WINDOWS IN ALGEBRAIC GEOMETRY AND APPLICATIONS TO MODULI

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by

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ABSTRACT

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We apply the theory of windows, as developed by Halpern-Leistner and by Ballard, Favero and Katzarkov, to study certain moduli spaces and their derived categories. Using quantization and other techniques we show that stable GIT quotients of $(\mathbb{P}^1)^n$ by PGL_2 over an algebraically closed field of characteristic zero satisfy a rare property called Bott vanishing, which states that $\Omega_Y^j \otimes L$ has no higher cohomology for every j and every ample line bundle L. Similar techniques are used to reprove the well known fact that toric varieties satisfy Bott vanishing. We also use windows to explore derived categories of moduli spaces of rank-two vector bundles on a curve. By applying these methods to Thaddeus' moduli spaces, we find a four-term sequence of semi-orthogonal blocks in the derived category of the moduli space of rank-two vector bundles on a curve of genus at least 3 and determinant of odd degree, a result in the direction of the Narasimhan conjecture.

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CHAPTER 1

INTRODUCTION

For an algebraic variety X, we consider the derived category of coherent sheaves $D^b(X)$, whose objects are bounded chain complexes of coherent sheaves, and whose morphisms include inverses of quasi-isomorphisms, in addition to the usual chain complex morphisms.

In recent years, exciting work has been done regarding derived categories in algebraic geometry, including applications to birational geometry, geometric invariant theory, moduli theory, among others. In [Kuz16], Kuznetsov suggests that the problem of whether a variety is rational can be attacked using techniques from derived categories and semi-orthogonal decompositions of those. Also, great advances have been made regarding Bridgeland stability conditions on derived categories of algebraic varieties, by Bayer, Macrì, Stellari, and several others (see e.g. [BMS16], [BLMS19]). Stability conditions have a big impact in the study moduli of spaces of stable sheaves.

Research done by Halpern-Leistner [HL15], as well as work by Ballard, Favero and Katzarkov [BFK19], study the derived category of a GIT quotient $X /\!\!/ G$ by a reductive group and how it changes under variation of GIT. One of the main results in these works shows how, under some mild hypotheses, the derived category of $X /\!\!/ G$ can be embedded into that of the ambient quotient stack $[X/G]$ in several different ways, each of them called a *window*. They also describe what happens with the derived categories when we move from one GIT chamber to another. These techniques can be used to describe derived categories of moduli spaces and also see how they vary under wall-crossing. As an example, Castravet and Tevelev have used the method of windows to prove a conjecture by Manin and Orlov, stating that the derived category of the moduli space $\overline{M}_{0,n}$ of stable rational curves with n marked points admits an S_n-equivariant full exceptional collection [CT17, CT20a, CT20b].

One feature of this theory is that it can be used to compute cohomology of sheaves on $X /\!\!/ G$ that descend from objects in the G-equivariant derived category $D^b_G(X)$. One of the main results in the present project uses this fact, together with other techniques in algebraic geometry, to prove that the geometric quotients $(\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ satisfy a rare property called Bott vanishing.

Specifically, this means that sheaves of the form $\Omega^j \otimes L$ have no higher cohomology, whenever L is an ample line bundle (see Theorem 2.1.1 below). The proof of this theorem involves the use of classical invariant theory, geometric syzygies, among other tools used toward cohomology calculations. Some of these techniques can also be applied in more general situations. Most remarkably, they yield a new proof of Bott vanishing for toric varieties in characteristic zero, a fact that has been known for a long time and with several different proofs.

As another application of the windows methods, we study the derived category of Thaddeus' moduli spaces M_σ of stable pairs (E, ϕ) on a smooth projective curve, where E is a rank two vector bundle and ϕ is a section, subject to a given stability condition. We find that, under certain conditions, the derived category of C embeds into that of M_σ using the Fourier-Mukai functor given by the universal family. This allows us to come up with a four-term sequence of semi-orthogonal blocks in the moduli space $M_C(2, \Lambda)$ of slope-semistable rank-two bundles with fixed determinant Λ of odd degree, on a curve C of genus at least 3.

In the present chapter, we describe the results from windows theory and derived categories of GIT quotients that will be needed in order to understand the subsequent work regarding cohomology vanishing and other applications to moduli spaces. This also requires some background from geometric invariant theory and derived categories. Bott vanishing is discussed in Chapter 2, including the main result about quotients of the form $(\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$. In Chapter 3 we discuss the derived category of moduli spaces of vector bundles over a curve, with applications of windows and wall-crossings to the study of the derived category of $M_C(2, \Lambda)$.

1.1 GIT quotients and Kempf-Ness stratifications

We will consider a smooth projective-over-affine variety X over an algebraically closed field $\Bbbk$ of characteristic 0, meaning a closed subvariety of $\mathbb{A}^r \times \mathbb{P}^d$, with a reductive group G acting on X.

Definition 1.1.1. A G-equivariant coherent sheaf F on X is a coherent sheaf F together with an isomorphism $\sigma^* F \cong p_2^* F$ where $\sigma, p_2 : G \times_{\Bbbk} X \to X$ are the action map and the second projection, respectively. A G-equivariant invertible sheaf $\mathcal{L}$ is also called a G-linearized line bundle, and can be seen as $\mathcal{L}$ together with an action of G on the total space of $\mathcal{L}$ that is compatible with the action on X and linear on the fibers.

Given an ample G-linearized line bundle $\mathcal{L}$ on X, we write $X = \operatorname{Proj} R$, where $R = \bigoplus_{d \geq 0} H^0(X, \mathcal{L}^{\otimes d})$, and the corresponding geometric invariant theory quotient is defined as

$\operatorname{Proj} R^G$, where R^G is the ring of invariants. We denote by X^u the unstable locus, that is

$$X^u = \bigcap_{\substack{\sigma \in H^0(X, \mathcal{L}^{\otimes d})^G \\ d \geq 0}} \{x \in X \mid \sigma(x) = 0\}.$$

The semi-stable locus is $X^{ss} = X \backslash X^u$ and the stable locus $X^s \subset X^{ss}$ consists of semi-stable points that also have a finite stabilizer and whose orbit is closed in X^{ss}. Denote by $Y = X /\!\!/_{\mathcal{L}} G$ be the corresponding GIT quotient, and $\pi : X^{ss} \twoheadrightarrow Y$ the quotient map from the semi-stable locus. The map π is affine, in particular π_* is exact, and we have $\pi_*(\mathcal{O}_{X^{ss}})^G = \mathcal{O}_Y$. The restriction to the stable locus gives a geometric quotient $X^s \to X^s /\!\!/ G$. We will be mostly interested in the cases where there is no strictly semi-stable locus, that is, $X^{ss} = X^s$.

Let $\lambda : \mathbb{G}_m \to G$ be a one-parameter subgroup. If F is a G-linearized line bundle on X and $y \in X^\lambda$ is a λ-fixed point, λ acts in the fiber F_y, with a given weight which we denote $\operatorname{weight}_\lambda F_y$. Similarly, if F is a G-equivariant vector bundle, its λ-weights on F_y are the eigenvalues of the action of λ on F_y. For a G-equivariant complex $F^\cdot \in D^b(X)$, we refer to the λ-weights of $\mathcal{H}^i(F^\cdot)$ for all i as the λ-weights of $F^\cdot$.

Suppose we have a G-linearized ample line bundle $\mathcal{L}$ for the action of G on X. The unstable locus $X^u = X \backslash X^{ss}$ can be described using the Hilbert-Mumford numerical criterion.

Theorem 1.1.2 (Hilbert-Mumford criterion)**.** *X^{ss} (resp. X^s) consists of the points x such that* $\operatorname{weight}_\lambda \mathcal{L}_y \geq 0$ *(resp. > 0) $\forall \lambda$ such that $y = \lim_{t \to 0} \lambda(t) x$ exists.*

Using the Hilbert-Mumford criterion, one can define what is called a Kempf-Ness (KN) stratification of the unstable locus, as described below (see [HL15, §2.1] for a more detailed discussion). For a given one-parameter subgroup $\lambda : \mathbb{G}_m \to G$, and a connected component Z of the fixed locus X^λ one can define the *blade* of Z, λ as

$$Y_{Z,\lambda} = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x \in Z\},$$

i.e., the points that are attracted to Z as $t \to 0$. The projection $q : Y_{Z,\lambda} \to Z$ sending a point to its limit as $t \to 0$ can be shown to be a bundle of affine spaces. Define also $\mu(Z, \lambda) = -\frac{1}{|\lambda|} \operatorname{weight}_\lambda \mathcal{L}|_Z$, where $|\lambda|$ is a norm over one-parameter subgroups given by a choice of some suitable inner product in the cocharacter lattice of a maximal torus of G. Then we can write a stratification of the unstable locus by iteratively selecting a pair $(Z_\alpha, \lambda_\alpha)$ such that μ is positive and maximal among those (Z, λ) for which Z is not contained in the previously defined strata. We may assume λ is a one-parameter subgroup of a maximal torus. Let $Z_\alpha^\circ \subset Z_\alpha$ be the open subset not intersecting any previous strata. We call $Y_\alpha = Y_{Z_\alpha^\circ, \lambda}$, the set attracted to Z_α°. Then the next stratum is $S_\alpha = G \cdot Y_\alpha$. It can be proved that this leads to an ascending sequence of finitely many G-equivariant open subvarieties $X^{ss} = X_0 \subset X_1 \subset \cdots \subset X$, where each $X_\alpha \backslash X_{\alpha-1} = S_\alpha$ is one of these strata.

For each stratum S_α, given by a pair Z_α, λ_α, one can define the Levi subgroup $L_\alpha \subset G$ given by elements $g \in G$ that centralize λ_α and satisfy $g(Z_\alpha) \subset Z_\alpha$; and the parabolic subgroup P_α as the $g \in G$ such that $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists and is in L_α. We have a short exact sequence

$$1 \to U_\alpha \to P_\alpha \to L_\alpha \to 1$$

where $U_\alpha = \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\}$. The inclusion $L_\alpha \hookrightarrow P_\alpha$ allows us to write P_α as a semidirect product $U_\alpha \rtimes L_\alpha$.

We have that the action map $G \times_{P_\alpha} Y_\alpha \to G \cdot Y_\alpha$ is an isomorphism, where $G \times_{P_\alpha} Y_\alpha \xrightarrow{\pi} G/P_\alpha$ is the fibered bundle with fiber isomorphic to Y_α. We also know that the λ_α-weights of the conormal bundle $\mathcal{N}^\vee_{S_\alpha/X}$ restricted to Z_α are positive. Also, it is not hard to see that the λ_α-weights of $T_{Y_\alpha}|_{Z_\alpha}$ are nonnegative (see [DH98, §1.3], [Kir84, §12-13] for details).

Example 1.1.3. Let $\mathbb{G}_m$ act on $X = \mathbb{A}^{n+1}$ by scalar multiplication. Then $\mathcal{O}_X = \Bbbk[x_0, \ldots, x_n]$ is acted on by $\mathbb{G}_m$ by $t \cdot p(x_0, \ldots, x_n) = p(t^{-1}x_0, \ldots, t^{-1}x_n)$. Now this action can be lifted to an action on $\mathcal{O}_X$ as an $\mathcal{O}_X$-module, by letting $t \cdot p(x_0, \ldots, x_n) = t \cdot p(t^{-1}x_0, \ldots, t^{-1}x_n)$. We denote by $\mathcal{O}_X(1)$ the trivial line bundle linearized in this way. The fixed locus by $\mathbb{G}_m$ is the origin, and it is destabilized by $\lambda : t \to t^{-1}$. The unstable locus is just $Z = \{0\}$, and the corresponding GIT quotient is $\mathbb{A}^{n+1} /\!\!/_{\mathcal{O}_X(1)} \mathbb{G}_m$ is $\mathbb{P}^n$. The same is true if we choose a linearization $\mathcal{O}_X(d)$ with $d > 0$.

Example 1.1.4. Let $\mathbb{G}_m$ act on $X = (\mathbb{P}^1)^n$ diagonally by $t \cdot (x : y) = (tx : t^{-1}y)$. On each $\mathbb{P}^1$, there is a natural way of linearizing the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ by $t \cdot (x : y) \times (x, y) = (tx : t^{-1}y) \times (tx, t^{-1}y)$. Doing this in each component and taking tensor products, we get a natural linearization for any ample line bundle $\mathcal{O}_X(d_1, \ldots, d_n)$. The $\mathbb{G}_m$-fixed locus consists of points $(z_1, \ldots, z_n)$ where z_i is either 0 or ∞. For a point z_I, where $z_i = \infty$ if $i \in I$ and $z_i = 0$ otherwise, we compute

$$\mu(\lambda, I) = -\mathrm{weight}_\lambda \mathcal{L}|_{z_I} = \sum_{i\in I} d_i - \sum_{i \notin I} d_i.$$

This is maximized at $(\infty, \ldots, \infty)$ and in fact this single point is the first stratum. The next stratum will be a projective line minus this point. For instance, if $d_1 \leq d_i$ for every i, then the second stratum will consist of points $(z, \infty, \ldots, \infty)$, $z \neq \infty$. Each subsequent strata will be a product of projective lines intersected with the complement of the previous strata. The whole unstable locus consists of points of the form $(z_1, \ldots, z_n)$ with $z_i = \infty$ for every $i \in I$, and where $\sum_{i\in I} d_i > \sum_{i \notin I} d_i$.

1.2 Derived categories

In the present section we give a brief description of the background that we will use from derived categories. Detailed definitions and statements can be found in the literature (see e.g. [Huy06]).

If $\mathcal{A}$ is an abelian category, a chain complex $F^{\cdot}$ is a collection of objects $F^i \in \mathcal{A}$, $i \in \mathbb{Z}$ with maps

$$\cdots \to F^{i-1} \xrightarrow{d^{i-1}} F^i \xrightarrow{d^i} F^{i+1} \to \cdots$$

such that $d^i \circ d^{i-1} = 0$ for every i. We denote by $\mathcal{H}^i(F^{\cdot})$ the i-th cohomology of this chain complex, that is, $\mathcal{H}^i(F^{\cdot}) = \ker d^i / \operatorname{im} d^{i-1}$, which is an object in $\mathcal{A}$. We say that $F^{\cdot}$ is bounded if there are integers m, M such that $\mathcal{H}^i(F^{\cdot}) = 0$ for $i < m$ and for $i > M$.

A morphism $F^{\cdot} \to G^{\cdot}$ between chain complexes is a collection of morphisms $F^i \to G^i$ that induce commutative diagrams with d^i_F and d^i_G. This allows us to define the category $\mathrm{Kom}^b(\mathcal{A})$, whose objects are bounded chain complexes of objects in $\mathcal{A}$, with morphisms of chain complexes. A morphism $\phi : F^{\cdot} \to G^{\cdot}$ induces morphisms between cohomologies, $\mathcal{H}^i(\phi) : \mathcal{H}^i(F^{\cdot}) \to \mathcal{H}^i(G^{\cdot})$. If ϕ induces isomorphisms $\mathcal{H}^i(F^{\cdot}) \cong \mathcal{H}^i(G^{\cdot})$ for every $i \in \mathbb{Z}$, then it is said to be a quasi-isomorphism.

Definition 1.2.1. The (bounded) derived category $D^b(\mathcal{A})$ is defined as the localization of $\mathrm{Kom}^b(\mathcal{A})$ at the class of quasi-isomorphisms. If V is an algebraic variety over $\Bbbk$, we denote by $D^b(V)$ the derived category of $\mathrm{coh}(V)$, the category of coherent sheaves on V.

Remark 1.2.2. To make a rigorous definition of $D^b(\mathcal{A})$, one first needs to define the homotopy category $K^b(\mathcal{A})$. In $K^b(\mathcal{A})$, the morphisms are $\mathrm{Hom}_{K^b(\mathcal{A})}(F^{\cdot}, G^{\cdot}) = \mathrm{Hom}_{\mathrm{Kom}^b(\mathcal{A})}(F^{\cdot}, G^{\cdot})/\sim$, where $\sim$ denotes homotopy equivalence. Then $K^b(\mathcal{A})$ is localized at quasi-isomorphisms, that is, quasi-isomorphisms become isomorphisms in $D^b(\mathcal{A})$. Details can be found in [Huy06, §2.1].

$D^b(\mathcal{A})$ is a triangulated category, with the shift functor $[1] : D^b(\mathcal{A}) \to D^b(\mathcal{A})$, $(F^{\cdot}[1])^i = F^{i+1}$, and exact triangles coming from $F^{\cdot} \xrightarrow{f} G^{\cdot} \to C^{\cdot}(f) \to F^{\cdot}[1]$, where $C^{\cdot}(f)$ is the mapping cone of f. The abelian category $\mathcal{A}$ itself can be seen as a full subcategory of $D^b(\mathcal{A})$, as every object $F \in D^b(\mathcal{A})$ can be thought of as a chain complex $\cdots \to 0 \to F \to 0 \to \cdots$ concentrated in degree 0, and every time there is a short exact sequence $0 \to F \to G \to K \to 0$ in $\mathcal{A}$, we get an exact triangle $F \to G \to K \to F[1]$ in $D^b(\mathcal{A})$.

Definition 1.2.3. A full triangulated subcategory $\mathcal{D}$ of $D^b(\mathcal{A})$ is said to be admissible if the inclusion functor $\mathcal{D} \hookrightarrow D^b(\mathcal{A})$ admits both a left and a right adjoint.

Remark 1.2.4. If V is a smooth projective variety, then the notions of having a right or a left adjoint can be shown to be equivalent, using Serre duality and Serre functors (see [Huy06, Theorem 3.12]).

Definition 1.2.5. We say that $D^b(\mathcal{A}) = \langle \mathcal{D}_1, \ldots, \mathcal{D}_r \rangle$ is a semi-orthogonal decomposition if $\mathcal{D}_i$ are admissible subcategories, with $\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{D}_i, \mathcal{D}_j) = 0$ for $i > j$ and such that for every object in $T \in D^b(\mathcal{A})$ there is a sequence of morphisms $0 = T_n \to T_{n-1} \to \cdots \to T_0 = T$, where the mapping cone of each morphism $T_i \to T_{i-1}$ is in $\mathcal{D}_i$. This last condition can be rephrased as saying that $D^b(\mathcal{A})$ is the smallest full triangulated subcategory containing all the subcategories $\mathcal{D}_i$.

Remark 1.2.6. If $\mathcal{D} \subset D^b(\mathcal{A})$ is any admissible subcategory, then there is a semi-orthogonal decomposition $D^b(\mathcal{A}) = \langle \mathcal{D}^\perp, \mathcal{D} \rangle$, where $\mathcal{D}^\perp$ is the right orthogonal to $\mathcal{D}^\perp$, that is, objects admitting no morphisms from $\mathcal{D}$. Similarly, $\langle \mathcal{D}, {}^\perp\mathcal{D} \rangle$ is also a semi-orthogonal decomposition.

Definition 1.2.7. A class of objects Ω in a triangulated category $\mathcal{D}$ is said to be a spanning class of $\mathcal{D}$ if the following two conditions hold.

(a) If $\mathrm{Hom}_{\mathcal{D}}(A, B[i])$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

(b) If $\mathrm{Hom}_{\mathcal{D}}(B[i], A)$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Lemma 1.2.8. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be admissible subcategories of $\mathcal{D}$ and Ω_1, Ω_2 spanning classes of $\mathcal{D}_1$, $\mathcal{D}_2$. If $\mathrm{Hom}_{\mathcal{D}}(A, B[i]) = 0$ for every $A \in \Omega_1$, $B \in \Omega_2$ and $i \in \mathbb{Z}$, then also $\mathrm{Hom}_{\mathcal{D}}(F, G) = 0$ for every $F \in \mathcal{D}_1$, $G \in \mathcal{D}_2$.*

Proof. We need to show that $\mathcal{D}_1 \subset {}^\perp\mathcal{D}_2$ or, equivalently, $\mathcal{D}_2 \subset \mathcal{D}_1^\perp$. First we see that $\Omega_1 \subset {}^\perp\mathcal{D}_2$. Let $A \in \Omega_1$. Since $\mathcal{D} = \langle \mathcal{D}_2, {}^\perp\mathcal{D}_2 \rangle$, we can fit A in a exact triangle $D \to A \to D' \to D[1]$ where $D \in {}^\perp\mathcal{D}_2$ and $D' \in \mathcal{D}_2$. Applying $\mathrm{Hom}(\cdot, B)$ for $B \in \Omega_2$ we get a long exact sequence where $\mathrm{Hom}(D, B[i]) = 0$ by definition and $\mathrm{Hom}(A, B[i]) = 0$ by hypothesis. Therefore $\mathrm{Hom}(D', B[i]) = 0$ for every i and every $B \in \Omega_2$, so $D' \cong 0$ since Ω_2 is a spanning class of $\mathcal{D}_2$. As a consequence, $A \cong D \in {}^\perp\mathcal{D}_2$.

Now let $G \in \mathcal{D}_2$. Similarly, there is an exact triangle $D \to G \to D' \to D[1]$ with $D \in \mathcal{D}_1$, $D' \in \mathcal{D}_1^\perp$. Applying $\mathrm{Hom}(A, \cdot)$ with $A \in \Omega_1$ we now see that $\mathrm{Hom}(A, D[i]) = \mathrm{Hom}(A, G[i]) = 0$ by the previuos discussion and therefore $D' \cong 0$. This implies $G \cong D \in \mathcal{D}_1^\perp$, as desired. □

If $\phi : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories, one can sometimes define *right derived functors* $R\phi : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ and $R^i\phi := \mathcal{H}^i(R\phi(\cdot)) : D^b(\mathcal{A}) \to \mathcal{B}$, so that $R^0\phi = \phi$ on $\mathcal{A}$ and every short exact sequence on $\mathcal{A}$ gives rise to a long exact sequence of $R^i\phi(\cdot)$ on $\mathcal{B}$. This can be done provided that for every object in $F \in \mathcal{A}$ we can find a quasi-isomorphism

$F \cong I^\cdot$ where every I^i is an injective object, or belongs to a suitable ϕ-adapted class of objects. Similarly, left derived functors can be defined from right exact functors, provided projective or other adapted resolutions exist.

Most importantly for our purposes, given a morphism of varieties $f : V \to W$, we get derived functors such as $Rf_* : D^b(V) \to D^b(W)$ and $Lf^* : D^b(W) \to D^b(V)$, and for the functor of global sections $\Gamma : \mathrm{coh}(V) \to \mathcal{A}b$ we get the sheaf cohomology functors $R^i\Gamma = H^i(V, \cdot)$, sometimes also called hypercohomology functors when applied to a chain complex $F^\cdot \in D^b(V)$. For $F \in \mathrm{coh}(V)$, one can define $\mathrm{Ext}^i(F, \cdot)$ as the i-th derived functor $R^i \operatorname{Hom}(F, \cdot)$, and $\mathscr{E}xt^i(F, \cdot)$ as the derived functor $R^i \mathscr{H}om(F, \cdot)$. We note that in general $\operatorname{Hom}_{D^b(\mathcal{A})}(F, G[i]) = \mathrm{Ext}^i(F, G)$. For computations with right derived functors, one can often use resolutions by locally free sheaves (see e.g. [Huy06, §3.3] for a thorough discussion).

Remark 1.2.9. If $\imath : V \hookrightarrow W$ is the inclusion of a closed subvariety, then $\imath_*$ is exact and the derived functors $R\Gamma_W \circ \imath_*$ and $R\Gamma_V$ coincide. We will denote both by $R\Gamma$. If $R\Gamma(F) = 0$ for an object $F \in D^b(V)$, we say that F is Γ-acyclic. This is equivalent to saying that $H^i(V, F) = 0\, \forall i$.

Notation 1.2.10. From now on, all functors between derived categories will be considered to be derived functors, unless stated otherwise. For instance, Rf_* will be denoted by just f_*.

Definition 1.2.11. An object $E \in D^b(\mathcal{A})$ is called exceptional if $R\operatorname{Hom}(E, E) = \Bbbk$, that is,

$$\operatorname{Hom}(E, E[l]) = \begin{cases} \Bbbk & \text{if} \quad l = 0 \\ 0 & \text{otherwise,} \end{cases}$$

An exceptional collection is a collection $E_1, \ldots, E_r$ of objects that are exceptional and such that $\operatorname{Hom}(E_i, E_j[l]) = 0$ if $i > j$. It is said to be full if $E_1, \ldots, E_r$ generate the whole derived category, that is, $D^b(\mathcal{A})$ is the smallest full triangulated subcategory containing all of the E_i.

Remark 1.2.12. It can be shown that the full triangulated subcategory $\langle E \rangle$ generated by an exceptional object E is always admissible. In fact, a full exceptional collection defines a semi-orthogonal decomposition.

1.2.1 Fourier-Mukai transforms

Definition 1.2.13. Let $P \in D^b(V \times W)$ and let $q : V \times W \to V$, $p : V \times W \to W$ be the projections. The Fourier-Mukai transform $\Phi_P : D^b(V) \to D^b(W)$ is the functor defined by $F \mapsto p_*(P \otimes q^*(F))$.

Recall that in our notation functors are assumed to be derived, so by $p_*(P \otimes q^*(F))$ we mean $Rp_*(P \otimes^L Lq^*(F))$, where $\otimes^L$ is the left derived tensor product. A Fourier-Mukai functor always admits left and right adjoints [Muk81] and in fact, the same is true for any exact functor between derived categories of smooth projective varieties [BvdB03]. By a theorem of Orlov, we know that any fully-faithful functor between derived categories of smooth projective varieties must be a Fourier-Mukai functor given by some object in the product [Orl03]. For a given Fourier-Mukai transform, one can check whether it is fully faithful using the following criterion of Bondal and Orlov.

Theorem 1.2.14 (Fully-faithfulness criterion [BO95]). *Let $P \in D^b(V \times W)$, where V and W are smooth projective varieties. The Fourier-Mukai transform Φ_P is fully faithful if and only if for any two closed points $x, y \in V$ one has*

$$\operatorname{Hom}_{D^b(W)}(\Phi_P(\mathcal{O}_x), \Phi_P(\mathcal{O}_y)[i]) = \begin{cases} \Bbbk & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim V. \end{cases}$$

The proof of this theorem uses the fact that the skyscraper sheaves $\mathcal{O}_x$ over closed points form a spanning class on $D^b(V)$ [Huy06, Proposition 3.17].

Let $q : \operatorname{Bl}_W V \to V$ be the blow-up of a smooth variety V along a smooth subvariety $W \subset V$ of codimension $c \geq 2$. Denote by E the exceptional divisor, with its inclusion $\imath : E \hookrightarrow \operatorname{Bl}_W V$ and projection $\pi = q|_E : E \to W$. For each integer k, one can define a functor

$$\Phi_k := \imath_* \circ (\mathcal{O}_E(kE) \otimes (\cdot)) \circ \pi^* : D^b(W) \to D^b(\operatorname{Bl}_W V).$$

Notice that Φ_k is the Fourier-Mukai transform given by $\mathcal{O}_E(kE)$ considered as an object in $D^b(W \times \operatorname{Bl}_W V)$, that is, supported in $E = W \times_W E \subset W \times \operatorname{Bl}_W V$. Orlov's blow-up formula tells us that the functors Φ_k are in fact fully-faithful, and they can be put into a semi-orthogonal decomposition of $D^b(\operatorname{Bl}_W V)$.

Theorem 1.2.15 (Orlov's blow-up formula [Orl92]). *For every k, the functor $\Phi_k = \Phi_{\mathcal{O}_E(kE)}$ is fully faithful, and defines an equivalence between $D^b(W)$ and an admissible subcategory $\mathcal{D}_k \subset D^b(\operatorname{Bl}_W V)$. The same is true for the functor $q^* : D^b(V) \hookrightarrow D^b(\operatorname{Bl}_W V)$. Moreover, the sequence of subcategories*

$$\mathcal{D}_{c-1}, \ldots, \mathcal{D}_1, q^* D^b(V)$$

defines a semi-orthogonal decomposition of $D^b(\operatorname{Bl}_W V)$.

1.3 Quotient stacks and descent

For a scheme X over S with an action by a reductive group G, the quotient stack $[X/G]$ is defined as follows.

Definition 1.3.1. The quotient stack $[X/G]$ is the category whose objects are principal G-bundles $P \to T$ together with a G-equivariant map $P \to X$, and whose morphisms consist of commutative diagrams

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

that are compatible with the G-equivariant maps $P \to X$, $P' \to X$.

Remark 1.3.2. The functor $[X/G] \to \mathrm{Sch}/S$ sending $P \to T$ to the scheme T makes $[X/G]$ a category over Sch/S fibered in groupoids. In fact, given a map $T \to T'$, a principal G-bundle over T' can be pulled back to a principal G-bundle over T.

Remark 1.3.3. The scheme X itself can also be seen as a quotient stack of X by the trivial group. It is not hard to see that, as a stack, it corresponds to the category of schemes over X. There is a canonical quotient map $q : X \to [X/G]$, which is the functor sending a morphism $P \xrightarrow{\phi} X$ to the trivial principal G-bundle $G \times P \to P$ together with the morphism $G \times P \to X$, $(g, p) \mapsto g \cdot \phi(p)$.

In our setting, we take X to be a projective-over-affine variety over $\Bbbk$, with the action of a reductive group G. We denote by $\mathfrak{X}$ the corresponding quotient stack $[X/G]$. We will work with coherent $\mathcal{O}_{\mathfrak{X}}$-modules, which are given by G-equivariant coherent $\mathcal{O}_X$-modules. Indeed $D^b(\mathfrak{X}) = D^b_G(X)$, that is, an object in $D^b(\mathfrak{X})$ is represented by a G-equivariant bounded chain complex in $D^b(X)$ (see e.g. [BFK19, Proposition 2.2.10]). Cohomology on $\mathfrak{X}$ is G-equivariant cohomology on X [BFK19, Lemma 2.2.8]. For a given G-linearized ample line bundle $\mathcal{L}$, denote by $\mathfrak{X}^{ss}$ the corresponding open substack $[X^{ss}/G]$, with its inclusion $\imath : \mathfrak{X}^{ss} \hookrightarrow \mathfrak{X}$. The quotient map π gives a map from the quotient stack $p : \mathfrak{X}^{ss} \to X /\!\!/ G$. We get a commutative diagram

$$\begin{tikzcd} X^{ss} \arrow[r, "q"] \arrow[rr, bend left, "\pi"] & \mathfrak{X}^{ss} \arrow[r, "p"] & X /\!\!/ G. \end{tikzcd}$$

If $X^{ss} = X^s$, $\mathfrak{X}^{ss}$ is a Deligne-Mumford stack [BFK19, Proposition 2.1.8], and the GIT quotient $Y = X /\!\!/_{\mathcal{L}} G$ is a coarse moduli space for $\mathfrak{X}^{ss}$. In this case, the map p is finite. If, further, the action is free on X^{ss}, then p is an isomorphism. This is because in this case $X^{ss} \to X /\!\!/ G$ is a principal G-bundle, and every principal G-bundle $P \to T$ with a G-equivariant morphism $P \to X$ defines a map from $T = P/G$ to $X /\!\!/ G$, getting a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & X /\!\!/ G. \end{array}$$

Notation 1.3.4. We denote by H^i the i-th hypercohomology of a complex in $D^b(X)$, that is, the i-th derived functor $R^i\Gamma$ of the functor of global sections. Also, denote by H^i_G the derived functor of invariant global sections Γ_G.

Remark 1.3.5. We assume G to be a reductive group, so taking G-invariants is an exact functor on finite dimensional representations. Therefore, for a complex $F^\cdot \in D^b(\mathfrak{X})$, $H^i(\mathfrak{X}, F^\cdot) = H^i_G(X, F^\cdot) = H^i(X, F^\cdot)^G$.

