Chapter 9

Combinatorics and Hessenberg varieties

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We discuss results and conjectures, which have appeared in the literature, involving immanants of Jacobi-Trudi matrices, chromatic symmetric and quasisymmetric functions, cohomology of Hessenberg varieties in the type A flag variety, and representations of the type A Hecke algebra. These topics are connected. The starting point in this area was a (still open) 1992 conjecture of Stembridge that monomial immanants of Jacobi-Trudi matrices are Schur positive. In 1993 Haiman formulated a conjecture on representations of the type A Hecke algebra, which he showed implies Stembridge's conjecture.

In 1993 Stanley and Stembridge made two conjectures that together imply Stembridge's conjecture. The first of these conjectures, about chromatic symmetric functions, was proved by Hikita in 2024, after a reduction found by Guay-Paquet in 2012. The second conjecture remains open. A refined version of the first Stanley-Stembridge conjecture involving chromatic quasisymmetric functions, made by ourselves in 2012, also remains open. We conjectured in addition a connection between chromatic quasisymmetric functions and cohomology of regular semisimple type A Hessenberg varieties. This conjecture was proved by Brosnan and Chow in 2018. Connections between chromatic quasisymmetric functions and representations of the type A Hecke algebra were found by Clearman-Hyatt-Shelton-Skandera in 2016.

This chapter is intended to introduce readers to the main ideas and results related to the conjectures mentioned above.

9.1 Introduction

We aim to familiarize the reader with a circle of ideas and conjectures connecting immanants of Jacobi–Trudi matrices, chromatic symmetric functions of graphs, Hessenberg varieties in the type A flag variety, and representations of type A Iwahori–Hecke algebras. All objects discussed in this introduction will be defined in following sections. All the results and conjectures we present here can be found in existing literature. We hope that a reader unfamiliar with one or more of the main topics will be able to get a clear picture of the basic ideas and will see how the various topics fit together. We have tried to provide instructive examples throughout the chapter.

We present a series of interrelated conjectures, all but two of which remain open. These are

- a 1992 conjecture of John Stembridge that monomial immanants of Jacobi–Trudi matrices are Schur-positive;
- a 1993 conjecture of Richard Stanley and John Stembridge that incomparability graphs of $(3+1)$-free posets have e-positive chromatic symmetric functions, which was proved by Tatsuyuki Hikita in 2024 after a

reduction by Mathieu Guay-Paquet to the case of posets that are both $(3+1)$-free and $(2+2)$-free;

- another 1993 conjecture of Stanley and Stembridge relating immanants and chromatic symmetric functions, the truth of which would, along with Hikita's Theorem, imply the truth of Stembridge's Immanant Conjecture;
- a 2012 e-positivity conjecture of our own refining the first Stanley–Stembridge Conjecture (as reduced by Guay-Paquet) through the introduction of a graded version of the chromatic symmetric function;
- another 2012 conjecture of ours stating that our graded chromatic symmetric functions are (up to a sign twist) the Frobenius characteristics of certain representations (introduced by Julianna Tyomczko) of symmetric groups on the cohomology groups of regular semisimple Hessenberg varieties, which was proved by Patrick Brosnan and Timothy Chow in 2018;
- a 1993 conjecture of Mark Haiman about Hecke algebra traces, which was shown by him to imply Stembridge's Monomial Immanant Conjecture, and was shown in 2016 by Samuel Clearman, Matt Hyatt, Brittany Shelton, and Mark Skandera to imply our graded version of the Stanley–Stembridge Conjecture.

There is much recent and ongoing work on the conjectures presented here and closely related ideas. We cannot discuss all of it and apologize to our friends and colleagues whose interesting work has been left out. If you think this chapter would be improved by a discussion of your own work (or someone else's), you are very likely correct.

9.2 Background and original motivation

We discuss in this section a small part of the work of Stembridge on immanants of Jacobi–Trudi matrices in [55]. This work, along with further work of Stanley and Stembridge in [54], was the original motivation behind the Stanley–Stembridge conjecture and Stanley's chromatic symmetric functions in [50]. These symmetric functions will be discussed in Section 9.3. Immanants are a natural generalization of determinants and permanents, and Jacobi-Trudi matrices are fundamental in symmetric function theory. In Section 9.2.2, we define these terms and discuss Stembridge's work. Prior to this, in Section 9.2.1, we introduce the various objects arising in algebraic combinatorics that will be needed in the discussion of Stembridge's work and in the rest of this chapter.

9.2.1 Symmetric functions and the Frobenius characteristic map

Let $\mathsf{Par}(n)$ be the set of **partitions** of a nonnegative integer n; that is, the set of weakly decreasing sequences of positive integers whose sum is n. Given a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathsf{Par}(n)$, the entries λ_i are called **parts** of λ and the number ℓ of parts of λ is called the **length** $\ell(\lambda)$ of λ. Define $|\lambda| := \sum_{i=1}^{\ell} \lambda_i = n$. At times we will need to view partitions as infinite sequences by appending 0′s; that is, we view λ as $(\lambda_1, \dots, \lambda_\ell, 0, 0, \dots)$. For each nonnegative integer n, let

$$[n] := \{1, 2, \dots, n\}.$$

A **symmetric function** is a formal power series of bounded degree in infinitely many pairwise commuting variables $x_1, x_2, \dots$ that is invariant under every permutation of the variables. Widely read texts on symmetric functions include [41], [44], and [53, Chapter 7]. We recall here some basic facts that can be found in all of these texts. The set $\Lambda_{\mathbb{F}}$ of symmetric functions with coefficients in a field $\mathbb{F}$ is an $\mathbb{Z}_{\geq 0}$-graded $\mathbb{F}$-algebra,

$$\Lambda_{\mathbb{F}} = \bigoplus_{n=0}^{\infty} \Lambda_{\mathbb{F}}^n,$$

where $\Lambda_{\mathbb{F}}^n$ consists of all homogeneous degree n symmetric functions (along with 0). We see that

$$\dim \Lambda_{\mathbb{F}}^n = |\mathsf{Par}(n)|.$$

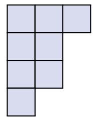
Indeed, given $\lambda \in \mathsf{Par}(n)$ and $j \in \mathbb{Z}_{>0}$, we write $a_j(\lambda)$ for the multiplicity of j as of part of λ, and let m_λ be the sum of all monomials in $\mathbf{x} := (x_1, x_2, \dots)$ in which each $j \in \mathbb{Z}_{>0}$ appears as an exponent exactly $a_j(\lambda)$ times. We observe that the set $\{m_\lambda : \lambda \in \mathsf{Par}(n)\}$ is a basis for $\Lambda_{\mathbb{F}}^n$. There are other interesting bases. For each $j \in \mathbb{Z}_{\geq 0}$, we write h_j for the sum of all monomials in $\mathbf{x}$ of degree j, e_j for the sum of all squarefree monomials in $\mathbf{x}$ of degree j, and p_j for $\sum_{i \geq 1} x_i^j$. Given $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathsf{Par}(n)$ and $b \in \{h, e, p\}$, we set

$$b_\lambda := \prod_{i=1}^{\ell} b_{\lambda_i}.$$

Then $\{b_\lambda : \lambda \in \mathsf{Par}(n)\}$ is a basis for $\Lambda_{\mathbb{F}}^n$. It follows that for $b \in \{h, e, p\}$, the algebra Λ is generated freely as a commutative $\mathbb{F}$-algebra by $\{b_n : n \in \mathbb{Z}_{>0}\}$, that is, Λ is isomorphic with the polynomial ring $\mathbb{F}[\{b_n : n \in \mathbb{Z}_{>0}\}]$. The symmetric functions m_λ, h_λ, e_λ, and p_λ are, respectively, the **monomial**, **complete homogeneous**, **elementary**, and **power sum** symmetric functions. For $b \in \{h, e, p\}$, we define b_α for any finite sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of nonnegative integers using multiplication as we did when defining b_λ for partitions. Such a sequence α is called a **weak composition**. As with partitions, at times we view weak compositions as infinite sequences by appending 0′s.

Another important basis for $\Lambda_{\mathbb{F}}^n$ is the Schur basis. The combinatorial definition of this basis involves Young diagrams. Associated with each partition

$\lambda \in \mathsf{Par}(n)$ is the **Young diagram** of shape λ, which is a left justified array of cells for which the number of cells in the ith row is λ_i for each $i \in [\ell(\lambda)]$. For example, the Young diagram of shape $(3, 2, 2, 1)$ is



A **semistandard tableau** of shape $\lambda \in \mathsf{Par}(n)$ is a filling of the Young diagram of shape λ with positive integers so that the entries of each row weakly increase from left to right and the entries of each column strictly increase from top down. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 5 \\ \hline 5 & 6 \\ \cline{1-2} 7 & 7 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

is a semistandard tableau of shape (3,2,2,1).

To each semistandard tableau T, we associate the monomial $x_T := x_1^{a_1} x_2^{a_2} \cdots$, where a_i is the number of times i appears in T. We call the weak composition $(a_1, a_2, \dots)$ the **type** of the semistandard tableau. For example, the type of the semistandard tableau above is $(1, 0, 0, 0, 3, 1, 2, 0, 1, 0, 0, \dots)$ and $x_T = x_1 x_5^3 x_6 x_7^2 x_9$. For $\lambda \in \mathsf{Par}(n)$, the **Schur function** s_λ is defined by

$$s_\lambda := \sum_T x_T,$$

where the sum is over all semistandard tableaux T of shape λ. It is not obvious that s_λ is symmetric, nor that $\{s_\lambda : \lambda \in \mathsf{Par}(n)\}$ is a basis for $\Lambda^n_{\mathbb{F}}$; proofs of these facts can be found in the references cited above. Since s_λ is symmetric, we have

$$\mathsf{s}_\lambda = \sum_{\mu \in \mathsf{Par}(n)} K_{\lambda,\mu} m_\mu,$$

where $K_{\lambda,\mu}$ is the number of semistandard tableaux of shape λ and type μ. The $K_{\lambda,\mu}$ are called **Kostka numbers**. Note that $s_{(n)} = h_n$ and $s_{1^n} = e_n = m_{1^n}$, where 1^n denotes the partition $(1, 1, \dots, 1)$ of length n. A semistandard tableau of shape λ and type 1^n is called a **standard Young tableau** of shape λ.

We assume now that $\mathbb{F} = \mathbb{C}$ and suppress $\mathbb{F}$ in our notation. Let us also discuss briefly some basic facts and notation from representation theory that we will use, often implicitly. Given a positive integer d, we write $M_d(\mathbb{C})$ for the algebra of $d \times d$ matrices with entries in $\mathbb{C}$ and we write $GL_d(\mathbb{C})$ for the group of invertible such matrices. A representation of S_n, that is, a homomorphism $\phi : S_n \to GL_d(\mathbb{C})$ for some d, can be extended by linearity to an algebra homomorphism $\Phi : \mathbb{C}[S_n] \to M_d(\mathbb{C})$, which makes $V := \mathbb{C}^d$ a $\mathbb{C}[S_n]$-module. Two $\mathbb{C}[S_n]$-modules V, W arising from respective representations ϕ, ψ are isomorphic if and only if ϕ and ψ are equivalent, which means that $V = W = \mathbb{C}^d$

for some d and there is some $g \in GL_d(\mathbb{C})$ such that $\psi(w) = g^{-1}\phi(w)g$ for all $w \in S_n$. The character χ_ϕ maps each $w \in S_n$ to the trace of $\phi(w)$. It is apparent that if ϕ and ψ are equivalent then $\chi_\phi = \chi_\psi$. The converse, that ϕ and ψ are equivalent if $\chi_\phi = \chi_\psi$, is a key and basic fact from representation theory. The upshot of all of this is that isomorphism classes of $\mathbb{C}[S_n]$-modules, equivalence classes of S_n-representations, and characters of S_n-representations are essentially the same things and we will use whichever of these objects (and the associated notation) that suits our purposes at any time.

The conjugacy classes of S_n are indexed conveniently by $\mathsf{Par}(n)$, using cycle type. A function $\chi : S_n \to \mathbb{C}$ that is constant on conjugacy classes of S_n is called a **class function**. Characters of representations of S_n are class functions. In fact, the characters of the irreducible representations form a basis for the vector space $\mathsf{CF}(n)$ of class functions (see, e.g., [35, 44, 53]).

It follows from well-known facts about representations of finite groups that S_n has $|\mathsf{Par}(n)|$ irreducible characters. Therefore, $\mathsf{CF} := \bigoplus_{n\geq 0} \mathsf{CF}(n)$ is isomorphic to Λ as an $\mathbb{Z}_{\geq 0}$-graded vector space. Moreover, CF is a graded $\mathbb{C}$-algebra with the induction product, which we describe now: first, given $m, n \in \mathbb{Z}_{\geq 0}$, we define $\iota : S_m \times S_n \hookrightarrow S_{m+n}$ so that $\iota((v, w))$ maps j to $v(j)$ if $j \in [m]$ and to $w(j-m)+m$ if $j \in [m+n] \setminus [m]$. Now given $\sigma \in \mathsf{CF}(m)$ and $\tau \in \mathsf{CF}(n)$, define the function $(\sigma \otimes \tau)^\circ$ on S_{m+n} by

$$(\sigma \otimes \tau)^\circ(x) = \begin{cases} \sigma(v)\tau(w) & \text{if } x = \iota(v, w), \\ 0 & otherwise. \end{cases}$$

Finally, define the **induced class function** $\sigma \circ \tau \in \mathsf{CF}(m+n)$ by

$$(\sigma \circ \tau)(x) := \frac{1}{m!n!} \sum_{g \in S_{m+n}} (\sigma \otimes \tau)^\circ(gxg^{-1}). \tag{9.1}$$

(If σ, τ are the respective characters χ_ϕ and χ_ψ of representations ϕ and ψ of S_m and S_n with associated modules V and W, then $\sigma \circ \tau$ is the character of the induced representation $(\phi \otimes \psi) \uparrow_{S_m \times S_n}^{S_{m+n}}$, which has associated $\mathbb{C}[S_{m+n}]$-module $\mathbb{C}[S_{m+n}] \otimes_{\mathbb{C}[S_m \times S_n]} (V \otimes W)$. The **induction product** on CF is the unique bilinear binary operation with restriction to $\mathsf{CF}(m) \times \mathsf{CF}(n)$ given by (9.1) for all m, n. This product makes CF an $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathsf{Par}(n)$. We embed the direct product $\prod_{i=1}^{\ell} S_{\lambda_i}$ into S_n, with the j^{th} factor S_{λ_j} acting as the stabilizer in S_n of all $t \in [n]$ other those satisfying $\sum_{i=1}^{j-1} \lambda_i < t \leq \sum_{i=1}^{j} \lambda_i$. The embedded subgroups, which will be denoted by Y_λ, are called **Young subgroups** or **parabolic subgroups**. Now S_n acts by translation on the set of cosets of Y_λ, giving rise to a $\mathbb{C}[S_n]$-module M^λ with $\mathbb{C}$-basis consisting of these cosets. Put another way, M^λ is the induction to S_n of the 1-dimensional complex trivial representation of Y_λ. Its character χ_{M^λ} is given by the induction product:

$$\chi_{M^\lambda} = 1_{\lambda_1} \circ 1_{\lambda_2} \circ \cdots \circ 1_{\lambda_\ell}. \tag{9.2}$$

One can use the modules M^λ to construct all the simple $\mathbb{C}[S_n]$-modules (equivalently, irreducible representations or irreducible characters of S_n). The construction relies on a certain partial order on $\mathsf{Par}(n)$, called **dominance**. Given partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of n (viewed as infinite sequences with 0's appended), we say that λ dominates μ, and write $\mu \preceq_{\mathsf{dom}} \lambda$, if $\sum_{i=1}^k \mu_i \leq \sum_{i=1}^k \lambda_i$ for all $k \geq 1$. For example $(3,3,2,1) \preceq_{\mathsf{dom}} (4,3,2)$.

Like the conjugacy classes, the irreducible representations of S_n are naturally indexed by partitions. Indeed, the representations of S_n on the spaces M^μ lead to an inductive construction of its irreducible representations S^λ. First, $S^{(n)} = M^{(n)}$ carries the trivial representation. Having defined S^λ for all $\lambda \succ_{\mathsf{dom}} \mu$, one can set

$$a_{\lambda,\mu} := \dim_{\mathbb{C}} \operatorname{Hom}(S^\lambda, M^\mu).$$

So, $\bigoplus_{\lambda \succ_{\mathsf{dom}} \mu} a_{\lambda,\mu} S^\lambda$ embeds in M^μ. It turns out that the quotient of M^μ by the image of this embedding is irreducible, and we call this quotient S^μ. It can be shown that

$$a_{\lambda,\mu} = K_{\lambda,\mu},$$

where $K_{\lambda,\mu}$ denotes the Kostka number discussed above. It follows that the decomposition of M^μ into irreducibles is given by the $\mathbb{C}[S_n]$-module isomorphism

$$M^\mu \cong \bigoplus_{\lambda \in \mathsf{Par}(n)} K_{\lambda,\mu} S^\lambda \tag{9.3}$$

since $K_{\lambda,\mu} = 0$ unless λ dominates μ.

Up to isomorphism, the set

$$\{S^\lambda : \lambda \in \mathsf{Par}(n)\}$$

consists of all simple $\mathbb{C}[S_n]$-modules (equivalently, all irreducible representations of S_n). We write χ^λ for the (irreducible) character of S^λ.

Given a partition $\lambda \in \mathsf{Par}(n)$, we define

$$z_\lambda := \prod_j a_j(\lambda)! j^{a_j(\lambda)},$$

where $a_j(\lambda)$ is the number of occurrences of j in λ, and remark that z_λ is the order of the centralizer in S_n of any element of cycle type λ. The **indicator function** $\delta_\lambda \in \mathsf{CF}(n)$ takes the value 1 on elements of cycle type λ and 0 elsewhere. The **Frobenius characteristic** $\mathsf{ch} : \mathsf{CF} \to \Lambda$ is the unique linear map satisfying

$$\mathsf{ch}(\delta_\lambda) = \frac{1}{z_\lambda} p_\lambda$$

for every partition λ. The map ch is an isomorphism of graded $\mathbb{C}$-algebras. Moreover, for each $n \in \mathbb{Z}_{\geq 0}$, $\mathsf{ch}^{-1}(h_n)$ is the trivial character 1_n of S_n and

$\mathsf{ch}^{-1}(e_n)$ is the alternating (or sign) character sgn_n. If we define $\omega : \Lambda \to \Lambda$ to be the unique algebra homomorphism satisfying

$$\omega(h_n) = e_n, \tag{9.4}$$

then the diagram

$$\begin{tikzcd} \mathsf{CF}(n) \arrow[r, "\otimes \mathsf{sgn}_n"] \arrow[d, "\mathsf{ch}"] & \mathsf{CF}(n) \arrow[d, "\mathsf{ch}"] \\ \Lambda^n \arrow[r, "\omega"] & \Lambda^n \end{tikzcd}$$

commutes for all $n \in \mathbb{Z}_{\geq 0}$.

Given a $\mathbb{C}[S_n]$-module V, it is convenient to let

$$\mathsf{ch}(V) := \mathsf{ch}(\chi_V),$$

where χ_V is the character of V. The importance of Schur functions stems from their connection with the irreducible representations as given in the following key fact:

$$s_\lambda = \mathsf{ch}(S^\lambda) \tag{9.5}$$

for all $\lambda \in \mathsf{Par}(n)$.

Let λ' denote the **conjugate** of $\lambda \in \mathsf{Par}(n)$, that is, the partition whose Young diagram is the transpose of the Young diagram of shape λ. Equivalently, $\lambda' = (\lambda'_1, \dots, \lambda'_k)$ where $k = \lambda_1$ and λ'_i equals the length of the ith column of the Young diagram of shape λ. For example if $\lambda = (3, 2, 2, 1)$ then $\lambda' = (4, 3, 1)$. Another important fact is

$$\omega(s_\lambda) = s_{\lambda'},$$

which is equivalent to

$$\mathsf{sgn}_n \otimes S^\lambda \cong S^{\lambda'}.$$

Equation (9.2) is equivalent to the following fact, which will be important to us:

$$h_\lambda = \mathsf{ch}(M^\lambda).$$

It therefore follows from (9.5) that (9.3) is equivalent to

$$h_\mu = \sum_{\lambda \in \mathsf{Par}(n)} K_{\lambda,\mu} s_\lambda, \tag{9.6}$$

for all $\mu \in \mathsf{Par}(n)$. On occasion μ will be a list of integers with a negative entry. In this case we define $K_{\lambda,\mu} = 0$ so that (9.6) still holds.

The space $\mathsf{CF}(n)$ admits an inner product

$$\langle \psi, \theta \rangle := \frac{1}{n!} \sum_{w \in S_n} \psi(w)\theta(w),$$

with respect to which the irreducible characters χ^λ form an orthonormal basis. An inner product $\langle\cdot,\cdot\rangle$ on Λ^n is defined so as to make ch an isometry:

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu} := \begin{cases} 1 & \lambda=\mu, \\ 0 & \text{otherwise.}\end{cases}$$

A key fact is that the bases $\{h_\lambda\}$ and $\{m_\lambda\}$ are dual with respect to this inner product:

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}.$$

Remark 9.2.1. *We could assume throughout that $\mathbb{F}=\mathbb{Q}$ rather than $\mathbb{F}=\mathbb{C}$. Indeed, all irreducible complex representations (and therefore all representations) of S_n are realizable over $\mathbb{Q}$. It is common to use $\mathbb{Q}$ when considering symmetric functions and the Frobenius characteristic. However, later in the chapter we will introduce various objects which are commonly studied using $\mathbb{C}$ rather than $\mathbb{Q}$; hence we use $\mathbb{C}$ here for consistency with what follows.*

The reader will notice that in fact in many settings all coefficients in sight are integers. This is worth keeping in mind. For example, if we assert that some symmetric function $f(\mathbf{x})$ is a non-negative linear combination of Schur functions, it is already implicit that the coefficients in question are real. If they are also integers, then $f(\mathbf{x}) = \mathsf{ch}(\chi)$ for the character χ of some S_n-representation.

9.2.2 The Immanant Conjecture and equivalent formulations

For any ring R and positive integer n, let $M_n(R)$ be the algebra of $n\times n$ matrices with entries in R. Each class function $\chi\in\mathsf{CF}(n)$ determines an **immanant** $\mathsf{Imm}_\chi : M_n(R)\to R$, defined by

$$\mathsf{Imm}_\chi(A) := \sum_{w\in S_n}\chi(w)\prod_{i=1}^n a_{i,w(i)}, \tag{9.7}$$

for $A=(a_{ij})\in M_n(R)$. While immanants have not been studied a lot, some attention has been given to the case where χ is an irreducible character. This is a natural choice, since if χ is the sign character then Imm_χ is the determinant, and if χ is the trivial character, then Imm_χ is the permanent. In what follows we take R to be the ring Λ of symmetric functions over $\mathbb{C}$.

A basic result in the theory of symmetric functions is the **Jacobi–Trudi identity**, which expresses each Schur function s_μ as the determinant of a matrix whose entries are complete homogeneous symmetric functions. Given a partition $\mu=(\mu,\ldots,\mu_n)$, we define the **Jacobi–Trudi matrix**

$$H_\mu := (h_{\mu_i+j-i})_{i,j=1}^n,$$

where, as always, $h_0=1$ and $h_m=0$ if $m<0$. For example,

$$H_{(3,2,1)} = \begin{bmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1\end{bmatrix}.$$

The **Jacobi–Trudi Identity** (see, e.g., [53, Theorem 7.16.1]) says that

$$s_\mu = \det(H_\mu).$$

So, for example, expanding $\det(H_{(3,2,1)})$ along the third row, we get

$$s_{(3,2,1)} = h_{(5,1)} - h_{(3,3)} + h_{(3,2,1)} - h_{(4,1,1)}.$$

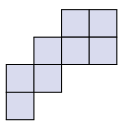
We observe that, as mentioned above,

- the determinant is the immanant associated to the sign character;
- the Frobenius characteristic of the sign character of S_n is the elementary symmetric function e_n; and
- by definition, $e_n = m_{(1,\ldots,1)}$, where there are n ones in the partition in the last subscript.

