

## DYNAMIC ANALYSIS AND OPTIMAL CONTROL OF A THREE-AGE-CLASS HIV/AIDS EPIDEMIC MODEL IN CHINA

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**ABSTRACT.** Based on the fact that HIV/AIDS manifests different transmission characteristics and pathogenesis in different age groups, and the proportions of youth and elderly HIV infected cases in total are increasing in China, we classify the whole population into three age groups, youth (15-24), adult (25-49), and elderly ( $\geq 50$ ), and establish a three-age-class HIV/AIDS epidemic model to investigate the transmission dynamics of HIV/AIDS in China. We derive the explicit expression for the basic reproduction number via the next generation matrix approach. Qualitative analysis of the model including the local, global behavior and permanence is carried out. In particular, numerical simulations are presented to reinforce these analytical results and demonstrate HIV epidemiological discrepancy among different age groups. We also formulate an optimal control problem and solve it using Pontryagin's Maximum Principle and an efficient iterative numerical methods. Our numerical results of optimal controls for the elderly group indicate that increasing the condom use and decreasing the rate of the formerly HIV infected persons converted to AIDS patients are important measures to control HIV/AIDS epidemic among elderly population.

**1. Introduction.** Since the discovery of HIV (human immunodeficiency virus) in the early 1980s, the disease has spread in successive waves to most regions around the global. In the world, about 36.9 million people are infected with HIV, and an estimated 1.1 million people died due to AIDS (acquired immune deficiency syndrome) in 2017 [46]. It is one of the top ten infectious diseases and a leading cause of death in mainland China, as reported by the China Center for Disease Control and Prevention (China, CDC). The cumulative total number of reported HIV/AIDS infection was 89,067 as of December 2004 [15], a figure that increased to 820, 756 as of June 2018 [6]. Meanwhile, HIV/AIDS manifests different transmission characteristics and pathogenesis in different age groups, and the proportions of youth and elderly HIV infected cases in total are increasing in China [48, 28, 27, 55].

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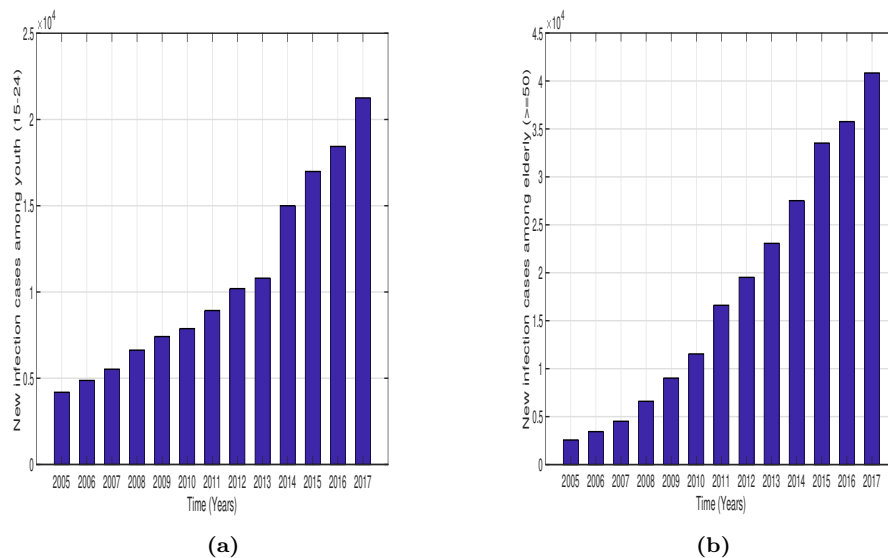
The elderly population, defined as people aged 50 and over in the HIV/AIDS literature [50, 3], has not been regarded as one of the major groups affected by HIV infection for a long time. More evidences, however, show that the rate of HIV infection has substantially increased among the elderly people [39, 22, 11]. Data reported by the Chinese National HIV/AIDS Information System indicate that both the total number of HIV/AIDS cases and the number of HIV/AIDS cases among the elderly population have been increasing in recent years ([16], Fig.1). In 2007, the elderly population accounted for 10.06% of the total cases of HIV/AIDS [50]. This proportion is quite close to the American level [3], higher than that in Asia, Africa, and Latin America, and even Australia. According to the annual statistics in 2012 [48], the fraction of elderly people with HIV/AIDS increased to 23.68% among the total HIV/AIDS reported cases, this ratio (see Table 2) was 25.61%, 26.60%, 29.24%, 28.71%, 30.35% in 2013, 2014, 2015, 2016, 2017, respectively.

Several facts contributed to the rapid transmission of HIV among the elderly population. Firstly, the fraction of heterosexual transmission increased from 35% in 2005 to almost 80% in 2012 [16, 5]. The rate of non-use of condoms was as high as 85% among the elderly people [14] and thus raised the likelihood of HIV infection. Secondly, the symptoms of HIV infection among the elderly population were easily confused with some geriatric diseases and thus the HIV virus was rarely tested [20]. The elderly people do not usually seek medical advice until evident symptoms of AIDS show up at the late stage of the disease, and the death rate of AIDS among the elderly population was 2.4 times the rate among people under 50 years of age [33]. Thirdly, chronic HIV infection usually advances to AIDS as time goes by. Statistics show that the development of AIDS from former HIV carriers increased from 10894 in 2012 [28] to 18231 in 2017 [27] in China.

In fact, there are many researches focus on HIV/AIDS transmission among adolescent group [18, 25, 37]. Compared with the elderly group, the number of HIV/AIDS reported among young people (15-24 years old) increased rapidly from 2012 to 2014 and has basically the same rate as the increasing prevalence of HIV/AIDS in China [23], the HIV/AIDS cases of young group increased from 4186 in 2005 [55] to 212500 in 2017 (see Fig.1 (a)). Different from the elderly group, the transmission rate of heterosexual and homosexual sexual behavior was 17% and 81% among young people, respectively [24]. These data show that the increase in the number of HIV infected adolescents in recent years is mainly due to homosexual sexual transmission. China National Health and Family Planning Commission data indicate that the proportion of students in HIV/AIDS infected among young people has risen from 5.8% in 2008 to 16.6% in 2014, and adolescent students have a higher prevalence than other populations [24]. To sum up, the epidemic transmission characteristics of HIV in different age groups are not the same. Therefore, it is very meaningful to study HIV transmission with different age groups.

Mathematical models have been used extensively to study the dynamics of HIV/AIDS among high risk groups [40, 2, 47, 53, 54]. Bacaer et al. used a mathematical model to study the transmission of HIV/AIDS among IDUs (injecting drugs users) and sex workers in Kunming, the provincial capital of Yunnan [2]. Xiao et al. constructed an HIV/AIDS model with 31 patches to understand the epidemic trend in China [47]. Zhang et al. divided the total population which is restricted within high risk population into two subgroups: IDUs and people engaged in PECS (commercial sex) which includes FSWs (female sex workers), and clients of FSWs. Due to this category, the predicting results have some differences with the actual

cases in Yunnan [53]. Zhang et al. developed a mathematical model on the transmission dynamics of HIV. In a case study for Yunnan, China, they divided the total population into four compartments: IDUs (injecting drugs users), FSWs (female sex workers), Clients of FSWs and MSM (men who have sex with men) [54]. Those models are all established to study HIV transmission among high-risk groups. Although HIV exhibits different transmission characteristics in different age groups, there has been very little research on modeling HIV transmission with different age groups.



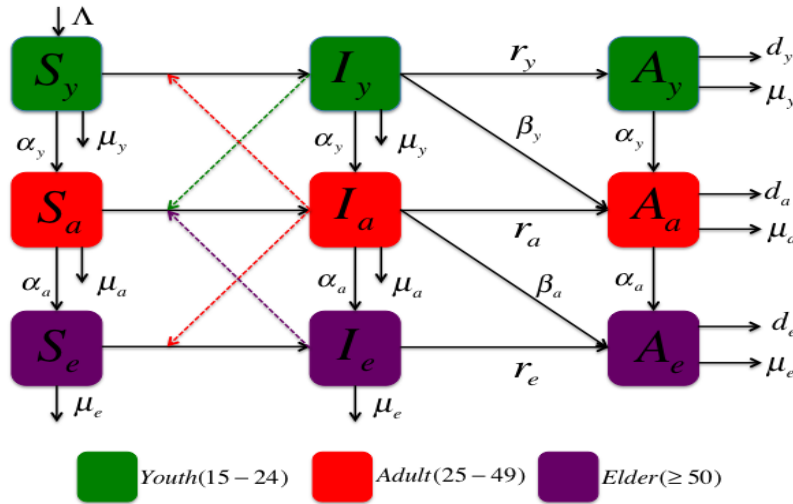
**Figure 1.** (a): The new HIV/AIDS infection cases among youth (15-24) from 2005 to 2017 (see Table 2), (b): The new HIV/AIDS infection cases among elderly ( $\geq 50$ ) from 2005 to 2017 (see Table 2).

In this paper, inspired by the above studies, we divide the whole population into three age compartments: youth group (15-24), adult group (25-49) and elderly group ( $\geq 50$ ), then propose a HIV/AIDS model with three-age-classes. After calculating the basic reproduction number  $R_0$ , we prove the global stability of the disease-free equilibrium when  $R_0 < 1$ , and analyze the persistence of the disease when  $R_0 > 1$ . In addition, under certain conditions, we show that there is a unique endemic equilibrium which is globally attractive if  $R_0 > 1$ . Furthermore, the optimal control for HIV transmission and AIDS among aged population is discussed. Simulations are also conducted to illustrate the theoretical results.

The paper is organized as follows. Section 2 presents the mathematical model. The stability of the disease-free equilibrium is proved in Section 3. The uniform persistence of the system and the global attractivity of the endemic equilibrium are discussed in Section 4 and Section 5, respectively. Section 6 focuses on the optimal control strategy under the objective function. Section 7 deals with numerical simulations and sensitivity analysis. Findings and conclusions are summarized in Section 8.

**2. Model formulation.** In order to take account of the variable properties of HIV transmission among different age groups in China, and based on the fact that the main transmission route of HIV is sexual transmission [28, 27], we divide the whole population into three groups: youth (15-24), adult (25-49) and elder ( $\geq 50$ ), the population sizes in those three classes are denoted by  $N_y, N_a, N_e$  respectively. In each class, the sub-population is composed of three compartments: susceptible ( $S_i$ ), infective without clinic symptom ( $I_i$ ) and AIDS (the people infected HIV with clinic symptom, i.e., the gradual loss of immune function, various opportunistic infections or malignant tumors appear in the body) with low sexual behavior ( $A_i$ ),  $i = y, a, e$ . In the column of Figure 2, the incidence rates from  $I_y, I_a, I_e$  to  $S_y, S_a, S_e$  are  $\beta_{11}I_y/N_y, \beta_{22}I_a/N_a$ , and  $\beta_{33}I_e/N_e$ , respectively. The incidence rate from  $I_y$  to  $S_a$  is  $\beta_{12}I_y/N_a$  (green imaginary line), the incidence rates from  $I_a$  to  $S_y, S_e$  are  $\beta_{21}I_a/N_y$  and  $\beta_{23}I_a/N_e$ , respectively. (red imaginary line), and the incidence rates from  $I_e$  to  $S_a$  is  $\beta_{32}I_e/N_a$  (purple imaginary line). Hence the term  $(\beta_{11}I_y + \beta_{21}I_a)S_y/N_y$  indicates the new infections in the  $S_y$  compartment. Similarly,  $(\beta_{22}I_a + \beta_{12}I_y)S_a/N_a$  and  $(\beta_{33}I_e + \beta_{23}I_a)S_e/N_e$  are indicate the new infection in  $S_a$  and  $S_e$  compartment, respectively.  $\alpha_y$  denotes transfer rate of individuals aging from the youth to the adult and  $\alpha_a$  represents the transfer rate of individuals aging from the adult to the elderly.

According to the character of HIV, the incubation period of HIV is 2-20 years or more [1],  $r_y, \beta_y$  represent the transfer rate from  $I_y$  to  $A_y$  and  $A_a$ , respectively. Similarly,  $r_a, r_e, \beta_a$  are defined. The parameters in model (1) are described in Table 1. Finally, through there were a few sexually transmitted HIV cases among children under 15 years old [1, 45], we do not consider this age group in our model.



**Figure 2.** The diagram of transmission among three epidemiological classes.

By the above notations and assumptions, the model is given by the following nine ordinary differential equations

$$\left\{ \begin{aligned} \frac{dS_y(t)}{dt} &= \Lambda - \alpha_y S_y(t) - \mu_y S_y(t) - \phi_y(t) S_y(t), \\ \frac{dI_y(t)}{dt} &= \phi_y(t) S_y(t) - \alpha_y I_y(t) - r_y I_y(t) - \mu_y I_y(t) - \beta_y I_y(t), \\ \frac{dA_y(t)}{dt} &= r_y I_y(t) - \mu_y A_y(t) - d_y A_y(t) - \alpha_y A_y(t), \\ \frac{dS_a(t)}{dt} &= \alpha_y S_y(t) - \alpha_a S_a(t) - \mu_a S_a(t) - \phi_a(t) S_a(t), \\ \frac{dI_a(t)}{dt} &= \phi_a(t) S_a(t) + \alpha_y I_y(t) - \alpha_a I_a(t) - r_a I_a(t) - \mu_a I_a(t) - \beta_a I_a(t), \\ \frac{dA_a(t)}{dt} &= \beta_y I_y(t) + r_a I_a(t) + \alpha_y A_y(t) - \mu_a A_a(t) - \alpha_a A_a(t) - d_a A_a(t), \\ \frac{dS_e(t)}{dt} &= \alpha_a S_a(t) - \mu_e S_e(t) - \phi_e(t) S_e(t), \\ \frac{dI_e(t)}{dt} &= \phi_e S_e(t) + \alpha_a I_a(t) - \mu_e I_e(t) - r_e I_e(t), \\ \frac{dA_e(t)}{dt} &= \beta_a I_a(t) + r_e I_e(t) + \alpha_a A_a(t) - \mu_e A_e(t) - d_e A_e(t), \end{aligned} \right. \quad (1)$$

where  $\phi_y(t) = (\beta_{11}I_y(t) + \beta_{21}I_a(t))/N_y(t)$ ,  $\phi_a(t) = (\beta_{22}I_a(t) + \beta_{12}I_y(t))/N_a(t)$ ,  $\phi_e(t) = (\beta_{33}I_e(t) + \beta_{23}I_a(t))/N_e(t)$ ,  $\beta_{11} = \delta_{11}(1 - c_{11})$ ,  $\beta_{12} = \delta_{12}(1 - c_{12})$ ,  $\beta_{21} = \delta_{21}(1 - c_{21})$ ,  $\beta_{22} = \delta_{22}(1 - c_{22})$ ,  $\beta_{23} = \delta_{23}(1 - c_{23})$ ,  $\beta_{33} = \delta_{33}(1 - c_{33})$ . Youth, adult, and elderly classes have different mortality rates  $\mu_y$ ,  $\mu_a$ ,  $\mu_e$ , and AIDS related death rate  $d_y$ ,  $d_a$ ,  $d_e$ , respectively. The other parameters are listed in Table 1.