For an object $\tilde{F} \in D^b(\mathfrak{X}^{ss})$, we say that it "descends" to $F \in D^b(X /\!\!/ G)$ if $p^*(F) \cong \tilde{F}$, that is, if there is a G-equivariant isomorphism $\pi^*(F) \cong \tilde{F}$, where p^*, π^* denote the derived pullbacks. Observe that given $\tilde{F}$, such F is unique up to isomorphism: it has to be the pushforward $p_*(\tilde{F}) = \pi_*(\tilde{F})^G$. In the case that G acts freely on X^{ss}, p is an isomorphism, so the categories $D^b(\mathfrak{X}^{ss})$ and $D^b(X /\!\!/ G)$ are equivalent, via $F \mapsto \pi_*(F)^G$. In general, for an object $\tilde{F} \in D^b(\mathfrak{X})$, we say that it descends to $F \in D^b(X /\!\!/ G)$ if its restriction $\tilde{F}|_{\mathfrak{X}^{ss}}$ does.

For a G-equivariant vector bundle on X, we have the following descent criterion (see [DN89, Theorem 2.3]).

Theorem 1.3.6 (Kempf's descent Lemma [DN89])**.** *A G-equivariant vector bundle $\mathcal{V}$ on X descends to a vector bundle on $X /\!\!/ G$ if and only if for every $x \in X^{ss}$, the stabilizer G_x acts trivially on the fiber $\mathcal{V}_x$.*

The action of G on X can be differentiated to obtain a G-equivariant morphism of vector bundles $s : \mathfrak{g} \otimes \mathcal{O}_X \to T_X$ (cf. [DH98, §2.1]). On a fiber over $x \in X$, this morphism looks as follows: the orbit map $G \to X$, $g \mapsto g \cdot x$ is differentiated at the identity $e \in G$, giving rise to $s_x : \mathfrak{g} \to T_{X,x}$. The map s can be viewed as a G-equivariant vector field $s \in H^0(X, T_X \otimes \mathfrak{g}^\vee)^G$. By abuse of notation, we will write $\mathfrak{g}$ for $\mathfrak{g} \otimes \mathcal{O}_X$. Taking the dual of s we get a two step complex $\Omega_X \to \mathfrak{g}^\vee$. For a two-step chain complex of flat modules $K = [A \to B]$, one can define its j-th derived exterior power $\Lambda^j K$ as the object $0 \to \Lambda^j A \to \Lambda^{j-1} A \otimes B \to \cdots \to S^j B \to 0$ where Λ^k and S^k denote exterior and symmetric powers, respectively (see [Wey03, §2.4], [Ill71, §I.4]).

Notation 1.3.7. We define the cotangent complex $L_{\mathfrak{X}} \in D^b(\mathfrak{X})$ to be the two-step G-linearized complex $\Omega_X \to \mathfrak{g}^\vee$, in degrees 0 and 1. We denote by $\Lambda^j L_{\mathfrak{X}}$ the j-th (derived) exterior power of this object, which can then be written as the Koszul complex

$$0 \to \Omega^j_X \to \Omega^{j-1}_X \otimes \mathfrak{g}^\vee \to \cdots \to S^j \mathfrak{g}^\vee \to 0,$$

concentrated in degrees 0 to j (see [Tot18] and the references therein).

Call $q : X^{ss} \to \mathfrak{X}^{ss}$ is the canonical quotient map to the quotient stack, and denote by $L_{\mathfrak{X}^{ss}}$ the restriction of $L_{\mathfrak{X}}$ to $\mathfrak{X}^{ss}$. We observe that if there is no strictly semi-stable locus $L_{\mathfrak{X}^{ss}}$ is (isomorphic to) a vector bundle, and the same is true for its exterior powers.

Lemma 1.3.8. *If $X^{ss} = X^s$, there is a short exact sequence $0 \to q^*\Omega_{\mathfrak{X}^{ss}} \to \Omega_{X^{ss}} \to \mathfrak{g}^\vee \to 0$. If, further, the action is free on X^{ss}, then $\Lambda^j L_{\mathfrak{X}}$ descends to Ω^j_Y for every j.*

Proof. For $x \in X$, the kernel of $\mathfrak{g} \to T_{X,x}$ is the Lie algebra of the stabilizer of x. Then the restriction of the map $\mathfrak{g} \to T_X$ to the stable locus is injective, since stabilizers are finite in X^s. This implies that if $X^{ss} = X^s$, we have a surjection $\Omega_{X^{ss}} \twoheadrightarrow \mathfrak{g}^\vee \otimes \mathcal{O}_{X^{ss}}$. But $\mathfrak{g}^\vee \otimes \mathcal{O}_{X^{ss}} = \Omega_{X^{ss}/\mathfrak{X}^{ss}}$, the relative cotangent bundle, since $\mathfrak{g} \otimes \mathcal{O}_{X^{ss}}$ is exactly $T_{X^{ss}/\mathfrak{X}^{ss}}$. Therefore, the relative sequence $0 \to q^*\Omega_{\mathfrak{X}^{ss}} \to \Omega_{X^{ss}} \to \mathfrak{g}^\vee \to 0$ is the exact sequence that we want.

If the action is free on X^{ss}, the GIT quotient Y is isomorphic to $\mathfrak{X}^{ss}$ and we have a G-equivariant short exact sequence of vector bundles $0 \to \pi^*\Omega_Y \to \Omega_{X^{ss}} \to \mathfrak{g}^\vee \to 0$. From this we see that in this case the restriction $L_{\mathfrak{X}^{ss}}$ is isomorphic to $\pi^*\Omega_Y$ in $D^b(\mathfrak{X}^{ss})$, that is, $L_{\mathfrak{X}}$ descends to Ω_Y and, for the same reason, each $\Lambda^j L_{\mathfrak{X}}$ descends to Ω^j_Y. □

1.4 Quantization and Windows

The Quantization Theorem states that cohomologies of a complex $F \in D^b(\mathfrak{X}^{ss})$ can be computed in $\mathfrak{X}$ if F is the restriction of some complex in $D^b(\mathfrak{X})$ satisfying a numerical condition related to the λ-weights in the unstable strata. It was proved by Teleman [Tel00] in the case where F is a vector bundle, and Halpern-Leistner [HL15, Theorem 3.29] proved it for an arbitrary object in the derived category. As usual, $\mathfrak{X}$ denotes the quotient stack $[X/G]$, where X is a smooth projective-over-affine variety over $\Bbbk$ and G is a reductive group.

Theorem 1.4.1 (Quantization Theorem [HL15])**.** *Let $\{S_\alpha\}$ be a KN stratification of the unstable locus, with the corresponding one-parameter subgroups λ_α and connected components Z_α of the fixed locus X^{λ_α}. Let*

$$\eta_\alpha = \mathrm{weight}_{\lambda_\alpha}(\det \mathcal{N}^\vee_{S_\alpha/X})|_{Z_\alpha}.$$

Let $\{w_\alpha\}$ be any collection of integers and suppose $\tilde{F}$, $\tilde{G} \in D^b(\mathfrak{X})$ restrict to F, $G \in D^b(\mathfrak{X}^{ss})$. If $\tilde{F}$, $\tilde{G}$ satisfy that, for every l and every α, all the λ_α-weights of $\mathcal{H}^l(\tilde{F})|_{Z_\alpha}$ are $\geq w_\alpha$ and all the λ_α-weights of $\mathcal{H}^l(\tilde{G})|_{Z_\alpha}$ are $< w_\alpha + \eta_\alpha$, then

$$\mathrm{Hom}_{D^b(\mathfrak{X})}(\tilde{F}, \tilde{G}) \cong \mathrm{Hom}_{D^b(\mathfrak{X}^{ss})}(F, G).$$

Remark 1.4.2. In particular, if $\tilde{F} \in D^b(\mathfrak{X})$ descends to $F \in D^b(\mathfrak{X}^{ss})$ and has λ_α-weights $< \eta_\alpha$, then $H^\cdot(\mathfrak{X}^{ss}, F) = H^\cdot(\mathfrak{X}, \tilde{F})$. If the action is free on X^{ss}, this computes the cohomologies $H^\cdot(Y, F)$, where Y is the GIT quotient.

Remark 1.4.3. The quantization theorem is a generalization of the original "quantization commutes with reduction" conjecture by Guillemin and Sternberg, who showed the equality between $\dim H^0(X, \mathcal{L})^G$ and $\dim H^0(X /\!\!/_{\mathcal{L}} G, \mathcal{L})$ and then conjectured the equality of the two holomorphic Euler characteristics [GS82].

Example 1.4.4. Write $\mathbb{P}^n$ as the GIT quotient of $X = \mathbb{A}^{n+1}$ by $G = \mathbb{G}_m$. Call $\mathcal{O}_X(d)$ the trivial line bundle on $\mathbb{A}^{n+1}$ with the linearization given by the character $t \mapsto t^d$, so that $\mathcal{O}_X(d)$ descends to $\mathcal{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n = \mathfrak{X}^{ss}$. The unstable locus is just the origin, and it is destabilized by $\lambda : t \mapsto t^{-1}$. We compute $\eta = n+1$ and $\operatorname{weight}_\lambda \mathcal{O}_X(d) = -d$. By the quantization theorem, $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = H^i_{\mathbb{G}_m}(\mathbb{A}^{n+1}, \mathcal{O}_X(d))$ as long as $d > -n-1$, so $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ whenever $i > 0$ and $d > -n-1$, and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ consists of homogeneous polynomials of degree d (cf. [Har77, III.5]).

In fact, Halpern-Leistner's work says much more. His Categorical Kirwan Surjectivity says that $D^b(\mathfrak{X}^{ss})$ can be embedded into $D^b(\mathfrak{X})$ in several different ways via different *windows*.

Definition 1.4.5. Consider a Kempf-Ness stratification $\{S_\alpha\}$ with inclusions $\sigma_\alpha : Z_\alpha \hookrightarrow X$. For any choice of integers $\{w_\alpha\}$, the full triangulated subcategory

$$G_w = \{F \in D^b(\mathfrak{X}) \mid \sigma_\alpha^*(F) \text{ is supported in weights } [w_\alpha, w_\alpha + \eta_\alpha)\}$$

is called a *window*.

Denote

$$D^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w} := \{F \in D^b_{\mathfrak{X}^u}(\mathfrak{X}) \mid \forall \alpha, \lambda_\alpha\text{-weights of } \mathcal{H}^\cdot(\sigma_\alpha^* F) \text{ are } \geq w_\alpha\}$$

where $D^b_{\mathfrak{X}^u}(\mathfrak{X})$ stands for the full triangulated subcategory of objects whose cohomology $\mathcal{H}^\cdot$ is supported in the unstable locus. Similarly, define

$$D^b_{\mathfrak{X}^u}(\mathfrak{X})_{< w} := \{F \in D^b_{\mathfrak{X}^u}(\mathfrak{X}) \mid \forall \alpha, \lambda_\alpha\text{-weights of } \mathcal{H}^\cdot(\sigma_\alpha^* F) \text{ are } < w_\alpha + \eta_\alpha\}.$$

Then we have the following theorem from [HL15, Theorem 2.10].

Theorem 1.4.6 (Derived Kirwan surjectivity [HL15])**.** *Given any choice of integers $w = \{w_\alpha\}$, the window G_w is equivalent to $D^b(\mathfrak{X}^{ss})$ via the restriction functor. Moreover, we have semi-orthogonal decompositions*

$$D^b_{\mathfrak{X}^u}(\mathfrak{X}) = \langle D^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w}, D^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w}\rangle$$
$$D^b(\mathfrak{X}) = \langle D^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w}, G_w, D^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w}\rangle.$$

The decomposition $D^b_{\mathfrak{X}^u}(\mathfrak{X}) = \langle D^b_{\mathfrak{X}^u}(\mathfrak{X})_{<w}, D^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w}\rangle$ can be refined as follows. For every stratum S_α, let $\mathfrak{S}_\alpha$ be the quotient stack $[S_\alpha/G]$, and let $\mathfrak{Z}_\alpha = [Z_\alpha/L_\alpha]$ where L_α is the

Levi subgroup of the corresponding one-parameter subgroup λ_α. Denote by $D^b(\mathfrak{Z}_\alpha)_v$ the full subcategory of complexes in $D^b(\mathfrak{Z}_\alpha)$ whose cohomologies are concentrated in weight v with respect to λ_α. Then the functor $j_* \circ q^* : D^b(\mathfrak{Z})_v \to D^b(\mathfrak{X})$ is fully faithful, where $q : \mathfrak{S}_\alpha \to \mathfrak{Z}_\alpha$ is the canonical projection and $j : \mathfrak{S}_\alpha \to \mathfrak{X}$ is the embedding. Using this one can obtain the following semi-orthogonal decompositions [HL15, BFK19]:

$$\begin{aligned}
D^b_{\mathfrak{X}^u}(\mathfrak{X})_{\geq w} = \langle \quad & D^b(\mathfrak{Z}_1)_{w_1}, D^b(\mathfrak{Z}_1)_{w_1+1}, \dots, \\
& D^b(\mathfrak{Z}_1)_{w_2}, D^b(\mathfrak{Z}_1)_{w_2+1}, \dots, \\
& \dots, \\
& D^b(\mathfrak{Z}_1)_{w_N}, D^b(\mathfrak{Z}_1)_{w_N+1}, \dots \rangle \\
D^b_{\mathfrak{X}^u}(\mathfrak{X})_{< w} = \langle \quad & \dots, D^b(\mathfrak{Z}_N)_{w_N-2}, D^b(\mathfrak{Z}_1)_{w_N-1}, \\
& \dots, \\
& \dots, D^b(\mathfrak{Z}_2)_{w_2-2}, D^b(\mathfrak{Z}_1)_{w_2-1}, \\
& \dots, D^b(\mathfrak{Z}_1)_{w_1-2}, D^b(\mathfrak{Z}_1)_{w_1-1} \rangle.
\end{aligned} \tag{1.4.1}$$

Example 1.4.7. Consider $\mathbb{P}^n$ as the GIT quotient of $X = \mathbb{A}^{n+1}$ by $\mathbb{G}_m$. In fact $\mathbb{P}^n$ is isomorphic to the open substack $\mathfrak{X}^{ss} \subset \mathfrak{X}$. By Hilbert's syzygy theorem, every finitely generated graded $\Bbbk[x_0, \dots, x_n]$-module has a finite resolution by free graded modules, so the ambient stack $[\mathbb{A}^{n+1}/\mathbb{G}_m]$ has an infinite full exceptional collection given by $\{\mathcal{O}_X(d)\}_{d\in\mathbb{Z}}$. In particular, the restrictions $\{\mathcal{O}_{\mathbb{P}^n}(d)\}_{d\in\mathbb{Z}}$ must generate the whole derived category $D^b(\mathbb{P}^n)$ (cf. [CT20b]). Derived Kirwan surjectivity shows that, in order to generate the whole category, it suffices to take the descent of objects in $D^b_{\mathbb{G}_m}(\mathbb{A}^{n+1})$ having λ-weights between w and $w+n$ at the origin. In fact, it can be shown that for any integer w the sequence $\mathcal{O}_{\mathbb{P}^n}(w), \dots, \mathcal{O}_{\mathbb{P}^n}(w+n)$ is a full exceptional collection [Beĭ78].

In order to get a decomposition of the form (1.4.1), consider the unstable locus $\mathfrak{X}^u = \mathfrak{Z} = [z/\mathbb{G}_m]$ where z is the origin. The object $\mathcal{O}_z(v) \in D^b(\mathfrak{Z})$ has λ-weight equal to $-v$, and in fact it is easy to see that $D^b(\mathfrak{Z})_v$ is generated by $\mathcal{O}_z(-v)$, so we have a semi-orthogonal decomposition

$$D^b(\mathfrak{Z}) = \langle \dots, \mathcal{O}_z(1), \mathcal{O}_z, \mathcal{O}_z(-1), \dots \rangle.$$

In fact, the objects $\mathcal{O}_z(v)$ are orthogonal to each other from both sides by Schur's Lemma, since each of them is an irreducible representation of $\mathbb{G}_m$. A full semi-orthogonal decomposition of $D^b_{\mathbb{G}_m}(\mathbb{A}^{n+1})$ is obtained as

$$D^b(\mathfrak{X}) = \langle \dots, \mathcal{O}_z(w+2), \mathcal{O}_z(w+1), \mathcal{O}_{\mathbb{P}^n}(w-n), \dots, \mathcal{O}_{\mathbb{P}^n}(w), \mathcal{O}_z(w), \mathcal{O}_z(w-1), \dots \rangle. \tag{1.4.2}$$

Here, by abuse of notation, we have written $\mathcal{O}_z(v)$ in place of $j_*\mathcal{O}_z(v)$, where $j : \{z\} \hookrightarrow X$ is the inclusion. Observe that the weights of $j_*\mathcal{O}_z(v)$ as an object in $D^b(\mathfrak{X})$ can be obtained by taking a ($\mathbb{G}_m$-equivariant) Koszul resolution of the origin $z \in X$

$$\Big[\mathcal{O}_X(-n-1) \to \cdots \to \mathcal{O}_X(-1)^{\oplus(n+1)} \to \mathcal{O}_X\Big] \cong j_*\mathcal{O}_z.$$

The restriction $j^*j_*\mathcal{O}_z$ to the unstable locus is therefore isomorphic to $\bigoplus_{d=0}^{n+1} \mathcal{O}_z(-d)[d]$ and this has weights $0, \ldots, n+1$. Similarly, for any $v \in \mathbb{Z}$, $j_*\mathcal{O}_z(-v)$ will have weights $v, \ldots, v+n+1$. If $v \geq w$, these weights are all $\geq w$ while, if $v < w$ the weights are all $< w + \eta$. This is consistent with the semi-orthogonal decomposition (1.4.2) as provided by Derived Kirwan surjectivity.

1.5 Variation of GIT

In [DH98], Dolgachev and Hu study the relationship between GIT quotients $X /\!\!/_{\mathcal{L}} G$ obtained using different choices of a G-linearized ample line bundle $\mathcal{L}$. The G-ample cone $C^G(X)$ is the convex cone in $NS^G(X) \otimes \mathbb{R}$ spanned by G-linearized line bundles with nonempty semi-stable locus, and it can be split into a system of finitely many walls and chambers where, for any $\mathcal{L}$ in the interior of a chamber we have $X^{ss}(\mathcal{L}) = X^s(\mathcal{L})$ and any two linearized ample line bundles within the same chamber give rise to the same GIT quotient. For linearizations located on a wall, there is strictly semi-stable locus, and they describe the birational transformation that occurs between two GIT quotients arising from adjacent chambers. This is called wall-crossing.

Results from [HL15] and [BFK19] describe how the derived category of a GIT quotient is changed when we move from one GIT chamber to another. Let $\mathcal{L}_0$ be a linearization lying on a wall, and suppose that, for another G-linearized line bundle $\mathcal{L}'$ and sufficiently small $\epsilon > 0$, the linearizations $\mathcal{L}_\pm := \mathcal{L}_0 \pm \epsilon\mathcal{L}'$ both lie in the interior of adjacent chambers, separated by a codimension-one wall containing $\mathcal{L}_0$. Points in the strictly semi-stable locus $X^{sss}(\mathcal{L}_0)$ change from being stable to unstable as one crosses the wall in one direction or the other. Other than that, points that are either stable or unstable for $\mathcal{L}_0$ will stay so for $\mathcal{L}_\pm$. In fact, using KN stratifications we can write $X^{ss}(\mathcal{L}_0)$ in two different ways

$$X^{ss}(\mathcal{L}_0) = S_1^\pm \cup \cdots \cup S_{m_\pm}^\pm \cup X^{ss}(\mathcal{L}_\pm) \tag{1.5.1}$$

where $S_\alpha^\pm$ are the KN strata of $X^u(\mathcal{L}_\pm)$ lying in $X^{ss}(\mathcal{L}_0)$. Further, the KN strata can be constructed from one-parameter subgroups $\lambda_\alpha^\pm$ in a way that $(Z_\alpha, \lambda_\alpha^+)$ appears in the KN stratification of $X^u(\mathcal{L}_+)$ and $(Z_\alpha, \lambda_\alpha^-)$ in that of $X^u(\mathcal{L}_-)$ [HL15, §5]. That is, from one side of the wall to the other, the maximal destabilizing one-parameter subgroup flips from λ_α^+ to $\lambda_\alpha^- : t \mapsto \lambda_\alpha^+(t)^{-1}$, and the unstable stratum flips from the descending manifold of Z_α to its ascending manifold. Let us denote $\lambda_\alpha := \lambda_\alpha^+$. We will use the following result (see [BFK19, Theorem 1] and [HL15, Proposition 4.5]).

Theorem 1.5.1 (Derived Categories under GIT wall-crossing [HL15, BFK19])**.** *Let $\mathfrak{X} = [X/G]$ and $\mathcal{L}_\pm = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$ as above, with ϵ sufficiently small, and where $\mathcal{L}_\pm$ lie in the interior of adjacent GIT chambers with non-empty semi-stable locus $X^{ss}(\mathcal{L}_\pm) = X^s(\mathcal{L}_\pm)$. Suppose $m_+ = m_-$ in (1.5.1). Consider the λ_α-weights of the canonical bundle ω_X restricted to Z_α, for every Z_α appearing in $X^{ss}(\mathcal{L}_0)$.*

(a) If these weights are all zero, there is an equivalence $D^b(\mathfrak{X}^{ss}(\mathcal{L}_+)) \cong D^b(\mathfrak{X}^{ss}(\mathcal{L}_-))$.

(b) If these weights are all < 0, there is a fully faithful functor $D^b(\mathfrak{X}^{ss}(\mathcal{L}_+)) \subset D^b(\mathfrak{X}^{ss}(\mathcal{L}_-))$.

(c) If these weights are all > 0, there is a fully faithful functor $D^b(\mathfrak{X}^{ss}(\mathcal{L}_-)) \subset D^b(\mathfrak{X}^{ss}(\mathcal{L}_+))$.

In other words, a window G_w corresponding to a given GIT chamber can sometimes be embedded in a bigger window corresponding to the derived category of another GIT chamber. Note that $\mathrm{weight}_\lambda\, \omega_X|_{Z_\alpha}$ is precisely $\eta_\alpha^+ - \eta_\alpha^-$, the difference between the widths of the windows on either side of the wall. We will be mostly interested in the cases when $\mathfrak{X}^{ss}(\mathcal{L}_\pm)$ is isomorphic to $X /\!\!/_{\mathcal{L}_\pm} G$, so Theorem 1.5.1 is a statement about the derived categories $D^b(X /\!\!/_{\mathcal{L}_\pm} G)$.

Example 1.5.2. Let $\mathbb{G}_m$ act on $X = \mathbb{A}^{n+1} \times \mathbb{A}^{m+1}$ by $t \cdot (z, w) = (tz, t^{-1}w)$. Let $\mathcal{L}_0 = \mathcal{O}_X$, $\mathcal{L}_+ = \mathcal{O}_X(1)$ and $\mathcal{L}_- = \mathcal{O}_X(-1)$. This is the standard flip. We have $X /\!\!/_{\mathcal{L}_0} \mathbb{G}_m = \mathrm{Spec} \bigoplus_{d \geq 0} H^0(\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(d, d))$, the affine cone over the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$, and there is a commutative diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X /\!\!/_{\mathcal{L}_+} \mathbb{G}_m \arrow[dr] & & X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m \arrow[dl] \\ & X /\!\!/_{\mathcal{L}_0} \mathbb{G}_m & \end{tikzcd}$$

where all the arrows are blow-ups (cf. [Tha96, Theorem 1.9]). $\tilde{X}$ is the blow-up of $X /\!\!/_{\mathcal{L}_0} \mathbb{G}_m$ at the origin, with exceptional locus $\mathbb{P}^n \times \mathbb{P}^m$, while $X /\!\!/_{\mathcal{L}_\pm} \mathbb{G}_m \to X /\!\!/_{\mathcal{L}_0} \mathbb{G}_m$ are small resolutions with exceptional loci $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively.

For the linearizations $\mathcal{L}_\pm$, we have

$$X^u(\mathcal{L}_+) = 0 \times \mathbb{A}^m, \quad X^u(\mathcal{L}_-) = \mathbb{A}^n \times 0.$$

The $\mathbb{G}_m$-fixed locus is the origin $Z = (0, 0)$, and the destabilizing one-parameter subgroup for $\mathcal{L}_+$ is $\lambda : t \mapsto t^{-1}$, whose weight on ω_X is $n - m$. The action is free on each semi-stable locus, so $\mathfrak{X}^{ss}(\mathcal{L}_\pm) \cong X /\!\!/_{\mathcal{L}_\pm} \mathbb{G}_m$. Then, Proposition 1.5.1 says that if $n < m$ we have an embedding $D^b(X /\!\!/_{\mathcal{L}_+} \mathbb{G}_m) \hookrightarrow D^b(X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m)$. If $n = m$, then both derived categories are equivalent.

The widths of the windows are $\eta_+ = m+1$, $\eta_- = n+1$. Suppose $n < m$. Then given a window $G_w^+ \subset D^b(X /\!\!/_{\mathcal{L}_+} \mathbb{G}_m)$, Theorem 1.4.6 together with (1.4.1) give a semi-orthogonal decomposition

$$D^b(X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m) \cong G_w^- \cong \langle G_w^+, D^b(\mathfrak{Z})_w, \ldots, D^b(\mathfrak{Z})_{w+m-n-1} \rangle$$

where $\mathfrak{Z} = [Z/\mathbb{G}_m]$ is the origin with the trivial action, and $D^b(\mathfrak{Z})_v$ is generated by the restriction of $\mathcal{O}_X(-v)$ to Z. Then $D^b(\mathfrak{Z})_v$ is embedded into $D^b(X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m)$ by $j_* \circ \pi^*$, where $\pi : 0 \times \mathbb{A}^{m+1} \to Z$ is the projection and $j : 0 \times \mathbb{A}^{m+1} \hookrightarrow \mathbb{A}^{n+1} \times \mathbb{A}^{m+1}$ the inclusion. Since $\mathcal{O}_X(-v)$ descends to $\mathcal{O}_{\mathbb{P}^m}(v)$ on $X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m$, where $\mathbb{P}^m$ is the exceptional locus of $X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m \to X /\!\!/_{\mathcal{L}_0} \mathbb{G}_m$, we get the following semi-orthogonal decomposition

$$D^b(X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m) = \langle D^b(X /\!\!/_{\mathcal{L}_-} \mathbb{G}_m), \mathcal{O}_{\mathbb{P}^m}(w), \ldots, \mathcal{O}_{\mathbb{P}^m}(w+m-n-1) \rangle.$$

CHAPTER 2

BOTT VANISHING

2.1 Introduction

We say that a smooth projective variety Y satisfies Bott vanishing if for every ample line bundle L we have

$$H^i(Y, \Omega_Y^j \otimes L) = 0 \quad \forall i > 0, \forall j \geq 0. \tag{2.1.1}$$

In [Tot20], Totaro gives a geometric interpretation of what it means for a K3 surface to have this property. In general, it is not clear what the geometric meaning of Bott vanishing is, although it is certainly useful, when it holds, to compute sections of some vector bundles.

This property turns out to be very restrictive. For instance, a Fano variety that satisfies Bott vanishing must be rigid, and even among rigid Fano varieties, the property is known to fail for quadrics of dimension at least 3 and Grassmannians other than $\mathbb{P}^n$ (see the discussion in [Tot20] and the references therein). Smooth toric varieties are among the few known examples of varieties satisfying Bott vanishing. Several different proofs can be found in [BC94], [BTLM97], [Fuj07], [Mus02]. In fact, vanishing (2.1.1) is shown for any projective toric variety, where Ω_Y^j is defined as the pushforward of $\Omega_{Y^0}^j$ from the smooth locus Y^0 (see e.g. [Fuj07]). Up until Totaro's paper [Tot20], there were no known non-toric examples of rationally connected varieties with this property. He proves that the quintic del Pezzo surface over any field satisfies Bott vanishing, as well as coming up with several other non-toric examples from K3 surfaces. Namely, he proves that Bott vanishing fails for K3 surfaces of degree less than 20 or equal to 22, while it holds for all K3 surfaces of degree 20 or at least 24 with Picard number 1. Recent work by Wang [Wan21] studies Bott vanishing for elliptic surfaces X with an elliptic fibration $\pi : X \to \mathbb{P}^1$, and how the property is affected by the geometric properties of the fibration, such as the existence of certain singular fibers.

The following theorem was motivated by [Tot20] and it continues the quest for non-toric examples of varieties satisfying Bott vanishing. Observe that the quintic del Pezzo surface can be realized as a GIT quotient of $(\mathbb{P}^1)^5$ by the diagonal action of PGL_2 with respect to the symmetric polarization $\mathcal{L} = \mathcal{O}(1, \ldots, 1)$. We prove that in fact Bott vanishing holds for every stable GIT quotient $(\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$, over an algebraically closed field of characteristic 0. In particular, this gives one new Fano example for each even dimension.

Theorem 2.1.1. *Let PGL_2 act diagonally on $(\mathbb{P}^1)^n$, over an algebraically closed field of characteristic zero. Suppose $\mathcal{L}$ is a PGL_2-linearized ample line bundle on $(\mathbb{P}^1)^n$ admitting no strictly semi-stable locus. Then the GIT quotient $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ satisfies Bott vanishing.*

To prove this, we use the results of Halpern-Leistner's to carry out computations in the derived category of GIT quotients [HL15], as described in §1. His Quantization Theorem will allow us to, roughly speaking, compute cohomologies $H^{\cdot}(X /\!\!/ G, F)$ on the GIT quotient as G-equivariant cohomologies $H^{\cdot}_G(X, \tilde{F})$ on the ambient quotient stack $[X/G]$, where $\tilde{F}$ must be some object in the derived category of $[X/G]$ descending to F and satisfying certain weights condition over the unstable locus. The stratification of the unstable locus associated to the action of PGL_2 on $(\mathbb{P}^1)^n$ is discussed in §2.5. We refer the reader to [CT17, CT20a, CT20b] for a description of the derived category of the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$ in terms of an equivariant full exceptional collection.

For our case, we use the object $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ described in §1.3, which descends to $\Omega^j_Y \otimes L$ in the GIT quotient. In §2.2 we check that this object satisfies the weights condition from the quantization theorem, and then devote most of the work to the corresponding computation of cohomology in the ambient quotient stack. We first see that, as a consequence of the Bott vanishing property on $X = (\mathbb{P}^1)^n$, this amounts to computing cohomologies of the complex of invariant global sections of the object $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ on $(\mathbb{P}^1)^n$ (see Lemma 2.3.1). Following Weyman's method of geometric syzygies [Wey03], we view these as sections of some sheaves in the product $X \times \mathbb{P}(\mathfrak{g})$, rather than sheaves on X. Let $M \subset X \times \mathbb{P}(\mathfrak{g})$ be the scheme-theoretic zero locus of the section $s : \Omega_X \to \mathfrak{g}^\vee$. Koszul resolution of M, together with an associated spectral sequence, yields then vanishing for the i-th cohomology in (2.1.1), for $i \neq 0, j$. This is discussed in §2.3. The techniques used up to this point do not require the particular context of PGL_2 acting on $(\mathbb{P}^1)^n$, and can be applied to other GIT quotients $X /\!\!/ G$ satisfying certain hypotheses. The main properties that we need are that of X itself satisfying Bott vanishing and M being a local complete intersection.

Next, we observe in §2.4 that in the case of an abelian group acting on a smooth affine variety, very similar techniques can be used to get a stronger vanishing result (see Theorem 2.4.3).