So, we can rewrite the Jacobi–Trudi Identity as

$$\mathsf{Imm}_{\mathsf{ch}^{-1}(m_{(1,\ldots,1)})}(H_\mu) = s_\mu.$$

There is a more general Jacobi–Trudi identity for skew Schur functions. Let $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_k)$ be partitions. We write $\nu \subseteq \mu$ if $k \leq n$ and $\nu_i \leq \mu_i$ for $1 \leq i \leq k$. For $\nu \subseteq \mu$, place the Young diagram of shape ν in the Young diagram of shape μ in a northwest justified manner and then delete the cells of the Young diagram for ν from that of μ. The resulting array is the skew diagram of skew shape μ/ν. For example, if $\mu = (4, 4, 2, 1)$ and $\nu = (2, 1)$ then μ/ν has skew diagram


.

Just as for straight shapes (i.e., shapes of the form $\mu/\emptyset$), a semistandard tableau of skew shape μ/ν is a filling of the skew diagram of shape μ/ν with positive integers so that the entries of each row weakly increase from left to right and the entries of each column strictly increase from top down. For example, a semistandard tableau of the skew shape $(4, 4, 2, 1)/(2, 1)$ is given by

$$\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 4 & 6 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 8 & 8 \\ \cline{1-4} 5 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\ .$$

The **skew Schur function** $s_{\mu/\nu}$ is defined by

$$s_{\mu/\nu} := \sum_T x_T,$$

where the sum is over all semistandard tableaux T of skew shape μ/ν and x_T is defined as in Section 9.2.1 for straight shapes.

We define

$$H_{\mu/\nu} := (h_{\mu_i-\nu_j-i+j})_{i,j=1}^n,$$

where ν is padded with 0's so that $\nu_i = 0$ when $i > k$. The matrices $H_{\mu/\nu}$ are also called Jacobi–Trudi matrices. The Jacobi–Trudi matrices defined earlier arise in the special case $\nu = \emptyset$. The Jacobi–Trudi identity for skew Schur functions says that

$$s_{\mu/\nu} = \det(H_{\mu/\nu}) = \mathsf{Imm}_{\mathsf{ch}^{-1}(m_{(1,1,\ldots,1)})}(H_{\mu/\nu}).$$

It follows from the Littlewood–Richardson rule (see, e.g., [53, Theorem A1.3.1]) that each skew Schur function is a nonnegative integer combination of Schur functions. Stembridge conjectured that the following generalization holds.

Conjecture 9.2.2 (Monomial Immanant Conjecture [55, Conjecture 4.1]). *Let $\lambda \in \mathsf{Par}(n)$ and let $\nu \subseteq \mu$ be partitions with $\ell(\mu) = n$. Then $\mathsf{Imm}_{\mathsf{ch}^{-1}(m_\lambda)}(H_{\mu/\nu})$ is a nonnegative integer combination of Schur functions (of degree $|\mu| - |\nu|$).*

Remark 9.2.3. *The formulation of the Jacobi–Trudi matrices given in [55] allows for μ with length less than n. By adding the vector $(1, 1, \ldots, 1)$ of length n to both μ and ν, we get $\nu^+ \subseteq \mu^+$ with skew shape μ^+/ν^+ the same as that of μ/ν. Moreover $\ell(\mu^+) = n$ and the Jacobi–Trudi matrix for μ/ν given in [55] is the same as our H_{μ^+/ν^+}. The upshot is that the conjecture formulated in [55] is equivalent to the one we have stated here.*

Stembridge refers to the immanants $\mathsf{Imm}_{\mathsf{ch}^{-1}(m_\lambda)}$ as **monomial immanants**. It is standard to refer to nonnegative combinations of Schur functions as **Schur-positive** (and to use similar nomenclature for nonnegative combinations of elements of various bases for Λ). So, one can state Stembridge's Conjecture as "Monomial immanants of Jacobi–Trudi matrices are Schur-positive."

Example 9.2.4. Say $\lambda = (3, 1)$, $\mu = (4, 3, 2, 1)$, and $\nu = (2, 1, 1)$. Then

$$H_{\mu/\nu} = \begin{bmatrix} h_2 & h_4 & h_5 & h_7 \\ 1 & h_2 & h_3 & h_5 \\ 0 & 1 & h_1 & h_3 \\ 0 & 0 & 0 & h_1 \end{bmatrix}. \tag{9.8}$$

Let us write $H_{\mu/\nu} = (a_{ij})_{i,j=1}^4$, Then, given $w \in S_4$, we see that $\prod_{i=1}^4 a_{i,w(i)} = 0$ unless $w(4) = 4$ and $w(3) \neq 1$. We compute directly that

$$m_{(3,1)} = p_{(3,1)} - p_{(4)} = 3\frac{p_{(3,1)}}{z_{(3,1)}} - 4\frac{p_{(4)}}{z_{(4)}},$$

So, coefficients arising when calculating $\mathsf{Imm}_{\mathsf{ch}^{-1}(m_{(3,1)})}$ take the value -4 on 4-cycles, 3 on elements of cycle type $(3,1)$ and 0 elsewhere. Since no 4-cycle $w \in S_4$ satisfies $w(4) = 4$ and the only element of cycle type $(3,1)$ in S_4 satisfying both $w(4) = 4$ and $w(3) \neq 1$ is (132), we see that

$$\mathsf{Imm}_{\mathsf{ch}^{-1}(m_\lambda)}(H_{\mu/\nu}) = 3a_{13}a_{21}a_{32}a_{44} = 3h_{(5,1)} = 3s_{(6)} + 3s_{(5,1)}.$$

Returning to the general case, we set

$$\delta = \delta_n := (n-1, n-2, \ldots, 2, 1, 0).$$

One can see that for any $\chi \in \mathsf{CF}(n)$ and any $\nu \subseteq \mu$ with μ having length n,

$$\mathsf{Imm}_\chi(H_{\mu/\nu}) = \sum_{w\in S_n} \chi(w) h_{\mu+\delta-w(\nu+\delta)}. \tag{9.9}$$

Here it is assumed that, if necessary, ν is padded with 0's so that ν can be viewed as a vector of length n for which $\nu + \delta$ makes sense. The permutation w acts on the set of compositions of length n by permuting entries and, as above, $h_m = 0$ for $m < 0$.

Example 9.2.5. With $\mu = (4,3,2,1)$ and $\nu = (2,1,1)$ as above, we get $\mu + \delta = (7,5,3,1)$ and $\nu + \delta = (5,3,2,0)$. If $w = 3124$ then

$$\mu + \delta - w(\nu + \delta) = (7,5,3,1) - (2,5,3,0) = (5,0,0,1)$$

and $h_{\mu+\delta-w(\nu+\delta)} = h_{(5,1)}$. (This is the $h_{(5,1)}$ appearing in the immanant computation above.) If $w = 2143$ then

$$\mu + \delta - w(\nu + \delta) = (7,5,3,1) - (3,5,0,2) = (4,0,3,-1)$$

and $h_{\mu+\delta-w(\nu+\delta)} = 0$.

We continue to follow Stembridge in [55]. Given $\nu \subseteq \mu$ with $\ell(\mu) = n$, set $N := |\mu| - |\nu|$. For each $\chi \in \mathsf{CF}(n)$, we have $\mathsf{Imm}_\chi(H_{\mu/\nu}) \in \Lambda^N$. It follows from (9.6) and (9.9) that for every such χ

$$\begin{aligned}
\mathsf{Imm}_\chi(H_{\mu/\nu}) &= \sum_{w\in S_n} \chi(w) h_{\mu+\delta-w(\nu+\delta)} \\
&= \sum_{w\in S_n} \chi(w) \sum_{\rho\in\mathsf{Par}(N)} K_{\rho,\mu+\delta-w(\nu+\delta)} s_\rho \\
&= \sum_{\rho\in\mathsf{Par}(N)} \left(\sum_{w\in S_n} \chi(w) K_{\rho,\mu+\delta-w(\nu+\delta)} \right) s_\rho.
\end{aligned} \tag{9.10}$$

We conclude that $\mathsf{Imm}_\chi(H_{\mu/\nu})$ is Schur-positive if and only if

$$\sum_{w\in S_n} \chi(w) K_{\rho,\mu+\delta-w(\nu+\delta)} \geq 0 \tag{9.11}$$

for each $\rho \in \mathsf{Par}(N)$. Now (9.11) can be rewritten in terms of inner products: having fixed $\nu \subseteq \mu$, define for each $\rho \in \mathsf{Par}(N)$ the class function $\Gamma^{\rho}_{\mu/\nu} \in \mathsf{CF}(n)$ by

$$\Gamma^{\rho}_{\mu/\nu}(w) := \frac{n!}{|\mathsf{Conj}(w)|} \sum_{v \in \mathsf{Conj}(w)} K_{\rho,\mu+\delta-v(\nu+\delta)}, \tag{9.12}$$

where $\mathsf{Conj}(w)$ is the S_n-conjugacy class of w. Then

$$\begin{aligned} \langle \chi, \Gamma^{\rho}_{\mu/\nu} \rangle &= \frac{1}{n!} \sum_{w \in S_n} \chi(w) \Gamma^{\rho}_{\mu/\nu}(w) \\ &= \sum_{w \in S_n} \frac{1}{|\mathsf{Conj}(w)|} \chi(w) \sum_{v \in \mathsf{Conj}(w)} K_{\rho,\mu+\delta-v(\nu+\delta)} \\ &= \sum_{w \in S_n} \chi(w) K_{\rho,\mu+\delta-w(\nu+\delta)}. \end{aligned}$$

Comparing with (9.10), we see that for all $\rho \in \mathsf{Par}(N)$, the coefficient of s_ρ in the Schur basis expansion of $\mathsf{Imm}_\chi(H_{\mu/\nu})$ equals $\langle \chi, \Gamma^{\rho}_{\mu/\nu} \rangle$; that is,

$$\mathsf{Imm}_\chi(H_{\mu/\nu}) = \sum_{\rho \in \mathsf{Par}(N)} \langle \chi, \Gamma^{\rho}_{\mu/\nu} \rangle s_\rho. \tag{9.13}$$

Hence $\mathsf{Imm}_\chi(H_{\mu/\nu})$ is Schur-positive if and only if

$$\langle \chi, \Gamma^{\rho}_{\mu/\nu} \rangle \geq 0$$

for all $\rho \in \mathsf{Par}(N)$. We specialize now to the case that $\chi = \mathsf{ch}^{-1}(m_\lambda)$ and observe that the coefficient of h_λ in $\mathsf{ch}(\Gamma^{\rho}_{\mu/\nu})$ is

$$\langle m_\lambda, \mathsf{ch}(\Gamma^{\rho}_{\mu/\nu}) \rangle = \langle \chi, \Gamma^{\rho}_{\mu/\nu} \rangle,$$

since the monomial and complete homogeneous bases for Λ^n are dual to each other. We can now conclude the following.

Theorem 9.2.6. *Let $\lambda \in \mathsf{Par}(n)$ and let $\nu \subseteq \mu$ be partitions with $\ell(\mu) = n$. Set $N = |\mu| - |\nu|$ and let $\rho \in \mathsf{Par}(N)$. Then the coefficient of s_ρ in the Schur basis expansion of $\mathsf{Imm}_{\mathsf{ch}^{-1}(m_\lambda)}(H_{\mu/\nu})$ equals the coefficient of h_λ in the h basis expansion of $\mathsf{ch}(\Gamma^{\rho}_{\mu/\nu})$.*

We see now that Stembridge's Monomial Immanant Conjecture (Conjecture 9.2.2) is equivalent to the following conjecture.

Conjecture 9.2.7 ([55, Conjecture 4.1′]). *Let $\nu \subseteq \mu$ be partitions with $\ell(\mu) = n$. Set $N = |\mu| - |\nu|$ and let $\rho \in \mathsf{Par}(N)$. Then $\mathsf{ch}(\Gamma^{\rho}_{\mu/\nu})$ is h-positive. Equivalently, the class function $\Gamma^{\rho}_{\mu/\nu}$ is the character of a permutation representation in which each point stabilizer is a Young subgroup of S_n.*

The Monomial Immanant Conjecture implies a weaker conjecture, due also to Stembridge and proved by Haiman, in which m_λ is replaced by s_λ. We state Haiman's result now and delay discussion of his proof to Section 9.6.2.

Theorem 9.2.8 (see Haiman [31, Theorem 1.4])**.** *Let $\lambda \in \mathsf{Par}(n)$ and let $\nu \subseteq \mu$ be partitions with $\ell(\mu) = n$. Then $\mathsf{Imm}_{\chi^\lambda}(H_{\mu/\nu})$ is Schur-positive.*

Since the Schur basis for Λ^n is self-dual, an argument similar to that of Theorem 9.2.6 can be used to obtain the following result.

Theorem 9.2.9. *Assume λ, μ, ν, ρ are as in Theorem 9.2.6. Then the coefficient of s_ρ in the Schur basis expansion of $\mathsf{Imm}_{\chi^\lambda}(H_{\mu/\nu})$ equals the coefficient of s_λ in the Schur basis expansion of $\mathsf{ch}(\Gamma^\rho_{\mu/\nu})$.*

Thus Haiman's Theorem 9.2.8 is equivalent to:

Theorem 9.2.10. *Let $\nu \subseteq \mu$ be partitions. Set $N = |\mu| - |\nu|$ and let $\rho \in \mathsf{Par}(N)$. Then $\mathsf{ch}(\Gamma^\rho_{\mu/\nu})$ is Schur-positive.*

9.3 Chromatic symmetric functions and immanants

Here we describe ideas and results from [54] and [50] on Stanley's chromatic symmetric functions and their connection to Stembridge's Monomial Immanant Conjecture (Conjecture 9.2.2).

9.3.1 Stanley's chromatic symmetric function

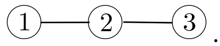
Let $G = (V, E)$ be a finite simple graph. A **proper coloring** of G is a function $\kappa : V \to \mathbb{Z}_{>0}$ satisfying $\kappa(i) \neq \kappa(j)$ whenever $\{i, j\} \in E$. We write $K(G)$ for the set of all proper colorings of G. The **chromatic symmetric function** of G is a formal power series in the set $\mathbf{x} = \{x_1, x_2, \ldots\}$ of variables,

$$X_G(\mathbf{x}) := \sum_{\kappa \in K(G)} \prod_{v \in V} x_{\kappa(v)}.$$

Indeed $X_G(\mathbf{x})$ is a symmetric function since permuting colors turns a proper coloring into another one. Moreover, each monomial appearing in $X_G(\mathbf{x})$ has degree $|V|$. Investigation of $X_G(\mathbf{x})$ by combinatorialists was stimulated by the seminal paper [50] of Stanley.

One can think of $X_G(\mathbf{x})$ as a finer graph invariant than the well-studied chromatic polynomial $P_G(x)$. We recall that for a positive integer k, $P_G(k)$ is the number of proper colorings κ of G such that $\kappa(v) \in [k]$ for all $v \in V$. Now $P_G(k)$ is obtained from $X_G(\mathbf{x})$ by setting $x_i = 1$ for $i \in [k]$ and $x_i = 0$ for $i > k$. Stanley's insight is that the structure of G might be related to the expansion of $X_G(\mathbf{x})$ in terms of the various bases for Λ.

Example 9.3.1. Say $V = \{1, 2, 3\}$ and $E = \{12, 23\}$. (When considering graphs, we will write ij for $\{i, j\} \in E$.) Then $G = (V, E)$ is the path graph



There are two types of proper colorings of G. First, for any three colors $i, j, k \in \mathbb{Z}_{>0}$ there are $3! = 6$ proper colorings κ with image $\{i, j, k\}$. Given two colors i, j, there is one proper coloring κ such that $\prod_{v\in V} x_{\kappa(v)} = x_i^2 x_j$, namely, $\kappa(1) = \kappa(3) = i$ and $\kappa(2) = j$. We conclude that

$$\begin{aligned} X_G(\mathbf{x}) &= 6m_{(1,1,1)} + m_{(2,1)} \\ &= 4s_{(3)} + s_{(2,1)} \\ &= 3e_{(3)} + e_{(2,1)}, \end{aligned}$$

with the last two equalities following from direct calculation. We observe that all coefficients in all three expansions above are nonnegative. It follows quickly from the definition that every $X_G(\mathbf{x})$ is m-positive.

Stanley showed that $\omega X_G(\mathbf{x})$ is always p-positive. This is a consequence of his symmetric function analog of a classical formula of Whitney [60] on chromatic polynomials. Indeed, given a graph G on vertex set $[n]$, let Π_G denote the bond lattice of G, that is, the induced subposet of the set partition lattice Π_n consisting of partitions of the set $[n]$ whose blocks induce connected subgraphs of G. Also let $\hat{0}$ denote its smallest element, which is the set partition with all singleton blocks.

Theorem 9.3.2 (Stanley [50, Theorem 2.6 and Corollary 2.7])**.** *Let G be any graph on $[n]$. Then*

$$X_G(\mathbf{x}) = \sum_{\pi\in\Pi_G} \mu_G(\hat{0}, \pi)\, p_{\lambda(\pi)},$$

where μ_G is the Möbius function of the poset Π_G and $\lambda(\pi)$ is the partition of n whose parts are the block sizes of the set partition π. Consequently $\omega X_G(\mathbf{x})$ is p-positive.

The consequence follows from the facts that (1) $\omega p_\lambda = (-1)^{n-\ell(\lambda)} p_\lambda$ for all partitions λ of n and (2) all bond lattices are **geometric lattices**. By a well-known property of geometric lattices, we have $(-1)^{n-|\pi|}\mu_G(\hat{0}, \pi) \geq 0$, where $|\pi|$ is the number of blocks of π.

Not all graphs have e-positive or even Schur-positive chromatic symmetric function. The question of which graphs have such positivity is the main topic of [50]. This will be discussed in the next subsection.

9.3.2 The Stanley–Stembridge Conjectures

We write $a + b$ for a poset that is the union of two chains of respective sizes a and b such that any two comparable elements lie together in one of these two chains. A poset P is called $(a + b)$**-free** if P has no induced subposet

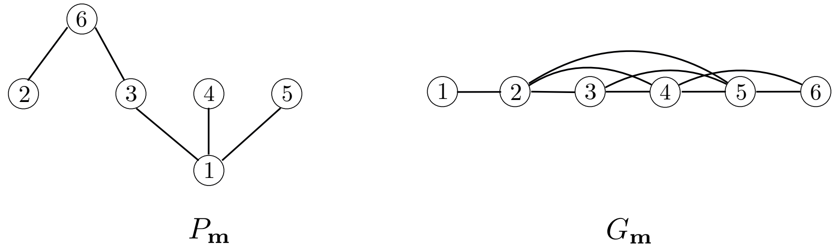


FIGURE 9.1: Naturally labeled unit interval order $P_{\mathbf{m}}$ and naturally labeled unit interval graph $G_{\mathbf{m}}$ for $\mathbf{m} = (2, 5, 5, 6, 6, 6)$.

isomorphic with $a + b$. A **unit interval** order is a poset that is both $(3 + 1)$-free and $(2 + 2)$-free. This term is used because such a poset is isomorphic to a poset whose elements are distinct unit intervals $I_1, I_2, \ldots, I_n$ on the real line, listed in increasing order of left endpoints, with order relation given by $I_i < I_j$ if I_i lies entirely to the left of I_j. A **naturally labeled unit interval order** is a unit interval order on $[n]$ for which the isomorphism takes j to I_j. Every unit interval order is isomorphic with a unique naturally labeled unit interval order (see [48, Proposition 4.2]). A graph G on $[n]$ whose edge set is $\{ij : i \neq j, I_i \cap I_j \neq \emptyset\}$ is called a **naturally labeled unit interval graph.**

Given a poset P, the **incomparability graph** $\mathsf{Inc}(P)$ has vertex set P and edges xy for all pairs x, y that are incomparable in P. Note that a graph is a naturally labeled unit interval graph if and only if it is the incomparability graph of a naturally labeled unit interval order.

For reasons that will become apparent in Section 9.5, it is convenient to call a weakly increasing sequence $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers satisfying $i \leq m_i \leq n$ for all i, a **Hessenberg vector**.

The first of several key points for us is the following lemma.

Lemma 9.3.3 (See [48, Proposition 4.1])**.** *A poset P on $[n]$ is a naturally labeled unit interval order if and only if there is some Hessenberg vector $\mathbf{m} = (m_1, \ldots, m_n)$ such that $i <_P j$ exactly when $j > m_i$. In this case, the edge set of the naturally labeled unit interval graph $\mathsf{Inc}(P)$ is*

$$\{ij : 1 \leq i < j \leq m_i\}.$$

We write $P_{\mathbf{m}}$ for the naturally labeled unit interval order corresponding to $\mathbf{m}$ and $G_{\mathbf{m}}$ for the naturally labeled unit interval graph corresponding to $\mathbf{m}$ as in Lemma 9.3.3. For example, see Figure 9.1 for both the Hasse diagram of the poset $P_{\mathbf{m}}$ and the graph $G_{\mathbf{m}}$ corresponding to $\mathbf{m} = (2, 5, 5, 6, 6, 6)$.

The next result connecting chromatic symmetric functions with monomial immanants was proved by Stanley and Stembridge in Section 5 of [54], although not stated exactly as it is here. Given a pair of partitions $\nu \subseteq \mu$ with $\ell(\mu) = n$, there is an associated Hessenberg vector $\mathbf{m}(\mu/\nu)$ of length n. In

Section 9.3.3, we will define $\mathbf{m}(\mu/\nu)$ and discuss the proof of the following theorem

Theorem 9.3.4. *Let $\nu \subseteq \mu$ be partitions with $|\mu| - |\nu| = N$. Then*

$$\mathsf{ch}(\Gamma^{(N)}_{\mu/\nu}) = \omega X_{G_{\mathbf{m}(\mu/\nu)}}(\mathbf{x}).$$

Now since $\omega(h_\lambda) = e_\lambda$, by Theorem 9.2.6 we have the following result.

Theorem 9.3.5. *Let $\nu \subseteq \mu$ be partitions with $\ell(\mu) = n$ and $|\mu| - |\nu| = N$. Then for all partitions λ of n, the coefficient of e_λ in the e basis expansion of $X_{G_{\mathbf{m}(\mu/\nu)}}(\mathbf{x})$ equals the coefficient of $s_{(N)}$ in the Schur basis expansion of $\mathsf{Imm}_{\mathsf{ch}^{-1}(m_\lambda)}(H_{\mu/\nu})$.*

It is observed in [54, Section 4] that every Hessenberg vector $\mathbf{m}$ is equal to $\mathbf{m}(\mu/\nu)$ for some partitions $\nu \subseteq \mu$. Hence Stembridge's Monomial Immanant Conjecture (Conjecture 9.2.2) implies the following conjecture, which is equivalent to Conjecture 9.2.7 for $\rho = (N)$.

Conjecture 9.3.6. *If P is a unit interval order then $X_{\mathsf{Inc}(P)}(\mathbf{x})$ is e-positive.*

Stanley and Stembridge proposed the following stronger version of Conjecture 9.3.6. This is known as the **Stanley–Stembridge Conjecture**.

Conjecture 9.3.7 (Stanley–Stembridge Conjecture, see [50, Conjecture 5.1] and [54, Conjecture 5.5])**.** *If P is a $(3+1)$-free poset, then $X_{\mathsf{Inc}(P)}(\mathbf{x})$ is e-positive.*

The following result, appearing in a preprint by Guay-Paquet, says that we can restrict our attention to Conjecture 9.3.6.

Theorem 9.3.8 (Guay-Paquet [29, Theorem 5.1])**.** *Conjecture 9.3.6 implies Conjecture 9.3.7.*

Conjecture 9.3.6 (and thus the Stanley–Stembridge Conjecture) has been proved in [33] by Hikita. We delay discussion of this proof to Section 9.4.5 since several additional definitions, introduced in Section 9.4, will be needed. Now we mention earlier results, due to Haiman and Gasharov, respectively, and two additional conjectures.

Although the next result is not stated explicitly in the paper [31] of Haiman, it is proved there implicitly since it is equivalent to the $\rho = (N)$ case of Theorem 9.2.10 (by Theorems 9.3.4).