### 3. Mathematical analysis.

**3.1. Basic properties.** In this subsection, the basic dynamical features of model (1) will be explored. We claim the following lemma.

**Lemma 3.1.** *The solution  $(S_y(t), I_y(t), A_y(t), S_a(t), I_a(t), A_a(t), S_e(t), I_e(t), A_e(t))$  of system (1) with nonnegative initial values eventually enters*

$$\mathcal{D} = \{(S_i, I_i, A_i) | S_i, I_i, A_i \geq 0, i = y, a, e, 0 \leq N_y \leq \bar{N}_y, 0 \leq N_a \leq \bar{N}_a, 0 \leq N_e \leq \bar{N}_e\}, \quad (2)$$

where

$$\begin{aligned} N_a &= S_a + I_a + A_a, \quad N_y = S_y + I_y + A_y, \quad N_e = S_e + I_e + A_e, \\ \bar{N}_y &= \frac{\Lambda}{\xi_1}, \quad \bar{N}_a = \frac{\Lambda m_1}{\xi_1 \xi_2}, \quad \bar{N}_e = \frac{\Lambda m_1 m_2}{\mu_e \xi_1 \xi_2}, \quad \xi_1 = \alpha_y + \mu_y, \quad \xi_2 = \alpha_a + \mu_a, \\ m_1 &= 3\alpha_y + \beta_y, \quad m_2 = 3\alpha_a + \beta_a. \end{aligned}$$

*Proof.* Adding the three equations in the youth class in model (1), we have  $\frac{dN_y}{dt} \leq \Lambda - \xi_1 N_y$ . According to the standard comparison theorem [8], there exists  $t_1 > 0$  such that  $N_y(t) \leq \frac{\Lambda}{\xi_1}$ , for  $t \geq t_1$ . It follows from system (1) that  $\frac{dN_a}{dt} \leq m_1 N_y - \xi_2 N_a$  for  $t \geq t_1$ . Then there exists  $t_2 > t_1$  such that  $N_a \leq \frac{\Lambda m_1}{\xi_1 \xi_2}$ , for  $t \geq t_2$ . Similarly, there exists  $t_3 > t_2$  such that  $N_e \leq \frac{\Lambda m_1 m_2}{\mu_e \xi_1 \xi_2}$ , for  $t \geq t_3$ . Therefore, solutions of system (1) are uniformly ultimately bounded. This completes the proof.  $\square$

In what follows, we consider only solutions with initial conditions inside the region  $\mathcal{D}$ .

3.2. **Disease-free equilibrium  $E_0$  and basic reproduction number  $R_0$ .** The system (1) always has the disease-free equilibrium  $E_0 = (S_y^0, 0, 0, S_a^0, 0, 0, S_e^0, 0, 0)$ , where  $S_y^0 = \bar{N}_y, S_a^0 = \frac{\alpha_y S_y^0}{\alpha_a + \mu_a}, S_e^0 = \frac{\alpha_a S_a^0}{\mu_e}$ .

We first denote  $z(t) = (I_y(t), A_y(t), I_a(t), A_a(t), I_e(t), A_e(t), S_y(t), S_a(t), S_e(t))$ . According to the concepts of the next generation matrix [9, 10], system (1) can be rewritten as follows:

$$\dot{z} = \mathcal{F}(z) - \mathcal{V}(z),$$

where

$$\mathcal{F} = \begin{pmatrix} \phi_y S_y \\ 0 \\ \phi_a S_a + \alpha_y I_y \\ 0 \\ \phi_e S_e + \alpha_a I_a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathcal{V} = \begin{pmatrix} (\alpha_y + r_y + \mu_y + \beta_y) I_y \\ (\mu_y + d_y + \alpha_y) A_y - r_y I_y \\ (\alpha_a + r_a + \mu_a + \beta_a) I_a - \alpha_y I_y \\ (\mu_a + d_a + \alpha_a) A_a - \beta_y I_y - r_a I_a - \alpha_y A_y \\ (\mu_e + r_e) I_e - \alpha_a I_a \\ (\mu_e + d_e) A_e - r_e I_e - \beta_a I_a - \alpha_a A_a \\ -\Lambda + \alpha_y S_y + \mu_y S_y + \phi_y S_y \\ -\alpha_y S_y + \alpha_a S_a + \mu_a S_a + \phi_a S_a \\ -\alpha_a S_a + \mu_e S_e + \phi_e S_e \end{pmatrix}.$$

The Jacobian matrices of  $\mathcal{F}$  and  $\mathcal{V}$  at  $E_0$  respective are

$$D\mathcal{F}(E_0) = \begin{pmatrix} \mathbf{F} & 0 \\ 0 & 0 \end{pmatrix}, D\mathcal{V}(E_0) = \begin{pmatrix} \mathbf{V} & 0 \\ * & * \end{pmatrix},$$

$$\mathbf{F} = \begin{pmatrix} \beta_{11} & 0 & \beta_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{12} + \alpha_y & 0 & \beta_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{23} + \alpha_a & 0 & \beta_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} \eta_1 & 0 & 0 & 0 & 0 & 0 \\ -r_y & \eta_2 & 0 & 0 & 0 & 0 \\ -\alpha_y & 0 & \eta_3 & 0 & 0 & 0 \\ -\beta_y & -\alpha_y & -r_a & \eta_4 & 0 & 0 \\ 0 & 0 & -\alpha_a & 0 & \eta_5 & 0 \\ 0 & 0 & -\beta_a & -\tau_a & -r_e & \eta_6 \end{pmatrix},$$

where

$$\eta_1 = \alpha_y + r_y + \mu_y + \beta_y, \eta_2 = \mu_y + d_y + \alpha_y, \eta_3 = \alpha_a + r_a + \mu_a + \beta_a, \\ \eta_4 = \mu_a + d_a + \alpha_a, \eta_5 = \mu_e + r_e, \eta_6 = \mu_e + d_e.$$

Hence

$$\mathbf{FV}^{-1} = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{31} & 0 & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$A_{11} = \frac{\beta_{11}}{\eta_1} + \frac{\beta_{21}\alpha_y}{\eta_1\eta_3}, \quad A_{13} = \frac{\beta_{21}}{\eta_3}, \quad A_{31} = \frac{(\beta_{12} + \alpha_y)}{\eta_1} + \frac{\beta_{22}\alpha_a}{\eta_1\eta_3}, \quad A_{33} = \frac{\beta_{22}}{\eta_3}, \quad A_{55} = \frac{\beta_{33}}{\eta_5}. \tag{3}$$

Then the characteristic equation of  $\mathbf{FV}^{-1}$  is derived as

$$\lambda^3(\lambda - A_{55})((\lambda - A_{11})(\lambda - A_{33}) - A_{31}A_{13}) = 0.$$

Then we obtain the basic reproduction number

$$R_0 = \max \{ R_0^{(1)}, R_0^{(2)} \},$$

where

$$R_0^{(1)} = A_{55}, \quad R_0^{(2)} = \frac{1}{2} \left( A_{11} + A_{33} + \sqrt{(A_{11} + A_{33})^2 + 4(A_{31}A_{13} - A_{11}A_{33})} \right). \tag{4}$$

**3.3. The global stability of  $E_0$ .** The main focus of this subsection is to analyze the local and global behavior of the disease-free equilibrium  $E_0$  of model (1). First, we state and prove the result about local asymptotic stability for the disease-free equilibrium  $E_0$ .

**Theorem 3.2.** *The disease-free equilibrium  $E_0$  is locally asymptotically stable provided that  $R_0 < 1$ .*

*Proof.* The  $9 \times 9$  Jacobian matrix  $J(E_0)$  can be presented as follows:

$$J(E_0) = \begin{pmatrix} M_{6 \times 6} & 0_{6 \times 3} \\ *_{3 \times 6} & N_{3 \times 3} \end{pmatrix}, \tag{5}$$

where

$$M_{6 \times 6} = \begin{pmatrix} -(\alpha_y + \mu_y) & -\beta_{11} & 0 & 0 & -\beta_{21} & 0 \\ 0 & \beta_{11} - \eta_1 & 0 & 0 & \beta_{21} & 0 \\ 0 & r_y & -\eta_2 & 0 & 0 & 0 \\ \sigma_y & -\beta_{12} & 0 & -(\alpha_a + \mu_a) & -\beta_{22} & 0 \\ 0 & \beta_{12} + \alpha_y & 0 & 0 & \beta_{22} - \eta_3 & 0 \\ 0 & \beta_y & \alpha_y & 0 & r_a & -\eta_4 \end{pmatrix}, \tag{6}$$

$$N_{3 \times 3} = \begin{pmatrix} -\mu_e & -\beta_{33} & 0 \\ 0 & \beta_{33} - \eta_5 & 0 \\ 0 & r_e & -(\mu_e + d_e) \end{pmatrix}. \tag{7}$$

The eigenvalues of (5) are determined by those of (6), (7), it suffices to prove that all eigenvalues of above matrices have negative real parts when  $R_0 < 1$ . The characteristic equation of  $N_{3 \times 3}$  in (7):  $(\lambda + \mu_e)(\lambda + \mu_e + r_e - \beta_{33})(\lambda + \mu_e + d_e) = 0$ , it is easy to show that all eigenvalues of the characteristic equation have negative real part from (3) and (4) when  $R_0^{(1)} < 1$ . Next, we calculate the eigenvalues of the following characteristic equation of  $M_{6 \times 6}$  in (6):

$$(\lambda + \eta_4)(\lambda + \alpha_a + \mu_a)(\lambda + \eta_2)(\lambda + \alpha_y + \mu_y)((\lambda + a_1)(\lambda + a_2) - \beta_{21}(\beta_{12} + \alpha_y)) = 0, \tag{8}$$

where  $a_1 = \eta_1 - \beta_{11}$ ,  $a_2 = \eta_3 - \beta_{22}$ . We claim that all roots of (8) have negative real parts. To achieve this goal, it suffices to show  $a_1, a_2 > 0$ ,  $a_1a_2 - \beta_{21}(\beta_{12} + \alpha_y) > 0$ .

If  $R_0^{(2)} < 1$ , it follows from the expression of  $R_0^{(2)}$  in (4) that

$$\begin{aligned} 1 > R_0^{(2)} &= \frac{1}{2} \left( A_{11} + A_{33} + \sqrt{(A_{11} + A_{33})^2 + 4(A_{31}A_{13} - A_{11}A_{33})} \right) \\ &= \frac{1}{2} \left( A_{11} + A_{33} + \sqrt{(A_{11} - A_{33})^2 + 4A_{31}A_{13}} \right) \\ &\geq \frac{1}{2} \left( A_{11} + A_{33} + \sqrt{(A_{11} - A_{33})^2} \right) \\ &\geq \max\{A_{11}, A_{33}\}, \end{aligned}$$

it implies that  $A_{11} < 1$  and  $A_{33} < 1$ . From (3), we obtain  $a_1 > 0$  and  $a_2 > 0$ .

As  $R_0^{(2)} < 1$ , it implies that  $A_{13}A_{31} < 1 - (A_{11} + A_{33}) + A_{11}A_{33}$ . From (3), we obtain that

$$\begin{aligned} a_1 a_2 - \beta_{21}(\beta_{12} + \alpha_y) &= (\eta_1 - \beta_{11})(\eta_3 - \beta_{22}) - \beta_{21}(\beta_{12} + \alpha_y) \\ &= \eta_1 \eta_3 \left( \left(1 - \frac{\beta_{11}}{\eta_1}\right) \left(1 - \frac{\beta_{22}}{\eta_3}\right) - \frac{\beta_{21}(\beta_{12} + \alpha_y)}{\eta_1 \eta_3} \right) \\ &\geq \eta_1 \eta_3 ((1 - A_{11})(1 - A_{33}) - A_{13}A_{31}) \\ &\geq \eta_1 \eta_3 (1 - (A_{11} + A_{33}) + A_{11}A_{33} - A_{13}A_{31}) > 0. \end{aligned}$$

Above all, the roots of (6) and (7) have negative real parts, it implies that  $E_0$  is locally asymptotically stable. This completes the proof.  $\square$

In the next, we prove the global stability of disease free equilibrium  $E_0$ . For this purpose, we first introduce the following system

$$\begin{cases} \frac{d\mathbb{X}}{dt} = P(\mathbb{X}, \mathbb{I}), \\ \frac{d\mathbb{I}}{dt} = G(\mathbb{X}, \mathbb{I}), \quad G(\mathbb{X}, 0) = 0, \end{cases} \quad (9)$$

where  $\mathbb{X} \in \mathbb{R}^n$  denotes (its components) the compartment of uninfected individuals and  $\mathbb{I} \in \mathbb{R}^m$  denotes (its components) the compartment of infected individuals including latent, infectious, etc.  $U(\mathbb{X}^*, 0)$  denotes the disease-free equilibrium of system (9).

By the similar arguments as those in [4], the following Lemma is valid.