Theorem 2.1.2. *Let G be an abelian reductive group acting on a smooth affine variety X, over an algebraically closed field of characteristic zero. Let $\mathcal{L}$ be a linearization with no strictly semi-stable locus and descending to a line bundle L in the GIT quotient $Y = X /\!\!/_{\mathcal{L}} G$. Suppose G*

acts freely on X^{ss} except for a subset of codimension at least 2. Then $H^i(Y, \Omega^j_Y \otimes L) = 0\ \forall i > 0, \forall j \geq 0$.

Observe that this is not the same as Bott vanishing, since the formula is only stated for the descent of the linearization $\mathcal{L}$, while Bott vanishing requires (2.1.1) to hold for any ample line bundle. However, this vanishing is essentially all that needs to be verified in the particular case that $X = \mathbb{A}^d$, where we have an explicit description of the G-ample cone and the ring of invariants, as detailed in [HK00]. As a consequence, we obtain yet another proof of Bott vanishing for the toric case in characteristic zero (see Theorem 2.4.6). More precisely, we show it for a $\mathbb{Q}$-factorial projective toric variety over an algebraically closed field of characteristic zero, using its description as a GIT quotient of the affine space due to Cox [Cox14]. We then hope that these techniques, using windows, may be applied to yield more examples of varieties satisfying Bott vanishing.

In §2.5 we finish the proof of Theorem 2.1.1. Here we mostly deal with the vanishing of $H^j(Y, \Omega^j_Y \otimes L)$, where $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$. Given the work developed in the previous sections, this amounts to computing cohomology of the complex of invariant global sections of the object $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ defined in Section 1.1. More precisely, we are left with the computation of the last cohomology of this complex, which is the same as investigating surjectivity of the map of invariant sections $H^0(X, \Omega_X \otimes S^{j-1}\mathfrak{g}^\vee)^G \to H^0(X, S^j\mathfrak{g}^\vee)^G$. To do this we use techniques that are particular to our case, that is, PGL_2 acting on $X = (\mathbb{P}^1)^n$. Namely, we handle invariant sections using the description of [HMSV05, HMSV09], where sections are identified with linear combinations of directed graphs with prescribed degrees on the vertices.

2.2 Weights and cohomology

Throughout the present Chapter, X will denote a smooth projective-over-affine variety over an algebraically closed field $\Bbbk$ of characteristic zero, with an action by a reductive group G. We will use the results described in §1.

In the following theorem and corollary, we check that we can apply the Quantization Theorem to $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$, where $\mathcal{L}$ is the G-linearized ample line bundle on X. In the holomorphic setting, this was observed in [Tel00, Theorem 7.1].

Theorem 2.2.1. *Let G be a reductive group acting on X, and $\{S_\alpha\}$ a KN stratification of the unstable locus as described in §1.1. The λ_α-weights of the complex $\Lambda^j L_{\mathfrak{X}}|_{Z_\alpha}$ are all $\leq \eta_\alpha$. If G is abelian, then the λ_α-weights of the individual terms $(\Lambda^j L_{\mathfrak{X}})^p|_{Z_\alpha} = (\Omega_X^{j-p} \otimes S^p\mathfrak{g}^\vee)|_{Z_\alpha}$ of the complex are all $\leq \eta_\alpha$.*

Proof. Let $Z = Z_\alpha$ correspond to the stratum S_α, and let $\lambda = \lambda_\alpha$ be the corresponding one-parameter subgroup. Since the weights condition is local, it is enough to compute them when we further restrict to an open affine $Z' \subset Z$. Consider the restriction

$$\mathfrak{g} \to T_X|_{Z'} \tag{2.2.1}$$

of the dual of $L_\mathfrak{X}$ to Z'. Include λ in a maximal torus of G and let $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Cartan subalgebra. We can write a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta.$$

Observe that $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\beta(d\lambda) \geq 0} \mathfrak{g}_\beta$ is precisely the Lie algebra of the parabolic subgroup $P = P_\alpha$. Call $\mathfrak{n}^- = \bigoplus_{\beta(d\lambda)<0} \mathfrak{g}_\beta$, so that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}^-$.

Let $Y = Y_\alpha^\circ$ be the corresponding blade. Using the isomorphism $G \times_P Y \cong G \cdot Y$, we get a short exact sequence of tangent sheaves:

$$0 \to G \times_P T_Y \to T_{G \cdot Y} \to \pi^* T_{G/P} \to 0.$$

When we restrict to Z', this sequence splits, since these are vector bundles and Z' is affine. Therefore we have $T_{G\cdot Y}|_{Z'} = T_Y|_{Z'} \oplus \pi^*(T_{G/P})|_{Z'}$. By the defnition of the blade, the first summand has only nonnegative λ-weights, while the second summand has all negative λ-weights. In fact, since $\mathfrak{n}^- = \mathfrak{g}/\mathfrak{p}$, we see that $(\mathfrak{n}^- \otimes \mathcal{O}_{G\cdot Y})|_{Z'} \xrightarrow{\sim} \pi^*(T_{G/P})|_{Z'}$.

Now the restriction of the sequence

$$0 \to T_{G\cdot Y} \to T_X|_{G \cdot Y} \to \mathcal{N}_{G \cdot Y/X} \to 0$$

to Z' splits, again because Z' is affine, so we obtain $T_X|_{Z'} = T_Y|_{Z'} \oplus (\pi^* T_{G/P})|_{Z'} \oplus (\mathcal{N}_{G\cdot Y/X})|_{Z'}$. Therefore, since the complex (2.2.1) is G-equivariant, it must split as the direct sum of the complexes

$$\mathfrak{n}^- \xrightarrow{\sim} (\pi^* T_{G/P})|_{Z'}$$

and

$$\mathfrak{p} \to T_Y|_{Z'} \oplus (\mathcal{N}_{S/X})|_{Z'}.$$

Similarly, the restricted complex $L_\mathfrak{X}|_{Z'} = [\Omega_X|_{Z'} \to \mathfrak{g}^\vee]$ is written as a direct sum of two complexes, namely the duals of the complexes above. Therefore, by [Wey03, Proposition 2.4.7], the exterior powers of $\Omega_X|_{Z'} \to \mathfrak{g}^\vee$ are quasi-isomorphic to those of the complex $\Omega_Y|_{Z'} \oplus (\mathcal{N}_{S/X}^\vee)|_{Z'} \to \mathfrak{p}^\vee$. Now the exterior and symmetric powers of $\mathfrak{p}$ and $T_Y|_{Z'}$ all have nonnegative λ-weights, so their duals have weights ≤ 0. On the other hand, the weights of $\mathcal{N}_{S/X}^\vee|_{Z'}$ are all positive and the sum of all of them is $\text{weight}_\lambda \det \mathcal{N}^\vee_{S/X}|_{Z'} = \eta_\alpha$. Combining all these, we see that for every j, the weights of $\Lambda^j(\Omega_Y|_{Z'} \oplus \mathcal{N}^\vee_{S/X}|_{Z'} \to \mathfrak{p}^\vee)$ are all $\leq \eta_\alpha$.

If G is abelian, then $G = P$, $\mathfrak{n}^- = 0$ and the weights of $\mathfrak{g}$ are all 0. Then it suffices to know that the exterior powers of $\Omega_Y|_{Z'} \oplus (\mathcal{N}^\vee_{S/X})|_{Z'}$ have weights $\leq \eta_\alpha$ for the reasons above. □

Corollary 2.2.2. *Let $\mathcal{L} \in D^b(\mathfrak{X})$ be a G-linearized ample line bundle giving a GIT quotient $Y = X /\!\!/_{\mathcal{L}} G$. Then the complex $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ satisfies the hypothesis of Theorem 1.4.1, so*

$$H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}).$$

If G is abelian, we also have $H^i(\mathfrak{X}^{ss}, (\Lambda^j L_{\mathfrak{X}^{ss}})^p \otimes \mathcal{L}) = H^i(\mathfrak{X}, (\Lambda^j L_{\mathfrak{X}})^p \otimes \mathcal{L})$ for each p.

Proof. Indeed, by definition of the stratification, $\text{weight}_{\lambda_\alpha} \mathcal{L}|_{Z_\alpha} < 0$ for every α, and weights are additive with respect to tensor product. □

2.3 The Koszul complex of sections

Let $\mathcal{L}$ be a G-linearized ample line bundle on a smooth projective-over-affine variety X and consider the complex $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$. We want to investigate the associated complex $F^\cdot$ of global sections,

$$F^\cdot = \left[0 \to H^0(X, \Omega^j_X \otimes \mathcal{L}) \to H^0(X, \Omega^{j-1}_X \otimes \mathcal{L}) \otimes \mathfrak{g}^\vee \to \cdots \to H^0(X, \mathcal{L}) \otimes S^j \mathfrak{g}^\vee \to 0\right], \tag{2.3.1}$$

concentrated in degrees 0 to j. For the remainder of the section, we extend the definition of Bott vanishing to a smooth projective-over-affine variety using equation (2.1.1).

Lemma 2.3.1. *Suppose X satisfies Bott vanishing. Then the hypercohomology of $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ equals the G-equivariant cohomology of $F^\cdot$, this is, $H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}) = \mathcal{H}^i(F^\cdot)^G$.*

Proof. Consider $\Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}$ as a complex of coherent sheaves on X. From a suitable bi-complex resolution we get a spectral sequence $E_1^{p,q} = H^q(X, (\Lambda^j L_{\mathfrak{X}})^p \otimes \mathcal{L})$ converging to the hypercohomology $H^{p+q}(X, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})$. Since X itself satisfies Bott vanishing, all the terms $E_1^{p,q}$ are equal to zero except for $q = 0$:

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow \cdots \longrightarrow E_1^{j,0} \longrightarrow 0.$$

Therefore the sequence degenerates at E_2. We get $E_\infty^{p,0} = E_2^{p,0} = \mathcal{H}^p(E_1^{\cdot,0})$. The hypercohomology $H^p(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})$ equals the invariant sections $H^p(X, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})^G = \mathcal{H}^p(E_1^{\cdot,0})^G$. But the complex $E_1^{\cdot,0}$ is precisely $F^\cdot$. □

Now write $\mathbb{P}^m = \mathbb{P}(\mathfrak{g})$ and let $W = X \times \mathbb{P}(\mathfrak{g})$, carrying a canonical G-action. Observe $H^0(\mathbb{P}(\mathfrak{g}), \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(l)) = S^l\mathfrak{g}^\vee$ for each $l \geq 0$. Given the action, the vector bundle $T_X \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ has a canonical G-equivariant global section $s \in H^0(X \times \mathbb{P}(\mathfrak{g}), T_X \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1))^G = (H^0(X, T_X) \otimes \mathfrak{g}^\vee)^G$, which is the one giving the map $\Omega_X \to \mathfrak{g}^\vee$. Let $M \subset W$ be the scheme-theoretic zero locus of s. Suppose this is a local complete intersection, that is, the section s is given locally by a regular sequence. By smoothness of $X \times (\mathbb{P}^1)^n$, this is equivalent to $\operatorname{codim} M = n$, where $n = \operatorname{rk}(T_X \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)) = \dim X$ (see e.g. [Har77, §II.8]). In this case, the associated augmented Koszul complex

$$K_s^{\cdot} = \left[0 \to \Omega_X^n \boxtimes \mathcal{O}_{\mathbb{P}(g)}(-n) \to \cdots \to \Omega_X \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(-1) \to \mathcal{O}_W \to \mathcal{O}_M \to 0\right] \tag{2.3.2}$$

is exact (see e.g. [Wei94, Corollary 4.5.4]). We consider this complex to be concentrated in degrees $-n$ to 1, this is, $K_s^p = \Omega_X^{-p} \otimes \mathcal{O}_{\mathbb{P}^m}(p)$ for $p \leq 0$ and $K_s^1 = \mathcal{O}_M$.

Proposition 2.3.2. *Suppose M is a local complete intersection and suppose X satisfies Bott vanishing. Then $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = 0$ for $i \neq 0, j$. If, in addition, $H^0(M, \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j)|_M)^G = 0$, then $H^j(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = 0$ too.*

Proof. From Corollary 2.2.2, we know $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})$. Therefore, by Lemma 2.3.1, it suffices to show $\mathcal{H}^i(F^{\cdot})^G = 0$ for $0 < i < j$. Since the Koszul complex $K_s^{\cdot}$ is exact, all its hypercohomologies vanish. The same is true for the complex $\tilde{K}_s^{\cdot} = K_s^{\cdot} \otimes (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j))$. Take the associated spectral sequence $E_1^{p,q} = H^q(X \times \mathbb{P}(\mathfrak{g}), \tilde{K}_s^p)$, converging to $H^{p+q}(X \times \mathbb{P}(\mathfrak{g}), \tilde{K}_s^{\cdot}) = 0$. Since X satisfies Bott vanishing, we have

$$H^q(X \times \mathbb{P}(\mathfrak{g}), \tilde{K}_s^p) = \begin{cases} H^0(X, \Omega_X^{-p} \otimes \mathcal{L}) \otimes S^{p+j}\mathfrak{g}^\vee & \text{if} \quad q = 0,\ -j \leq p \leq 0 \\ H^0(X, \Omega_X^{-p} \otimes \mathcal{L}) \otimes H^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(j+p))^\vee & \text{if} \quad q = m,\ p \leq -j-m-1 \\ H^q(M, (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^m}(j))|_M) & \text{if} \quad p = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the sequence has the following shape:

$$\begin{array}{ccc}
\cdots \longrightarrow E_1^{-j-m-2,m} \longrightarrow E_1^{-j-m-1,m} \longrightarrow 0 \longrightarrow \cdots & & \cdots \longrightarrow 0 \longrightarrow E_1^{1,m} \\
\cdots \longrightarrow 0 \longrightarrow \cdots & & \cdots \longrightarrow 0 \longrightarrow E_1^{1,m-1} \\
\vdots & & \vdots \qquad \vdots \\
\cdots \longrightarrow 0 \longrightarrow \cdots & & \cdots \longrightarrow 0 \longrightarrow E_1^{1,1} \\
\cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow E_1^{-j,0} & \longrightarrow \cdots \longrightarrow & E_1^{0,0} \longrightarrow E_1^{1,0}.
\end{array}$$

Note that the complex $F^{\cdot}$ is precisely the naive truncation of the shifted complex $E_1^{\cdot,0}[-j]$

obtained by omitting the last term, since the differentials are determined precisely by the section $s : \Omega_X \to \mathfrak{g}^\vee$.

From the description of the spectral sequence, we see that for $q = 0$ and $-j+1 \leq p \leq 0$, it degenerates at E_2 and we get $0 = H^{i-j}(X \times \mathbb{P}(\mathfrak{g}), \tilde{K}_s^\cdot) = \mathcal{H}^i(F^\cdot)$, for $1 \leq i < j$ (even before taking invariants). By the same reason we find that indeed $H^q(M, (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^m}(j))|_M) = 0$ if $q > 0$, although we do not need this. Now if, further, $\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j)|_M$ has no invariant global sections, then the complex of G-invariants $(E_1^{\cdot,0}[-j])^G$ is precisely $(F^\cdot)^G$, so in this case, $\mathcal{H}^j((F^\cdot)^G) = 0$. Since taking invariant sections is an exact functor, this is the same as saying that $\mathcal{H}^j(F^\cdot)^G = 0$. □

2.3.1 Vanishing on the GIT quotient

Observe that Proposition 2.3.2 applies to the quotient stack $\mathfrak{X}^{ss}$. Now, if the action of G on X^{ss} is free, this result can be interpreted in terms of its coarse moduli space, namely the GIT quotient $Y = X /\!\!/_{\mathcal{L}} G$. Indeed, if G acts freely on X^{ss}, then $\Lambda^j L_{\mathfrak{X}^{ss}}$ descends to Ω_Y^j, as observed in Lemma 1.3.8. Suppose further that $\mathcal{L}$ descends to a line bundle L on Y. By exactness of $p_* = \pi_*(\cdot)^G$, we have $H^i(Y, \Omega_Y^j \otimes L) = H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L})$, so if the latter vanishes, then so does $H^i(Y, \Omega_Y^j \otimes L)$.

In the case of a quotient $Y = (\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$ without strictly semi-stable locus, we can check that the hypotheses of Proposition 2.3.2 are satisfied, so $H^i(Y, \Omega_Y^j \otimes L) = 0$ for $i \neq 0, j$, where L is the descent of the linearization $\mathcal{L}$. In Section 2.5, we will see that, in order to show Bott vanishing for $(\mathbb{P}^1)^n /\!\!/_{\mathcal{L}} PGL_2$, the only line bundle we need to consider is precisely the descent of $\mathcal{L}$. The rest of that section will be devoted to prove the vanishing of the j-th cohomology.

More generally, when the action is not free on X^{ss} and the coarse moduli space Y is not smooth, we introduce the following notation.

Notation 2.3.3. Let $Y^0 \subset Y$ be the nonsingular locus, with inclusion $\imath : Y^0 \hookrightarrow Y$. Then for each j, Ω_Y^j will denote the (non-derived) pushforward $\imath_*(\Omega_{Y^0}^j)$. We call $X' \subset X^{ss}$ the locus where G acts freely.

Note that $X' \subset \pi^{-1}(Y^0)$, where $\pi : X^{ss} \to Y$ is the quotient map. Recall $\Lambda^j L_{\mathfrak{X}^{ss}}$ is a vector bundle provided $X^{ss} = X^s$. We are interested in the cases when $\pi_*(\Lambda^j L_{\mathfrak{X}^{ss}})^G = \Omega_Y^j$. Suppose this holds and $\mathcal{L}$ descends to L, that is, there is a G-equivariant isomorphism $\pi^*(L) \cong \mathcal{L}|_{\mathfrak{X}^{xx}}$. In this situation, the projection formula yields $\pi_*(\Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L})^G = \Omega_Y^j \otimes L$ and Proposition 2.3.2 can be interpreted as vanishing of cohomologies $H^i(Y, \Omega_Y^j \otimes L)$.

Proposition 2.3.4. *With the notation as above, suppose $X^{ss} = X^s$ and $X^{ss} \backslash X' \subset X^{ss}$*

has codimension at least 2. If $\mathcal{L}$ descends to a line bundle L on Y, then $H^i(Y, \Omega_Y^j \otimes L) = H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L})$ for every i, j.

Proof. Let $Y' = \pi(X')$ and consider the open inclusions $X' \hookrightarrow X^{ss}$ and $\iota : Y' \hookrightarrow Y$,

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "\pi"] & X^{ss} \arrow[d, "\pi"] \\ Y' \arrow[r, hook, "\iota"] & Y \end{tikzcd}$$

where $Y' \subset Y^0 \subset Y$ and $X' \subset \pi^{-1}(Y^0) \subset X^{ss}$. We first observe that, since $X^{ss} = X^s$, π is equidimensional and so $Y \backslash Y' \subset Y$ has codimension at least 2. The same is true for $Y^0 \backslash Y' \subset Y^0$. Write ι as a composition

$$Y' \overset{\imath'}{\hookrightarrow} Y^0 \overset{\imath}{\hookrightarrow} Y.$$

By smoothness of X, Y has to be normal, and then we see that $\iota_* \mathcal{O}_{Y'} = \mathcal{O}_Y$, while $\imath'_* \Omega^j_{Y'} = \Omega^j_{Y^0}$, by the codimension condition. Using $\iota = \imath \circ \imath'$, we get $\iota_* \Omega^j_{Y'} = \Omega^j_Y$.

G acts freely on X', so we have a G-equivariant short exact sequence $0 \to \pi^* \Omega_{Y'} \to \Omega_{X'} \to \mathfrak{g}^\vee \to 0$. Therefore, the restriction of $L_{\mathfrak{X}^{ss}}$ to X' descends to $\Omega_{Y'}$, and similarly for their j-th exterior powers. We then have $\pi_*(\Lambda^j L_{\mathfrak{X}^{ss}}|_{X'})^G = \Omega^j_{Y'}$. On the other hand, it is not difficult to see that $\pi_*(\Lambda^j L_{\mathfrak{X}^{ss}}|_{X'})^G = \pi_*(\Lambda^j L_{\mathfrak{X}^{ss}})^G|_{Y'}$. That is,

$$\iota^* \pi_*(\Lambda^j L_{\mathfrak{X}^{ss}})^G = \Omega^j_{Y'}. \tag{2.3.3}$$

Using the projection formula on (2.3.3) and the fact that $\iota_* \mathcal{O}_{Y'} = \mathcal{O}_Y$, we get $\pi_*(\Lambda^j L_{\mathfrak{X}^{ss}})^G = \iota_* \Omega^j_{Y'} = \Omega^j_Y$. Therefore,

$$\pi_*(\Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L})^G = \Omega^j_Y \otimes L$$

again by the projection formula. By exactness of $p_* = \pi_*(\cdot)^G$, we obtain $H^i(Y, \Omega^j_Y \otimes L) = H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L})$. $\square$

Remark 2.3.5. If $X^{ss} = X^s$ and the action is free on $\pi^{-1}(Y^0)$, we have that $X' = \pi^{-1}(Y^0)$ and the condition on the codimension of $X^{ss} \backslash X' \subset X^{ss}$ is automatically satisfied. Indeed, $Y \backslash Y^0 \subset Y$ has codimension ≥ 2 by normality of Y. Equidimensionality of π guarantees that the same is true for $X^{ss} \backslash \pi^{-1}(Y^0) \subset X^{ss}$.

2.4 The case of X affine and G abelian

In the case that G is an abelian group and X is a smooth affine variety, we get a stronger version of Proposition 2.3.2, provided there is no strictly semi-stable locus. To do this, we apply very similar techniques to the ones used in Section 2.3. The difference is that, in this case, we can take advantage of Corollary 2.2.2 by working on the semi-stable locus from the beginning. Also, an affine variety X automatically satisfies the Bott vanishing condition.

As usual, $\mathcal{L}$ denotes a G-linearized ample line bundle on a smooth projective-over-affine variety X with a G-action. Consider the augmented Koszul complex $K_s^{\cdot}$ on $X \times \mathbb{P}(\mathfrak{g})$ defined in (2.3.2), and let $\bar{K}_s^{\cdot}$ be its restriction to $X^{ss} \times \mathbb{P}(\mathfrak{g})$. We first observe that this restriction is exact if $X^{ss} = X^s$. This is because the projection of M to X lands entirely on the unstable locus.

Lemma 2.4.1. *If $X^{ss} = X^s$, then $M \cap (X^{ss} \times \mathbb{P}(\mathfrak{g})) = \emptyset$. In particular, the restriction $\bar{K}_s^{\cdot}$ is acyclic in this case.*

Proof. For a pair (x, l) in M, l must be a line in $\mathfrak{g} = \mathbb{A}^{m+1}$ contained in the Lie algebra of the stabilizer G_x, so x cannot be stable. By the assumption, $x \notin X^{ss}$. As a consequence, the restriction of $K_s^{\cdot}$ to $X^{ss} \times \mathbb{P}(\mathfrak{g})$ is acyclic since $M \cap (X^{ss} \times \mathbb{P}(\mathfrak{g})) = \emptyset$ is a local complete intersection. $\square$

Now suppose G is abelian and X is affine. Let $\bar{F}^{\cdot}$ the complex of global sections of $\Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}$,

$$\bar{F}^{\cdot} = \Big[0 \to H^0(X^{ss}, \Omega^j_{X^{ss}} \otimes \mathcal{L}) \to H^0(X^{ss}, \Omega^{j-1}_{X^{ss}} \otimes \mathcal{L}) \otimes \mathfrak{g}^{\vee} \to \cdots \to H^0(X^{ss}, \mathcal{L}) \otimes S^j \mathfrak{g}^{\vee} \to 0\Big], \tag{2.4.1}$$

concentrated in degrees 0 to j. Using a similar argument to the one in Lemma 2.3.1, we can show the complex of invariants $(\bar{F}^{\cdot})^G$ computes the hypercohomologies of $\Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}$.

Lemma 2.4.2. *If G is abelian and X is affine, we have $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = \mathcal{H}^i(\bar{F}^{\cdot})^G$.*

Proof. First, we see that by Corollary 2.2.2, $H^i(\mathfrak{X}^{ss}, (\Lambda^j L_{\mathfrak{X}^{ss}})^p \otimes \mathcal{L}) = H^i(\mathfrak{X}, (\Lambda^j L_{\mathfrak{X}})^p \otimes \mathcal{L})$. For $i > 0$ this is zero since X is affine. Now take the spectral sequence $E_1^{p,q} = H^q(\mathfrak{X}^{ss}, (\Lambda^j L_{\mathfrak{X}^{ss}})^p \otimes \mathcal{L})$, which converges to $H^{p+q}(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L})$. By the previous observation, we see that $E_1^{p,q} = 0$ for $q \neq 0$, and so $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = \mathcal{H}^i(E_1^{\cdot,0})$ for every i. But the complex $E_1^{\cdot,0}$ is precisely $(\bar{F}^{\cdot})^G$. $\square$

Following the ideas from Proposition 2.3.2, we obtain the following vanishing result. Here Ω^j_Y and X' are as in Notation 2.3.3.

Theorem 2.4.3. *Suppose G is abelian, X is affine and $X^{ss} = X^s$. Then $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = 0$ for every $i > 0, j \geq 0$. Further, if $X^{ss} \backslash X'$ has codimension at least 2 and $\mathcal{L}$ descends to L, we have*

$$H^i(Y, \Omega^j_Y \otimes L) = 0 \quad \forall i > 0, j \geq 0.$$

Proof. Let $\bar{K}_s^{\cdot}$ be the restriction of $K_s^{\cdot}$ to $X^{ss} \times \mathbb{P}(\mathfrak{g})$. By Lemma 2.4.1, $\bar{K}_s^{\cdot}$ is an acyclic complex, being the (augmented) Koszul resolution of $M \cap (X \times \mathbb{P}(\mathfrak{g})) = \emptyset$.

Take now the spectral sequence $E_1^{p,q} = H^q(X^{ss} \times \mathbb{P}(\mathfrak{g}), \bar{K}_s^p \otimes \mathcal{L})^G$, converging to $H^{p+q}(X^{ss} \times \mathbb{P}(\mathfrak{g}), \bar{K}_s^{\cdot} \otimes \mathcal{L})^G = 0$. Since $H^i(X^{ss}, (\Lambda^j L_{\mathfrak{X}^{ss}})^p \otimes \mathcal{L})^G = 0$ for $i > 0$, we find

$$E_1^{p,q} = \begin{cases} (H^0(X^{ss}, \Omega_{X^{ss}}^{-p} \otimes \mathcal{L}) \otimes S^{p+j}\mathfrak{g}^\vee)^G & \text{if} \quad q = 0,\ -j \leq p \leq 0 \\ (H^0(X^{ss}, \Omega_{X^{ss}}^{-p} \otimes \mathcal{L}) \otimes H^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(j+p))^\vee)^G & \text{if} \quad q = m,\ p \leq -j-m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the complex of invariants $(\bar{F}^{\cdot})^G$ is precisely the shifted complex $E_1^{\cdot,0}[-j]$. For $q = 0$ and $-j+1 \leq p \leq 0$, the sequence degenerates at E_2 and we get $0 = H^{i-j}(X^{ss} \times \mathbb{P}(\mathfrak{g}), \bar{K}_s^{\cdot} \otimes \mathcal{L}) = \mathcal{H}^i(\bar{F}^{\cdot})^G$, for $i \geq 1$. From Lemma 2.4.2, we conclude $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = 0$ for $i > 0$. The last part of the statement is a direct consequence of Proposition 2.3.4. $\square$

2.4.1 The toric case

Now let Y be a $\mathbb{Q}$-factorial projective toric variety. From [Cox14], we know Y is the GIT quotient of an affine space $X = \mathbb{A}^d$ by the abelian reductive group $G = \mathrm{Hom}(\mathrm{Cl}\, Y, \mathbb{G}_m)$, with $X^s = X^{ss}$ and $X^{us} \subset X$ has codimension at least 2. The character group of G is canonically identified with $\mathrm{Cl}\, Y$. If we call Σ the fan in $N \cong \mathbb{Z}^n$ determining the toric variety and $d = |\Sigma(1)|$ the number of 1-dimensional cones, then we have a surjection $\mathbb{Z}^d = \langle e_\rho,\ \rho \in \Sigma(1)\rangle_{\mathbb{Z}} \twoheadrightarrow \mathrm{Cl}\, Y$ and we can write $X = \mathrm{Spec}\, R$, where $R = \Bbbk[x_1, \ldots, x_d] = \bigoplus_{v \in \mathrm{Cl}\, Y} R_v$ is the Cox ring, and each graded piece $R_v \cong H^0(Y, \mathcal{O}_Y(D))$, for $v = [D]$. We have a short exact sequence

$$0 \to M \to \mathbb{Z}^d \to \mathrm{Cl}\, Y \to 0, \tag{2.4.2}$$

with the map on the left being $m \mapsto \sum \langle m, n_\rho \rangle e_\rho$, where $n_\rho \in N$ is the vector corresponding to the 1-dimensional cone $\rho \in \Sigma(1)$. This way, the action of G is described by the short exact sequence

$$1 \to G \to (\mathbb{G}_m^\vee)^d \to T \to 1$$

obtained by applying $\mathrm{Hom}(\cdot, \mathbb{G}_m)$ to (2.4.2). Here $T = N \otimes \mathbb{G}_m^\vee$ is the torus acting on Y. Using the usual description of Y as $\bigcup u_\sigma$, where $u_\sigma = \mathrm{Spec}\, \Bbbk[\sigma^\vee \cap M]$, this quotient is described locally by $u_\sigma = U_\sigma / G$, where $U_\sigma = \{z \in \mathbb{A}^d \mid z_\rho \neq 0 \quad \forall \rho \notin \sigma(1)\}$ (see [Cox14], [CLS11, §14] for details). In the case that Y is smooth, then $G = \mathbb{G}_m^{d-n}$ is a torus and the action is free on X^{ss}.

In what follows we will see that, from Theorem 2.4.3 we can recover Bott vanishing for the toric case, over $\Bbbk$. For the remainder of the present section X will denote $\mathbb{A}^d$, G will denote

$\mathrm{Hom}(\mathrm{Cl}\,Y, \mathbb{G}_m)$ and $Y = X /\!\!/_{\mathcal{L}} G$ will be a projective $\mathbb{Q}$-factorial toric variety obtained as a GIT quotient given by a linearization $\mathcal{L}$.

We first see that, in order to show Bott vanishing, the only ample line bundle we need to consider is the descent of $\mathcal{L}$ (cf. [HK00, Proposition 2.9]).

Lemma 2.4.4. *With the notation as above, let L be an ample line bundle on Y. Then L is the descent of a linearization $\mathcal{L}'$ such that $Y = X /\!\!/_{\mathcal{L}'} G$.*

Proof. Since X is an affine space and X^{us} has codimension ≥ 2, we see $\mathrm{Pic}\,X = \mathrm{Pic}\,X^{ss}$ are trivial and the G-equivariant Picard group is $\mathrm{Pic}^G X = \mathrm{Pic}^G X^{ss} = \mathrm{Cl}\,Y$, the character group of G. The map $\mathrm{Pic}\,Y \to \mathrm{Pic}^G X^{ss}$, $L \mapsto \pi^* L$, is the inclusion $\mathrm{Pic}\,Y \hookrightarrow \mathrm{Cl}\,Y$. That is, every given line bundle L on Y is the descent of $\mathcal{L}_v$, which is the trivial line bundle on X linearized by the character $v = L \in \mathrm{Cl}\,Y$. Further, given a linearization $\mathcal{L}_w$, for some $w \in \mathrm{Cl}\,Y$, we see that R^G is precisely $\bigoplus_{k \geq 0} R_{kw} = \bigoplus_{k\geq 0} H^0(Y, L^{\otimes k})$, for $L = w$. If L is an ample line bundle on Y, then $Y = \mathrm{Proj} \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})$, so that L is the descent of a linearization $\mathcal{L}' \in \mathrm{Pic}^G X$ such that $Y = X /\!\!/_{\mathcal{L}'} G$. □

We also check that the action is free on the preimage of the smooth locus $Y^0 \subset Y$.