Theorem 9.3.9 (Haiman, see [31, Theorem 1.4])**.** *If P is a unit interval order, then $X_{\mathsf{Inc}(P)}(\mathbf{x})$ is Schur-positive.*

Gasharov proves the following more general result by giving a combinatorial interpretation of the coefficients in the Schur basis expansion. We discuss this interpretation in Section 9.4.3.

Theorem 9.3.10 (Gasharov [22, Theorem 2])**.** *If P is a $(3+1)$-free poset, then $X_{\mathsf{Inc}(P)}(\mathbf{x})$ is Schur-positive.*

Note that a poset P is $(3+1)$-free if and only if $\mathsf{Inc}(P)$ is claw-free. (A graph G is said to be **claw-free** if no induced subgraph of G is a complete bipartite graph $K_{1,3}$.) The next conjecture is attributed to Gasharov by Stanley in [51] and to Stanley by Gasharov in [23].

Conjecture 9.3.11. *If G is a claw-free graph, then $X_G(\mathbf{x})$ is Schur-positive.*

Although Conjecture 9.3.6 (Hikita's Theorem) addresses Conjecture 9.2.7 only in the case $\rho = (N)$, there is another conjecture of Stanley and Stembridge whose truth, along with Hikita's result, would imply Conjecture 9.2.7 (and thus Stembridge's Monomial Immanant Conjecture). By using the connection between chromatic symmetric functions and immanants given in (9.16) below, we rephrase this conjecture here.

Conjecture 9.3.12 (Stanley and Stembridge [54, Conjecture 5.1]). *Let $\nu \subseteq \mu$ be partitions with $\ell(\mu) = n$ and $|\mu| - |\nu| = N$. If ρ is a partition of N, then there exists a multiset T of length n Hessenberg vectors satisfying*

$$\mathsf{ch}(\Gamma^{\rho}_{\mu/\nu}) = \sum_{\mathbf{m}\in T} \omega X_{G_{\mathbf{m}}}(\mathbf{x}).$$

Conjecture 9.3.12 remains open. It is proved when the Young diagram of μ/ν contains no 2×2 square by Stanley and Stembridge in [54] and when ρ is a hook shape by Nate Lesnevich in [40].

9.3.3 Connection with immanants: the proof of Theorem 9.3.4

Recall that Theorem 9.3.4 connects chromatic symmetric functions with monomial immanants. This subsection is devoted to the proof of Theorem 9.3.4 obtained by Stanley and Stembridge in [54].

Given partitions $\nu \subseteq \mu$ with $\ell(\mu) = n$, we write

$$H_{\mu/\nu} = (a_{ij})_{i,j=1}^{n}.$$

Our first goal is to understand when $a_{ij} = 0$, which will give us a description of those $w \in S_n$ satisfying $\prod_{i=1}^{n} a_{i,w(i)} \neq 0$. By definition,

$$a_{ij} = 0 \text{ if and only if } \mu_i - \nu_j < i - j.$$

This allows the following observations (viewing partitions as infinite sequences by appending 0's):

- If $1 \le i \le j \le n$, then

$$\mu_i - \nu_j \ge \mu_i - \mu_j \ge 0 \ge i - j.$$

- If $\mu_i - \nu_j < i - j$ then
  - $\mu_{i+1} - \nu_j \le \mu_i - \nu_j < i + 1 - j$ whenever $i < n$, and
  - $\mu_i - \nu_{j-1} \le \mu_i - \nu_j < i - (j-1)$ whenever $j > 1$.

It follows that the zeros in $H_{\mu/\nu}$ all live below the main diagonal and form a southwest-justified set of entries. In other words, there exists a Hessenberg vector

$$\mathbf{m}(\mu/\nu) = (m_1, \ldots, m_n)$$

such that

$$a_{ij} = 0 \text{ if and only if } i > m_j.$$

Example 9.3.13. If $\mu = (4,3,2,1)$ and $\nu = (2,1,1)$ hold, then (as seen in (9.8)) $\mathbf{m}(\mu/\nu) = (2,3,3,4)$ and $a_{31} = a_{41} = a_{42} = a_{43} = 0$.

As above, we set $N = |\mu| - |\nu|$. It is clear from the definition of the Kostka numbers that if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is any sequence of integers satisfying $\sum_{j=1}^{n} \alpha_j = N$, then

$$K_{(N),\alpha} = \begin{cases} 1 & \text{if all } \alpha_i \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{9.14}$$

Having obtained Hessenberg vector $\mathbf{m}$ from μ/ν, we define

$$S_n(\mathbf{m}) := \{v \in S_n : v(j) \leq m_j \text{ for all } j \in [n]\}.$$

It follows directly from the definitions that $v \in S_n(\mathbf{m})$ if and only if $\mu + \delta - v(\nu + \delta)$ has no negative entries. So, combining (9.14) with the definition (9.12), we obtain

$$\Gamma^{(N)}_{\mu/\nu}(w) = \frac{n!}{|\mathsf{Conj}(w)|} |\mathsf{Conj}(w) \cap S_n(\mathbf{m})|.$$

Given $w \in S_n$, let us write $\lambda(w)$ for the cycle type of w. (So, for example, if $w = 364152$ in one-line notation, then $w = (134)(26)$ in cycle notation and $\lambda(w) = (3,2,1)$.) It follows from applying the orbit-stabilizer theorem to the action of S_n on itself by conjugation that

$$\frac{n!}{|\mathsf{Conj}(w)|} = z_{\lambda(w)}.$$

We conclude that

$$\mathsf{ch}(\Gamma^{(N)}_{\mu/\nu}) = \sum_{w \in S_n(\mathbf{m})} p_{\lambda(w)}. \tag{9.15}$$

Next we use a general result in [54] on cycle indicators of permutations with restricted position to show that for all Hessenberg vectors $\mathbf{m}$ of length n,

$$\sum_{w \in S_n(\mathbf{m})} p_{\lambda(w)} = \omega X_{G_{\mathbf{m}}}(\mathbf{x}), \tag{9.16}$$

thereby completing the proof of Theorem 9.3.4. To this end, let B be any subset of $[n] \times [n]$ and let

$$S_n(B) := \{w \in S_n : (j, w(j)) \in B \text{ for all } j \in [n]\}.$$

Define the cycle indicator

$$Z_n[B] := \sum_{w \in S_n(B)} p_{\lambda(w)}.$$

We can view any subset S of $[n] \times [n]$ as a directed graph with vertex set $[n]$ and directed edge set $\{(i,j) : (i,j) \in S\}$. Suppose R is a nonattacking rook placement, that is, a subset of $[n] \times [n]$ with at most one element in each row and each column. Clearly every connected component of R is either a path or a cycle. Define the *type* of R to be the pair of partitions (α, β) where the parts of α are the sizes of the connected components that are paths and the parts of β are the sizes of the connected components that are cycles. Given a subset S of $[n] \times [n]$, let $r^n_{\alpha,\beta}(S)$ be the number of nonattacking rook placements of type (α, β) in S.

Theorem 9.3.14 ([54, Theorem 3.2]). *For any $B \subseteq [n] \times [n]$,*

$$Z_n[B] = \sum_{\alpha,\beta} (-1)^{|\beta|} (\prod_{i \geq 1} a_i(\alpha)!) r^n_{\alpha,\beta}(\bar{B})\ \omega(m_\alpha)\, p_\beta,$$

where the sum is over all pairs of partitions (α, β) with $|\alpha| + |\beta| = n$ and $a_i(\alpha)$ is the multiplicity of i in α and $\bar{B}$ is the complement of B in $[n] \times [n]$.

Given a Hessenberg vector $\mathbf{m}$ of length n, let

$$B_{\mathbf{m}} := \{(i,j) \in [n] \times [n] : i \leq m_j\}.$$

Note that $S_n(B_{\mathbf{m}}) = S_n(\mathbf{m})$ and so the cycle indicator on the left hand side of (9.16) is $Z_n[B_{\mathbf{m}}]$, which means we can apply Theorem 9.3.14. Recall that $P_{\mathbf{m}}$ is the poset on $[n]$ defined by $i >_P j$ if and only if $i > m_j$. It is not difficult to see that

$$r_{\alpha,\beta}(\overline{B_{\mathbf{m}}}) = \begin{cases} c_\alpha(P_{\mathbf{m}}) & \text{if } \beta = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

where $c_\alpha(P)$ is the number of ways to partition a poset P into chains whose lengths are given by the parts of α. It therefore follows from Theorem 9.3.14 that

$$\sum_{w \in S_n(\mathbf{m})} p_{\lambda(w)} = \sum_{\alpha \in \mathsf{Par}(n)} (\prod_i a_i(\alpha)!)\, c_\alpha(P_{\mathbf{m}})\, \omega(m_\alpha).$$

The right hand side is equal to $\omega X_{\mathsf{Inc}(P_{\mathbf{m}})}(x)$ since a proper coloring of the incomparability graph of a poset P can be associated with an ordered partition of P into chains. Thus (9.16) holds.

9.4 A quasisymmetric refinement

Here we discuss a refinement of the chromatic symmetric function, which was introduced in our own work (see [47] and [48]). This leads us to a refinement of the Stanley–Stembridge Conjecture (more precisely of Conjecture 9.3.6), which remains open. We present results supporting the refined conjecture as well as more general conjectures. We also discuss Hikita's proof of the Stanley Stembridge conjecture, in which our refinement appears.

9.4.1 Quasisymmetric functions

We review here the definition of the algebra of quasisymmetric functions. The application of quasisymmetric functions to problems in enumerative combinatorics was initiated by Stanley and Ira Gessel (see [24]). We will not discuss this here. .

Given $n \in \mathbb{Z}_{>0}$ and $S \subseteq [n-1]$, let $D(n,S)$ be the set of all nonincreasing functions $f : [n] \to \mathbb{Z}_{>0}$ such that $f(i) > f(i+1)$ whenever $i \in S$. Now define the **fundamental quasisymmetric function**

$$F_{n,S} := \sum_{f \in D(n,S)} \prod_{i=1}^{n} x_{f(i)}.$$

So, for example,

$$F_{3,\{2\}} = \sum_{i \geq j > k} x_i x_j x_k.$$

(Our definition of a fundamental quasisymmetric function differs from the original one appearing in the work [25] of Gessel, but this has no effect on what follows.) Given a ring R, we write Ω_R^n for the R-submodule of $R[[x_1, x_2, x_3, \ldots]]$ generated by $\{F_{n,S} : S \subseteq [n-1]\}$. We define $\Omega_R^0 = R$ and set

$$\Omega_R = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Omega_R^n.$$

It is straightforward to show that Ω_R is closed under multiplication and hence is a $\mathbb{Z}_{\geq 0}$-graded R-algebra, called the **algebra of quasisymmetric functions**. Moreover, the ring Λ_R of symmetric functions with coefficients in R is contained in Ω_R. Indeed, as shown in [25], Ω_R consists of all $f \in R[[\{x_i : i \in \mathbb{Z}_{>0}\}]]$ of bounded degree satisfying

(Q) If $i_1 > \cdots > i_k$ and $j_1 > \cdots > j_k$ and $a_1, \ldots, a_k \in \mathbb{Z}_{>0}$ then the coefficients of $\prod_{\ell=1}^{k} x_{i_\ell}^{a_\ell}$ and $\prod_{\ell=1}^{k} x_{j_\ell}^{a_\ell}$ in f are equal.

Every symmetric function satisfies condition (Q).

9.4.2 The chromatic quasisymmetric function

Given graph $G = ([n], E)$ and proper coloring $\kappa \in K(G)$, we define the **ascent set**

$$\mathsf{Asc}(\kappa) := \{ij \in E : i < j \text{ and } \kappa(i) < \kappa(j)\}$$

and

$$\mathsf{asc}(\kappa) := |\mathsf{Asc}(\kappa)|.$$

So, for example, if $n = 4$ and $E = \{12, 13, 23, 34\}$, $\kappa(1) = \kappa(4) = 3$, $\kappa(2) = 7$, and $\kappa(3) = 2$, then $\mathsf{Asc}(\kappa) = \{12, 34\}$ and $\mathsf{asc}(\kappa) = 2$.

The **chromatic quasisymmetric function** of G is

$$X_G(\mathbf{x}; t) := \sum_{\kappa \in K(G)} t^{\mathsf{asc}(\kappa)} \prod_{i=1}^{n} x_{\kappa(i)}.$$

Two observations can be confirmed with a little thought:

- the chromatic quasisymmetric function is indeed quasisymmetric, lying in $\Omega_{\mathbb{C}[t]}$, and
- if one sets $t = 1$, then $X_G(\mathbf{x}; t)$ becomes the chromatic symmetric function $X_G(\mathbf{x})$.

We present here two examples that appeared in [48, Example 3.2]. In both examples, $n = 3$.

Example 9.4.1. First, consider the case $E = \{12, 23\}$. If κ is a proper coloring of G that uses three different colors, then

$$\mathsf{asc}(\kappa) = \begin{cases} 0 & \kappa(1) > \kappa(2) > \kappa(3), \\ 2 & \kappa(1) < \kappa(2) < \kappa(3), \\ 1 & \text{otherwise,} \end{cases}$$

while if $\kappa(1) = \kappa(3) \neq \kappa(2)$ then $\mathsf{asc}(\kappa) = 1$. We conclude that

$$\begin{aligned} X_G(\mathbf{x}; t) &= (1 + 4t + t^2)m_{111} + tm_{21} \\ &= s_{111} + t(2s_{111} + s_{21}) + t^2 s_{111} \\ &= e_3 + t(e_3 + e_{21}) + t^2 e_3. \end{aligned}$$

In particular, $X_G(\mathbf{x}; t)$ is a symmetric function with coefficients in $\mathbb{C}[t]$, equivalently a polynomial in t with coefficients in Λ. Moreover, using the second interpretation, the coefficient of each power of t in $X_G(\mathbf{x}; t)$ is e-positive. We will return to this phenomenon shortly.

Example 9.4.2. Now consider the case $E = \{13, 23\}$. One can compute that

$$X_G(\mathbf{x}; t) = 2(1 + t + t^2)m_{111} + \sum_{i<j} x_i x_j^2 + t^2 \sum_{i<j} x_i^2 x_j.$$

Comparing Examples 9.4.1 and 9.4.2, we see that, unlike $X_G(\mathbf{x})$, the refinement $X_G(\mathbf{x}; t)$ is not determined by the isomorphism type of G. Moreover, if we think of $X_G(\mathbf{x}; t)$ as an element of $\Omega_{\mathbb{C}}[t]$, then the coefficient of some t^j, which is quasisymmetric, might or might not be symmetric. While this seems discouraging, restricting the structure of G leads to interesting results.

9.4.3 A refinement of the Stanley–Stembridge Conjecture

A key result in [48] is as follows.

Theorem 9.4.3 ([48, Theorem 4.5]). *If G is a naturally labeled unit interval graph then $X_G(\mathbf{x};t) \in \Lambda_{\mathbb{C}[t]}$. Equivalently, $X_G(\mathbf{x};t) \in \Lambda[t]$.*

We mention both rings $\Lambda_{\mathbb{C}[t]}$ and $\Lambda[t]$ in Theorem 9.4.3 because it can be useful to think of $X_G(\mathbf{x};t)$ as a symmetric function with coefficients in $\mathbb{C}[t]$ or as a polynomial in t whose coefficients are symmetric functions.

We sketch the proof of Theorem 9.4.3. It suffices to show that for each $a \in \mathbb{Z}_{>0}$ there is an involution ψ_a on $K(G)$ such that for each $\kappa \in K(G)$,

- $|\kappa^{-1}(a)| = |\psi_a(\kappa)^{-1}(a+1)|$,
- $\psi_a(\kappa)^{-1}(b) = \kappa^{-1}(b)$ if $b \notin \{a, a+1\}$,
- $\mathsf{asc}(\psi_a(\kappa)) = \mathsf{asc}(\kappa)$.

Indeed, if such ψ_a exists then $X_G(\mathbf{x};t)$ is invariant under the action of the transposition exchanging x_a and x_{a+1} and fixing all other x_j.

Given κ and a, let $G_{\kappa,a}$ be the subgraph of G induced on $\kappa^{-1}(a) \cup \kappa^{-1}(a+1)$. Using the fact that G is a naturally labeled unit interval graph, one can show that every connected component of $G_{\kappa,a}$ is a path on which, with the correct choice, vertex labels increase as one moves from one endpoint to the other. We obtain $\psi_a(\kappa)$ by exchanging colors a and $a+1$ on all components of $G_{\kappa,a}$ with oddly many vertices.

With Theorem 9.4.3 in hand, it is natural to ask for the expansion of $X_G(\mathbf{x};t)$ with respect to the various familiar bases of $\Lambda_{\mathbb{C}[t]}$ when G is a naturally labeled unit interval graph. We begin with the promised refinement of the Stanley–Stembridge Conjecture.

Conjecture 9.4.4 (Shareshian and Wachs [47, Conjecture 4.9] (see also [48, Conjecture 5.1])). *If G is a naturally labeled unit interval graph then for every $j \in \mathbb{Z}_{\geq 0}$ the coefficient of t^j in $X_G(\mathbf{x};t) \in \Lambda[t]$ is e-positive.*

Remark 9.4.5. *Conjecture 4.9 in [47] (see also Conjecture 5.1 in [48]) is stronger than Conjecture 9.4.4. With G as in the conjecture, the degree in t of $X_G(\mathbf{x};t)$ is $|E(G)|$. According to [48, Corollary 2.8], $X_G(\mathbf{x};t)$ is* **palindromic** *in t, that is, the coefficients of t^j and $t^{|E(G)|-j}$ are equal for all $0 \leq j \leq |E(G)|$. We conjectured that $X_G(\mathbf{x};t)$ is also* e-**unimodal**; *that is, for $0 \leq j < \frac{|E(G)|}{2}$, we obtain an e-positive symmetric function upon subtracting the coefficient of t^j from the coefficient of t^{j+1}.*

Conjecture 9.4.4 remains open, as does the associated unimodality conjecture. Both are known to be true in certain cases, which include the path graphs discussed in Example 9.4.6 below, and incomparability graphs of naturally labeled unit interval posets that have no induced 3-chain, see [13] and [32].

Example 9.4.6. For $n \in \mathbb{Z}_{>0}$, set $\mathbf{m}(n) = (2, 3, \ldots, n, n)$, that is, $m_i = i + 1$ for $i \in [n-1]$. In this case $G_{\mathbf{m}(n)}$ is a path with edges $\{i, i+1\}$. It is shown in [46] (see also [48, Appendix C]) that

$$\sum_{n\geq 0} X_{G_{\mathbf{m}(n)}}(\mathbf{x};t)z^n = \frac{(1-t)\sum_{n\geq 0} e_n z^n}{\sum_{n\geq 0} e_n (zt)^n - t\sum_{n\geq 0} e_n z^n}. \tag{9.17}$$

(This equation is a t-analog of [50, Proposition 5.3].) Direct calculation shows that e-positivity and e-unimodality hold when $G = G_{\mathbf{m}(n)}$. We remark that the formula given in (9.17) was first derived in [46] for a symmetric function analog in $\Lambda[t]$ of the classical Eulerian polynomial in $\mathbb{Z}_{\geq 0}[t]$. Stanley observed that by P-partition reciprocity, one of our characterizations of this symmetric function was the same (after application of ω) as a descent enumerator of words with distinct adjacent letters; see Section 7 of [46]. This descent enumerator was, in turn, the same as what would later become the chromatic quasisymmetric function of the path. In fact, this descent enumerator was the motivation behind our introduction of chromatic quasisymmetric functions in the first place.

There are precise results about the expansion of $X_G(\mathbf{x};t)$ (with G as in Conjecture 9.4.4) in the Schur and power sum bases. Our result on Schur expansion involves a type of tableau introduced by Gasharov.

Definition 9.4.7 (Gasharov [22]). *Given a poset P on $[n]$ and a partition λ of n, a **P-tableau** of shape λ is a filling of the Young diagram of λ with the elements of $[n]$, each appearing once, in such a way that, for $i, j \in [n]$,*

- *if j appears directly to the right of i then $i <_P j$, and*
- *if j appears directly below i then $i \not>_P j$.*

Write $\mathcal{T}_P(\lambda)$ for the set of P-tableaux of shape λ.

Example 9.4.8. If $\mathbf{m} = (3, 5, 5, 6, 7, 8, 8, 8)$ and $P = P_{\mathbf{m}}$, then

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 3 & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$ is a P-tableau of shape $(3, 2, 2, 1)$;

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$ is not a P-tableau, since $1 \not<_P 3$;

$$\begin{array}{|c|c|}\hline 2 & 8 \\ \hline 1 & 5 \\ \hline 4 & 7 \\ \hline 3 & 6 \\ \hline\end{array}$$ is not a P-tableau since $8 >_P 5$.

Definition 9.4.9. *Given a graph G on $[n]$ and a poset P on $[n]$, a* G**-inversion** *of a P-tableau T is an edge ij of G such that*

- *$i < j$ in the usual order on $\mathbb{Z}_{>0}$;*
- *i appears in a lower row of T than does j.*

We write $\mathrm{inv}_G(T)$ for the number of G-inversions of T.

Example 9.4.10. With $\mathbf{m}$ as in Example 9.4.8 and $G = G_{\mathbf{m}}$, if T is the first tableau appearing in the example, the G-inversions of T are $23,\ 25,\ 35,\ 45,\ 46,\ 68,$ and 78. So, $\mathrm{inv}_G(T) = 7$.

The next theorem reduces to Gasharov's Schur basis expansion in [22] of the chromatic symmetric functions for unit interval orders when t is set equal to 1; see Theorem 9.3.10.

Theorem 9.4.11 ([48, Theorem 6.3])**.** *Let G be a naturally labeled unit interval graph on $[n]$ and let P be such that $\mathsf{Inc}(P) = G$. Then*

$$X_G(\mathbf{x}; t) = \sum_{\lambda \in \mathsf{Par}(n)} \left(\sum_{T \in \mathcal{T}_P(\lambda)} t^{\mathrm{inv}_G(T)} \right) s_\lambda.$$

Consequently, the coefficient of each power of t in $X_G(\mathbf{x}; t) \in \Lambda[t]$ is Schur-positive.

Example 9.4.12. Say $\mathbf{m} = (2, 3, 3)$. Then $G := G_{\mathbf{m}}$ is the path with edges 12 and 23 and the only relation in $P := P_{\mathbf{m}}$ is $1 < 3$. The P-tableaux appear in the table below.

T	λ	$\mathrm{inv}_G(T)$
$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$	$(1,1,1)$	0
$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array}$	$(1,1,1)$	1
$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline 3\\\hline\end{array}$	$(1,1,1)$	1
$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$(1,1,1)$	2
$\begin{array}{\|c\|c\|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array}$	$(2,1)$	1

We conclude that

$$X_G(\mathbf{x};t) = (1+2t+t^2)s_{111} + ts_{21},$$

in agreement with Example 9.4.1.

The proof of Theorem 9.4.11 uses the same idea as Gasharov's proof of Theorem 9.3.10, although key details are different due to the necessity of keeping track of inversions. We use the Jacobi–Trudi identity and a sign-reversing involution. Here is a short sketch. Write $c_\lambda(t)$ for the coefficient of s_λ in the Schur decomposition of $X_G(\mathbf{x};t)$. We aim to show that

$$c_\lambda(t) = \sum_{T\in\mathcal{T}_P(\lambda)} t^{\mathrm{inv}_G(T)}. \tag{9.18}$$

Extending the inner product $\langle\cdot,\cdot\rangle$ from Λ to $\Lambda[t]$ and applying the Jacobi–Trudi identity, we get

$$\begin{aligned} c_\lambda(t) &= \langle X_G(\mathbf{x};t), s_\lambda\rangle \\ &= \langle X_G(\mathbf{x};t), \det H_\lambda\rangle. \end{aligned}$$

Expanding $\det H_\lambda$ as a signed sum of complete homogeneous symmetric functions and recalling that the bases $\{h_\lambda\}$ and $\{m_\mu\}$ are dual with respect to $\langle\cdot,\cdot\rangle$, we see that we can determine $c_\lambda(t)$ by calculating a signed sum of coefficients of certain m_μ in the monomial decomposition of $X_G(\mathbf{x};t)$. Such coefficients are determined in turn by counting certain proper colorings of G. We can associate to each such proper coloring an $n\times n$ array filled with elements of $[n]\cup\{0\}$ such that each element of $[n]$ appears exactly once and the nonzero entries in any row are left justified, form a chain in P, and appear in increasing order. (The nonzero elements in row i are those vertices that receive color i in the corresponding coloring.) We define $\mathrm{inv}_G(A)$ for such filled arrays as it is defined for P-tableaux, ignoring zeros. Finally, there is a sign-reversing and inv_G-preserving involution on the set of such arrays with fixed-point set consisting of those arrays that become P-tableaux upon removing zeros. Such arrays have positive sign and the truth of (9.18) follows.