**Lemma 3.3.** *The disease-free equilibrium  $U(\mathbb{X}^*, 0)$  of system (9) is globally asymptotically stable provided that  $R_0 < 1$ , and the following assumptions are satisfied*

(C<sub>1</sub>) *For system  $\frac{d\mathbb{X}}{dt} = P(\mathbb{X}, 0)$ ,  $\mathbb{X}^*$  is globally asymptotically stable,*

(C<sub>2</sub>)  *$\bar{G}(\mathbb{X}, \mathbb{I}) = H\mathbb{I} - G(\mathbb{X}, \mathbb{I})$ ,  $\bar{G}(\mathbb{X}, \mathbb{I}) \geq 0$  for  $(\mathbb{X}, \mathbb{I}) \in \Omega$ ,  $H = D_{\mathbb{I}}G(\mathbb{X}^*, 0)$  (the derivative of  $G(\mathbb{X}, \mathbb{I})$  with respect to  $\mathbb{I}$  at  $U(\mathbb{X}^*, 0)$ ), and  $-H$  is a non-singular M-matrix.*

**Theorem 3.4.** *The disease-free equilibrium  $E_0$  of system (1) is globally asymptotically stable provided that  $R_0 < 1$ .*

*Proof.* In order to prove this result, it suffices to show that system (1) satisfies the above two conditions in Lemma 3.3. System (1) can be re-written in the form of (9) by using the similar method in [4], then  $\mathbb{X} = (S_y, S_a, S_e)$ ,  $\mathbb{I} = (I_y, A_y, I_a, A_a, I_e, A_e)$ .  $U(\mathbb{X}^*, 0) = E_0$  (the disease-free equilibrium of system (1)), where  $\mathbb{X}^* = (S_y^0, S_a^0, S_e^0)$ . Furthermore,  $P(\mathbb{X}, 0), G(\mathbb{X}, \mathbb{I})$  can be expressed as



follows

$$P(\mathbb{X}, 0) = \begin{pmatrix} \Lambda - \alpha_y S_y - \mu_y S_y \\ \alpha_y S_y - \alpha_a S_a - \mu_a S_a \\ \alpha_a S_a - \mu_e S_e \end{pmatrix}, \tag{10}$$

$$G(\mathbb{X}, \mathbb{I}) = \begin{pmatrix} \phi_y S_y - \alpha_y I_y - r_y I_y - \mu_y I_y - \beta_y I_y \\ r_y I_y - \mu_y A_y - d_y A_y - \alpha_y A_y \\ \phi_a S_a + \alpha_y I_y - \alpha_a I_a - r_a I_a - \mu_a I_a - \beta_a I_a \\ \beta_y I_y + r_a I_a + \alpha_y A_y - \mu_a A_a - \alpha_a A_a - d_a A_a \\ \phi_e S_e + \alpha_a I_a - \mu_e I_e - r_e I_e \\ \beta_a I_a + r_e I_e + \alpha_a A_a - \mu_e A_e - d_e A_e \end{pmatrix}. \tag{11}$$

From (10), the global stability of  $\mathbb{X}^*$  of the system  $\frac{d\mathbb{X}}{dt} = P(\mathbb{X}, 0)$  is easy to check. Hence, condition  $(C_1)$  is satisfied.

Next, we prove that model (1) satisfies condition  $(C_2)$ . From (11), the expressions of  $-H, \bar{G}(\mathbb{X}, \mathbb{I})$  are given by

$$-H = \begin{pmatrix} -(\beta_{11} - \eta_1) & 0 & -\beta_{21} & 0 & 0 & 0 \\ -r_y & \eta_2 & 0 & 0 & 0 & 0 \\ -(\beta_{12} + \alpha_y) & 0 & -(\beta_{22} - \eta_3) & 0 & 0 & 0 \\ -\beta_y & -\alpha_y & -r_a & \eta_4 & 0 & 0 \\ 0 & 0 & -(\beta_{23} + \alpha_a) & 0 & -(\beta_{33} - \eta_5) & 0 \\ 0 & 0 & -\beta_a & -\alpha_a & -r_e & \eta_6 \end{pmatrix}, \bar{G}(\mathbb{X}, \mathbb{I}) = \begin{pmatrix} (N_y - S_y)\phi_y \\ 0 \\ (N_a - S_a)\phi_a \\ 0 \\ (N_e - S_e)\phi_e \\ 0 \end{pmatrix}. \tag{12}$$

Since  $0 \leq S_i \leq N_i$  ( $i = y, a, e$ ), it follows from (12) that  $\bar{G}(\mathbb{X}, \mathbb{I}) \geq 0$ . Further, it suffices to show that  $-H$  in (12) is a non-singular M-matrix when  $R_0 < 1$ . In fact, it is obvious that  $-H$  has positive diagonal entries and negative off-diagonal entries. Next, we prove that all eigenvalues of  $-H$  have positive real parts when  $R_0 < 1$ . The characteristic equation of  $-H$  is derived as

$$(\lambda - \eta_2)(\lambda - \eta_4)(\lambda - \eta_6)(\lambda - (\eta_5 - \beta_{33}))(\lambda^2 - (a_1 + a_2)\lambda + a_1 a_2 - \beta_{21}(\beta_{12} + \alpha_y)) = 0. \tag{13}$$

It follows from the process of the proof for Theorem 3.4 that all eigenvalues of (13) have positive real parts. Thus, we obtain that  $-H$  is a non-singular M-matrix. This completes the proof of Theorem 3.4.  $\square$

**4. Permanence of the disease.** In this section, we investigate permanence of the disease for system (1). We first introduce the following notations which will be used throughout this section

$$\begin{aligned} X &= \{(S_y, I_y, A_y, S_a, I_a, A_a, S_e, I_e, A_e) | S_i, I_i, A_i \geq 0, i = y, a, e\}, \\ X_0 &= \{(S_y, I_y, A_y, S_a, I_a, A_a, S_e, I_e, A_e) \in X | I_i, A_i > 0, i = y, a, e\}, \\ \partial X_0 &= X \setminus X_0. \end{aligned}$$

It is obvious that  $X$  and  $X_0$  are positivity invariant of system (1). In the following, we will show that system (1) is uniformly persistent with respect to  $(X_0, \partial X_0)$ . For this purpose, we denote

$$M_\partial = \{(S_i(0), I_i(0), A_i(0)) \in \partial X_0 | \Phi_t(S_i(0), I_i(0), A_i(0)) \in \partial X_0, \forall t \geq 0, i = y, a, e\}, \tag{14}$$

here  $\Phi_t : X \rightarrow X$  is the semiflow defined by system (1). Now, we give the following Lemma

**Lemma 4.1.**  $\{(S_y, 0, 0, S_a, 0, 0, S_e, 0, 0) | S_y, S_a, S_e \geq 0\} = M_\partial$ .

*Proof.* For convenience, let  $\mathbb{M} = \{(S_y, 0, 0, S_a, 0, 0, S_e, 0, 0) | S_y, S_a, S_e \geq 0\}$ . We will show that  $\mathbb{M} \subset M_\partial$  and  $M_\partial \subset \mathbb{M}$ . For the former part, it is obvious, so we will prove the latter part, which means that if  $(S_y(0), I_y(0), A_y(0), S_a(0), I_a(0), A_a(0), S_e(0), I_e(0), A_e(0)) \in M_\partial$ , then  $I_y(0) = A_y(0) = I_a(0) = A_a(0) = I_e(0) = A_e(0) = 0$ . By contradiction, assume at least one of  $I_i(0)$  or  $A_i(0)$  ( $i = y, a, e$ ) is greater than zero, for example,  $I_y(0) > 0$ , then we can get  $A_y, I_a(t), A_a(t), I_e(t)$ , and  $A_e(t)$  are all greater than zero in certain interval, such as  $[0, T_1]$ . In fact, for  $t \in [0, T_1]$ , from the inequality  $\frac{dI_y(t)}{dt}|_{(1)} \geq -\eta_1 I_y(t)$ , we have

$$I_y(t) \geq q_1 > 0, \text{ for } t \in [0, T_1],$$

where  $q_1 = I_y(0)$ . Substituting  $q_1$  into equations for  $I_a(t)$  and  $A_y(t)$ , for  $t \in [0, T_1]$ , yields

$$\frac{dI_a(t)}{dt}|_{(1)} \geq \alpha_y q_1 - \eta_3 I_a(t), \quad \frac{dA_y(t)}{dt}|_{(1)} \geq r_y q_1 - \eta_2 A_y(t),$$

then  $I_a(t) \geq q_2 > 0$ ,  $A_y(t) \geq q_3 > 0$ , for  $t \in [0, T_1]$ , where  $q_2 = \frac{\alpha_y q_1}{\eta_3}(1 - e^{-\eta_3 T_1})$ ,  $q_3 = \frac{r_y q_1}{\eta_2}(1 - e^{-\eta_2 T_1})$ . It follows from the inequality  $\frac{dA_a(t)}{dt}|_{(1)} \geq \beta_y q_1 + r_a q_2 + \alpha_y q_3 - \eta_4 A_a(t)$  that  $A_a(t) \geq \frac{\beta_y q_1 + r_a q_2 + \alpha_y q_3}{\eta_4}(1 - e^{-\eta_4 T_1}) := q_4 > 0$ . Similarly, we have  $I_e(t) \geq \frac{\alpha_a q_2}{\eta_5}(1 - e^{-\eta_5 T_1}) := q_5 > 0$ ,  $A_e(t) \geq \frac{\beta_a q_2 + r_e q_5 + \alpha_a q_4}{\eta_6}(1 - e^{-\eta_6 T_1}) := q_6 > 0$ .

Consequently,  $I_y(0) > 0$  implies that  $A_y > 0$ ,  $I_a(t) > 0$ ,  $A_a(t) > 0$ ,  $I_e(t) > 0$ , and  $A_e(t) > 0$  for  $t \in [0, T_1]$ . From the definition of  $M_\partial$  in (14), we know that any point in  $\partial X_0$  with  $I_y(t) > 0$  can not belong to  $M_\partial$ . The similar idea and procedure show that any point in  $\partial X_0$  other than  $(S_y, 0, 0, S_a, 0, 0, S_e, 0, 0)$  can not belong to  $M_\partial$ . Hence, it is now obvious that Lemma 4.1 holds.  $\square$

**Lemma 4.2.** *The disease-free equilibrium  $E_0$  of system (1) is weak repeller for  $X_0$ , i.e.*

$$\limsup_{t \rightarrow \infty} \text{dist}(\Phi(t), E_0) > 0,$$

where  $\Phi(t) = (S_y(t), I_y(t), A_y(t), S_a(t), I_a(t), A_a(t), S_e(t), I_e(t), A_e(t))$  is an arbitrary solution of system (1) with any initial value in  $X_0$ .

*Proof.* By Leenheer and Smith (Proof of Lemma 3.5 in [19]), we only need prove  $W^s(E_0) \cap X_0 = \emptyset$ , where  $W^s(E_0)$  is stable manifold of  $E_0$ . If it is not true, then there exists a solution  $(S_y, I_y, A_y, S_a, I_a, A_a, S_e, I_e, A_e)$  in  $X_0$  such that

$$S_i(t) \rightarrow S_i^0, \quad I_i(t) \rightarrow 0, \quad A_i(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad i = y, a, e. \quad (15)$$

Thus, by (15) there exists a  $T$  and small enough  $\bar{\varepsilon} > 0$  such that  $A_i(t) < \bar{\varepsilon}$ ,  $I_i(t) < \bar{\varepsilon}$  for all  $t > T$ . The equation

$$\frac{d\tilde{S}_y(t)}{dt} = \Lambda - (\alpha_y + \mu_y)\tilde{S}_y - \bar{\varepsilon}(\beta_{11} + \beta_{21}), \quad (16)$$

has an equilibrium  $\tilde{S}_y^*(\bar{\varepsilon})$ , which is globally stable and  $\lim_{\bar{\varepsilon} \rightarrow 0} \tilde{S}_y^* = S_y^0$ . Then for any solution of (16), there exists a  $T_4 > T$  and small enough positive  $\varepsilon_1$  such that  $\tilde{S}_y(t) \geq \tilde{S}_y^*(\bar{\varepsilon}) - \varepsilon_1$  holds when  $t \geq T_4$ . By the comparison principle, there exists a small enough  $\varepsilon_2 > 0$  such that  $S_y(t) \geq \tilde{S}_y(t) \geq S_y^0 - \varepsilon_2$ .

For the equation

$$\frac{d\tilde{S}_a(t)}{dt} = \alpha_y(S_y^0 - \varepsilon_2) - (\alpha_a + \mu_a)\tilde{S}_a - \bar{\varepsilon}(\beta_{12} + \beta_{22}),$$

there also exists  $T_5 > T_4$  and small enough  $\varepsilon_3$  such that  $S_a(t) \geq \frac{\alpha_y S_y^0}{\mu_a + \alpha_a} - \varepsilon_3 = S_a^0 - \varepsilon_3$  for  $t > T_5$ . The same produce applies to  $S_e(t)$  yielding that  $S_e(t) \geq S_e^0 - \varepsilon_4$  for  $t > T_6$  ( $T_6 > T_5$ ) and small enough positive  $\varepsilon_4$ .

In order to use comparison theorem, we replace  $I_i, A_i$  ( $i = 1, 2, 3$ ) with some new variables  $x_j$  ( $j = 1, 2, 3, 4, 5, 6$ ) in system (1), respectively. Accordingly, for any large enough  $t > 0$ , we consider the following auxiliary system which is associated with  $\mathcal{D}$ .

$$\begin{cases} \frac{dx_1}{dt} = (\beta_{11}x_1 + \beta_{21}x_3) \left(1 - \frac{\varepsilon_2}{S_y^0}\right) - \alpha_y x_1 - r_y x_1 - \mu_y x_1 - \beta_y x_1, \\ \frac{dx_2}{dt} = r_y x_1 - \mu_y x_2 - d_y x_2 - \alpha_y x_2, \\ \frac{dx_3}{dt} = (\beta_{12}x_1 + \beta_{22}x_3) \left(1 - \frac{\varepsilon_3}{S_a^0}\right) + \alpha_y x_1 - \alpha_a x_3 - r_a x_3 - \mu_a x_3 - \beta_a x_3, \\ \frac{dx_4}{dt} = \beta_y x_1 + r_a x_3 + \alpha_y x_2 - \mu_a x_4 - \alpha_a x_4 - d_a x_4, \\ \frac{dx_5}{dt} = (\beta_{23}x_2 + \beta_{33}x_5) \left(1 - \frac{\varepsilon_4}{S_e^0}\right) + \alpha_a x_2 - \mu_e x_5 - r_e x_5, \\ \frac{dx_6}{dt} = \beta_a x_4 + r_e x_5 + \alpha_a x_4 - \mu_e x_6 - d_e x_6, \end{cases} \tag{17}$$

The Jacobian matrix of system (17) at the disease-free equilibrium  $E_0$  is

$$J = J_0 - \frac{\varepsilon_2}{S_y^0} Q_1 - \frac{\varepsilon_3}{S_a^0} Q_2 - \frac{\varepsilon_4}{S_e^0} Q_3,$$

where

$$\begin{aligned} J_0 &= \begin{pmatrix} \beta_{11} - \eta_1 & 0 & \beta_{21} & 0 & 0 & 0 \\ r_y & -\eta_2 & 0 & 0 & 0 & 0 \\ (\beta_{12} + 2\alpha_y) & 0 & \beta_{22} - \eta_3 & 0 & 0 & 0 \\ \beta_y & \alpha_y & r_a & -\eta_4 & 0 & 0 \\ 0 & 0 & (\beta_{23} + 2\alpha_a) & 0 & \beta_{33} - \eta_5 & 0 \\ 0 & 0 & \beta_a & \alpha_a & r_e & -\eta_6 \end{pmatrix}, \\ Q_1 &= \begin{pmatrix} \beta_{11} & 0 & \beta_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{12} & 0 & \beta_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{23} & 0 & 0 & \beta_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

