PRACTICE PROBLEMS FOR THE FINAL EXAM

1. Let $C$ be the portion of the helix $r(t) = \cos t \, i + \sin t \, j + 4t \, k$ between $t = 0$ and $t = \pi$, and let $F = x \, i + z \, k$. Evaluate $\int_C F \cdot dr$.

2. Compute work done by vector field $F = (-yx^2 + e^{x^2}) \, i + (xy^2 - e^{y^2}) \, j$ in moving an object along the circle $x^2 + y^2 = 1$ once in the counterclockwise direction.

3. Let $F = ye^{xy} \, i + (xe^{xy} - z^3) \, j + (2\sin z - 3yz^2) \, k$. (a) Evaluate curl $F$. (b) Find a function $f$ such that $F = \nabla f$. (c) Find work done by $F$ along the straight line segment from $(0, 5, 0)$ to $(1, 0, \pi)$.

4. Find the work done by the force field $F(x, y, z) = z \, i + x \, j + y \, k$ in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ (a) along a straight line and (b) along the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$. Is this force field conservative? Justify your answer.

5. Consider the vector field

$$F = \frac{-y}{x^2 + y^2} \, i + \frac{x}{x^2 + y^2} \, j.$$  

(a) Evaluate directly the line integral of $F$ along the unit circle, once around in the counterclockwise direction. Is $F$ conservative?  
(b) Compute the curl of $F$. Why does your answer not contradict Green’s theorem?

6. Let $C_1$ be the unit circle $x^2 + y^2 = 1$ and $C_2$ the concentric circle of radius two. Orient both $C_1$ and $C_2$ counterclockwise. Suppose that $F = P \, i + Q \, j$ is a vector field in plane such that

$$\int_{C_1} F \cdot n \, ds = 10 \quad \text{and} \quad \int_{C_2} F \cdot dr = 17.$$  

(a) If $F$ is smooth in the plane, compute $\iint_D \text{div} \, F \, dA$, where $D$ is the domain defined by the inequality $x^2 + y^2 \leq 1$.  
(b) If $F$ is smooth on the annulus bounded by $C_1$ and $C_2$ and $Q_x = P_y$ everywhere on the annulus, compute $\int_{C_2} F \cdot dr$. 
7. Find the center of mass:
   (a) of the thin wire of density 1 bent into the shape of cycloid \( \mathbf{r}(t) = (t - \sin t) \mathbf{i} + (1 - \cos t) \mathbf{j}, \ 0 \leq t \leq 2\pi. \)
   (b) of the part of the spherical surface \( x^2 + y^2 + z^2 = 25 \) above the plane \( z = 3 \),
   assuming that its density at \((x, y, z)\) equals \( z \).

8. Evaluate surface integral \( \iint_S x^2y \, dS \) over the portion \( S \) of the cylinder \( x^2+y^2 = 4 \) between the planes \( z = 0 \) and \( z = 3 \).

9. Find the flux of \( \mathbf{F} = x \mathbf{i} + x \mathbf{j} + y \mathbf{k} \) across the hemisphere \( x^2 + y^2 + z^2 = 25 \),
   \( y \geq 0 \), oriented in the direction of the positive \( y \)-axis.

10. Use Stokes’ Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \),
    where \( \mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + xz \mathbf{k} \) and \( C \) is the triangle with the vertices \((1,0,0)\), \((0,1,0)\) and \((0,0,1)\) oriented
    counterclockwise as viewed from above.

11. Use the divergence theorem to calculate the outward flux of \( \mathbf{F}(x, y, z) = x^3 \mathbf{i} + y^3 \mathbf{j} + z^3 \mathbf{k} \)
    across the surface of the solid bounded by the cylinder \( x^2 + y^2 = 1 \) and
    the planes \( z = 0 \) and \( z = 2 \).

12. Compute the outward flux of \( \mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \)
    across the ellipsoid \( 4x^2 + 9y^2 + 6z^2 = 36 \). (Hint: wouldn’t it be easier to compute
    the flux across a sphere?)

**SOLUTIONS**

1. \( 8\pi^2 \). 2. \( \pi/2 \). 3. (a) 0; (b) \( e^{xy} - yz^3 - 2 \cos z + C; \) (c) 4.

4. (a) \( (3\pi - 9)/2 \). (b) \( -3\pi/4 \). The force field is not conservative because \( \int_C \mathbf{F} \cdot d\mathbf{r} \)
   is not path independent.

5. (a) \( 2\pi \), \( \mathbf{F} \) is not conservative. (b) \( \text{curl} \ \mathbf{F} = 0 \). This does not contradict Green’s
   Theorem because \( \mathbf{F} \) is not continuous at the origin.

6. (a) 10; (b) 17. 7. (a) \( (\pi, 4/3) \); (b) \( (0,0,49/12) \). 8. 0. 9. \( 250\pi/3 \).

10. \( -1/2 \). 11. \( 11\pi \). 12. \( 4\pi \).