It follows from the discussion directly below Conjecture 9.4.4, that the polynomial $c_\lambda(t) = \sum_{T\in\mathcal{T}_P(\lambda)} t^{\mathrm{inv}_G(T)}$ in (9.18) is palindromic with center of symmetry $\frac{|E(G)|}{2}$. We will see in Section 9.5.5 that unimodality (and palindromicity) of the coefficients of $c_\lambda(t)$, and therefore Schur-unimodality of $X_G(\mathbf{x};t)$), follows from a connection with Hessenberg varieties. Although palindromicity follows from combinatorial considerations, we do not have a direct combinatorial proof of unimodality.

Example 9.4.13. For $\mathbf{m} = (1,2,\ldots,n)$, the poset $P_\mathbf{m}$ is the natural total order on $[n]$ and its incomparability graph $G_\mathbf{m}$ is the graph on $[n]$ with no edges.

Note that for this special case, Theorem 9.4.11 reduces to the well-known expansion in the Schur basis

$$\mathbf{e}_{1^n} = \sum_{\lambda \in \mathsf{Par}(n)} f^\lambda \mathsf{s}_\lambda, \tag{9.19}$$

where f^λ is the number of standard Young tableaux of shape λ.

The famous Robinson–Schensted–Knuth (RSK) correspondence (see [53, Chapter 7]) provides a direct proof of (9.19) by giving a bijection from words to pairs of tableaux (one standard and the other semistandard). A long-standing open problem (initially for $t = 1$) is to find a RSK-like bijection that proves the general case of the identity in Theorem 9.4.11. Some progress has been made on this problem by Thomas Sundquist, David Wagner, and Julian West in [56], and Timothy Chow in [15] for $t = 1$, and by Dongkwan Kim and Pavlo Pylyavskyy in [39] and Kyle Celano in [11], [12] for general t.

We observe that Theorem 9.4.11 supports Conjecture 9.4.4 somewhat, since every e-positive symmetric function is Schur-positive. A bit of further support is provided by the following result.

Theorem 9.4.14 ([48, Corollary 7.2]). *Let G be a natural unit interval graph on $[n]$. Then the coefficient of $e_{(n)}$ in the e-basis expansion of $X_G(\mathbf{x};t)$ is $[n]_t \prod_{i=1}^{n-1}[m_i - i]_t$.*

Theorem 9.4.14 is equivalent to Theorem 7.1 of [48], which gives the coefficient of $p_{(n)}$ in the power-sum expansion of $X_G(\mathbf{x};t)$. Next we discuss general coefficients in the power sum expansion of $X_G(\mathbf{x};t)$, thereby providing further support for Conjecture 9.4.4. For this purpose it will be convenient to apply the involution ω. Let us first restate the conjecture with ω involved.

Conjecture 9.4.15. *If G is a naturally labeled unit interval graph then $\omega X_G(\mathbf{x};t)$ is h-positive.*

Every h-positive symmetric function is p-positive; hence our goal is to show p-positivity of $\omega X_G(\mathbf{x};t)$. In order to state our result, we need the following definitions.

Definition 9.4.16. *For a permutation $w \in S_n$ and a graph G, let $\mathrm{inv}_G(w)$ be the number of edges ij of G such that*

- *$i < j$;*
- *i appears to the right of j in w (written in one-line notation).*

Note that for any graph G and P-tableau T,

$$\mathrm{inv}_G(T) = \mathrm{inv}_G(w(T)),$$

where $w(T)$ is the permutation obtained by reading the rows of T from left to right and from top down.

Example 9.4.17. Say $n = 4$ and $G = ([4], E)$, where $E = \{12, 13, 23, 34\}$. If $w = 3142$ then $\mathrm{inv}_G(w) = 2$, since 13 and 23 lie in E and 3 appears before 1 and 2 in w (and no other edges ij satisfy the given conditions).

Definition 9.4.18. *For a partition $\mu = (\mu_1, \dots \mu_\ell)$ of n and a naturally labeled unit interval order P on $[n]$,*

- *$\mathcal{N}_{P,\mu}$ consists of those $w \in S_n$ such that when the one-line representation for w is broken into segments of lengths $\mu_1, \dots, \mu_\ell$, each segment contains no entries w_i, w_{i+1} such that $w_i >_P w_{i+1}$ and each segment contains no nontrivial left-to-right P-maximum. (A nontrivial left-to-right P-maximum is some w_j that does not appear first in its segment but satisfies $w_j >_P w_i$ for all $i < j$ such that w_i appears in the same segment as w_j.)*

- *$\widetilde{\mathcal{N}}_{P,\mu}$ consists of those $w \in S_n$ such that when the one-line representation for w is broken into segments of lengths $\mu_1, \dots, \mu_\ell$, each segment starts with its smallest element and contains no entries w_i, w_{i+1} such that $w_i <_P w_{i+1}$.*

Example 9.4.19. Say $\mu = (4, 2, 2)$, $\mathbf{m} = (4, 5, 5, 6, 7, 7, 8, 8)$ and $P = P_{\mathbf{m}}$. The permutation $w = 46285713$ is broken by μ into segments 4628, 57 and 13. We see that $w \notin \mathcal{N}_{P,\mu}$ since $6 >_P 2$. Similarly, $w = 24785613$ is broken into segments 2478, 56 and 13 and does not lie in $\mathcal{N}_{P,\mu}$ since $7 >_P 2$ and $7 >_P 4$. The permutation $w = 53784621$ lies in $\mathcal{N}_{P,\mu}$.

Example 9.4.20. With $\mu = (4, 2, 2)$, $\mathbf{m} = (4, 5, 5, 6, 7, 7, 8, 8)$, and $P = P_{\mathbf{m}}$ as above, $15782346 \notin \widetilde{\mathcal{N}}_{P,\mu}$ since $1 <_P 5$, and $31245678 \notin \widetilde{\mathcal{N}}_{P,\mu}$ since 3 is not the smallest element of $\{1, 2, 3, 4\}$. On the other hand, $46781325 \in \widetilde{\mathcal{N}}_{P,\mu}$.

The first expression for the expansion of $\omega X_{G_{\mathbf{m}}}(\mathbf{x}; t)$ in the power sum basis appearing in the next result is Theorem 4 in the paper [7] of Christos Athanasiadis (and was Conjecture 7.6 in [48]). The case $\mu = (n)$ of this result was Lemma 7.4 of [48]. The equality of the two given expressions is Proposition 7.8 of [48]. Athanasiadis' proof involves a technique of Ron Adin and Yuval Roichman [4] for obtaining an expansion of a symmetric function in the power-sum symmetric functions from an expansion in the fundamental quasisymmetric basis. Athanasiadis applies this technique to our expansion [48, Theorem 3.1] of chromatic quasisymmetric functions of incomparability graphs in the fundamental quasisymmetric basis.

Theorem 9.4.21 ([7, Theorem 4] and [48, Proposition 7.8])**.** *Let P be a naturally labeled unit interval order on $[n]$ with incomparability graph G and write*

$$\omega X_G(\mathbf{x}; t) = \sum_{\mu \in \mathsf{Par}(n)} d_\mu(t) \frac{p_\mu}{z_\mu}.$$

Then for each $\mu = (\mu_1, \dots, \mu_\ell) \in \mathsf{Par}(n)$,

$$d_\mu(t) = \sum_{w \in \mathcal{N}_{P,\mu}} t^{\mathrm{inv}_G(w)}$$

$$= \prod_{i=1}^{\ell}[\mu_i]_t \sum_{w\in\widetilde{\mathcal{N}}_{P,\mu}} t^{\mathrm{inv}_G(w)},$$

where for all $j \geq 0$,

$$[j]_t := 1 + t + \cdots + t^{j-1}.$$

Example 9.4.22. Say $n = 3$ and $\mathbf{m} = (2,3,3)$. So the only relation in $P := P_{\mathbf{m}}$ is $1 < 3$ and the edge set of $G := G_{\mathbf{m}}$ is $\{12, 23\}$. First we calculate $\mathrm{inv}_G(w)$ for each $w \in S_3$.

w	$\mathrm{inv}_G(w)$
123	0
132	1
213	1
231	1
312	1
321	2

Next we determine $\mathcal{N}_{P,\mu}$ and $\widetilde{\mathcal{N}}_{P,\mu}$ for each partition μ of 3.

μ	$\mathcal{N}_{P,\mu}$	$\widetilde{\mathcal{N}}_{P,\mu}$
$(1,1,1)$	S_3	S_3
$(2,1)$	$\{123, 213, 231, 321\}$	$\{123, 231\}$
(3)	$\{123, 213, 321\}$	$\{123\}$

We conclude that

$$\omega X_G(\mathbf{x};t) = (1+4t+t^2)\frac{p_{111}}{6} + (1+2t+t^2)\frac{p_{21}}{2} + (1+t+t^2)\frac{p_3}{3}.$$

It is straightforward to check that this is consistent with Example 9.4.1.

Note that the chromatic quasisymmetric function in Example 9.4.22 is p-unimodal. The question of p-unimodality of $\omega X_G(\mathbf{x};t)$ in general remains open. Such p-unimodality would follow from e-unimodality of $X_G(\mathbf{x};t)$.

9.4.4 Chromatic quasisymmetric functions that are symmetric

In her Ph.D. dissertation [20] (see also [19]), Brittney Ellzey extends the domain of applicability of Theorem 9.4.21 to graphs that are not incomparability graphs of posets. She proves that if G is any labeled graph whose chromatic quasisymmetric function $X_G(\mathbf{x};t)$ is symmetric then $\omega X_G(\mathbf{x};t)$ is p-positive. Moreover, she gives the coefficients in the power-sum expansion of $\omega X_G(\mathbf{x};t)$. However, these coefficients have a different formulation than that of Theorem 9.4.21, which involves a poset P.

In order to state Ellzey's result, we need the following definitions. Let $G = ([n], E)$ be any labeled graph.

- For a permutation $w = w_1 w_2 \cdots w_n, \in S_n$ (in one line notation),
  - assign a rank rk to each entry w_i of w as follows: $\mathrm{rk}(w_i)$ is the length of the longest subword $w_{i_1} w_{i_2} \cdots w_{i_k}$ of w such that $w_{i_k} = w_i$ and $w_{i_j} w_{i_{j+1}}$ is an edge of G for all $j \in [k-1]$.
  - define a lexicographic order $<_G$ on the entries of w by letting $w_i <_G w_j$ if $\mathrm{rk}(w_i) < \mathrm{rk}(w_j)$ or if $\mathrm{rk}(w_i) = \mathrm{rk}(w_j)$ and $w_i < w_j$.
- For a partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of n, the set $\mathcal{N}_{G,\mu}$ consists of all $w \in S_n$ such that when the one-line representation for w is broken into segments $v_1, \ldots, v_\ell$ of respective lengths $\mu_1, \ldots, \mu_\ell$, each v_k satisfies:
  - v_k contains no entries w_i, w_{i+1} such that $w_i >_G w_{i+1}$;
  - for each entry w_j of v_k, if w_j is not the leftmost entry of v_k then there is an entry w_i of v_k for which $i < j$ and $w_i w_j$ is an edge of G.

Example 9.4.23. Let $\mu = (3, 2)$ and $G = ([5], \{12, 23, 34, 45, 15)\})$.

- If $w = 51432$ then $w \in \mathcal{N}_{G,\mu}$. Indeed, the segmented w, with ranks given as superscripts, is $5^1 1^2 4^2 \mid 3^3 2^4$. From this we see that both conditions of the definition of $\mathcal{N}_{G,\mu}$ hold.
- If $w = 54132$ then $w \notin \mathcal{N}_{G,\mu}$ since the segmented ranked w, which is $5^1 4^2 1^2 \mid 3^3 2^4$, indicates that the first condition of the definition is violated by $4 >_G 1$.
- If $w = 13524$ then $w \notin \mathcal{N}_{G,\mu}$ since the segmented ranked w, which is $1^1 3^1 5^2 \mid 2^2 4^3$, indicates that the second condition of the definition is violated in the first segment v_1 by 3 and in the second segment v_2 by 4.

Ellzey proves, as a special case of Theorem 9.4.24 below, that for any labeled graph G on $[n]$, if $X_G(\mathbf{x}, t)$ is symmetric (i.e., the coefficient of each t^j is symmetric) then

$$\omega X_G(\mathbf{x}; t) = \sum_{\mu \in \mathsf{Par}(n)} \left(\sum_{w \in \mathcal{N}_{G,\mu}} t^{\mathrm{inv}_G(w)} \right) \frac{p_\mu}{z_\mu}, \tag{9.20}$$

where inv_G is defined in Definition 9.4.16. Ellzey's result begs the question of whether there are actually any labeled graphs outside the class of natural unit interval graphs, whose chromatic quasisymmetric function is symmetric. In [21, Section 3] Ellzey and Wachs prove that the labeled cycle

$$C_n := ([n], \{12, 23, \ldots, (n-1)n, 1n\})$$

is such a labeled graph and that $X_{C_n}(\mathbf{x}; t)$ is e-positive. More recently, a more general class of such graphs was obtained by Maria Gillespie, Joseph Pappe, and Kyle Salois in [26] with e-positivity (and Schur-positivity) left open.

Ellzey shows that the domain of applicability of Theorem 9.4.21 can be extended much further. One can view a labeled graph $G = ([n], E)$ as an acyclic directed graph $\hat{G} = ([n], \hat{E})$ by orienting each edge $ij \in E$ as (i, j), when $i < j$. That is, $\hat{E} = \{(i, j) \in [n] \times [n] : i < j \text{ and } ij \in E\}$. This leads to a wider class of chromatic quasisymmetric functions, which we describe next.

Given a directed graph D with vertex set $V(D)$ and edge set $E(D)$, let $\bar{D}$ be the undirected graph obtained from D by removing the directions on edges. A **proper coloring** of the digraph D is defined to be the same as that of the underlying graph $\bar{D}$. The **ascent set** of a proper coloring κ of D is now

$$\mathsf{Asc}_D(\kappa) := \{(u, v) \in E(D) : \kappa(u) < \kappa(v)\}$$

and let $\mathsf{asc}_D(\kappa) = |\mathsf{Asc}_D(\kappa)|$. The **chromatic quasisymmetric function** of the digraph D is defined as,

$$X_D(\mathbf{x}; t) := \sum_{\kappa \in K(\bar{D})} t^{\mathsf{asc}_D(\kappa)} \prod_{v \in V(D)} x_{\kappa(v)}.$$

Note that if G is a labeled graph on $[n]$ then $X_G(\mathbf{x}; t) = X_{\hat{G}}(\mathbf{x}; t)$; thus this directed graph version provides a larger class of chromatic quasisymmetric functions.

Equation (9.20) is the special case of the following result for acyclic digraphs.

Theorem 9.4.24 (Ellzey [20, Theorem 5.3.5]). *Let D be a directed graph on $[n]$. If $X_D(\mathbf{x}; t)$ is symmetric then*

$$\omega X_D(\mathbf{x}; t) = \sum_{\mu \in \mathsf{Par}(n)} \left(\sum_{w \in \mathcal{N}_{\bar{D},\mu}} t^{\mathrm{inv}_D(w)} \right) \frac{p_\mu}{z_\mu},$$

where $\mathrm{inv}_D(w)$ is the number of directed edges (w_j, w_i) of D such that $i < j$. Consequently $\omega X_D(\mathbf{x}; t)$ is p-positive.

Ellzey's proof is based on Athanasiadis' proof for the special case of incomparability graphs. As mentioned above, our expansion of $X_{\mathsf{Inc}(P)}(\mathbf{x}; t)$ in the fundamental basis played a role in Athanasiadis' proof. However since our expansion applied only to incomparability graphs, Ellzey needed to obtain such an expansion that worked for all directed graphs and she needed a sign-reversing involution that wasn't needed for the special case. By following the ideas in Ellzey's proof, a subsequent quasisymmetric generalization of Theorem 9.4.24 was obtained in [6].

Ellzey considers a class of digraphs called **circular indifference digraphs** and shows that these digraphs have symmetric chromatic quasisymmetric functions. The circular indifference digraphs are a circular analog of the natural unit interval graphs. (Unit interval graphs also called **indifference graphs**.) Moreover, this class includes the natural unit interval graphs since the natural unit interval graphs are essentially the **acyclic** circular indifference digraphs. It also includes the directed cycle

$$DC_n := ([n], \{(1,2), (2,3), \dots, (n-1,n), (n,1)\}),$$

but not the labeled cycle C_n, nor the more general labeled graphs obtained in [26]. In the original paper [50] in which the chromatic symmetric functions were first introduced, Stanley also speculates (in Section 5) that the undirected version of these graphs, which are called **circular indifference graphs**, might have e-positive chromatic symmetric function and gives an affirmative answer for the n-cycle. Ellzey does the same for the refined version with the following result, whose $t = 1$ case was obtained by Stanley in [50, Proposition 5.4].

Theorem 9.4.25 (Ellzey [20, Theorem 5.5.2])**.** *We have*

$$\sum_{n\geq 2} X_{DC_n}(\mathbf{x};t)z^n = \frac{\sum_{k\geq 2} kt[k-1]_t e_k z^k}{1 - \sum_{k\geq 2} t[k-1]_t e_k z^k}.$$

Consequently, $X_{DC_n}(\mathbf{x};t)$ is e-positive and e-unimodal.

Stanley's proof of Theorem 9.4.25 for the $t = 1$ case and Ellzey's more involved proof for the general case make use of a technique called the **Transfer-Matrix Method**. A subsequent alternative proof was given by Per Alexandersson and Greta Panova in [5].

We do not know of any digraph whose chromatic quasisymmetric function is symmetric but not e-positive. This, together with results discussed above, suggests that the following question might have a positive answer. A positive answer would imply that our refinement of the Stanley–Stembridge conjecture is true, or more generally that every circular indifference digraph has e-positive chromatic quasisymmetric function, which would answer Stanley's original question on circular indifference graphs.

Question 9.4.26. *Let D be a digraph. Is symmetry of $X_D(\mathbf{x};t)$ a sufficient condition for e-positivity?*

In Section 9.5, we discuss a geometric interpretation (involving Hessenberg varieties) of the chromatic quasisymmetric functions of natural unit interval orders. We ask here whether this interpretation can be extended in some way to include other symmetric chromatic quasisymmetric functions. We ask a similar question for the connection with Hecke algebra traces discussed in Section 9.6.

9.4.5 The modular law of Abreu and Nigro, and Hikita's proof of the Stanley–Stembridge Conjecture

As mentioned earlier, the Stanley–Stembridge Conjecture (Conjecture 9.3.6) has been proved by Hikita. The chromatic quasisymmetric function $X_G(\mathbf{x};t)$ is used in his proof. However, the refined Conjecture 9.4.4 remains open.

A key component of Hikita's proof is the modular law for $X_G(x,t)$ of Alex Abreu and Antonio Nigro, which we discuss here. A version of the modular law that applies when $t = 1$ was given earlier by Guay-Paquet in [29] and used therein to prove Theorem 9.3.8.

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a Hessenberg vector with associated naturally labeled unit interval graph $G := G_{\mathbf{m}}$. We observe that the transposition $(i, i+1) \in S_n$ induces an automorphism of G if and only if

$$N_G(i) \setminus \{i+1\} = N_G(i+1) \setminus \{i\}, \tag{9.21}$$

where $N_G(j)$ is the neighborhood of j in G. Equivalently, $(i, i+1)$ induces an automorphism of G if and only if $m_i = m_{i+1}$ and there is no $k \in [n]$ such that $m_k = i$.

Example 9.4.27. If $\mathbf{m} = (3,4,4,7,7,8,8,8)$ then the transpositions $(2,3)$ and $(6,7)$ induce automorphisms of $G_{\mathbf{m}}$. Although $m_4 = m_5 = 7$, the transposition $(4,5)$ does not induce an automorphism, since 2 (like 3) is a neighbor of 4 but not a neighbor of 5.

Definition 9.4.28. *Suppose (9.21) holds for $G = G_{\mathbf{m}}$ and $i \in [n-1]$.*

- *If $m_{i-1} < m_i$ and $m_{i+1} > i+1$ then let*
  - *$G^-_{u(i)}$ be the graph obtained from G by removing the edge $\{i, m_{i+1}\}$,*
  - *$G^{--}_{u(i)}$ be the graph obtained from $G^-_{u(i)}$ by removing the edge $\{i+1, m_{i+1}\}$,*

  and call $(G, G^-_{u(i)}, G^{--}_{u(i)})$ an **upward modular triple with respect to** i.
- *If there is some $j < i$ with $m_{j-1} < m_j = i+1$ (we set $m_0 = 0$) then let*
  - *$G^-_{d(i)}$ be the graph obtained from G by removing the edge $\{j, i+1\}$,*
  - *$G^{--}_{d(i)}$ be the graph obtained from $G^-_{d(i)}$ by removing the edge $\{j, i\}$,*

  and call $(G, G^-_{d(i)}, G^{--}_{d(i)})$ a **downward modular triple with respect to** i.
- *A modular triple with respect to $i \in [n-1]$ is defined to be a triple (G, G^-, G^{--}) of graphs on vertex set $[n]$ that is either an upward modular triple or a downward modular triple with respect to i*

We make the following observations.

- If (G, G^-, G^{--}) is a modular triple, then there exist Hessenberg vectors $\mathbf{m}, \mathbf{m}^-, \mathbf{m}^{--}$ such that $G = G_{\mathbf{m}}$, $G^- = G_{\mathbf{m}^-}$, and $G^{--} = G_{\mathbf{m}^{--}}$. Indeed, $\mathbf{m}$ has been given. If the given triple is upward modular with respect to i, then $\mathbf{m}^-$ is obtained from $\mathbf{m}$ by reducing m_i by 1 and m^{--} is obtained from $\mathbf{m}$ by reducing both m_i and m_{i+1} by 1. If the given triple is downward modular with respect to i and $m_{j-1} < m_j = i+1$

then $\mathbf{m}^-$ is obtained from $\mathbf{m}$ by reducing m_j by 1 and $\mathbf{m}^{--}$ is obtained from $\mathbf{m}$ by reducing m_j by 2. (We have used the conditions $m_{i-1} < m_i$ and $m_{j-1} < m_j$.)

- If (G, G^-, G^{--}) is a modular triple with respect to i then $(i, i+1)$ induces an automorphism of G^{--}, since G^{--} is obtained from G by removing some vertex from both $N_G(i)$ and $N_G(i+1)$.

A basic t-version of the modular law, due to Abreu and Nigro (see [1]), is as follows.

Theorem 9.4.29. *If (G, G^-, G^{--}) is a modular triple, then*

$$(1+t)X_{G^-}(\mathbf{x};t) = tX_{G^{--}}(\mathbf{x};t) + X_G(\mathbf{x};t). \tag{9.22}$$

Example 9.4.30. If $\mathbf{m} = (3,3,3)$, then $G = G_{\mathbf{m}}$ is the complete graph K_3 and has automorphism group S_3. If we take $i = 1$, we get an upward modular triple with $E(G^-) = \{12, 23\}$ and $E(G^{--}) = \{12\}$. We have calculated already that

$$X_{G^-}(\mathbf{x};t) = e_3 + (e_3 + e_{21})t + e_3t^2.$$

Direct inspection shows that

$$X_G(\mathbf{x};t) = e_3(1 + 2t + 2t^2 + t^3)$$

and

$$X_{G^{--}}(\mathbf{x};t) = e_{21}(1+t).$$

A straightforward calculation shows that Theorem 9.4.29 holds in this case.

If we take $i = 2$ (hence $j = 1$) then we get a downward modular triple with G^- as in the case $i = 1$ and $E(G^{--}) = \{23\}$. Calculations that show Theorem 9.4.29 holds are the same as those in the case $i = 1$.

