0$ then $\wedge_s(L)$ is the ordinary exterior algebra $\wedge(L)$.

We now consider the vector spaces

$$\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G) \otimes \wedge_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$$

$$\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G) \otimes \wedge_s^r(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$$

$$\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G) \otimes \wedge_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)$$

$$\mathrm{Sym}_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G) \otimes \wedge_s^r(\mathcal{L}(N_0 \times G, N_1 \times G)^G).$$

The notion of type is defined for these spaces by adding the types of the factors. Also the group $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$ acts on the 1^n-component of these spaces by acting on each factor in a tensor product. The third $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$ module is isomorphic to the first and the fourth is isomorphic to the second by Proposition 7.5. Let

$$\mathcal{E}\wedge_{N_0,N_1}(G) = \Big(\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G) \otimes \wedge_s(\mathcal{L}(N_0 \times G, N_1 \times G)^G)\Big)\,|_{1^n}$$

and

$$\mathcal{E}\wedge_{N_0,N_1}^r(G) = \Big(\mathcal{E}(\mathcal{L}(N_0 \times G, N_1 \times G)^G) \otimes \wedge_s^r(\mathcal{L}(N_0 \times G, N_1 \times G)^G)\Big)\,|_{1^n}.$$

There is an $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module decomposition

$$\mathcal{E}\wedge_{N_0,N_1}(G) = \bigoplus_{r=0}^{n} \mathcal{E}\wedge^r_{N_0,N_1}(G).$$

For any labeled forest pair $(A; B_1, \ldots, B_r)$ define $\beta(A; B_1, \ldots, B_r) \in \mathcal{E}\wedge^r_{N_0,N_1}(G)$ to be

$$\beta(A; B_1, \ldots, B_r) = \beta(A) \ \otimes \ (\beta(B_1) \wedge \ldots \wedge \beta(B_r)).$$

The following theorem follows from Theorem 6.1.

Theorem 8.3 *Fix $r = 0, 1, \ldots, n$. The set*

$$\{\beta(F_1; F_2; w) : F_1F_2 \in \mathcal{F}_n,\ t(F_2) = r,\ w \in \mathcal{S}_n(G)\}$$

is a generating set for $\mathcal{E}\wedge^r_{N_0,N_1}(G)$ subject only to the following relations:

- *(β-Fixed Point Relations) Let $(A_1, \ldots, A_k; A_{k+1}, \ldots, A_{k+r})$ be a labeled forest pair with n leaves. Then for $g \in G$ and $j \in [k+r]$ we have*

  $$\beta(A_1, \ldots, A_j, \ldots, A_{k+r}) = \beta(A_1, \ldots, (A_j \cdot g), \ldots, A_{k+r})$$

  regardless of whether A_j occurs in the first or second forest.

- *(β-anticommuting relations) Let $(\cdots(A^\frown B)\cdots)$ be a labeled forest pair with n leaves. Then*

  $$\beta(\cdots(A^\frown B)\cdots) + (-1)^{d(A)d(B)}\ \beta(\cdots(B^\frown A)\cdots) = 0$$

  regardless of whether the subtree occurs in a tree of the first or second forest.

- *(β-Jacobi Relations) Let $(\cdots(A^\frown(B^\frown C))\cdots)$ be a labeled forest pair with n leaves. Then*

  $$(-1)^{d(A)d(C)}\ \beta(\cdots(A^\frown(B^\frown C))\cdots) + (-1)^{d(B)d(C)}\ \beta(\cdots(C^\frown(A^\frown B))\cdots)$$
  $$+(-1)^{d(A)d(B)}\ \beta(\cdots(B^\frown(C^\frown A))\cdots) = 0.$$

  regardless of whether the subtree occurs in a tree of the first or second forest.

- *(β-Enveloping Relations) Let $(A_1, \ldots, A_k; B)$ be a labeled forest pair with n leaves. Then for $j \in [k-1]$*

  $$\beta(A_1, \ldots, A_j, A_{j+1}, \ldots, A_k; B)$$
  $$-(-1)^{d(A_j)d(A_{j+1})}\ \beta(A_1, \ldots, A_{j+1}, A_j, \ldots, A_k; B)$$
  $$-\sum_{g \in G} \beta(A_1, \ldots, (A_j{}^\frown(A_{j+1}) \cdot g), \ldots, A_k; B) = 0.$$

- *(β-Superexterior Relations) Let $(A; B_1, \ldots, B_r)$ be a labeled forest pair with n leaves. Then for $j \in [r-1]$*

  $$\beta(A; B_1, \ldots, B_j, B_{j+1}, \ldots, B_r) = (-1)^{d(B_j)d(B_{j+1})+1}\, \beta(A; B_1, \ldots, B_{j+1}, B_j, \ldots, B_r).$$

Moreover, the action of $\mathcal{S}_n(G)$ on $\mathcal{E}\wedge^r_{N_0,N_1}(G)$ can be described on the generators as

$$u \cdot \beta(F_1; F_2; w) = \beta(F_1; F_2; u \cdot w). \qquad \square$$

Define a *double comb forest* to be a labeled forest pair $(A; B)$ where A and B are both comb forests. One can easily extend the proofs of Theorem 5.6 and Corollary 6.3 to obtain the following.