Let  $s(J) = \max\{\Re\lambda \mid \lambda \in \sigma(J)\}$ , where  $\Re\lambda$  denotes the real part of  $\lambda$ , and  $\sigma(J)$  is the set of eigenvalues of matrix  $J$ . From (8), (18), we know that if  $R_0 > 1$ , then  $s(J_0) > 0$ , and  $s(J) > 0$  for  $\varepsilon_2, \varepsilon_3, \varepsilon_4$ . In fact,  $J$  is a quasi-positive matrix (the matrix has nonnegative off-diagonal entries), and there exists a vector  $\rho > 0$  such that  $J\rho = s(J)\rho$  (See Corollary 3.2 in [38]). Then,  $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t)) \rightarrow \infty$  as  $t \rightarrow \infty$ . By the comparison principle, we have  $(I_y(t), A_y(t), I_a(t), A_a(t), I_e(t), A_e(t)) \rightarrow \infty$  as  $t \rightarrow \infty$ , which is a contradiction. This completes the proof of Lemma 4.2.  $\square$

**Theorem 4.3.** *If  $R_0 > 1$ , then system (1) is uniform persistence with respect to  $(X_0, \partial X_0)$ , i.e., there exists  $\varsigma > 0$  such that the solution of system (1) with any initial values in  $X_0$  satisfies*

$$\liminf_{t \rightarrow \infty} I_i(t) > \varsigma, \quad \liminf_{t \rightarrow \infty} A_i(t) > \varsigma, \quad i = y, a, e.$$

*Proof.* It is obvious that both  $X$  and  $X_0$  are positively invariant and  $\Phi_t$  is compact and point dissipative, which implies that there exists a global attractor for  $\Phi_t$ , then condition  $(C_1)$  in [56] is satisfied. Lemma 4.1 implies that  $M_\partial$  is the maximal compact invariant set of  $\Phi_t$  in  $\partial X_0$ . Choosing the Morse decomposition (See Definition 1.2.4 in [56]) of  $M_\partial$  as  $\{E_0\}$ , thus  $\cup_{X \in M_\partial} \omega(X) = \{E_0\}$ , which means  $\{E_0\}$  is isolated. It is easy to see that weak uniform persistence of  $\Phi_t$  in Lemma 4.2, which implies the solution of system (1) from  $X_0$  can not run to the boundary, i.e.,  $W^s(E_0) \cap X_0 = \emptyset$ , then condition  $(C_2)$  is also satisfied. By Theorem 1.3.1 in [56] with  $L = X_0$ , system (1) is uniformly persistent with respect to  $(X_0, \partial X_0)$ . This completes the proof of the Theorem.  $\square$

To sum up, by Lemma 3.1 and Theorem 4.3, we obtain the permanence of the disease.

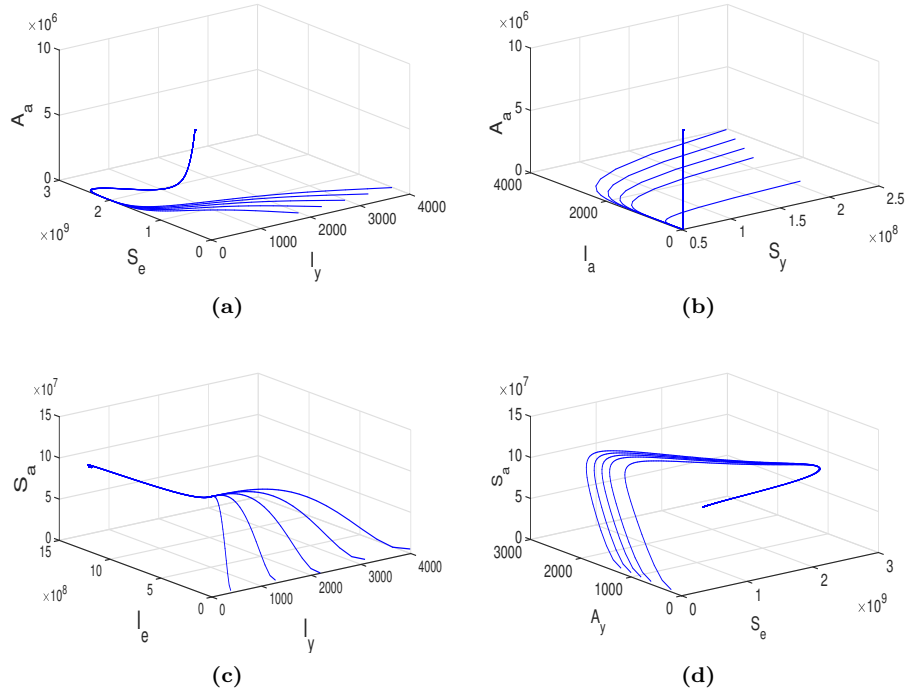
**Theorem 4.4.** *If  $R_0 > 1$ , then system (1) has at least one endemic equilibrium.*

*Proof.* For the existence of endemic equilibrium in system (1), from those equations in system (1), it revealed that  $X_0$  is a positively invariant set for  $\Phi_t$ , so  $\Phi_t(X_0) \subset X_0$  for  $t \geq 0$ . Furthermore,  $\Phi_t$  is point dissipative (it is equal to the solution of system (1) is ultimately bounded, this conclusion has been proved in Lemma 3.1),  $\Phi_t$  is compact for each  $t > 0$ , and  $\Phi_t$  is uniformly persistent in regard to  $(X_0, \partial X_0)$  (it has been obtained in Theorem 4.3). Based on the above results,  $\Phi_t$  has a stationary coexistence state in  $X_0$ , i.e., there exists at least one endemic equilibrium for  $\Phi_t$  (Theorem 1.3.7 in [56]). This completes the proof Theorem 4.4.  $\square$

Figure 3 (a), (b), (c), (d) show that the solutions of system (1) with different initial values eventually converge when  $R_0 > 1$ , which give us the hint that there may be one and only one endemic equilibrium and that it may be globally attractive if  $R_0 > 1$ .

**5. The global attractivity of the endemic equilibrium: A special case.** We first assume that some parameters of system (1) satisfy the following assumption in this section.

**Assumption 5.1.** *Assume that  $\beta_{21} = \beta_{12} = \beta_{23} = \beta_{32} = 0$ ,  $\beta_y < \alpha_y$ , and  $\beta_a < \alpha_a$  in system (1).*



**Figure 3.** Dynamics of the solutions of system (1) when  $R_0 = 1.73415$ , where the variables and parameters are given in Table 1.

**Theorem 5.2.** *If Assumption 5.1 and  $\tilde{R}_0 > 1$  are satisfied, then there exists a unique endemic equilibrium  $E^* = (S_y^*, I_y^*, A_y^*, S_a^*, I_a^*, A_a^*, S_e^*, I_e^*, A_e^*)$  of system (1) and  $E^*$  is globally attractive, where*

$$\tilde{R}_0 = \max\{R_1, R_2, R_3\}, \quad R_1 = \frac{\beta_{11}}{\bar{\eta}_1}, \quad R_2 = \frac{\beta_{22}}{\bar{\eta}_3}, \quad R_3 = \frac{\beta_{33}}{\eta_5},$$

$$\bar{\eta}_1 = r_y + \mu_y + \beta_y, \quad \bar{\eta}_3 = r_a + \mu_a + \beta_a.$$

*Proof.* Substituting  $E^*$  into system (1) under the Assumption 5.1 yields

$$\begin{cases} \Lambda - \alpha_y S_y^* - \mu_y S_y^* - \frac{\beta_{11} I_y^* S_y^*}{N_y^*} = 0, & \frac{\beta_{11} I_y^* S_y^*}{N_y^*} - \alpha_y I_y^* - r_y I_y^* - \mu_y I_y^* - \beta_y I_y^* = 0, \\ r_y I_y^* - \mu_y A_y^* - d_y A_y^* - \alpha_y A_y^* = 0, & \alpha_y S_y^* - \alpha_a S_a^* - \mu_a S_a^* - \frac{\beta_{22} I_a^* S_a^*}{N_a^*} = 0, \\ \frac{\beta_{22} I_a^* S_a^*}{N_a^*} - r_a I_a^* - \mu_a I_a^* - \beta_a I_a^* = 0, & \beta_y I_y^* + r_a I_a^* + \alpha_y A_y^* - \mu_a A_a^* - \alpha_a A_a^* - d_a A_a^* = 0, \\ \alpha_a S_a^* - \mu_e S_e^* - \frac{\beta_{33} I_e^* S_e^*}{N_e^*} = 0, & \frac{\beta_{33} I_e^* S_e^*}{N_e^*} - \mu_e I_e^* - r_e I_e^* = 0, \\ \beta_a I_a^* + r_e I_e^* + \alpha_a A_a^* - \mu_e A_e^* - d_e A_e^* = 0, \end{cases} \tag{19}$$

where  $N_i^* = S_i^* + I_i^* + A_i^*$ ,  $i = y, a, e$ . After a careful calculation for the first, second and third equations of (19), we obtain

$$A_y^* = \frac{r_y I_y^*}{\eta_2}, I_y^* = \frac{\Lambda - (\alpha_y + \mu_y)S_y^*}{\bar{\eta}_1}, S_y^* = \frac{\Lambda}{\frac{\bar{\eta}_1(R_1-1)}{1+r_y/\eta_2} + \alpha_y}.$$

Then  $S_a^*, A_a^*$  can be derived from the forth, fifth and sixth equations of (19), namely,  $A_a^* = (\beta_y I_y^* + r_a I_a^* + \alpha_y A_y^*)/\eta_4$ ,  $S_a^* = (\alpha_y S_y^* + \alpha_y I_y^* + \bar{\eta}_3)/(\alpha_a + \mu_a)$ , where  $I_a^*$  is the root of the following quadratic equation

$$a_0(I_a^*)^2 + a_1 I_a^* + a_2 = 0, \tag{20}$$

where

$$\begin{aligned} a_0 &= \bar{\eta}_3 \left( \frac{\beta_{22}}{\alpha_a + \mu_a} + 1 + \frac{r_a}{\eta_4} - \frac{\bar{\eta}_3}{\alpha_a + \mu_a} \right) > 0, \\ a_1 &= - \left( \frac{\beta_{22}\alpha_y S_y^*}{\alpha_a + \mu_a} + \frac{\bar{\eta}_3\beta_{12}I_y^*}{\alpha_a + \mu_a} + \left( \frac{\beta_y I_y^* + \alpha_y A_y^*}{\eta_4} + \frac{\alpha_y S_y^*}{\alpha_a + \mu_a} \right) \bar{\eta}_3 \right), \\ a_2 &= - \left( \beta_{12} + \alpha_y \left( 1 + \frac{\beta_y I_y^* + \alpha_y A_y^*}{\eta_4} \right) \right) < 0. \end{aligned}$$

It is obvious that equation (20) has a unique positive root  $I_a^*$ , hence  $A_a^*$  is positive. Similarly,  $S_e^*, I_e^*, A_e^*$  can be determined. Thus, we complete the proof of the existence and uniqueness of the endemic equilibrium.

Next, we need to show that  $E^*$  is globally attractive when  $\tilde{R}_0 > 1$ . For this purpose, we construct a Lyapunov function  $V(t) = V_y(t) + V_a(t) + V_e(t)$ , where

$$\begin{aligned} V_i(t) &= N_i - N_i^* - N_i^* \ln \frac{N_i}{N_i^*} + \frac{(\beta_i + \alpha_i + 2\mu_i)N_i^*}{\beta_{kk}(I_i^* + A_i^*)} \left( I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*} \right) \\ &\quad + \frac{(\beta_i + \alpha_i + 2\mu_i)}{2r_i} \left( 1 + \frac{S_i^*}{I_i^* + A_i^*} \right) \frac{(A_i - A_i^*)^2}{N_i}, \quad i = y, a, e, k = 1, 2, 3. \end{aligned}$$

The time derivative of  $V_y(t)$  computed along the solutions of system (1) under Assumption 5.1 is

$$\begin{aligned} \frac{dV_y(t)}{dt} &= \left( 1 - \frac{N_y^*}{N_y} \right) (\Lambda - (\alpha_y + \mu_y)S_y - (\mu_y + \beta_y)I_y - (\mu_y + d_y + \alpha_y)A_y) \\ &\quad + \frac{(I_y - I_y^*)(\beta_y + \alpha_y + 2\mu_y)N_y^*}{(I_y^* + A_y^*)} \left( \frac{S_y}{N_y} - \frac{S_y^*}{N_y^*} \right) + \frac{(\beta_y + \alpha_y + 2\mu_y)(A_y - A_y^*)N_y^*}{r_y N_y (I_y^* + A_y^*)} \frac{dA_y}{dt} \\ &\quad - \frac{(\beta_y + \alpha_y + 2\mu_y)}{2r_y} \left( \frac{N_y^*}{I_y^* + A_y^*} \right) \frac{(A_y - A_y^*)^2}{(N_y)^2} \frac{dN_y}{dt}. \end{aligned}$$

