ADDITIONAL PROBLEMS FOR MTH 531

1. Let (X, d) be a metric space and $A \subset X$ its subspace. Define the function $d_A : X \to \mathbb{R}$ by the formula $d_A(x) = \inf \{d(x, a) \,|\, a \in A\}$. Prove that d_A is continuous, and that $d_A(x) = 0$ if and only if $x \in \bar{A}$.

2. Let (X, d) be a metric space. Prove that $A \subset X$ is bounded if and only if $\operatorname{diam}(A)$ is finite.

3. Let $GL(2, \mathbb{R})$ denote the set of all 2×2 matrices A with real entries such that $\det A \neq 0$. Let $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ denote the matrices with $\det A = 1$. Give $GL(2, \mathbb{R})$ and $SL(2, \mathbb{R})$ topologies by regarding them as subsets of $\mathbb{R}^4$ via

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longrightarrow \quad (a, b, c, d) \in \mathbb{R}^4.$$

Determine whether $GL(2, \mathbb{R})$ and $SL(2, \mathbb{R})$ are compact.

4. Let (X, d) be a metric space whose diameter equals 4, and consider its open covering

$$X = \bigcup_{x \in X} (X - \{x\}).$$

Prove that $\delta = 3$ is a Lebesgue number for this covering.

5. Let $D^2 \subset \mathbb{R}^2$ be the closed unit disk given by $x^2 + y^2 \leq 1$ and let $X = D^2 - \{0\}$ with the subspace topology. Show that there exists an open covering of X which does not have Lebesgue number.

6. Prove that the plane $\mathbb{R}^2$ with the dictionary order topology is metrizable but not second–countable.

7. Let A be a subspace of a regular space X and consider the partition of X which consists of A and the singletons $\{x\}$ for all $x \notin A$. This partition defines an equivalence relation on X whose quotient space (with quotient topology) is called X/A. Prove that X/A is Hausdorff if and only if A is closed.

8. The second–countability is an essential part of the definition of a manifold: show that $\mathbb{R}^2$ with the dictionary order topology satisfies all the conditions for a 1–manifold except for second–countability.