Theorem 8.4 *Let C^r_n be the set of double comb forests with n leaves and r combs in the second forest. Then the set $\{c(A; B) : (A; B) \in C^r_n\}$ is a basis for $WH^{n-r}(Q_n(G))$ and the set $\{\beta(A; B) : (A; B) \in C^r_n\}$ is a basis for $\mathcal{E}\wedge^r_{N_0,N_1}(G)$.* $\square$

We now describe the isomorphisms between the $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-modules. First we need to define the sign of an ordered pair of forests $(T_1, \ldots, T_k; T_{k+1}, \ldots, T_{k+r})$ by

$$\mathrm{sgn}(T_1, \ldots, T_k; T_{k+1}, \ldots, T_{k+r}) = (-1)^{\sum_{j=1}^{k} i(T_j) \,+\, \sum_{j=1}^{\lfloor r/2 \rfloor} i(T_{k+2j})}\, \mathrm{sgn}(T_1) \cdots \mathrm{sgn}(T_{k+r}),$$

where $\mathrm{sgn}(T_i)$ is the sign of a tree as defined in (11).

Theorem 8.5 *There is an $(\mathcal{S}_{N_0} \times \mathcal{S}_{N_1})(G)$-module isomorphism*

$$\varphi_{N_0,N_1} : \mathcal{E}\wedge^r_{N_0,N_1}(G) \to WH^{n-r}(Q_n(G)) \otimes \mathrm{sgn}_{N_0}$$

determined by

$$\varphi_{N_0,N_1}(\beta(F_1; F_2; w)) = \mathrm{sgn}_{N_0}(w)\ \mathrm{sgn}(F_1; F_2)\ c(F_1; F_2; w), \tag{28}$$

where $F \in \mathcal{F}_n$, $w \in \mathcal{S}_n(G)$.

Proof: Just as in the proof of Theorem 7.3, we must show that relations are mapped to corresponding relations. We leave this to the reader. $\square$

Corollary 8.6 *The $\mathcal{S}_n(G)$-module $WH^{n-r}(Q_n(G))$ is isomorphic to each of the following $\mathcal{S}_n(G)$-modules*

$$\mathcal{E}\wedge^r_{\emptyset,[n]}(G)$$
$$\mathcal{E}\wedge^r_{[n],\emptyset}(G) \otimes \mathrm{sgn}$$
$$\left(\mathrm{Sym}_s(\mathcal{L}(\emptyset, [n] \times G)^G) \otimes \wedge^r_s(\mathcal{L}(\emptyset, [n] \times G)^G)\right)|_{1^n}$$
$$\left(\mathrm{Sym}(\mathcal{L}([n] \times G)^G) \otimes \wedge^r(\mathcal{L}([n] \times G)^G)\right)|_{1^n} \otimes \mathrm{sgn}.$$

$\square$

Remark: Corollary 8.6 reduces to the original result of Bergeron [Be] when $G = \mathbf{Z}/2\mathbf{Z}$. Bergeron works with the Orlik-Solomon algebra which is module-isomorphic to Whitney homology. Since any representation of the hyperoctahedral group is isomorphic to its dual, Bergeron's result is a special case of Corollary 8.6.

When G is a cyclic group, $Q_n(G)$ is the intersection lattice of a complex hyperplane arrangement discussed in [Or-Te] and described below. We shall use the Orlik-Solomon theory of hyperplane arrangements [Or-So] and Corollary 8.6 to describe the action of $\mathcal{S}_n(G)$ on the cohomology of the complement of the arrangement.