Using (19) and noting that

$$\begin{aligned} \frac{S_y}{N_y} - \frac{S_y^*}{N_y^*} &= \frac{(I_y^* + A_y^*)(S_y - S_y^*) - S_y^*(I_y - I_y^*) - S_y^*(A_y - A_y^*)}{N_y N_y^*}, \\ \Lambda_y &= (\alpha_y + \mu_y)S_y^* + (\mu_y + \beta_y + r_y)I_y^*, \\ \frac{dA_y}{dt} &= r_y(I_y - I_y^*) - (\mu_y + d_y + \alpha_y)(A_y - A_y^*), \end{aligned}$$

we have

$$\begin{aligned} \frac{dV_y(t)}{dt} &= -(\alpha_y + \mu_y) \frac{(S_y - S_y^*)^2}{N_y} - (\mu_y + d_y + \alpha_y) \frac{(A_y - A_y^*)^2}{N_y} \\ &\quad - \frac{(\beta_y + \alpha_y + 2\mu_y)}{2r_y(N_y^*)^2} \left(1 + \frac{S_y^*}{I_y^* + A_y^*}\right) (A_y - A_y^*)^2 \left(2(\mu_y + d_y + \alpha_y)N_1 + \frac{dN_y}{dt}\right) \\ &\quad - \left(\mu_y + \beta_y + \frac{S_y^*(\beta_y + \alpha_y + 2\mu_y)}{N_y(I_y^* + A_y^*)}\right) \frac{(I_y - I_y^*)^2}{N_y} \\ &= -(\alpha_y + \mu_y) \frac{(S_y - S_y^*)^2}{N_y} - (\mu_y + d_y + \alpha_y) \frac{(A_y - A_y^*)^2}{N_y} \\ &\quad - \left(\mu_y + \beta_y + \frac{S_y^*(\beta_y + \alpha_y + 2\mu_y)}{N_y(I_y^* + A_y^*)}\right) \frac{(I_y - I_y^*)^2}{N_y} \\ &\quad - \frac{(\beta_y + \alpha_y + 2\mu_y)}{2r_y(N_y^*)^2} \left(1 + \frac{S_y^*}{I_y^* + A_y^*}\right) (A_y - A_y^*)^2 \Theta(t), \end{aligned}$$

where  $\Theta(t) = \Lambda_y + (\alpha_y + \mu_y)(S_y + A_y) + (\mu_y + 2\alpha_y - \beta_y)I_y$ . It is obvious that  $\frac{dV_y}{dt} \leq 0$ , and the equality  $\frac{dV_y}{dt} = 0$  holds if and only if  $S_y = S_y^*$ ,  $I_y = I_y^*$ ,  $A_y = A_y^*$ . Hence

$$\lim_{t \rightarrow \infty} S_y(t) = S_y^*, \quad \lim_{t \rightarrow \infty} I_y(t) = I_y^*, \quad \lim_{t \rightarrow \infty} A_y(t) = A_y^*. \tag{21}$$

The time derivative of  $V_a(t)$  computed along the solutions of system (1) under Assumption 5.1 is

$$\begin{aligned} \frac{dV_a(t)}{dt} &= \left(1 - \frac{N_a^*}{N_a}\right) (\alpha_y(S_y - S_y^*) + (\alpha_a + \mu_a)(S_a - S_a^*) \\ &\quad + (\mu_a + \beta_a)(I_a - I_a^*) + \beta_y(I_y - I_y^*) + \alpha_y(A_y - A_y^*)) \\ &\quad + \frac{(\beta_a + \alpha_a + 2\mu_a)N_a^*}{(I_a^* + A_a^*)} \left(\frac{S_a}{N_a} - \frac{S_a^*}{N_a^*}\right) \\ &\quad + \frac{(\beta_a + \alpha_a + 2\mu_a)}{r_a N_a} \left(1 + \frac{S_a^*}{I_a^* + A_a^*}\right) (A_a - A_a^*) \frac{dA_a}{dt} \\ &\quad - \frac{(\beta_a + \alpha_a + 2\mu_a)}{2r_a} \left(1 + \frac{S_a^*}{I_a^* + A_a^*}\right) \frac{(A_a - A_a^*)^2}{(N_a)^2} \frac{dN_a}{dt}, \\ &= -(\alpha_a + \mu_a) \frac{(S_a - S_a^*)^2}{N_a} - \left(\mu_a + \beta_a + \frac{S_a^*(\beta_a + \alpha_a + 2\mu_a)}{N_a(I_a^* + A_a^*)}\right) \frac{(I_a - I_a^*)^2}{N_a} \\ &\quad - (\mu_a + d_a + \alpha_a) \frac{(A_a - A_a^*)^2}{N_a} - \frac{(\beta_a + \alpha_a + 2\mu_a)}{2r_a(N_a^*)(I_a^* + A_a^*)} (A_a - A_a^*)^2 \Gamma_1(t) \\ &\quad + \frac{(S_a - S_a^*) + (I_a - I_a^*) + (A_a - A_a^*)}{N_a} \Gamma_2(t) \\ &\quad + \frac{(\beta_a + \alpha_a + 2\mu_a)}{r_a(N_a^*)^2} \left(1 + \frac{S_a^*}{I_a^* + A_a^*}\right) (A_a - A_a^*) \Gamma_3(t), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Gamma_1(t) &= \alpha_y S_y + \beta_y I_y + \alpha_y A_y + (\alpha_a + \mu_a)(S_a + A_a) + (\mu_a + 2\alpha_a - \beta_a)I_a, \\ \Gamma_2(t) &= \alpha_y((S_y - S_y^*) + \beta_y(I_y - I_y^*) + \alpha_y(A_y - A_y^*)), \\ \Gamma_3(t) &= \beta_y(I_y - I_y^*) + r_a(I_a - I_a^*) + \alpha_y(A_y - A_y^*). \end{aligned} \tag{23}$$

In fact, if (21) holds, then there exists a large enough  $t_1$  such that  $\frac{dV_a}{dt} \leq 0$  for  $t > t_1$ , and the equality  $\frac{dV_a}{dt} = 0$  holds if and only if  $S_a = S_a^*$ ,  $I_a = I_a^*$ ,  $A_a = A_a^*$ . Hence

$$\lim_{t \rightarrow \infty} S_a(t) = S_a^*, \lim_{t \rightarrow \infty} I_a(t) = I_a^*, \lim_{t \rightarrow \infty} A_a(t) = A_a^*.$$

Similarly, it follows from Eq. (22)-(23) that

$$\begin{aligned} \frac{dV_e(t)}{dt} &= -\frac{\mu_e(S_e - S_e^*)^2}{N_e} - \left( \mu_e + \beta_a + \frac{2S_e^*\mu_a}{N_e(I_e^* + A_e^*)} \right) \frac{(I_e - I_e^*)^2}{N_a} - (\mu_e + d_e) \frac{(A_e - A_e^*)^2}{N_e} \\ &\quad - \frac{2\mu_e}{2r_e(N_e^*)^2} \left( 1 + \frac{S_e^*}{I_e^* + A_e^*} \right) (A_e - A_e^*)^2 (\mu_e N_e + \alpha_a S_a + \beta_a I_a + \alpha_a A_a) \\ &\quad + \frac{(S_e - S_e^*) + (I_e - I_e^*) + (A_e - A_e^*)}{N_e} (\alpha_a(S_a - S_a^*) + \beta_a(I_a - I_a^*) + \alpha_a(A_a - A_a^*)) \\ &\quad + \frac{(\beta_a + 2\mu_e)}{r_e(N_e^*)^2} \left( 1 + \frac{S_e^*}{I_e^* + A_e^*} \right) (A_e - A_e^*) (\beta_a(I_a - I_a^*) + r_e(I_e - I_e^*) + \alpha_a(A_a - A_a^*)), \end{aligned}$$

and there exists  $t_2 > t_1$  such that  $\frac{dV_e}{dt} \leq 0$  for  $t > t_2$ , with  $\frac{dV_e}{dt} = 0$  at  $S_e = S_e^*$ ,  $I_e = I_e^*$ ,  $A_e = A_e^*$ . Hence,  $\lim_{t \rightarrow \infty} S_e(t) = S_e^*$ ,  $\lim_{t \rightarrow \infty} I_e(t) = I_e^*$ ,  $\lim_{t \rightarrow \infty} A_e(t) = A_e^*$ . Above all, we claim that  $E^*$  is globally attractive. This completes the proof of the Theorem.  $\square$

One of the most significant concerns about HIV is its ability to infect susceptible population, we show in Theorem 3.4 that the population remains in the absence of HIV and the disease will be eventually extinct if  $R_0 < 1$ , which is a critical threshold to determine the condition for the disease outbreak. In Theorem 4.3, we claim that the disease will be always persistent when  $R_0 > 1$ . Finally, the disease probably becomes epidemic at certain level, which corresponds to the endemic equilibrium in Theorem 4.4 and Theorem 5.2.

**6. Optimal control.** In this section, we propose and analyze an optimal control problem applied to the HIV dynamics described by system (1). Compared with new youth cases, the new elderly infected cases increase rapidly, which is shown in Table 2. The proportion of the elderly cases in total increases from 6.30% in 2005 to 30.35% in 2017. This indicates that older people have increasingly risk of HIV/AIDS infection. Further, in the aged group, the condom use rate just reaches 15% or less [14] at the time of sexual behavior, which increases HIV infection in the elderly people group dramatically. The increasing number of HIV infected individuals moves to and becomes AIDS patients also results in the raise of elderly HIV/AIDS infected cases [28, 27]. Hence, some measures must be taken for the HIV transmission of the elderly. For this purpose, we take optimal control measures to reduce the number of HIV infected and AIDS patients among elderly, namely our goal here is to find one optimal control strategy for  $I_a$  individuals move to  $A_e$  individuals that will minimize the transfer rate, the other optimal control strategy for decrease HIV transmission, such as increases condom use. We introduce time-dependent controls  $u_1(t)$  and  $u_2(t)$ . This results in the following system



$$\left\{ \begin{aligned} \frac{dS_y(t)}{dt} &= \Lambda - \alpha_y S_y(t) - \mu_y S_y(t) - \phi_y(t) S_y(t), \\ \frac{dI_y(t)}{dt} &= \phi_y(t) S_y(t) - \alpha_y I_y(t) - r_y I_y(t) - \mu_y I_y(t) - \beta_y I_y(t), \\ \frac{dA_y(t)}{dt} &= r_y I_y(t) - \mu_y A_y(t) - d_y A_y(t) - \alpha_y A_y(t), \\ \frac{dS_a(t)}{dt} &= \alpha_y S_y(t) - \alpha_a S_a(t) - \mu_a S_a(t) - \phi_a(t) S_a(t), \\ \frac{dI_a(t)}{dt} &= \phi_a(t) S_a(t) + \alpha_y I_y(t) - \alpha_a I_a(t) - r_a I_a(t) - \mu_a I_a(t) - \beta_a(1 - u_2(t)) I_a(t), \\ \frac{dA_a(t)}{dt} &= \beta_y I_y(t) + r_a I_a(t) + \alpha_y A_y(t) - \mu_a A_a(t) - \alpha_a A_a(t) - d_a A_a(t), \\ \frac{dS_e(t)}{dt} &= \alpha_a(t) S_a(t) - \mu_e S_e(t) - \phi_e(t) S_e(t) (1 - u_1(t)), \\ \frac{dI_e(t)}{dt} &= \phi_e(t) S_e(t) (1 - u_1(t)) + \alpha_a I_a(t) - \mu_e I_e(t) - r_e I_e(t), \\ \frac{dA_e(t)}{dt} &= \beta_a(1 - u_2(t)) I_a(t) + r_e I_e(t) + \alpha_a A_a(t) - \mu_e A_e(t) - d_e A_e(t), \end{aligned} \right. \tag{24}$$

where the parameters are as defined in Table (1). The control functions  $u_1$  and  $u_2$  are bounded Lebesgue integrable functions and represent the condom use and remove from  $I_e(t)$  to  $A_e(t)$  suppressing control measures, respectively. The coefficient  $(1 - u_1(t))$ , represents the control effect that reduces HIV transmission of elderly, while the coefficient  $(1 - u_2(t))$  gives the effect of treatment that reduces the transfer from  $I_e(t)$  to  $A_e(t)$ . If  $u_1 = 0, u_2 \neq 0$ , then there is only enhance treatment for HIV infection elderly and no inhibition of transmission. If  $u_1 \neq 0, u_2 = 0$ , then there is only inhibition of transmission and no control measure for enhance treatment of HIV infection individuals among elderly. If  $u_1 = 0$  and  $u_2 = 0$ , then there is no inhibition of transmission and remove from  $I_e(t)$  to  $A_e(t)$ .

Note that for nonnegative initial conditions and bounded Lebesgue measurable controls, the state system admits nonnegative bounded solutions. We formulate an similar form in [34, 32] objective functional for the control system (24), with the goal of minimizing HIV infection individuals and AIDs patients among elderly over a finite time horizon  $[0, t_f]$

$$J(I_e, A_e, u) = \int_0^{t_f} \left( w_1 I_e^2(t) + w_2 A_e^2(t) + \frac{w_3}{2} u_1^2(t) + \frac{w_4}{2} u_2^2(t) \right) dt, \tag{25}$$

where  $u(t) = (u_1(t), u_2(t))$ , positive constants  $w_1$  and  $w_2$  represent the balancing factors associated to the total numbers of new HIV infections, AIDs patients among the elderly group, respectively. The balancing factors associated to the cost component  $u_1^2(t)$  and  $u_2^2(t)$ , are denoted by a positive constant  $w_3$  and  $w_4$ , respectively.