Lemma 2.4.5. *G acts freely on $\pi^{-1}(Y^0)$.*

Proof. The smooth locus of Y is given by $\bigcup_{\sigma \text{ smooth}} u_\sigma$ (see e.g. [CLS11, Proposition 11.1.2]). It suffices to check that G acts freely on U_σ for $\sigma \in \Sigma$ a smooth cone. Consider the map $h : \mathbb{Z}^d \twoheadrightarrow \mathrm{Cl}\,Y$ from (2.4.2) and suppose $g \in G = \mathrm{Hom}(\mathrm{Cl}\,Y, \mathbb{G}_m)$ is in the stabilizer of some $z \in U_\sigma$. Since $z_\rho \neq 0$ for every $\rho \notin \sigma(1)$, this implies $g(v) = 1$ for every $v \in h(\langle e_\rho,\, \rho \notin \sigma(1)\rangle_{\mathbb{Z}})$.

But in fact, if σ is a smooth cone, the restriction of h to the span of $\{e_\rho,\, \rho \notin \sigma(1)\}$ is still surjective. This is because we can complete $\{n_\rho,\, \rho \in \sigma(1)\}$ to a $\mathbb{Z}$-basis $\{n_\rho\} \cup \{n'_\alpha\}$ of N, choose a dual basis $\{m_\rho, \rho \in \sigma(1)\} \cup \{m'_\alpha\}$ and see that under the map $f : M \to \mathbb{Z}^d$ from (2.4.2), $m_\rho \mapsto e_\rho + w$, some $w \in \langle e_\rho,\, \rho \notin \sigma(1)\rangle_{\mathbb{Z}}$. As a consequence, every vector $w \in \mathbb{Z}^d$ can be written as $w' + w''$, with $w' \in \mathrm{im}(f) = \ker(h)$, $w'' \in \langle e_\rho, \rho \notin \sigma(1)\rangle_{\mathbb{Z}}$. We conclude $\langle e_\rho, \rho \notin \sigma(1)\rangle_{\mathbb{Z}} \to \mathrm{Cl}\,Y$ is surjective. Therefore $g(v) = 1$ for every $v \in \mathrm{Cl}\,Y$, so $g = 1$. □

Finally, we get a new proof of the following well-known result.

Theorem 2.4.6 (Bott vanishing for toric varieties)**.** *Let Y be a $\mathbb{Q}$-factorial projective toric variety over $\Bbbk$ and L an ample line bundle on Y. Then $H^i(Y, \Omega_Y^j \otimes L) = 0$ for every $i > 0, j \geq 0$. In particular, a smooth projective toric variety over $\Bbbk$ satisfies Bott vanishing.*

Proof. Let L be an ample line bundle on Y. By the discussion above L is the descent of a linearization $\mathcal{L}'$ such that $Y = X /\!\!/_{\mathcal{L}'} G$, so we can assume L is the descent of the linearization

$\mathcal{L}$. By Lemma 2.4.5, the non-free locus $X^{ss}\backslash X'$ has codimension ≥ 2 (see Remark 2.3.5), so Theorem 2.4.3 implies $H^i(Y, \Omega_Y^j \otimes L) = 0$ for $i > 0, j \geq 0$. If Y is smooth, then $Y^0 = Y$ and this is Bott vanishing. $\square$

2.5 The case of $X = (\mathbb{P}^1)^n$ and $G = PGL_2$

Now we consider the diagonal action of PGL_2 on $(\mathbb{P}^1)^n$, so throughout this section G will denote PGL_2, X will denote $(\mathbb{P}^1)^n$ and $\mathfrak{g}$ will denote $\mathfrak{sl}_2$. For a given ample line bundle $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$, $d_i > 0$, where $\sum d_i$ is even, there is a unique PGL_2-linearization, giving rise to a GIT quotient $Y = X /\!\!/_{\mathcal{L}} PGL_2$, a projective variety. Variation of GIT is described, for instance, in [Has03, §8]

A maximal torus of $\mathfrak{g}$ is one-dimensional, so to get a KN stratification it essentially suffices to consider a single one-parameter subgroup. We consider $\lambda : \mathbb{G}_m \to PGL_2$ given by

$$\lambda(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

The fixed locus of λ is the union of the points z_I where, for every $I \subset \{1, \ldots, n\}$, z_I has coordinates $z_i = \infty$ if $i \in I$ and $z_i = 0$ otherwise. We use the convention $0 = (0 : 1)$, $\infty = (1 : 0)$. One can compute

$$\mu(\lambda, I) = -\text{weight}_\lambda \mathcal{L}|_{z_I} = \sum_{i \in I} d_i - \sum_{i \notin I} d_i$$

(cf. Example 1.1.4) and we can get a KN stratification of the unstable locus indexed by the subsets I for which $\mu(\lambda, I) > 0$. Indeed, a point $z = (z_1, \ldots, z_n) \in X$ is unstable if and only if there is an $I \subset \{1, \ldots, n\}$ with $\sum_{i \in I} d_i > \sum_{i \notin I} d_i$ such that $z_i = \alpha$ for every $i \in I$. Also, it can be computed that $\eta_{\lambda, I} = 2(|I| - 1)$ (see [CT20b] for details). A linearization $\mathcal{O}_X(d_1, \ldots, d_n)$ is in a GIT wall if and only if there is a splitting $I \cup I^c = \{1, \ldots, n\}$ such that $\sum_{i \in I} d_i = \sum_{i \notin I} d_i$. Since the ambient space $X = (\mathbb{P}^1)^n$ is a smooth projective toric variety, it satisfies Bott vanishing and then results from Section 2.3 can be applied.

Note that in our case, the cotangent sheaf is a direct sum of line bundles, namely $\Omega_X = \bigoplus_{i=1}^n \mathcal{O}_X(0, \ldots, -2, \ldots, 0)$, each summand having a -2 in the i-th position and zeros elsewhere. The section $s \in H^0(T_X \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1))^G$ associated to the map $\Omega_X \to \mathfrak{g}^\vee$ is then the sum of n sections s_i, where each $s_i \in (H^0(X, \mathcal{O}_X(0, \ldots, 2, \ldots, 0)) \otimes \mathfrak{g}^\vee)^G$.

Let $\{E, H, F\}$ be the usual basis of $\mathfrak{g}$, where $[E, H] = -2E$, $[E, F] = H$, $[H, F] = -2F$. If we choose a basis $\{X_0, Y_0, Z_0\}$ of $\mathfrak{g}^\vee = H^0(\mathbb{P}(\mathfrak{g}), T_{\mathbb{P}(\mathfrak{g})})$ that is dual to the basis $\{-E, H, F\}$, then we can explicitly compute s_i in coordinates. For this, we use the isomorphism $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, $\partial/\partial(x/y) \mapsto -y^2$, where $(x:y)$ are coordinates in $\mathbb{P}^1$. Writing

$$E = \frac{\partial}{\partial t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\Bigg|_{t=0}, \quad H = \frac{\partial}{\partial t}\frac{1}{1-t^2}\begin{pmatrix} 1+t & 0 \\ 0 & 1-t \end{pmatrix}\Bigg|_{t=0}, \quad F = \frac{\partial}{\partial t}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\Bigg|_{t=0}$$

and using the chart $y \neq 0$ we find that the action of PGL_2 on $\mathbb{P}^1$ determines the map $\mathfrak{g} \to T_{\mathbb{P}^1}$ that sends

$$E \mapsto \frac{\partial}{\partial(x/y)}, \quad H \mapsto \frac{2x}{y}\frac{\partial}{\partial(x/y)}, \quad F \mapsto -\frac{x^2}{y^2}\frac{\partial}{\partial(x/y)}.$$

Combining all these and using $\partial/\partial(x_i/y_i) \mapsto -y_i^2$ on each i-th component $\mathbb{P}^1_{(x_i:y_i)}$, we find $s_i = x_i^2 Z_0 - 2x_i y_i Y_0 + y_i^2 X_0$. Observe that we can also identify $\mathfrak{g} \cong \mathfrak{g}^\vee$ as $\mathfrak{g}$-representations by sending the basis $\{E, H, F\}$ to $\{Z_0, 2Y_0, -X_0\}$.

Remark 2.5.1. Consider the diagonal action of SL_2 on X. Then any ample line bundle $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$ carries a unique SL_2-linearization, giving rise to a GIT quotient $Y = X /\!\!/_{\mathcal{L}} SL_2$. If $\sum d_i$ is even, then $\mathcal{L}$ also admits a unique PGL_2-linearization, and $X /\!\!/_{\mathcal{L}} SL_2 = X /\!\!/_{\mathcal{L}} PGL_2$. In any case, $\mathcal{L}^{\otimes 2}$ admits a PGL_2-linearization and $X /\!\!/_{\mathcal{L}} SL_2$ is canonically isomorphic to the quotient $X /\!\!/_{\mathcal{L}^{\otimes 2}} SL_2 = X /\!\!/_{\mathcal{L}^{\otimes 2}} PGL_2$.

Proposition 2.5.2. *Consider $X = (\mathbb{P}^1)^n$, $G = PGL_2$, $\mathfrak{g} = \mathfrak{sl}_2$ as above. Let $M \subset X \times \mathbb{P}(\mathfrak{g})$ be the vanishing locus of the section $s \in H^0(T_X \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1))^G$ associated to the map $\Omega_X \to \mathfrak{g}^\vee$. Then $M = \bigcap(s_i = 0)$ is a local complete intersection.*

Proof. The section s is the direct sum of the n sections $s_i \in H^0(\mathcal{O}_X(0, \ldots, 2, \ldots, 0) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))^G$ given by $s_i = x_i^2 Z_0 - 2x_i y_i Y_0 + y_i^2 X_0$, as noted above. By smoothness of X, it suffices to check that $\dim M = 2$. Consider the map $p : M \to \mathbb{P}^2$ given by the projection on the second component. We show p is a finite map. Indeed, since p is projective, it suffices to show that it has finite fibers. We note that a given point $(x_1 : y_1; \ldots; x_n : y_n) \times (X_0 : Y_0 : Z_0)$ is in M if and only if for every i, $(X_0 : Y_0 : Z_0) \in \mathbb{P}^2$ is in the line that is tangent to the rational normal curve $(X_0 Z_0 - Y_0^2 = 0) \subset \mathbb{P}^2$ at the point $(x_i^2 : x_i y_i : y_i^2)$. Since every point is in at most 2 lines tangent to a given conic, we have $|p^{-1}(X_0 : Y_0 : Z_0)| \leq 2^n$. Therefore p is finite and $\dim M = 2$. □

Corollary 2.5.3. *For a PGL_2-linearized ample line bundle $\mathcal{L}$ on X, we have $H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}) = 0$ for $i \neq 0, j$.*

Proof. This follows from Proposition 2.3.2, as M is a local complete intersection and $X = (\mathbb{P}^1)^n$ has the Bott vanishing property. Recall $H^i(\mathfrak{X}^{ss}, \Lambda^j L_{\mathfrak{X}^{ss}} \otimes \mathcal{L}) = H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})$ by Corollary 2.2.2. □

2.5.1 The ring of invariants

Let $Y = (\mathbb{P}^1)^n /\!/ PGL_2$ be given by a polarization $\mathcal{L} = \mathcal{O}(d_1, \ldots, d_n)$, $d_i > 0$. We will assume $X^{ss} = X^s$. This implies that the action of $G = PGL_2$ is free in X^{ss} and Y is smooth.

Remark 2.5.4. The condition $X^{ss} = X^s$ is equivalent to the following condition: there is no partition $I \sqcup I^c = \{1, \ldots, n\}$ such that $\sum_{i \in I} d_i = \sum_{i \notin I} d_i$. This is a consequence of the Hilbert-Mumford criterion and the description of the unstable locus (see e.g. [CT20b, §4]).

If we consider the action of the torus $(\mathbb{G}_m)^n$ on the Grassmannian $\mathrm{Gr}(2, n)$ and linearize the ample line bundle $\mathcal{O}_{G(2,n)}(1)$ of the Plücker embedding using some character $(l_1, \ldots, l_n)$, we have Gelfand-MacPherson correspondence [Kap93, Theorem 2.4.7]:

$$\bigoplus_{d \geq 0} H^0((\mathbb{P}^1)^n, \mathcal{O}(dl_1, \ldots, dl_n))^{PGL_2} = \bigoplus_{d \geq 0} H^0(\mathrm{Gr}(2, n), \mathcal{O}(d))^{(\mathbb{G}_m)^n}.$$

That is, $\bigoplus_{d \geq 0} H^0((\mathbb{P}^1)^s, \mathcal{O}(dl_1, \ldots, dl_n))^{PGL_2}$ can be seen as a subring of the homogeneous coordinate ring of the Grassmannian, $\Bbbk[p_{ik}]/(p_{ik}p_{rl} - p_{ir}p_{kl} + p_{il}p_{kr})$, where $p_{ik} = x_i y_k - x_k y_i$ are the Plücker minors. The d-th graded piece corresponds to polynomials in p_{ik} having multi-degree $dl_1, \ldots, dl_n$ in $x_1, y_1; \ldots; x_n, y_n$.

Lemma 2.5.5. *Suppose we have a linearization $\mathcal{L}$ giving $X^{ss} = X^s$ and with an unstable locus having an irreducible component of codimension 1. Then $Y = (\mathbb{P}^1)^n /\!/_{\mathcal{L}} PGL_2$ is a smooth projective toric variety.*

Proof. Given that $X^s = X^{ss}$, Y is the (smooth) geometric quotient X^{ss}/G. By the description of the unstable locus, we can assume $d_1 + d_2 > \sum_{i \geq 3} d_i$ without loss of generality. That is, X^{ss} does not intersect the big diagonal $\{p_1 = p_2\} \subset (\mathbb{P}^1)^n$. Call $V = 0 \times \infty \times (\mathbb{P}^1)^{n-2}$ and consider $\mathbb{G}_m$ as the subgroup of PGL_2 given by

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}. \tag{2.5.1}$$

Observe $\mathbb{G}_m$ acts on V, and the linearization $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$ restricts to a $\mathbb{G}_m$-linearization of $\mathcal{L}|_V = \mathcal{O}_V(d_3, \ldots, d_n)$, which corresponds to the character $t \mapsto t^{d_1 - d_2}$. By the stability condition, we see that every stable G-orbit intersects V. Further, $V \cap X^{ss}$ is precisely the semi-stable locus $V^{ss} = V^s$ for the $\mathbb{G}_m$-linearization of $\mathcal{L}|_V$. In fact, $Z = (0, \infty, z_3, \ldots, z_n) \in V$ is unstable if and only if there is some $I' \subset \{3, \ldots, n\}$ such that one of the following holds:

(a) $d_2 + \sum_{i \in I'} d_i > d_1 + \sum_{i \notin I'} d_i$ and $z_i = \infty$ for all $i \in I'$, or

(b) $d_2 + \sum_{i \in I'} d_i < d_1 + \sum_{i \notin I'} d_i$ and $z_i = 0$ for all $i \notin I'$.

Since $d_1 + d_2 > \sum_{i\geq 3} d_i$, this is the same stability condition for the PGL_2 action. Observe $\mathbb{G}_m$ acts freely on V^{ss}, and $V^{ss} /\!\!/ \mathbb{G}_m$ is a geometric quotient.

We note that in fact the GIT quotients $X /\!\!/ G$ and $V /\!\!/ \mathbb{G}_m$ coincide. To see this, we can look at the coordinate rings of invariants. First, observe that $p_{i1} = x_i y_1 - y_i x_1$ restricts to x_i on V, while p_{2i} restricts to y_i. Call $2\delta = d_1 + d_2 - \sum_{i\geq 3} d_i > 0$. Then the restriction $\bigoplus_{k\geq 0} H^0(X, \mathcal{L}^{\otimes k})^{PGL_2} \to \bigoplus_{k\geq 0} H^0(V, \mathcal{L}|_V^{\otimes k})^{\mathbb{G}_m}$ is an isomorphism of graded rings, with inverse given in degree k by

$$R(x_i, y_i) \mapsto p_{21}^{\delta k} R(p_{i1}, p_{2i}),$$

for a polynomial $R(x_3, y_3; \ldots; x_n, y_n) \in H^0(V, \mathcal{O}_V(kd_3, \ldots, kd_n))^{\mathbb{G}_m}$.

Now let the torus $T = (\mathbb{G}_m)^{n-2}$ act on $V = (\mathbb{P}^1)^{n-2}$, by (2.5.1) in each component. Then $V^{ss} = X^{ss} \cap V$ is invariant under the action of T. In fact, suppose $z = (0, \infty, z_3, \ldots, z_n) \in X^{us}$. Then $z_i = \alpha$ for every $i \in I$, for some I such that $\sum_{i\notin I} d_i > \sum_{i\in I} d_i$. Since $d_1 + d_2 > \sum_{i\geq 3} d_i$, either $1 \in I$, in which case $\alpha = 0$, or $2 \in I$, in which case $\alpha = \infty$. But both 0 and ∞ are fixed by $\mathbb{G}_m$, so $t \cdot z$ will still be unstable for any $t \in T$.

Further, T acts on V^{ss} with an open dense orbit, say $T \cdot (0, \infty, 1, \ldots, 1)$. We conclude that the $(n-3)$-dimensional torus $T/\mathbb{G}_m$ acts on $Y = V^{ss}/\mathbb{G}_m$ with an open dense orbit. Therefore Y is a toric variety. $\square$

Another proof of the previous lemma can be found in [Sch17, Theorem 2], using the point of view of variation of GIT.

Now suppose X^{us} has codimension ≥ 2. We claim that any ample line bundle on Y is the descent of an ample line bundle $\mathcal{O}_X(d'_1, \ldots, d'_n)$ on X living in the same GIT chamber as $\mathcal{L}$, in the sense of [DH98].

Lemma 2.5.6. *Suppose X^{us} has codimension at least 2 and let L be an ample line bundle on $Y = X /\!\!/_{\mathcal{L}} PGL_2$, where $\mathcal{L}$ is such that $X^s = X^{ss}$. Then L is the descent of an ample line bundle $\mathcal{L}' = \mathcal{O}_X(d'_1, \ldots, d'_n)$ such that $Y = X /\!\!/_{\mathcal{L}'} PGL_2$.*

Proof. Since the action of G is free on $X^{ss} = X^s$, by Kempf's descent lemma, every line bundle on X descends to a line bundle on Y, and in fact π^* is an isomorphism from $\operatorname{Pic} Y$ to the G-equivariant Picard group $\operatorname{Pic}^G X^{ss} = \operatorname{Pic} \mathfrak{X}^{ss}$, with inverse $\mathcal{L}' \mapsto \pi_*(\mathcal{L}')^G$. Further, every PGL_2-linearized line bundle on X^{ss} extends uniquely to a PGL_2-linearized line bundle on X by the codimension hypothesis (see e.g. [Dol03, §7]).

For any $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, PGL_2 acts naturally on the global sections $H^0(X, \mathcal{O}_X(v))$. Let R be the $\mathbb{Z}^n$-graded ring $R = \bigoplus_{v\in\mathbb{Z}^n} R_v$, where $R_v = H^0(X, \mathcal{O}_X(v))^{PGL_2}$. Notice $R_v = 0$ if $\sum v_i$ is odd or if some $v_i < 0$. If $\mathcal{L} = \mathcal{O}_X(w)$ is the linearization, then by definition

$Y = X /\!\!/_{\mathcal{L}} PGL_2 = \operatorname{Proj} \bigoplus_{k \geq 0} R_{kw}$. From the previous observation, every PGL_2-linearized line bundle $\mathcal{O}_X(v)$ descends to a line bundle $\mathcal{L}_v$ on Y, and by the codimension hypothesis, $R_v = H^0(Y, \mathcal{L}_v)$. On the other hand, given a line bundle L on Y, L must be $\mathcal{L}_{w'}$ for some $w' \in \mathbb{Z}^n$. If $\mathcal{L}_{w'}$ is ample, then $Y = \operatorname{Proj} \bigoplus_{k \geq 0} (Y, \mathcal{L}_{w'}^{\otimes k}) = \operatorname{Proj} \bigoplus_{k \geq 0} R_{kw'}$. From this we see that the w'_i are nonnegative, and in fact $w'_i > 0$ since $\dim Y = n - 3$. That is, $\mathcal{L}' = \mathcal{O}_X(w')$ is ample and $Y = X /\!\!/_{\mathcal{L}'} PGL_2$. □

Remark 2.5.7. In the situation of Lemma 2.5.6, we have $\operatorname{codim} X^{us} \geq 2$ and $\operatorname{Pic} Y = \operatorname{Pic}^G X$, so the quotient $X /\!\!/_{\mathcal{L}'} PGL_2$ can only be isomorphic to $Y = X /\!\!/_{\mathcal{L}} PGL_2$ if $\mathcal{L}$ and $\mathcal{L}'$ live in the same GIT chamber (cf. [Has03, Proposition 5.1]). Observe that such $\mathcal{L}'$, descending to an ample line bundle on Y, cannot be in one of the GIT walls, because the ample cone of Y is open in $\operatorname{Pic} Y = \operatorname{Pic}^G X$. In particular, $\mathcal{L}'$ admits no strictly semi-stable locus.

If we want to show vanishing for $H^i(Y, \Omega_Y^j \otimes L)$, by Corollary 2.2.2 and Lemma 2.3.1, we need to compute $H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L}) = \mathcal{H}^i(F^\cdot)^G$, where $F^\cdot$ is given by (2.3.1). From Corollary 2.5.3, we know $\mathcal{H}^i(F^\cdot)^G = 0$ for $i \neq 0, j$, so it remains to show that the maps of G-invariant global sections $(H^0(X, \Omega_X \otimes \mathcal{L}) \otimes S^{j-1}\mathfrak{g}^\vee)^G \to (H^0(X, \mathcal{L}) \otimes S^j \mathfrak{g}^\vee)^G$ are surjective. The following two propositions show this holds when $j = 1$ and 2.

Proposition 2.5.8. *Let $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$ be a linearization with no strictly semi-stable locus. The map $H^0(X, \Omega_X \otimes \mathcal{L})^G \to (H^0(X, \mathcal{L}) \otimes \mathfrak{g}^\vee)^G$ is surjective.*

Proposition 2.5.9. *Let $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$ be a linearization with no strictly semi-stable locus. The map $(H^0(X, \Omega_X \otimes \mathcal{L}) \otimes \mathfrak{g}^\vee)^G \to (H^0(X, \mathcal{L}) \otimes S^2\mathfrak{g}^\vee)^G$ is surjective.*

In order to prove these two propositions, we will first investigate invariant global sections. Observe that for a given line bundle $\mathcal{O}_{(\mathbb{P}^1)^s}(l_1, \ldots, l_s)$ on $(\mathbb{P}^1)^s$, global sections can be written as $H^0((\mathbb{P}^1)^s, \mathcal{O}_{(\mathbb{P}^1)^s}(l_1, \ldots, l_s)) = V_{l_1} \otimes \cdots \otimes V_{l_s}$, where V_l is the irreducible $(l+1)$-dimensional representation of $\mathfrak{sl}_2$. We can also identify V_l with the space of degree l polynomials in two variables, $V_l = \langle x^l, x^{l-1}y, \ldots, y^l \rangle$, with the action given by $g \cdot p(x, y) = p(g^{-1} \cdot (x, y))$, for $g \in PGL_2$. In particular, from Gelfand-MacPherson correspondence, the vector space $(V_{l_1} \otimes \cdots \otimes V_{l_s})^{PGL_2}$ can be identified with the elements of multi-degree $(l_1, \ldots, l_s)$ in the homogeneous coordinate ring of the Grassmannian $\Bbbk[p_{ik}]/(p_{ik}p_{rl} - p_{ir}p_{kl} + p_{il}p_{kr})$.

Remark 2.5.10. For $l = 2$, write $V_2 = \langle x_0^2, x_0 y_0, y_0^2 \rangle$. We have $\mathfrak{g} \cong V_2$ as $\mathfrak{g}$-representations, by identifying the bases $\{E, H, F\}$ and $\{y_0^2, 2x_0y_0, -x_0^2\}$. If we further use the isomorphism of $\mathfrak{g}$-representations $\mathfrak{g} \cong \mathfrak{g}^\vee$, we get $\{X_0, Y_0, Z_0\} = \{x_0^2, x_0y_0, y_0^2\}$.

Let us use this identification of $\mathfrak{g}^\vee \cong V_2$. The map $\Omega_X \to \mathfrak{g}^\vee$ is then determined by the n sections $s_i = x_i^2 Z_0 - 2x_iy_iY_0 + y_i^2 X_0 = (x_0y_i - x_iy_0)^2 \in (H^0(X, \mathcal{O}_X(0, \ldots, 2, \ldots, 0)) \otimes \mathfrak{g}^\vee)^G$, by taking $X_0 = x_0^2$, $Y_0 = x_0y_0$, $Z_0 = y_0^2$, where $\{X_0, Y_0, Z_0\}$ is the basis of $\mathfrak{g}^\vee$ dual to $\{-E, H, F\}$, and $H^0(X, \mathcal{O}_X(0, \ldots, 2, \ldots, 0)) = V_2 = \langle x_i^2, x_iy_i, y_i^2 \rangle$.

We see further that the symmetric powers $S^m\mathfrak{g}^\vee$ split canonically as $V_{2m} \oplus S^{m-2}\mathfrak{g}^\vee$ as $\mathfrak{g}$-representations, for $m \geq 2$. Indeed, let $\mathbb{P}^2 = \mathbb{P}(\mathfrak{g})$, so that $S^m\mathfrak{g}^\vee = H^0(\mathbb{P}(\mathfrak{g}), \mathcal{O}_{\mathbb{P}^2}(m))$, and let $C = \mathbb{P}^1$ be the G-invariant conic in $\mathbb{P}^2$ defined by $X_0Z_0 - Y_0^2 = 0$. The curve C is given in coordinates by the rational normal curve embedding $(x^2 : xy : y^2)$. Using the tautological short exact sequence and tensoring with $\mathcal{O}_{\mathbb{P}^2}(m)$, we get

$$0 \to \mathcal{O}_{\mathbb{P}^2}(m-2) \to \mathcal{O}_{\mathbb{P}^2}(m) \to \mathcal{O}_C(2m) \to 0. \tag{2.5.2}$$

Taking global sections we get $0 \to S^{m-2}\mathfrak{g}^\vee \to S^m\mathfrak{g}^\vee \to V_{2m} \to 0$. By semisimplicity of $\mathfrak{sl}_2$, this splits in a unique way. Observe that the map $S^{m-2}\mathfrak{g}^\vee \to S^m\mathfrak{g}^\vee$ is multiplication by $X_0Z_0 - Y_0^2$, while the map $S^m\mathfrak{g}^\vee \to V_{2m}$ sends precisely $\{X_0, Y_0, Z_0\}$ to $\{x_0^2, x_0y_0, y_0^2\}$, where we write $V_{2m} = \langle x_0^{2m}, \ldots, y_0^{2m} \rangle$. Now consider again the complex $F^\cdot$ from (2.3.1), with its differentials $H^0(\Omega_X^{j-m+1} \otimes \mathcal{L}) \otimes S^{m-1}\mathfrak{g}^\vee \to H^0(\Omega_X^{j-m} \otimes \mathcal{L}) \otimes S^m\mathfrak{g}^\vee$. Using the splittings $S^r\mathfrak{g}^\vee = V_{2r} \oplus S^{r-2}\mathfrak{g}^\vee$, compose with the inclusion $V_{2m-2} \hookrightarrow S^{m-1}\mathfrak{g}^\vee$ and the projection $S^m\mathfrak{g}^\vee \to V_{2m}$ to get a map $H^0(\Omega_X^{j-m+1} \otimes \mathcal{L}) \otimes V_{2m-2} \to H^0(\Omega_X^{j-m} \otimes \mathcal{L}) \otimes V_{2m}$. That is, the map making the following diagram commute

$$\begin{array}{ccc} H^0(\Omega_X^{j-m+1} \otimes \mathcal{L}) \otimes S^{m-1}\mathfrak{g}^\vee & \longrightarrow & H^0(\Omega_X^{j-m} \otimes \mathcal{L}) \otimes S^m\mathfrak{g}^\vee \\ \downarrow\circlearrowright & & \downarrow\circlearrowright \\ H^0(\Omega_X^{j-m+1} \otimes \mathcal{L}) \otimes V_{2m-2} & \longrightarrow & H^0(\Omega_X^{j-m} \otimes \mathcal{L}) \otimes V_{2m}. \end{array}$$

This way we get a new complex

$$\bar{F}^\cdot = \left[0 \to H^0(X, \Omega_X^j \otimes \mathcal{L}) \to H^0(X, \Omega_X^{j-1} \otimes \mathcal{L}) \otimes V_2 \to \cdots \to H^0(X, \mathcal{L}) \otimes V_{2j} \to 0\right], \tag{2.5.3}$$

which we can think of as a "partial" version of $F^\cdot$. Observe that, by commutativity of the diagram above, $\bar{F}^\cdot$ is indeed a chain complex.

By the discussion above, the differential maps in $\bar{F}^\cdot$ correspond to multiplication by $s_i = (x_0y_i - x_iy_0)^2$, where x_i, y_i are coordinates in the i-th component, and x_0, y_0 correspond to the terms $V_{2m} = \langle x_0^{2m}, \ldots, y_0^{2m} \rangle$. In what follows next, we will study the complex $\bar{F}^\cdot$, and then we will see that from this we can get back some information about the original complex $F^\cdot$ from (2.3.1).

2.5.2 Computations in $(\mathbb{P}^1)^{n+1}$

Now for $j > 0$, we consider the diagonal action of PGL_2 on $(\mathbb{P}^1)^{n+1} = \mathbb{P}^1 \times X$. Using coordinates $x_0, y_0; x_i, y_i$, take $s_i = (x_0y_i - x_iy_0)^2 \in H^0(\mathbb{P}^1 \times X, \mathcal{O}(2; 0, \ldots, 2, \ldots, 0))^{PGL_2}$ for $i = 1, \ldots, n$. We choose the polarization $\mathcal{V} = \mathcal{O}(2j; d_1, \ldots, d_n) = \mathcal{O}_{\mathbb{P}^1}(2j) \boxtimes \mathcal{L}$.

Proposition 2.5.11. *Suppose $\mathcal{L} = \mathcal{O}_X(d_1, \dots, d_n)$ is a polarization on X with no strictly semi-stable locus, and let $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(2j) \boxtimes \mathcal{L}$ as above. Let M be the scheme-theoretic intersection $\bigcap(s_i = 0) \subset \mathbb{P}^1 \times X$. Then M is a local complete intersection and $H^0(M, \mathcal{V}|_M)$ has no PGL_2-invariants.*

Proof. Write $D_i = (x_0y_i - x_iy_0 = 0)$ so that $M = \bigcap 2D_i$, while $\bigcap D_i$ is the small diagonal $\mathbb{P}^1 \subset (\mathbb{P}^1)^{n+1}$. Then $M = \bigcap 2D_i$ is a local complete intersection, having codimension n. For a reduced divisor $D \subset V$, we have a tautological short exact sequence

$$0 \to \mathcal{O}_D(-D) \to \mathcal{O}_{2D} \to \mathcal{O}_D \to 0. \tag{2.5.4}$$

Claim. For every $0 \le m \le n$ and every $I \subset \{1, \dots, m\}$, the sheaf

$$\mathcal{O}_{\bigcap_{i \le m} D_i \cap \bigcap_{i>m} 2D_i}(-\sum_{i \in I} D_i) \otimes \mathcal{V} \tag{2.5.5}$$

has no PGL_2-invariant global sections.