Fix a positive integer m and let ω be a primitive mth root of unity. Let $\mathcal{A}_{n,m}$ be the set of hyperplanes in $\mathbf{C}^n$ of the form $z_j = 0$ or $z_j - \omega^h z_k = 0$ where $j, k \in [n]$ and $h \in [m]$. Let $\mathbf{Z}_m = \mathbf{Z}/m\mathbf{Z}$. The group $\mathcal{S}_n(\mathbf{Z}_m)$ acts on $\mathbf{C}^n$ on the right by

$$(z_1, \ldots, z_n) \cdot (\sigma, f) = (\omega^{f_1} z_{\sigma(1)}, \ldots, \omega^{f_n} z_{\sigma(n)})$$

where $\sigma \in \mathcal{S}_n$ and $f = (f_1, \ldots, f_n) \in (\mathbf{Z}_m)^n$. Since $\mathcal{S}_n(\mathbf{Z}_m)$ permutes the set $\mathcal{A}_{n,m}$ of hyperplanes, $\mathcal{S}_n(\mathbf{Z}_m)$ has a representation on the singular cohomology $H^r(M_{n,m})$ where $M_{n,m}$ is the complement $\mathbf{C}^n - \cup_{A \in \mathcal{A}_{n,m}} A$ of the union of the hyperplanes. When $m = 1$, $\mathcal{A}_{n,m}$ is the complexified braid arrangement (also known as the Coxeter arrangement of type A_n) whose intersection lattice is the partition lattice. When $m = 2$, $\mathcal{A}_{n,m}$ is the complexified Coxeter arrangement of type B_n whose intersection lattice is the signed partition lattice. It is easy to see that for general m, the Dowling lattice $Q_n(\mathbf{Z}_m)$ is the lattice of intersections of the arrangement $\mathcal{A}_{n,m}$.

Corollary 8.7 *The $\mathcal{S}_n(\mathbf{Z}_m)$-module $H^{n-r}(M_{n,m})$ is isomorphic to the dual of the $\mathcal{S}_n(\mathbf{Z}_m)$-modules of Corollary 8.6. In particular we have the $\mathcal{S}_n(\mathbf{Z}_m)$-module isomorphisms*

$$\begin{aligned} H^*(M_{n,m})^{\mathrm{dual}} &= \Big(\mathrm{Sym}_s(\mathcal{L}(\emptyset, [n] \times \mathbf{Z}_m)^{\mathbf{Z}_m}) \otimes \wedge_s(\mathcal{L}(\emptyset, [n] \times \mathbf{Z}_m)^{\mathbf{Z}_m})\Big) \mid_{1^n} \\ &= \Big(\mathrm{Sym}(\mathcal{L}([n] \times \mathbf{Z}_m)^{\mathbf{Z}_m}) \otimes \wedge(\mathcal{L}([n] \times \mathbf{Z}_m)^{\mathbf{Z}_m})\Big) \mid_{1^n} \otimes \, \mathrm{sgn}. \end{aligned}$$

Proof: Since the lattice of intersections of $\mathcal{A}_{m,n}$ is the Dowling lattice $Q_n(\mathbf{Z}_m)$, the Orlik-Solomon formula [Or-So] tells us that the $\mathcal{S}_n(\mathbf{Z}_m)$-module $H^{n-r}(M_{n,m})$ is isomorphic to $WH_{n-r}(Q_n(\mathbf{Z}_m))$, the dual of the $\mathcal{S}_n(\mathbf{Z}_m)$-module $WH^{n-r}(Q_n(\mathbf{Z}_m))$. □

9 Splitting Basis for Homology

In [Wa1] (see also [Wa2]) a basis for the homology of the partition lattice is constructed by splitting permutations in $\mathcal{S}_n$ to obtain partitions. This construction was put in a geometric framework and generalized to intersection lattices of real hyperplane arrangements by Björner and Wachs [Bj-Wa] producing a splitting basis for the signed partition lattices. Since the only Dowling lattices that are intersection lattices of real hyperplane arrangements are the partition lattice and the signed partition lattice, the construction of Björner and Wachs does not apply to general Dowling lattices. In this section we show that although the results of Björner and Wachs don't apply, the splitting bases for the homology of the partition lattice and the signed partition lattice can still be generalized to Dowling lattices.

To *split* an element $(\sigma, g) \in \mathcal{S}_n(G)$ at positions $i_1 < \cdots < i_k$ in $[n]$ means to form the G-partition with G-blocks $\{0, \sigma(1), \ldots, \sigma(i_1-1)\}$, $\overline{\{(\sigma(i_1), g_{i_1}), \ldots, (\sigma(i_2-1), g_{i_2-1})\}}$, $\ldots, \overline{\{(\sigma(i_k), g_{i_k}), \ldots, (\sigma(n), g_n)\}}$. If the set of positions is empty then the G-partition $\hat{1}$ is obtained. For each $w \in \mathcal{S}_n(G)$, let Q_w be the induced subposet of $Q_n(G)$ consisting of all G-partitions obtained by splitting w at any set of positions. It is easy to see that Q_w is isomorphic to the lattice of subsets of $[n]$. It follows that the order complex $\Delta(Q_w)$ is the barycentric subdivision of the boundary of an $(n-1)$-simplex. Choose a fundamental cycle of the spherical complex $\Delta(Q_w)$ and let ρ_w be the element of $H_*(Q_n(G))$ $(= H_{n-2}(Q_n(G)))$ obtained by attaching $\hat{0}$ and $\hat{1}$ to each chain of the fundamental cycle. So ρ_w is a linear combination of all the maximal chains of Q_w where the coefficient of each chain is ± 1. We will show that for an appropriate set of w the ρ_w form a basis for $H_*(Q_n(G))$.