We consider the following set of admissible (bounded) control functions

$$\Omega = \{u = (u_1, u_2) \in L^\infty(0, t_f) \times L^\infty(0, t_f) | u_1(t), u_2(t) \in [0, 1], \forall t \in [0, t_f]\},$$

where  $u_1(t)$  and  $u_2(t)$  are Lebesgue measurable with upper bounds. Thus, the optimal control problem consists of determining the vector function  $(\bar{S}_y, \bar{I}_y, \bar{A}_y, \bar{S}_a, \bar{I}_a, \bar{A}_a, \bar{S}_e, \bar{I}_e, \bar{A}_e)$  associated with an admissible control pair  $u_1^*, u_2^*$  on the

time interval  $[0, t_f]$  minimizing the objective functional (25), i.e.,

$$J(I_e(u^*), A_e(u^*), u^*) = \min_{\Omega} J(I_e(u), A_e(u), u). \tag{26}$$

According to the Pontryagin Maximum Principle [35], the Hamiltonian is constructed from system (24) with the underlying state dynamic attached

$$\begin{aligned} \mathcal{H} = & w_1 I_e^2(t) + w_2 A_e^2(t) + \frac{w_3}{2} (u_1)^2 + \frac{w_4}{2} (u_2)^2 + \lambda_1 \frac{dS_y}{dt} + \lambda_2 \frac{dI_y}{dt} + \lambda_3 \frac{dA_y}{dt} \\ & + \lambda_4 \frac{dS_a}{dt} + \lambda_5 \frac{dI_a}{dt} + \lambda_6 \frac{dA_a}{dt} + \lambda_7 \frac{dS_e}{dt} + \lambda_8 \frac{dI_e}{dt} + \lambda_9 \frac{dA_e}{dt}. \end{aligned}$$

**Theorem 6.1.** *There exists a pair of optimal controls  $u_1^*$ ,  $u_2^*$  and corresponding solution vector  $(\bar{S}_y, \bar{I}_y, \bar{A}_y, \bar{S}_a, \bar{I}_a, \bar{A}_a, \bar{S}_e, \bar{I}_e, \bar{A}_e)$  that minimizes  $J(I_e, A_e, u)$  over  $\Omega$ . Furthermore, there exist adjoint functions  $\bar{\lambda}_k$ ,  $k = 1, 2, \dots, 9$ , with transversality conditions  $\bar{\lambda}_k(t_f) = 0$ ,  $k = 1, 2, \dots, 9$ , as follows:*

$$\left\{ \begin{aligned} \frac{d\lambda_1}{dt} &= (\alpha_y + \mu_y) \bar{\lambda}_1 + (\bar{\lambda}_1 - \bar{\lambda}_2) f_1 - \bar{\lambda}_4 \alpha_y, \\ \frac{d\lambda_2}{dt} &= (\bar{\lambda}_1 - \bar{\lambda}_2) \beta_{11} \bar{S}_y f_2 + \bar{\lambda}_2 (\alpha_y + r_y + \mu_y + \beta_y) - r_y \bar{\lambda}_3 - \beta_y \bar{\lambda}_6 \\ &\quad + (\bar{\lambda}_4 - \bar{\lambda}_5) \beta_{12} f_3 - \alpha_y \bar{\lambda}_5, \\ \frac{d\lambda_3}{dt} &= (\bar{\lambda}_2 - \bar{\lambda}_1) f_4 + (\mu_y + d_y + \alpha_y) \bar{\lambda}_3 - \alpha_y \bar{\lambda}_6, \\ \frac{d\lambda_4}{dt} &= (\alpha_a + \mu_a) \bar{\lambda}_4 + (\bar{\lambda}_4 - \bar{\lambda}_5) f_5 - \alpha_a \bar{\lambda}_7, \\ \frac{d\lambda_5}{dt} &= (\alpha_a + r_e + \mu_e + \beta_a (1 - u_2(t))) \bar{\lambda}_5 + (\bar{\lambda}_4 - \bar{\lambda}_5) f_6 - r_a \bar{\lambda}_6 \\ &\quad - \alpha_a \bar{\lambda}_8 - \beta_a (1 - u_2(t)) \bar{\lambda}_9 + (\bar{\lambda}_1 - \bar{\lambda}_2) f_7 + (\bar{\lambda}_7 - \bar{\lambda}_8) f_8, \\ \frac{d\lambda_6}{dt} &= (\mu_a + d_a + \alpha_a) \bar{\lambda}_6 - \alpha_a \bar{\lambda}_9 + (\bar{\lambda}_5 - \bar{\lambda}_4) f_9, \\ \frac{d\lambda_7}{dt} &= \mu_e \bar{\lambda}_7 + (\bar{\lambda}_7 - \bar{\lambda}_8) f_{10}, \\ \frac{d\lambda_8}{dt} &= (\bar{\lambda}_7 - \bar{\lambda}_8) f_{11} + (\mu_e + r_e) \bar{\lambda}_8 - r_e \bar{\lambda}_9 + 2w_1 \bar{I}_e, \\ \frac{d\lambda_9}{dt} &= (\mu_e + d_e) \bar{\lambda}_9 + 2w_2 \bar{A}_e + (\bar{\lambda}_8 - \bar{\lambda}_7) f_{12}, \end{aligned} \right.$$

where  $\bar{N}_i = \bar{S}_i + \bar{I}_i + \bar{A}_i$ ,  $i = y, a, e$ .

$$\begin{aligned} f_1 &= \frac{(\beta_{11} \bar{I}_y + \beta_{21} \bar{I}_a) (\bar{N}_y - \bar{S}_y)}{\bar{N}_y^2}, \quad f_2 = \frac{\bar{N}_y - \bar{I}_y}{\bar{N}_y^2}, \quad f_3 = \frac{\bar{S}_a}{\bar{N}_a}, \quad f_4 = \frac{(\beta_{11} \bar{I}_y + \beta_{21} \bar{I}_a) \bar{S}_y}{\bar{N}_y^2}, \\ f_5 &= \frac{(\beta_{12} \bar{I}_y + \beta_{22} \bar{I}_a) (\bar{N}_a - \bar{S}_a)}{\bar{N}_a^2}, \quad f_6 = \frac{\beta_{22} \bar{S}_a \bar{N}_a - (\beta_{12} \bar{I}_y + \beta_{22} \bar{I}_a) \bar{S}_a}{\bar{N}_a^2}, \quad f_7 = \frac{\beta_{21} \bar{S}_y}{\bar{N}_y}, \\ f_8 &= \frac{\beta_{23} (1 - u_1(t)) \bar{S}_e}{\bar{N}_e}, \quad f_9 = \frac{(\beta_{12} \bar{I}_y + \beta_{22} \bar{I}_a) \bar{S}_a}{\bar{N}_a^2}, \quad f_{10} = \frac{(\beta_{33} \bar{I}_e + \beta_{23} \bar{I}_a) (1 - u_1(t)) (\bar{N}_e - \bar{S}_e)}{\bar{N}_e^2}, \\ f_{11} &= \frac{\beta_{33} (1 - u_1(t)) \bar{S}_e (\bar{N}_e - \beta_{23} \bar{I}_a - \beta_{33} \bar{I}_e)}{\bar{N}_e^2}, \quad f_{12} = \frac{(\beta_{23} \bar{I}_a + \beta_{33} \bar{I}_e) (1 - u_1(t)) \bar{S}_e}{\bar{N}_e^2}. \end{aligned}$$

Also, the optimal control functions  $u_1^*, u_2^*$  are given by

$$\begin{aligned} u_1^*(t) &= \min \left\{ \max \left\{ 0, \frac{\bar{S}_e}{w_3 \bar{N}_e} (\bar{\lambda}_8 - \bar{\lambda}_7) (\beta_{23} \bar{I}_a + \beta_{33} \bar{I}_e) \right\}, 1 \right\}, \\ u_2^*(t) &= \min \left\{ \max \left\{ 0, \frac{\beta_a}{w_4} (\bar{\lambda}_9 - \bar{\lambda}_5) \bar{I}_a \right\}, 1 \right\}. \end{aligned} \tag{27}$$

*Proof.* The existence of optimal control  $u(t)$  due to the convexity of the integrand of (25) with respect to  $u_1(t), u_2(t)$ , a priori boundedness of the state solutions is given by Corollary 4.1 [13], and the Lipschitz property of the state system with respect to the state variables. We obtain the adjoint and control system by Pontryagin’s Maximum principle.

The control system

$$\begin{aligned} \frac{dS_y}{dt} &= \frac{\partial \mathcal{H}}{\partial \lambda_1}, \quad \frac{dI_y}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_2}, \quad \frac{dA_y}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_3}, \quad \frac{dS_a}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_4}, \quad \frac{dI_a}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_5}, \\ \frac{dA_a}{dt} &= \frac{\partial \mathcal{H}}{\partial \lambda_6}, \quad \frac{dS_e}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_7}, \quad \frac{dI_e}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_8}, \quad \frac{dA_e}{dt} = \frac{\partial \mathcal{H}}{\partial \lambda_9}. \end{aligned}$$

The adjoint system

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial S_y}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial I_y}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial \mathcal{H}}{\partial A_y}, \quad \frac{d\lambda_4}{dt} = -\frac{\partial \mathcal{H}}{\partial S_a}, \quad \frac{d\lambda_5}{dt} = -\frac{\partial \mathcal{H}}{\partial I_a}, \\ \frac{d\lambda_6}{dt} &= -\frac{\partial \mathcal{H}}{\partial A_a}, \quad \frac{d\lambda_7}{dt} = -\frac{\partial \mathcal{H}}{\partial S_e}, \quad \frac{d\lambda_8}{dt} = -\frac{\partial \mathcal{H}}{\partial I_e}, \quad \frac{d\lambda_9}{dt} = -\frac{\partial \mathcal{H}}{\partial A_e}, \end{aligned}$$

with zero final time conditions. To get the characterizations of the optimal control given by (27), we obtain

$$\frac{\partial \mathcal{H}}{\partial u_1} = 0, \quad \frac{\partial \mathcal{H}}{\partial u_2} = 0.$$

□

Using the bounds on the controls, we obtain the desired characterization (27). We study numerically the optimal control problem (26) in the next section.

**7. Application to the control of HIV/AIDS among the aged group in China.** In this section, we use model (1) to predict the HIV/AIDS trend among three-age-classes in China. Firstly, we collect and settle the HIV/AIDS data in China, estimate the parameters and initial values of model (1). Secondly, we perform some sensitivity analyses to analyze the effects of the parameter values on the basic reproduction of system (1). At last, we solve the optimal control problem in (25) by numerical simulation.

**7.1. Estimation of the model parameters and initial values.** National death rates of youth, adult, and elderly are respective taken to be  $\mu_y = 0.765$ ,  $\mu_a = 1.852$ , and  $\mu_e = 2.07$  in [7]. Average incubation period of youth, adult, and elderly are respective taken to be  $r_y = 1/12.5, r_a = 1/10, r_e = 1/7.9$  per year in [1], corresponding to the incubation expectancies of youth, adult, and elderly equal to  $1/r_y = 12.5, 1/r_a = 10, 1/r_e = 7.9$  years, respectively.

AIDS-related death rate among youth, adult, and elderly are respective assumed to be  $d_y = 2.3\%, d_a = 9.7\%, d_e = 16\%$  in [12] and [52].

Condom use rates among youth, adult, and elderly are taken to be  $c_{11} = c_{12} = 57.5\%, c_{21} = c_{22} = c_{23} = 34.5\%, c_{33} = 20.0\%$ , respectively. (see [51] and [17]).

Infection rate of  $i$  group from youth, adult, and elderly are respective chosen to be  $\delta_{11} = \delta_{12} = 0.012\%$ ,  $\delta_{21} = 2.8\%$ ,  $\delta_{22} = 4.2\%$ ,  $\delta_{23} = 6.6\%$ ,  $\delta_{33} = 7.2\%$  in [23], [42], and [21]. Those papers all focus on statistic and Mate analysis based on the real demographic and epidemiological data in China, hence our parameter values are reasonable and credible.

The annual average birth rate is 12.4% over one thousand in 2005, which is obtained from the National Bureau of Statistics of China [7], then recruiting number of susceptible youth is  $248016514 \times 1.24\% = 3075405$ , namely,  $\Lambda = 3075405$ .

Notice that all the parameters in Table 1 have already been fixed except  $\alpha_y$ ,  $\alpha_a$ ,  $\beta_y$ ,  $\beta_a$ . In order to estimate those parameters, we simulate system (1) starting from the initial conditions.

The population of youth, adult, and elder groups are 248020700, 583080200 and 337676900 in 2005, respectively, which is obtained from the National Bureau of Statistics of China [7]. Thus, we have the initial values of susceptible:  $S_y(0) = 248020700 - 4186 = 248016514$ ,  $S_a(0) = 583080200 - 33430 = 583046770$ ,  $S_e(0) = 337676900 - 2563 = 337674337$ .

From [48, 5], we have the annual HIV/AIDS cases in 2005 among youth, elderly populations:  $I_y(0) = 3980$ ,  $A_y(0) = 1206$ ,  $I_e(0) = 2238$ ,  $A_e(0) = 1199$ . Furthermore, the total annual HIV and AIDS cases are 33161, and 7550 in 2005 [43], respectively. Hence, we obtain  $I_a(0) = 26943$ ,  $A_a(0) = 5145$ .

A relatively good fit to new HIV/AIDS infection cases among youth, adult, and elderly (see Figure 5 and Figure 4) was obtain with  $\alpha_y = 0.025$ ,  $\alpha_a = 0.043$ ,  $\beta_y = 0.03$ ,  $\beta_a = 0.041$ .

All the parameter values have now been fixed and are summarized in Table 1.

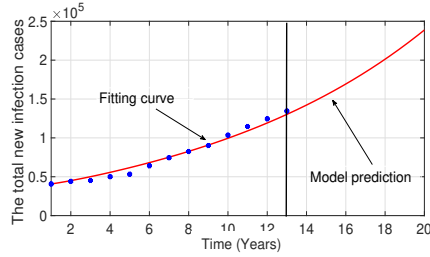
**Table 1.** The parameters and numerical values

Parameters	Description	Range (%)	Value ( $year^{-1}$ )	Source
$\Lambda$	Recruitment of the youth class	-	3075405	Assume
$\mu_y$	Natural death rate of youth	[0.066-0.087]	0.765‰	[7]
$\mu_a$	Natural death rate of adult	[0.1-0.327]	1.852‰	[7]
$\mu_e$	Natural death rate of elderly	[0.19-0.45]	2.07‰	[7]
$r_y$	Average remove rate from $I_y$ to $A_y$	-	1/12.5	[1]
$r_a$	Average remove rate from $I_a$ to $A_a$	-	1/10	[1]
$r_e$	Average remove rate from $I_e$ to $A_e$	-	1/7.9	[1]
$d_y$	AIDS-related death rate among youth	[1.0-4.3]	2.3%	[12]
$d_a$	AIDS-related death rate among adult	[5.03-12.1]	9.7%	[12]
$d_e$	AIDS-related death rate among elderly	[12.1-17.5]	16%	[52]
$c_{11}, c_{12}$	Condom use rate of youth	[34-68]	57.5%	[51]
$c_{21}, c_{22}, c_{23}$	Condom use rate of adult	[30-40]	34.5%	[51]
$c_{33}$	Condom use rate among elderly	[17.5-37.5]	20.0%	[17]
$\alpha_y$	Transfer rate of youth group	-	0.043	Fit
$\alpha_a$	Transfer rate of adult group	-	0.031	Fit
$\delta_{11}, \delta_{12}$	Infected rate among youth	[0.012-0.065]	0.014%	[23]
$\delta_{21}$	Infected rate from adult to young	[0.13 - 9.9]	2.8%	[42]
$\delta_{22}$	Infected rate among adult	[0.13 - 9.9]	4.2%	[42]
$\delta_{23}$	Infected rate from adult to elderly	[0.13 - 9.9]	6.6%	[42]
$\delta_{33}$	Infected rate among elderly	[2.7-30.6]	25%	[21]
$\beta_y$	Transfer rate from $I_y$ to $A_a$	-	0.036	Fit
$\beta_a$	Transfer rate from $I_a$ to $A_e$	-	0.041	Fit

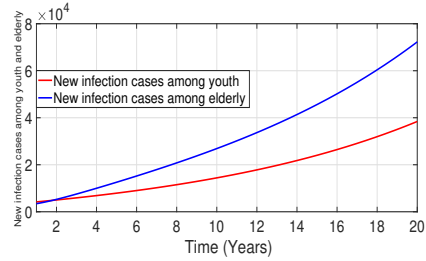
The annual HIV/AIDS reported data in China from 2005 ( $t = 0$ ) to 2017 in Table 1 show that the HIV/AIDS cases among elderly population increase rapidly. We use system (1) to simulate youth, the elderly and total HIV/AIDS cases, the numerical fitted curves of HIV/AIDS cases based on the statistical data in Table 2 are presented in Figure 5, 4, it can be seen that our model fit the reported cases very well, and the predictions of HIV/AIDS cases among groups in the next few years are given in Figure 5, 4. The fitting results show that the model is reasonable. Especially, it is observed that the growth rate of HIV infected in old people will be faster than young people in Figure 4 (b), which reveals that the elderly group will become HIV high risk group and HIV/AIDS cases among the elderly group remain increase rapidly unless some effective measures are taken. Sensitivity analysis of the main parameters, analysis of optimal control and prevention measures are given in Figure 6, 3, 8, 9, 10.