Remark 9.4.31. *It follows from Theorem 9.4.29 that when (G, G^-, G^{--}) is a modular triple, $tX_{G^{--}}(\mathbf{x};t) + X_G(\mathbf{x};t)$ is divisible by $1+t$ in $\Lambda_{\mathbb{Z}[t]}$. In fact, both $X_{G^{--}}(\mathbf{x};t)$ and $X_G(\mathbf{x};t)$ are divisible by $1+t$. This follows from Theorem 9.4.11: if $(i, i+1)$ induces an automorphism of $\mathsf{Inc}(P)$ and T is a P-tableau, then we obtain another P-tableau of the same shape by exchanging the positions of i and $i+1$, and the two P-tableaux have inversion numbers that differ by 1.*

It turns out that Example 9.4.30 is representative of the general case. It is straightforward to calculate that, for any n,

$$X_{K_n}(\mathbf{x};t) = e_n[n]_t!, \tag{9.23}$$

where $[n]_t! := [n]_t[n-1]_t \cdots [1]_t$. We can extend the definition of $X_G(\mathbf{x};t)$ so that its domain is the set of all graphs with totally ordered vertex set. In this setting, it is straightforward to show that if G has connected components $G^1, \ldots, G^k$ then

$$X_G(\mathbf{x};t) = \prod_{i=1}^{k} X_{G^i}(\mathbf{x};t). \tag{9.24}$$

Abreu and Nigro proved the following result, which we will also call the modular law.

Theorem 9.4.32 (See [1, Theorem 1.1]). *Let $\mathbb{M}(n)$ be the set of Hessenberg vectors of length n and set $\mathbb{M} = \bigcup_{n=0}^{\infty} \mathbb{M}(n)$. If a function $\Phi : \mathbb{M} \to \Lambda_{\mathbb{C}[t]}$ satisfies all of the conditions*

- $\Phi((n, n, \dots, n)) = e_n[n]_t!$ *for all n;*
- *if $G_{\mathbf{m}}$ has connected components $G_{\mathbf{m}}^1 \cdots G_{\mathbf{m}}^k$ and each $G_{\mathbf{m}}^j$ is isomorphic with some $G_{\mathbf{m}_j}$, then $\Phi(\mathbf{m}) = \prod_{j=1}^{k} \Phi(\mathbf{m}_j)$;*
- *the equality $(1+t)\Phi(\mathbf{m}^-) = t\Phi(\mathbf{m}^{--}) + \Phi(\mathbf{m})$ holds whenever $(G_{\mathbf{m}}, G_{\mathbf{m}^-}, G_{\mathbf{m}^{--}})$ is a modular triple,*

then

$$\Phi(\mathbf{m}) = X_{G_{\mathbf{m}}}(\mathbf{x}; t)$$

for all $\mathbf{m} \in \mathbb{M}$.

Hikita's proof of Conjecture 9.3.6, given the modular law, is clever and elementary. It begins with a change in the notation used to describe a natural unit interval graph. Given a Hessenberg vector $\mathbf{m} = (m_1, \dots, m_n)$, Hikita uses in its place the vector $\mathbf{e} := \mathbf{e}(\mathbf{m}) = (\mathsf{e}(1), \dots, \mathsf{e}(n))$, where

$$\mathsf{e}(i) := n - m_{n+1-i}.$$

For example, if $\mathbf{m} = (2,3,5,5,5)$ then $\mathbf{e}(\mathbf{m}) = (0,0,0,2,3)$. Hikita then defines[1] the polynomials $c_\lambda(\mathbf{e}; t)$ by

$$X_{G_{\mathbf{m}}}(\mathbf{x}; t) = \sum_{\lambda \in \mathsf{Par}(n)} c_\lambda(\mathbf{e}; t) e_\lambda;$$

and sets

$$|\mathbf{e}| := \sum_{i=1}^{n} \mathsf{e}(i)$$

and

$$|\mathbf{e}_\lambda| := \sum_{1 \le i < j \le \ell(\lambda)} \lambda_i \lambda_j.$$

A key observation (proved in [34]) is that

$$\sum_{\lambda \in \mathsf{Par}(n)} t^{|\mathbf{e}| - |\mathbf{e}_\lambda|} \frac{c_\lambda(\mathbf{e}; t)}{\prod_i [\lambda_i]_t!} = 1. \tag{9.25}$$

[1] Hikita writes $\mathcal{X}(\mathbf{e}; q)$ for what we call $X_{G_{\mathbf{m}}}(\mathbf{x}; t)$

Example 9.4.33. If $\mathbf{m} = (2,3,3)$, then $\mathbf{e} = (0,0,1)$; hence $|\mathbf{e}| = 1$. We know already that $c_{(3)}(\mathbf{e};t) = 1+t+t^2$, $c_{(2,1)}(e;t) = t$, and $c_{(1,1,1)}(\mathbf{e};t) = 0$. Since $|\mathbf{e}_{(3)}| = 0$ and $|\mathbf{e}_{(2,1)}| = 2$, we get

$$\begin{aligned}\sum_{\lambda\in\mathsf{Par}(n)} t^{|\mathbf{e}|-|\mathbf{e}_\lambda|}\frac{c_\lambda(\mathbf{e};t)}{\prod_i[\lambda_i]_t!} &= t\frac{(1+t+t^2)}{(1+t)(1+t+t^2)} + t^{-1}\frac{t}{1+t} + 0 \\ &= \frac{t}{1+t} + \frac{1}{1+t} \\ &= 1.\end{aligned}$$

Equation (9.25) suggests that, for a choice of positive $t \in \mathbb{R}$, the numbers $t^{|\mathbf{e}|-|\mathbf{e}_\lambda|}\frac{c_\lambda(\mathbf{e};t)}{\prod_i[\lambda_i]_t!}$ determine a probability distribution on $\mathsf{Par}(n)$. To show that the numbers $t^{|\mathbf{e}|-|\mathbf{e}_\lambda|}\frac{c_\lambda(\mathbf{e};t)}{\prod_i[\lambda_i]_t!}$ indeed determine a probability distribution, Hikita defines a dynamic random process (dependent on $\mathbf{e}$ and t) that starts with the empty partition and then builds a Young tableau with n entries, one square at a time. He shows that the probability that the resulting Young tableau has shape λ is $t^{|\mathbf{e}|-|\mathbf{e}_\lambda|}\frac{c_\lambda(\mathbf{e};t)}{\prod_i[\lambda_i]_t!}$ as desired, using the modular law.

Conjecture 9.3.6 and therefore the first Stanley–Stembridge Conjecture (Conjecture 9.3.7) follow from taking $t = 1$: since both $1^{|\mathbf{e}|-|\mathbf{e}_\lambda|}$ and $\prod_i \lambda_1!$ are positive, $c_\lambda(\mathbf{e};1)$ must be nonnegative for each $\lambda \in \mathsf{Par}(n)$.

9.5 Hessenberg varieties and chromatic quasisymmetric functions

In this section we discuss a connection between chromatic quasisymmetric functions and geometry. The study of this connection has its roots in our work in [46] on a symmetric function analog in $\Lambda[t]$ of the classical Eulerian polynomial in $\mathbb{Z}_{\geq 0}[t]$. We showed that this symmetric function is equal to the Frobenius characteristic of the graded representation of the symmetric group on cohomology of a certain toric variety; see Section 7 of [46]. As was discussed in Example 9.4.6, this symmetric function motivated our study of chromatic quasisymmetric functions and would become $\omega X_G(\mathbf{x},t)$, where G is the path. We discuss this further in Example 9.5.4.

It follows from Haiman's Schur-positivity theorem (Theorem 9.3.9) that if $G = G_{\mathbf{m}}$ is a unit interval graph associated to Hessenberg vector $\mathbf{m} = (m_1,\dots,m_n)$, then $X_G(\mathbf{x})$ is the Frobenius characteristic of (the character of) an S_n representation. Similarly, it follows from our Theorem 9.4.11 that $X_G(\mathbf{x};t)$ is the Frobenius characteristic of a graded S_n representation, with t determining the grading. The question of whether this (graded) representation occurs "in nature" arises immediately, and turns out to have a positive answer. As we conjectured in [47, 48] and Brosnan and Chow proved in [9], $\omega X_G(\mathbf{x};t)$ is the Frobenius characteristic of a representation of S_n on the cohomology of a projective variety determined by $\mathbf{m}$ and a generic $n \times n$ matrix. This representation, which was defined in [58] by Tymoczko, generalizes the representation on cohomology of the toric variety mentioned above.

9.5.1 The type A flag variety

Given a positive integer n and a field $\mathbb{F}$ we define the **flag variety** $\mathrm{Fl}_n = \mathrm{Fl}_n(\mathbb{F})$ in two ways. Set $\mathcal{G} = GL_n(\mathbb{F})$ and let $\mathcal{B} \leq \mathcal{G}$ consist of all upper triangular elements of $\mathcal{G}$. In the **coset model**, Fl_n is the set $\mathcal{G}/\mathcal{B}$ of all cosets $g\mathcal{B}$ of $\mathcal{B}$ in $\mathcal{G}$. In the **flag model**, Fl_n consists of all full flags

$$V_\bullet : 0 = V_0 < V_1 < \cdots, V_{n-1} < V_n = \mathbb{F}^n$$

of subspaces of $\mathbb{F}^n$. (We observe that $\dim V_i = i$ for all i.) The natural action of $\mathcal{G}$ on $\mathbb{F}^n$ induces a transitive action of $\mathcal{G}$ on the set of all flags. In this action, $\mathcal{B}$ is the stabilizer of the flag $E_\bullet$ in which E_i is spanned by the standard basis vectors $e_1, \dots, e_i$. Therefore, the map sending the coset $g\mathcal{B}$ to $g(E_\bullet)$ is a bijection between the coset model and the flag model. The set of all flags has a natural interpretation as a projective variety over $\mathbb{F}$, see for example [8, Section 10.3]. We will examine certain subvarieties.

9.5.2 Type A Hessenberg varieties

Given a Hessenberg vector $\mathbf{m} = (m_1, \dots, m_n)$ and field $\mathbb{F}$, we define the **Hessenberg space** $\mathbf{HS}(\mathbf{m})$ to be the subspace of $M_n(\mathbb{F})$ consisting of all $C = (c_{ij}) \in M_n(\mathbb{F})$ such that $c_{ij} = 0$ whenever $i > m_j$. Given also $x \in M_n(\mathbb{F})$, the **Hessenberg variety** $\mathbf{Hess}(x, \mathbf{m})$ is defined in the respective models as follows:

$$\mathbf{Hess}(x, \mathbf{m}) := \{g\mathcal{B} \in \mathcal{G}/\mathcal{B} : g^{-1}xg \in \mathbf{HS}(\mathbf{m})\} \tag{9.26}$$

and

$$\mathbf{Hess}(x, \mathbf{m}) := \{V_\bullet : xV_i \leq V_{m_i} \text{ for all } i \in [n]\}. \tag{9.27}$$

It is straightforward to see that the image of the set of cosets described in (9.26) under the bijection sending gB to $g(E_\bullet)$ is the set of flags described in (9.27). In what follows we will use whichever model suits our purposes at the time. From here on, we will assume $\mathbb{F} = \mathbb{C}$ and consider topological properties of $\mathbf{Hess}(x, \mathbf{m})$.

Example 9.5.1. If $\mathbf{m} = (n, n, \dots, n)$ or x is a scalar matrix then we have $\mathbf{Hess}(x, \mathbf{m}) = \mathrm{Fl}_n$. If $\mathbf{m} = (1, 2, \dots, n)$ and x is nonsingular then $\mathbf{Hess}(x, \mathbf{m})$ consists of all flags $V_\bullet$ such that $x(V_\bullet) = V_\bullet$. In particular, if x is **regular**, that is, every eigenvalue of x has multiplicity one, then $\mathbf{Hess}(x, \mathbf{m})$ is finite.

Example 9.5.2. Say $n = 3$ and $\mathbf{m} = (2, 3, 3)$. Given $x \in M_3(\mathbb{C})$, we see that $\mathbf{Hess}(x, \mathbf{m})$ consists of all flags $V_\bullet$ satisfying $xV_1 \leq V_2$. Let us assume further that x is regular. So, x fixes finitely many 1-dimensional subspaces of $\mathbb{C}^3$. We consider the map $\pi : \mathbf{Hess}(x, \mathbf{m}) \to \mathbb{P}^2$ that sends $V_\bullet$ to V_1. Say $V \in \mathbb{P}^2$ is not an eigenspace for x. Then a flag $0 < V < W < \mathbb{C}^3$ lies in $\mathbf{Hess}(x, \mathbf{m})$ if and only if $W = V + xV$; hence $\pi^{-1}(V)$ is a point. On the other hand, if $V \in \mathbb{P}^2$ is an eigenspace for x then $0 < V < W < \mathbb{C}^3 \in \mathbf{Hess}(x, \mathbf{m})$ whenever W is a 2-space containing V. In this case, the map sending $0 < V < W < \mathbb{C}^3$ to W/V is a bijection from the set of all $V_\bullet \in \mathbf{Hess}(x, \mathbf{m})$ satisfying $V_1 = V$ to the set of 1-spaces from $\mathbb{C}^3/V$. It follows that $\pi^{-1}(V) \cong \mathbb{P}^1$. We see now that if x has k distinct eigenspaces then $\mathbf{Hess}(x, \mathbf{m})$ is isomorphic with the variety obtained from $\mathbb{P}^2$ by blowing up k points. So, the nonzero Betti numbers of $\mathbf{Hess}(x, \mathbf{m})$ are $\beta_0 = 1$, $\beta_2 = k + 1$, and $\beta_4 = 1$.

The systematic study of Hessenberg varieties as defined above was initiated by Filippo de Mari and Mark Shayman in [18]. There are more general definitions. In [17], de Mari, Procesi, and Shayman consider a complex reductive group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$ and Borel subgroup $\mathcal{B} \leq \mathcal{G}$ with Lie algebra $\mathfrak{b} \subseteq \mathfrak{g}$. A Hessenberg space in $\mathfrak{g}$ is a subspace H containing $\mathfrak{b}$ that is invariant under the adjoint action of $\mathcal{B}$. Given such H and $x \in \mathfrak{g}$, the Hessenberg variety $\mathbf{Hess}(x, H)$ consists of all cosets $g\mathcal{B} \in \mathcal{G}/\mathcal{B}$ satisfying $\mathsf{Ad}(g^{-1})x \in H$. (Here $\mathsf{Ad} : \mathcal{G} \to GL(\mathfrak{g})$ is the adjoint representation.) When $\mathcal{G} = GL_n(\mathbb{C})$, the possible Hessenberg spaces in $\mathfrak{g} = M_n(\mathbb{C})$ are exactly the $\mathbf{HS}(\mathbf{m})$ discussed earlier and this definition is the same as the one given in (9.27).

Even more generally, one can start with an arbitrary representation $\phi : \mathcal{G} \to GL(V)$ of a reductive group, a parabolic subgroup $\mathcal{P} \leq \mathcal{G}$, a $\mathcal{P}$-invariant subspace $U \leq V$, and $v \in V$ and consider the variety of all $g\mathcal{P} \in \mathcal{G}/\mathcal{P}$ such that $\phi(g^{-1})v \in U$. Such varieties are examined by Mark Goresky, Robert Kottwitz, and Robert MacPherson in [28].

We mention here one result from [18]. We use different language than that found in [18] and the result here is more general than stated by de Mari and Shayman, but their methods yield our result. It is appropriate to attribute the result to them. Given a Hessenberg vector $\mathbf{m} = (m_1, \ldots, m_n)$, recall that $G_\mathbf{m}$ is the naturally labeled unit interval graph on $[n]$ associated with $\mathbf{m}$ given in Lemma 9.3.3. Also recall the definition, given in Definition 9.4.16, of $\mathrm{inv}_G(w)$, where G is a graph on $[n]$ and $w \in S_n$.

Theorem 9.5.3 (See [18, Theorem 5.3]). *If $\mathbf{m} = (m_1, \ldots, m_n)$ is a Hessenberg vector and $s \in M_n(\mathbb{C})$ is regular semisimple (that is, diagonalizable with n distinct eigenvalues) then the cohomology of $\mathbf{Hess}(s, \mathbf{m})$ is concentrated in even degrees, and the Poincaré polynomial $P_{\mathbf{Hess}(s,\mathbf{m})}(t)$ satisfies*

$$P_{\mathbf{Hess}(s,\mathbf{m})}(t^{1/2}) = \sum_{w \in S_n} t^{\mathrm{inv}_{G_\mathbf{m}}(w)}. \tag{9.28}$$

Comparing Theorems 9.4.21 and 9.5.3, we see that $P_{\mathbf{Hess}(s,\mathbf{m})}(t^{1/2})$ is the coefficient of $z_{1^n}^{-1}p_{1^n}$ in the power sum expansion of $X_{G_\mathbf{m}}(\mathbf{x}; t)$. This coefficient gives the dimensions of the graded pieces of the inverse Frobenius characteristic $\mathsf{ch}^{-1}X_{G_{\mathbf{m}}}(\mathbf{x};t)$. We will see that this equality is a shadow of a much more interesting one. Let us give a motivating example here.

Example 9.5.4. Assume $\mathbf{m} = (2,3,\ldots,n-1,n,n)$. As shown in [17], if $s \in M_n(\mathbb{C})$ is regular semisimple then $\mathbf{Hess}(s,\mathbf{m})$ is the **permutahedral variety** ; that is, the toric variety associated to the complete fan in $\mathbb{R}^{n-1}$ determined by the Weyl chamber decomposition of type A_{n-1} as described in [43]. The action of S_n on this fan determines an action on $\mathbf{Hess}(s,\mathbf{m})$ which determines in turn a graded representation on cohomology $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$. This representation was determined by Claudio Procesi in [43] and its Frobenius characteristic was determined by Stanley in [49]. Since Stanley's formula is the same as the formula for $\omega X_{G_{\mathbf{m}}}(\mathbf{x};t)$ discussed in Example 9.4.6, we have

$$\omega \sum_{j=0}^{n-1} \mathsf{ch} H^{2j}(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})t^j = X_{G_{\mathbf{m}}}(\mathbf{x};t). \tag{9.29}$$

Example 9.5.4 led us to believe that a generalization of (9.29) should hold for all regular semisimple Hessenberg varieties $\mathbf{Hess}(s,\mathbf{m})$. In order to formulate a precise conjecture, we needed a representation of S_n on $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ for every Hessenberg vector $\mathbf{m}$. There is no known action of S_n on arbitrary $\mathbf{Hess}(s,\mathbf{m})$. So, another approach to finding a representation was necessary. Fortunately, Julianna Tymoczko had already described such a representation on cohomology in [58]. This representation and the generalization of (9.29) are the subjects of the next two subsections.

9.5.3 GKM Theory

Let $s \in M_n(\mathbb{C})$ be regular semisimple and let $\mathbf{m} = (m_1,\ldots,m_n)$ be a Hessenberg vector. The centralizer $T := C_{GL_n(\mathbb{C})}(s)$ in the adjoint action acts by translation on $\mathbf{Hess}(s,\mathbf{m})$. Indeed, if $t \in T$ (so $ts = st$) and $g^{-1}sg \in \mathbf{HS}(\mathbf{m})$ then

$$(tg)^{-1}s(tg) = g^{-1}t^{-1}stg = g^{-1}sg \in \mathbf{HS}(\mathbf{m}).$$

Under our assumption that s is regular semisimple, T is a (maximal) torus in $GL_n(\mathbb{C})$.

There is a well-developed theory of torus actions on varieties. In the case at hand the work of Goresky, Kottwitz, and MacPherson in [27] can be applied. A gentle introduction to this work is provided by Tymoczko in [59]. As shown in [18, Theorem 3.4], $\mathbf{Hess}(s,\mathbf{m})$ is a smooth projective variety. (As mentioned above, a smaller class of Hessenberg varieties than those we are considering is studied in [18], but the methods used by de Mari and Shayman apply to all regular semisimple Hessenberg varieties. We will not mention this again.) The action of T on $\mathbf{Hess}(s,\mathbf{m})$ is algebraic. It follows (see [27, Theorem 14.1(7)]) that the action of T on $\mathbf{Hess}(s,\mathbf{m})$ is equivariantly formal, as defined in [27].

Equivariant formality allows the computation of T-equivariant and singular cohomology using only information about T-fixed points and 1-dimensional T-orbits. Assume (as holds in the cases we consider) that the numbers of such fixed points and orbits are finite. The **GKM graph** $\Gamma(Y,T)$ for an equivariantly formal T-action on a projective variety Y has one vertex for each T-fixed point and one edge for each 1-dimensional orbit. A 1-dimensional T-orbit $\mathcal{O}$ is a torus whose closure in Y is a sphere consisting of $\mathcal{O}$ and two T-fixed points. The edge $e_{\mathcal{O}}$ in $\Gamma(Y,T)$ corresponding to $\mathcal{O}$ has as its endpoints the two vertices corresponding to the T-fixed points in $\overline{\mathcal{O}}$.

The edges in $\Gamma(Y,T)$ are labeled by linear forms. We describe this labeling L. Let $\mathfrak{t}$ be the Lie algebra of T. Given a 1-dimensional T-orbit $\mathcal{O}$ in Y, let $S \leq T$ be the stabilizer of a point in $\mathcal{O}$. Since T is abelian, S is the kernel of the action of T on $\mathcal{O}$; hence the choice of point does not matter. Now S is a closed subgroup of the algebraic group T, corresponding to a subalgebra $\mathfrak{s}$ of $\mathfrak{t}$. As $\mathcal{O}$ is 1-dimensional, $\mathfrak{s}$ has codimension one in $\mathfrak{t}$ and is therefore the kernel of a linear form $\ell \in \mathfrak{t}^*$. The edge $e_{\mathcal{O}}$ corresponding to $\mathcal{O}$ is labeled with $L(e_{\mathcal{O}}) = \ell$.

Now let $\mathbf{S}$ be the symmetric algebra on $\mathfrak{t}^*$. So, if $\mathfrak{t}^*$ has basis $t_1, \dots, t_n$ then $\mathbf{S}$ is the polynomial ring $\mathbb{C}[t_1, \dots, t_n]$. Assume $\Gamma(Y,T)$ has vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. We say that a function $f : \mathcal{V} \to \mathbf{S}$ is $\Gamma(Y,T)$**-compatible** if

$$L(e_{\mathcal{O}}) \text{ divides } f(v) - f(w) \text{ whenever } e_{\mathcal{O}} = \{v, w\} \in \mathcal{E}. \tag{9.30}$$

Let us write $\mathbf{S}^{\mathcal{V}}(Y,T)$ for the set of all $\Gamma(Y,T)$-compatible functions. We observe that $\mathbf{S}^{\mathcal{V}}(Y,T)$ is a graded $\mathbf{S}$-algebra, with pointwise addition and multiplication. (The grading is $\mathbf{S}^{\mathcal{V}}(Y,T) = \bigoplus_{d \geq 0} \mathbf{S}^{\mathcal{V}}(Y,T)_d$ with $\mathbf{S}^{\mathcal{V}}(Y,T)_d$ being the set of all $\Gamma(Y,T)$-compatible f such that for all $v \in \mathcal{V}$ $f(v)$ is homogeneous of degree d or $f(v) = 0$.) Let us write $\mathbf{I}$ for the ideal in $\mathbf{S}$ generated by the variables $t_1, \dots, t_n$. With all of this notation in hand, we can give the following explicit description of the T-equivariant and ordinary cohomology rings of Y.

Theorem 9.5.5 (See [27, (1.2.3) and (1.2.4)]). *Given an equivariantly formal action of torus T on projective variety Y with the cohomology of Y vanishing in odd degrees, there are $\mathbf{S}$-algebra isomorphisms*

1. $H_T^*(Y;\mathbb{C}) \cong \mathbf{S}^{\mathcal{V}}(Y,T)$ *and*
2. $H^*(Y;\mathbb{C}) \cong \mathbf{S}^{\mathcal{V}}(Y,T)/\mathbf{I} \cdot \mathbf{S}^{\mathcal{V}}(Y,T)$.

Moreover, the isomorphism in (1) maps $H_T^{2j}(Y,\mathbb{C})$ to the j^{th} graded part of $\mathbf{S}^{\mathcal{V}}(Y,T)$ and the isomorphism in (2) preserves the gradings.

