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Dynamics of a time-periodic two-strain SIS epidemic model with diffusion and latent period

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ABSTRACT

Coinfection of hosts with multiple strains or serotypes of the same agent, such as different influenza virus strains, different human papilloma virus strains, and different dengue virus serotypes, is not only a very serious public health issue but also a very challenging mathematical modeling problem. In this paper, we study a time-periodic two-strain SIS epidemic model with diffusion and latent period. We first define the basic reproduction number R_0^i and introduce the invasion number $\hat{R}_0^i$ for each strain i $(i = 1, 2)$, which can determine the ability of each strain to invade the other single-strain. The main question that we investigate is the threshold dynamics of the model. It is shown that if $R_0^i \leqslant 1 (i = 1, 2)$, then the disease-free periodic solution is globally attractive; if $R_0^i > 1 \geqslant R_0^j (i \neq j, i, j = 1, 2)$, then competitive exclusion, where the jth strain dies out and the ith strain persists, is a possible outcome; and if $\hat{R}_0^i > 1 (i = 1, 2)$, then the disease persists uniformly. Finally we present the basic framework of threshold dynamics of the system by using numerical simulations, some of which are different from that of the corresponding multi-strain SIS ODE models.

1. Introduction

There are many examples of simultaneous infection of hosts with multiple strains or serotypes of the same agent, such as coinfection of different influenza virus strains (Greenbaum et al. [1]), different human papilloma virus strains (Chaturvedi et al. [2], and different dengue virus serotypes (Ferguson et al. [3]). It is very key to study the dynamics of coinfection since antimicrobials, which can be used to treat one infection, may affect the others. Various mathematical models have been proposed to study the dynamics of coinfection of multiple strains (Bremermann and Thieme [4], Martcheva [5]), in particular of two strains (Alizon [6], Allen et al. [7], Blyuss and Kyrychko [8], Gao et al. [9]).

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With the purpose of understanding the influence of spatial heterogeneity of environment and spatial movement to the geographic spread of infectious diseases, reaction–diffusion equations have been frequently used. These studies mainly focus on the existence, uniqueness, the asymptotic profile of the steady states, existence and asymptotic spreading speed of traveling waves, and so on. We refer to the monograph of Murray [10], the surveys of Fitzgibbon and Langlais [11] and Ruan and Wu [12], Wang et al. [13] and the references cited therein for related results and references.

To incorporate diffusion and spatial heterogeneity explicitly, Tuncer and Martcheva [14] considered the following two-strain diffusive SIS epidemic model with space-dependent transmission parameters

$$\begin{cases} \frac{\partial}{\partial t}S = d_S\Delta S - \frac{(\beta_1(x)I_1(x,t)+\beta_2(x)I_2(x,t))S(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} \\ \qquad\qquad +\gamma_1(x)I_1(x,t)+\gamma_2(x)I_2(x,t),\ x\in\Omega,\quad t>0, \\ \frac{\partial}{\partial t}I_1 = d_1\Delta I_1 + \frac{\beta_1(x)S(x,t)I_1(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} - \gamma_1(x)I_1(x,t),\ x\in\Omega,\quad t>0, \\ \frac{\partial}{\partial t}I_2 = d_2\Delta I_2 + \frac{\beta_2(x)(x,t)I_2(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} - \gamma_2(x)I_2(x,t),\ x\in\Omega,\quad t>0 \end{cases} \tag{1.1}$$

under the non-flux boundary conditions

$$\frac{\partial}{\partial n}S = \frac{\partial}{\partial n}I_1 = \frac{\partial}{\partial n}I_2 = 0,\ x\in\partial\Omega,\ t>0. \tag{1.2}$$

They firstly defined the basic reproduction number for each strain by $R_0^i(i=1,2)$ and introduced the invasion numbers of the two strains $\hat{R}_0^i(i=1,2)$, respectively. Then they showed that the disease-free equilibrium (DFE) is globally stable if $R_0 := \max\{R_0^1, R_0^2\} < 1$ and conversely unstable if $R_0 > 1$. They also showed that if both $\hat{R}_0^1 > 1$ and $\hat{R}_0^2 > 1$ hold, then there is a coexistence steady state. Finally, they investigated various competition exclusion scenario between the two strains and the stability of the coexistence equilibrium by numerical investigations.

Ackleh et al. [15] considered the threshold dynamics of (1.1)–(1.2) with a bilinear disease transmission term. They took into account the following two cases: (a) all coefficients are spatially homogeneous. In this case they showed that the DFE is globally attractive if $R_0 < 1$, and one strain may outcompete the other one and cause it to extinction if $R_0 > 1$; (b) all coefficients are spatially inhomogeneous and the diffusion rates are equal; namely, $d_S = d_1 = d_1 = d$. In this situation, both competitive exclusion and coexistence may occur. Further studies of model (1.1)–(1.2) can be found in Wu et al. [16], who studied, among other things, what characteristics of the model imply coexistence, and the model exhibits competitive exclusion under what conditions.

As reported by Altizer et al. [17], host–pathogen interactions can be affected by the seasonality, for example, contact rates, host social behavior, host immune response, and host births and deaths. Thus, it is crucial to introduce temporal heterogeneity into epidemic models, which can be described by non-autonomous evolution equations. Peng and Zhao [18] established the spatial dynamics of a time-periodic SIS epidemic model with diffusion in terms of the basic reproduction number and showed that temporal periodicity and spatial heterogeneity can strengthen the persistence of the infectious disease. Wang et al. [19] took into account the dynamics of an almost periodic SIS epidemic model with diffusion. Their results emphasized that due to the interaction of temporal almost periodicity and spatial heterogeneity, the persistence of the disease may be strengthened.

Latent period refers to the period between the moment of being infected and the moment of becoming infectious. Many infectious diseases have a latent period (such as chicken pox, cholera, measles, influenza etc.); that is, other susceptible individuals are not infected by the infected ones until some time later. How the latent period of an infectious disease affects the transmission dynamics of the disease, in particular the spatial spread of the disease, is a challenging and very interesting problem. There are many papers focusing on reaction–diffusion epidemic models with fixed latent period, see Bai et al. [20], Guo et al. [21], Li and

Zou [22], Liang et al. [23,24], Lou and Zhao [25], Wang and Zhao [26], Zhang and Wang [27,28], Zhang et al. [29], Zhao et al. [30,31], Zhao et al. [32], Zhao et al. [33] and the references therein.

In the paper, we incorporate the demographic structure, spatial diffusion, temporal heterogeneity and latent period into system (1.1)–(1.2) and then study the threshold dynamics of the generalized model. In Section 2, we propose a time-periodic two-strain SIS epidemic model with diffusion and latent period (2.5). In Section 3, we consider the threshold dynamics of (2.5); namely, if the basic reproduction number of the ith strain R_0^i is less than or equal to 1 for $i=1,2$, then the disease-free periodic solution is globally attractive; if $R_0^i > 1 \geqslant R_0^j$ $(i \neq j, i, j = 1,2)$, then competitive exclusion, where the jth strain dies out and the ith strain persists, is a possible outcome; and if $\hat{R}_0^i > 1 (i=1,2)$, then the disease is uniformly persistent. In Section 4, we give the basic framework of threshold dynamics of the system by numerical simulations, in which all coefficients are only dependent upon the time variable t (time-periodic). At last, a brief discussion is given.

2. Model formulation

Assume that the pathogenic microorganism and its genetic variant or subtype spread in one population, which leads to two different infectious classes. Assume that the population lives in a bounded domain $\Omega \in \mathbb{R}^n$ and the boundary $\partial\Omega$ is smooth. One supposes that a susceptible individual can be infected by only one virus strain and a recovered individual does not have immunity and can be infected again. Moreover, the two different infectious classes consist of two different latent groups and two different infective groups, respectively. Thus, we divide the population into five compartments: the susceptible group $S(x,t)$, two latent groups $L_i(x,t)(i=1,2)$, and two infective groups $I_i(x,t)(i=1,2)$, where x is the position and t represents the time.

Denote the densities of the two different infectious classes with infection age $a \geqslant 0$ and at position $x \in \bar{\Omega}$ and time $t \geqslant 0$ by $E_1(x,a,t)$ and $E_2(x,a,t)$, respectively. The constant D_i denotes the diffusion rate of the ith infectious class for $i=1,2$; $d(x,t)$ is the natural death rate at location x and time t; the functions $\delta_i(x,a,t)(i=1,2)$ represent the recovery rates of the two infectious classes with infection age a at location x and time t; $\kappa_i(x,a,t)(i=1,2)$ represent the mortality rates induced by the disease. We take into account the following model

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) E_i = D_i \Delta E_i - \left(d(x,t) + \kappa_i(x,a,t) + \delta_i(x,a,t)\right) E_i(x,a,t), \\ \qquad\qquad x \in \Omega, a > 0,\ t > 0, \\ \frac{\partial}{\partial n} E_i = 0, \quad x \in \partial\Omega, a > 0,\ t > 0, \end{cases} \tag{2.1}$$

where $i=1,2$, n denotes the outward normal. Suppose that $\tau_i(i=1,2)$ are the average latency periods of the two different infectious diseases, respectively. It follows from the definitions of L_i and I_i that

$$L_i(x,t) = \int_0^{\tau_i} E_i(x,a,t)da, \quad I_i(x,t) = \int_{\tau_i}^{+\infty} E_i(x,a,t)da, \quad i=1,2. \tag{2.2}$$

Furthermore, suppose that κ_i and δ_i satisfy

$$\kappa_i(x,a,t) = \kappa_i(x,t),\ \forall x \in \Omega,\ t \geqslant 0,\ a \in [0,\infty)$$

and

$$\delta_i(x,a,t) = \begin{cases} 0, & \forall x \in \Omega,\ a \in [0,\tau_i),\ t \geqslant 0, \\ \delta_i(x,t), & \forall x \in \Omega,\ a \in [\tau_i, +\infty),\ t \geqslant 0, \end{cases} \qquad i=1,2.$$

Integrating (2.1) on a and using (2.2) yield

$$\frac{\partial}{\partial t} I_i = D_i \Delta I_i - \left(d(x,t) + \kappa_i(x,t) + \delta_i(x,t)\right) I_i(x,t) + E_i(x,\tau_i,t) - E_i(x,\infty,t)$$

and

$$\frac{\partial}{\partial t}L_i = D_i\Delta L_i - (d(x,t)+\kappa_i(x,t))\,L_i(x,t) + E_i(x,0,t) - E_i(x,\tau_i,t),$$

respectively. Assume that $E_i(x,\infty,t)=0$. In particular, we adopt $E_i(x,0,t)$ with the following form

$$E_i(x,0,t) = \frac{\beta_i(x,t)S(x,t)I_i(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)}, \quad i=1,2,$$

due to the fact that the contact of the susceptible and infectious individuals yields the new infected individuals, where $\beta_i(x,t)\geqslant 0$ is the infection rate.

Suppose the growth of a population $N(x,t)$ is described by the demographic equation

$$\frac{\partial}{\partial t}N = D_N\Delta N + \mu(x,t) - d(x,t)N(x,t),$$

where D_N, $\mu(\cdot,\cdot)$ and $d(\cdot,\cdot)$ denote the diffusion, recruiting and death rates, respectively. Furthermore, we suppose that the disease does not transmit vertically. Thus, we can use the following system to describe the infection dynamics

$$\begin{cases} \frac{\partial}{\partial t}S = D_S\Delta S + \mu(x,t) - d(x,t)S(x,t) + \delta_1(x,t)I_1(x,t) + \delta_2(x,t)I_2(x,t) \\ \qquad - \frac{\beta_1(x,t)S(x,t)I_1(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} - \frac{\beta_2(x,t)S(x,t)I_2(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)}, \\ \frac{\partial}{\partial t}L_i = D_i\Delta L_i - (d(x,t)+\kappa_i(x,t))\,L_i(x,t) - E_i(t,\tau_i,x) + \frac{\beta_i(x,t)S(x,t)I_i(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)}, \\ \qquad i=1,2, \\ \frac{\partial}{\partial t}I_i = D_i\Delta I_i - (d(x,t)+\kappa_i(x,t)+\delta_i(x,t))\,I_i(x,t) + E_i(x,\tau_i,t), \quad i=1,2. \end{cases} \tag{2.3}$$

The following assumptions are needed in the sequel:

(H) Assume that $D_S, D_i > 0$ for $i=1,2$; the functions $d(\cdot,\cdot)$, $\mu(\cdot,\cdot)$, $\delta_i(\cdot,\cdot)(i=1,2)$, $\kappa_i(\cdot,\cdot)(i=1,2)$ and $\beta_i(\cdot,\cdot)(i=1,2)$ are Hölder continuous functions on $\bar{\Omega}\times\mathbb{R}$ and are periodic in time with period $T>0$. $d(\cdot,\cdot)$ is positive and $\mu(\cdot,\cdot)$, $\delta_i(\cdot,\cdot)(i=1,2)$, $\kappa_i(\cdot,\cdot)(i=1,2)$ and $\beta_i(\cdot,\cdot)(i=1,2)$ are non-negative non-trivial on $\bar{\Omega}\times\mathbb{R}$.

Next, we derive the functions $E_i(t,\tau_i,x)(i=1,2)$. Letting $v_i(x,a,\xi)=E_i(x,a,a+\xi)(i=1,2)$, we investigate solutions of (2.1) along the line $t=a+\xi$ for any $\xi\geqslant 0$. For $a\in(0,\tau_i]$, we have

$$\begin{cases} \frac{\partial}{\partial a}v_i = D_i\Delta v_i - (d(x,a+\xi)+\kappa_i(x,a+\xi))\,v_i(x,a,\xi), \\ v_i(x,0,\xi) = E_i(x,0,\xi) = \frac{\beta_i(x,\xi)S(x,\xi)I_i(x,\xi)}{S(x,\xi)+I_1(x,\xi)+I_2(x,\xi)}. \end{cases}$$

It follows that

$$v_i(x,a,\xi) = \int_\Omega \Gamma_i(x,y,\xi+a,\xi)\frac{\beta_i(y,\xi)S(y,\xi)I_i(y,\xi)}{S(y,\xi)+I_1(y,\xi)+I_2(y,\xi)}dy, \quad i=1,2,$$

where $\Gamma_i(x,y,\xi+a,\xi)$ with $x,y\in\Omega$ and $t>s\geqslant 0$ denotes the fundamental solution of the operator $\partial_t - D_i\Delta - (d(\cdot,t)+\kappa_i(\cdot,t))$ with no-flux boundary condition. Since $d(\cdot,\cdot+T)=d(\cdot,\cdot)$ and $\kappa_i(\cdot,\cdot+T)=\kappa_i(\cdot,\cdot)(i=1,2)$ in $\overline{\Omega}\times[0,\infty)$, one has $\Gamma_i(x,y,t,s)=\Gamma_i(x,y,t+T,s+T)$ for all $x,y\in\Omega$ and $t>s\geqslant 0$. Consequently, we have

$$E_i(x,a,t) = \int_\Omega \Gamma_i(x,y,t,t-a)\frac{\beta_i(y,t-a)S(y,t-a)I_i(y,t-a)}{S(y,t-a)+I_1(y,t-a)+I_2(y,t-a)}dy$$

due to $E_i(x,a,t)=v_i(x,a,t-a)$. Let $a=\tau_i$, then

$$E_i(x,\tau_i,t) = \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\frac{\beta_i(y,t-\tau_i)S(y,t-\tau_i)I_i(y,t-\tau_i)}{S(y,t-\tau_i)+I_1(y,t-\tau_i)+I_2(y,t-\tau_i)}dy. \tag{2.4}$$

Substituting (2.4) into (2.3), one gets the following system

$$\begin{cases} \frac{\partial}{\partial t}S = D_S\Delta S + \mu(x,t) - d(x,t)S(x,t) + \delta_1(x,t)I_1(x,t) + \delta_2(x,t)I_2(x,t) \\ \qquad\qquad - \frac{\beta_1(x,t)S(x,t)I_1(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} - \frac{\beta_2(x,t)S(x,t)I_2(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)}, \quad x\in\Omega,\ t>0, \\ \frac{\partial}{\partial t}I_i = D_i\Delta I_i - \left(d(x,t)+\kappa_i(x,t)+\delta_i(x,t)\right)I_i(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\frac{\beta_i(y,t-\tau_i)S(y,t-\tau_i)I_i(y,t-\tau_i)}{S(y,t-\tau_i)+I_1(y,t-\tau_i)+I_2(y,t-\tau_i)}dy, \quad x\in\Omega,\ t>0\ i=1,2, \\ \frac{\partial}{\partial n}S = \frac{\partial}{\partial n}I_i = 0, \quad x\in\partial\Omega,\ t>0,\ i=1,2, \end{cases} \tag{2.5}$$

where we omit the $L_i(x,t)(i=1,2)$ equations from (2.3) because they can be decoupled.