Given a permutation $\sigma \in \mathcal{S}_n$, $\sigma(i)$ is said to be a *left-to-right minimum* if $\sigma(i) < \sigma(j)$ for all $j < i$. Let

$$\mathcal{W}_n(G) = \{(\sigma, g) \in \mathcal{S}_n(G) : \forall i \in [n],\ \sigma(i) \text{ is a left to right minimum of } \sigma \Rightarrow g_i = e\}.$$

Note that if $(F; w)$ is a comb forest with n leaves then $w \in \mathcal{W}_n(G)$. Conversely if $w \in \mathcal{W}_n(G)$ then there is a unique forest F such that $(F; w)$ is a comb forest. Set $m_w = c(F; w)$. For example let $G = \mathbf{Z}_2$ and $w = 5\bar{8}\bar{6}471\bar{2}3 \in \mathcal{W}_n(G)$. Then the left to right minima of $\tilde{w}$ are $5, 4, 1$. The comb forest $(F; w)$ is $((5 \frown \bar{8}) \frown \bar{6}), (4 \frown 7), ((1 \frown \bar{2}) \frown 3)$. Hence the chain m_w from top to bottom is

$$\hat{1} > 058647/1\bar{2}3 > 058647/1\bar{2}/3 > 058647/1/2/3 > 0586/47/1/2/3 >$$

$$0586/4/7/1/2/3 > 0/5\bar{8}\bar{6}/4/7/1/2/3 > 0/5\bar{8}/6/4/7/1/2/3 > 0/5/8/6/4/7/1/2/3.$$

(For the sake of simplicity of notation we have just given equivalence class representatives of the G-blocks here.) This is clearly a maximal chain of Q_w. It is also a maximal chain of Q_u where $u = 5\bar{8}\bar{6}4731\bar{2}$ or $u = 5\bar{8}\bar{6}741\bar{2}3$ or $u = 5\bar{8}\bar{6}7431\bar{2}$. In fact there are no other $u \in \mathcal{W}_n(G)$ for which w is a maximal chain in Q_u. For all of these u note that $\tilde{w} <_L \tilde{u}$, where $<_L$ denotes lexicographical order on $\mathcal{S}_n$. This example illustrates the general situation described in the next lemma whose proof we leave to the reader.

Lemma 9.1 *For all $w \in \mathcal{W}_n(G)$, m_w is a maximal chain of Q_w. Moreover if m_w is a maximal chain of Q_u where $u \in \mathcal{W}_n(G)$ and $u \neq w$, then $\tilde{w} <_L \tilde{u}$.* $\square$

Theorem 9.2 *(Splitting basis) The set $\{\rho_w : w \in \mathcal{W}_n(G)\}$ is a basis for $H_*(Q_n(G))$ and the set $\{m_w : w \in \mathcal{W}_n(G)\}$ is a basis for $H^*(Q_n(G))$.*

Proof: Lexicographical order on $\mathcal{S}_n$ induces a partial order on $\mathcal{W}_n(G)$ given by $u <_L v$ if $\tilde{u} <_L \tilde{v}$. Let $w_1, w_2, \ldots, w_d$, where $d = \prod_{j=1}^{n-1}(1 + j|G|)$, be any linear extension of this partial order. It follows from Lemma 9.1 that the matrix

$$M = (\langle \rho_{w_i}, m_{w_j} \rangle)_{i,j \in [d]}$$

is lower triangular with each diagonal entry equaling ± 1. Hence M is invertible and the result follows from Proposition 2.1. $\square$

Remark: Since the matrix M is invertible over the integers,the splitting basis $\{\rho_w : w \in \mathcal{W}_n(G)\}$ is also a basis for integral homology of $Q_n(G)$ and the comb forest basis $\{m_w : w \in \mathcal{W}_n(G)\}$ is also a basis for integral cohomology of $Q_n(G)$. Other bases for integral homology and cohomology of $Q_n(G)$ are discussed in [Go].

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