**Table 2.** Numbers of youth, adult, the elderly and total new reporting HIV/AIDS cases in China (2005 – 2017). Adult cases are calculated by the total cases minus other groups cases, where the data on children under 15 years old from 2005 to 2017 are counted in [45].

Year	Total cases	Source	The youth cases (proportion)	Source	The adult cases (proportion)	The elderly cases (proportion)	Source
2005	40711	[43]	4186 (10.28%)	[55]	33430 (82.12%)	2563 (6.30%)	[48]
2006	44070	[43]	4872 (11.06%)	[55]	34227 (77.67%)	3437 (7.80%)	[16]
2007	45151	[43]	5524 (12.23%)	[55]	34449 (76.30%)	4515 (10.06%)	[16]
2008	50081	[43]	6628 (13.23%)	[55]	36064 (72.01%)	6599 (13.18%)	[44]
2009	53249	[43]	7416 (13.93%)	[55]	35916 (67.45%)	9016 (16.93%)	[44]
2010	64108	[43]	7875 (13.28%)	[55]	44696 (69.72%)	11537 (18.00%)	[44]
2011	74517	[36]	8925 (11.98%)	[55]	48983 (65.73%)	16609 (22.30%)	[44]
2012	82434	[28]	10195 (12.37%)	[55]	52718 (63.95%)	19521 (23.68%)	[44]
2013	90119	[29]	10800 (13.49%)	[36]	56253 (62.42%)	23066 (25.61%)	[44]
2014	103501	[30]	15000 (14.61%)	[36]	60139 (58.10%)	27520 (26.60%)	[44]
2015	114656	[31]	16986 (14.81%)	[49]	63308 (55.22%)	33522 (29.24%)	[41]
2016	124555	[26]	18437 (15.00%)	[49]	70356 (56.49%)	35762 (28.71%)	[41]
2017	134551	[27]	21250 (15.79%)	[49]	72468 (53.86%)	40833 (30.35%)	[41]

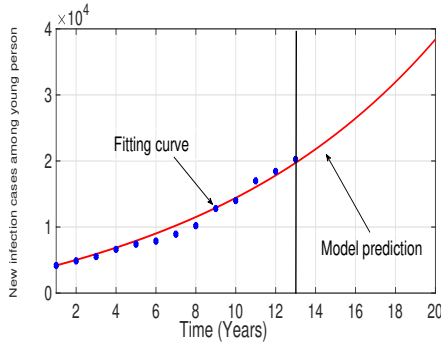


(a) The fitting curve of total new HIV/AIDS infection cases.

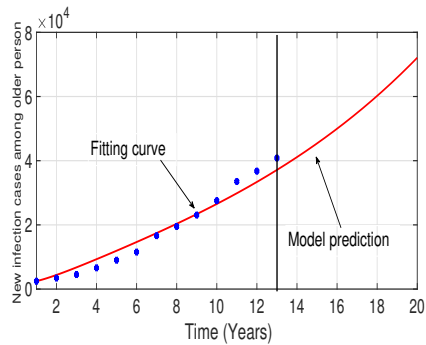


(b) The fitting curves of new HIV/AIDS infection cases among youth and elderly.

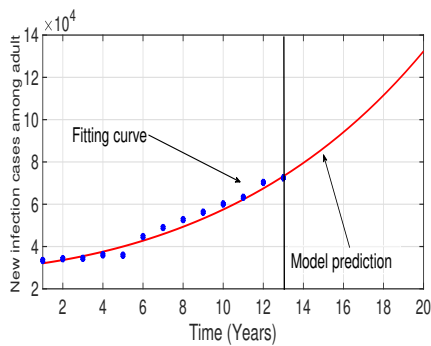
**Figure 4.** The fitting curves of total new HIV/AIDS infection cases.



(a) The fitting curve of new HIV/AIDS infection cases among youth (15-24).



(b) The fitting curve of new HIV/AIDS infection cases among elderly ( $\geq 50$ ).



(c) The fitting curve of new HIV/AIDS infection cases among adult (25-49).

**Figure 5.** The fitting curves of new HIV/AIDS infection cases among youth, elderly, adult, respectively.

**7.2. Sensitivity analysis.** In order to analyze the effects of the parameter values on the basic reproduction number of system (1), we perform sensitivity analysis by

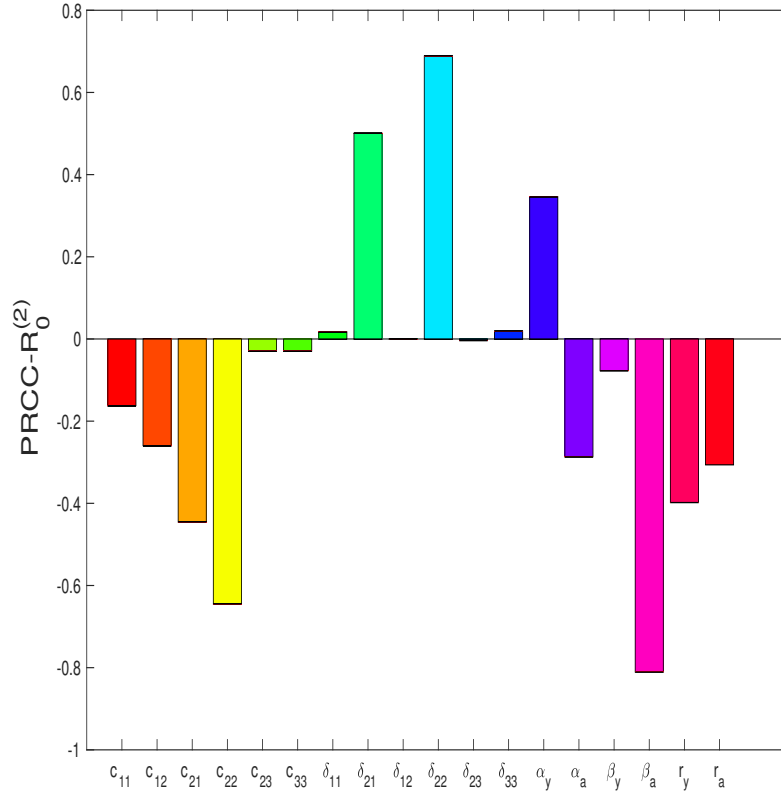


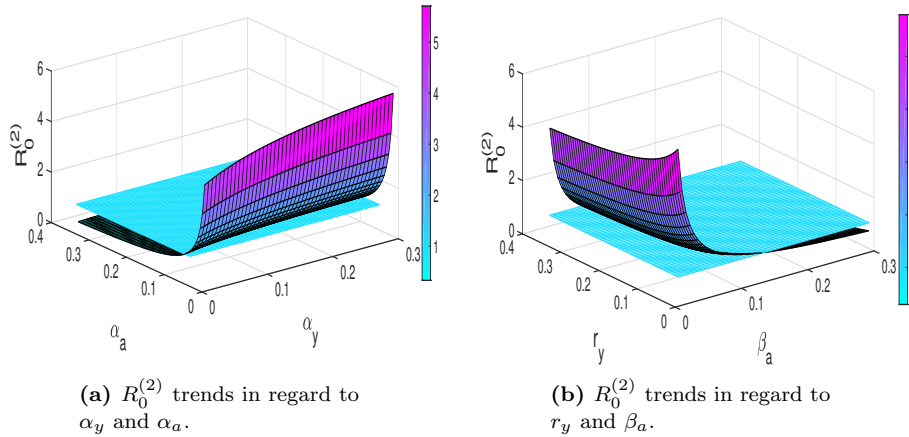
Figure 6.  $PRCC - R_0^{(2)}$

Latin square sampling and partial rank correlation coefficient (PRCC) methods. In the absence of available data on the distribution functions, we choose a uniform distribution for all input parameters with the minimum and maximum values shown in Table 1 and tested for significant PRCCs for all parameters of  $R_0^{(2)}$  (PRCCs is given in Table 3). Figure 6 shows PRCCs values of parameters against the basic reproduction number, which indicates that the parameters of model (1) except  $\delta_{11}$  and  $\delta_{12}$  have significant impact on the basic reproduction number  $R_0^{(2)}$ . These results indicate that decreasing the transmission rate of sexual or increasing the condom use proportion are the most effective measures to reduce the basic reproduction number. That is, the implement of highly condom use is a critical control measure. So, studying the effects of changing these two parameters on the basic reproduction number is of significantly applied value. Besides, the transfer rates  $\beta_a$  and  $\alpha_a$  have important impact on  $R_0^{(2)}$ , it means that an important reason for HIV/AIDS cases increasing vigorously among old people is due to the HIV cases among adult become to HIV/AIDS among the elderly group [28, 27].

We carry out some sensitivity analyses to investigate the influence of parameters on  $R_0^{(2)}$  by three-dimensional diagram (see Figure 7, 8), the parameters values are

Parameters	p value	PRCC	Parameters	p value	PRCC
$c_{11}$	0.3721	-0.1622	$\delta_{22}$	0.6886	0.7230
$c_{12}$	0.6879	-0.2595	$\alpha_y$	0.6152	0.3462
$c_{21}$	0.2277	-0.4457	$\alpha_a$	0.7502	-0.2868
$c_{22}$	0.8075	-0.6452	$\beta_y$	0.5915	-0.0771
$\delta_{11}$	0.4428	0.0172	$\beta_a$	0	-0.8105
$\delta_{21}$	0.3971	0.5004	$r_y$	0.2674	-0.3981
$\delta_{12}$	0.8916	-0.0008	$r_a$	0.2412	-0.3063
$\delta_{23}$	0.8646	-0.0036	$c_{23}$	0.1928	-0.0292
$\delta_{33}$	0.3956	0.0190	$c_{33}$	0.5424	-0.0135

**Table 3.** The PRCC of the parameters in model (1).

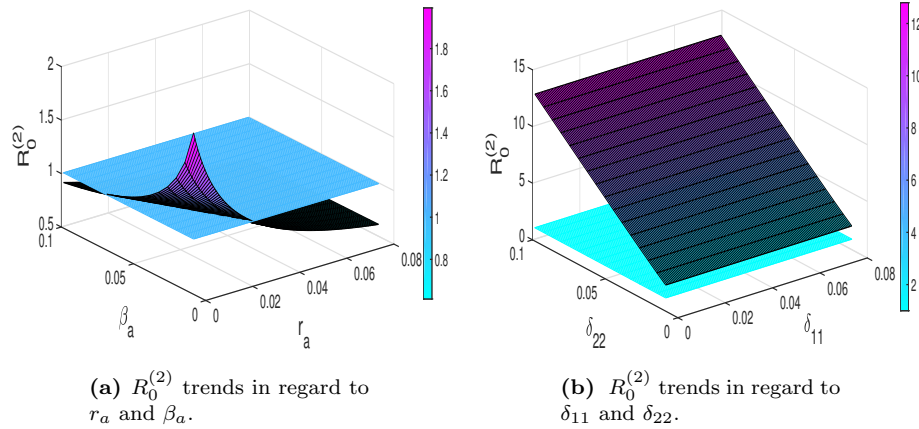


**Figure 7.**  $R_0^{(2)}$  trends with respect to different parameters, the other parameters are listed in Table 1.

the same as the above cases besides  $\alpha_a, \alpha_y$  in Figure 7 (a). Similarly, in Figure 7 (b),  $r_y, \beta_a$  influences on  $R_0^{(2)}$ , it is shown that  $R_0^{(2)}$  is an increasing function of  $\alpha_y$  and  $r_y$ , respectively, and decreasing function of  $\beta_a, r_a$  (see Figure 8 (a)). Furthermore, from Figure 8 (b) we can see that  $R_0^{(2)}$  is always greater than 1, it means that the population of infected HIV among youth and adult increasing is mainly due to sexual transmission.

**7.3. Optimal control strategies.** We now solve the optimal control problem proposed in Section 6 for  $w_1 = 1, w_2 = 2, w_3 = 1000, w_4 = 950, t_f = 20$ , and the parameters, initial conditions in Table 1 and subsection 7.1. The optimal controls  $u_1^*$  and  $u_2^*$  take the maximum values for interval  $[0, 4.444], [0, 10.36]$  years, respectively (see Figure 9 (a)). The optimal controls  $u_1(t)$  and  $u_2(t)$  are decreasing functions in intervals  $[4.44, 18], [10.3, 18]$ , respectively. We obtain that  $u_1^*(18) = 0.06, u_2^*(18) = 0.04924$  from Figure 9 (a). From Figure 9 (b), it is revealed that the number of AIDS individuals among elderly associated with the optimal control strategies decreases from 1460 to 460 individuals fleetly, and at the time of  $t = 18$  years, the number of AIDS patient among elder people associated with the optimal controls



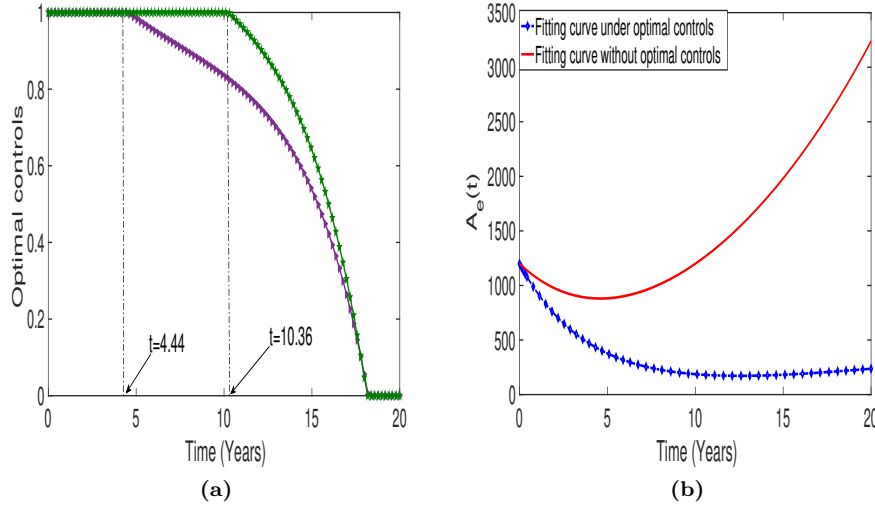


**Figure 8.**  $R_0^{(2)}$  trends with respect to different parameters, the other parameters are listed in Table 1.

is almost 430. We obtain that the strategy associated with controls leads to significant decrease on the number of AIDS individuals among elderly. The maximum value of the number of  $A_e$  also decreases significantly when the control strategies are applied. The optimal control  $u_2(t)$  implies a significant transfer of HIV infected among adult to the AIDS people among elderly. Meanwhile, we analyze the impact of initial values of  $I_y(t)$ ,  $I_a(t)$ ,  $I_e(t)$  on the optimal control strategies. In Figure 10(a), (b), it is shown that decrease of  $I_a(0)$ ,  $I_y(0)$  will shorten the duration of the maximum values of optimal controls  $u_1^*$ ,  $u_2^*$ . Conversely, the increase in the initial value of infected individuals in the three groups will prolong the duration of the maximum control intensity, wherein the change in the initial value of the elderly group has great impact on the control intensity (see Figure 10 (c)). These results indicated that the difficulty of control and the cost of control increase as the initial value of infected individuals among the three groups. In other words, it is necessary to take timely control measures for HIV transmission, which can improve control efficiency while reducing control costs.