Let $\tau=\max\{\tau_1,\tau_2\}$. Denote $\mathbb{X}:=C(\bar\Omega,\mathbb{R}^3)$ with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Let $\mathbb{C}_\tau := C([-\tau,0],\mathbb{X})$ with the norm $\|\phi\| = \max_{\theta\in[-\tau,0]}\|\phi(\theta)\|_{\mathbb{X}}$, $\forall\phi\in\mathbb{C}_\tau$. Let $\mathbb{X}^+ := C(\bar\Omega,\mathbb{R}^3_+)$ and $\mathbb{C}^+_\tau := C([-\tau,0],\mathbb{X}^+)$. Clearly, the Banach spaces $(\mathbb{X},\mathbb{X}^+)$ and $(\mathbb{C}_\tau,\mathbb{C}^+_\tau)$ are strongly ordered. For a function $u(t):[-\tau,\sigma)\to\mathbb{X}$ with $\sigma>0$, denote $u_t\in\mathbb{C}_\tau$ by

$$u_t(\theta)=u(t+\theta), \quad \theta\in[-\tau,0].$$

Similarly, define $\mathbb{Y}:=C(\bar\Omega,\mathbb{R})$ and $\mathbb{Y}^+:=C(\bar\Omega,\mathbb{R}^+)$. Moreover, consider the equation

$$\begin{cases} \frac{\partial}{\partial t}w = D_S\Delta w - d(x,t)w(x,t), \quad x\in\Omega,\ t>0, \\ \frac{\partial}{\partial n}w = 0, \quad x\in\partial\Omega,\ t>0, \\ w(x,0)=\phi_S(x), \quad x\in\Omega,\ \phi_S\in\mathbb{Y}^+, \end{cases} \tag{2.6}$$

where $D_S>0$ and $d(x,t)$ is positive and Hölder continuous on $\bar\Omega\times\mathbb{R}$, and T-periodic in t. By Hess [34, Chapter II] with (2.6), we have that there is an evolution operator $V_S(t,s):\mathbb{Y}\to\mathbb{Y}$ for $0\leqslant s\leqslant t$ satisfying $V_S(t,t)=I$, $V_S(t,s)V_S(s,\rho)=V_S(t,\rho)$ for $0\leqslant\rho\leqslant s\leqslant t$, and $V_S(t,0)(\phi_S)(x)=w(x,t;\phi_S)$ for $x\in\Omega$, $t\geqslant 0$ and $\phi_S\in\mathbb{Y}$. Here $w(x,t;\phi_S)$ is the solution of (2.6). Similarly, we consider the equation

$$\begin{cases} \frac{\partial}{\partial t}\bar w_i = D_i\Delta\bar w_i - (d(x,t)+\kappa_i(x,t)+\delta_i(x,t))\bar w_i(x,t), \quad x\in\Omega,\ t>0, \quad i=1,2, \\ \frac{\partial}{\partial n}\bar w_i = 0, \quad x\in\partial\Omega,\ t>0, \quad i=1,2, \\ \bar w_i(x,0)=\phi_i(x), \quad x\in\Omega,\ \phi_i\in\mathbb{Y}^+, \quad i=1,2, \end{cases} \tag{2.7}$$

where $D_i>0$, $d(x,t)$, $\kappa_i(x,t)$ and $\delta_i(x,t)$ satisfy the assumption (H). System (2.7) can determine the evolution operators $V_i(t,s)(i=1,2)$, which have the similar properties as $V_S(t,s)$. By [35, Lemma 6.1], we have $V_S(t,s)=V_S(t+T,s+T)$ and $V_i(t,s)=V_i(t+T,s+T)$ for $(t,s)\in\mathbb{R}^2_+$ with $t\geqslant s$ and $i=1,2$ due to the periodicity of coefficients. In addition, $V_S(t,s)$ and $V_i(t,s)(t>s)$ are compact, strongly positive and analytic operators on $\mathbb{Y}^+$. Together with [35, Theorem 6.6] with $\alpha=0$, we know that there are $\mathcal{Q}\geqslant 1$ and $c_0\in\mathbb{R}$ satisfying

$$\|V_S(t,s)\|, \|V_i(t,s)\| \leqslant \mathcal{Q}e^{-c_0(t-s)} \quad \text{for all} \ \ t,s\in\mathbb{R} \text{ with } t\geqslant s, \quad i=1,2.$$

Let $F=(F_S,F_1,F_2):[0,\infty)\times\mathbb{C}^+_\tau\to\mathbb{X}$ be

$$\begin{aligned} F_S(t,\phi) =& \mu(\cdot,t) - \frac{\beta_1(\cdot,t)\phi_S(\cdot,0)\phi_1(\cdot,0)}{\phi_S(\cdot,0)+\phi_1(\cdot,0)+\phi_2(\cdot,0)} - \frac{\beta_2(\cdot,t)\phi_S(\cdot,0)\phi_2(\cdot,0)}{\phi_S(\cdot,0)+\phi_1(\cdot,0)+\phi_2(\cdot,0)} \\ &+ \delta_1(\cdot,t)\phi_1(\cdot,0)+\delta_2(\cdot,t)\phi_2(\cdot,0), \\ F_i(t,\phi) =& \int_\Omega \Gamma_i(\cdot,y,t,t-\tau_i)\frac{\beta_i(y,t-\tau_i)\phi_S(y,-\tau_i)\phi_i(y,-\tau_i)}{\phi_S(y,-\tau_i)+\phi_1(y,-\tau_i)+\phi_2(y,-\tau_i)}dy \end{aligned}$$

for $\phi=(\phi_S,\phi_1,\phi_2)\in\mathbb{C}^+_\tau$, $t>0$, $x\in\bar\Omega$ and $i=1,2$. Let

$$U(t,s) := \begin{pmatrix} V_S(t,s) & 0 & 0 \\ 0 & V_1(t,s) & 0 \\ 0 & 0 & V_2(t,s) \end{pmatrix}.$$

Clearly, $U(t,s)$ is an evolution operator from $\mathbb{X}$ to $\rightarrow \mathbb{X}$ for $(t,s) \in \mathbb{R}^2$ with $t \geqslant s$. Define $A_S(t)$ and $A_i(t)$ by

$$D(A_S(t)) = \left\{ \psi \in C^2(\bar{\Omega}) \mid \frac{\partial}{\partial n}\psi = 0 \text{ on } \partial\Omega \right\},$$
$$A_S(t)\psi(x) = D_S \Delta\psi(x) - d(x,t)\psi(x), \quad \forall \psi \in D(A_S(t))$$

and

$$D(A_i(t)) = \left\{ \psi \in C^2(\bar{\Omega}) \mid \frac{\partial}{\partial n}\psi = 0 \text{ on } \partial\Omega \right\},$$
$$A_i(t)\psi(x) = D_i \Delta\psi(x) - (d(x,t) + \kappa_i(x,t) + \delta_i(x,t))\psi(x), \quad \forall \psi \in D(A_i(t)),$$

respectively, where $i = 1, 2$. Let

$$A(t) := \begin{pmatrix} A_S(t) & 0 & 0 \\ 0 & A_1(t) & 0 \\ 0 & 0 & A_2(t) \end{pmatrix}$$

and $u(x,t) := (S(x,t), I_1(x,t), I_2(x,t))$. Then (2.5) can be rewritten into the abstract equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = A(t)u(x,t) + F(t,u_t), \quad x \in \Omega, \ t > 0, \\ u(x,\zeta) = \phi(x,\zeta), \quad x \in \Omega, \ \zeta \in [-\tau, 0]. \end{cases} \tag{2.8}$$

It is also expressed as the integral form

$$u(t,\phi) = U(t,0)\phi(0) + \int_0^t U(t,s)F(s,u_s)ds, \quad t \geqslant 0, \quad \phi \in \mathbb{C}_\tau^+. \tag{2.9}$$

We call a solution of (2.9) a mild solution of (2.8).

Theorem 2.1. *For each $\phi \in \mathbb{C}_\tau^+$, system (2.5) admits a unique mild solution $u(t,\phi)$ on $[0,+\infty)$ with $u_0 = \phi$. Moreover, system (2.5) generates a T-periodic semiflow $\Phi_t(\cdot) := u_t(\cdot) : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$, namely, $\Phi_t(\phi)(x,s) = u_t(\phi)(x,s) = u(x,t+s;\phi)$ for any $\phi \in \mathbb{C}_\tau^+$, $t \geqslant 0$, $x \in \bar{\Omega}$ and $s \in [-\tau, 0)$. In addition, $\Phi_T : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ admits a global compact attractor.*

Proof. We firstly prove the existence of mild solutions $u(t,\phi)$ of (2.5). Consider the following system:

$$\begin{cases} \frac{\partial}{\partial t}v_S^+ = D_S\Delta v_S^+ + \mu(x,t) - d(x,t)v_S^+(x,t) + \delta_1(x,t)v_1^+(x,t) + \delta_2(x,t)v_2^+(x,t), \\ \qquad\qquad x \in \Omega, \ t > 0, \\ \frac{\partial}{\partial t}v_i^+ = D_i\Delta v_i^+ - \left(d(x,t) + \kappa_i(x,t) + \delta_i(x,t)\right) v_i^+(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)v_i^+(y,t-\tau_i)dy, \ x \in \Omega, \ t > 0, \ i = 1,2, \\ \frac{\partial}{\partial n}v_S^+ = \frac{\partial}{\partial n}v_i^+ = 0, \ x \in \partial\Omega, \ t > 0, \ i = 1,2, \\ v_S^+(x,s) = \phi_S(x,s), \ v_i^+(x,s) = \phi_i(x,s), \ x \in \Omega, \ s \in [-\tau, 0], \ i = 1,2. \end{cases} \tag{2.10}$$

According to Fitzgibbon [36, Theorem 4.2], one has that (2.10) admits a unique mild solution $v^+(x,t;\phi) := (v_S^+(x,t;\phi), v_1^+(x,t;\phi), v_2^+(x,t;\phi))^T$ on $t \geq -\tau$, where

$$v_S^+(x,s;\phi) = \phi_S(x,s), \quad v_i^+(x,s;\phi) = \phi_i(x,s), \ \forall x \in \bar{\Omega}, \ s \in [-\tau, 0).$$

For each $\varphi = (\varphi_S, \varphi_1, \varphi_2) \in \mathbb{C}_\tau^+$, define $B^+(t,\varphi) = (B_S^+, B_1^+, B_2^+)^T$ by

$$B_S^+(t,\varphi) := \mu(\cdot,t) + \delta_1(\cdot,t)\varphi_1(\cdot,0) + \delta_2(\cdot,t)\varphi_2(\cdot,0),$$
$$B_i^+(t,\varphi) := \int_\Omega \Gamma_i(\cdot,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\varphi_i(y,-\tau_i)dy.$$

Similarly, define $B^-(t,\varphi) = (B_S^-, B_1^-, B_2^-)^T$ by

$$B_S^-(t,\varphi) := -\frac{\beta_1(\cdot,t)\varphi_S(\cdot,0)\varphi_1(\cdot,0)}{\varphi_S(\cdot,0)+\varphi_1(\cdot,0)+\varphi_2(\cdot,0)} - \frac{\beta_2(\cdot,t)\varphi_S(\cdot,0)\varphi_2(\cdot,0)}{\varphi_S(\cdot,0)+\varphi_1(\cdot,0)+\varphi_2(\cdot,0)}$$

and $B_i^-(t,\varphi) \equiv 0$ for $i = 1, 2$. Then $v^+(t)$ satisfies

$$v^+(t) = U(t,s)v^+(s) + \int_s^t U(t,r)B^+(r, v_r)dr.$$

Let $w(x,t) \equiv (0,0,0),\ \forall (x,t) \in \bar{\Omega} \times [-\tau, \infty)$. Similarly, we define $w_t \in \mathbb{C}_\tau$ by

$$w_t(\theta) = w(t+\theta),\ \forall \theta \in [-\tau, 0].$$

Then the function $w(t)$ satisfies

$$w(t) = U(t,s)w(s) + \int_s^t U(t,r)B^-(r, w_r)dr.$$

Let $B(t,\varphi) = F(t,\varphi)$. Thus, for every $t > 0$ and $w_t(\cdot,\cdot) \leqslant \varphi(\cdot,\cdot) \leqslant v_t^+(\cdot,\cdot)$ on $\bar{\Omega} \times [-\tau, 0]$, one has

$$\lim_{h\to 0^+} \frac{1}{h}\mathrm{dist}\left(v^+(t) - \varphi(0) + h[B^+(t, v_t^+) - B(t,\varphi)],\ \mathbb{X}^+\right) = 0$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\mathrm{dist}\left(\varphi(0) - w(t) + h[B(t,\varphi) - B^-(t, w_t)],\ \mathbb{X}^+\right) = 0.$$

Due to Martain and Smith [37, Proposition 3], we know that (2.5) admits a unique mild solution $u(x,t;\phi)$ on $t \in [0,\infty)$ with $u_0(\cdot,\cdot;\phi) = \phi$. We also have that $u(x,t;\phi)$ is classic for $t > \tau$ by the analyticity of $U(t,s)$.

Let

$$P(t) = \int_\Omega \left(S(x,t) + \sum_{i=1}^{2}(L_i(x,t) + I_i(x,t))\right)dx.$$

Then one has

$$\begin{aligned}\frac{dP(t)}{dt} &\leqslant \int_\Omega \mu(x,t)dx - \bar{d}_{min}\int_\Omega \left(S(x,t) + \sum_{i=1}^{2}(L_i(x,t) + I_i(x,t))\right)dx \\ &\leqslant \bar{\mu}_{max} - \bar{d}_{min}P(t),\ t > \tau,\end{aligned} \tag{2.11}$$

where $\bar{\mu}_{max} = \sup\{\mu(t,x)\}|\Omega|$ ($|\Omega|$ is a measure of Ω) and $\bar{d}_{min} = \inf_{(x,t)\in\Omega\times[\tau,\tau+T]} d(x,t)$. Using the comparison principle for the above equation (2.11), we obtain that there is a constant $M_0 = \frac{\bar{\mu}_{\max}}{\bar{d}_{\min}} > 0$ so that for each $\phi \in \mathbb{C}_\tau^+$, there exists an $l = l(\phi) \in \mathbb{N}$ large enough satisfying

$$P(t) \leqslant M_0 + 1,\ \forall t \geqslant lT + \tau.$$

By the uniform boundedness of $\Gamma_i(x,y,t,t-\tau_i)$ and $\beta_i(x,t)$ for any $x, y \in \Omega$ and $t \in [\tau_i, \tau_i + T]$, one has

$$\begin{cases} \frac{\partial}{\partial t}I_i \leqslant D_i\Delta I_i - \big(d(x,t) + \kappa_i(x,t) + \delta_i(x,t)\big)I_i(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_{i1}(y,t-\tau_i)I_i(y,t-\tau_i)dy \\ \qquad \leqslant D_i\Delta I_i(x,t) - \big(d(x,t) + \kappa_i(x,t) + \delta_i(x,t)\big)I_i(x,t) + \tilde{B}\int_\Omega I_i(y,t-\tau_i)dy \\ \qquad x \in \Omega, \quad t > lT + \tau, \\ \frac{\partial}{\partial n}I_i = 0, \quad x \in \partial\Omega, \quad t > lT + \tau, \end{cases}$$

where

$$\tilde{B} := \sup_{t\in[\tau_i,\tau_i+T],\ x,y\in\Omega} \Gamma_i(x,y,t,t-\tau_i)\beta_{i1}(y,t-\tau_i).$$

By the comparison principle, there is a $B > 0$, which is independent upon the initial value $\phi \in \mathbb{C}_\tau^+$, such that for each $\phi \in \mathbb{C}_\tau^+$, there is $\tilde{l}(\phi) \in \mathbb{N}$ with $\tilde{l}(\phi) \gg l(\phi)$ large enough satisfying $I_i(x, t; \phi) \leqslant B$ for $x \in \bar{\Omega}$, $t \geqslant \tilde{l}T + \tau$ and $i = 1, 2$.

