

MGF 1106: Exam 2 Solutions

1. (15 points) A coin and a die are tossed together onto a table.

a. What is the sample space for this experiment?

**Solution:** For example, one possible outcome is "heads on the coin and 4 on the die", which we may write as " $H4$ ". The set of all possible outcomes may similarly be written as:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

Note that  $n(S) = 12$ .

b. What is the probability of getting: tails on the coin, three on the die?

**Solution:**

$$P(T3) = \frac{n(\{T3\})}{n(S)} = \frac{1}{12}$$

c. What is the probability of getting: heads on the coin, even number on the die?

**Solution:**

$$P(\text{heads, even}) = \frac{n(\{H2, H4, H6\})}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

2. (10 points) Ten comedians show up for an open-mic night. Due to time constraints, only seven can perform that night.

a. In how many ways can the promoter choose three comedians to leave out?

**Solution:** The promoter must choose 3 of the 10 comedians to leave out. Since the order in which this is done is irrelevant, the number of ways to do this is

$${}_{10}C_3 = 120$$

b. In how many ways can the promoter choose the line-up for the seven that will perform?

**Solution:** Having asked 3 comedians to leave, the promoter must then choose a line-up, i.e., *order* for the seven remaining comedians to perform in. The total number of different ways to do this is

$$7! = 5,040$$

This, of course, comes from the fundamental counting principle, as the promoter has 7 choices for the first slot, 6 remaining choices for the second slot, 5 remaining choices for the third slot, and so on, or, as we've pointed out, in general,

the number of ways of arranging  $n$  objects in order =  $n!$

3. (10 points) How many different permutations (rearrangements) of the letters in each word are possible?

a. "WORD"

**Solution:** Again, recall that the number of ways to arrange  $n$  (distinct) objects in order is  $n!$ . Here, we have 4 distinct objects, the letters W,O,R, and D, so the number of different arrangements, or permutations, is

$$4! = 24$$

b. "LETTER"

**Solution:** Here we have 6 total objects, but they are not all distinct. The two E's are identical and the two T's are identical. In this case, we must compensate for the fact that, say if someone hands us an arrangement like "TETLER", we may rearrange the E's and/or T's and hand it back to them, without actually having changed anything. To do this, we divide out by the number of ways of rearranging the identical objects. Since we have 2 E's and 2 T's, the number of ways of rearranging the E's and T's is  $2! \cdot 2!$ , which is just 4.

$$\frac{6!}{2! \cdot 2!} = \frac{720}{4} = 180$$

In general, the number of different arrangements of  $n$  objects,  $p$  of which are identical, another group of  $q$  are identical, another group of  $r$  are identical, ..., is:

$$\frac{n!}{p! \cdot q! \cdot r! \cdot \dots}$$

4. (20 points) The local pizza shop makes pizzas in three sizes, with two kinds of crust, and offers nine different toppings.

a. How many different *plain* (no toppings) pizzas can be ordered?

**Solution:** To order a plain pizza, you have two choices to make:

- 1) Choose size. You have 3 options on size.
- 2) Choose crust. You have 2 options on crust.

Hence, by the fundamental counting principle (FCP), the total number of ways of ordering a plain pizza is

$$3 \cdot 2 = 6$$

b. How many different *one-topping* pizzas can be ordered?

**Solution:** To order a one-topping pizza, you have three choices to make:

- 1) Choose size. You have 3 options on size.
- 2) Choose crust. You have 2 options on crust.
- 3) Choose one topping. You have 9 options.

Hence, by FCP, the total number of different one-topping pizzas is

$$3 \cdot 2 \cdot 9 = 54$$

c. Forgetting size and crust, in how many ways can you select *two* toppings?

**Solution:** Of course, the order in which we select two toppings is irrelevant, so the total number of ways of selecting two toppings is

$${}_9C_2 = 36$$

d. How many different *two*-topping pizzas can be ordered? (Hint: You have 3 choices to make: size, crust, and the two toppings. Use FCP and your answer from part c.)