We aim to apply Theorem 9.5.5 to the action of $T = C_{GL_n(\mathbb{C})}(s)$ to the Hessenberg variety $\mathbf{Hess}(s, \mathbf{m})$. We observe that $GL_n(\mathbb{C})$ acts on $\mathrm{Fl}_n(\mathbb{C})$ (by translation in the coset model and through its natural action on $\mathbb{C}^n$ in the flag model) and that for every $g \in GL_n(\mathbb{C})$ and every $x \in M_n(\mathbb{C})$,

$$\mathbf{Hess}(g^{-1}xg, \mathbf{m}) = g\mathbf{Hess}(x, \mathbf{m});$$

hence $\mathbf{Hess}(g^{-1}xg, \mathbf{m}) \cong \mathbf{Hess}(x, \mathbf{m})$. So, there is no loss in assuming our regular semisimple $s \in M_n(\mathbb{C})$ is diagonal (with n distinct eigenvalues). We continue under this assumption. Now T is the set of all nonsingular $n \times n$ diagonal matrices. The Lie algebra $\mathfrak{t}$ of T is the set of all $n \times n$ diagonal matrices. We write t_i for the element of $\mathfrak{t}^*$ that picks out the i^{th} diagonal entry of $g \in T$. So, $(t_1, \ldots, t_n)$ is a basis for $\mathfrak{t}^*$ and $\mathbf{S} = \mathbb{C}[t_1, \ldots, t_n]$ as above.

Our first step is to understand fixed points and 1-dimensional orbits of T on $\mathrm{Fl}_n(\mathbb{C})$. (This is posed as an exercise in [58].) We use the flag model. Given $w \in S_n$, we define the **permutation flag**

$$V_\bullet(w) : 0 < V_1(w) < \cdots < V_{n-1}(w) < \mathbb{C}^n,$$

where $V_j(w)$ is spanned by $\{e_{w(i)} : i \in [j]\}$.

Lemma 9.5.6. *The T-fixed points in $\mathbf{Hess}(s, \mathbf{m})$ are the permutation flags$V_\bullet(w)$.*

Proof. Note that T fixes a flag $V_\bullet$ if and only if T fixes every V_i. Since T is abelian and reductive, every T-invariant subspace of $\mathbb{C}^n$ is a direct sum of T-invariant 1-spaces. A 1-space X is T-invariant if and only if it is spanned by some $e_i \in X$. So, $V_\bullet \in \mathrm{Fl}_n(\mathbb{C})$ is fixed by T if and only if each V_i is spanned by standard basis vectors. Therefore the T-fixed points in $\mathrm{Fl}_n(\mathbb{C})$ are exactly the permutation flags. Finally, since the Hessenberg vector $\mathbf{m}$ satisfies $m_i \geq i$ for all i, each permutation flag lies in $\mathbf{Hess}(s, \mathbf{m})$. □

Lemma 9.5.7. *A flag $V_\bullet \in Fl_n(\mathbb{C})$ lies in a 1-dimensional T-orbit if and only if there exist $1 \leq j_0 < j_1 \leq n$ along with $k, \ell \in [n]$ and $\alpha \in \mathbb{C}^*$, such that all of*

1. *for $1 \leq i < j_0$, V_i is spanned by i standard basis vectors, none of which is e_k or e_ℓ;*
2. *for $j_0 \leq i < j_1$, V_i is spanned by $i - 1$ standard basis vectors, none of which is e_k or e_ℓ, along with $e_k + \alpha e_\ell$;*
3. *for $j_1 \leq i \leq n$, V_i is spanned by i standard basis vectors, two of which are e_k and e_ℓ*

hold.

Example 9.5.8. The flag

$$V_\bullet : 0 < \mathbb{C}\{e_3\} < \mathbb{C}\{e_3, e_2 + ie_5\} < \mathbb{C}\{e_1, e_3, e_2 + ie_5\} < \mathbb{C}\{e_1, e_2, e_3, e_5\} < \mathbb{C}^5$$

satisfies the conditions of the lemma with $j_0 = 2$, $j_1 = 4$, $k = 2$, $\ell = 5$, and $\alpha = i$.

Proof. Let $\mathcal{O}$ be the T-orbit containing $V_\bullet$. Assume first that $\mathcal{O}$ is 1-dimensional. Since $V_\bullet$ is not T-invariant, there is some $j \in [n]$ such that V_j is not spanned by standard basis vectors. Let j_0 be the smallest such j. There exists some

$A \subseteq [n]$ with $|A| = j_0 - 1$ such that $B := \{e_i : i \in A\}$ is a basis for V_{j_0-1}. We pick $v \in V_{j_0}$ such that $B \cup \{v\}$ is a basis for V_{j_0} and

$$v = \sum_{i \in [n] \setminus A} \alpha_i e_i, \tag{9.31}$$

with each $\alpha_i \in \mathbb{C}$.

We will see that, after appropriate scaling, there exist $k, \ell \in [n] \setminus A$ such that $v = e_k + \alpha e_\ell$ with $\alpha \neq 0$. We aim to apply the Orbit-Stabilizer Theorem. Let R, S be the respective stabilizers of $V_\bullet$ and V_{j_0} in T. We observe that

- $R \leq S$;
- $\dim T - \dim R = \dim \mathcal{O} = 1$.

Let C be the set of all $i \in [n] \setminus A$ such that $\alpha_i \neq 0$ in (9.31). Since V_{j_0} is not spanned by standard basis vectors, $|C| > 1$. Let

$$\tau = \mathsf{diag}(\gamma_1, \ldots, \gamma_n) \in T.$$

Note that every element of V_{j_0} supported only on C is a scalar multiple of v. Therefore τ fixes V_{j_0} if and only if v is an eigenvector for τ if and only if there is some λ such that $\gamma_i = \lambda$ for all $i \in C$. We conclude that

$$1 = \dim T - \dim R \geq \dim T - \dim S = |C| - 1; \tag{9.32}$$

hence $|C| = 2$. Say $C = \{k, \ell\}$. Then we can choose $v = e_k + \alpha e_\ell$ (with $\alpha \neq 0$ determined by V_{j_0}). Moreover,

$$S = \{\mathsf{diag}(\gamma_1, \ldots, \gamma_n) \in T : \gamma_k = \gamma_\ell\}, \tag{9.33}$$

and by (9.32) $R = S$.

As S is reductive and abelian, each V_i is spanned by S-invariant 1-spaces. The S-invariant 1-spaces are those spanned by standard basis vectors along with those spanned by $e_k + \beta e_\ell$ for $\beta \in \mathbb{C}^*$. Find the smallest j_1 such that $e_k \in V_{j_1}$. We observe that $j_1 > j_0$ and that if $j_0 \leq i < j_1$ there is no $\beta \neq \alpha$ such that $e_k + \beta e_\ell \in V_i$, since this would force $e_k \in V_i$. We conclude that $V_\bullet$ satisfies conditions (1)-(3) of the lemma.

Conversely, if $V_\bullet$ satisfies conditions (1)-(3) we can confirm that the stabilizer of $V_\bullet$ in T is S as in (9.33), and $\dim \mathcal{O} = 1$ by the orbit-stabilizer theorem. □

Lemma 9.5.9. *Let $V_\bullet \in Fl_n$ satisfy conditions (1)-(3) of Lemma 9.5.7 with respect to fixed j_0, j_1, k, ℓ, and α. The T-orbit of $V_\bullet$ consists of all flags*

$$V_\bullet^\beta : 0 < V_1^\beta < \cdots < V_{n-1}^\beta$$

such that $\beta \in \mathbb{C}^*$ and both of

1. *if* $i < j_0$ *or* $i \geq j_1$ *then* $V_i^\beta = V_i$*;*
2. *if* $j_0 \leq i < j_1$ *then* V_i^β *is spanned by the standard basis vectors in* V_i *and* $e_k + \beta e_\ell$

hold.

Proof. This follows from the facts that for $\tau = \mathsf{diag}(\gamma_1, \ldots, \gamma_n) \in T$, $\tau e_i = \gamma_i e_i$ and $\tau(e_k + \alpha e_\ell) = \gamma_k e_k + \gamma_\ell \alpha e_\ell$. □

Example 9.5.10. If $V_\bullet$ is as in Example 9.5.8 then the T-orbit of $V_\bullet$ consists of all flags

$$V_\bullet^\beta : 0 < \mathbb{C}\{e_3\} < \mathbb{C}\{e_3, e_2{+}\beta e_5\} < \mathbb{C}\{e_1, e_3, e_2{+}\beta e_5\} < \mathbb{C}\{e_1, e_2, e_3, e_5\} < \mathbb{C}^5$$

such that $\beta \in \mathbb{C}^*$.

Definition 9.5.11. *Given* $V_\bullet \in Fl_n(\mathbb{C})$ *satisfying conditions (1)-(3) of Lemma 9.5.7 with respect to* j_0, j_1, k, ℓ*, and* α*, we define permutation flags* $V_\bullet(w^k)$ *and* $V_\bullet(w^\ell)$ *by*

- *if* $i < j_0$ *or* $i \geq j_1$ *then* $V_i(w^k) = V_i(w^l) = V_i$*;*
- *if* $j_0 \leq i < j_1$ *then* $V_i(w^k)$ *is spanned by the standard basis vectors in* V_i *along with* e_k *and* $V_i(w^\ell)$ *is spanned by the standard basis vectors in* V_i *along with* e_ℓ*.*

Note that $w^k(j_0) = w^\ell(j_1) = k$, $w^k(j_1) = w^\ell(j_0) = \ell$, and $w^k(i) = w^\ell(i)$ for all $i \neq j_0, j_1$. Hence the permutation w^ℓ is obtained from the permutation w^k by transposing the letters k and ℓ. That is,

$$w^\ell = (k, \ell) \cdot w^k = w^k \cdot (j_0, j_1). \tag{9.34}$$

Example 9.5.12. If $V_\bullet$ is as in Example 9.5.8 then we have

$$V_\bullet(w^2) : 0 < \mathbb{C}\{e_3\} < \mathbb{C}\{e_2, e_3\} < \mathbb{C}\{e_1, e_2, e_3\} < \mathbb{C}\{e_1, e_2, e_3, e_5\} < \mathbb{C}^5$$

and

$$V_\bullet(w^5) : 0 < \mathbb{C}\{e_3\} < \mathbb{C}\{e_3, e_5\} < \mathbb{C}\{e_1, e_3, e_5\} < \mathbb{C}\{e_1, e_2, e_3, e_5\} < \mathbb{C}^5.$$

Lemma 9.5.13. *Assume* $V_\bullet$ *satisfies conditions (1)-(3) of Lemma 9.5.7 with respect to* j_0, j_1, k, ℓ*, and* α *and let* $\mathcal{O}$ *be the* T*-orbit of* $V_\bullet$ *in* $Fl_n(\mathbb{C})$*. The closure* $\overline{\mathcal{O}}$ *of* $\mathcal{O}$ *in* $Fl_n(\mathbb{C})$ *is* $\mathcal{O} \cup \{V_\bullet(w^k), V_\bullet(w^\ell)\}$*. Moreover, if* $V_\bullet$ *lies in the Hessenberg variety* $\mathbf{Hess}(s, \mathbf{m})$ *then* $\overline{\mathcal{O}} \subseteq \mathbf{Hess}(s, \mathbf{m})$*.*

Proof. If (β_i) is a sequence from $\mathbb{C}^*$ that converges to 0 then the sequence $(V_\bullet^{\beta_i})$ converges to $V_\bullet(w^k)$ in $\mathrm{Fl}_n(\mathbb{C})$. Similarly, if $(|\beta_i|)$ converges to ∞ then $(V_\bullet^{\beta_i})$ converges to $V_\bullet(w^\ell)$. The map from $\mathcal{O} \cup \{V_\bullet(w^k), V_\bullet(w^\ell)\}$ to $\mathbb{P}^1$ sending $V_\bullet^\beta$ to β, $V_\bullet(w^k)$ to 0, and $V_\bullet(w^\ell)$ to ∞ is a homeomorphism. If $V_\bullet \in \mathbf{Hess}(s, \mathbf{m})$ then $\mathcal{O} \subseteq \mathbf{Hess}(s, \mathbf{m})$ since $\mathbf{Hess}(s, \mathbf{m})$ is T-invariant. Both $V_\bullet(w^k)$ and $V_\bullet(w^\ell)$ lie in $\mathbf{Hess}(s, \mathbf{m})$ by Lemma 9.5.6. □

Lemma 9.5.14. *Assume $V_\bullet$ satisfies conditions (1)-(3) of Lemma 9.5.7 with respect to j_0, j_1, k, ℓ, and α. Let $\mathcal{O}$ be the T-orbit of $V_\bullet$ and let $\mathbf{m}$ be a Hessenberg vector. The closure $\overline{\mathcal{O}}$ lies in $\mathbf{Hess}(s, \mathbf{m})$ if and only if $m_{j_0} \geq j_1$.*

Proof. By Lemma 9.5.13, it suffices to determine whether $V_\bullet \in \mathbf{Hess}(s, \mathbf{m})$. If $i < j_0$ or $i \geq j_1$ then $sV_i = V_i$. If $j_0 \leq i < j_1$ then $V_i + sV_i$ is spanned by the standard basis vectors in V_i along with e_k and e_ℓ. Indeed, since s is diagonal and regular,

$$\mathbb{C}\{e_k + \alpha e_\ell, s(e_k + \alpha e_\ell)\} = \mathbb{C}\{e_k, e_\ell\}.$$

It follows that for $j_0 \leq i < j_1$, $V_i + sV_i$ is contained in V_{j_1} but not contained in V_p if $p < j_1$. Since $\mathbf{m}$ is weakly increasing, the Lemma follows. □

Proposition 9.5.15. *Assume $s \in M_n(\mathbb{C})$ is regular and diagonal and that $\mathbf{m} = (m_1, \ldots, m_n)$ is a Hessenberg vector. Let $T = C_{GL_n(\mathbb{C})}(s)$. The GKM graph $\Gamma(\mathbf{Hess}(s, \mathbf{m}), T)$ has vertex set $\mathcal{V}$ consisting of the permutation flags, that is*

$$\mathcal{V} = \{V_\bullet(w) : w \in S_n\}.$$

The edge set $\mathcal{E}$ of $\Gamma(\mathbf{Hess}(s, \mathbf{m}), T)$ consists of all $\{V_\bullet(u), V_\bullet(v)\}$ such that $v = u \cdot (j_0, j_1)$, where $1 \leq j_0 < j_1 \leq n$ and $m_{j_0} \geq j_1$. Assuming, without loss of generality, that $u(j_0) < u(j_1)$, the edge $\{V_\bullet(u), V_\bullet(v)\}$ is labeled with $t_k - t_\ell$, where $k = u(j_0)$ and $\ell = u(j_1)$.

Proof. This follows from Lemmas 9.5.6, 9.5.7, 9.5.9, 9.5.13, and 9.5.14, and equation (9.34). □

Example 9.5.16. Say $\mathbf{m} = (2, 3, 3)$ and $s \in M_3(\mathbb{C})$ is diagonal with three distinct eigenvalues. The fixed points of the diagonal torus T are the six permutation flags $V_\bullet(w)$, and these form the vertex set $\mathcal{V}$ of $\Gamma(\mathbf{Hess}(s, \mathbf{m}), T)$. The edge set $\mathcal{E}$ of $\Gamma(\mathbf{Hess}(s, \mathbf{m}), T)$ is described in the table below, with the following notation:

- We will write w for the permutation flag $V_\bullet(w)$.
- We will write $V_1 < V_2$ for a flag $0 < V_1 < V_2 < \mathbb{C}^3$.
- We will describe each edge by giving a representative of the 1-dimensional T-orbit corresponding to the edge.

Orbit Representative	Edge Endpoints	Edge Label
$\mathbb{C}\{e_1 + e_2\} < \mathbb{C}\{e_1, e_2\}$	$123, 213$	$t_1 - t_2$
$\mathbb{C}\{e_1\} < \mathbb{C}\{e_1, e_2 + e_3\}$	$123, 132$	$t_2 - t_3$
$\mathbb{C}\{e_2\} < \mathbb{C}\{e_2, e_1 + e_3\}$	$213, 231$	$t_1 - t_3$
$\mathbb{C}\{e_1 + e_3\} < \mathbb{C}\{e_1, e_3\}$	$132, 312$	$t_1 - t_3$
$\mathbb{C}\{e_2 + e_3\} < \mathbb{C}\{e_2, e_3\}$	$231, 321$	$t_2 - t_3$
$\mathbb{C}\{e_3\} < \mathbb{C}\{e_3, e_1 + e_2\}$	$312, 321$	$t_1 - t_2$

We list in the next table six elements of $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)$, which are indexed by the permutations in S_3. These elements manifestly form a linearly independent set over $\mathbb{C}$. The reader can confirm that the set of cosets represented by these elements in $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)/\mathbf{I}\cdot\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)$ remains linearly independent and therefore corresponds to a linearly independent set in $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$. Counting elements of each degree among this set and comparing with the Betti numbers of $\mathbf{Hess}(s,\mathbf{m})$ as determined in Example 9.5.2, we see that the given set of cosets corresponds to a (homogeneous) basis of $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$. This is an example of a "flow-up" basis, as described in [59].

f	$f(123)$	$f(132)$	$f(213)$	$f(231)$	$f(312)$	$f(321)$
f_{123}	1	1	1	1	1	1
f_{132}	0	t_2-t_3	0	0	t_2-t_1	0
f_{213}	0	0	t_1-t_2	t_3-t_2	0	0
f_{231}	0	0	0	t_1-t_3	0	t_1-t_2
f_{312}	0	0	0	0	t_1-t_3	t_2-t_3
f_{321}	0	0	0	0	0	$(t_1-t_2)(t_2-t_3)$

Note that w is the shortest element (with respect to the usual length function on S_3) in the support of f_w.

9.5.4 Tymoczko's dot action and its connection with chromatic quasisymmetric functions

Let us revisit Proposition 9.5.15, changing the language slightly. As in Example 9.5.16, we will write w for a vertex $V(w)$ of the GKM graph $\Gamma := \Gamma((\mathbf{Hess}(s,\mathbf{m}),T)$. Recall that we assume that permutations are composed from right to left. If $t=(i,j)\in S_n$ is a transposition such that $i<j\le m_i$, then for each $x\in S_n$, with $x(i)<x(j)$, the pair $\{x,xt\}$ lies in the edge set $\mathcal{E}$ of Γ. Every edge in $\mathcal{E}$ arises in this fashion. If $xtx^{-1}=(k,\ell)$ then the edge $\{x,xt\}$ is labeled with t_k-t_ℓ. Now S_n acts on itself by left multiplication and acts on $\{t_1,\dots,t_n\}$ by permuting indices. Combining these two actions, we obtain an action of S_n on the labeled graph Γ by automorphisms. That is, if $\{x,xt\}\in\mathcal{E}$ and $g\in S_n$, then $\{gx,gxt\}\in\mathcal{E}$. Moreover, if $xtx^{-1}=(k,\ell)$ then $(gx)t(gx)^{-1}=(g(k),g(\ell))$; hence the label on the edge $\{gx,gxt\}$ is obtained from that on $\{x,xt\}$ by applying g to the appropriate indices.

The action of S_n on Γ determines a linear action on the algebra $\mathbf{S}^{\mathcal{V}}(\Gamma)$ of Γ-compatible functions: given $f\in\mathbf{S}^{\mathcal{V}}(\Gamma)$ and $w\in S_n$, $f(w)$ is a polynomial in $t_1,\dots,t_n$. Given such f and $v\in S_n$, we define $vf\in\mathbf{S}^{\mathcal{V}}(\Gamma)$ by (for each $u\in S_n$)

$$(vf)(u)(t_1,\dots,t_n) := f(v^{-1}u)(t_{v(1)},\dots,t_{v(n)}).$$

Colloquially, applying v to f "moves" a polynomial that "appears at w" so that it "appears at vw" with its variables permuted according to v. The reader can confirm that this is indeed a well-defined action. Linearity is apparent from the definition. The action preserves the grading. This action was first studied closely by Tymoczko, who called it the "dot action," in [58]. The terms "dot action" and "dot representation" are now standard.

Example 9.5.17. We continue with the set-up of Example 9.5.16 and apply the transposition $(1,2)$ to each f_w.

f	$f(123)$	$f(132)$	$f(213)$	$f(231)$	$f(312)$	$f(321)$
$(1,2)f_{123}$	1	1	1	1	1	1
$(1,2)f_{132}$	0	0	0	t_1-t_3	0	t_1-t_2
$(1,2)f_{213}$	t_2-t_1	t_3-t_1	0	0	0	0
$(1,2)f_{231}$	0	t_2-t_3	0	0	t_2-t_1	0
$(1,2)f_{312}$	0	0	0	0	t_1-t_3	t_2-t_3
$(1,2)f_{321}$	0	0	0	0	$(t_2-t_1)(t_1-t_3)$	0

We see that in its action on $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T) \cong H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$, $(1,2)$ fixes f_{312} and exchanges f_{132} and f_{231}. We are more interested in what happens in the quotient $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)/\mathbf{I}\cdot\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T) \cong H^*(\mathbf{Hess}(s,\mathbf{m}))$. We calculate that in $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)$,

$$(1,2)f_{213} = (t_2-t_1)f_{123} - f_{132} + f_{213} + f_{231}.$$

Therefore, abusing notation and writing f_w for the coset $f_w + \mathbf{I}\cdot\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}))$, we see that in $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)/\mathbf{I}\cdot\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)$,

$$(1,2)f_{213} = -f_{132} + f_{213} + f_{231}.$$

A similar computation shows that in $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)/\mathbf{I}\cdot\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)$, $(1,2)$ fixes f_{321}. After identifying $\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)/\mathbf{I}\cdot\mathbf{S}^{\mathcal{V}}(\mathbf{Hess}(s,\mathbf{m}),T)$ with $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ we see that $(1,2)$ acts trivially on $H^0(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ and $H^4(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ and that the matrix for the action of $(1,2)$ on $H^2(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ with respect to the ordered basis $(f_{132}, f_{213}, f_{231}, f_{312})$ is

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reader can calculate directly that the matrix for the action of $(1,2,3)$ on $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ with respect to the same ordered basis is

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Let us write $\chi_{s,\mathbf{m}}$ for the graded character of the representation of S_n on $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ that we have described in this example. Looking at the matrices above, we see that this character is as in the table below.

Cycle type of w	$(1,1,1)$	$(2,1)$	(3)
$\chi_{s,\mathbf{m}}(w)$	$1+4t+t^2$	$1+2t+t^2$	$1+t+t^2$

This gives the following result for the Frobenius characteristic of $\chi_{s,\mathbf{m}}$:

$$\begin{aligned}\mathsf{ch}\chi_{s,\mathbf{m}} &= (1+4t+t^2)\frac{p_{111}}{6}+(1+2t+t^2)\frac{p_{21}}{2}+(1+t+t^2)\frac{p_3}{3}\\ &= s_3+t(2s_3+s_{21})+t^2s_3\\ &= h_3+t(h_3+h_{21})+t^2h_3,\end{aligned}$$

the second and third equalities following from direct computation.

Comparing Examples 9.4.12 and 9.5.17, we see that if $\mathbf{m}=(2,3,3)$ and $s\in M_3(\mathbb{C})$ is regular semisimple then

$$\mathsf{ch}\chi_{s,\mathbf{m}}=\omega X_{G_{\mathbf{m}}}(\mathbf{x};t). \tag{9.35}$$

This is the $n=3$ case of Example 9.5.4.

The key result of this section is that (9.35) and its generalization (9.29) hold for all Hessenberg vectors. This was conjectured by us in [47, Conjecture 5.3] and [48, Conjecture 10.1] and proved by Patrick Brosnan and Timothy Chow in [9].

Theorem 9.5.18 ([9, Theorem 129]). *If $\mathbf{m}=(m_1,\dots,m_n)$ is a Hessenberg vector and $s\in M_n(\mathbb{C})$ is regular semisimple, then the graded character $\chi_{s,\mathbf{m}}$ of the dot action of S_n on $H^*(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ satisfies*

$$\mathsf{ch}\chi_{s,\mathbf{m}}=\omega X_{G_{\mathbf{m}}}(\mathbf{x};t).$$

It follows from Theorem 9.5.18 that Conjecture 9.4.4 (the refined Stanley–Stembridge Conjecture) is equivalent to the following conjecture.