For $B > 0$ given above, consider the following equation

$$\begin{cases} \frac{\partial}{\partial t} u_S = D_S \Delta u_S + \mu(x,t) - d(x,t) u_S(x,t) + B(\delta_1(x,t) + \delta_2(x,t)), \\ \qquad x \in \Omega,\ t > lT + \tau, \\ \frac{\partial}{\partial n} u_S = 0,\ x \in \partial\Omega,\ t > \tilde{l}T + \tau. \end{cases} \tag{2.12}$$

Since the S-equation of system (2.5) can be dominated by (2.12) for any $t > \tilde{l}T + \tau$, there is $B_s > 0$ such that for each $\phi \in \mathbb{C}_\tau^+$, there is $l_s = l_s(\phi) \in \mathbb{N}$ with $l_s > \tilde{l}(\phi)$ satisfying $S(x, t; \phi) \leqslant B_s$ for $x \in \bar{\Omega}$ and $t \geqslant l_s T + \tau$.

Define $\Phi_t : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ by $\Phi_t(\phi)(x, s) = u_t(\phi)(x, s) = u(x, t + s; \phi)$ for $x \in \bar{\Omega}$, $t > 0$ $s \in [-\tau, 0]$ and $\phi \in \mathbb{C}_\tau^+$. Similar to the proof of Zhang et al. [29, Lemma 2.1] one can show that $\{\Phi_t\}_{t \geqslant 0}$ is a T-periodic semiflow on $\mathbb{C}_\tau^+$. According to the above discussion, one knows that Φ_t is point dissipative. Let $n_0 T > 2\tau$. Then $\Phi_T^{n_0} = u_{n_0 T}$ is compact. Following from Magal and Zhao [38, Theorem 2.9], we have that $\Phi_T : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ admits a compact global attractor. $\square$

Remark 2.2. In the following, we always denote $\tau := \max\{\tau_1, \tau_2\}$ and let $n_0 \in \mathbb{N}$ satisfy $n_0 T > 2\tau$.

3. Threshold dynamics

In this section, we firstly analyze the dynamics of single-strain SIS epidemic models and then study the dynamics of the two-strain SIS model (2.5).

3.1. Threshold dynamics of single-strain SIS epidemic models

We investigate the dynamics of single-strain SIS models in this subsection. We fix $i \in \{1, 2\}$ and let $I_j(x, t) \equiv 0$, $\forall (x, t) \in \bar{\Omega} \times \mathbb{R}^+$, $j = 1, 2$ and $j \neq i$. Then system (2.5) reduces to the following single-strain model

$$\begin{cases} \frac{\partial}{\partial t} S = D_S \Delta S + \mu(x,t) - d(x,t) S(x,t) + \delta_i(x,t) I_i(x,t) \\ \qquad - \frac{\beta_i(x,t) S(x,t) I_i(x,t)}{S(x,t) + I_i(x,t)},\ t > 0,\ x \in \Omega, \\ \frac{\partial}{\partial t} I_i = D_i \Delta I_i - (d(x,t) + \kappa_i(x,t) + \delta_i(x,t))\, I_i(x,t) \\ \qquad + \int_\Omega \Gamma_i(x, y, t, t - \tau_i) \frac{\beta_i(y, t-\tau_i) S(y, t-\tau_i) I_i(y, t-\tau_i)}{S(y, t-\tau_i) + I_i(y, t-\tau_i)} dy,\ x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial n} S = \frac{\partial}{\partial n} I_i = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.1}$$

Let $\mathcal{C}_T(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ be the set of all continuous and T-periodic functions from $\bar{\Omega} \times \mathbb{R}$ to $\mathbb{R}$ with norm $\|\rho\|_{\mathcal{C}_T} = \max_{x \in \bar{\Omega}, t \in [0,T]} |\rho(x, t)|$ for any $\rho \in \mathbb{C}_T$. Let

$$\mathcal{C}_T^+ := \{\rho \in \mathcal{C}_T : \rho(t)(x) \geqslant 0, \forall t \in \mathbb{R}, x \in \bar{\Omega}\},$$

which is the positive cone of $\mathcal{C}_T$. Define $\mathbb{Q} = C([-\tau, 0], \mathbb{Y})$ with the norm $\|\rho\|_{\mathbb{Q}} := \max_{\theta \in [-\tau, 0]} \|\rho(\theta)\|_{\mathbb{Y}}$ for any $\rho \in \mathbb{Q}$. Set $\mathbb{Q}^+ := C([-\tau, 0], \mathbb{Y}^+)$, then $(\mathbb{Q}, \mathbb{Q}^+)$ is a strongly ordered Banach space. Let $\mathbb{P} := C(\bar{\Omega}, \mathbb{R}^2)$ with the supremum norm $\|\cdot\|_{\mathbb{P}}$. Let $\mathbb{D}_\tau := C([-\tau, 0], \mathbb{P})$ with the norm $\|\phi\| = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_{\mathbb{P}}$, $\forall \phi \in \mathbb{D}_\tau$. Let $\mathbb{P}^+ := C(\Omega, \mathbb{R}_+^2)$ and $\mathbb{D}_\tau^+ := C([-\tau, 0], \mathbb{P}^+)$, then $(\mathbb{P}, \mathbb{P}^+)$ and $(\mathbb{D}_\tau, \mathbb{D}_\tau^+)$ are strongly ordered Banach spaces.

Setting $I_i(x, t) \equiv 0$ on $\bar{\Omega} \times \mathbb{R}^+$, we get the following equation for $S(x, t)$:

$$\begin{cases} \frac{\partial}{\partial t} S = D_S \Delta S + \mu(x,t) - d(x,t) S(x,t), \quad x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial n} S = 0, \quad x \in \partial\Omega.\ t > 0, \end{cases} \tag{3.2}$$

By Lemma 2.1 of [29], there is a unique positive solution $S^*(x,t)$ of (3.2) which is T-periodic with respect to $t\in\mathbb{R}$ and globally asymptotically stable. Consequently, we call the function $(S^*,0)$ the disease-free periodic solution of (3.1). Linearizing the second equation of system (3.1) at $(S^*,0)$, we get the linear equation

$$\begin{cases} \frac{\partial}{\partial t}\omega_i = D_i\Delta\omega_i - r_i(x,t)\omega_i(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\omega_i(y,t-\tau_i)dy,\ x\in\Omega,\ t>0, \\ \frac{\partial}{\partial n}\omega_i = 0,\ x\in\partial\Omega,\ t>0, \end{cases} \tag{3.3}$$

where $r_i(x,t)=d(x,t)+\kappa_i(x,t)+\delta_i(x,t)$. Let $\psi_i(x,s)\in\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})$ be the initial distribution of infectious individuals of the ith component at the spatial position $x\in\bar\Omega$ and time $s\in\mathbb{R}$. One defines an operator $C_i:\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})\to\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})$ by

$$(C_i\psi_i)(x,t)=\int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i,)\psi_i(y,t-\tau_i)dy.$$

Fix $t\in\mathbb{R}$. Due to the synthetical influence of mobility, recovery and mortality, the term $V_i(t-\tau_i,s)\psi_i(s)(x)(s<t-\tau_i)$ indicates the density of those infective individuals at location x who were infective at time s and retain infective at time $t-\tau_i$ when time evolved from s to $t-\tau_i$. Furthermore, $\int_{-\infty}^{t-\tau_i}(V_i(t-\tau_i,s)\psi_i(s))(x)ds$ denotes the density distribution of the accumulative infective individuals at position x and time $t-\tau_i$ for all previous time $s<t-\tau_i$. Hence, the term

$$\begin{aligned} &\int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\psi_i(s)(y)dsdy \\ &\quad = \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\int_{\tau_i}^{+\infty}(V_i(t-\tau_i,t-s)\psi_i(t-s))(y)dsdy \\ &\quad = \int_{\tau_i}^{+\infty}\int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)(V_i(t-\tau_i,t-s)\psi_i(t-s))(y)dsdy \end{aligned}$$

represents the density of new infected individuals at time t and location x. Consequently, the next generation infection operator can be defined by

$$\mathcal{L}_i(\psi_i)(x,t)=\int_{\tau_i}^{+\infty}\int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\left(V_i(t-\tau_i,t-s)\psi_i(t-s)\right)(y)dsdy,\quad i=1,2.$$

Obviously, $\mathcal{L}_i$ is a bounded and positive linear operator on $\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})$. Similar to [39–41], define the basic reproduction number R_0^i for strain i by

$$R_0^i := r(\mathcal{L}_i), \tag{3.4}$$

where $r(\mathcal{L}_i)$ denotes the spectral radius of $\mathcal{L}_i$. Here we would like to refer readers to [20,23,24,42] for the latest progress on the theory of the basic reproduction number of the time-periodic reaction–diffusion epidemic models with latent period.

We define another operator $\hat{\mathcal{L}}_i(\psi_i)(t,x):\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})\to\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})$ by

$$\hat{\mathcal{L}}_i(\psi_i)(x,t)=\int_{\tau_i}^{\infty}V_i(t,t-s)\left(C_i\psi_i\right)(t-s)(x)ds,\ t\in\mathbb{R},\ s\geqslant 0. \tag{3.5}$$

Clearly, the linear operator $\hat{\mathcal{L}}_i$ defined on $\mathcal{C}_T(\bar\Omega\times\mathbb{R},\mathbb{R})$ is bounded, positive and compact. Let

$$\tilde{A}_i(\psi_i)(x,t)=(C_i\psi_i)(x,t)\ \text{ and }\ \tilde{B}_i(\psi_i)(x,t)=\int_{\tau_i}^{\infty}\left(V_i(t,t-s+\tau_i)\psi_i(t-s+\tau_i)\right)(x)ds.$$

Since $\mathcal{L}_i=\tilde{A}_i\tilde{B}_i$ and $\hat{\mathcal{L}}_i=\tilde{B}_i\tilde{A}_i$, it follows that $R_0^i=r(\mathcal{L}_i)=r(\hat{\mathcal{L}}_i)$ for $i=1,2$.

Using a similar argument as above, there are constants $\mathcal{Q}_i > 1$ and $c_i \in \mathbb{R}$ satisfying

$$\|V_i(t,s)\| \leqslant \mathcal{Q}_i e^{c_i(t-s)} \text{ for all } t,s \in \mathbb{R} \text{ with } t \geqslant s.$$

Obviously, $c_i^* := \bar{\omega}(V_i) \leqslant c_i$, where $\bar{\omega}(V_i)$ denotes the exponential growth bound of the operator $V_i(t,s)$, namely,

$$\bar{\omega}(V_i) = \inf\{\omega \mid \exists M \geqslant 1 : \forall s \in \mathbb{R},\ t \geqslant 0 : \|V_i(t+s,s)\| \leqslant Me^{\omega t}\}.$$

Since the operator $V_i(t,s)$ is strongly positive and compact on $\mathbb{Y}$ for $t > s$, then the Krein–Rutman theorem implies that the principle eigenvalue of $V_i(T,0)$, defined by $r(V_i(T,0))$, is positive. Following from Hess [34, Lemma 14.2], one has $r(V_i(T,0)) < 1$. By Thieme [43, Proposition 5.6], we further have $c_i^* < 0$. For each $\sigma \in (c_i^*, \infty)$, for $\psi_i \in \mathcal{C}_T(\mathbb{R} \times \bar{\Omega}, \mathbb{R})$ define

$$\left(\hat{\mathcal{L}}_\sigma^i \psi_i\right)(x,t) := \int_{\tau_i}^{\infty} e^{-\sigma s}(V_i(t,t-s)\left(C_i\psi_i\right)(t-s))(x)ds, \tag{3.6}$$

which is a linear operator. Obviously, $\hat{\mathcal{L}}_0^i = \hat{\mathcal{L}}_i$. By [34], we know that for $\sigma \in (c_i^*, \infty)$, the operator $\hat{\mathcal{L}}_\sigma^i$ is bounded. In addition, the operator $\hat{\mathcal{L}}_\sigma^i$ is also compact since $V_i(t,s)$ with $t > s$ is compact. For $\sigma \in (c_i^*, \infty)$, let $\rho_i(\sigma)$ be the spectral radius of $\hat{\mathcal{L}}_\sigma^i$. It is obvious that $R_0^i = r(\mathcal{L}_i) = r(\hat{\mathcal{L}}_i) = \rho_i(0)$. The following lemma gives some properties of $\rho_i(\sigma)$.

Lemma 3.1. *One has*

(i) $\rho_i(\sigma)$ *is non-increasing and continuous in* $\sigma \in (c_i^*, \infty)$*;*

(ii) $\rho_i(\infty) = 0$*;*

(iii) $\rho_i(\sigma) = 1$ *has at most one solution on* $\sigma \in (c_i^*, \infty)$*;* ρ_i *is either strictly decreasing in* $\sigma \in (c_i^*, \infty)$*, or strictly decreasing in* $\sigma \in (c_i^*, b_i)$ *for some* $b_i > c_i^*$*, and* $\rho_i(\sigma) = 0$ *in* $\sigma \in [b_i, \infty)$*.*

The proof of the lemma is similar to [44, Lemma 1] and [29, Lemma 3.2] and we omit the details.

For $\epsilon > 0$, consider equation

$$\begin{cases} \frac{\partial}{\partial t}\underline{\omega}_i^\epsilon = D_i \Delta \underline{\omega}_i^\epsilon - r_i(x,t)\underline{\omega}_i^\epsilon(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\left(\beta_i(y,t-\tau_i)+\epsilon\right)\underline{\omega}_i^\epsilon(y,t-\tau_i)dy,\ x \in \Omega,\ t > 0, \\ \underline{\omega}_i^\epsilon(x,s) = \phi_i(x,s), \phi_i \in \mathbb{Q},\ x \in \Omega,\ s \in [-\tau_i, 0], \\ \frac{\partial}{\partial n}\underline{\omega}_i^\epsilon = 0,\ x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.7}$$

Define the Poincaré map $\mathcal{P}_i^\epsilon : \mathbb{Q} \to \mathbb{Q}$ of (3.7) by $\mathcal{P}_i^\epsilon(\psi_i) = \underline{\omega}_{i,T}^\epsilon(\psi_i)$ for any $\psi_i \in \mathbb{Q}$, where $\underline{\omega}_{i,t}^\epsilon$ is the solution map of (3.7), and $\underline{\omega}_{i,T}^\epsilon(\psi_i)(x,s) = \underline{\omega}_i^\epsilon(x, s+T; \psi_i)$ for any $(x,s) \in \bar{\Omega} \times [-\tau, 0]$. Define $(\mathcal{P}_i^\epsilon)^{n_0} : \mathbb{Q} \to \mathbb{Q}$ by $(\mathcal{P}_i^\epsilon)^{n_0}(\psi_i) = \underline{\omega}_i^\epsilon(x, n_0T + s; \psi_i)$ for any $(x,s) \in \bar{\Omega} \times [-\tau, 0]$. By using arguments similar to Jin and Zhao [45, Proposition 3], we have that $\underline{\omega}_{i,t}^\epsilon(x,s;\psi_i) > 0$ for $t > \tau$, $\psi_i \in \mathbb{Q}^+$ with $\psi_i \not\equiv 0$ and $\underline{\omega}_{i,t}^\epsilon(\cdot,\cdot;\psi_i)$ is strongly positive for $t > 2\tau$. In addition, $\underline{\omega}_{i,t}^\epsilon$ is compact on $\mathbb{Q}^+$ for $t > 2\tau$. Therefore, $(\mathcal{P}_i^\epsilon)^{n_0} = (\underline{\omega}_{i,T}^\epsilon)^{n_0}$ is strongly positive and compact. According to [46, Lemma 3.1], one has that $\mathcal{P}_i^\epsilon$ admits a positive and simple eigenvalue defined by $r_0^{i,\epsilon}$ and a strongly positive eigenfunction denoted by ψ_i, such that $\mathcal{P}_i^\epsilon(\psi_i) = r_0^{i,\epsilon}\psi_i$ and the modulus of any other eigenvalue is less than $r_0^{i,\epsilon}$. Especially, we substitute $\mathcal{P}_i$ and r_0^i with $\mathcal{P}_i^0$ and $r_0^{i,0}$ if $\epsilon = 0$, respectively. Let $\underline{\omega}_i^\epsilon(x,t;\psi_i)$ be the solution of (3.7) with $\underline{\omega}_i^\epsilon(\cdot,\cdot;\psi_i) = \psi_i(\cdot,\cdot)$ on $\bar{\Omega} \times [-\tau, 0]$. Due to the strong positivity of ψ_i, we can get $\underline{\omega}_i^\epsilon(\cdot,\cdot;\psi_i) \gg 0$. Let $\mu^{i,\epsilon} = \frac{\ln r_0^{i,\epsilon}}{T}$ and $\mathcal{V}_i^\epsilon(x,t) = e^{-\mu^{i,\epsilon}t}\underline{\omega}_i^\epsilon(x,t;\psi_i)$ for $x \in \bar{\Omega}$ and $t > -\tau$. Similar to Jin and Zhao [45, Lemma 3.2] and Xu and Zhao [47, Theorem 2.1], we conclude that $e^{\mu^{i,\epsilon}t}\mathcal{V}_i^\epsilon(x,t)$ is a solution of (3.7) and $\mathcal{V}_i^\epsilon(x,t)$ is a nonnegative and nontrivial T-periodic function. Moreover, one has $\mathcal{V}_i^\epsilon(x,t) > 0$ for all $x \in \bar{\Omega}$ and $t \in \mathbb{R}$ because $V_i(t,s)$ is strongly positive. To sum up the above argument, one has the following lemma.