Given the claim, the proposition is proved by taking $m = 0$ in (2.5.5). To prove the claim we use (2.5.4) on $\bigcap_{i \le m+1} D_i \cap \bigcap_{i>m+1} 2D_i \subset \bigcap_{i \le m} D_i \cap \bigcap_{i>m+1} 2D_i$, to get

$$0 \to \mathcal{O}_{\bigcap_{i \le m+1} D_i \cap \bigcap_{i>m+1} 2D_i}(-D_{m+1}) \to \mathcal{O}_{\bigcap_{i \le m} D_i \cap \bigcap_{i>m} 2D_i} \to \mathcal{O}_{\bigcap_{i \le m+1} D_i \cap \bigcap_{i>m+1} 2D_i} \to 0.$$

Now tensor with $\mathcal{V}(-\sum_{i \in I} D_i)$ and take PGL_2-invariant global sections. The claim will then be proved if we show $\mathcal{O}_{\bigcap_{i \le m+1} D_i \cap \bigcap_{i>m+1} 2D_i}(-\sum_{i \in I'} D_i)$ has no invariant global sections for every $I' \subset \{1, \dots, m+1\}$. That is, the claim is true for m if it is true for $m+1$. Therefore, we can do induction on $n - m$, so that all we need to show is that

$$H^0(\mathcal{O}_{\bigcap_{i \le n} D_i}(-\sum_{i \in I} D_i) \otimes \mathcal{V})^G = 0$$

for any $I \subset \{1, \dots, n\}$.

Recall $\bigcap_{i \le n} D_i = \mathbb{P}^1$ is the small diagonal, and $\mathcal{O}_{(\mathbb{P}^1)^{n+1}}(-D_i) = \mathcal{O}(-1; 0, \dots, -1, \dots, 0)$, so that $\mathcal{O}_{\bigcap_{i \le n} D_i}(-\sum_{i \in I} D_i) \otimes \mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(2j + \sum_{i \le n} d_i - 2|I|)$. The PGL_2-invariant global sections of this sheaf are homogeneous polynomials in x and y of degree $2j + \sum_{i \le n} d_i - 2|I|$ that are restrictions to the small diagonal of polynomials in $p_{ik} = x_iy_k - x_ky_i$. Of course, any such polynomial will restrict to 0 in the diagonal, unless it has degree 0. But $2j + \sum_{i \le n} d_i - 2|I|$ cannot be zero. This follows from the following claim.

Claim. $\sum_{i=1}^n d_i \ge 2n$. In particular $2j + \sum_{i=1}^n d_i - 2|I| > 0$ for every $I \subset \{1, \dots, n\}$.

Let us prove this claim. Without loss of generality, we may assume $d_1 \le \ldots \le d_n$. Choose $0 \le m \le n$ such that $d_1 = \ldots = d_m = 1$, $d_{m+2} \ge 2$ and m has the same parity as n. Observe that, since $\sum d_i$ is even and $\mathcal{L}$ has no strictly semi-stable locus, as a consequence of Remark 2.5.4 we must have

$$d_n + d_{n-2} + \ldots + d_{m+2} > d_{n-1} + \ldots + d_{m+3} + d_{m+1} + m. \tag{2.5.6}$$

In fact, if $d_n + d_{n-2} + \ldots + d_{m+2} - (d_{n-1} + \ldots + d_{m+3} + d_{m+1}) = r \leq m$, we would have $d_n + d_{n-2} + \ldots + d_{m+2} = (d_{n-1} + \ldots + d_{m+3} + d_{m+1}) + (d_1 + \ldots + d_r)$, and then writing each of the remaining $d_{r+1} = d_{r+2} = \ldots = d_m = 1$ at either side of this equation we would get $\sum_{i \notin I} d_i = \sum_{i \in I} d_i$, where $I = \{1, \ldots, r\} \cup \{r+1, r+3, \ldots\} \cup \{m+1, m+3, \ldots, n-1\}$, a contradiction. In particular, from (2.5.6) we have $d_n > d_{m+1} + m$. Then $\sum d_i = m + d_{m+1} + \sum_{i=m+2}^{n-1} d_i + d_n > 2(m + d_{m+1}) + \sum_{i=m+2}^{n-1} d_i$. Since $d_{m+1} \geq 1$ and $d_{n-1} \geq \ldots \geq d_{m+2} \geq 2$, this is at least $2m + 2 + 2(n - m - 2) = 2n - 2$. Thus $\sum_{i=1}^{n} d_i > 2n - 2$, so in fact $\sum_{i=1}^{n} d_i \geq 2n$. This completes the proof. $\square$

Corollary 2.5.12. *With the same hypotheses, $\mathcal{H}^i(\bar{F}^{\cdot})^G = 0$ for $i > 1$.*

Proof. Since $M = \bigcap (s_i = 0)$ is a local complete intersection, the augmented Koszul complex determined by $s_1, \ldots, s_n$,

$$0 \to \mathcal{O}(-\sum_{i=1}^{n} 2D_i) \to \cdots \to \bigoplus \mathcal{O}(-2D_i) \to \mathcal{O} \to \mathcal{O}_M \to 0$$

is acyclic, and so is the complex

$$K^{\cdot} = \left[0 \to \mathcal{V}(-\sum 2D_i) \to \cdots \to \bigoplus \mathcal{V}(-2D_i) \to \mathcal{V} \to \mathcal{V} \otimes \mathcal{O}_M \to 0\right]. \tag{2.5.7}$$

We consider this complex having nonzero terms in degrees $-n$ to 1. This means that for $-n \leq p \leq 0$, the term K^p is precisely

$$\bigoplus_{|I|=-p} \mathcal{V}(-\sum_{i \in I} 2D_i) = \mathcal{O}_{\mathbb{P}^1}(2j + 2p) \boxtimes (\Omega_X^{-p} \otimes \mathcal{L}).$$

Take the spectral sequence $E_1^{pq} = H^q((\mathbb{P}^1)^{n+1}, K^p)$, which converges to $H^{p+q}((\mathbb{P}^1)^{n+1}, K^{\cdot}) = 0$. We get

$$H^q((\mathbb{P}^1)^{n+1}, K^p) = \begin{cases} H^0(X, \Omega_X^{-p} \otimes \mathcal{L}) \otimes V_{2j+2p} & \text{if} \quad q = 0,\ -j \leq p \leq 0 \\ H^0(X, \Omega_X^{-p} \otimes \mathcal{L}) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2j + 2p)) & \text{if} \quad q = 1,\ p < -j \\ H^q(M, \mathcal{V}|_M) & \text{if} \quad p = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.5.8}$$

and the sequence has the following shape

$$\cdots \longrightarrow E_1^{-j-2,1} \longrightarrow E_1^{-j-1,1} \longrightarrow 0 \longrightarrow \cdots \qquad \cdots \longrightarrow 0 \longrightarrow H^1(M, \mathcal{V}|_M)$$

$$\cdots \longrightarrow 0 \longrightarrow E_1^{-j,0} \longrightarrow E_1^{-j+1,0} \longrightarrow \cdots \longrightarrow E_1^{0,0} \longrightarrow H^0(M, \mathcal{V}|_M).$$

The complex $\bar{F}^\cdot$ from (2.5.3) is the same as the (shifted) naive truncation of $E_1^{\cdot,0}[-j]$ obtained by omitting the last term $H^0(M, \mathcal{V}|_M)$ of $E_1^{\cdot,0}$, since the differentials are determined precisely by the sections s_i.

We see that for $q = 0$ and $p > -j+1$, the sequence degenerates at E_2 and we get $0 = H^{i-j}(\mathbb{P}^1 \times X, K^\cdot) = \mathcal{H}^i(\bar{F}^\cdot)$, for $1 < i < j$ (even before taking invariants). Further, since $H^0(M, \mathcal{V}|_M)^G = 0$ by the previous proposition, the complex of G-invariants $(E_1^{\cdot,0}[-j])^G$ is precisely $(\bar{F}^\cdot)^G$, so $\mathcal{H}^j((\bar{F}^\cdot)^G) = 0$ too, that is, $\mathcal{H}^j(\bar{F}^\cdot)^G = 0$. $\square$

2.5.3 Directed graphs as invariant sections

Given a G-linearized ample line bundle $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$ on X, let us use the identifications $H^0(X, \mathcal{L}) = V_{d_1} \otimes \cdots \otimes V_{d_n}$, and $H^0(X, \Omega_X \otimes \mathcal{L}) = \bigoplus_{i=1}^n V_{d_1} \otimes \cdots \otimes V_{d_i-2} \otimes \cdots \otimes V_{d_n}$. We also use $\mathfrak{g}^\vee = V_2$ and $S^2\mathfrak{g}^\vee = V_0 \oplus V_4$ as $\mathfrak{g}$-representations. Then to show Propositions 2.5.8 and 2.5.9, we need to investigate the maps

$$\bigoplus_{i=1}^n (V_{d_1} \otimes \cdots \otimes V_{d_i-2} \otimes \cdots \otimes V_{d_n})^G \xrightarrow{t_1} (V_2 \otimes V_{d_1} \otimes \cdots \otimes V_{d_n})^G \tag{2.5.9}$$

and

$$\bigoplus_{i=1}^n (V_2 \otimes V_{d_1} \otimes \cdots \otimes V_{d_i-2} \otimes \cdots \otimes V_{d_n})^G \xrightarrow{t_2} (V_4 \otimes V_{d_1} \otimes \cdots \otimes V_{d_n})^G \oplus (V_0 \otimes V_{d_1} \otimes \cdots \otimes V_{d_n})^G \tag{2.5.10}$$

and show that both are surjective. In view of Gelfand-MacPherson correspondence, we will work with these invariants using the language of graphs (as in [HMSV05] and [HMSV09]).

Notation 2.5.13. Let J be a directed graph with vertices $V(J)$ and edges $E(J)$. Let $w \in V(J)$ be a vertex. By $\deg(w)$ we mean the number of edges touching w. We say that two vertices w and v are adjacent if there is an edge between them. An edge going from w to v will be denoted by $w \to v$.

A directed graph J can be represented by a $2 \times m$ tableau, where $m = |E(J)|$. A diagram

$$\begin{bmatrix} a_1 & \ldots & a_m \\ b_1 & \ldots & b_m \end{bmatrix}$$

represents the graph with edges $a_i \to b_i$.

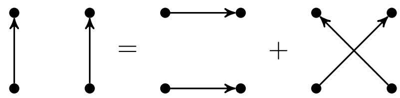
Definition 2.5.14. Let $l = (l_1, \ldots, l_r) \in \mathbb{Z}^r_{\geq 0}$, with $\sum l_i$ even. We call $\mathfrak{F}_l$ the free vector space generated by directed graphs J having r vertices, say $V(J) = \{w_1, \ldots, w_r\}$, with degrees $\deg(w_i) = l_i$. We denote by $\mathfrak{F}'_l$ the quotient of $\mathfrak{F}_l$ by the following relations:

(a) If K is obtained from J by reversing the direction of one edge, then $K = -J$. In particular, any graph having a self-loop is equal to zero in $\mathfrak{F}'_l$.

(b) The relation $J = H + K$, whenever H and K are obtained by replacing a 2×2 submatrix as follows:

$$\begin{bmatrix} \cdots & a & \cdots & b & \cdots \\ \cdots & c & \cdots & d & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & a & \cdots & c & \cdots \\ \cdots & b & \cdots & d & \cdots \end{bmatrix} + \begin{bmatrix} \cdots & a & \cdots & b & \cdots \\ \cdots & d & \cdots & c & \cdots \end{bmatrix}$$

We observe that the space $\mathfrak{F}'_l$ is exactly identified with the ring of invariants $(V_{l_1} \otimes \cdots \otimes V_{l_r})^{PGL_2}$. A Plücker minor $p_{ik} = x_i y_k - x_k y_i$ corresponds to an edge $w_i \to w_k$, and the relations defining $\mathfrak{F}'_l$ are precisely $p_{ik} = -p_{ki}$ and the Plücker relations. Plücker relation is drawn as follows:



If $\sum l_i$ is odd or if one $l_i < 0$, we just set $\mathfrak{F}'_l = 0$. Then for fixed r, we can put all the spaces $\mathfrak{F}'_l$ together in a $\mathbb{Z}^r$-graded ring $\mathfrak{F}' = \bigoplus_{l \in \mathbb{Z}^r} \mathfrak{F}'_l$. This is the same construction as the ring R defined in the proof of Lemma 2.5.6. In this language, the product of two graphs J_1 and J_2 consists of a graph having edges $E(J_1 J_2) = E(J_1) \cup E(J_2)$. An element $J \in \mathfrak{F}'_l$ is a graph if it is written as a product of Plücker minors p_{ik}. In general, an element of $\mathfrak{F}'_l$ is a polynomial in p_{ik}, this is, a linear combination of graphs.

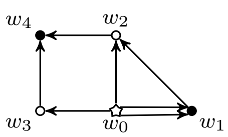
We will be mostly interested in the spaces $\mathfrak{F}'_l$ when $l = (2m, d_1, \ldots, d_n)$. For the graphs in $\mathfrak{F}'_l$, we label the $n+1$ vertices as $w_0, w_1, \ldots, w_n$, so that $\deg w_0 = 2m$, $\deg w_i = d_i$ for $i \geq 1$. We call $V(J)_0$ the set of vertices adjacent to w_0, and for w_i we call $e(w_0, w_i)$ the number of edges between w_0 and w_i.

Definition 2.5.15. Let $l = (2m, d_1, \ldots, d_n)$ and let J be a directed graph in $\mathfrak{F}_l$, as above. A 2-coloring of J is an assignment $c : V(J) - \{w_0\} \to \{0, 1\}$ such that $c(a) \neq c(b)$ for every two adjacent vertices a and b, and also $\sum_{w_i \in c^{-1}(0)} e(w_0, w_i) = \sum_{w_i \in c^{-1}(1)} e(w_0, w_i) = m$.

Example 2.5.16. The graph given by

$$\begin{bmatrix} w_0 & w_0 & w_0 & w_0 & w_1 & w_2 & w_3 \\ w_1 & w_1 & w_2 & w_3 & w_2 & w_4 & w_4 \end{bmatrix}$$

admits a 2-coloring:



If $m = 1$, we can think of a 2-coloring as a bipartition of the graph obtained by deleting w_0 and replacing the edges coming from it by an edge joining the two vertices $w_{i_1}, w_{i_2} \in V(J)_0$. In this bipartition w_{i_1} and w_{i_2} must be in different blocks. In particular, if a graph $J \in \mathfrak{F}'_{(2,d_1,\ldots,d_n)}$ has a double edge coming from w_0, this is, if $w_{i_1} = w_{i_2}$, then J cannot admit a 2-coloring. For coloring purposes, the directions of the edges are irrelevant.

Remark 2.5.17. Suppose $\mathcal{L} = \mathcal{O}_X(d_1, \ldots, d_n)$ is such that $X^{ss} = X^s$. Then no graph $J \in \mathfrak{F}_{(2m,d_1,\ldots,d_n)}$ admits a 2-coloring. Indeed, if J had a 2-coloring, then we can call $I = \{i \mid c(w_i) = 0\} \subset \{1, \ldots, n\}$, so that $\sum_{i \in I} d_i = \sum_{i \notin I} d_i$.

Lemma 2.5.18. *The image of the map t_1 from (2.5.9) consists of the vector subspace generated by graphs having a double edge coming from w_0.*

Proof. By the explicit description of $\bar{F}^{\cdot}$ in (2.5.3), we know the maps $V_{d_1} \otimes \cdots \otimes V_{d_i - 2} \otimes \cdots \otimes V_{d_n} \to V_2 \otimes V_{d_1} \otimes \cdots \otimes V_{d_n}$ are given by multiplication by $s_i = (x_0 y_i - x_i y_0)^2$. Taking invariants we get maps $\mathfrak{F}'_{(d_1,\ldots,d_i-2,\ldots,d_n)} \to \mathfrak{F}'_{(2,d_1,\ldots,d_n)}$. We identify $\mathfrak{F}'_{(d_1,\ldots,d_i-2,\ldots,d_n)} = \mathfrak{F}'_{(0,d_1,\ldots,d_i-2,\ldots,d_n)}$ by adding an extra vertex w_0 with $\deg w_0 = 0$. Then multiplication of a graph J by s_i corresponds to adding two extra edges to J, both going from w_0 to w_i. $\square$

Notation 2.5.19. Let $l = (2m, d_1, \ldots, d_n)$. A cycle is a sequence of vertices $w_{i_1}, \ldots, w_{i_r}$ such that each w_{i_k} is adjacent to $w_{i_{k+1}}$, and w_{i_r} is adjacent to w_{i_1}. We say that the cycle is central if it involves the vertex w_0. We call r the length of the cycle. A subgraph C determined by the cycle $w_{i_1}, \ldots, w_{i_r}$ will be denoted by $(w_{i_1}, \ldots, w_{i_m})$ if the signs of the edges are given by

$$C = \begin{bmatrix} w_{i_1} & w_{i_2} & \cdots & w_{i_{r-1}} & w_{i_r} \\ w_{i_2} & w_{i_3} & \cdots & w_{i_r} & w_{i_1} \end{bmatrix}.$$

For a cycle we do not require that all the vertices w_{i_k} be different. We observe that rotating the indices $i_1, \ldots, i_r$ does not change the cycle, while reversing an arrow switches the sign.

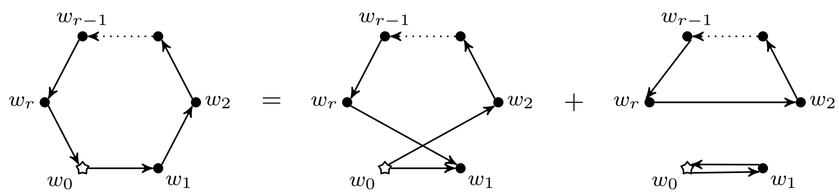
Remark 2.5.20. Let $l = (0, d_1, \ldots, d_n)$ and J a graph in $\mathfrak{F}'_l$. It is a well-known fact that J admits a 2-coloring if and only if it does not contain a cycle of odd length. This fact is sometimes referred to as Kőnig's Theorem.

Lemma 2.5.21. *Suppose $J \in \mathfrak{F}'_{(2,d_1,\ldots,d_n)}$ is a graph having a central cycle of even length. Then J is in the image of the map t_1 from (2.5.9).*

Proof. We can assume $w_0, \ldots, w_r$ is a cycle in J, where r is odd. Let J_0 be the subgraph given by the cycle, $J_0 = (w_0, \ldots, w_r)$, so that J is a multiple of J_0, say $J = J_0 H$. It suffices to show that J_0 can be written as a linear combination of graphs having a double edge from w_0. Consider the Plücker relation

$$\begin{bmatrix} w_r & w_1 \\ w_0 & w_2 \end{bmatrix} = \begin{bmatrix} w_r & w_0 \\ w_1 & w_2 \end{bmatrix} + \begin{bmatrix} w_r & w_1 \\ w_2 & w_0 \end{bmatrix}$$

or



so that we have $J_0 = J_1 + J_1'$, where J_1' has a double edge between w_0 and w_1. Then $J \equiv J_1 H$ mod $\operatorname{im}(t_1)$. On the other hand, J_1 is equivalent to the cycle $-(w_1, w_0, w_2, \ldots, w_r)$. Similarly, given a cycle $J_i = (-1)^i (w_1, \ldots, w_i, w_0, w_{i+1}, \ldots, w_r)$, we use the Plücker relation on

$$\begin{bmatrix} w_i & w_{i+1} \\ w_0 & w_{i+2} \end{bmatrix}$$

to obtain $J \equiv (-1)^{i+1} J_{i+1} H \mod \operatorname{im}(t_1)$. We conclude $J \equiv (-1)^r J_r H \equiv -J \mod \operatorname{im}(t_1)$, since r is odd and $J_r = (w_1, \ldots, w_r, w_0) = J_0$. Then $J \in \operatorname{im}(t_1)$, as desired. $\square$

Now we have the tools to show Proposition 2.5.8.

Proof of Proposition 2.5.8. Let $K^\cdot$ be the complex (2.5.7), where $\mathcal{V}$ is the sheaf $\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{L} = \mathcal{O}_{(\mathbb{P}^1)^{n+1}}(2; d_1, \ldots, d_n)$, and consider the spectral sequence $E_1^{pq} = H^q((\mathbb{P}^1)^{n+1}, K^p)$ from (2.5.8):

$$\cdots \longrightarrow E_1^{-3,0} \longrightarrow E_1^{-2,0} \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow E_1^{-1,0} \longrightarrow E_1^{0,0} \longrightarrow H^0(M, \mathcal{V}|_M).$$

The restriction of $d_1^{-1,0}$ to invariant sections is t_1. We need to show it is surjective. The second page of the spectral sequence has the following shape:

$$\begin{tikzcd}
\cdots & E_2^{-3,0} \arrow[drr, "d_2"] & E_2^{-2,0} \arrow[drr, "d_2"] & 0 & \cdots & \\
& \cdots & 0 & E_2^{-1,0} & E_2^{0,0} & H^0(M, \mathcal{V}|_M)
\end{tikzcd}$$

We want to describe the restriction of the map $d_2^{-2,0}$ to invariant sections, that is $(d_2^{-2,0})^G : (E_2^{-2,0})^G \to (E_2^{0,0})^G$. Observe $(E_2^{0,0})^G = (E_1^{0,0})^G / \operatorname{im}(t_1)$ since $H^0(M, \mathcal{V}|_M)^G = 0$. The whole sequence degenerates at E_3, so $d_2^{-2,0}$ must be an isomorphism, in particular surjective. Therefore, any $J \in (E_1^{0,0})^G$ can be written as a sum $J' + J''$, where $J' \in \operatorname{im}(t_1)$ and $J'' \in \operatorname{im}((d_2^{-2,0})^G)$.

The map $d_2^{-2,0}$ is obtained by doing a bi-complex resoultion of $K^{-2} \to K^{-1} \to K^0$ that computes cohomologies of K^p, and then chasing the diagram. Since, for each q and p, $H^q(\mathbb{P}^1 \times X, \mathcal{O}_{\mathbb{P}^1}(2+2p) \boxtimes (\Omega_X^{-p} \otimes \mathcal{L})) = H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2+2p)) \otimes H^0(X, \Omega^{-p} \otimes \mathcal{L})$, it suffices to use resolutions of $\mathcal{O}_{\mathbb{P}^1}(2+2p)$ and then tensor with $H^0(X, \Omega_X^{-p} \otimes L)$, for $p = -2, -1, 0$. We use the usual Čech resolution, given by $S_{x_0} \times S_{y_0} \to S_{x_0 y_0}$, where $S = \Bbbk[x_0, y_0]$ and that, when restricted to rational functions of a given degree l, it computes the cohomologies of $\mathcal{O}_{\mathbb{P}^1}(l)$ (see e.g. [Har77, §III.5.1]). We have

$$\begin{tikzcd}
(S_{x_0y_0})_{-2} \otimes H^0(X, \Omega_X^2 \otimes \mathcal{L}) \arrow[r, "f"] & (S_{x_0y_0})_0 \otimes H^0(X, \Omega_X \otimes \mathcal{L}) \arrow[r, "f"] & (S_{x_0y_0})_2 \otimes H^0(X, \mathcal{L}) \\
(S_{x_0} \times S_{y_0})_{-2} \otimes H^0(X, \Omega_X^2 \otimes \mathcal{L}) \arrow[r, "f"] \arrow[u, "h"] & (S_{x_0} \times S_{y_0})_0 \otimes H^0(X, \Omega_X \otimes \mathcal{L}) \arrow[r, "f"] \arrow[u, "h"] & (S_{x_0} \times S_{y_0})_2 \otimes H^0(X, \mathcal{L}). \arrow[u, "h"]
\end{tikzcd}$$

Write $H^0(X, \Omega_X^2 \otimes \mathcal{L}) = \bigoplus_{l>k} V_{d_1} \otimes \cdots \otimes V_{d_k-2} \otimes \cdots \otimes V_{d_l-2} \otimes \cdots \otimes V_{d_n}$ and $H^0(X, \Omega_X \otimes \mathcal{L}) = \bigoplus_i V_{d_1} \otimes \cdots \otimes V_{d_i-2} \otimes \cdots \otimes V_n$, so that the map $K^{-2} \to K^{-1}$ is $\sum f^{kl}$, where each f^{kl} is multiplication by s_l onto the k-th component and multiplication by $-s_k$ onto the l-th component. The map $t : K^{-1} \to K^0$ is $\sum t^i$ where t^i is multiplication by s_i. Recall $s_i = p_{0i}^2 = (x_0 y_i - x_i y_0)^2$.

Let $u \in (S_{x_0y_0})_{-2} \otimes (V_{d_1} \otimes \cdots \otimes V_{d_k-2} \otimes \cdots \otimes V_{d_l-2} \otimes \cdots \otimes V_{d_n})$. We have $f^{kl}(u) = (\ldots, s_k u, \ldots, -s_l u, \ldots)$, with zeros in the remaining coordinates. Write

$$u = \frac{P}{x_0^m} + \frac{Q}{y_0^m} + \frac{R}{x_0 y_0}$$

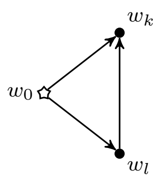
for some polynomials P, Q, R, whose homogeneous degrees with respect to x_0, y_0 are $m-2$, $m-2$, 0, respectively. Then $s_k u = h(v)$ for some $v \in (S_{x_0} \times S_{y_0})_0 \otimes (V_{d_1} \otimes \cdots \otimes V_{d_l-2} \otimes \cdots \otimes V_n)$, and we can choose

$$v = \left(Q\frac{s_k}{x_0^m} + R\frac{x_k^2 y_0}{x_0} - R x_k y_k, -P\frac{s_k}{y_0^m} - R\frac{x_0 y_k^2}{y_0} + R x_k y_k \right).$$

Similarly, we find v' such that $-s_l u = h(v')$. To find $d_2^{-2,0}(u)$ we then need to compute $f(v) + f(v') = s_l v + s_k v'$. We get $s_l v + s_k v' = (b, b)$, where

$$b = R\left(\frac{y_0}{x_0}(s_l x_k^2 - s_k x_l^2) + s_k x_l y_l - s_l x_k y_k \right).$$

Simplifying we get $b = R(x_0 y_l - x_l y_0)(x_0 y_k - x_k y_0)(x_l y_k - x_k y_l) = p_{0l} p_{0k} p_{lk} R$. Now $H^0(\mathbb{P}^1 \times X, \mathcal{V})$ is identified with the diagonal of $(S_{x_0} \times S_{y_0})_2 \otimes H^0(X, \mathcal{L})$ so, if we call $\bar{v} \in E_2^{-2,0}$ the class represented by v, we have $d_2^{-2,0}(\bar{v}) = p_{0l} p_{0k} p_{lk} R$, a multiple of $p_{0l} p_{0k} p_{lk}$. If $\bar{v}$ was invariant, then $d_2^{-2,0}(\bar{v})$ is a linear combination of graphs having $p_{0l} p_{0k} p_{lk}$ as a subgraph, this is, graphs that have a central cycle w_0, w_l, w_k of length three:

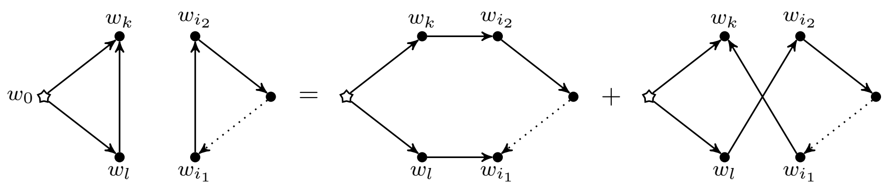


Therefore, modulo $\mathrm{im}(t_1)$, every $J \in \mathfrak{F}'_{(2,d_1,\ldots,d_n)}$ is a linear combination of graphs of the form $p_{0l} p_{0k} p_{lk} R$. Then it suffices to show that all such graphs are in the image of t_1. Given

$J = p_{0l}p_{0k}p_{lk}R$, call $J' \in \mathfrak{F}'_{(0,d_1,\ldots,d_n)}$ the graph obtained by replacing the two edges $w_0 \to w_k$, $w_0 \to w_l$ by an extra $w_l \to w_k$, this is, $J' = p_{lk}^2 R$. Observe that a 2-coloring of J' would need to have $c(w_k) \neq c(w_l)$, so it would give a 2-coloring on J. Since $\mathcal{L}$ has no strictly semi-stable locus, J and J' do not admit a 2-coloring and by Remark 2.5.20 J' must contain some odd cycle, say $(w_{i_1}, \ldots, w_{i_r})$. Note that any two vertices that are adjacent on J' are adjacent on J too, so in fact $(w_{i_1}, \ldots, w_{i_r})$ is an odd cycle in J, that is not central. Apply the Plücker relation

$$\begin{bmatrix} w_l & w_{i_1} \\ w_k & w_{i_2} \end{bmatrix} = \begin{bmatrix} w_l & w_k \\ w_{i_1} & w_{i_2} \end{bmatrix} + \begin{bmatrix} w_l & w_{i_1} \\ w_{i_2} & w_k \end{bmatrix}$$

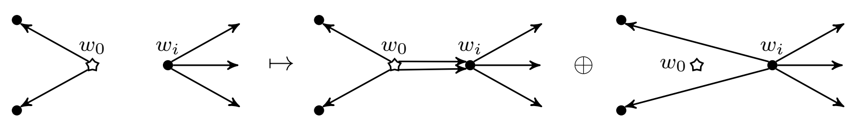
or



to get $J = H + K$, where H contains the cycle $w_0, w_k, w_{i_2}, \ldots, w_{i_r}, w_{i_1}, w_l$ and K contains the cycle $w_0, w_l, w_{i_2}, \ldots, w_{i_r}, w_{i_1}, w_k$, both of even length $r + 3$. By Lemma 2.5.21, $J \in \operatorname{im}(t_1)$. We conclude t_1 is surjective. $\square$

Next, we investigate the map t_2 from (2.5.10). According to the splitting $S^2\mathfrak{g}^\vee = V_4 \oplus V_0$ we write $t_2 = (t, t')$, and further $t = \sum t^i$, $t' = \sum t'^i$, where $t^i : \mathfrak{F}'_{(2,d_1,\ldots,d_i-2,\ldots,d_n)} \to \mathfrak{F}'_{(4,d_1,\ldots,d_n)}$ and $t'^i : \mathfrak{F}'_{(2,d_1,\ldots,d_i-2,\ldots,d_n)} \to \mathfrak{F}'_{(0,d_1,\ldots,d_n)}$. Let us describe these maps in terms of graphs with vertices $\{w_0, w_1, \ldots, w_n\}$.

Lemma 2.5.22. *Let J be a graph in $\mathfrak{F}'_{(2,d_1,\ldots,d_n)}$. Write J as a polynomial, $J = p_{0k}p_{0l}H$. Then $t^i(J) = p_{0i}^2 J$, while $t'^i(J) = \frac{2}{3}p_{ik}p_{il}H$. This is, t^i adds a double edge $w_0 \to w_i$ to the graph while, up to a constant, t'^i replaces the edges $w_0 \to w_k$, $w_0 \to w_l$ by $w_i \to w_k$ and $w_i \to w_l$:*



Proof. By the explicit description of $\bar{F}^{\cdot}$ from (2.5.3), we know t^i is multiplication by $(x_0y_i - x_iy_0)^2 = p_{0i}^2$, which corresponds to adding two edges, both from w_0 to w_i.