Conjecture 9.5.19. *If $\mathbf{m}=(m_1,\dots,m_n)$ is a Hessenberg vector and $s\in M_n(\mathbb{C})$ is regular semisimple, then the dot action of S_n on each cohomology group $H^{2j}(\mathbf{Hess}(s,\mathbf{m});\mathbb{C})$ gives rise to a $\mathbb{C}[S_n]$-module isomorphic to a direct sum of modules M^λ arising from translation actions on cosets of Young subgroups Y_λ.*

The proof of Theorem 9.5.18 given in [9] uses high-powered geometric machinery along with facts about symmetric functions and S_n-representations. Given a Hessenberg vector $\mathbf{m}=(m_1,\dots,m_n)$ a key point is to consider the variety

$$\mathcal{H}(\mathbf{m}):=\{(V_\bullet,x)\in \mathrm{Fl}_n(\mathbb{C})\times M_n(\mathbb{C}) : V_\bullet\in\mathbf{Hess}(x,\mathbf{m})\}.$$

The fibers of the projection pr_2 of $\mathcal{H}(\mathbf{m})$ onto $M_n(\mathbb{C})$ are the Hessenberg varieties $\mathbf{Hess}(x,\mathbf{m})$. The projection pr_1 onto the first factor makes $\mathcal{H}(\mathbf{m})$ a vector bundle over $\mathrm{Fl}_n(\mathbb{C})$. This facilitates certain computations. Let M^{rs} be the set of regular semisimple elements in $M_n(\mathbb{C})$. The restriction of pr_2 to $\mathsf{pr}_2^{-1}(M^{\mathsf{rs}})$ is a covering map; hence $pr_2^{-1}(M^{\mathsf{rs}})$ admits an action of the fundamental group $\pi := \pi_1(M^{\mathsf{rs}}, s_0)$, the **monodromy action**. From this action arises a representation of π on the cohomology of any fiber of pr_2. Now π is a braid group and so has S_n as a quotient. The monodromy action factors through this quotient and hence determines a representation of S_n. Corollary 125 of [9] says that the monodromy and dot actions of S_n on $H^*(\mathbf{Hess}(s, \mathbf{m}); \mathbb{C})$ are the same. This allows investigation of the dot action using various general results about monodromy actions.

The number of Young subgroups Y_λ of S_n is the number of partitions of n, which is in turn the number of conjugacy classes of elements of S_n. Since Y_λ is not conjugate to Y_μ if $\lambda \neq \mu$, it follows that a representation $\phi : S_n \to GL(V)$ is determined by the dimensions of the fixed spaces V^{Y_λ} for all $\lambda \in \mathsf{Par}(n)$. (All of this is equivalent, through the Frobenius characteristic and Frobenius reciprocity, to the fact that the symmetric functions h_λ, for all $\lambda \in \mathsf{Par}(n)$, form a basis for Λ^n.)

Given $\lambda \in \mathsf{Par}(n)$, pick an element $x_\lambda \in M_n(\mathbb{C})$ in Jordan form such that the parts of λ are the sizes of the Jordan blocks of x_λ and no two different Jordan blocks of x_λ have the same diagonal entries; so x_λ is regular. Theorems 4 and 127 of [9] say that

$$\dim(H^j(\mathbf{Hess}(s, \mathbf{m}); \mathbb{C}))^{Y_\lambda} = \dim H^j(\mathbf{Hess}(x_\lambda, \mathbf{m}); \mathbb{C})$$

for all $\lambda \in \mathsf{Par}(n)$ and all j.

On the other hand, basic facts about symmetric functions imply that, given $\lambda \in \mathsf{Par}(n)$ and a representation $\phi : S_n \to GL(V)$ with character χ, $\dim V^{Y_\lambda}$ is the coefficient of m_λ in the monomial expansion of $\mathsf{ch}\chi$. So, to prove Theorem 9.5.18, it suffices to show that the Betti numbers of the various $\mathbf{Hess}(x_\lambda, \mathbf{m})$ are the same as the coefficients of the corresponding m_λ in $X_G(\mathbf{x}; t)$. Combinatorial formulas for the Betti numbers were given by Tymoczko in [57]. Brosnan and Chow use previous work of Chow in [14] to show that the coefficients of the m_λ are given by the same formulas. The reader who also reads [9] will likely agree with us that what we have just described is easier said than done.

Additional proofs of Theorem 9.5.18 have been given, by Guay-Paquet in [30], by Young-Hoon Kiem and Donggun Lee in [38] and [37], and by Martha Precup and Eric Sommers in [42].

9.5.5 Some consequence of the connection

The connection between chromatic quasisymmetric functions and regular semisimple Hessenberg varieties given by Theorem 9.5.18 allows for combinatorial methods to be used to address questions about such varieties and allows for geometric methods to be used to solve combinatorial problems. We give examples here.

First, Tymoczko asked in [58] for the decomposition of the dot representation into irreducible representations of S_n. One might also ask for a description of the character of the dot representation. With Theorem 9.5.18 in hand, Theorems 9.4.11 and 9.4.21, along with our understanding of the Frobenius characteristic map ch allow us to answer both questions.

Theorem 9.5.20. *Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a Hessenberg vector and let $s \in M_n(\mathbb{C})$ be regular semisimple.*

1. *For $\lambda \in \mathsf{Par}(n)$ and $j \geq 0$, the multiplicity of the irreducible representation S^λ in the dot representation on $H^{2j}(\mathbf{Hess}(\mathsf{s}, \mathbf{m}))$ is the number of $P_{\mathbf{m}}$-tableaux T of shape λ' (the partition conjugate to λ) satisfying $\mathrm{inv}_{G_{\mathbf{m}}}(T) = j$.*

2. *If $w \in S_n$ has cycle type $\lambda \in \mathsf{Par}(n)$, then, using the notation from Theorem 9.4.21,*
$$\sum_{j \geq 0} \mathrm{Trace}(w, H^{2j}(\mathbf{Hess}(s, \mathbf{m})))t^j = \sum_{v \in \mathcal{N}_{P_{\mathbf{m}}, \lambda}} t^{\mathrm{inv}_{G_{\mathbf{m}}}(v)}.$$

Second, a regular semisimple Hessenberg variety $\mathbf{Hess}(s, \mathbf{m})$ is nonsingular. Moreover, if $m_i > i$ for all $i < n$ then $\mathbf{Hess}(s, \mathbf{m})$ is irreducible (see [18, Theorem 3.4 and Lemma 4.1]). So, assuming $m_i > i$ for all $i < n$, the hard Lefschetz Theorem applies: if $\dim \mathbf{Hess}(s, \mathbf{m}) = d$ then one can pick $\omega \in H^2(\mathbf{Hess}(s, \mathbf{m}))$ such that multiplication by ω^{d-i} determines an isomorphism between $H^i(\mathbf{Hess}(s, \mathbf{m}))$ and $H^{2d-i}(\mathbf{Hess}(s, \mathbf{m}))$ for all i. It follows that the Betti numbers of $\mathbf{Hess}(s, \mathbf{m})$ form a unimodal and palindromic sequence. As communicated to us by Tymoczko, she and Robert MacPherson have proved that one can choose ω so that multiplication by ω commutes with the dot action on $H^*(\mathbf{Hess}(s, m))$. (One can find a proof that such an ω exists at the end of the proof of Theorem 4.11 in the paper [10] of Ana Bălibanu and Peter Crooks.) From this and Theorem 9.5.18 we get an equivariant unimodality theorem, which can be shown to hold even if $m_i = i$ for some $i < n$:

Theorem 9.5.21 (MacPherson–Tymoczko). *Let $\mathbf{m}$ be a Hessenberg vector of length n and let $s \in M_n(\mathbb{C})$ be regular semisimple. If $0 \leq j < \frac{1}{2}\sum_{i=1}^n (m_i - i)$ then the dot action makes $H^{2j}(\mathbf{Hess}(s, \mathbf{m}))$ a $\mathbb{C}[S_n]$-submodule of $H^{2j+2}(\mathbf{Hess}(s, \mathbf{m}))$.*

Remark 9.5.22. *We can rephrase the e-unimodality conjecture discussed in Remark 9.4.5 to say that, for $0 \leq j < \frac{1}{2}\sum_{i=1}^n (m_i - i)$, the dot action makes the quotient $H^{2j+2}(\mathbf{Hess}(s, \mathbf{m}))/H^{2j}(\mathbf{Hess}(s, \mathbf{m}))$ a permutation module in which each point stabilizer is a Young subgroup of S_n.*

We obtain combinatorial consequences from Theorems 9.5.18 and 9.5.21.

Theorem 9.5.23 (See [48, Proposition 10.2]). *If $\mathbf{m}$ is a Hesseneberg vector and $G = G_{\mathbf{m}}$ then $X_G(\mathbf{x}; t)$ is Schur-unimodal (as well as palindromic) as a polynomial in t. That is, for $0 \leq j < \frac{|E(G)|}{2}$, upon subtracting the coefficient of t^j from that of t^{j+1}, we get a Schur-positive symmetric function.*

Given a poset P on $[n]$ and a graph G on $[n]$, define the polynomial

$$A_{P,G}(q,t) := \sum_{w \in S_n} q^{\mathrm{maj}_P(w)} t^{\mathrm{inv}_G(w)},$$

where

$$\mathrm{maj}_P(w) := \sum_{i : w(i) >_P w(i+1)} i$$

and recall

$$\mathrm{inv}_G(w) := |\{w(i)w(j) \in E(G) : i < j, w(i) > w(j)\}|.$$

For any Hessenberg vector $\mathbf{m}$, let

$$A_{\mathbf{m}}(q,t) := A_{P_{\mathbf{m}},G_{\mathbf{m}}}(q,t).$$

For $\mathbf{m} = (1, 2, \ldots, n)$, we have

$$A_{\mathbf{m}}(q,t) = \sum_{w \in S_n} q^{\mathrm{maj}(w)} = [n]_q!,$$

and for $\mathbf{m} = (n, n, \ldots, n)$, we have

$$A_{\mathbf{m}}(q,t) = \sum_{w \in S_n} t^{\mathrm{inv}(w)} = [n]_t!,$$

where maj and inv are the classical Mahonian permutation statistics **major index** and **inversion number** : $\mathrm{maj}(w) := \sum_{i:w(i)>w(i+1)} i$ and $\mathrm{inv}(w) := |\{(i,j) : 1 \le i < j \le n, w(i) > w(j)\}|$.

Recall from Theorem 9.5.3 that when s is regular semisimple $A_{\mathbf{m}}(1,t)$ is the Poincaré polynomial of $\mathbf{Hess}(s,\mathbf{m})$. In [18] de Mari and Shayman refer to $A_{\mathbf{m}}(1,t)$ as a **generalized Eulerian polynomial** because when $\mathbf{m} = (2, 3, \ldots, n, n)$, the polynomial $A_{\mathbf{m}}(1,t)$ is the classical Eulerian polynomial $A_n(t)$. The Eulerian polynomial has two well-known combinatorial formulations:

$$A_n(t) = \sum_{w \in S_n} t^{\mathrm{des}(w)} \quad \text{and} \quad A_n(t) = \sum_{w \in S_n} t^{\mathrm{exc}(w)},$$

where $\mathrm{des}(w) = |\{i : w(i) > w(i+1)\}|$ and $\mathrm{exc}(w) = |\{i : w(i) > i\}|$. It is well-known that $A_n(t)$ is palindromic and unimodal. Stanley notes in [52, Exercise 50 f] that by the hard Lefschetz Theorem, the generalized Eulerian polynomials $A_{\mathbf{m}}(1,t)$ are also palindromic and unimodal.

It follows from [46, Theorems 1.1 and 7.2] (see also [48, Theorem 9.7]) that for $\mathbf{m} = (2, 3, \ldots, n, n)$, the polynomial $A_{\mathbf{m}}(q,t)$ is

- equal to a natural q-analog of the Eulerian polynomials, that is,

$$A_{\mathbf{m}}(q,t) = A_n(q,t) := \sum_{w \in S_n} q^{\mathrm{maj}(w) - \mathrm{exc}(w)} t^{\mathrm{exc}(w)},$$

- palindromic as a polynomial in t centered at $\frac{n-1}{2}$,
- q-unimodal, that is, for $0 \leq j < \frac{n-1}{2}$, upon subtracting the coefficient of t^j from that of t^{j+1}, we get a polynomial in $\mathbb{Z}_{\geq 0}[q]$.

In [48, Theorem 9.3] we show that for general Hessenberg vectors $\mathbf{m}$, the polynomial $A_{\mathbf{m}}(q,t)$ is obtained by taking the stable principal specialization of $\omega X_{G_{\mathbf{m}}}(\mathbf{x},t)$, that is, by setting $x_i = q^{i-1}$ for each i and then multiplying by $\prod_{j=1}^{n}(1-q^j)$. The following generalization of the results just discussed is therefore a consequence of Theorem 9.5.23 since taking the stable principal specialization of a Schur function yields a polynomial in $\mathbb{Z}_{\geq 0}[q]$.

Theorem 9.5.24 (See [48, Conjecture 9.6])**.** *Let $\mathbf{m}$ be a Hessenberg vector. Then as a polynomial in t, the generalized q-Eulerian polynomial $A_{\mathbf{m}}(q,t)$ is palindromic and q-unimodal.*

Remark 9.5.25. *Our efforts to prove Theorem 9.5.24 are what led us to formulate the conjecture relating chromatic quasisymmetric functions and Hessenberg varieties ([47, Conjecture 5.3]), which was subsequently proved by Brosnan and Chow (see Theorem 9.5.18). We don't know of a way to prove the purely combinatorial Theorem 9.5.24 without relying on Theorems 9.5.18 and 9.5.21.*

9.6 Hecke algebras, immanants, and chromatic quasisymmetric functions

We review here some of the work of Haiman in [31], of Clearman, Hyatt, Shelton, and Skandera in [16], and Abreu and Nigro in [2]. Haiman used Kazhdan-Lusztig theory for the type A Hecke algebra to study various immanant conjectures of Stembridge. He made several very interesting conjectures and proved that the "$q = 1$" case of the first of these conjectures ([31, Conjecture 2.1]) implies Stembridge's Monomial Immanant Conjecture and thus implies the Stanley–Stembridge e-Positivity Conjecture Clearman, Hyatt, Shelton, and Skandera strengthened Haiman's result: they showed that [31, Conjecture 2.1] implies our refined Stanley–Stembridge e-Positivity Conjecture (Conjecture 9.4.4).

Haiman's second conjecture ([31, Conjecture 3.1]) implies the second conjecture of Stanley and Stembridge (Conjecture 9.3.12). Abreu and Nigro found a counterexample to [31, Conjecture 3.1] and proposed a weaker version of this conjecture, which in the case $q = 1$ is the same as Haiman's second conjecture. In the general case it reduces Haiman's first conjecture to our refined Stanley–Stembridge e-Positivity Conjecture.

9.6.1 Hecke algebras traces on the Kazhdan–Lusztig basis

We begin by stating some basic facts about the type A Hecke algebra that can be found in, for example, [24]. Given positive integer n, let $\mathbb{S}$ be the set of simple transpositions in S_n,

$$\mathbb{S} := \{(i, i+1) : i \in [n-1]\} \subseteq S_n.$$

The **Iwahori–Hecke algebra of type A**, denoted by $H_n(q)$, is the unital $\mathbb{C}(q^{1/2}, q^{-1/2})$-algebra generated by the set $\{T_s : s \in \mathbb{S}\}$, subject to the following relations:

(R1) $T_s^2 = (q-1)T_s + q1$ (where 1 is the multiplicative identity);

and if $r = (i, i+1)$, $s = (j, j+1)$, and $i \neq j$, then

(R2) $T_r T_s = T_s T_r$ if $|i-j| > 1$,

(R3) $T_r T_s T_r = T_s T_r T_s$ if $|i-j| = 1$.

Given $w \in S_n$ and any reduced word $w = s_1, \dots, s_\ell$ (so w is the product of the s_i, each $s_i \in \mathbb{S}$, and ℓ as small as possible) we define

$$T_w := T_{s_1} \cdots T_{s_\ell} \in H_n(q).$$

One can confirm that T_w does not depend on the choice of reduced word. The set $\{T_w : w \in S_n\}$ is a $\mathbb{C}(q^{1/2}, q^{-1/2})$-basis for $H_n(q)$. Another basis, introduced by David Kazhdan and George Lusztig in [36] is the **Kazhdan–Lusztig basis** $\{C'_w : w \in S_n\}$, where

$$C'_w := q^{-\ell(w)/2} \sum_{v \leq w} P_{v,w}(q) T_v.$$

Here $\leq$ is the Bruhat order on S_n and ℓ is the usual length function. The polynomials $P_{v,w}(q)$ are **Kazhdan–Lusztig polynomials**. For the $w \in S_n$ we will care about most, each $P_{v,w}(q) = 1$. We will be interested in an adjusted basis $\{C_w : w \in S_n\}$, where

$$C_w := q^{\ell(w)/2} C'_w = \sum_{v \leq w} P_{v,w}(q) T_v.$$

If one sets $q = 1$, the Hecke algebra $H_n(q)$ becomes the group algebra $\mathbb{C}[S_n]$ through the identification of T_w with w. It turns out that the $\mathbb{C}(q^{1/2}, q^{-1/2})$-representations of $H_n(q)$ are very closely related to the $\mathbb{C}$-representations of S_n. In particular, the irreducible representations S_q^λ of $H_n(q)$ are indexed by partitions of n and obtained from representations induced from 1-dimensional representations of parabolic subalgebras in the same way that the irreducible representations S^λ of S_n are obtained from the permutation representations M^λ. (The Kostka numbers play the same role.) This allows us to define for each partition λ of n, a **trace** function or **character** χ_q^λ on $H_n(q)$: one takes the irreducible representation S_q^λ of $H_n(q)$ and sets $\chi_q^\lambda(T)$ to be the trace of $T \in H_n(q)$ on S_q^λ. Given $w \in S_n$, if one evaluates $\chi_q^\lambda(T_w)$ at $q = 1$, one obtains $\chi^\lambda(w)$. We define the space of traces on $H_n(q)$ to be the $\mathbb{C}$-vector space spanned by the χ_q^λ.

Let us now recall some notation found in [16]. We will define elements η_q^λ, ϵ_q^λ, ψ_q^λ, ϕ_q^λ of the trace space as follows.

- Give a partition λ of n, set
  - $\eta^\lambda := \mathsf{ch}^{-1} h_\lambda$;
  - $\epsilon^\lambda := \mathsf{ch}^{-1} e_\lambda$;
  - $\phi^\lambda := \mathsf{ch}^{-1} m_\lambda$;
  - $\psi^\lambda := \mathsf{ch}^{-1} p_\lambda$.

  (We observe that η^λ and ϵ^λ are characters of S_n while ϕ^λ and ψ^λ are virtual characters but in general are not actual characters.)

- For $\theta \in \{\eta, \epsilon, \phi, \psi\}$ and partition λ, find the integers $c_{\mu,\lambda}$ (where μ runs over all partitions of n) such that $\theta^\lambda = \sum_\mu c_{\mu,\lambda}\chi^\mu$. (So, the $c_{\mu,\lambda}$ tell us how to expand $\mathsf{ch}\theta^\lambda$ in the Schur basis.)

- Now set $\theta_q^\lambda := \sum_\mu c_{\mu,\lambda}\chi_q^\mu$.

Example 9.6.1. We consider the case $n = 3$. First we find the irreducible representations of $H_3(q)$. From what we know about S_3 these should have dimensions $1, 2, 1$, the respective dimensions of $S^{(3)}$, $S^{(2,1)}$ and $S^{(1,1,1)}$.

Let us consider first the 1-dimensional representations. Such a representation ϕ assigns to T_{213} and T_{132} respective scalars $x_1, x_2 \in \mathbb{C}(q^{1/2}, q^{-1/2})$, each satisfying

$$x_i^2 - (q-1)x_i - q = 0,$$

since T_{213} and T_{132} satisfy (R1). So, $x_1, x_2 \in \{-1, q\}$. Since T_{213} and T_{132} satisfy (R3), we must have $x_1 = x_2$. If $x_i = q$ for both i then $\phi(T_w) = q^{\ell(w)}$ for all $w \in S_n$, yielding the character $\chi_q^{(3)}$, while if $x_i = -1$ for both i then $\phi(T_w)$ is the sign of w for all w, yielding $\chi_q^{(1,1,1)}$.

It remains to find a 2-dimensional irreducible representation. So, we seek matrices $A_{213}, A_{132} \in M_2(\mathbb{C}(q^{[1/2}, q^{-1/2}))$ satisfying the analogues of (R1) and (R3). The reader can confirm that the choices

$$M_{213} = \begin{pmatrix} q & q^{1/2} \\ 0 & -1 \end{pmatrix} \text{ and } M_{132} = \begin{pmatrix} -1 & 0 \\ q^{1/2} & q \end{pmatrix}$$

do the trick. We calculate also that

$$M_{213}M_{132} = \begin{pmatrix} 0 & q^{3/2} \\ -q^{1/2} & -q \end{pmatrix} \text{and } M_{132}M_{231} = \begin{pmatrix} -q & -q^{1/2} \\ q^{3/2} & 0 \end{pmatrix},$$

and that

$$M_{213}M_{132}M_{213} = M_{132}M_{213}M_{132} = \begin{pmatrix} 0 & -q^{3/2} \\ -q^{3/2} & 0 \end{pmatrix}.$$

Taking traces, we see that $\chi_q^{(2,1)}$ is as in the table below.

λ	$\chi_q^\lambda(T_{123})$	$\chi_q^\lambda(T_{132})$	$\chi_q^\lambda(T_{213})$	$\chi_q^\lambda(T_{231})$	$\chi_q^\lambda(T_{312})$	$\chi_q^\lambda(T_{321})$
(3)	1	q	q	q^2	q^2	q^3
$(2,1)$	2	$q-1$	$q-1$	$-q$	$-q$	0
$(1,1,1)$	1	-1	-1	1	1	-1

It turns out that in S_3, each Kazhdan–Lusztig polynomial satisfies $P_{v,w}(q) = 1$; hence $C_w = \sum_{v\le w} T_v$ for all w. This allows us to compute quickly the following table.

λ	$\chi_q^\lambda(C_{123})$	$\chi_q^\lambda(C_{132})$	$\chi_q^\lambda(C_{213})$	$\chi_q^\lambda(C_{231})$	$\chi_q^\lambda(C_{312})$	$\chi_q^\lambda(C_{321})$
(3)	1	$[2]_q$	$[2]_q$	$[2]_q[2]_q$	$[2]_q[2]_q$	$[2]_q[3]_q$
$(2,1)$	2	$[2]_q$	$[2]_q$	q	q	0
$(1,1,1)$	1	0	0	0	0	0

The values in this table can be used to determine the trace functions θ_q^λ for $\theta \neq \chi$. For example, since $m_3 = s_3 - s_{21} + s_{111}$, we get

$$\phi_q^{(3)}(C_{231}) = (q^2 + 2q + 1) - q + 0 = q^2 + q + 1.$$

We note in passing that the first table in Example 9.6.1 is not what is usually referred to as the character table for $H_3(q)$. Knowledge of the value of a trace θ_q on T_w as w runs through a set of conjugacy class representatives of shortest possible length is sufficient to determine θ_q (see , e.g., [24]). Recording such information for all χ_q^λ one obtains a square character table.

9.6.2 Connection with immanants

In [31] Haiman proposes Kazhdan–Lusztig theory for the Hecke algebra $H_n(q)$ as a natural setting for Stembridge's Monomial Immanant Conjecture and related conjectures. He presents the following conjecture and shows that it implies Stembridge's Monomial Immanant Conjecture.

Conjecture 9.6.2 ([31, Conjecture 2.1]). *If $\lambda \in \mathsf{Par}(n)$ and $w \in S_n$ then the trace $\phi_q^\lambda(C_w) \in \mathbb{Z}_{\ge 0}[q]$. Moreover, these polynomials are unimodal.*

Remark 9.6.3. *Haiman shows that for all $w \in S_n$ the trace $\phi_q^\lambda(C_w)$ is a palindromic polynomial with center of symmetry $\ell(w)/2$.*

The variable q does not appear in the Monomial Immanant Conjecture. Haiman shows that by setting $q = 1$ in Conjecture 9.6.2, one is able to obtain the Monomial Immanant Conjecture. The novelty in Haiman's approach is that the introduction of the parameter q can be helpful, which is the same as the novelty of our approach to the Stanley–Stembridge Conjecture in Section 9.4.