Lemma 3.2. *Let $\mu^{i,\epsilon} = \frac{\ln r_0^{i,\epsilon}}{T}$. Then there is a T-periodic function $\mathcal{V}_i^\epsilon(x,t)$, which is strongly positive, such that $e^{\mu^{i,\epsilon}t}\mathcal{V}_i^\epsilon(x,t)$ is a solution of (3.7).*

Similar to Zhang et al. [29, Lemmas 3.3 and 3.4], we can obtain the following lemma and theorem.

Lemma 3.3. *Let $\mu^i = \frac{\ln r_0^i}{T}$. If $r_0^i > r(V_i(\omega,0))$, then $\rho(\mu^i) = 1$.*

Theorem 3.4. *one has*

(i) *$R_0^i > 1$ if and only if $r_0^i > 1$;*

(ii) *$R_0^i = 1$ if and only if $r_0^i = 1$;*

(iii) *$R_0^i < 1$ if and only if $r_0^i < 1$.*

In the following, the threshold dynamics of system (3.1) are established.

Lemma 3.5. *For the initial value $\psi := (\psi_S, \psi_i) \in \mathbb{D}_\tau^+$, suppose that $(S(x,t;\psi), I_i(x,t;\psi))$ is the solution of (3.1). Then*

(i) *If $I_i(x,t_0;\psi) \not\equiv 0$ for some $t_0 \geqslant 0$, then one has $I_i(x,t;\psi) > 0$ for all $x \in \bar{\Omega}$ and $t > t_0$;*

(ii) *For any $\psi \in \mathbb{D}_\tau^+$, one has $S(\cdot,t;\psi) > 0$, $\forall t > 0$, and $\liminf_{t\to\infty} S(x,t;\psi) \geqslant \mathcal{Q}$ uniformly for $x \in \bar{\Omega}$, where the constant $\mathcal{Q} > 0$ is independent of ψ.*

Proof. By Theorem 2.1, it is clear that $I_i(x,t;\psi)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} I_i \geqslant D_i \Delta I_i - r_i(x,t) I_i(x,t), & x \in \Omega, \quad t > 0, \\ \frac{\partial}{\partial n} I_i = 0, & x \in \partial\Omega, \quad t > 0. \end{cases}$$

If $I_i(x,t;\psi) \not\equiv 0$ for some $t_0 \geqslant 0$ and $i = 1,2$, then by the maximum principle [34, Proposition 13.1], one has $I_i(x,t;\psi) > 0$ for any $x \in \bar{\Omega}$ and $t > t_0$.

Assume that $w(x,t)$ solves the equation

$$\begin{cases} \frac{\partial}{\partial t} w = D_S \Delta w + \mu(x,t) - (d(x,t) + \beta_1(x,t) + \beta_2(x,t)) w(x,t), & x \in \Omega, \quad t > 0, \\ \frac{\partial}{\partial n} w = 0, \quad x \in \partial\Omega, \quad t > 0, \\ w(x,0) = \phi_S(x,0), \quad x \in \Omega. \end{cases} \tag{3.8}$$

Then $S(x,t) \geqslant w(x,t)$ for any $x \in \bar{\Omega}$ and $t > 0$ by the comparison principle. Let $w^*(x,t)$ be the unique positive T-periodic solution of (3.8). Then according to [29, Lemma 2.1], one has

$$\liminf_{t\to\infty} S(x,t) \geqslant \liminf_{t\to\infty} w^*(x,t) \quad \text{uniformly for } x \in \bar{\Omega}. \quad \square$$

Based on Theorem 2.1, there is $B_s > 0$ so that for every $\psi \in \mathbb{C}_\tau^+$, there exists a $l_s \in \mathbb{N}$ large enough satisfying $S(x,t;\psi) \leqslant B_s$ for any $x \in \bar{\Omega}$, $t > l_s T + \tau$. Consider equation

$$\begin{cases} \frac{\partial}{\partial t} w_i = D_i \Delta w_i - r_i(x,t) w_i(x,t) + \int_\Omega \Gamma_i(x,y,t,t-\tau_i) \frac{B_s \beta_i(y,t-\tau_i) w_i(y,t-\tau_i)}{B_s + w_i(y,t-\tau_i)} dy, \\ \qquad\qquad x \in \Omega, \quad t > 0, \\ \frac{\partial}{\partial n} w_i = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases} \tag{3.9}$$

Lemma 3.6. *Assume that $w_i(x,t;\psi_i)$ is the solution of (3.9) with $w_i(x,s) = \psi_i(x,s)$, $\forall x \in \bar{\Omega}$, $s \in [-\tau, 0]$, $\psi_i \in \mathbb{Q}^+$. If $R_0^i = 1$ and $\beta_i(x,t) > 0$ for any $x \in \bar{\Omega}$ and $t \in \mathbb{R}^+$, then $w_i(x,t) \equiv 0$ is globally attractive on $\bar{\Omega} \times \mathbb{R}^+$.*

Proof. By a straightforward computation, one has that (3.9) is dominated by (3.7). Note that ${\mathcal{P}_i}^{n_0}$ is defined as before. If $\beta_i(x,t) > 0$, $\forall x \in \bar{\Omega}$, $t \in \mathbb{R}$, then the Poincaré map ${\mathcal{P}_i}^{n_0}$ is strongly positive and compact. Then ${\mathcal{P}_i}^{n_0}$ has a simple eigenvalue denoted by $(r_0^i)^{n_0}$ and a strongly positive eigenfunction defined by $\psi_i \in \mathbb{Q}$ and the modulus of any other eigenvalue is less than $(r_0^i)^{n_0}$. In view of Theorem 3.4, we have $r_0^i = 1$ if $R_0^i = 1$. In particular, we have $\mu_i = 0$ if $R_0^i = 1$. As the previous argument (see Lemma 3.2), there is a strongly positive periodic T-periodic function $\mathcal{V}_i^*(x,t)$ satisfying equation (3.7) with $\epsilon = 0$. Then for each initial value $\psi_i(x,s) \in \mathbb{Q}$ ($x \in \Omega, s \in [-\tau, 0]$), there is a number $\kappa > 0$ satisfying $\psi_i(x,s) \leqslant \kappa \mathcal{V}_i^*(x,s)$ for any $x \in \Omega$ and $s \in [-\tau, 0]$. By the parabolic comparison principle, one has $w_i(x,t;\psi_i) \leqslant \kappa \mathcal{V}_i^*(x,t)$ for any $(x,t) \in \Omega \times \mathbb{R}^+$. Especially, $S_i^{n_0}(\phi_i) := w_i(x, n_0T + s; \psi_i) \subset [0, \kappa\mathcal{V}_i^*]_{\mathbb{Q}}, \forall x \in \Omega,\ s \in [-\tau, 0]$, where

$$[0, \kappa\mathcal{V}_i^*]_{\mathbb{Q}} := \{u \in \mathbb{Q} : 0 \leqslant u(x,s) \leqslant \kappa\mathcal{V}_i^*(x,s),\ \forall x \in \bar{\Omega},\ s \in [-\tau, 0]\}.$$

It is clear that the positive orbit $\gamma_+(\phi_i) := \{S_i^{kn_0}(\phi_i) : \forall k \in \mathbb{N}\}$ of $S_i^{n_0}(\cdot)$ is precompact. In addition, $S_i^{n_0}$ maps $[0, \kappa\mathcal{V}_i^*]_{\mathbb{Q}}$ into $[0, \kappa\mathcal{V}_i^*]_{\mathbb{Q}}$ and $S_i^{n_0}(\cdot)$ is monotone. Then applying Zhao [48, Theorem 2.2.2], we obtain the conclusion. □

Theorem 3.7. *Suppose that $(S(x,t;\psi), I_i(x,t;\psi))$ is the solution of (3.1) with the initial value $\psi = (\psi_S, \psi_i) \in \mathbb{D}_\tau$. Then one has:*

(1) *if $R_0^i < 1$, then the T-periodic solution $(S^*, 0)$ is globally attractive;*

(2) *if $R_0^i = 1$ and $\beta_i(x,t) > 0$ for $x \in \bar{\Omega}$ and $t \in \mathbb{R}$, then the T-periodic solution $(S^*, 0)$ is globally attractive;*

(3) *if $R_0^i > 1$, then there is a $M > 0$ so that for any $\psi \in \mathbb{D}_\tau^+$, one has*

$$\liminf_{t\to\infty} S(x,t;\psi) \geqslant M \quad \text{and} \quad \liminf_{t\to\infty} I_i(x,t;\psi) \geqslant M \qquad \text{uniformly for } x \in \bar{\Omega}.$$

Proof. 1 Assume that $R_0^i < 1$. By Theorem 3.4, one has $r_0^i < 1$. For $\epsilon > 0$, consider equation

$$\begin{cases} \frac{\partial}{\partial t}\omega_i^\epsilon = D_i \Delta \omega_i^\epsilon - r_i(x,t)\omega_i^\epsilon(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)(\beta_i(y,t-\tau_i) + \epsilon)\omega_i^\epsilon(y,t-\tau_i)dy, \ x \in \Omega, \quad t > 0, \\ \frac{\partial}{\partial n}\omega_i^\epsilon = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{3.10}$$

See Lemma 3.2 for the definitions of μ_i^ϵ and $r_0^{i,\epsilon}$. Since $r_0^i < 1$, there is a constant $\epsilon_0 > 0$ satisfying $r_0^{i,\epsilon} < 1$ for $\epsilon \in [0, \epsilon_0)$. Fix $\epsilon \in [0, \epsilon_0)$. Then one has $\mu^{i,\epsilon} := \frac{\ln r_0^{i,\epsilon}}{T} < 0$. It follows from Lemma 3.2 that there is a function $\mathcal{V}_i^\epsilon(x,t)$, which is T-periodic and strongly positive, such that $\omega_i^\epsilon(x,t) = e^{\mu^{i,\epsilon}t}\mathcal{V}_i^\epsilon(x,t)$ satisfies (3.10).

For $x \in \Omega$ and $t \geqslant 0$, one has

$$\begin{cases} \frac{\partial}{\partial t} I_i \leqslant D_i \Delta I_i - r_i(x,t) I_i(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)[\beta_i(y,t-\tau_i) + \epsilon] I_i(y,t-\tau_i)dy, \ x \in \Omega, \ t > 0, \\ \frac{\partial}{\partial n} I_i = 0, \ x \in \partial\Omega, \quad t > 0. \end{cases}$$

For any given initial distribution $\psi \in \mathbb{D}_\tau^+$, due to the boundedness of $I_i(x,t;\psi)$, there is some $\alpha > 0$ such that $I_i(x,t;\psi) \leqslant \alpha e^{\mu_i^\epsilon t}\mathcal{V}_i^\epsilon(x,t)$ for $x \in \bar{\Omega}$ and $t \in [-\tau, 0]$. By the comparison theorem (see Martin and Smith [37, Proposition 3]), we have $I_i(x,t;\psi) \leqslant \alpha\ e^{\mu_i^\epsilon t}\mathcal{V}_i^\epsilon(x,t)$ for any $x \in \bar{\Omega}$ and $t > 0$. Because of $\mu_i^\epsilon < 0$, we have that $I_i(x,t;\psi) \to 0$ as $t \to \infty$ uniformly $x \in \bar{\Omega}$. Therefore, the S equation in (2.5) is asymptotic to (3.2). By Zhang et al. [29, Lemma 2.1], we get that the solution $S^*(x,t)$ of (3.2) is globally attractive. Consequently, similar to these arguments of [29, Theorem 4.3 (i)], we have

$$\lim_{t\to\infty} \|S(\cdot,t;\psi) - S^*(\cdot,t)\|_{C(\bar{\Omega})} = 0.$$

(2) Suppose that $R_0^i = 1$ and $\beta_i(x,t) > 0$ for $x \in \bar{\Omega},\ t \in \mathbb{R}$. Using (3.9) and the I_i-equation of system (3.1), one gets $w_i(x,t) \geqslant I_i(x,t;\psi)$ for $x \in \bar{\Omega}$ and $t > l_s T + \tau$, where l_s is defined by Theorem 2.1. Using (3.9), one has that $I_i(x,t;\psi) \to 0$ as $t \to \infty$ uniformly $x \in \bar{\Omega}$. Similar to (i), one has

$$\lim_{t\to\infty} \|S(\cdot,t;\psi) - S^*(\cdot,t)\|_{C(\bar{\Omega})} = 0.$$

(3) Suppose $R_0^i > 1$. Then $r_0^i > 1$. Let

$$\mathbb{W}_0^i = \{\psi = (\psi_S, \psi_i) \in \mathbb{D}_\tau^+ : \psi_i(\cdot,0) \not\equiv 0\}$$

and

$$\partial\mathbb{W}_0^i := \mathbb{D}_\tau^+ \backslash \mathbb{W}_0 = \{\psi = (\psi_S, \psi_i) \in \mathbb{D}_\tau^+ : \psi_i(\cdot,0) \equiv 0\}.$$

Assume $\psi \in \mathbb{W}_0^i$. By Lemma 3.5, one gets $I_i(x,t;\psi) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Thus, for any $k \in \mathbb{N}$, one has $\Psi_{n_0T}^k(\mathbb{W}_0^i) \subseteq (\mathbb{W}_0^i)$, where $\Psi_t : \mathbb{D}_\tau^+ \to \mathbb{D}_\tau^+$ is defined by $\Psi_t(\psi)(x,s) = (S(x,t+s;\psi), I_i(x,t+s;\psi))$ and $(S(x,t;\psi), I_i(x,t;\psi))$ is the solution of (3.1) with initial data $\psi = (\psi_S, \psi_i) \in \mathbb{D}_\tau$. Define

$$M_\partial := \{\psi \in \partial\mathbb{W}_0^i : \Psi_{n_0T}^k(\psi) \in \partial\mathbb{W}_0^i, \forall k \in \mathbb{N}\}.$$

Let $\mathcal{M} := (S^*, \hat{0})$. Here $\hat{0}$ denotes a function which is identical to 0 for $[-\tau, 0]$. Denote $\omega(\psi)$ be the omega limit set of the orbit $\gamma^+ := \{\Psi_{n_0T}^k(\psi) : \forall k \in \mathbb{N}\}$. For any $\psi \in M_\partial$, one has $\Psi_{n_0T}^k(\psi) \in \partial\mathbb{W}_0^i$, $\forall k \in \mathbb{N}$. Thus, one concludes that $I_i(x,t;\psi) \equiv 0$ for $x \in \bar{\Omega}$ and $t \geqslant 0$. Following from [29, Lemma 2.1], one has $\lim_{t\to\infty} \|S(\cdot,t;\psi) - S^*(\cdot,t)\|_{C(\bar{\Omega})} = 0$. Consequently, one gets $\omega(\psi) = \{\mathcal{M}\}$, $\forall \psi \in M_\partial$.