Consider the splitting $S^2\mathfrak{g}^\vee = V_4 \oplus V_0$ obtained from (2.5.2). Here V_0 is the one-dimensional vector space with the trivial action. The projection $\pi : S^2\mathfrak{g} \to V_0$ is the unique $\mathfrak{g}$-equivariant map that satisfies $\pi \circ \imath = \mathrm{Id}_{V_0}$, where $\imath : V_0 \hookrightarrow S^2\mathfrak{g}^\vee$ is the inclusion from (2.5.2), namely, $\imath$ is multiplication by $X_0Z_0 - Y_0^2$. We find π explicitly, and it is defined as follows: for $P = \alpha Y_0^2 + \beta X_0 Z_0 + \ldots \in S^2\mathfrak{g}^\vee$, $\pi(P) = \frac{1}{3}(2\beta - \alpha)$. It is easy to check that π is indeed a $\mathfrak{g}$-equivariant map: observe, for instance, that $E \cdot (X_0Y_0) = F \cdot (Y_0Z_0) = -X_0Z_0 - 2Y_0^2$, and

$\pi(-X_0Z_0 - 2Y_0^2) = 0$. Indeed, this together with the fact that all monomials other than X_0Z_0 and Y_0^2 map to zero ensures that $\pi(g \cdot P) = 0\ \forall g \in \mathfrak{g}$, $P \in S^2\mathfrak{g}^\vee$, so π is a map of representations. Further, we see that $\pi(X_0Z_0 - Y_0^2) = 1$, and then $\pi \circ \imath = \mathrm{Id}_{V_0}$. By uniqueness, π must be the desired map.

Now look at $t' = \sum t'^i$. Each t'^i is given by multiplication by $x_i^2 Z_0 - 2x_iy_iY_0 + y_i^2 X_0$ followed by the projection π from $S^2\mathfrak{g}^\vee$. Suppose $J \in (V_2 \otimes V_{d_1} \otimes \cdots \otimes V_{d_i-2} \otimes \cdots \otimes V_{d_n})^{PGL_2} = \mathfrak{F}'_{(2,d_1,\ldots,d_i-2,\ldots,d_n)}$ is a directed graph, written as $J = (Ax_0^2 + Bx_0y_0 + Cy_0^2)H = (AX_0 + BY_0 + CZ_0)H$ for some polynomials A, B, C and H. Multiplying by $x_i^2Z_0 - 2x_iy_iY_0 + y_i^2X_0$ and looking at the terms involving X_0Z_0 and Y_0^2, we find $t'^i(J) = \frac{2}{3}(Ax_i^2 + Bx_iy_i + Cy_i^2)H$. Now, since J is actually a PGL_2-invariant section, it has to be of the form $J = (x_0y_k - x_ky_0)(x_0y_l - x_ly_0)H$. That is, J is a graph where the two edges coming from w_0 are $w_0 \to w_k$ and $w_0 \to w_l$ (up to sign). Then we have $A = y_ky_l$, $B = -(y_kx_l + y_lx_k)$, $C = x_kx_l$ and we compute

$$t'^i(J) = \frac{2}{3}(x_iy_l - x_ly_i)(x_iy_k - x_ky_i)H.$$

That is, up to multiplication by $2/3$, the map t'^i precisely erases the edges $w_0 \to w_k$, $w_0 \to w_l$, and replaces them by $w_i \to w_k$, $w_i \to w_l$. $\square$

Since multiplying everything in $\mathfrak{F}'_{(0,d_1,\ldots,d_n)}$ by a constant does not change the image of the map t_2, from now on we just ignore the constant $2/3$ appearing in t'. Now we can prove Proposition 2.5.9.

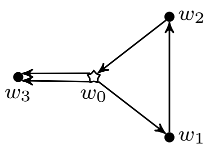
Proof of Proposition 2.5.9. Write $t_2 = (t, t')$, according to the decomposition in (2.5.10). By Corollary 2.5.12, t is surjective. Then it suffices to show that, for any graph $H \in \mathfrak{F}'_{(0,d_1,\ldots,d_n)} = \mathfrak{F}'_{(d_1,\ldots,d_n)}$, we have $(0, H) \in \mathrm{im}(t_2)$.

Step 1. Let J be a graph in $\mathfrak{F}'_{(4,d_1,\ldots,d_n)}$ having a subgraph B of the form

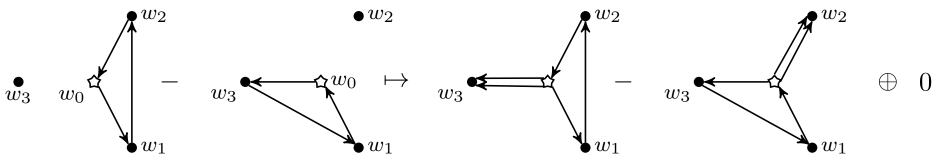
$$B_{1,2,3} = \begin{bmatrix} w_0 & w_1 & w_2 & w_0 & w_0 \\ w_1 & w_2 & w_0 & w_3 & w_3 \end{bmatrix}.$$

That is, $B_{1,2,3}$ has a cycle (w_0, w_1, w_2) and a double edge between w_0 and w_3 (and similarly, B_{i_1,i_2,i_3} denotes a permutation of indices in the expression above).

$B_{1,2,3} \quad =$



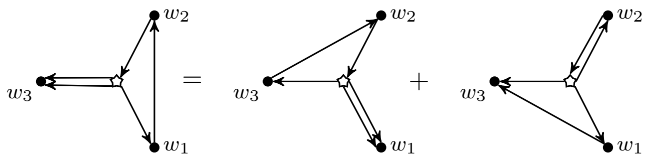
Then we show $(J, 0) \in \mathrm{im}(t_2)$, or in other words, $J \in t_2(\ker t')$. For this, write $J = B_{1,2,3}H$ and let $P = (w_0, w_1, w_2) \in \mathfrak{F}'_{(2,2,2,0,\ldots,0)}$ and $P' = (w_0, w_3, w_1) \in \mathfrak{F}'_{(2,2,0,2,\ldots,0)}$. We see that $t_2(PH - P'H) = (B_{1,2,3}H - B_{3,1,2}H, 0)$, this is, $B_{1,2,3}H \equiv B_{3,1,2}H \mod t_2(\ker t')$.



On the other hand, take $B_{1,2,3}$ and apply the Plücker relation to the edges

$$\begin{bmatrix} w_0 & w_1 \\ w_3 & w_2 \end{bmatrix}$$

to obtain $B_{1,2,3} = B_{3,2,1} + B_{1,3,2}$.

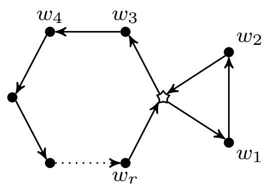


Also, we know $B_{1,2,3} = -B_{2,1,3}$ by reversing the arrows. Combining all these, we get that $B_{1,2,3}H \equiv B_{3,2,1}H + B_{1,3,2}H \equiv 2B_{3,2,1}H \equiv -2B_{2,3,1}H \equiv -2B_{1,2,3}H \mod t_2(\ker t')$. Thus we obtain $3B_{1,2,3}H \in t_2(\ker t')$, so $(J,0) \in \mathrm{im}(t_2)$.

Step 2. Let J be a graph in $\mathfrak{F}'_{(4,d_1,\dots,d_n)}$ having a subgraph C of the form

$$C_{1,\dots,r} = \begin{bmatrix} w_0 & w_1 & w_2 & w_0 & w_3 & \cdots & w_r \\ w_1 & w_2 & w_0 & w_3 & w_4 & \cdots & w_0 \end{bmatrix}$$

for r odd. That is, $C_{1,\dots,r}$ has cycles (w_0, w_1, w_2) and $(w_0, w_3, \dots, w_r)$.

$C_{1,\dots,r} =$ 

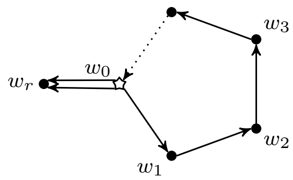
Then we see $(J,0) \in \mathrm{im}(t_2)$. Indeed, by Lemma 2.5.21, the even cycle $(w_0, w_3, \dots, w_r)$ can be written as a sum of graphs having double edges coming from w_0. Using this, $C_{1,\dots,r}$ is written as a sum of graphs containing subgraphs of the form B from Step 1. By Step 1, we get $J \in t_2(\ker t')$.

Step 3. Let J be a graph in $\mathfrak{F}'_{(4,d_1,\dots,d_n)}$ having a subgraph B of the form

$$B_{1,\dots,r} = \begin{bmatrix} w_0 & w_1 & \cdots & w_{r-1} & w_0 & w_0 \\ w_1 & w_2 & \cdots & w_0 & w_r & w_r \end{bmatrix} \tag{2.5.11}$$

with r odd. That is, $B_{1,\dots,r}$ has an odd cycle $(w_0, \dots, w_{r-1})$ and a double edge between w_0 and w_r.

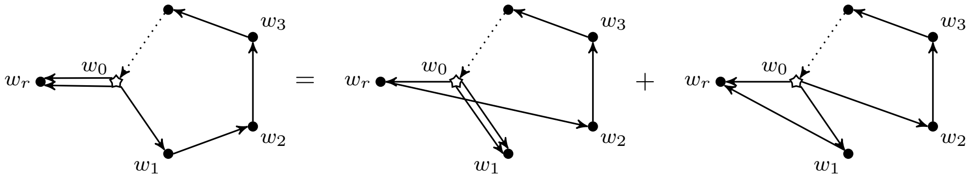
$B_{1,\dots,r}$ =



We show $(J,0) \in \operatorname{im}(t_2)$. If $r = 3$, this is Step 1. Now suppose this is true for $r-2$. Write $J = C_{1,\dots,r}H$ and do the Plücker relation to the edges

$$\begin{bmatrix} w_0 & w_1 \\ w_r & w_2 \end{bmatrix}$$

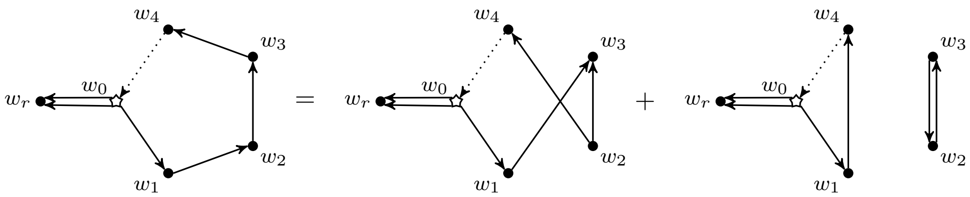
to obtain $B_{1,\dots,r} = B_{r,2,\dots,r-1,1} + C$, where C is a graph of the form given in Step 2.



Therefore, $B_{1,\dots,r}H \equiv B_{r,2,\dots,r-1,1}H \mod t_2(\ker t')$. On the other hand, if we use Plücker on

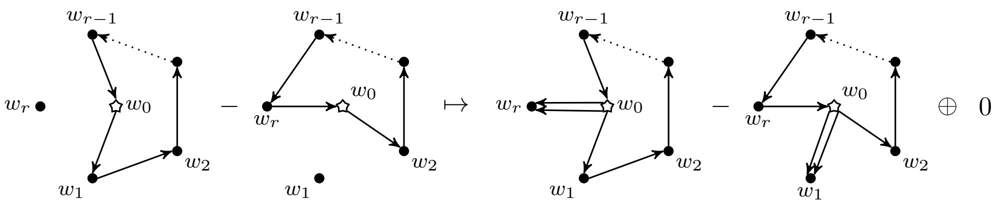
$$\begin{bmatrix} w_1 & w_3 \\ w_2 & w_4 \end{bmatrix}$$

we get $B_{1,\dots,r} = -B_{1,3,2,4,\dots,r} + B'$, where B' is a graph containing $B_{1,4,5,\dots,r}$ as a subgraph.



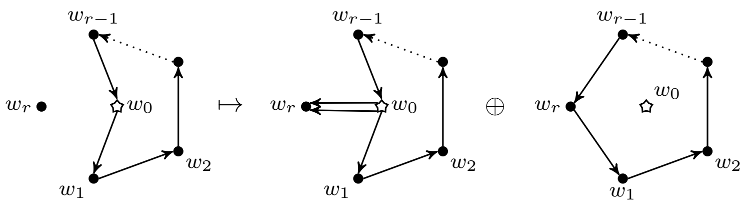
By induction hypothesis, $B_{1,\dots,r}H \equiv -B_{1,3,2,4,\dots,r}H \mod t_2(\ker(t'))$.

Now, using the same argument as in Step 1, let $P = (w_0, \dots, w_{r-1})$, $P' = (w_0, w_2, \dots, w_r)$, and we see that $t_2(PH - P'H) = (B_{1,\dots,r}H - B_{2,\dots,r,1}H, 0)$, so that $B_{1,\dots,r}H \equiv B_{2,\dots,r,1}H \mod t_2(\ker t')$.



We combine all the equivalences above to obtain $B_{1,\dots,r}H \equiv B_{2,1,3,\dots,r}H \equiv -B_{1,2,3,\dots,r}H \mod t_2(\ker t')$, and then $(J,0) \in \operatorname{im}(t_2)$.

Step 4. Now let $H \in \mathfrak{F}'_{(0,d_1,\dots,d_n)}$ be any graph. Then $(J, H) \in \mathrm{im}(t_2)$ for some J containing a subgraph B of the form (2.5.11) from Step 3. Indeed, since H does not admit a 2-coloring, by Remark 2.5.20 it must contain an odd cycle, say $C = (w_1, \dots, w_r)$ is a subgraph of H, $H = CP$ for some P. But then $(B_{1,\dots,r}P, H) = t_2(C'P)$, where C' is the cycle $(w_0, w_1, \dots, w_{r-1})$.



The graph $J = B_{1,\dots,r}P$ is in $t_2(\ker t')$ by Step 3. Finally, since both (J, H) and $(J, 0) \in \mathrm{im}(t_2)$, we obtain $(0, H) \in \mathrm{im}(t_2)$, so this concludes the proof. □

2.5.4 Main result

Now we can prove the main result.

Proof of Theorem 2.1.1. If X^{us} has codimension 1, then we are done by Lemma 2.5.5 and the fact that every smooth projetive toric variety satisfies Bott vanishing (see the references given in §1 or Theorem 2.4.6). Otherwise, by Lemma 2.5.6 it suffices to show vanishing for $\Omega^j_Y \otimes L$, where L is the descent of the polarization $\mathcal{L}$. If $j = 0$, then $H^i(Y, L) = H^i(X, \mathcal{L})^G$ which is certainly 0 for $i > 0$. Assume $j \geq 1$.

From Corollary 2.2.2, $H^i(Y, \Omega^j_Y \otimes \mathcal{L}) = H^i(\mathfrak{X}, \Lambda^j L_{\mathfrak{X}} \otimes \mathcal{L})$. This is zero for $i \neq 0, j$ by Corollary 2.5.3. By Lemma 2.3.1, we need to show $\mathcal{H}^j(F^\cdot)^G = 0$, where $F^\cdot$ is given by (2.3.1). That is, we need to show that the map

$$(H^0(X, \Omega_X \otimes \mathcal{L}) \otimes S^{j-1}\mathfrak{g}^\vee)^G \xrightarrow{t_j} (H^0(X, \mathcal{L}) \otimes S^j \mathfrak{g}^\vee)^G$$

is surjective for every j. Propositions 2.5.8 and 2.5.9 show this is true for $j = 1$ and $j = 2$. Now we do induction on j. Let $j \geq 3$. Consider the short exact sequence from (2.5.2), giving rise to the splitting $S^m\mathfrak{g}^\vee = V_{2m} \oplus S^{m-2}\mathfrak{g}^\vee$ for $m \geq 2$. We use (2.5.2) for $m = j$ and $m = j - 1$. Take its pullback to $X \times \mathbb{P}(\mathfrak{g})$ and tensor with the pullbacks of $\Omega_X \otimes \mathcal{L}$ and $\mathcal{L}$, respectively. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & (\Omega_X \otimes \mathcal{L}) \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j-3) & \longrightarrow & (\Omega_X \otimes \mathcal{L}) \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j-1) & \longrightarrow & (\Omega_X \otimes \mathcal{L}) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2j-2) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j-2) & \longrightarrow & \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(j) & \longrightarrow & \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(2j) & \longrightarrow 0.
\end{array}$$

The vertical maps are given by the section $s \in H^0(X \times \mathbb{P}(\mathfrak{g}), T_X \boxtimes \mathfrak{g}^\vee)$ defining the map $\Omega_X \to \mathfrak{g}^\vee$, as usual. Taking global sections we see that the map $H^0(X, \Omega_X \otimes \mathcal{L}) \otimes S^{j-1}\mathfrak{g}^\vee \xrightarrow{d_j}$

$H^0(X,\mathcal{L})\otimes S^j\mathfrak{g}^\vee$ splits as

$$\begin{tikzcd}
H^0(X,\Omega_X\otimes\mathcal{L})\otimes V_{2(j-1)} \arrow[d, "t"] \arrow[drr, "t'"] & \oplus & H^0(X,\Omega_X\otimes\mathcal{L})\otimes S^{j-3}\mathfrak{g}^\vee \arrow[d, "t_{j-2}"] \\
H^0(X,\mathcal{L})\otimes V_{2j} & \oplus & H^0(X,\mathcal{L})\otimes S^{j-2}\mathfrak{g}^\vee.
\end{tikzcd}$$

By induction hypothesis, the restriction t^G_{j-2} of t_{j-2} to invariant sections is surjective, while the restriction t^G is surjective by Corollary 2.5.12. As a consequence, $t^G_j = (t^G, t'^G + t^G_{j-2})$ is surjective. This completes the proof. □

2.6 A note on Fano varieties

As mentioned in the beginning of the present Chapter, (non-toric) Fano varieties satisfying Bott vanishing are particularly interesting. We have $T_Y = \Omega_Y^{n-1}\otimes K_Y^\vee$, so if $K_Y^\vee$ is ample and Y satisfies Bott vanishing, then $H^1(Y,T_Y)$ must be zero, and Y must be rigid. In particular, Bott vanishing holds for at most finitely many smooth complex Fano varieties in each dimension.

If $Y = (\mathbb{P}^1)^n /\!\!/ PGL_2$ is as in Theorem 2.1.1 and it is non-toric, then $\operatorname{Pic} Y$ is the G-ample cone of $(\mathbb{P}^1)^n$ (see the proof of Lemma 2.5.6) and K_Y is the descent of $K_X = \mathcal{O}_X(-2,\dots,-2)$. If Y is Fano, then by Lemma 2.5.6 it has to be the quotient $(\mathbb{P}^1)^n /\!\!/_{\mathcal{O}_X(2,\dots,2)} PGL_2$. Observe that $\mathcal{O}_X(2,\dots,2)$ has no strictly semi-stable locus if and only if n is odd. In other words, Theorem 2.1.1 provides us with exactly one non-toric example of a Fano variety satisfying Bott vanishing in each even dimension. In the case of dimension 2, this was the quintic del Pezzo surface.

An interesting non-example comes from a Fano threefold that contains the quintic del Pezzo surface as a hyperplane section. Let M be the Fano threefold over $\mathbb{C}$ of index 2 and degree 5, with Picard number 1. The canonical line bundle is $K_M = \mathcal{O}_M(-2)$, where $\mathcal{O}_M(1)$ is the ample generator of the Picard group. M is a rigid Fano threefold, isomorphic to a linear section of the Grassmannian $\mathrm{Gr}(2,5)\subset\mathbb{P}^9$ by a subspace $\mathbb{P}^6\subset\mathbb{P}^9$. The quintic del Pezzo surface V can be realized as a divisor in the linear system $|\mathcal{O}_M(1)|$. It can be computed that the Hodge numbers of M are $h^{0,0}(M) = h^{1,1}(M) = 1$ and zero otherwise, in particular $h^{1,2}(M) = 0$. The description of M can be found in [KPS18, §5.1] or [Muk88, §4].

This variety M does not satisfy Bott vanishing. Indeed, we claim that $H^1(M,\Omega_M^2(1))$ has dimension at least 3. Observe that by Serre duality, this is the same as saying that $H^2(M,\Omega_M(-1))$ has dimension ≥ 3. To show this, we follow a strategy similar to [JR07, Lemma 1.2], using the dualized tangent sequence

$$0\to\mathcal{O}_V(-1)\to\Omega_M|_V\to\Omega_V\to 0 \tag{2.6.1}$$

and the ideal sequence tensored with Ω_M

$$0 \to \Omega_M(-1) \to \Omega_M \to \Omega_M|_V \to 0. \tag{2.6.2}$$

By the Kodaira-Akizuki-Nakano vanishing theorem [Laz04, Theorem 4.2.3], we know that $H^1(M, \Omega_M(-1)) = 0$. Using the fact that $h^{0,0}(M) = h^{1,1}(M) = 1$, $h^{1,2}(M) = 0$ and sequence (2.6.2), we get an exact sequence

$$0 \to H^1(M, \Omega_M) \to H^1(V, \Omega_M|_V) \to H^2(M, \Omega_M(-1)) \to 0,$$

so $h^2(\Omega_M(-1)) = h^1(\Omega_M|_V) - 1$. It suffices to check that $h^1(\Omega_M|_V) \geq 4$.

Now take sequence (2.6.1) and observe $\mathcal{O}_V(-1) = K_V$ by adjunction. Since $h^{2,1}(V) = h^{1,2}(V) = 0$, we get

$$0 \to H^1(V, \Omega_M|_V) \to H^1(V, \Omega_V) \to H^2(V, K_V) \to H^2(V, \Omega_M|_V) \to 0.$$

From the Hodge numbers of V, $h^{1,1}(V) = 10 - 5 = 5$, $h^{2,2}(V) = 1$, we see

$$h^2(\Omega_M|_V) + 5 = h^1(\Omega_M|_V) + 1.$$

In particular, $h^1(\Omega_M|_V) \geq 4$, proving the claim and the fact that M does not satisfy Bott vanishing.

2.7 Further open questions

It is worth asking whether similar techniques can be applied to find vanishing results in other spaces. The fact that we can recover a new proof for the toric case seems to be especially encouraging.

A question that arises immediately is what can be said about GIT quotients of $X = (\mathbb{P}^m)^n$ by the action of PGL_{m+1}. The main hurdle would occur when dealing with chain complexes of invariant sections of the form $H^0(X, \mathcal{O}_X(d_1, \ldots, d_n))^{PGL_{m+1}}$. Gelfand-MacPherson correspondence is still valid, and these sections occur in the coordinate ring of a Grassmannian $\mathrm{Gr}(m+1, n)$, but it is unclear a priori how to manage them, since here we cannot use the description of $\mathfrak{sl}_2$-representations. In particular, we do not have the description of these sections as graphs. It would be interesting to see what can be said in this and other similar cases.

Another interesting question is the following: If $f : X \to Y$ is a birational morphism of smooth projective varieties and X satisfies Bott vanishing, is it true that Y satisfies Bott vanishing? In [HK99], Hu and Keel show that any such morphism can be realized as a variation of GIT. It would be good to know if the methods used can be applied to study this question.

The theory of windows [HL15, BFK19] does not only allow to compute cohomology spaces in GIT quotients via quantization, but it provides a description of the whole derived category of $X /\!\!/ G$, and its relationships with the derived category of the quotient stack $[X/G]$. As we have seen an application to toric varieties, it is worth asking whether something can be said about the derived categories of other Mori Dream Spaces. As we know by Hu and Keel's work [HK00], a Mori Dream Space is the GIT quotient of the spectrum of its Cox ring. Halpern-Leistner's results can be applied to the GIT quotient of a singular variety X, provided that X satisfies a technical condition. This condition is stated in terms of a Kempf-Ness stratification of the unstable locus. Namely, given a one-parameter subgroup λ, its fixed locus Z and the closed immersion $\sigma : Z \hookrightarrow S$ into the corresponding Kempf-Ness stratum S, it is required that the restriction of the relative cotangent complex $\sigma^* L^{\cdot}_{S/X}$ have non-negative weights with respect to λ (see [HL15, §2.1] for details). It would be relevant to determine which Mori Dream Spaces have a Cox ring that satisfies this hypothesis, and then apply the theory of windows to study those spaces.

CHAPTER 3

MODULI OF BUNDLES ON A CURVE

3.1 Moduli of slope-stable rank-two bundles on a curve.

A vector bundle V over a smooth projective curve C is said to be slope-semistable (or just semistable) is for every nontrivial vector sub-bundle $W \subset V$ one has

$$\frac{\deg W}{\operatorname{rk} W} \leq \frac{\deg V}{\operatorname{rk} V}.$$

The bundle is stable if the inequality is strict. For a given line bundle Λ, there is a moduli space $M_C(2, \Lambda)$ parametrizing semi-stable vector bundles on rank 2 on C [Ses67]. If $\deg \Lambda$ is odd, this is a smooth projective variety and it carries a universal family $\mathcal{E}$, also called a Poincaré bundle [Ram73, Definition 2.10], and its Picard group is isomorphic to $\mathbb{Z}$ [DN89].

Remark 3.1.1. The space $M_C(2, \Lambda)$ only depends on the parity of $\deg \Lambda$. Indeed, twisting V by a line bundle Λ' produces a vector bundle V' with determinant $\Lambda \otimes (\Lambda')^2$. We will be interested in the case that Λ is of odd degree.

Fix a line bundle Λ of degree one. If C has genus $g \geq 2$, it is known that there is a fully faithful functor from the derived category of C to that of $M_C(2, \Lambda)$. This embedding is achieved by the Fourier-Mukai transform $\Phi_{\mathcal{E}}$ associated to the Poincaré bundle $\mathcal{E}$ on $C \times M_C(2, \Lambda)$, normalized so that $\Theta = c_1(\mathcal{E}|_{x \times M})$ is an ample generator of $\operatorname{Pic} M_C(2, \Lambda)$. This result was proved by Fonarev and Kuznetsov [FK18] in the case of a generic curve, and by Narasimhan [Nar17, Nar18] in the general case. Moreover, the blocks given by $\Theta^{-1}, \mathcal{O}_{M_C(2,\Lambda)}, \Phi_{\mathcal{E}}(D^b(C))$ constitute the start of a semi-orthogonal decomposition. Narasimhan conjectured that $D^b(M_C(2, \Lambda))$ has a semi-orthogonal decomposition consisting of blocks of the form

$$D^b(\mathrm{pt}),\ D^b(\mathrm{pt}),\ D^b(C),\ D^b(C), \ldots,\ D^b(C^{(g-2)}),\ D^b(C^{(g-2)}),\ D^b(C^{(g-1)}) \tag{3.1.1}$$

where $C^{(i)}$ denotes the i-th symmetric power of C. In [Lee18], Lee proves a decomposition of the motive of $M_C(2, \Lambda)$ that is compatible with this conjecture. In [BM19], Belmans and Mukhopadhyay find four terms that are the start of a semi-orthogonal decomposition of $M_C(2, \Lambda)$ for a curve of genus $g \gg 0$.

3.2 Thaddeus' spaces of bundles on a curve.

Let C be a smooth projective curve of genus at least 2 over $\mathbb{C}$. In [Tha94], Thaddeus introduces moduli spaces $M_\sigma(\Lambda)$ that parametrize pairs (E,ϕ), where E is a rank-two vector bundle with determinant Λ, and $\phi \in H^0(E)$ is a section, satisfying the following stability condition: for every line subbundle $L \subset E$, one must have

$$\begin{aligned} \deg L \leq & \frac{1}{2}\deg E - \sigma & & \text{if } \phi \in H^0(L), \\ \deg L \leq & \frac{1}{2}\deg E + \sigma & & \text{if } \phi \notin H^0(L). \end{aligned}$$

It can be shown that for a given line bundle Λ of degree d and $\sigma \in (0, d/2]$ the moduli space $M_\sigma(\Lambda)$ exists as a projective variety and, in the case there is no strictly semi-stable locus, it is smooth and it carries a universal bundle F with a universal section $\tilde{\phi} : C \times M_\sigma(\Lambda) \to F$.

The moduli spaces $M_\sigma(\Lambda)$ can be obtained as GIT quotients as follows. If $d \gg 0$, a bundle E in a stable pair is generated by global sections, and we call $\chi = H^0(E) = d+2-2g$. Then $M_\sigma(\Lambda)$ is a GIT quotient of $U \times \mathbb{P}\mathbb{C}^\chi$ by SL_χ, where $U \subset \mathrm{Quot}$ is the locally closed subscheme of the Grothendieck Quot scheme [Gro95] corresponding to locally free quotients $\mathcal{O}_C^\chi \twoheadrightarrow E$ inducing an isomorphism $s : \mathbb{C}^\chi \xrightarrow{\sim} H^0(E)$ and such that $\Lambda^2 E = \Lambda$. Such an isomorphism $s : \mathbb{C}^\chi \xrightarrow{\sim} H^0(E)$ induces a map $\Lambda^2\mathbb{C}^\chi \to H^0(\Lambda)$, and we get an inclusion $U \times \mathbb{P}\mathbb{C}^\chi \hookrightarrow \mathbb{P}\operatorname{Hom}(\Lambda^2\mathbb{C}^\chi, H^0(\Lambda)) \times \mathbb{P}\mathbb{C}^\chi$, where a quotient $s : \mathcal{O}_C^\chi \twoheadrightarrow E$ on the left is sent to the induced map in the first coordinate. Then $M_\sigma(\Lambda)$ can be seen as the GIT quotient of a closed subset of $\mathbb{P}\operatorname{Hom} \times \mathbb{P}\mathbb{C}^\chi$ by SL_χ, where the linearization is given by $\mathcal{O}(\chi + 2\sigma, 4\sigma)$. Here we write $\mathbb{P}\operatorname{Hom}$ for $\mathbb{P}\operatorname{Hom}(\Lambda^2\mathbb{C}^\chi, H^0(\Lambda))$.

For arbitrary d, we pick any effective divisor D on C with $\deg D \gg 0$, and $M_\sigma(\Lambda)$ can be seen as the closed subset of $M_\sigma(\Lambda(2D))$ consisting of pairs (E,ϕ) such that $\phi|_D = 0$. This way, $M_\sigma(\Lambda)$ a GIT quotient by SL_χ, with $\chi' = d + 2 - 2g + 2\deg D$, now of the closed subset of $U' \times \mathbb{P}\mathbb{C}^{\chi'}$ determined by the condition that ϕ vanishes along D. [Tha94, §1].

Remark 3.2.1. Scalar matrices in SL_χ act trivially on $U \times \mathbb{P}\mathbb{C}^\chi$, so the action factors through the quotient $SL_\chi \to PGL_\chi$. If we replace $\mathcal{O}(\chi + 2\sigma, 4\sigma)$ by its χ-th power, this bundle carries a PGL_χ-linearization and $M_\sigma(\Lambda)$ can also be written as a GIT quotient $X /\!\!/ PGL_\chi$. For the quotient stacks we will always use PGL_χ instead of SL_χ, that is, $\mathfrak{X}$ will denote $[X/PGL_\chi]$.

Thaddeus also describes how $M_\sigma(\Lambda)$ varies with σ. For fixed Λ, these spaces are all GIT quotients of the same scheme, with different stability conditions. The GIT walls occur when $\sigma \in d/2 + \mathbb{Z}$, and for $0 \leq i \leq w = \lfloor (d-1)/2 \rfloor$ we have different GIT chambers with moduli spaces $M_0, M_1, \ldots, M_w$, where $M_i = M_\sigma(\Lambda)$ for $\sigma \in (\max(0, d/2 - i - 1), d/2 - i)$. These M_i are smooth projective rational varieties of dimension $d + g - 2$. Indeed, $M_0 = \mathbb{P}H^1(\Lambda^{-1})$ is a projective space, M_1 is a blow-up of M_0 along a copy of C embedded by the complete linear system of $\omega_C \otimes \Lambda$, and the remaining ones are small modifications of M_1.