**Solution:** To order a two-topping pizza, you can think of having three choices to make:

- 1) Choose size. You have 3 options on size.
- 2) Choose crust. You have 2 options on crust.
- 3) Choose a pair of toppings. You have  ${}_9C_2 = 36$  options.

Hence, by FCP, the total number of different two-topping pizzas is

$$3 \cdot 2 \cdot 36 = 216$$

5. (25 points) A committee of six people is to be selected from a group of five men and seven women.

a. How many different committees are possible?

**Solution:** All together, there are 12 people total. Since order is not preset in a 'committee' (with no distinction between its members), The total number of ways of selecting 6 people from 12 total is

$${}_{12}C_6 = 924$$

b. How many different committees are 'half and half', three men and three women?

**Solution:** To choose a committee with of 3 men and 3 women, we must:

- 1) Choose 3 men. There are  ${}_5C_3 = 10$  ways to do this.
- 2) Choose 3 women. There are  ${}_7C_3 = 35$  ways to do this.

By FCP, the total number of ways to form a committee of 3 men and 3 women is

$${}_5C_3 \cdot {}_7C_3 = 10 \cdot 35 = 350$$

c. If the committee is chosen randomly, what is the probability that it is half and half?

**Solution:**

$$P(3M, 3W) = \frac{\text{number of committees with 3 men and 3 women}}{\text{number of committees with any 6 people}} = \frac{350}{924} \approx 0.38 \approx 38\%$$

d. If the committee is chosen randomly, what is the probability that it is all women?

**Solution:**

$$P(6W) = \frac{\text{number of committees with 6 women}}{\text{number of committees with any 6 people}} = \frac{{}_7C_6}{{}_{12}C_6} = \frac{7}{924} \approx 0.0076 \approx 0.76\%$$

Of course, for the top, we can also just choose which woman to leave out. This is another way to see that there are 7 different all-woman committees.

e. If the committee is chosen randomly, what is the probability that all the men are chosen?

**Solution:** That all the men get chosen, means that the committee consists of the 5 men and 1 of the women. How many different committees with all 5 men and 1 woman? Well, the only choice is who we pick for the 1 woman, so there are 7 such committees.

$$P(5M, 1W) = \frac{\text{number of committees with all 5 men and 1 woman}}{\text{number of committees with any 6 people}} = \frac{7}{924} \approx 0.0076 \approx 0.76\%$$

6. (20 points) A four-card hand is to be drawn from a standard deck.

a. In how many ways can four hearts be drawn?

**Solution:** In a "hand" of cards, order does not matter. Once the cards are in our hands, we can rearrange them as we like. Thus, the number of ways we can select 4 hearts from 13 total is:

$${}_{13}C_4 = 715$$

b. In how many ways can one card of each suit be drawn?

**Solution:** To select one card of each suit, you must:

- 1) Choose a heart. You have 13 options.
- 2) Choose a diamond. You have 13 options.
- 3) Choose a spade. You have 13 options.
- 4) Choose a club. You have 13 options.

By FCP, the total number of ways of choosing one card of each suit is:

$$13 \cdot 13 \cdot 13 \cdot 13 = 28,561$$

c. If the cards are drawn randomly, what is the probability of getting all hearts?

**Solution:**

$$P(\{\heartsuit, \heartsuit, \heartsuit, \heartsuit\}) = \frac{\text{number of ways to select 4 hearts}}{\text{number of ways to select any 4 cards}} = \frac{{}^{13}C_4}{{}^{52}C_4} = \frac{715}{270,725} \approx 0.00264$$

d. If the cards are drawn randomly, what is the probability of getting two hearts and two spades?

**Solution:**

$$P(\{\heartsuit, \heartsuit, \spadesuit, \spadesuit\}) = \frac{\text{number of ways to select 2 hearts and 2 spades}}{\text{number of ways to select any 4 cards}} = \frac{{}^{13}C_2 \cdot {}^{13}C_2}{{}^{52}C_4} = \frac{6,084}{270,725} \approx 0.02247$$