Now, we have a claim as follows:

Claim. *$\mathcal{M}$ is a uniform weak repeller for $\mathbb{W}_0^i$ in the sense that, for some $\theta_0 > 0$,*

$$\limsup_{k\to\infty} \|\Psi_{n_0T}^k(\psi) - \mathcal{M}\| \geqslant \theta_0, \quad \forall \psi \in \mathbb{W}_0^i.$$

We firstly take into account a linear equation with a parameter $\theta > 0$ as follows

$$\begin{aligned}\frac{\partial}{\partial t} v_i^\theta =& D_i \Delta v_i^\theta - r_i(x,t) v_i^\theta(x,t) \\ &+ \int_\Omega \Gamma_i(x,y,t,t-\tau_i) \frac{\beta_i(y,t-\tau_i)(S^*(y,t-\tau_i) - \theta)}{S^*(y,t-\tau_i)} v_i^\theta(y,t-\tau_i) dy, \\ & x \in \Omega,\ t > 0.\end{aligned} \tag{3.11}$$

Define the Poincaré map $(\mathcal{E}_\theta^i)^{n_0} : \mathbb{Q}^+ \to \mathbb{Q}^+$ of (3.11) by $(\mathcal{E}_\theta^i)^{n_0}(\psi_i) = v_{i,n_0T}^\theta(\psi_i)$, where $v_{i,n_0T}^\theta(\psi_i)(x,s) = v_i^\theta(x, s+n_0T; \psi_i)$ for $(x,s) \in \bar{\Omega} \times [-\tau, 0]$, and $v_i^\theta(x,t;\psi_i)$ is the solution of (3.11) with $v_i^\theta(x,s) = \psi_i(x,s)$ for all $(x,s) \in \bar{\Omega} \times [-\tau, 0]$. The map $(\mathcal{E}_\theta^i)^{n_0}$ is positive and compact. The Krein–Rutman theorem (see, for example Hess [34, Theorem 7.1]) implies that $(\mathcal{E}_\theta^i)^{n_0}$ admits a positive and simple eigenvalue defined by $\tilde{r}_\theta^i$ and a positive eigenfunction denoted by $\tilde{\varphi}_i \in \mathbb{Q}$ satisfying $(\mathcal{E}_\theta^i)^{n_0}(\tilde{\varphi}_i) = \tilde{r}_\theta^i \tilde{\varphi}_i$. Since $r_0^i > 1$ and hence $(r_0^i)^{n_0} > 1$, there is $\theta_1 > 0$ satisfying $\tilde{r}_\theta^i > 1$ for $\theta \in (0, \theta_1)$. Fix $\bar{\theta} \in (0, \theta_1)$. In view of the continuous dependence of solutions on the initial data, there is $\theta_0 \in (0, \theta_1)$ satisfying

$$\|(S(\cdot,\cdot), I_i(\cdot,\cdot)) - (S^*(\cdot,\cdot), 0)\|_{C(\bar{\Omega},\ t\in[0,n_0T])} < \bar{\theta} \tag{3.12}$$

if $|(\phi_S(x,s), \phi_i(x,s)) - (S^*(x,s), 0)| < \theta_0,\ \forall x \in \bar{\Omega},\ s \in [-\tau, 0]$.

Next, we show the claim by contradiction. Suppose

$$\limsup_{k\to\infty} \|\Psi_{n_0T}^k(\psi) - \mathcal{M}\| < \theta_0$$

for some $\psi \in \mathbb{W}_0^i$; that is, there is a $\tilde{k}_1 \in \mathbb{N}$ satisfying

$$|(S(x, kn_0T + s; \psi), I_i(x, kn_0T + s; \psi)) - (S^*(x, s), 0)| < \theta_0$$

for any $x \in \bar{\Omega}$, $s \in [-\tau, 0]$ and $k > \tilde{k}_1$. Following from (3.12), one has that $0 < I_i(x, t, \psi) < \bar{\theta}$ and $S^*(x, t; \psi) - \bar{\theta} < S(x, t; \psi) < S^*(x, t; \psi)$ for any $x \in \bar{\Omega},\ t \geqslant \tilde{k}_1 n_0 T + \tau$. Consequently, $I_i(x, t; \psi)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t} I_i \geqslant\ & D_i \Delta I_i - r_i(x, t) I_i(x, t) \\ & + \int_{\Omega} \Gamma_i(x, y, t, t - \tau_i) \frac{\beta_i(y, t - \tau_i)(S^*(y, t - \tau_i) - \bar{\theta}) I_i(y, t - \tau_i)}{S^*(y, t - \tau_i)} dy\end{aligned}$$

for any $x \in \Omega,\ t > (\tilde{k}_1 + 1) n_0 T$. Since $I_i(x, t + s; \psi) > 0$ for $x \in \bar{\Omega},\ t > \tau$ and $s \in [-\tau, 0]$, there exists $\kappa > 0$ satisfying

$$I_i((x, \tilde{k}_1 + 1) n_0 T + s; \psi) \geqslant \kappa \tilde{\varphi}_i(x, s), \quad x \in \bar{\Omega},\ s \in [-\tau, 0],$$

where $\tilde{\varphi}_i$ is the eigenfunction corresponding to the eigenvalue $\tilde{r}^i_{\bar{\theta}}$. Due to the comparison principle, we have

$$I_i(x, t + s; \psi) \geqslant \kappa v_i(x, t - (\tilde{k}_1 + 1) n_0 T + s; \tilde{\varphi}_i), \quad \forall x \in \bar{\Omega},\ t \geqslant (\tilde{k}_1 + 1) n_0 T,\ s \in [-\tau, 0], .$$

It follows that

$$I_i(x, kn_0T + s; \psi) \geqslant \kappa v_i(x, (k - \tilde{k}_1 - 1) n_0 T + s; \tilde{\varphi}_i)) = \kappa (\tilde{r}^i_{\bar{\theta}})^{(k - \tilde{k}_1 - 1)} \tilde{\varphi}_i(x, s) \tag{3.13}$$

for any $x \in \bar{\Omega},\ k > \tilde{k}_1 + 1,\ s \in [-\tau, 0]$. Since $\tilde{\varphi}_i(x, s)$ is positive for $(x, s) \in \bar{\Omega} \times [-\tau, 0]$, we can get $\tilde{\varphi}_i(x_i, s_i) > 0$ for some $(x_i, s_i) \in \bar{\Omega} \times [-\tau, 0]$. Thus, it follows from (3.13) that $I_i(x_i, kT + s_i; \psi) \to +\infty$ as $k \to \infty$. Clearly, there is a contradiction due to the boundedness of $I_i(x, t + s; \psi)$. The claim is proved.

Based on the above claim, it is observed that $\mathcal{M}$ is isolated and invariant for Ψ_{n_0T} in $\mathbb{W}_0^i$, and $W^s(\mathcal{M}) \cap \mathbb{W}_0^i = \emptyset$, where $W^s(\mathcal{M})$ denotes the stable set of $\mathcal{M}$. According to the acyclicity theorem on uniform persistence for maps (see, for example [48, Theorem 1.3.1 and Remark 1.3.1]), one has that $\Psi_{n_0T} : \mathbb{D}_\tau^+ \to \mathbb{D}_\tau^+$ is uniformly persistent on $(\mathbb{W}_0^i, \partial \mathbb{W}_0^i)$; that is, there is a $\tilde{\delta} > 0$ satisfying

$$\liminf_{k \to \infty} d(\Psi^k_{n_0T}, \partial \mathbb{W}_0^i) \geqslant \tilde{\delta},\ \forall \phi \in \mathbb{W}_0^i.$$

By [48, Theorem 3.1.1], we further have that the semiflow $\Psi_t : \mathbb{D}_\tau^+ \to \mathbb{D}_\tau^+$ is uniformly persistent on $(\mathbb{W}_0^i, \partial \mathbb{W}_0^i)$. Then by the compactness of Ψ_{n_0T}, Ψ_{n_0T} is a κ-condensing. Therefore, Magal and Zhao [38, Theorem 4.5] implies that $\Psi_{n_0T} : \mathbb{W}_0^i \to \mathbb{W}_0^i$ with $\rho(\varphi) = d(\varphi, \partial \mathbb{W}_0^i)$ has a global attractor $\mathcal{Z}_0$.

Next, define $p : \mathbb{Q}^+ \to [0, \infty)$ by

$$p(\psi_i) = \min_{x \in \bar{\Omega}} \psi_i(x, 0), \quad \forall \psi_i \in \mathbb{Q}^+.$$

Since $\mathcal{Z}_0 = \Psi_{n_0T}(\mathcal{Z}_0)$, we have that $\psi_i(\cdot, 0) > 0,\ \psi \in \mathcal{Z}_0$. Let $\mathcal{B}_i := \bigcup_{t \in [0, n_0T]} \Psi_t(\mathcal{Z}_0)$. It then follows that $\mathcal{B}_i \subset \mathbb{W}_0^i$ and $\lim_{t \to \infty} d(\Psi_t(\psi), \mathcal{B}_i) = 0$ for all $\psi \in \mathbb{W}_0^i$. Since $\mathcal{B}_i$ is a compact subset of $\mathbb{W}_0^i$, one has $\min_{\psi \in \mathcal{B}_i} p(\psi) > 0$. Thus, there is a $\delta^* > 0$ satisfying $\liminf_{t \to \infty} I_i(\cdot, t; \psi) \geqslant \delta^*$. Combining Lemma 3.5, we get the persistence statement in the second result. The proof is completed. □

Remark 3.8. Consider the following equation

$$\begin{cases} \frac{\partial}{\partial t} \bar{S}_i = D_S \Delta \bar{S}_i + \mu(x, t) - d(x, t) \bar{S}_i(x, t) - \beta_i(x, t) \bar{S}_i(x, t), & x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial n} \bar{S}_i = 0,\ x \in \partial \Omega,\ t > 0. \end{cases} \tag{3.14}$$

By [29, Lemma 2.1], we know that Eq. (3.14) has a unique positive and globally attractive T-periodic solution $S_i^*(x,t)$ in $\mathbb{Y}^+$. Since $S(x,t;\psi)$ satisfies the inequality

$$\begin{cases} \frac{\partial}{\partial t}S \geq D_S\Delta S + \mu(x,t) - d(x,t)S(x,t) - \beta_i(x,t)S(x,t), & x\in\Omega,\ t>0,\\ \frac{\partial}{\partial n}S = 0,\ x\in\partial\Omega,\ t>0, \end{cases}$$

then the comparison principle implies

$$\liminf_{t\to\infty}(S(x,t)-S_i^*(x,t))\geq 0 \qquad \text{uniformly for } x\in\bar{\Omega}.$$

Moreover, by Theorem 2.1, there is a $B>0$ such that for each $\phi\in\mathbb{C}_\tau^+$, there is $l_i\in\mathbb{N}$ satisfying

$$I_i(x,t;\phi)\leqslant B \quad \text{for all } x\in\bar{\Omega} \text{ and } t\geqslant l_iT+\tau. \tag{3.15}$$

For the sake of convenience, let $\mathcal{F}_i(x,t)=S_i^*(x,t)+B$ for $(x,t)\in\bar{\Omega}\times\mathbb{R}^+$ and $i=1,2$.

3.2. Threshold dynamics of a two-strain SIS model

In this subsection, we study the dynamics of (2.5). The following lemma is similar to Lemma 3.5.

Lemma 3.9. *Suppose that $(S(\cdot,\cdot;\phi),I_1(\cdot,\cdot;\phi),I_2(\cdot,\cdot;\phi))$ is a solution of (2.5) with the initial data $\phi:=(\phi_S,\phi_1,\phi_2)\in\mathbb{C}_\tau^+$. Then*

(i) *If there is some $t_0\geqslant 0$ satisfying $I_1(x,t_0;\phi)\not\equiv 0$ ($I_2(x,t_0;\phi)\not\equiv 0$), then*

$$I_1(x,t;\phi)>0\ (I_2(x,t;\phi)>0),\quad \forall x\in\bar{\Omega},\ t>t_0;$$

(ii) *For any $\phi\in\mathbb{C}_\tau^+$, one has $S(\cdot,t;\phi)>0$, $\forall t>0$, and $\liminf_{t\to\infty}S(x,t;\phi)\geqslant\mathcal{Q}$ uniformly for $x\in\bar{\Omega}$, where the constant $\mathcal{Q}>0$ is independent of ϕ.*

Next, we define the so-called invasion numbers $\hat{R}_0^i$ for the ith strain ($i=1,2$). Consider the following system

$$\begin{cases} \frac{\partial}{\partial t}u_i = D_i\Delta u_i - r_i(x,t)u_i(x,t)\\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\frac{S_j^*(y,t-\tau_i)}{\mathcal{F}_j(y,t-\tau_i)}u_i(y,t-\tau_i)dy, \quad x\in\Omega,\ t>0,\\ \frac{\partial}{\partial n}u_i=0,\ x\in\partial\Omega,\ t>0 \end{cases} \tag{3.16}$$

for $i\neq j$ and $i,j=1,2$. Define the Poincaré map $\hat{\mathcal{P}}_i:\mathbb{Q}\to\mathbb{Q}$ of (3.16) by $\hat{\mathcal{P}}_i(\phi_i)=u_{i,T}(\phi_i)$ for all $\phi_i\in\mathbb{Q}$, where $u_{i,T}(\phi_i)(x,s)=u_i(x,s+T;\phi_i)$ for any $(x,s)\in\bar{\Omega}\times[-\tau,0]$, and $u_i(x,t;\phi_i)$ is the solution of (3.16) with initial value $\phi_i\in\mathbb{Q}$. Denote the spectral radius of $\hat{\mathcal{P}}_i$ by ρ_i^0. Let

$$\begin{aligned}(\tilde{\mathcal{L}}_i\phi_i)(x,t) = \int_{\tau_i}^{+\infty}\int_\Omega \Gamma_i(x,y,t,t-\tau_i)\beta_i(y,t-\tau_i)\frac{S_j^*(y,t-\tau_i)}{\mathcal{F}_j(y,t-\tau_i)}\\ \times V_i(t-\tau_i,t-s)\phi_i(t-s)(y)dsdy,\quad \forall\phi_i\in\mathbb{C}_T(\bar{\Omega}\times\mathbb{R},\mathbb{R}).\end{aligned}$$

Obviously, the linear operator $\tilde{\mathcal{L}}_i$ on $\mathcal{C}_T(\bar{\Omega}\times\mathbb{R},\mathbb{R})$ is bounded and positive. Following [39–41], we define the invasion number for the ith strain by

$$\hat{R}_0^i := r(\tilde{\mathcal{L}}_i). \tag{3.17}$$

Similar to Zhang et al. [29, Lemma 3.4], one has the following results.

Theorem 3.10. *One has*

(i) $\hat{R}_0^i > 1$ *if and only if* $\rho_i^0 > 0$;

(ii) $\hat{R}_0^i = 1$ *if and only if* $\rho_i^0 = 0$;

(iii) $\hat{R}_0^i < 1$ *if and only if* $\rho_i^0 < 0$.

Proposition 3.11. *If* $\hat{R}_0^i > 1$, *then* $R_0^i > 1$ *for* $i = 1, 2$.

3.2.1. Competitive exclusion and coexistence

Now we present the competitive exclusion and coexistence of (2.5) by virtue of R_0^1, R_0^2, $\hat{R}_0^1$ and $\hat{R}_0^2$.

Theorem 3.12. *Assume that* $R_0^2 > 1 > R_0^1$. *Suppose that* $(S(\cdot,\cdot;\phi), I_1(\cdot,\cdot;\phi), I_2(\cdot,\cdot;\phi))$ *is the solution of* (2.5) *with the initial data* $\phi := (\phi_S, \phi_1, \phi_2) \in \mathbb{C}_\tau^+$. *Then there is a* $\mathcal{P} > 0$ *such that for each* $\phi \in \mathbb{C}_\tau^+$ *with* $\phi_2(\cdot, 0) \not\equiv 0$, *one has* $\lim_{t\to\infty} I_1(x,t;\phi) = 0$ *and*

$$\liminf_{t\to\infty} I_2(x,t;\phi) \geqslant \mathcal{P} \qquad \text{uniformly for } x \in \bar{\Omega}. \tag{3.18}$$

Proof. As that in Theorem 3.7(1), we can prove $\lim_{t\to\infty} I_1(x,t;\phi) = 0$ by virtue of $R_0^1 < 1$. In the following we show (3.18).