More precisely, for each $0 \leq i \leq w = \lfloor (d-1)/2 \rfloor$ there are projective bundles $\mathbb{P}W_i^+$ and $\mathbb{P}W_i^-$ over the symmetric product $C^{(i)}$, of (projective) ranks $d+g-2i-2$, $i-1$, respectively, with embeddings $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$, and such that $\mathbb{P}W_i^+$ parametrizes the pairs (E, ϕ) appearing in M_i but not in M_{i-1}, while $\mathbb{P}W_i^-$ parametrizes those appearing in M_{i-1} but not in M_i. We have a diagram

$$\begin{tikzcd}[column sep=small]
& \tilde{M}_2 \arrow[dl] \arrow[dr] & & \tilde{M}_3 \arrow[dl] \arrow[dr] & & \tilde{M}_w \arrow[dl] \arrow[dr] & \\
M_1 \arrow[d] & & M_2 & & \cdots & & M_w \arrow[d] \\
M_0 & & & & & & N
\end{tikzcd} \tag{3.2.1}$$

where $\tilde{M}_i$ is both the blow-up of M_{i-1} along $\mathbb{P}W_i^-$ and the blow-up of M_i along $\mathbb{P}W_i^+$, N is the moduli space of ordinary slope-semi-stable vector bundles and the map $M_w \to N$ is an "Abel-Jacobi" map with fiber $\mathbb{P}H^0(E)$ over a vector bundle E. If $d \geq 2g-1$ this last map is surjective, and if $d = 2g-1$ then is a birational morphism (see [Tha94, §3] for details).

The Picard group of $M_1 = \mathrm{Bl}_C M_0$ is generated by a hyperplane section H in $M_0 = \mathbb{P}^{d+g-2}$ and the exceptional divisor E_1. Since the maps $M_i \dashrightarrow M_{i+1}$ are small birational modifications for each $i \geq 1$, there are natural isomorphisms $\operatorname{Pic} M_1 \cong \operatorname{Pic} M_i$, $i \geq 1$.

Notation 3.2.2. For each m, n, $\mathcal{O}_1(m,n)$ will denote the line bundle $\mathcal{O}_{M_1}((m+n)H - nE_1)$, and $\mathcal{O}_i(m,n)$ will denote the image of $\mathcal{O}_1(m,n)$ under the isomorphism $\operatorname{Pic} M_1 \cong \operatorname{Pic} M_i$.

Suppose $d \gg 0$. Then for $\sigma \notin d/2 + \mathbb{Z}$ the universal bundle F on M_i descends from $\mathcal{F}(1)$ on $U \times \mathbb{P}\mathbb{C}^\chi \times C$, where $\mathcal{O}^\chi \twoheadrightarrow \mathcal{F}$ is the universal quotient over $U \times C$, and the universal section $\tilde{\phi}$ comes from the universal section of $\mathcal{F}(1)$. Let $\pi : C \times M_i \to M_i$ be the projection. For every $i \geq 1$, the line bundle $\det \pi_! F$ descends from $\mathcal{O}(0, \chi)$ on $\mathbb{P}\operatorname{Hom} \times \mathbb{P}\mathbb{C}^\chi$. On M_1, $\det \pi_! F$ corresponds to $\mathcal{O}_{M_1}((g-d-1)H + (d-g)E_1)$. For $x \in C$, call $F_x = F|_{\{x\} \times M}$. The line bundle $\det F_x = \Lambda^2 F_x$ does not depend on x, and it is the descent of $\mathcal{O}(1,2)$. On M_1, it corresponds to $\mathcal{O}_{M_1}(E_1 - H)$ [Tha94, §5].

Definition 3.2.3. If V is a vector bundle over $Y \times T$ and $\pi : Y \times T \to T$ is the projection, the determinant of cohomology of V is defined as the line bundle $\det \pi_! V$. It can be shown that this definition extends to a morphism from the K-group, $K(Y \times T) \to \operatorname{Pic} Y$, so that if $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of sheaves, then $\det \pi_! V \cong \det \pi_! V' \otimes \det \pi_! V''$ [KM76].

Notation 3.2.4. We will denote $\zeta^{-1} := \det \pi_! F = \mathcal{O}_i(-1, g-d)$, and $\Lambda_M := \Lambda^2 F_x = \mathcal{O}_i(0,-1)$. Also, call $\theta := \zeta^2 \otimes \Lambda_M^\chi = \mathcal{O}_i(2, d-2)$, where $\chi = d + 2 - 2g$ (cf. [Nar17, Proposition 2.1]).

3.2.1 Semi-orthogonal blocks on M_1

As before, let $E_1 \subset M_1$ be the exceptional locus of the blow-up $M_1 \to M_0$ along $C \subset M_0$. By Orlov's blow-up formula (Theorem 1.2.15), for any integer k we have a fully faithful functor $\Phi_k : D^b(C) \hookrightarrow D^b(M_1)$, corresponding to the Fourier-Mukai transform given by $\mathcal{O}_Z(kE_1)$, where $Z = C \times_C E_1$. Observe this is supported precisely on the zero locus of the universal section $\tilde{\phi} : \mathcal{O}_{C\times M_1} \to F$, and it is a local complete intersection. Indeed, pairs (E, ϕ) in $\mathbb{P}W_1^+ = E_1$ consist of extensions

$$0 \to \mathcal{O}_C(x) \to E \to \Lambda(-x) \to 0$$

with the canonical section $\phi \in H^0(C, \mathcal{O}_C(x))$ vanishing on $x \in C$ [Tha94, §3.2], and in fact $\tilde{\phi}$ cannot have zeros outside this locus, since $M_1 \backslash E_1$ consists of extensions $0 \to \mathcal{O}_C \to E \to \Lambda \to 0$ together with a (constant) section $\phi \in H^0(C, \mathcal{O}_C)$ [Tha94, §3.1]. Therefore we can write a Koszul resolution

$$\left[\Lambda^2 F^\vee \to F^\vee \xrightarrow{\tilde{\phi}} \mathcal{O}_{C\times M_1}\right] \xrightarrow{\sim} \mathcal{O}_Z. \tag{3.2.2}$$

Now consider a different functor, determined by the universal bundle F on $C \times M_1$ and the corresponding Fourier-Mukai transform $\Phi_F = p_*(q^*(\cdot) \otimes F) : D^b(C) \to D^b(M_1)$.

Proposition 3.2.5. *The functor Φ_F is fully-faithful.*

We need a few lemmas first.

Lemma 3.2.6. *Suppose $0 < k \leq d+g-2$ and $0 \leq l \leq d+g-4$. Then $H^i(M_1, \mathcal{O}_1(-kH+lE_1)) = 0 \;\forall i \geq 0$.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_{M_1} \to \mathcal{O}_{M_1}(E_1) \to \mathcal{O}_\pi(-1) \to 0, \tag{3.2.3}$$

where $E_1 = \mathbb{P}W_1^+$ and $\pi : E_1 \to C$ is the $\mathbb{P}^r$-bundle, $r = d + g - 4$. $\mathcal{O}_{M_1}(-kH)$ is Γ-acyclic provided $0 < k \leq d + g - 2 = \dim M_1$. Then twisting (3.2.3) by $\mathcal{O}_{M_1}(-kH)$ and taking a long exact sequence in cohomology gives Γ-acyclicity of $\mathcal{O}_{M_1}(-kH + E_1)$ for such k. Similarly, twisting by powers of $\mathcal{O}_{M_1}(E_1)$ and using induction, we get that $R\Gamma(\mathcal{O}_{M_1}(-kH + lE)) = 0$ as well, since $\mathcal{O}_\pi(-l)$ is Γ-acyclic for $0 < l \leq d + g - 4$. □

Lemma 3.2.7. $R\Gamma(\Lambda_M^{-1}) = 0$.

Proof. Recall $\Lambda_M^{-1} = \mathcal{O}_{M_1}(H - E_1)$. Since the embedding $C \xhookrightarrow{|\omega_C \otimes \Lambda|} M_0 = \mathbb{P}^{3g-3}$ is given by a complete linear system [Tha94, §3.4], the image of C is not contained in any hyperplane and thus $H^0(M_1, \mathcal{O}_{M_1}(H - E_1)) = 0$. Now use the exact sequence

$$0 \to \mathcal{O}_{M_1}(H - E_1) \to \mathcal{O}_{M_1}(H) \to \mathcal{O}_{E_1}(H) \to 0. \tag{3.2.4}$$

Observe that $H^i(M_1, \mathcal{O}_{E_1}(H)) = H^i(C, \omega_C \otimes \Lambda)$, because $C \hookrightarrow M_0$ is given by $|\omega_C \otimes \Lambda|$. Since $\deg \omega_C \otimes \Lambda > \deg \omega_C$, we have $H^1(C, \omega_C \otimes \Lambda) = 0$. On the other hand, $H^0(M_1, \mathcal{O}_{M_1}(H)) = \mathbb{C}^{d+g-1}$ and $H^{>0}(M_1, \mathcal{O}_{M_1}(H)) = 0$. Taking a long exact sequence in cohomology from (3.2.4) we get

$$0 \to H^0(M_1, \mathcal{O}_{M_1}(H)) \to H^0(C, \omega_C \otimes \Lambda) \to H^1(M_1, \Lambda_M^{-1}) \to 0 \tag{3.2.5}$$

and $H^i(M_1, \Lambda_M^{-1}) = 0$ for $i \neq 1$. Now by Riemann-Roch $H^0(C, \omega_C \otimes \Lambda) = \mathbb{C}^{d+g-1}$, so the first map in (3.2.5) is an isomorphism and $H^1(M_1, \Lambda_M^{-1}) = 0$, proving the lemma. □

Lemma 3.2.8. *Let $x \in C$. Then $R\Gamma(F_x^\vee) = 0$, while $R\Gamma(F_x) = \mathbb{C}$, with $H^0(M_1, F_x) = \mathbb{C}$ given by restriction of the universal section $\tilde{\phi} : \mathcal{O}_{C \times M_1} \to F$ to $x \times M_1$.*

Proof. Consider the resolution (3.2.2) and restrict to $x \times M_1$ to get

$$\left[\Lambda_M^{-1} \to F_x^\vee \to \mathcal{O}_{M_1}\right] \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^r_x} \tag{3.2.6}$$

where $\mathbb{P}^r_x$ is the fiber over $x \in C \subset M_0$ along the blow-up $\pi : M_1 \to M_0$. We twist by $\Lambda_M = \mathcal{O}_{M_1}(E_1 - H)$ to get

$$\left[\mathcal{O}_{M_1} \xrightarrow{\tilde{\phi}} F_x \to \Lambda_M\right] \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^r_x}(-1), \tag{3.2.7}$$

using the facts that $F_x^\vee \otimes \Lambda_M = F_x$ and that $\mathcal{O}_{M_1}(H)$ restricts trivially to the fiber $\mathcal{O}_{\mathbb{P}^r_x}$. Since the right hand side is Γ-acyclic we see that, applying $R\Gamma$ or, equivalently, taking a long exact sequence in cohomology, yields $H^0(M_1, F_x) = \mathbb{C}$, $H^{>0}(M_1, F_x) = 0$. In other words, $R\Gamma(F_x) = \mathbb{C}$. Further, the isomorphism $H^0(M_1, \mathcal{O}_{M_1}) \xrightarrow{\tilde{\phi}} H^0(M_1, F_x)$ from (3.2.7) is provided precisely by the universal section.

To show that $R\Gamma(F_x^\vee) = 0$, we apply $R\Gamma$ to (3.2.6). We know that $R\Gamma(\mathcal{O}_{M_1}) = R\Gamma(\mathcal{O}_{\mathbb{P}^r}) = \mathbb{C}$, while $R\Gamma(\Lambda_M^{-1}) = 0$ by Lemma 3.2.7. Then the claim will be proved if we show that $H^0(M_1, F_x^\vee) = 0$, as this would also imply that $H^1(M_1, F_x^\vee) = 0$. Any global section $s \in H^0(M_1, F_x^\vee)$, composed with $F_x^\vee \to \mathcal{O}_{M_1}$ gives a constant section $\mathcal{O}_{M_1} \to \mathcal{O}_{M_1}$ vanishing along the locus Z, hence identically 0. But then by exactness of $0 \to \Lambda_M^{-1} \to F_x^\vee \to \mathcal{O}_{M_1}$, the section $s : \mathcal{O}_{M_1} \to F_x^\vee$ must lift to a section $\mathcal{O}_{M_1} \to \Lambda_M^{-1}$. But Λ_M^{-1} has no global sections, again by Lemma 3.2.7. □

Proof of Proposition 3.2.5. By Bondal-Orlov's criterion (Theorem 1.2.14), in order to show fully faithfulness of Φ_F we only need to consider the sheaves $\Phi_F(\mathcal{O}_x) = F_x$ for closed points $x \in C$.

On the other hand, consider the functor Φ_1 from Theorem 1.2.15. Observe that the Fourier-Mukai kernel of Φ_1 is $\mathcal{O}_Z(E_1)$, so we can compute $\Phi_1(\mathcal{O}_x) = \Phi_{\mathcal{O}_Z(E_1)}(\mathcal{O}_x)$ for a point $x \in C$ by restricting (3.2.2) to $x \times M_1$ and twisting by $\mathcal{O}_{M_1}(E_1)$. As before, let $\mathbb{P}^r_x$ denote the fiber over $x \in C \subset M_0$ along the blow-up. The fact that $\mathcal{O}_{M_1}(H)$ restricts trivially to this fiber implies that both Λ_M and $\mathcal{O}_{M_1}(E_1)$ restrict to $\mathcal{O}_{\mathbb{P}^r_x}(-1)$ there and we get

$$\begin{aligned}\Phi_{\mathcal{O}_Z(E_1)}(\mathcal{O}_x) &\cong [\mathcal{O}_{M_1} \to F_x \to \Lambda_M] \\ &\cong \mathcal{O}_{\mathbb{P}^r_x}(-1),\end{aligned}$$

as in (3.2.7).

Since we already know that Φ_1 is fully faithful, we have

$$\mathrm{Hom}_{D^b(M_1)}(\Phi_1(\mathcal{O}_x), \Phi_1(\mathcal{O}_y)[i]) = \begin{cases} 0 & \text{if } x \neq y \\ 0 & \text{if } x = y \text{ and } k \neq 0, 1 \\ \mathbb{C} & \text{if } x = y \text{ and } k = 0, 1. \end{cases} \tag{3.2.8}$$

But $R\,\mathrm{Hom}_{D^b(M_1)}(\Phi_1(\mathcal{O}_x), \Phi_1(\mathcal{O}_y)) \cong R\Gamma \circ R\,\mathscr{H}\!om(\Phi_1(\mathcal{O}_x), \Phi_1(\mathcal{O}_y))$ can also be obtained as follows: take $R\,\mathscr{H}\!om(\Phi_1(\mathcal{O}_x), \Phi_1(\mathcal{O}_y)) \cong \Phi_1(\mathcal{O}_x)^\vee \otimes^L \Phi_1(\mathcal{O}_y)$ as an inner tensor product obtained from the double complex

$$\begin{tikzcd} \mathcal{O}_{M_1} \arrow[r] & F_x^\vee \otimes \Lambda_M \arrow[r] & \Lambda_M \\ \Lambda_M^{-1} \otimes F_y \arrow[r] \arrow[u] & F_x^\vee \otimes F_y \arrow[r] \arrow[u] & F_y \arrow[u] \\ \Lambda_M^{-1} \arrow[r] \arrow[u] & F_x^\vee \arrow[r] \arrow[u] & \mathcal{O}_{M_1} \arrow[u] \end{tikzcd} \tag{3.2.9}$$

which produces the total complex

$$\left[\Lambda_M^{-1} \to F_x^\vee \oplus F_y^\vee \to \mathcal{O}_{M_1}^{\oplus 2} \oplus (F_x^\vee \otimes F_y) \to F_x \oplus F_y \to \Lambda_M\right] \cong \Phi_1(\mathcal{O}_x)^\vee \otimes^L \Phi_1(\mathcal{O}_y), \tag{3.2.10}$$

again using $F_x \cong F_x^\vee \otimes \Lambda_M$.

The hypercohomology $R\Gamma$ of (3.2.10) can be computed taking a spectral sequence, by applying $R\Gamma$ to each individual term. On the other hand, we know that $R\Gamma$ of this complex is given by (3.2.8). We will combine these to show that

$$R\Gamma(F_x^\vee \otimes F_y) = \begin{cases} 0 & \text{if } x \neq y \\ \mathbb{C} \oplus \mathbb{C}[-1] & \text{if } x = y. \end{cases}$$

By Lemma 3.2.6, $R\Gamma(\Lambda_M) = 0$, and by Lemma 3.2.7 $R\Gamma(\Lambda_M^{-1}) = 0$. Also, Lemma 3.2.8 computes hypercohomology of both F_x and $F_x^\vee$. Summing up, applying $R\Gamma$ to (3.2.10) yields a spectral sequence $E_2^{p,q}$ of the form

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & 0 & \longrightarrow & H^1(F_x^\vee \otimes F_y) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & H^0(\mathcal{O}_{M_1})^{\oplus 2} \oplus H^0(F_x^\vee \otimes F_y) & \longrightarrow & H^0(F_x) \oplus H^0(F_y) & \longrightarrow & 0,
\end{array}
$$

where the map $H^0(M_1, \mathcal{O}_{M_1})^{\oplus 2} \to H^0(M_1, F_x) \oplus H^0(M_1, F_y)$ is the isomorphism $\mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2$ given by the universal section in each coordinate, by Lemma 3.2.8. Since this spectral sequence must converge to (3.2.8), we must have that, if $x \neq y$, $H^i(M_1, F_x^\vee \otimes F_y) = 0\ \forall i$, while $H^0(M_1, F_x^\vee \otimes F_x) = H^1(M_1, F_x^\vee \otimes F_x) = \mathbb{C}$ and $H^i(M_1, F_x^\vee \otimes F_x) = 0$ for $i \neq 0, 1$. This completes the proof. $\square$

Let us denote by $\mathcal{D}$ the essential image of Φ_F in $D^b(M_1)$. By Proposition 3.2.5, $\mathcal{D} \cong D^b(C)$ is an admissible subcategory of $D^b(M_1)$ (cf. §1.2.1), and the same will be true for the image of $\Phi_{F \otimes p^*(L)}$, where L is any line bundle on M_1, since this image is just $\mathcal{D} \otimes L$. Also, the fact that M_1 is a rational variety ensures that $H^i(M_1, \mathcal{O}_{M_1}) = 0$ for $i \neq 0$, so that every line bundle on M_1 is an exceptional object. Moreover, if $g \geq 3$ we can use these to find a sequence of blocks $D^b(\mathrm{pt})$, $D^b(C)$, $D^b(\mathrm{pt})$, $D^b(C)$ that are part of a semi-orthogonal decomposition.

Proposition 3.2.9. *Suppose $d = 2g-1$ and $g \geq 3$ and call $\mathcal{D} = \Phi_F(D^b(C))$. Then the sequence*

$$\theta^{-1},\ \mathcal{D} \otimes \zeta \otimes \theta^{-1},\ \mathcal{O}_{M_1},\ \mathcal{D} \otimes \zeta$$

is part of a semi-orthogonal decomposition of $D^b(M_1)$.

Proof. All these are admissible subcategories by Proposition 3.2.5 and by the fact that line bundles are exceptional objects on M_1. Here by abuse of notation we write L for the full triangulated subcategory $\langle L \rangle$. Since skyscraper sheaves $\mathcal{O}_x \in D^b(C)$ of closed points are a spanning class of $D^b(C)$ [Huy06, Proposition 3.17], by Lemma 1.2.8 all we need to check is that the sequence of bundles

$$\theta^{-1},\ F_x \otimes \zeta \otimes \theta^{-1},\ \mathcal{O}_{M_1},\ F_y \otimes \zeta$$

is semi-orthogonal for every two closed points $x, y \in C$, that is, that there are no Ext groups from right to left. Using $R\operatorname{Hom} = R\Gamma \circ R\mathscr{H}\!om$, this is equivalent to showing that the following objects are Γ-acyclic:

(a) $F_x^\vee \otimes \zeta^{-1}$

(b) $F_x \otimes \zeta \otimes \theta^{-1}$

(c) θ^{-1}

(d) $F_y^\vee \otimes \zeta^{-1} \otimes \theta^{-1}$

(e) $F_x^\vee \otimes F_y \otimes \theta^{-1}$

(f) $F_y^\vee \otimes \zeta^{-1}$.

Recall the definitions of Λ_M, ζ and θ (see Notation 3.2.4). When $d = 2g-1$, we have

$$\begin{aligned}\Lambda_M &= \mathcal{O}_{M_1}(-H+E_1)\\ \zeta &= \mathcal{O}_{M_1}(gH-(g-1)E_1)\\ \theta &= \zeta^2 \otimes \Lambda_M = \mathcal{O}_{M_1}((2g-1)H-(2g-3)E_1).\end{aligned}$$

We also see that $\theta^{-1}\otimes\Lambda_M^{-1}\otimes\zeta^{-1} = \mathcal{O}_{M_1}(-(3g-2)H+(3g-5)E_1) = \omega_{M_1}$, the canonical bundle. Thus, by Serre duality, Γ-acyclicity of (d) $F_y^\vee\otimes\zeta^{-1}\otimes\theta^{-1}$ is equivalent to that of $F_y\otimes\Lambda_M^{-1} = F_y^\vee$, and that was established in Lemma 3.2.8. Observe also that (f) is redundant with (a), which is in turn equivalent to (b), simply because $F_x^\vee\otimes\zeta^{-1} = F_x\otimes\Lambda_M^{-1}\otimes\zeta^{-1} = F_x\otimes\zeta\otimes\theta^{-1}$.

Therefore, it remains to prove that (a) $F_x^\vee\otimes\zeta^{-1}$, (c) θ^{-1} and (e) $F_x^\vee\otimes F_y\otimes\theta^{-1}$ are Γ-acyclic. That $R\Gamma(\theta^{-1}) = 0$ follows from Lemma 3.2.6, since $0 < 2g-1 \le 3g-3$ and $0 \le 2g-3 \le 3g-5$ whenever $g \ge 2$. For the remaining ones, we use the Koszul resolution (3.2.2) and its restriction to $x \times M_1$. Twisting by ζ^{-1}, we get

$$\left[\Lambda_{M_1}\otimes\zeta^{-1} \to F_x^\vee\otimes\zeta^{-1} \to \zeta^{-1}\right] \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_x^r}(1-g),$$

where $r = d+g-4 = 3g-5$. The right hand side is Γ-acyclic, since $g-1 \le 3g-5$ provided $g \ge 3$. On the other hand, we see that $\Lambda_M\otimes\zeta^{-1} = \mathcal{O}_{M_1}(-(g+1)H+gE_1)$ is Γ-acyclic by Lemma 3.2.6, because $g \ge 3$ ensures that both $0 < g+1 \le 3g-3$ and $0 \le g \le 3g-5$. Similarly, we obtain that $\zeta^{-1} = \mathcal{O}_{M_1}(-gH+(g-1)E_1)$ is also Γ-acyclic, and therefore we conclude that $R\Gamma(F_x^\vee\otimes\zeta^{-1}) = 0$.

Finally, to show that $R\Gamma(F_x^\vee\otimes F_y\otimes\theta^{-1}) = 0$ for any two points $x, y \in C$, we use the resolutions

$$\begin{aligned}\left[\Lambda_M^{-1} \to F_x^\vee \to \mathcal{O}_{M_1}\right] &\xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_x^r}\\ \left[\theta^{-1} \to F_y\otimes\theta^{-1} \to \Lambda_M\otimes\theta^{-1}\right] &\xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_y^r}(2-2g),\end{aligned}$$

both of which follow from (3.2.2). The (derived) tensor product $\mathcal{O}_{\mathbb{P}_x^r}\otimes^L\mathcal{O}_{\mathbb{P}_y^r}(2-2g)$ can then be computed as the total complex of

$$\begin{tikzcd} \theta^{-1} \arrow[r] & F_y\otimes\theta^{-1} \arrow[r] & \Lambda_M\otimes\theta^{-1} \\ F_x^\vee\otimes\theta^{-1} \arrow[r] \arrow[u] & F_x^\vee\otimes F_y\otimes\theta^{-1} \arrow[r] \arrow[u] & F_x\otimes\theta^{-1} \arrow[u] \\ \Lambda_M^{-1}\otimes\theta^{-1} \arrow[r] \arrow[u] & F_y^\vee\otimes\theta^{-1} \arrow[r] \arrow[u] & \theta^{-1}. \arrow[u] \end{tikzcd} \tag{3.2.11}$$

All the terms in (3.2.11) other than $F_x^\vee \otimes F_y \otimes \theta^{-1}$ can be proved to be Γ-acyclic. For instance, we already know that $R\Gamma(\theta^{-1}) = 0$, while $\Lambda_M \otimes \theta^{-1} = \mathcal{O}_{M_1}(-2gH + (2g-2)E_1)$ and $\Lambda_M^{-1} \otimes \theta^{-1} = \mathcal{O}_{M_1}(2 - 2gH + (2g-4)E_1)$ can easily be seen to be Γ-acyclic from Lemma 3.2.6, given $g \geq 3$. Γ-acyclicity of $F_x^\vee \otimes \theta^{-1}$ is, by Serre duality, equivalent to Γ-acyclicity of (a) $F_x^\vee \otimes \zeta^{-1}$, which has already been proved. And from

$$\left[\theta^{-1} \to F_x \otimes \theta^{-1} \to \Lambda_M \otimes \theta^{-1}\right] \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}_x^r}(2-2g)$$

we obtain $R\Gamma(F_x \otimes \theta^{-1}) = 0$, since $g \geq 3$ implies $2g - 2 \leq 3g - 5$ and so $R\Gamma(\mathcal{O}_{\mathbb{P}_x^r}(2-2g)) = 0$.

From this analysis we conclude that $R\Gamma(F_x^\vee \otimes F_y \otimes \theta^{-1}) = R\Gamma(\mathcal{O}_{\mathbb{P}_x^r} \otimes^L \mathcal{O}_{\mathbb{P}_y^r}(2-2g))$, so the proposition will be proved if we show the right hand side is zero. If $x \neq y$, then $\mathcal{O}_{\mathbb{P}_x^r}$ and $\mathcal{O}_{\mathbb{P}_y^r}(2-2g)$ have disjoint supports, so the corresponding tensor product is zero. If $x = y$, observe that, since $\mathcal{O}_{\mathbb{P}_x^r} \cong \left[\Lambda_M^{-1} \to F_x^\vee \to \mathcal{O}_{M_1}\right]$, the (derived) dual $\mathcal{O}_{\mathbb{P}_x^r}^\vee$ is isomorphic to the (shifted) complex $\mathcal{O}_{M_1} \to F_x \to \Lambda_M$ concentrated in degrees 0, 1 and 2. Then

$$\mathcal{O}_{\mathbb{P}_x^r}^\vee \otimes \Lambda_M^{-1}[2] = \mathcal{O}_{\mathbb{P}_x^r}$$

and

$$\begin{aligned}\mathcal{O}_{\mathbb{P}_x^r} \otimes^L \mathcal{O}_{\mathbb{P}_x^r}(2-2g) &= R\operatorname{Hom}(\mathcal{O}_{\mathbb{P}_x^r}, \mathcal{O}_{\mathbb{P}_x^r}(2-2g) \otimes \Lambda_M^{-1}[2]) \\ &= R\operatorname{Hom}(\mathcal{O}_{\mathbb{P}_x^r}, \mathcal{O}_{\mathbb{P}_x^r}(3-2g))[2].\end{aligned}$$

But $R\operatorname{Hom}(\mathcal{O}_{\mathbb{P}_x^r}, \mathcal{O}_{\mathbb{P}_x^r}(3-2g)) = R\Gamma_{\mathbb{P}^r}(\mathcal{O}_{\mathbb{P}^r}(3-2g)) = 0$ as long as $0 < 2g - 3 \leq r = 3g - 5$, which is true in our case. This completes the proof. □

3.2.2 Wall-crossing between the spaces M_i

Consider the diagram (3.2.1). The wall between two consecutive chambers M_{i-1} and M_i occurs at $\sigma = d/2 - i$. The birational transformation $M_{i-1} \dashrightarrow M_i$ is an isomorphism outside of the loci $\mathbb{P}W_i^- \subset M_{i-1}$, $\mathbb{P}W_i^+ \subset M_i$, where W_i^- and W_i^+ are vector bundles over the symmetric product $C^{(i)}$ of rank i and $d + g - 1 - 2i$, respectively. We have a diagram

$$\begin{tikzcd} & \tilde{M} \arrow[dl] \arrow[dr] & \\ M_{i-1} = M_{\sigma+\epsilon} \arrow[dr] & & M_{\sigma-\epsilon} = M_i \arrow[dl] \\ & M_\sigma & \end{tikzcd}$$

where $\tilde{M}$ is both the blow-up of $M_{\sigma+\epsilon} = M_{i-1}$ along $\mathbb{P}W_i^-$ and the blow-up of $M_{\sigma-\epsilon} = M_i$ along $\mathbb{P}W_i^+$. M_σ is a singular space, obtained from the contraction to $C^{(i)}$ of the exceptional locus $\mathbb{P}W_i^- \times_{C^{(i)}} \mathbb{P}W_i^+ \subset \tilde{M}$.

When $d \gg 0$, $M_{\sigma\pm\epsilon}(\Lambda)$ and $M_\sigma(\Lambda)$ are obtained as a GIT quotient of $U \times \mathbb{PC}^\chi$, with $\chi = d + 2 - 2g$. If we call $\mathcal{L}_\pm$ the corresponding linearizations, we can write $M_{\sigma\pm\epsilon}(\Lambda)$ as GIT quotients of $X \subset U \times \mathbb{PC}^\chi$, where X is the union of the three semi-stable loci, so that $X = X^{ss}(\mathcal{L}_+) \cup X^{ss}(\mathcal{L}_-) \sqcup X^{sss}(\mathcal{L}_0)$ and $X^{sss}(\mathcal{L}_0) = X^u(\mathcal{L}_+) \cap X^u(\mathcal{L}_-)$. For $\mathcal{L}_\pm$ there is no strictly semi-stable locus and in fact PGL_χ acts freely on the semi-stable locus [Tha94, §1.6], so $X^{ss} /\!\!/_{\mathcal{L}_\pm} SL_\chi$ is isomorphic to the quotient stack $\mathfrak{X}^{ss}(\mathcal{L}_\pm)$ (cf. Remark 3.2.1). If d is arbitrary, one can fix an effective divisor D on C of large degree so that these spaces are GIT quotients of a closed subset X of $U' \times \mathbb{PC}^{\chi'}$, $\chi' = d + 2 - 2g + 2\deg D$, determined by the condition that in the pair (E', ϕ') the section ϕ' vanishes along D, and a similar analysis can be done.

Using techniques from windows, and especially Theorem 1.5.1, we can obtain the following result (cf. [Pot16, Corollary 8.1]).