Haiman considers the ring $\mathbb{C}[S_n] \otimes \Lambda_{\mathbb{C}}$, that is, the group ring of S_n with coefficients in the ring of symmetric functions with complex coefficients. One can extend any class function on S_n to a function on $\mathbb{C}[S_n] \otimes \Lambda_{\mathbb{C}}$, by linearity. With these extensions in mind, Haiman defines for partitions $\nu \subseteq \mu$ with $\ell(\mu) = n$, the element

$$I_{\mu/\nu} := \sum_{w \in S_n} w h_{\mu+\delta-w(\nu+\delta)}.$$

of $\mathbb{C}[S_n] \otimes \Lambda_{\mathbb{C}}$. (Here $\delta = (n-1, n-2, \ldots, 1, 0)$ as in (9.9).) Our notation differs from that used by Haiman so as to be consistent with that which we have used throughout this chapter. Haiman's results are, in fact, a bit more general than those we present.

Haiman then observes (cf. (9.9)) that if χ is a class function on S_n, extended as above,

$$\mathsf{Imm}_\chi(H_{\mu/\nu}) = \chi(I_{\mu/\nu}), \tag{9.36}$$

which allows the study of immanants through the examination of Hecke algebra characters. Let us write $C_w(1)$ for the element of the group algebra $\mathbb{C}[S_n]$ obtained from the Kazhdan–Lusztig basis element C_w in the Hecke algebra upon identifying each T_w with w, that is,

$$C_w(1) := \sum_{v \leq w} P_{v,w}(1)\, w \in \mathbb{C}[S_n].$$

A $\mathbb{C}$-basis for the k^{th} homogeneous piece of the graded ring $\mathbb{C}[S_n] \otimes \Lambda_{\mathbb{C}}$ is

$$\mathcal{B}_{n,k} := \{C_w(1)s_\lambda : \ w \in S_n, \ \lambda \in \mathsf{Par}(k)\}. \tag{9.37}$$

Haiman proves the following result by using a generalization of the Steinberg tensor product theorem to infinite-dimensional irreducible highest weight modules over a simple complex Lie algebra.

Theorem 9.6.4 (See [31, Theorem 1.5]). *Let $\nu \subseteq \mu$ be partitions with $\ell(\lambda) = n$ and $|\mu| - |\nu| = k$. Then all coefficients in the expansion of $I_{\mu/\nu}$ in terms of the basis $\mathcal{B}_{n,k}$ are nonnegative integers.*

Theorem 9.6.4 and Conjecture 9.6.2 together imply Stembridge's Monomial Immanant Conjecture (Conjecture 9.2.2). Indeed, given partitions $\nu \subseteq \mu$ with $\ell(\mu) = n$ and $|\mu| - |\nu| = N$, we expand

$$I_{\mu/\nu} = \sum_{\substack{w \in S_n, \\ \rho \in \mathsf{Par}(N)}} a_{w,\rho}\, C_w(1) s_\rho \tag{9.38}$$

in terms of $\mathcal{B}_{n,N}$ with each $a_{w,\rho} \in \mathbb{Z}_{\geq 0}$. Now for $\lambda \in \mathsf{Par}(n)$,

$$\begin{aligned}\mathsf{Imm}_{\mathsf{ch}^{-1}(m_\lambda)}(H_{\mu/\nu}) &= \phi^\lambda(I_{\mu/\nu}) \\ &= \sum_{\rho \in \mathsf{Par}(N)} \left(\sum_{w \in S_n} a_{w,\rho}\, \phi^\lambda(C_w(1)) \right) s_\rho \qquad (9.39) \\ &= \sum_{\rho \in \mathsf{Par}(N)} \left(\sum_{w \in S_n} a_{w,\rho} \left[\phi_q^\lambda(C_w)\right]_{q=1} \right) s_\rho .\end{aligned}$$

If Conjecture 9.6.2 holds then

$$a_{w,\rho} \left[\phi_q^\lambda(C_w)\right]_{q=1} \geq 0$$

for all pairs (w, ρ) and Conjecture 9.2.2 follows.

In a similar vein, Haiman proves Stembridge's weaker immanant conjecture (Theorem 9.2.8) by using the $q = 1$ case of the following result.

Theorem 9.6.5 ([31, Lemma 1.1]). *If* $\lambda \in \mathsf{Par}(n)$ *and* $w \in S_n$ *then the trace* $\chi_q^\lambda(C_w) \in \mathbb{Z}_{\geq 0}[q]$. *Moreover, these polynomials are palindromic and unimodal.*

9.6.3 Connection with chromatic quasisymmetric functions

In the first table in Table 9.1, we list the chromatic quasisymmetric function of $G_{\mathbf{m}}$, for each Hessenberg vector $\mathbf{m}$ of length 3. In the second table in Table 9.1, we list the symmetric function $\sum_\lambda \chi_q^\lambda(C_w) s_\lambda$, for each permutation w in S_3. One can compute the chromatic quasisymmetric functions by hand and use the second table of Example 9.6.1 to obtain the coefficients $\chi_q^\lambda(C_w)$. We use q rather than t for the grading variable here, for reasons that will become clear.

TABLE 9.1: Chromatic quasisymmetric functions of $G_{\mathbf{m}}$ for $\mathbf{m}$ length 3 and symmetric functions for permutations in S_3.

$\mathbf{m}$	$X_{G_{\mathbf{m}}}(\mathbf{x}; q)$
$(1,2,3)$	$s_3 + 2s_{21} + s_{111}$
$(1,3,3)$	$(q+1)(s_{21} + s_{111})$
$(2,2,3)$	$(q+1)(s_{21} + s_{111})$
$(2,3,3)$	$qs_{21} + (q^2+2q+1)s_{111}$
$(3,3,3)$	$(q^3+2q^2+2q+1)s_{111}$

w	$\sum_\lambda \chi_q^\lambda(C_w) s_\lambda$
123	$s_3 + 2s_{21} + s_{111}$
132	$(q+1)(s_3 + s_{21})$
213	$(q+1)(s_3 + s_{21})$
231	$(q^2+2q+1)s_3 + qs_{21}$
312	$(q^2+2q+1)s_3 + qs_{21}$
321	$(q^3+2q^2+2q+1)s_3$

We see that for each $\mathbf{m} = (m_1, m_2, m_3)$ there is some $w \in S_3$ such that

$$X_{G_{\mathbf{m}}}(\mathbf{x}; q) = \sum_\lambda \chi_q^\lambda(C_w) s_{\lambda'},$$

with λ' being the conjugate partition to λ. It is shown in [16] that this phenomenon occurs for all n. It is probably not yet clear to the reader what the correspondence between $\mathbf{m}$ and w should be.

For a positive integer n, the number of Hessenberg vectors of length n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is much smaller than $n!$. However, there is a well-known subset of S_n with size C_n. Given $\sigma \in S_\ell$ and $w \in S_n$, we say w **contains** σ if there exist $i_1 < \cdots < i_\ell$ such that, for $1 \leq j < k \leq \ell$, $w(i_j) < w(i_k)$ if and only if $\sigma(j) < \sigma(k)$. For example, $w = 64372158$ contains 312: we can take $i_1 = 4$, $i_2 = 5$, and $i_3 = 7$, and compare the relations among $7, 2, 5$ with those among $3, 1, 2$. We say that w **avoids** σ if w does not contain σ. The number of $w \in S_n$ avoiding 312 is C_n (see [53, Exercise 6.19(ff)]). Here is an imprecise version of the above-mentioned theorem of Clearman, Hyatt, Shelton, and Skandera.

Theorem 9.6.6 (See [16, Corollary 7.5]). *For each positive integer n, there is a bijection β from the set of 312-avoiding permutations in S_n to the set of Hessenberg vectors of length n such that*

$$\begin{aligned}
X_{G_{\beta(w)}}(\mathbf{x};q) &= \sum_{\lambda \in \mathsf{Par}(n)} \epsilon_q^\lambda(C_w) m_\lambda \\
&= \sum_{\lambda \in \mathsf{Par}(n)} \eta_q^\lambda(C_w)\, \omega m_\lambda \\
&= \sum_{\lambda \in \mathsf{Par}(n)} \chi_q^\lambda(C_w) s_{\lambda'} \\
&= \sum_{\lambda \in \mathsf{Par}(n)} (-1)^{\ell(\lambda)} \psi_q^\lambda(C_w) \frac{p_\lambda}{z_\lambda} \\
&= \sum_{\lambda \in \mathsf{Par}(n)} \phi_q^\lambda(C_w) e_\lambda
\end{aligned}$$

for all 312-avoiding $w \in S_n$.

The last four equalities of Theorem 9.6.6 follow from the first one and the construction of the various traces θ_q^λ described above. We will attempt to give a very brief idea of what is involved in the proof given in [16] of the first equality. Using tableaux, a combinatorial formula for $\epsilon_q^\lambda(C_w)$ is given for 312-avoiding $w \in S_n$. It is shown that this combinatorial formula is the same as one derived from straightforward calculations for the coefficient of m_λ in $X_{G_{\beta(w)}}(\mathbf{x};t)$. To obtain the combinatorial formula for $\epsilon_q^\lambda(C_w)$, Clearman, Hyatt, Shelton, and Skandera associate with w a descending star network F_w, which is a type of acyclic planar directed graph that can be embedded in a disc with boundary vertices labeled 1 to $2n$ in a clockwise direction. The boundary vertices labeled $1, \ldots, n$ are the sources and the boundary vertices labeled $n+1, \ldots, 2n$ are the sinks. They then use q-analogs of the following facts:

- For all 312-avoiding $w \in S_n$ and class function θ,

$$\theta(C_w(1)) = \mathsf{Imm}_\theta(B_{F_w}),$$

  where B_{F_w} is the path matrix of F_w, that is, the matrix that gives the number of paths from each source to each sink in F_w.

- for any $n \times n$ matrix X,

$$\mathsf{Imm}_{\epsilon^\lambda}(X) = \sum_{(I_1,\ldots,I_r)} \det X_{I_1} \cdots \det X_{I_r}$$

  summed over all ordered set partitions $(I_1, \ldots, I_r)$ of $[n]$, where X_{I_j} is the submatrix of X consisting only of the rows and columns indexed by elements of I_j.

The first fact can be found in [16, Equation (3.6)] and its q-analog was proved in [16, Proposition 3.8]. The second fact can be found in [16, Equation (2.1)] and its q-analog can be found in [16, equation (2.5)].

Since F_w is a descending star network there is exactly one path from the ith source to the ith sink for each i, and these paths have a natural partial order. The map taking the ith path to i is an isomorphism from the partial order on the paths to the naturally labeled unit interval order $P_{\beta(w)}$. The q-analogs of the two facts listed above play an essential role in the proof of Theorem 6.4 of [16], which states that

$$\epsilon_q^\lambda(C_w) = \sum_T q^{\mathrm{inv}_{G_{\beta(w)}}(\sigma(T))} \tag{9.40}$$

summed over all row-strict $P_{\beta(w)}$-tableaux T of shape λ, where **row strict** means that only the first condition of Definition 9.4.7 has to be satisfied, and $\sigma(T)$ denotes the permutation obtained from T by concatenating the rows of T from top to bottom. (Recall that $\mathrm{inv}_G(w)$ was defined in Definition 9.4.16.) The right hand side of (9.40) is also equal to the coefficient of m_λ in $X_{G_{\beta(w)}}(\mathbf{x}, q)$; cf. [16, Proposition 7.3]. This is proved by associating with each row-strict $P_{\beta(w)}$-tableaux T, the proper coloring of $G_{\beta(w)}$ obtained by coloring all entries of the ith row of T (viewed as vertices of the graph) with i, for each i. Hence we can conclude that the first equality holds.

Remark 9.6.7. *Recall that for any naturally labeled unit interval graph G, Schur-positivity of the chromatic quasisymmetric function $X_G(\mathbf{x}; t)$ was first obtained in our work through a combinatorial description of the expansion coefficients; see Theorem 9.4.11. Schur-unimodality was obtained as a consequence of the bridge between chromatic quasisymmetric functions and Hessenberg varieties discussed in Section 9.5.5; see Theorem 9.5.23. It turns out that an alternative bridge can be used to obtain this result, namely the bridge between chromatic quasisymmetric functions and Hecke algebra traces established in Theorem 9.6.6. Indeed, it follows from the third equality of Theorem 9.6.6 and Theorem 9.6.5 that Schur-positivity and Schur-unimodality hold.*

Combining Theorem 9.6.6 and Theorem 9.5.18, we obtain the following result.

Corollary 9.6.8. *With n and β as in Theorem 9.6.6, the dot representations on the cohomology groups of regular semisimple Hessenberg varieties satisfy*

$$\begin{aligned}
\mathrm{ch}\chi_{s,\beta(w)} &= \sum_{\lambda\in\mathrm{Par}(n)} \epsilon_q^{\lambda}(C_w)\,\omega m_\lambda \\
&= \sum_{\lambda\in\mathrm{Par}(n)} \eta_q^{\lambda}(C_w) m_\lambda \\
&= \sum_{\lambda\in\mathrm{Par}(n)} \chi_q^{\lambda}(C_w) s_\lambda \\
&= \sum_{\lambda\in\mathrm{Par}(n)} \psi_q^{\lambda}(C_w) \frac{p_\lambda}{z_\lambda} \\
&= \sum_{\lambda\in\mathrm{Par}(n)} \phi_q^{\lambda}(C_w) h_\lambda
\end{aligned}$$

for all 312-avoiding $w \in S_n$.

It remains to give a precise description of the bijection β. It turns out that a nice description exists: given 312-avoiding $w = w(1)\cdots w(n) \in S_n$, we define $\beta(w) = (m_1, \ldots, m_n)$ one entry at a time: $m_1 = w(1)$ and, for $i > 1$, $m_i = \max\{w(i), m_{i-1}\}$. So, for example, if $w = 24315$ then $\beta(w) = (2, 4, 4, 4, 5)$.

Given our interest in $X_{G_\mathbf{m}}$ and $\chi_{s,\mathbf{m}}$, it is more useful for us to describe β^{-1}. This is done in a clever and attractive manner in [16], using descending star networks. We give here a different description that appears in the honors thesis [45] of Ryan Schneider. We use the colexicographic order on transpositions: given (i, j) and (k, ℓ) in S_n with $i < j$ and $k < \ell$, we declare $(i, j) \prec (k, \ell)$ if either $j < \ell$ or $j = \ell$ and $i < k$ (in the usual order on $\mathbb{Z}$). So, for example, in S_4 we have

$$(1, 2) \prec (1, 3) \prec (2, 3) \prec (1, 4) \prec (2, 4) \prec (3, 4).$$

Now given Hessenberg vector $\mathbf{m} = (m_1, \ldots, m_n)$, we write all transpositions $(i, j) \in S_n$ satisfying $i < j \le m_i$ from left to right in colexicographic order and compose the transpositions from right to left to obtain $\beta^{-1}(\mathbf{m})$. For example, if $\mathbf{m} = (3, 3, 4, 4)$ then

$$\beta^{-1}(\mathbf{m}) = (1, 2)(1, 3)(2, 3)(3, 4) = (1, 3, 4) = 3241,$$

which is indeed 312-avoiding. We list below $\beta^{-1}(\mathbf{m})$ for each Hessenberg vector $\mathbf{m}$ of length three. The reader can compare Table 9.2 with Table 9.1 to confirm that Theorem 9.6.6 holds when $n = 3$ with the described choice of β^{-1}.

It is natural to ask whether the geometric interpretation of C_w given in Corollary 9.6.8 for 312-avoiding permutations can be lifted to one for all

TABLE 9.2: Images of $\beta^{-1}(\mathbf{m})$ for each Hessenberg vector $\mathbf{m}$.

$\mathbf{m}$	$\beta^{-1}(\mathbf{m})$
$(1,2,3)$	$1 = 123$
$(1,3,3)$	$(2,3) = 132$
$(2,2,3)$	$(1,2) = 213$
$(2,3,3)$	$(1,2)(2,3) = (1,2,3) = 231$
$(3,3,3)$	$(1,2)(1,3)(2,3) = (1,3) = 321$

permutations. As discussed by Abreu and Nigro in [2, 3], there is such an interpretation in which a certain subvariety of the flag variety called a Lusztig variety plays the role of a Hessenberg variety. Indeed, for all $w \in S_n$, the symmetric function $\sum_{\lambda \in \mathsf{Par}(n)} \chi_q^\lambda(C_w) s_\lambda$ is equal to the Frobenius characteristic of a natural representation of S_n on the intersection cohomology of the type A regular semisimple **Lusztig variety**

$$Y(s,w) := \{V_\bullet : \dim(sV_i \cup V_j) \geq r_{i,j}(w)\},$$

where s is regular semisimple and $r_{i,j}(w) := |\{k \in [n] : k \leq i, w(k) \leq j\}|$. The representation arises from a certain monodromy action. It is not difficult to show that when w is 312-avoiding,

$$Y(s,w) = \mathbf{Hess}(s, \beta(w)).$$

When $Y(s,w)$ is smooth, its intersection cohomology is its ordinary cohomology. It turns out that when s is regular semisimple and w avoids 312, Lusztig's representation of S_n on the cohomology of $Y(s,w)$ is the same as the representation on the cohomology of $\mathbf{Hess}(s,w)$ arising from the monodromy action discussed in Section 9.5.4. As mentioned in that section, Brosnan and Chow showed in [9] that this induced representation on $\mathbf{Hess}(s,\beta(w))$ is the same as the dot representation $\chi_{s,\beta(w)}$.

Theorem 9.6.6 shows that Conjecture 9.4.4 (together with its associated e-unimodality conjecture given in Remark 9.4.5) and Conjecture 9.5.19 (together with the conjecture on quotients given in Remark 9.5.22) are equivalent to the following special case of Conjecture 9.6.2.

Conjecture 9.6.9 (Haiman [31]). *If $w \in S_n$ avoids 312, then for every partition λ of n, the trace $\phi_q^\lambda(C_w) \in \mathbb{Z}_{\geq 0}[q]$. Moreover, these polynomials are unimodal.*

The following result, which was conjectured by Haiman in [31, Conjecture 4.1], shows that Conjecture 9.6.9 holds for $\lambda = (n)$. It is equivalent to our Theorem 9.4.14 as a consequence of Theorem 9.6.6. A direct proof is also given in [16].

Theorem 9.6.10. *If $w \in S_n$ avoids 312, then*

$$\phi_q^{(n)}(C_w) = [n]_q \prod_{i=1}^{n-1} [\beta(w)_i - i]_q.$$

Another conjecture of Haiman reduces Conjecture 9.6.2 to the case of 312-avoiding permutations given in Conjecture 9.6.9.

Conjecture 9.6.11 (Haiman [31, Conjecture 3.1]). *For each $w \in S_n$, there exist 312-avoiding $w_1, \ldots, w_k \in S_n$ such that for all Hecke algebra traces θ,*

$$\theta(C_w) = \sum_{i=1}^{k} q^{\frac{\ell(w)-\ell(w_i)}{2}} \theta(C_{w_i}).$$

In [16], Clearman, Hyatt, Shelton, and Skandera show that this conjecture is true in a special case.

Theorem 9.6.12 (Clearman, Hyatt, Shelton, and Skandara [16]). *If $w \in S_n$ avoids the patterns 3412 and 4231 then there exists a 312-avoiding $u \in S_n$ such that $\ell(w) = \ell(u)$ and for all Hecke algebra traces θ,*

$$\theta(C_w) = \theta(C_u).$$

For example, by looking at the second table at the beginning of this subsection, one can see that the permutation 312 has the same trace evaluation as a permutation that avoids the pattern 312, namely 231.

To prove Theorem 9.6.12, Clearman, Hyatt, Shelton, and Skandera used combinatorial methods similar to the ones used in their proof of Theorem 9.6.6. More recently, in [2] Abreu and Nigro gave geometric proofs of both results. Moreover, they showed that Conjecture 9.6.11 is not true in general by proving that $w = 62754381$ is a counterexample; see [2, Theorem 1.8]. They also proposed the following weaker version of Conjecture 9.6.11.

Conjecture 9.6.13 (Abreu and Nigro [2]). *If $w \in S_n$ then there exist 312-avoiding $w_1, \ldots, w_k \in S_n$ such that for all Hecke algebra traces θ, the evaluation $\theta(C_w)$ is a combination of $\theta(C_{w_i})$, with coefficients in $\mathbb{Z}_{\geq 0}[q]$.*

This weaker version still serves to reduce the first assertion of Conjecture 9.6.2 to the first assertion of the special case Conjecture 9.6.9, which, as was mentioned above, is equivalent (by Theorem 9.6.6) to our refinement of the Stanley-Stembridge e-Positivity Conjecture (Conjecture 9.4.4). One can ask if Conjecture 9.6.13 can be strengthened by asserting that for each i, the coefficient of $\theta(C_{w_i})$ is a palindromic and unimodal polynomial in $\mathbb{Z}_{>0}[q]$ with center of symmetry $\frac{\ell(w)-\ell(w_i)}{2}$. This strengthened version of the conjecture would reduce all of Conjecture 9.6.2 to its special case or equivalently to the version of our refinement of the first Stanley-Stembridge conjecture that includes e-unimodality (as given in Remark 9.4.5).

As discussed in [31, Section 3], the $q = 1$ case of Conjecture 9.6.11 would imply the second Stanley–Stembridge Conjecture (Conjecture 9.3.12), which reduces the Monomial Immanant Conjecture to the unit interval case of the first Stanley–Stembridge conjecture (Conjecture 9.3.6), now proved by Hikita.

Since the $q = 1$ case of Conjecture 9.6.13 is essentially the same as the $q = 1$ case of Conjecture 9.6.11, the weaker Conjecture 9.6.13 would still imply Conjecture 9.3.12.

To show that the $q = 1$ case of Conjecture 9.6.13 implies Conjecture 9.3.12, let $\mu/\nu, n$ and N be as in Conjecture 9.3.12. By Theorem 9.2.6, for each $\lambda \in \mathsf{Par}(n)$ and $\rho \in \mathsf{Par}(N)$, the coefficient of h_λ in the h-expansion of $\mathrm{ch}(\Gamma^{\rho}_{\mu/\nu})$ is equal to the coefficient of s_ρ in the Schur expansion of $\mathsf{Imm}_{\phi^\lambda}(H_{\mu/\nu})$, which we now compute. By (9.39) the coefficient of s_ρ is $\sum_{w \in S_n} a_{w,\rho}\phi^\lambda(C_w(1))$, where $a_{w,\rho} \in \mathbb{Z}_{\geq 0}$. We can rewrite this coefficient as

$$\sum_{w \in M_\rho} \phi^\lambda(C_w(1)), \tag{9.41}$$

where M_ρ is a multiset of permutations in S_n. Now the q=1 case of Conjecture 9.6.13 implies that M_ρ can be replaced in (9.41) by a multiset $M_\rho(312)$ of 312-avoiding permutations in S_n. Therefore, by Theorem 9.2.6,

$$\begin{aligned}
\mathrm{ch}(\Gamma^{\rho}_{\mu/\nu}) &= \sum_{\lambda \in \mathsf{Par}(n)} \sum_{w \in M_\rho(312)} \phi^\lambda(C_w(1)) h_\lambda \\
&= \sum_{w \in M_\rho(312)} \sum_{\lambda \in \mathsf{Par}(n)} \phi^\lambda(C_w(1)) h_\lambda \\
&= \sum_{w \in M_\rho(312)} \omega X_{G_{\beta(w)}}(\mathbf{x}),
\end{aligned}$$

with the last equality following from the $q = 1$ case of Theorem 9.6.6, which is also implicit in Haiman's work in [31]. Hence, Conjecture 9.3.12 is a consequence of the $q = 1$ case of Conjecture 9.6.13.

Haiman's work and that of Clearman–Hyatt–Shelton–Skandera and Abreu–Negro complete a circle of ideas and conjectures connecting immanants, chromatic symmetric functions, Hessenberg varieties, and Hecke algebras.

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