Let $\mathbb{W}_0 = \{\phi \in \mathbb{C}_\tau^+ : \phi_2(\cdot, 0) \not\equiv 0\}$ and $\partial\mathbb{W}_0 := \mathbb{C}_\tau^+ \setminus \mathbb{W}_0 = \{\phi \in \mathbb{C}_\tau^+ : \phi_2(\cdot, 0) \equiv 0\}$. Define $M_\partial := \{\phi \in \partial\mathbb{W}_0 : \Phi_{n_0T}^k(\phi) \in \partial\mathbb{W}_0, \forall k \in \mathbb{N}\}$, where $\Phi_{n_0T}^k(\phi)(\cdot,\cdot) = u(\cdot, \cdot + n_0T; \phi)$ on $\bar{\Omega} \times [-\tau, 0)$ and $u(x,t;\phi)$ is the solution of (2.5). Denote the omega limit set of the orbit $\gamma^+ := \{\Phi_{n_0T}^k(\phi) : \forall k \in \mathbb{N}\}$ by $\omega(\phi)$. Let $\mathcal{E} := (S^*, \hat{0}, \hat{0})$. For any $\phi \in M_\partial$, one has $\Phi_{n_0T}^k(\phi) \in \partial\mathbb{W}_0$, $\forall k \in \mathbb{N}$. Therefore, $I_2(\cdot,\cdot;\phi) \equiv 0$ for any $\phi \in M_\partial$. Following from Zhang et al. [29, Lemma 2.1], one has that

$$\lim_{t\to\infty} \|S(\cdot, t, \phi) - S^*(\cdot, t)\| = 0.$$

Thus, $\omega(\phi) = \{\mathcal{E}\}$ for any $\phi \in M_\partial$.

Now we give a claim as follows.

Claim. $\mathcal{E}$ *is a uniform weak repeller for* $\mathbb{W}_0$ *in the sense that for some* $\epsilon_0 > 0$,

$$\limsup_{k\to\infty} \|\Phi_{n_0T}^k(\phi) - \mathcal{E}\| \geqslant \epsilon_0, \quad \forall \phi \in \mathbb{W}_0.$$

For $\epsilon > 0$, consider the equation

$$\begin{cases} \frac{\partial}{\partial t}\bar{u}_2^\epsilon = & D_2 \Delta \bar{u}_2^\epsilon - r_2(x,t)\bar{u}_2^\epsilon(x,t) \\ & \quad + \int_\Omega \Gamma_2(x,y,t,t-\tau_2)\beta_2(y,t-\tau_2)\bar{u}_2^\epsilon(y,t-\tau_2)\frac{S^*(y,t-\tau_2)-\epsilon}{S^*(y,t-\tau_2)+\epsilon}dy, \\ & \quad x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial n}\bar{u}_2^\epsilon = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.19}$$

Let $\bar{u}_2^\epsilon(x,t;\phi_2)$ be the solution of (3.19) with $\bar{u}_2^\epsilon(\cdot,\cdot) = \phi_2(\cdot,\cdot)$ on $\bar{\Omega} \times [-\tau, 0]$. Define the Poincaré map $\mathcal{T}_2^\epsilon : \mathbb{Q}^+ \to \mathbb{Q}^+$ of (3.19) by $(\mathcal{T}_2^\epsilon)^{n_0}(\phi_2) = \bar{u}_{2,n_0T}^\epsilon(\phi_2)$, where $\bar{u}_{2,n_0T}^\epsilon(\phi_2)(\cdot,\cdot) = \bar{u}_2^\epsilon(\cdot, \cdot + n_0T; \phi_2)$ on $\bar{\Omega} \times [-\tau, 0]$. Obviously, $(\mathcal{T}_2^\epsilon)^{n_0}$ is positive and compact. The Krein–Rutman theorem once again implies that $(\mathcal{T}_2^\epsilon)^{n_0}$ admits a positive and simple eigenvalue $\hat{r}_\epsilon^2$ and a positive eigenfunction $\hat{\varphi}_2 \in \mathbb{Y}$, such that $(\mathcal{T}_2^\epsilon)^{n_0}(\hat{\varphi}_2) = \hat{r}_\epsilon^2 \hat{\varphi}_2$. Due to $R_0^2 > 1$ (hence, $\left(r_0^2\right)^{n_0} > 1$), there is $\bar{\epsilon} > 0$ satisfying $\hat{r}_\epsilon^2 > 1$ for $\epsilon \in (0, \bar{\epsilon})$. Fix $\hat{\epsilon} \in (0, \bar{\epsilon})$. Then there is $\epsilon_0 \in (0, \bar{\epsilon})$ satisfying

$$\|(S(\cdot,\cdot), I_1(\cdot,\cdot), I_2(\cdot,\cdot)) - (S^*(\cdot,\cdot), 0, 0)\|_{C\left(\bar{\Omega},\ t\in[0,n_0T]\right)} < \hat{\epsilon}$$

if $|\phi(x,s) - (S^*(x,s), 0, 0)| < \epsilon_0$, $\forall x \in \bar{\Omega}$, $s \in [-\tau, 0]$.

To show the claim, we suppose by contradiction that $\limsup_{k\to\infty} \|\Phi^k_{n_0T}(\phi) - \mathcal{E}\| < \epsilon_0$ for some $\phi \in \mathbb{W}_0$. Then there is an integer $\tilde{k}_1 \in \mathbb{N}$ satisfying $S(x,t;\phi) > S^*(x,t;\phi) - \hat{\epsilon}$ and $0 < I_i(x,t;\phi) < \hat{\epsilon}$ for $x \in \bar{\Omega}$, $t \geqslant \tilde{k}_1 n_0 T - \tau$ and $i = 1,2$. Consequently, $I_2(\cdot,\cdot;\phi)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} I_2 \geqslant D_2 \Delta I_2 - r_2(x,t) I_2(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_2(x,y,t,t-\tau_2)\beta_2(y,t-\tau_2) \frac{S^*(y,t-\tau_2)-\epsilon_0}{S^*(y,t-\tau_2)+\epsilon_0} I_2(y,t-\tau_2)dy, \ x \in \bar{\Omega}, \\ \frac{\partial}{\partial n} I_2 = 0, \quad x \in \partial\Omega, \ t > \tilde{k}_1. \end{cases}$$

for any $t > \tilde{k}_1 n_0 T$. Since $I_2(x,t;\phi) > 0$ for $x \in \bar{\Omega}$ and $t > \tau$, there is a constant $\kappa > 0$ satisfying

$$I_2(x, (\tilde{k}_1 + 1)n_0 T + s; \phi) \geqslant \kappa \hat{\varphi}_2(x,s), \quad x \in \bar{\Omega}, \ s \in [-\tau, 0].$$

Here $\hat{\varphi}_2$ is the eigenfunction to the eigenvalue $\hat{r}^2_{\hat{\epsilon}}$. According to the comparison principle, one has

$$I_2(x, t+s;\phi) \geqslant \kappa \bar{u}^{\hat{\epsilon}}_2(x, t - (\tilde{k}_1+1)n_0 T + s; \hat{\varphi}_2), \quad \forall x \in \bar{\Omega}, \ t > (\tilde{k}_1+1)n_0 T.$$

It follows that

$$I_2(x, kT+s;\phi) \geqslant \kappa \bar{u}^{\hat{\epsilon}}_2(x, (k-\tilde{k}_1-1)n_0 T + s; \hat{\varphi}_2) = \kappa (\hat{r}^2_{\hat{\epsilon}})^{(k-\tilde{k}_1-1)} \hat{\varphi}_2(x,s), \tag{3.20}$$

where we selected $k > (\tilde{k}_1 + 1)$. By the positivity of $\hat{\varphi}_2(x,s)$ on $(x,s) \in \bar{\Omega} \times [-\tau, 0]$, we can obtain that $\hat{\varphi}_2(x_2, s_2) > 0$ for some $x_2 \in \bar{\Omega}$ and $s_2 \in [-\tau, 0]$. Thus, (3.20) implies that $I_2(x_2, kT + s_2; \phi) \to +\infty$ as $k \to \infty$. By the boundedness of $I_2(\cdot;\phi)$, there is a contradiction.

Denote the stable set of $\mathcal{E}$ by $W^s(\mathcal{E})$. From the claim, $\mathcal{E}$ is invariant and isolated for Φ_{n_0T} in $\mathbb{W}_0$ and $W^s(\mathcal{E}) \cap \mathbb{W}_0 = \emptyset$. Theorem 1.3.1 and Remark 1.3.1 of [48]) indicate that $\Phi_{n_0T} : \mathbb{C}^+_\tau \to \mathbb{C}^+_\tau$ is uniformly persistent on $(\mathbb{W}_0, \partial\mathbb{W}_0)$; namely, there is a $\tilde{\delta} > 0$ satisfying

$$\liminf_{k\to\infty} d(\Phi^k_{n_0T}(\phi), \partial\mathbb{W}_0) \geqslant \tilde{\delta}, \ \forall \phi \in \mathbb{W}_0.$$

Due to [48, Theorem 3.1.1], the semiflow $\Phi_t : \mathbb{C}^+_\tau \to \mathbb{C}^+_\tau$ is also uniformly persistent on $(\mathbb{W}_0, \partial\mathbb{W}_0)$. In addition, Φ_{n_0T} is a κ-condensing map due to the compactness of Φ_{n_0T}. Therefore, by [38, Theorem 4.5], one has that $\Phi_{n_0T} : \mathbb{W}_0 \to \mathbb{W}_0$ with $\rho(x) = d(x, \partial\mathbb{W}_0)$ has a global attractor $\mathcal{Z}_0$.

Clearly, $\phi_1(\cdot,\cdot) \equiv 0$ on $\bar{\Omega} \times \mathbb{R}$ for any $\phi \in \mathcal{Z}_0$. In order to prove the theorem, we use the argument similar to Lou and Zhao [49, Theorem 4.1]. Define $p : \mathbb{Q} \to \mathbb{R}_+$ by

$$p(\phi_2) = \min_{x\in\bar{\Omega}} \phi_2(x, 0), \quad \forall \phi \in \mathbb{Q}.$$

Since $\mathcal{Z}_0 = \Phi_{n_0T}(\mathcal{Z}_0)$, one has that $\phi_2(\cdot, 0) > 0, \ \phi \in \mathcal{Z}_0$. Let $\mathcal{B}_0 := \bigcup_{t\in[0,n_0T]} \Phi_t(\mathcal{Z}_0)$. It then follows that $\mathcal{B}_0 \subset \mathbb{W}_0$ and $\lim_{t\to\infty} d(\Phi_t(\phi), \mathcal{B}_0) = 0, \ \forall \phi \in \mathbb{W}_0$. Because $\mathcal{B}_0$ is compact, there is $\min_{\phi\in\mathcal{B}_0} p(\phi_2) > 0$. Then, by Lemma 3.5, there is a $\mathcal{P} > 0$ satisfying $\liminf_{t\to\infty} I_2(\cdot,t;\phi) \geqslant \mathcal{P}$. $\square$

Theorem 3.13. *Suppose that $R^2_0 > 1 = R^1_0$ and $\beta_1(\cdot,\cdot) > 0$ on $\bar{\Omega} \times [0,\infty)$. Then there is a $\mathcal{P} > 0$ such that, if $\phi \in \mathbb{C}^+_\tau$ satisfies $\phi_2(\cdot, 0) \not\equiv 0$, then*

$$\lim_{t\to\infty} I_1(x,t;\phi) = 0 \quad \text{and} \quad \liminf_{t\to\infty} I_2(x,t;\phi) \geqslant \mathcal{P} \qquad \text{uniformly for } x \in \bar{\Omega}.$$

Proof. The proof is similar to that in Theorem 3.12 and the details are omitted. $\square$

The following theorems are similar to Theorems 3.12 and 3.13.

Theorem 3.14. *Suppose that $R_0^1 > 1 > R_0^2$. Then there is a $\mathcal{P}_1 > 0$ such that, if any $\phi \in \mathbb{C}_\tau^+$ satisfies $\phi_1(\cdot, 0) \not\equiv 0$, then $\lim_{t\to\infty} I_2(x, t; \phi) = 0$ and $\liminf_{t\to\infty} I_1(x, t; \phi) \geqslant \mathcal{P}_1$ uniformly for $x \in \bar{\Omega}$.*

Theorem 3.15. *Suppose that $R_0^1 > 1 = R_0^2$ and $\beta_2(\cdot, \cdot) > 0$ on $\bar{\Omega} \times [0, \infty)$. Then there is a $\mathcal{P}_1 > 0$ such that, if $\phi \in \mathbb{C}_\tau^+$ satisfies $\phi_1(\cdot, 0) \not\equiv 0$, then $\lim_{t\to\infty} I_2(x, t; \phi) = 0$ and $\lim_{t\to\infty} \inf I_1(x, t; \phi) \geqslant \mathcal{P}_1$ uniformly for $x \in \bar{\Omega}$.*

Finally, one establishes the persistence of (2.5).

Theorem 3.16. *Assume $\hat{R}_0^1 > 1$ and $\hat{R}_0^2 > 1$. Then there is an $\eta > 0$ such that, if any $\phi = (\phi_S, \phi_1, \phi_2) \in \mathbb{C}_\tau^+$ satisfies $\phi_1(\cdot, 0) \not\equiv 0$ and $\phi_2(\cdot, 0) \not\equiv 0$, then*

$$\liminf_{t\to\infty} S(x, t) \geqslant \eta, \quad \liminf_{t\to\infty} I_i(x, t) \geqslant \eta, \quad i = 1, 2, \quad \text{uniformly for } x \in \bar{\Omega}.$$

Proof. Due to Proposition 3.11 and $\hat{R}_0^i > 1 (i = 1, 2)$, one has $R_0^i > 1 (i = 1, 2)$. Let

$$\mathbb{Z}_0 := \{\phi \in \mathbb{C}_\tau^+ | \phi_1(\cdot, 0) \not\equiv 0 \text{ and } \phi_2(\cdot, 0) \not\equiv 0\}, \quad \partial\mathbb{Z}_0 := \{\phi \in \mathbb{C}_\tau^+ | \phi_1(\cdot, 0) \equiv 0 \text{ or } \phi_2(\cdot, 0) \equiv 0\}$$

and

$$\mathbb{Z}_\partial := \{\phi \in \partial\mathbb{Z}_0 : \Phi_{n_0T}^k(\phi) \in \partial\mathbb{Z}_0, \ \forall k \in \mathbb{N}\}.$$

Denote $E_0 = \{(S^*, \hat{0}, \hat{0})\}$, $E_1 = \{(\phi_S, \phi_1, \hat{0}) | \forall(\phi_S, \phi_1) \in \mathcal{B}_1\}$, $E_2 = \{(\phi_S, \hat{0}, \phi_2) \ | \forall(\phi_S, \phi_2) \in \mathcal{B}_2\}$, where $\mathcal{B}_i (i = 1, 2)$ is defined in Theorem 3.7 and $\hat{0}$ denotes the constant function identically zero in $\mathbb{Y}$. It can be seen that $\Phi_t(\mathbb{Z}_0) \in \mathbb{Z}_0, \ \forall t > 0$. Next, we show the following claims.

Claim 1. *$\bigcup_{\phi\in\mathbb{Z}_\partial} \bar{\omega}(\phi) = E_0 \bigcup E_1 \bigcup E_2, \ \forall\phi \in \mathbb{Z}_\partial$, where $\bar{\omega}(\phi)$ is the omega limit set of the orbit $\gamma^+(\phi) := \{\Phi_{n_0T}^k(\phi) : \forall k \in \mathbb{N}\}$ of system (2.5) for $\phi \in \mathbb{Z}_\partial$.*

From the definition of $\mathbb{Z}_\partial$, we have $u(kn_0T + \cdot; \phi) := \Phi_{n_0T}^k(\phi) \in \partial\mathbb{Z}_0, \ \forall k \in \mathbb{N}$, which further implies that either $I_1(\cdot, \cdot; \phi) \equiv 0$ or $I_2(\cdot, \cdot; \phi) \equiv 0$ on $\mathbb{R}^+ \times \bar{\Omega}$. In fact, if there exist $t_i > 0$ such that $I_i(\cdot, t_i; \phi) \not\equiv 0$ on $x \in \bar{\Omega}$, $i = 1, 2$, then the strong positivity of $V_i(t, s)(t > s)$ implies that $I_i(x, t; \phi) > 0$ for all $x \in \bar{\Omega}$ and $t > t_i$, which contradicts the fact that $\Phi_{n_0T}^k \in \partial\mathbb{Z}_0, \ \forall k \in \mathbb{N}$. If $I_1(\cdot, \cdot; \phi) \equiv 0$ on $\bar{\Omega} \times \mathbb{R}^+$, then by Theorem 3.7, $\bar{\omega}(\phi) \in E_0 \bigcup E_2$. If $I_2(\cdot, \cdot; \phi) \equiv 0$ on $\bar{\Omega} \times \mathbb{R}^+$, then one has $\bar{\omega}(\phi) \in E_0 \bigcup E_1$. Therefore, Claim 1 holds.