Proposition 3.2.10. *Let $\sigma = d/2 - i$. For $1 \leq i \leq (d + g - 1)/3$, there is an embedding $D^b(M_{i-1}) \hookrightarrow D^b(M_i)$. For $i \geq (d + g - 1)/3$, there is an embedding the other way, $D^b(M_i) \hookrightarrow D^b(M_{i-1})$. Moreover, when $1 < i \leq (d + g - 1)/3$ there is a semi-orthogonal decomposition $D^b(M_i) = \langle D^b(M_{i-1}), D^b(C^{(i)}), \ldots, D^b(C^{(i)})\rangle$, with $d + g - 3i - 1$ copies of $D^b(C^{(i)})$.*

Proof. Take an effective divisor D of large degree, so that $M_\sigma \hookrightarrow M'_\sigma := M_\sigma(\Lambda(2D))$, where M'_σ is a GIT quotient of $X' \subset U' \times \mathbb{PC}^{\chi'}$, $\chi' = d+2-2g+2\deg D$, as in the discussion above. Then write M_σ as a GIT quotient of $X \subset X'$ by $SL_{\chi'}$ with strictly semi-stable locus Z corresponding to pairs (E', ϕ'), where E' splits as $E' = L' \oplus M'$, with $\deg L' = i + \deg D$, $\deg M' = d - i + \deg D$, and $\phi' \in H^0(L')$ vanishes along D. The map $\mathcal{O}_C^{\chi'} \twoheadrightarrow E'$ is given by a block-diagonal matrix $(\mathcal{O}_C^a \twoheadrightarrow L') \oplus (\mathcal{O}_C^b \twoheadrightarrow M')$.

We can write $M_{i-1} = M_{\sigma+\epsilon}(\Lambda) = X /\!\!/_{\mathcal{L}_+} SL_{\chi'}$ and $M_i = M_{\sigma-\epsilon}(\Lambda) = X /\!\!/_{\mathcal{L}_-} SL_{\chi'}$. Note that both $[X/PGL_{\chi'}]$ and $[X'/PGL_{\chi'}]$ are smooth quotient stacks of dimension $d + g - 2$ and $d + g - 2 + 2\deg D$ [Tha94], and thus X and X' are both smooth. Since $X \subset X'$ is cut out by the $2\deg D$ conditions imposed by the vanishing of a section along D, then it is a local complete intersection. Also, $M_{\sigma\pm\epsilon}$ is isomorphic to the quotient stack $[X^{ss}(\mathcal{L}_\pm)/PGL_{\chi'}]$ because $PGL_{\chi'}$ has no stabilizers on $X^{ss}(\mathcal{L}_\pm)$.

The KN stratification of the unstable locus with respect to $\mathcal{L}_\pm$ has a unique stratum $S_\pm$, consisting of the vector bundle $W_i^\pm$ over Z. The stabilizer of Z is $\lambda = \mathbb{G}_m$, acting on $L' \oplus M'$ by (t^b, t^{-a}), where $a = h^0(L')$, $b = h^0(M')$, and one can show that the λ-weights of $\mathcal{N}^\vee_{S/X'}$ are all $\pm(a + b) = \pm\chi'$ or 0 (see [Pot16, §7]). Then the weights of $\mathcal{N}^\vee_{S_\pm/X}$ are all $\pm\chi'$ and $\eta_\pm = \mathrm{weight}_{\lambda_\pm} \mathcal{N}_{S_\pm/X}$ is just the codimension of $S_\pm \subset X$.

Since $S_\pm$ is the bundle $W_i^\pm$ on Z, we have $\mathrm{codim}(S_\pm \subset X) = \mathrm{rk}\, W_i^\mp$, so that $\eta_+ = i\chi'$ and $\eta_- = (d+g-1-2i)\chi'$ and then $\mathrm{weight}_\lambda\, \omega_X|_Z = \eta_- - \eta_+ = (d+g-1-3i)\chi'$. By Theorem 1.5.1, and since $M_{\sigma\pm\epsilon} \cong \mathfrak{X}^{ss}(\mathcal{L}_\pm)$, we get a window embedding $D^b(M_{\sigma+\epsilon}) \subset D^b(M_{\sigma-\epsilon})$ if $\eta_+ \leq \eta_-$ and the other way around if $\eta_+ \geq \eta_-$.

Moreover, if $G_w^+ = D^b(M_{\sigma+\epsilon})$ is a window, determined by the range of weights $[w, w+\eta_+) \subset [w, w+\eta_-)$, then Theorem 1.4.6 and (1.4.1) give semi-orthogonal blocks $D^b(\mathfrak{Z})_v$, so that

$$D^b(M_{\sigma-\epsilon}) = \langle G_w^+, D^b(\mathfrak{Z})_w, \ldots, D^b(\mathfrak{Z})_{w+\mu-1}\rangle, \tag{3.2.12}$$

where $\mu = \eta_- - \eta_+$. In our case, $\mathfrak{Z} = [Z/L]$, where L is the Levi subgroup, i.e. the centralizer of λ in $PGL_{\chi'}$, acting on Z. We have a short exact sequence of groups

$$1 \to \mathbb{G}_m \to L \to PGL_a \times PGL_b \to 1$$

with $\mathbb{G}_m = \lambda$ acting on Z trivially and $[Z/PGL_a \times PGL_b] \cong C^{(i)}$. Then $\mathfrak{Z} \cong [C^{(i)}/\mathbb{G}_m]$, with the trivial action of $\mathbb{G}_m$, and $D^b(\mathfrak{Z}) = D^b_{\mathbb{G}_m}(C^{(i)})$, so the blocks in (3.2.12) are given by the fully faithful images of $j_*(\pi^*(\cdot) \otimes \mathcal{O}_\pi(l)) : D^b(C^{(i)}) \to D^b(M_i)$ for $l \in [w, w+\mu)$, where $\pi : \mathbb{P}W_i^+ \to C^{(i)}$ is the projection and $j : \mathbb{P}W_i^+ \hookrightarrow M_i$ the inclusion (cf. Example 1.5.2). □

Corollary 3.2.11. *If $d = 2g-1$, then $D^b(M_{i-1}) \subset D^b(M_i)$ for every $1 \leq i \leq g-1$.*

Proof. Indeed, for $1 \leq i \leq g-1$, the inequality $i < (3g-2)/3$ always holds. □

Now suppose $d = 2g-1$ and take the four semi-orthogonal blocks from Proposition 3.2.9. Using Theorem 1.4.1 we can get semi-orthogonal blocks in all of the spaces M_i. For the next lemma, we take $\sigma = d/2 - i$ and write M_{i-1}, M_i as GIT quotients $X /\!\!/_{\mathcal{L}_\pm} SL_{\chi'}$, as above. Let F denote the universal bundle on each M_i, and let Λ_M, ζ and θ be defined on each M_i as in Notation 3.2.4.

Lemma 3.2.12. *The objects of the form F_x, Λ_M, ζ, θ on both M_{i-1} and M_i are the descent of objects $\tilde{F}_x$, $\tilde{\Lambda}_M$, $\tilde{\zeta}$, $\tilde{\theta}$ on $D^b(\mathfrak{X})$ such that, up to rescaling by a constant, have λ-weights*

(a) $\mathrm{weight}_\lambda\, \tilde{F}_x|_Z = 0, -1$

(b) $\mathrm{weight}_\lambda\, \tilde{\Lambda}_M|_Z = -1$

(c) $\mathrm{weight}_\lambda\, \tilde{\zeta}|_Z = g - i$

(d) $\mathrm{weight}_\lambda\, \tilde{\theta}|_Z = 2g - 2i - 1$,

and the windows have widths $\eta_+ = i$ and $\eta_- = 3g - 2 - 2i$.

Proof. Let $\sigma = d/2 - i$ and embed $\imath : M_\sigma(\Lambda) \hookrightarrow M'_\sigma = M_\sigma(\Lambda(2D))$ for an effective divisor D, $\deg D \gg 0$. $M'_{\sigma\pm\epsilon} = M_\sigma(\Lambda(2D))$ are GIT quotients of X' by $SL_{\chi'}$ and the universal bundle F' on $M'_{\sigma\pm\epsilon}$ is the descent of $\mathcal{F}'(1)$ on $C \times X' \subset C \times U' \times \mathbb{P}\mathbb{C}^{\chi'}$, where $\mathcal{F}'$ is the universal family on $C \times U'$ [Tha94, §1.20]. The σ-strictly semi-stable locus $Z' \subset X'$ corresponds to split bundles $L' \oplus M'$ together with a section $\phi' \in H^0(L')$, and the action of $SL_{\chi'}$ on $H^0(E')$ is given by (t^b, t^{-a}), where $a = h^0(L')$, $b = h^0(M')$ and $a + b = h^0(L' \oplus M') = \chi'$.

Let us compute the λ-weights of $\mathcal{F}'_x(1)$ on Z', for a point $x \in C$. The fiber of $\mathcal{F}'_x$ over $L' \oplus M'$ is $L'_x \oplus M'_x$, which is acted on with weights b in the first component and $-a$ in the second. Since the λ-weight of $\mathcal{O}_{\mathbb{PC}^{\chi'}}(1)$ over the section $(\phi', 0)$ is $-b$, we get that the weights of $\mathcal{F}'_x(1)$ are 0 and $-a - b = -\chi'$.

The bundle $\det \pi_! F'$ descends from $\det \pi_! \mathcal{F}'(1)$. On the fiber of $\pi_! \mathcal{F}'$ over $L' \oplus M'$, λ acts on $H^0(L') \oplus H^0(M')$ with weights b and $-a$, with multiplicities $h^0(L') = a$ and $h^0(M') = b$, respectively. Taking tensor product with $\mathcal{O}_{\mathbb{PC}^{\chi'}}(1)$ shifts each weight by $-b$, and then taking determinant we get $\text{weight}_\lambda \det \pi_! \mathcal{F}'(1)|_{Z'} = 0 \cdot a + (-a - b) \cdot b = -b\chi'$.

For $\det F'_x$, which is the descent of $\det \mathcal{F}'_x(1)$, we see that λ acts with weights $b, -a$ on $L'_x \oplus M'_x$ and then shifting by $-b$ and taking determinant we get $\text{weight}_\lambda \det \mathcal{F}'_x(1)|_{Z'} = -a - b = -\chi'$.

Now for the universal bundle F over $M_{\sigma \pm \epsilon}(\Lambda)$, we have the following short exact sequence [Tha94, Remark 1.19]

$$0 \to F \to \imath^* F' \to \imath^* F'|_{D \times M_{\sigma \pm \epsilon}} \to 0.$$

From this we see that $\Lambda_M = \det F_x \cong \det F'_x$ is the descent of an object with λ-weight equal to $-\chi'$. Also, since $\det \pi_! F'|_{D \times M_{\sigma \pm \epsilon}} = \det \bigoplus_{x \in D} F'_x = (\det F'_x)^{\deg D}$, we get $\zeta^{-1} = \det \pi_! F = \det \pi_! F' \otimes (\det F'_x)^{-\deg D}$ (cf. Definition 3.2.3) is the descent of an object with λ-weight equal to $-b\chi' + \deg D \chi'$. Recall $\deg L' = i + \deg D$, $\deg M' = d - i + \deg D$, so by Riemann-Roch $b = h^0(M') = 2g - 1 - i + \deg D + 1 - g$ and the weight is $\chi'(\deg D - b) = \chi'(i - g)$. As for $\theta = \zeta^2 \otimes \Lambda_M$, the weights must be $(2(g - i) - 1)\chi' = (2i - 2g + 1)\chi'$.

Summing up, the bundles F_x, Λ_M, ζ, θ on $M_{\sigma \pm \epsilon}$ are the descent of $SL_{\chi'}$-equivariant bundles $\tilde{F}_x$, $\tilde{\Lambda}_M$, $\tilde{\zeta}$, $\tilde{\theta}$ on X having weights that, rescaling everything by $1/\chi'$, are precisely

(a) $\text{weight}_\lambda \tilde{F}_x|_Z = 0, -1$

(b) $\text{weight}_\lambda \tilde{\Lambda}_M|_Z = -1$

(c) $\text{weight}_\lambda \tilde{\zeta}|_Z = g - i$

(d) $\text{weight}_\lambda \tilde{\theta}|_Z = 2g - 2i - 1$.

After rescaling by the same constant, the windows have widths $\eta_+ = i$ and $\eta_- = 3g - 2 - 2i$, as computed before. □

Proposition 3.2.13. *Let $d = 2g - 1$ and $g \geq 3$. For $i = 1, \ldots, g-1$, the sequence*

$$\theta^{-1},\ \Phi_F(D^b(C)) \otimes \zeta \otimes \theta^{-1},\ \mathcal{O}_{M_i},\ \Phi_F(D^b(C)) \otimes \zeta$$

is the start of a semi-orthogonal decomposition of $D^b(M_i)$.

Proof. By Proposition 3.2.9 we know this is true on M_1, so we will transfer these blocks across the walls using the windows. Recall that Corollary 3.2.11 provides fully faithful functors from $D^b(M_1)$ all the way into $D^b(M_{g-1})$.

If A, B are objects in $\mathfrak{X}$, descending to both $M_{\sigma \pm \epsilon}$ and with $\lambda = \lambda_+$-weights such that

$$-\eta_- < \mathrm{weight}_\lambda B|_Z - \mathrm{weight}_\lambda A|_Z < \eta_+ \tag{3.2.13}$$

then Theorem 1.4.1 implies that

$$R\operatorname{Hom}_{M_{\sigma+\epsilon}}(A, B) = R\operatorname{Hom}_{\mathfrak{X}}(A, B) = R\operatorname{Hom}_{M_{\sigma-\epsilon}}(A, B).$$

Indeed, the first equality follows directly from the Quantization Theorem applied on $M_{\sigma+\epsilon}$, while the second is the same theorem applied on $M_{\sigma-\epsilon}$, following the fact that

$$\mathrm{weight}_{\lambda_-} B|_Z - \mathrm{weight}_{\lambda_-} A|_Z = -(\mathrm{weight}_\lambda B|_Z - \mathrm{weight}_\lambda A|_Z).$$

Fully faithfulness of Φ_F then follows from the fact that, between objects of the form F_x, the difference in weights, as given by Lemma 3.2.12, are in $[0, 1]$, which is always both $< \eta_+ = i$ and $> -\eta_- = 2 + 2i - 3g$ for $2 \leq i \leq g - 1$. This way we get that

$$R\operatorname{Hom}_{M_1}(F_x, F_y) = R\operatorname{Hom}_{\mathfrak{X}}(F_x, F_y) = R\operatorname{Hom}_{M_2}(F_x, F_y),$$

so by Theorem 1.2.14 and fully faithfullness on M_1, which was proved in Proposition 3.2.5, we obtain fully faithfullness of Φ_F on M_2 and similarly, by induction, $\Phi_F : D^b(C) \to D^b(M_i)$ is fully faithful for every i. In particular, its image is an admissible subcategory, equivalent to $D^b(C)$.

Now similarly, the objects objects of the form

(a) θ^{-1}

(b) $F_x \otimes \zeta \otimes \theta^{-1}$

(c) $\mathcal{O}_{M_i}$

(d) $F_y \otimes \zeta$

descend from objects in X having the following weights

(a) $2i - 2g + 1$

(b) $i-g+1$, $i-g$

(c) 0

(d) $g-i$, $g-i-1$,

given by Lemma 3.2.12. Within each block the difference in weights is always within $[0,1]$, so it always fits in a window for both M_{i-1} and M_i, and we can transfer fully faithful images of these blocks from M_1 to M_2 and in general between M_{i-1} and M_i. Now, to see that these blocks remain semi-orthogonal when we cross from one chamber to the next, we use Quantization Theorem again. Recall that semi-orthogonality can be checked on closed points only (see Lemma 1.2.8 and [Huy06, Proposition 3.17]), so it suffices to verify that, when taking morphisms from bottom to top between the objects above, all the differences in the λ-weights satisfy the inequalities (3.2.13).

These differences are computed to be the numbers

(a) $i-g$

(b) $i-g+1$

(c) $2i-2g+1$

(d) $3i-3g+1$

(e) $3i-3g+2$

(f) $2i-2g$

(g) $2i-2g+2$

and it is easy to check that they are all $> 2i+2-3g$ and $< i$ whenever $2 \leq i \leq g-1$. Indeed,

(a) $i-g < i$ trivially.

(b) $i-g+1 < i$ trivially.

(c) $2i-2g+1 < i$ is equivalent to $i < 2g-1$, which is true for $i \leq g-1$.

(d) $3i-3g+1 < i$ is equivalent to $2i < 3g-1$, which is true for $i \leq g-1$.

(e) $3i-3g+2 < i$ is equivalent to $2i < 3g-2$, which is true for $i \leq g-1$.

(f) $2i-2g < i$ is equivalent to $i < 2g$, which is true for $i \leq g-1$.

(g) $2i-2g+2 < i$ is equivalent to $i \leq 2(g-1)$, which is true for $i \leq g-1$, and

(a) $i - g > 2i + 2 - 3g$ is equivalent to $i < 2(g-1)$, which is true for $i \leq g-1$.

(b) $i - g + 1 > 2i + 2 - 3g$ is equivalent to $i < 2g - 1$, which is true for $i \leq g - 1$.

(c) $2i - 2g + 1 > 2i + 2 - 3g$ trivially, as $g > 0$.

(d) $3i - 3g + 1 > 2i + 2 - 3g$ is equivalent to $i > 1$.

(e) $3i - 3g + 2 > 2i + 2 - 3g$ is equivalent to $i > 0$.

(f) $2i - 2g > 2i + 2 - 3g$ is equivalent to $g > 2$.

(g) $2i - 2g + 2 > 2i + 2 - 3g$ is equivalent to $g > 0$.

In conclusion, using induction we get that these four blocks are admissible subcategories and define a semi-orthogonal sequence on every M_i. □

Call $\xi : M_{g-1} \to N$ the last map in (3.2.1), where $N = M_C(2, \Lambda)$ is the space of stable rank-two vector bundles on a curve with determinant Λ, and $\deg \Lambda = 2g - 1$. The Picard group of N is generated by an ample line bundle θ_N, such that $\xi^* \theta_N = \theta$ [Tha94, §5.8, 5.9]. Let $\mathcal{E}$ be the universal bundle on $C \times N$, normalized so that $\det \pi_! \mathcal{E} = \mathcal{O}_N$ and $\det \mathcal{E}_x = \theta_N$ (cf. [Nar17]). Then we have the following corollary.

Corollary 3.2.14. *Let $\mathcal{E}$ be the Poincaré bundle on the moduli space $N = M_C(2, \Lambda)$ over a curve of genus ≥ 3, normalized as above. Then the sequence*

$$\theta^{-1},\ \Phi_{\mathcal{E}}(D^b(C)) \otimes \theta_N^{-1},\ \mathcal{O}_N,\ \Phi_{\mathcal{E}}(D^b(C))$$

is the start of a semi-orthogonal decomposition.

Proof. Observe that ξ^* is fully faithful. Indeed, since ξ is a projective birational morphism with N normal, we have $\xi_*(\mathcal{O}_{M_{g-1}}) = \mathcal{O}_N$ [Tha94, Lemma 5.12] and then by adjointness $\mathrm{Hom}_{D^b(M_{g-1})}(\xi^* A, \xi^* B) = \mathrm{Hom}_{D^b(N)}(A, \xi_* \xi^* B) = \mathrm{Hom}_{D^b(N)}(A, B)$. The pullback $\xi^*(\mathcal{E})$ is a family of vector bundles on $C \times M_{g-1}$ whose fiber over each point $x \times (E, \phi) \in C \times M_{g-1}$ is exactly the fiber E_x. Thus, it has to coincide with the universal bundle F up to twist by a line bundle on M_{g-1}, so that $\xi^* \mathcal{E} = F \otimes L$. Then $\xi^* \det \mathcal{E}_x = \Lambda_M \otimes L^2$, which by the normalization chosen it must equal $\xi^* \theta_N = \theta$, so $L = \zeta$. This shows that $\xi^*(\mathcal{E}) = F \otimes \zeta$, and the result then follows from Proposition 3.2.13. □

Remark 3.2.15. If $g = 2$, $M_{g-1} = M_1$ and a similar argument still gives a sequence of three semi-orthogonal blocks in N. In fact, it can be shown that this is a full semi-orthogonal decomposition [BO95, Theorem 2.9], but our approach does not address fullness.

3.3 Moduli of parabolic bundles on $\mathbb{P}^1$.

Definition 3.3.1. A rank 2 quasi-parabolic vector bundle on $(\mathbb{P}^1; p_1, \ldots, p_n)$ is the data of a vector bundle V and a one-dimensional subspace F_j of each of the fibers V_j over p_j. A parabolic vector bundle has the additional data of weights $a_{j,2} > a_{j,1} > 0$ on each of these points. We may assume $a_{j,2} + a_{j,1} = 1$ [Bau91].

A quasi-parabolic bundle is equivalent to giving a subsheaf $V' \subset V$ that is also locally free of rank 2 and with $\det V' = \det V \otimes \mathcal{O}_C(-p_1 - \cdots - p_n)$. In fact, given $(V, F_1, \ldots, F_n)$, one can define

$$0 \to V' \xrightarrow{\beta} V \xrightarrow{\gamma} \bigoplus_{j=1}^{n} (V_j/F_j) \otimes \mathcal{O}_{p_j} \to 0$$

and we have $F_j = \operatorname{im} \beta_{p_j}$ [Cas15].

For a line subbundle $L \subset V$, call $L' = L \cap V'$. A *parabolic* structure on L is given by attaching weights

$$l_j = \begin{cases} a_{j,1} & \text{if } L_{p_j} \neq F_j \\ a_{j,2} & \text{if } L_{p_j} = F_j. \end{cases}$$

The *parabolic degree* of a parabolic vector bundle is defined in such a way that, if $V' \subset V$ is a rank two parabolic vector bundle and $L' \subset L$ a line subbundle, we have

$$\operatorname{Pardeg}(V' \subset V) = \deg V + \sum (a_{j,1} + a_{j,2})$$
$$\operatorname{Pardeg}(L' \subset L) = \deg L + \sum l_j.$$

Definition 3.3.2. $V' \subset V$ is stable if for every line subbundle $L \subset V$ we have $\operatorname{Pardeg}(L' \subset L) < \frac{1}{2} \operatorname{Pardeg}(V' \subset V)$.

Parabolic vector bundles were introduced by Mehta and Seshadri [MS80, Ses77], in order to generalize to curves with cusps the Narasimhan-Seshadri correspondence between stable vector bundles on smooth projective curves and unitary representations of their fundamental groups [NS65]. By a result of Mehta and Seshadri [MS80] and subsequent work by Bauer [Bau91], for any set of weights $\alpha = \{\alpha_j\}$ there is a moduli space $\mathcal{N}_\alpha$ of parabolic bundles of rank 2 over with trivial determinant that are semi-stable with respect to those weights, and it has the structure of a normal projective variety. The open subvariety of stable bundles is smooth. When $C = \mathbb{P}^1$, these spaces have been extensively studied and well described. In the case that the weights are all $(0, 1/2)$ and n is odd, we have the following description by Casagrande.

Theorem 3.3.3. [Cas15] *Let $C = \mathbb{P}^1$ and fix weights $a_{j,1} = 0$, $a_{j,2} = 1/2$, with $n = 2g + 1$. Then $\mathcal{N}_\alpha$ is the intersection of two quadrics $Q_1 \cap Q_2 \subset \mathbb{P}^{2g}$, where*

$$Q_1 = \left\{ \sum x_j^2 = 0 \right\}, \qquad Q_2 = \left\{ \sum \lambda_j x_j^2 = 0 \right\}$$

and $p_j = (\lambda_j : 1)$.

An interesting problem is to study the derived categories of $\mathcal{N}_\alpha$ in the case where $C = \mathbb{P}^1$ and understand how these vary when we change the weight system. The spaces $\mathcal{N}_\alpha$ for different sets of weights are related to each other by GIT wall-crossing, as described by Bauer [Bau91]. It worth asking whether one can carry out computations analogous to those in §3.2.2, by embedding these spaces in a quotient stack $\mathfrak{X}$ and analyzing what happens under wall-crossing. For instance, if all $\alpha_j = 1/2$, then $\mathcal{N}_\alpha$ is Fano, and the minimal model program can be carried out by flipping some $\mathbb{P}^k$'s in every step. When running the anti-canonical minimal model program and we move toward the Fano model, perhaps one can find consecutive embeddings of the derived categories, with orthogonal complements analogous to those in Proposition 3.2.10. If one wants to find a semi-orthogonal decomposition similar to (3.1.1), a possible candidate would be a weighted projective line $C(\lambda, d)$, as defined by Geigle and Lenzing in [GL87], which can be roughly thought of as a projective line with n marked points and weights attached to them. Variation of the stability conditions could possibly correspond to variation of the parameters d in $C(p, d)$.

BIBLIOGRAPHY

[BFK19] M. Ballard, D. Favero, and L. Katzarkov, *Variation of geometric invariant theory quotients and derived categories*, J. Reine Angew. Math. **746** (2019), 235–303.

[Bau91] S. Bauer, *Parabolic bundles, elliptic surfaces and* SU(2)*-representation spaces of genus zero Fuchsian groups*, Math. Ann. **290** (1991), no. 3, 509–526.

[BC94] V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), no. 2, 293–338.

[BM19] P. Belmans and S. Mukhopadhyay, *Admissible subcategories in derived categories of moduli of vector bundles on curves*, Adv. Math. **351** (2019), 653–675.

[BLMS19] A. Bayer, M. Lahoz, E. Macrì, and P. Stellari, *Stability conditions on Kuznetsov components* (2019), available at `https://arxiv.org/abs/1703.10839`.

[BMS16] A. Bayer, E. Macrì, and P. Stellari, *The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds*, Invent. Math. **206** (2016), no. 3, 869–933.

[Beĭ78] A. A. Beĭlinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69 (Russian).

[BO95] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties* (1995), available at `https://arxiv.org/abs/alg-geom/9506012`.

[BvdB03] A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[BTLM97] A. Buch, J. F. Thomsen, N. Lauritzen, and V. Mehta, *The Frobenius morphism on a toric variety*, Tohoku Math. J. (2) **49** (1997), no. 3, 355–366.

[Cas15] C. Casagrande, *Rank* 2 *quasiparabolic vector bundles on* $\mathbb{P}^1$ *and the variety of linear subspaces contained in two odd-dimensional quadrics*, Math. Z. **280** (2015), no. 3-4, 981–988.

[CT17] A.-M. Castravet and J. Tevelev, *Derived category of moduli of pointed curves - I*, to appear in *Algebraic Geometry* (2017), available at `https://arxiv.org/abs/1708.06340`.

[CT20a] A.-M. Castravet and J. Tevelev, *Exceptional collections on certain Hassett spaces*, Épijournal Géom. Algébrique **4** (2020), Art. 20, 34 (English, with English and French summaries).

[CT20b] A.-M. Castravet and J. Tevelev, *Derived category of moduli of pointed curves - II* (2020), available at `https://arxiv.org/abs/2002.02889`.

[Cox14] D. A. Cox, *Erratum to "The homogeneous coordinate ring of a toric variety" [MR1299003]*, J. Algebraic Geom. **23** (2014), no. 2, 393–398.

[CLS11] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[Dol03] I. V. Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003.

[DH98] I. V. Dolgachev and Y. Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56. With an appendix by Nicolas Ressayre.

[DN89] J.-M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), no. 1, 53–94.

[FK18] A. Fonarev and A. Kuznetsov, *Derived categories of curves as components of Fano manifolds*, J. Lond. Math. Soc. (2) **97** (2018), no. 1, 24–46.

[Fuj07] O. Fujino, *Multiplication maps and vanishing theorems for toric varieties*, Math. Z. **257** (2007), no. 3, 631–641.

[GL87] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297.

[Gro95] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki 1960-61, Soc. Math. France, Paris, 1995, pp. Exp. 221.

[GS82] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538.

[HL15] D. Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Has03] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352.

[HMSV05] B. Howard, J. Millson, A. Snowden, and R. Vakil, *The projective invariants of ordered points on the line* (2005), available at `https://arxiv.org/abs/math/0505096`.

[HMSV09] B. Howard, J. Millson, A. Snowden, and R. Vakil, *The equations for the moduli space of n points on the line*, Duke Math. J. **146** (2009), no. 2, 175–226.

[HK99] Y. Hu and S. Keel, *A GIT proof of Włodarczyk's weighted factorization theorem* (1999), available at `https://arxiv.org/abs/math/9904146`.

[HK00] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348.

[Huy06] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.

[Ill71] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin-New York, 1971.

[JR07] P. Jahnke and I. Radloff, *Fano threefolds with sections in* $\Omega^1_V(1)$, Math. Nachr. **280** (2007), no. 1-2, 127–139.

[Kap93] M. M. Kapranov, *Chow quotients of Grassmannians. I*, I. M. Gelfand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110.

[Kir84] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984.

[KM76] F. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55.

[Kuz16] A. Kuznetsov, *Derived categories view on rationality problems*, Rationality problems in algebraic geometry, Lecture Notes in Math., vol. 2172, Springer, Cham, 2016, pp. 67–104.

[KPS18] A. G. Kuznetsov, Y. G. Prokhorov, and C. A. Shramov, *Hilbert schemes of lines and conics and automorphism groups of Fano threefolds*, Jpn. J. Math. **13** (2018), no. 1, 109–185.

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Vol. 48, Springer-Verlag, Berlin, 2004.

[Lee18] K.-S. Lee, *Remarks on motives of moduli spaces of rank 2 vector bundles on curves* (2018), available at `https://arxiv.org/abs/1806.11101`.

[MS80] V. B. Mehta and C. S. Seshadri, *Moduli of Vector Bundles on Curves with Parabolic Structures*, Mathematische Annalen **248** (1980), 205-240.

[Muk81] S. Mukai, *Duality between* $D(X)$ *and* $D(\hat{X})$ *with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[Muk88] S. Mukai, *Curves,* $K3$ *surfaces and Fano* 3*-folds of genus* ≤ 10, Algebraic geometry and commutative algebra, Vol. I, Kinokuniya, Tokyo, 1988, pp. 357–377.

[Mus02] M. Mustaţă, *Vanishing theorems on toric varieties*, Tohoku Math. J. (2) **54** (2002), no. 3, 451–470.

[Nar17] M. S. Narasimhan, *Derived categories of moduli spaces of vector bundles on curves*, J. Geom. Phys. **122** (2017), 53–58.

[Nar18] M. S. Narasimhan, *Derived categories of moduli spaces of vector bundles on curves II*, Geometry, algebra, number theory, and their information technology applications, Springer Proc. Math. Stat., vol. 251, Springer, Cham, 2018, pp. 375–382.

[NS65] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540–567.

[Orl92] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 4, 852–862.

[Orl03] D. O. Orlov, *Derived categories of coherent sheaves and equivalences between them*, Russian Math. Surveys **58** (2003), no. 3, 511–591.

[Pot16] N. Potashnik, *Derived Categories of Moduli Spaces of Semistable Pairs over Curves*, Ph.D. thesis, Columbia University, 2016.

[Ram73] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69–84.

[Ses67] C. S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **85** (1967), 303–336.

[Ses77] C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Bull. Amer. Math. Soc. **83** (1977), no. 1, 124–126.

[Sch17] J. Schmitt, *A correspondence of good G-sets under partial geometric quotients*, Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry **59** (2017), no. 2, 343–360.

[Tel00] C. Teleman, *The quantization conjecture revisited*, Ann. of Math. (2) **152** (2000), no. 1, 1–43.

[Tha94] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), no. 2, 317–353.

[Tha96] M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.

[Tot18] B. Totaro, *Hodge theory of classifying stacks*, Duke Math. J. **167** (2018), no. 8, 1573–1621.

[Tot20] B. Totaro, *Bott vanishing for algebraic surfaces*, Trans. Amer. Math. Soc. (2020).

[Wan21] C. Wang, *Bott vanishing for elliptic surfaces* (2021), available at `https://arxiv.org/abs/2105.03575`.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wey03] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003.