From an epidemiological point of views, the control measures must be taken when the disease outbreaks, i.e., the basic reproduction number larger than 1 in this paper. Hence, one interesting question is how the basic reproduction  $R_0^{(1)}$  of elderly in (4) influences on the optimal control strategies  $u_1(t)$  and  $u_2(t)$ . To do this, we choose the parameter  $\delta_{33} = 0.172, 0.25, 0.4$ ,  $r_e = 1/8.2, 1/7.9, 1/7.2$ , and other parameters are the same in Table 1. After simple calculation, we obtain  $R_0^{(1)} = 1.075, 1.56, 2.5$ , respectively. Corresponding the optimal control strategies  $u_1^*$  and  $u_2^*$  are showed in Fig. 11. From Fig. 11 (a) – (b), it can be seen that the intensity of the optimal control strategies  $u_1^*$  and  $u_2^*$  will be strengthen when the basic reproduction number  $R_0^{(1)}$  increases. This is indicated that the increase in the basic reproduction will have a negative effect on the control strategies. In other words, the optimal controls should be strengthen when the prevalence of HIV increases among elderly.

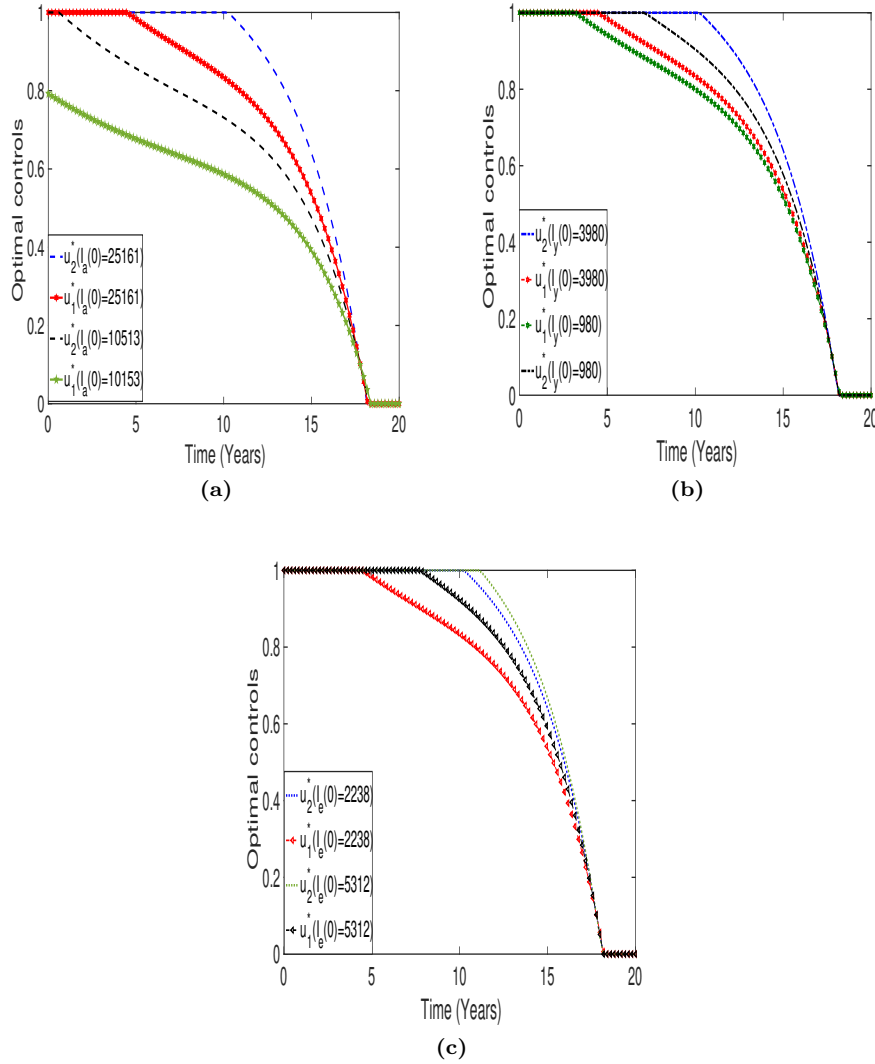
Finally, in order to effectively control the HIV/AIDS infection among the elderly, we discuss the numerical solutions of the optimal system and the corresponding



**Figure 9.** (a): Optimal solutions for the system (1),  $u_2^*$  in green full line and  $u_1^*$  in purple full line. (b):  $A_e(t)$  trends in regard to the optimal controls, where  $A_e$  without control strategy in red solid line,  $A_e$  under the optimal control in blue dashed line. Other parameters are the same in Table 1.

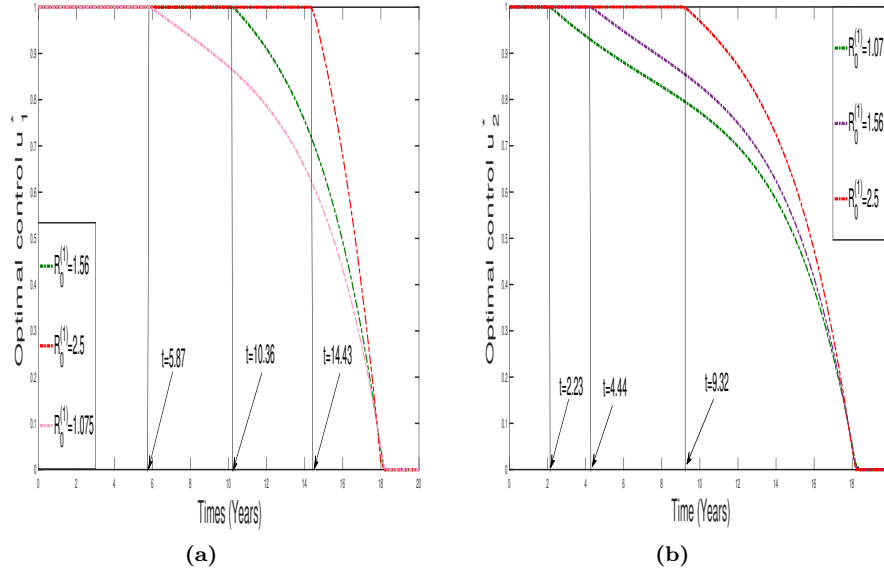
results of the optimal controls  $u_1$  and  $u_2$ . In Figure 9 (b), the impact of the optimal controls of the AIDS cases among the elderly is presented, it can be seen that AIDS decreases observably among the elderly under the optimal controls. As a comparison with the optimal controls we take in Figure 9 (b), we show the counterfactual AIDS cases among the old people under the optimal strategy  $u_1$  or  $u_2$  only are shown in Figure 12 and Figure 13. We observe in Figure 12 (b) that the number of old AIDS patients  $A_e(t)$  is eventually decreasing but rising at the beginning under the optimal control  $u_2(t)$  only. This result is due to a lack of intervention in the sexual behaviors among the elderly. In Figure 13 (b), it shows that the number of  $A_e(t)$  is initially controlled significantly but start rising again at about the 8th year. This may be connected to the fact that the development of chronic HIV infection to AIDS is an important cause of the increase in the number of the aged AIDS patients. The above results, however, do not correspond to what we purpose and expect. On the contrary, the result depicted in Figure 9 (b) clearly suggests that the strategy we propose in Section 5 is very efficient and effective for the control of the number of AIDS patients among the elderly.

**8. Conclusion and discussion.** In this paper, the dynamics of a HIV/AIDS epidemic model with three-age-levels are analyzed. In fact, compartment models for HIV transmission among high risk groups ( FSWs, MSM) have been studies by many researchers [40, 2, 47, 53, 54]. However, to our knowledge, there are no mathematical models established to analyze HIV transmission among different age groups although the spread and duration of HIV/AIDS vary considerably among people of different ages. Our main concerns are to study the HIV transmission among different ages and to study the effect of optimal control strategies on HIV/AIDS among

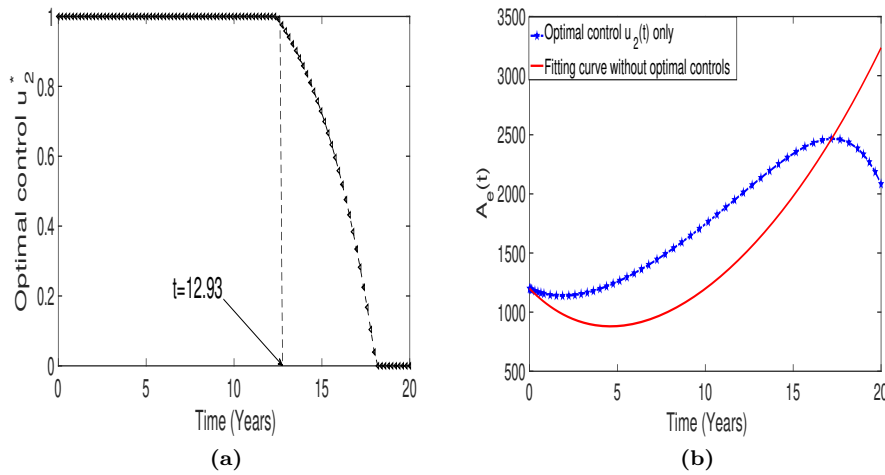


**Figure 10.** (a): Optimal control with respect to  $I_a(0)$ , (b): Optimal control with respect to  $I_y(0)$ , (c): Optimal control with respect to  $I_e(0)$ . Other parameters are the same in Table 1.

the elderly population. For this purpose, the total population is divided into three compartments including youth (15-24 years old), adult (25-49 years old) and elderly ( $\geq 50$  years old) in our model. We deduce the formula of the basic reproduction number  $R_0$  and show that the disease-free equilibrium is globally stable if  $R_0 < 1$  and the disease is persistent when  $R_0 > 1$ . Meanwhile, we prove the existence, uniqueness and global attractivity of the endemic equilibrium in a special case. At last, the optimal controls are given to control HIV/AIDS transmission among elderly individuals.

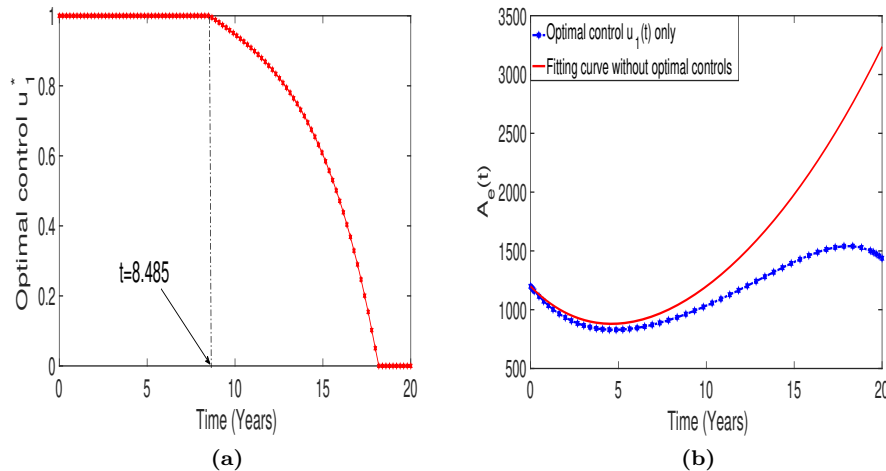


**Figure 11.** (a): The influence of  $R_0^{(1)}$  on  $u_1^*$ . (b): The influence of  $R_0^{(1)}$  on  $u_2^*$ , other parameters are the same in Table 1.



**Figure 12.** (a): Optimal control strategy  $u_2(t) \neq 0$ ,  $u_1(t) = 0$ . (b): The impact on  $A_e(t)$  with the optimal control strategy  $u_2(t) \neq 0$ ,  $u_1(t) = 0$ , other parameters are the same in Table 1.

The numerical analyses reflect the trend of HIV/AIDS among different age groups in China based on fitting and prediction of the annual HIV/AIDS infected cases among these groups from 2005 to 2017. The model predicts consistent increases in the numbers of HIV/AIDS infections among all age groups in the next a few years,



**Figure 13.** (a): Optimal control strategy with  $u_1(t) \neq 0$ ,  $u_2(t) = 0$ .  
 (b): The impact on  $A_e(t)$  with the optimal control strategy  $u_1(t) \neq 0$ ,  $u_2(t) = 0$ , other parameters are the same in Table 1.

with a significantly rapid increase among the elderly population. This finding indicates that the elderly people have become a new high-risk group, which is in lines with [48]. In the part of sensitivity analyses, we analyze the effects of the parameter values on the basic reproduction number  $R_0^{(2)}$  of system (1) by Latin square sampling and partial rank correlation coefficient (PRCC) methods, these results indicate that decreasing the transmission rate of sexual behaviors and increasing condom use are the two most effective measures to reduce the basic reproduction number. More specifically, the numerical results of the optimal controls show that strengthening treatment to reduce HIV infection individuals move to AIDS patients and encouraging condom use is critical to the control of HIV transmission among elderly group.

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