Claim 2. *E_0 is a uniformly weak repeller for $\mathbb{Z}_0$ in the sense that*

$$\limsup_{k\to\infty} \| \Phi_{n_0T}^k(\phi) - E_0\| \geqslant \theta_0, \quad \forall\phi \in \mathbb{Z}_0$$

for some $\theta_0 > 0$ small enough.

The proof of the claim is completely similar to those in Theorems 3.7 and 3.12, so we omit it.

Claim 3. *Each E_i $(i = 1, 2)$ is a uniformly weak repeller for $\mathbb{Z}_0$ in the sense that there is a $\epsilon_0 > 0$ small enough satisfying*

$$\limsup_{k\to\infty} \| \Phi_{n_0T}^k(\phi) - E_i\| \geqslant \epsilon_0, \quad \forall\phi \in \mathbb{Z}_0.$$

We only give the proof for E_1. For $\epsilon > 0$, consider the following equation

$$\begin{cases} \frac{\partial}{\partial t}\bar{u}_2^\epsilon = D_2\Delta\bar{u}_2^\epsilon + \int_\Omega \Gamma_2(x, y, t, t - \tau_2)\beta_2(y, t - \tau_2)\frac{S_1^*(y,t-\tau_2)-\epsilon}{\mathcal{F}_1(y,t-\tau_2)} \\ \qquad\qquad u_2^\epsilon(y, t - \tau_2)dy - r_2(x, t)\bar{u}_2^\epsilon(x, t), \ \forall x \in \Omega, \ t > 0, \\ \frac{\partial}{\partial n}\bar{u}_2^\epsilon = 0, \ \forall x \in \partial\Omega, \ t > 0, \end{cases} \tag{3.21}$$

where $u^*_{\mathcal{S}}(x,t)$ is the unique positive periodic solution of (3.14) and $B>0$ satisfies (3.15). Let $\bar{u}^{\epsilon}_2(x,t;\phi_2)$ be the solution of (3.21) with $\bar{u}^{\epsilon}_2(\cdot,\cdot)=\phi_2(\cdot,\cdot)$ on $\bar{\Omega}\times[-\tau,0]$. Define the Poincaré map $\mathcal{A}^{n_0,\epsilon}_2:\mathbb{Q}\to\mathbb{Q}$ of (3.21) by $\mathcal{A}^{n_0,\epsilon}_2(\phi_2)=\bar{u}^{\epsilon}_{2,n_0T}(\phi_2)$, where $\bar{u}^{\epsilon}_{2,n_0T}(\phi_2)(x,s)=\bar{u}^{\epsilon}_2(x,s+n_0T;\phi_2)$ for $(x,s)\in[-\tau,0]\times\bar{\Omega}$. Clearly, $\mathcal{A}^{n_0,\epsilon}_2$ is positive and compact. Then $\mathcal{A}^{n_0,\epsilon}_2$ admits a positive and simple eigenvalue $\breve{r}^2_{\epsilon}$ and a positive eigenfunction $\breve{\varphi}_2\in\mathbb{Q}$ satisfying $\mathcal{A}^{n_0,\epsilon}_2(\breve{\varphi}_2)=\breve{r}^2_{\epsilon}\breve{\varphi}_2$. Due to $\rho^2_0>0$ (since $\hat{R}^0_i>1$, see Theorem 3.10), there is $\hat{\epsilon}>0$ satisfying $\breve{r}^2_{\epsilon}>0$ for $\epsilon\in(0,\hat{\epsilon})$. Given $\bar{\epsilon}\in(0,\hat{\epsilon})$. Then there exists $\epsilon_0\in(0,\hat{\epsilon})$ satisfying

$$\|\Phi^k_{n_0T}(\phi)-E_1\|<\frac{1}{2}\bar{\epsilon},\ \forall t\in[0,n_0T],\quad x\in\bar{\Omega}, \tag{3.22}$$

if $\|\phi(x,s)-E_1\|<\epsilon_0,\ \forall x\in\bar{\Omega},\ s\in[-\tau_i,0]$.

Now we show that E_1 is a uniformly weak repeller for $\mathbb{Z}_0$ by contradiction. Suppose on the contrary that

$$\limsup_{t\to\infty}\|\Phi^k_{n_0T}(\phi)-E_1\|<\epsilon_0$$

for some $\phi\in\mathbb{Z}_0$. Combining (3.22) and Remark 3.8, we then have that there is $\mathcal{K}_0\in\mathbb{N}$ large enough satisfying

$$S(x,t;\phi)\geq S^*_1(x,t)-\bar{\epsilon},\quad 0<I_1(x,t;\phi)<B,\quad 0<I_2(x,t;\phi)<\bar{\epsilon} \tag{3.23}$$

for any $x\in\bar{\Omega}$ and $t\geqslant\mathcal{K}_0n_0T$. On the other hand, I_2 satisfies

$$\begin{aligned}\frac{\partial}{\partial t}I_2\geqslant&D_2\Delta I_2-r_2(x,t)I_2(x,t)\\&+\int_{\Omega}\Gamma_2(x,y,t,t-\tau_2)\frac{\beta_2(y,t-\tau_2)\left(S^*_1(y,t-\tau_2)-\bar{\epsilon}\right)}{\mathcal{F}_1(y,t-\tau_2)}I_2(y,t-\tau_2)dy\end{aligned}$$

for $t>\mathcal{K}_0n_0T$. As the proofs of Theorem 3.7(3) and 3.12, there exists $(x_2,s_2)\in\overline{\Omega}\times[-\tau,0]$ such that $I_2(x_2,kT+s_2;\phi)\to\infty$ if $k\to\infty$, which contradicts with (3.23). As a consequence, Claim 3 holds.

By the above discussion, $\mathcal{E}:=E_0\bigcup E_1\bigcup E_2$ is isolated and invariant for Φ_{n_0T} in $\mathbb{Z}_0$ and $W^s(\mathcal{E})\bigcap\mathbb{Z}_0=\emptyset$, where $W^s(\mathcal{E})$ denotes the stable set of $\mathcal{E}$. It follows from the acyclicity theorem on uniform persistence for maps (see [48]) that $\Phi_{n_0T}:\mathbb{C}^+_{\tau}\to\mathbb{C}^+_{\tau}$ is uniformly persistent on $(\mathbb{Z}_0,\partial\mathbb{Z}_0)$. Thus, there is $\tilde{\delta}>0$ so that

$$\liminf_{k\to\infty}d(\Phi^k_{n_0T}(\phi),\partial\mathbb{Z}_0)\geqslant\tilde{\delta},\ \forall\phi\in\mathbb{Z}_0.$$

Furthermore, the semiflow $\Phi_t:\mathbb{C}^+_{\tau}\to\mathbb{C}^+_{\tau}$ is also uniformly persistent on $(\mathbb{Z}_0,\partial\mathbb{Z}_0)$. It follows that Φ_{n_0T} is α-condensing. Thus, Φ_{n_0T} has a global attractor $\mathcal{A}_0$ in $\mathbb{Z}_0$.

We further apply the similar argument as that in [49, Theorem 4.1] to complete the proof of the conclusion. Define $p:\mathbb{C}^+_{\tau}\to\mathbb{R}^+$ by

$$p(\phi)=\min\left\{\min_{x\in\bar{\Omega}}\phi_1(x,0),\ \min_{x\in\bar{\Omega}}\phi_2(x,0)\right\},\quad\forall\phi\in\mathbb{C}^+_{\tau}.$$

Since $\mathcal{A}_0=\Phi_{n_0T}(\mathcal{A}_0)$, where $\mathcal{A}_0$ is a global attractor of Φ_{n_0T}, we have that $\phi_i(\cdot,0)>0$ for all $\phi\in\mathcal{A}_0$. Let $\mathcal{B}_0:=\bigcup_{t\in[0,n_0T]}\Phi_t(\mathcal{A}_0)$. It follows that $\mathcal{B}_0\subset\mathcal{A}_0$ and

$$\lim_{t\to\infty}d(\Phi_t(\phi),\mathcal{B}_0)=0\quad\forall\ \phi\in\mathcal{A}_0.$$

Since $\mathcal{B}_0$ is compact, one gets $\min_{\phi\in\mathcal{B}_0}p(\phi)>0$. Therefore, there is $\delta^*>0$ satisfying $\liminf_{t\to\infty}I_i(x,t;\phi)\geqslant\delta^*$. This completes the proof. $\square$

3.2.2. Global extinction

Finally, we show that the periodic solution $(S^*,0,0)$ of (2.5) is globally attractive if $R_0^i \leqslant 1$ for all $i=1,2$. We give the following three theorems without proofs.

Theorem 3.17. *Suppose that $R_0^i < 1$ for $i=1,2$. Then the periodic solution $(S^*,0,0)$ of (2.5) is globally attractive.*

Theorem 3.18. *Suppose that $R_0^i = 1$ and $\beta_i(\cdot,\cdot) > 0$ on $\bar{\Omega} \times [0,\infty)$ for both $i=1,2$. Then the periodic solution $(S^*,0,0)$ of (2.5) is globally attractive.*

Theorem 3.19. *Let $R_0^i < 1$, $R_0^j = 1$ and $\beta_j(x,t) > 0$ on $\bar{\Omega} \times [0,\infty)$, $i,j=1,2$, $i \neq j$. Then the periodic solution $(S^*,0,0)$ of (2.5) is globally attractive.*

4. Numerical simulations

In this section, we give some numerical simulations to illustrate the results established in Section 3. More precisely, we show the potential dynamical outcomes of system (2.5), in which all coefficients are only dependent upon the time variable t. Furthermore, we always choose $D := 0.04 = D_S = D_1 = D_2$, $\tau := 1.1 = \tau_1 = \tau_2$ and the initial conditions to be as follows:

$$S(s,x) = 2 + 0.3\cos x, \quad I_1(s,x) = 2 + 0.2\cos x, \quad I_2(s,x) = 2 + 0.1\sin x,$$

where $s \in [-\tau, 0]$ and $x \in \Omega$. In the following we always take $\Omega = [0,100]$. In addition, we truncate the time domain $\mathbb{R}^+$ by $[0,300]$. Define the uniform partition of domain $\Omega = [0,100]$ by:

$$0 = z_1 < z_2 < z_3 < \cdots < z_{2n-1} < z_{2n} < z_{2n+1} = 100,$$

where $h = 100/n$ and $z_i = z_1 + (i-1)h$, $i = 1,2,\ldots,2n+1$. The time domain $[0,300]$ can be treated similarly.

Table 1 lists the potential dynamical outcomes of system (2.5).

(i) Cases 1 and 2. Here we only simulate the results in Case 1. Case 2 can be treated similarly. Take the coefficients of system (2.5) as follows:

$$\begin{aligned}
&\mu = d = 0.7,\ \delta_1 = 0.1,\ \delta_2 = 0.13,\ \kappa_1 = 0.01,\ \kappa_2 = 0.04,\\
&\beta_1 = 0.5(1 + 0.4 \times \sin(0.2 * t\pi + 1)),\ \beta_2 = (7 + 6.5 \times \cos(0.2 * t\pi + 1)), \quad \forall t \in [0,300].
\end{aligned}$$

It is clear that the above coefficients are 10-periodic in time t. Due to these parameters and the discussion of Section 2, we have

$$\begin{aligned}
\Gamma_i(x,y,\tau_i) =& 1/100 \times e^{-(\mu+\kappa_i)\tau}\\
&\times[1 + \sum_{n=1}^{\infty}(\cos(n\pi/100(x-y)) + \cos(n\pi/100(x+y)))e^{-((n\pi/100)^2 D_i\tau)}]
\end{aligned}$$

for $i=1,2$ (see [21, Section 4] and [50, Section 6]). Let

$$f_i(y,t-\tau_i) = \frac{\beta_i(t-\tau_i)S(y,t-\tau_i)I_i(y,t-\tau_i)}{S(y,t-\tau_i) + I_1(y,t-\tau_i) + I_2(y,t-\tau_i)}, \quad i = 1,2.$$

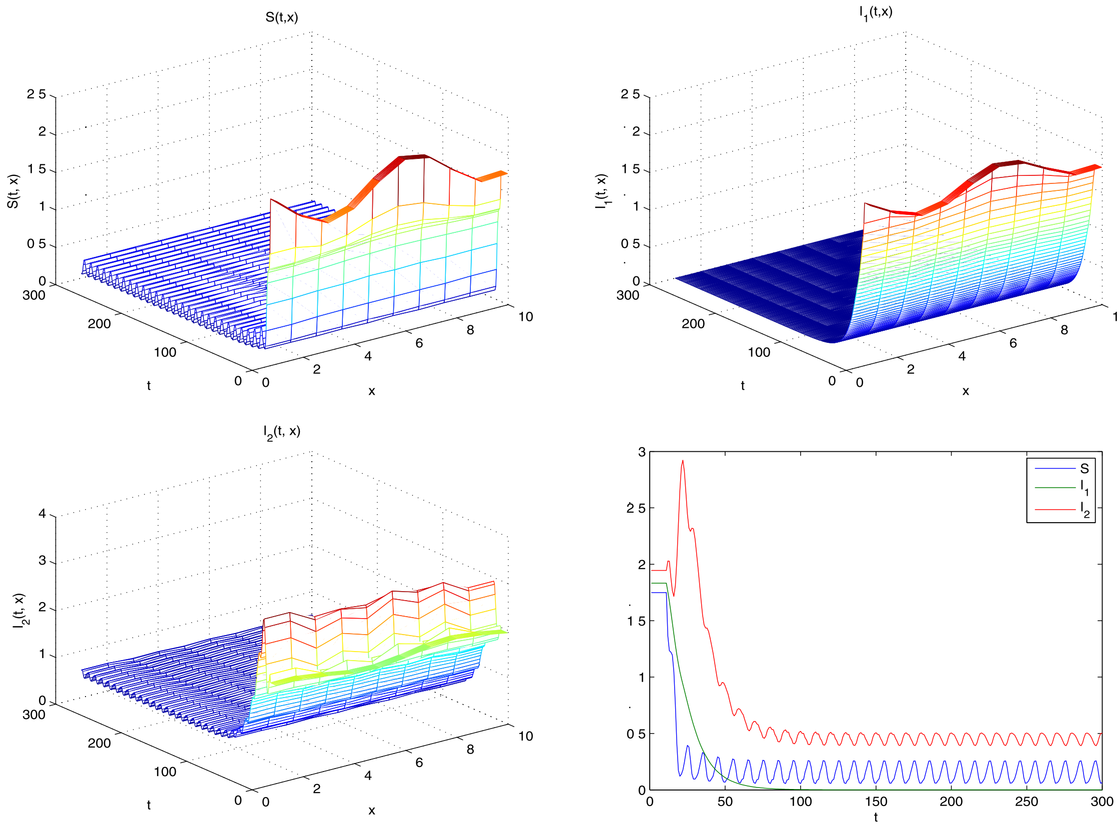

Fig. 1. The dynamic behavior of system (2.5) when $R_0^1 < 1$ and $R_0^2 > 1$.

Then it further follows from the composite Simpson's rule ([50, A.1]) that

$$\begin{aligned}&\int_0^{100} \Gamma_i(x,y,\tau_i) f_i(y,t-\tau_i)dy \\ =&h/3\Bigg(f_i(z_1,t-\tau_i)\Gamma_i(x,z_1,\tau_i) + 4\sum_{m=1}^{n} f_i(z_{2m},t-\tau_i)\Gamma_i(x,z_{2m},\tau_i) \\ &+2\sum_{m=1}^{n} f_i(z_{2m},t-\tau_i)\Gamma_i(x,z_{2m},\tau_i) + f_i(z_{2n+1},t-\tau_i)\Gamma_i(x,z_{2n+1},\tau_i)\Bigg).\end{aligned}$$

In addition, we have the basic reproduction number of Strain 1

$$R_0^1 = \frac{\int_0^T \beta_1(t)dt}{\int_0^T (\mu+\delta_1+\kappa_1)dt} = \frac{\int_0^{10} 0.5(1+0.4\times\sin(0.2*t\pi+1))dt}{\int_0^{10}(0.7+0.1+0.01)dt} < 1$$

and the basic reproduction number of Strain 2

$$R_0^2 = \frac{\int_0^T \beta_2(t)dt}{\int_0^T (\mu+\delta_2+\kappa_2)dt} = \frac{\int_0^{10} (7+6.5\times\cos(0.2*t\pi+1))dt}{\int_0^{10}(0.7+0.13+0.04)dt} > 1.$$

Fig. 1 presents the simulations of the conclusions of Theorem 3.12. Clearly, the infectious disease of Strain 1 becomes extinct and the infectious disease of Strain 2 is persistent in this case. In particular, from Fig. 1 we find that the solution of system (2.5) converges 10-periodic and non-negative functions as $t \to \infty$.

Table 1
The potential dynamical outcomes of system (2.5).

Case	*Strain 1*	*Strain 2*	*Expected outcome*	*Theorem(s)*
1	$R_0^1 \leqslant 1$	$R_0^2 > 1$	$I_1 \to 0$, I_2 persists	3.12 and 3.13
2	$R_0^1 > 1$	$R_0^2 \leqslant 1$	I_1 persists, $I_2 \to 0$	3.14 and 3.15
3	$R_0^1 > 1$	$R_0^2 > 1$	(A) I_1 persists, $I_2 \to 0$	–
			(B) $I_1 \to 0$, I_2 persists	–
			(C) I_1 and I_2 persist	3.16
4	$R_0^1 \leqslant 1$	$R_0^2 \leqslant 1$	$I_1 \to 0$, $I_2 \to 0$	3.17–3.19

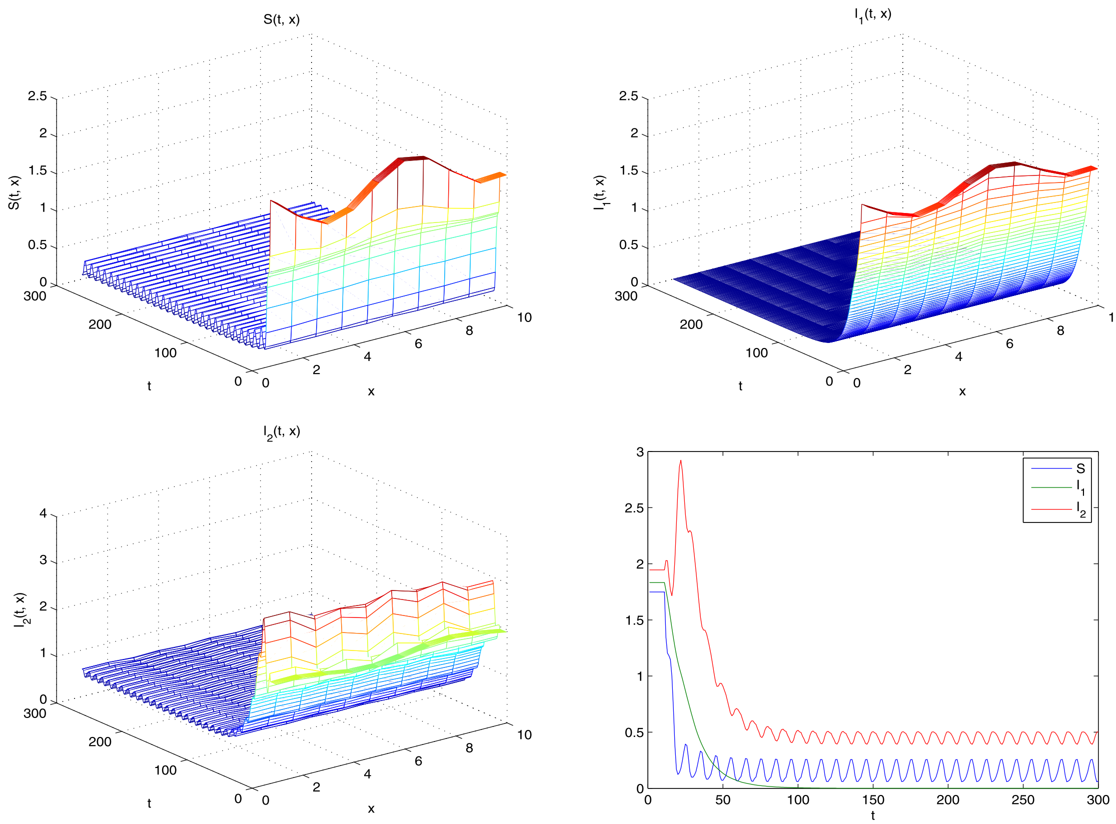

Fig. 2. The dynamic behavior of system (2.5) when $R_0^1 = 1$ and $R_0^2 > 1$.

Furthermore, the results of Theorem 3.13 can also be simulated. We set the following coefficients in system (2.5):

$$\mu = d = 0.7,\ \delta_1 = 0.1,\ \delta_2 = 0.13,\ \kappa_1 = 0.01,\ \kappa_2 = 0.04,$$
$$\beta_1 = 0.5(1.62 + 0.4 \times \sin(0.2 * t\pi + 1)),\ \beta_2 = 2(7 + 6.5 \times \cos(0.2 * t\pi + 1)),\quad \forall t \in [0, 300].$$

Obviously, we have $R_0^1 = 1$ and $R_0^2 > 1$. Fig. 2 shows that the infectious disease of Strain 1 becomes extinct and the infectious disease of Strain 2 persists if $R_0^1 = 1$ and $R_0^2 > 1$.

(ii) Case 3. There are three potential dynamical outcomes: (A) I_1 persists and I_2 becomes extinct; (B) I_1 becomes extinct and I_2 persists; (C) both I_1 and I_2 persist. Here, we show (A) and (C) ((B) can be treated similarly and thus we omit it).

Outcome (A). We take the parameters of system (2.5) as follows:

$$\mu = d = 0.7,\ \delta_1 = 0.1,\ \delta_2 = 0.13,\ \kappa_1 = 0.01,\ \kappa_2 = 0.04,$$
$$\beta_1 = 5 + 4.4 \times \sin(0.2 * t\pi + 1),\ \beta_2 = 1.74 + 0.5 \times \cos(0.2 * t\pi + 1),\quad \forall t \in [0, 300].$$

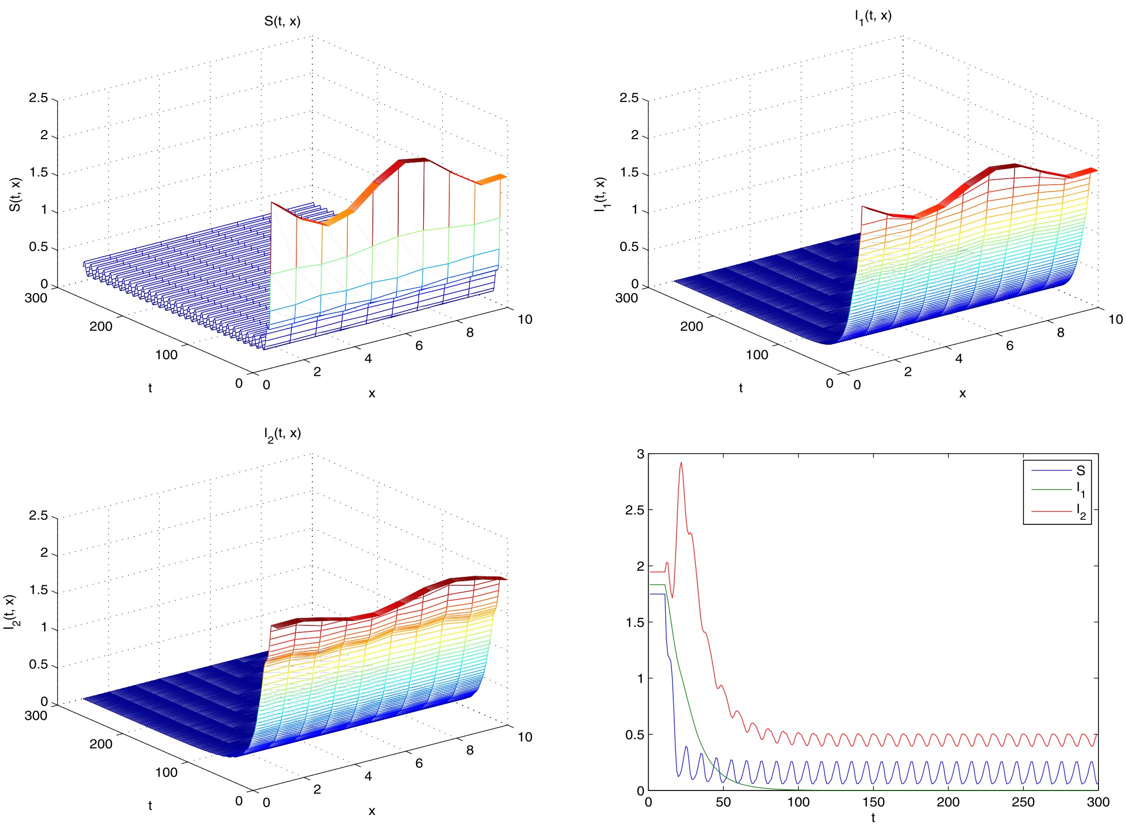

Fig. 3. Outcome (A) of dynamic behavior of system (2.5) when $R_0^i > 1, \quad i = 1, 2$.

Direct computations give that $R_0^i > 1$ for $i = 1, 2$. Fig. 3 shows that the infectious disease of Strain 1 persists and the infectious disease of Strain 2 becomes extinct.

Outcome (C). Set the parameters of system (2.5) as follows:

$$\mu = d = 0.6, \ \delta_1 = 0.1, \ \delta_2 = 0.13, \ \kappa_1 = 0.01, \ \kappa_2 = 0.04,$$
$$\beta_1 = 6 + 4.8 \times \sin(0.2 * t\pi + 1), \ \beta_2 = 6.2 + 4.5 \times \sin(0.2 * t\pi + 1), \quad \forall t \in [0, 300].$$

It is easy to see that $R_0^i > 1, \ i = 1, 2$. Fig. 4 shows that both I_1 and I_2 persist, which is completely different from the outcome (A) though there hold $R_0^1 > 1$ and $R_0^2 > 1$, too.

(iii) Case 4. Firstly, we choose the following parameters of system (2.5):

$$\mu = d = 0.7, \ \delta_1 = 0.1, \ \delta_2 = 0.13, \ \kappa_1 = 0.01, \ \kappa_2 = 0.04,$$
$$\beta_1 = 0.81 + 0.2 \times \sin(0.2 \times t\pi + 1), \ \beta_2 = 0.87 + 0.25 \times \cos(0.2 \times t\pi + 1), \quad \forall t \in [0, 300].$$

Direct calculations show that $R_0^i = 1$ for $i = 1, 2$. Secondly, we choose the parameters in (2.5) as below:

$$\mu = d = 0.7, \ \delta_1 = 0.1, \ \delta_2 = 0.13, \ \kappa_1 = 0.01, \ \kappa_2 = 0.04,$$
$$\beta_1 = 0.5 \times (1 + 0.4 \sin(0.2 \times t\pi + 1)), \ \beta_2 = 0.6 \times (1 + 0.5 \cos(0.2 \times t\pi + 1)), \quad \forall t \in [0, 300].$$

In this case we have that $R_0^i < 1$ for $i = 1, 2$. Figs. 5 and 6 show that the disease-free periodic solution $(1, 0, 0)$ is globally attractive in these two subcases respectively, namely, both I_1 and I_2 become extinct when $R_0^i \leq 1$, $i = 1, 2$.

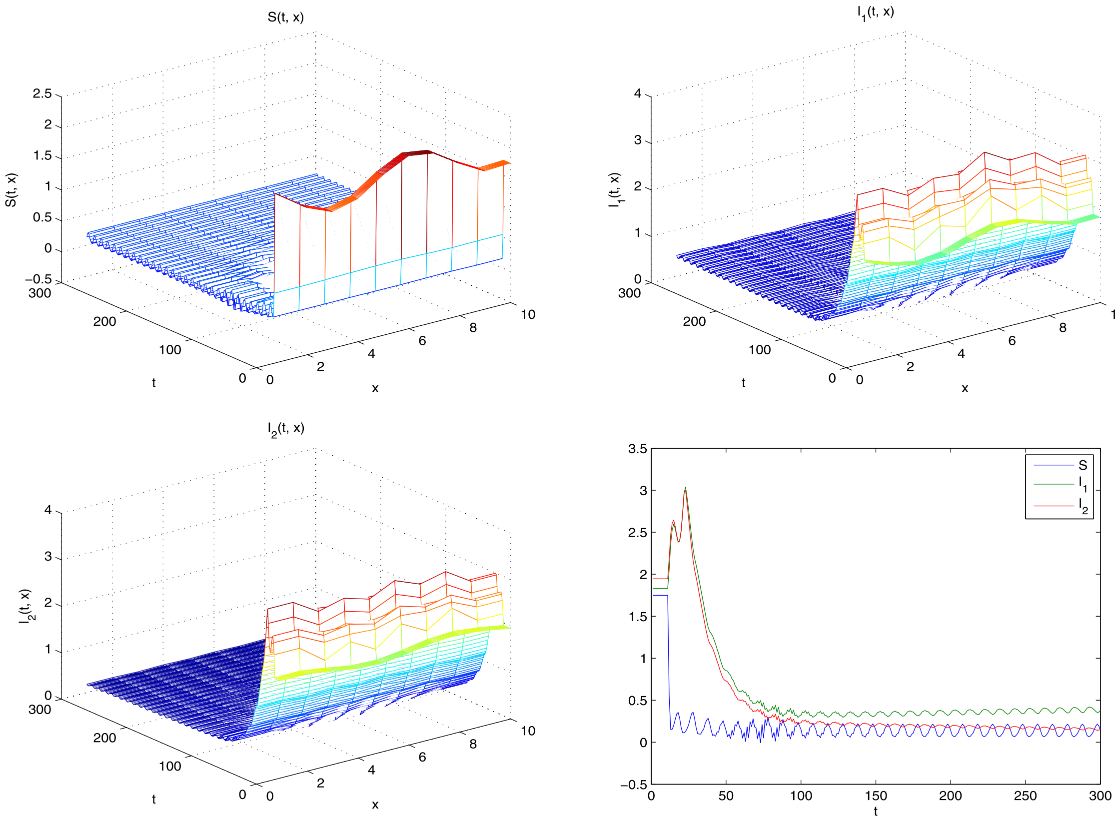

Fig. 4. Outcome (C) of dynamic behavior of system (2.5) when $R_0^i > 1, \quad i = 1, 2$.

5. Discussion

In this paper we proposed and studied the following time-periodic two-strain SIS epidemic model

$$\begin{cases} \frac{\partial}{\partial t}S = D_S \Delta S + \mu(x,t) - d(x,t)S(x,t) + \delta_1(x,t)I_1(x,t) + \delta_2(x,t)I_2(x,t) \\ \qquad\qquad - \frac{\beta_1(x,t)S(x,t)I_1(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} - \frac{\beta_2(x,t)S(x,t)I_2(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)}, \quad x \in \Omega, \ t > 0, \\ \frac{\partial}{\partial t}I_i = D_i \Delta I_i - \left(d(x,t) + \kappa_i(x,t) + \delta_i(x,t)\right) I_i(x,t) \\ \qquad\qquad + \int_\Omega \Gamma_i(x,y,t,t-\tau_i) \frac{\beta_i(y,t-\tau_i)S(y,t-\tau_i)I_i(y,t-\tau_i)}{S(y,t-\tau_i)+I_1(y,t-\tau_i)+I_2(y,t-\tau_i)} dy, \ x \in \Omega, \ t > 0 \ i = 1,2, \\ \frac{\partial}{\partial n}S = \frac{\partial}{\partial n}I_i = 0, \quad x \in \partial\Omega, \ t > 0, \ i = 1,2. \end{cases}$$

In Section 3.1, we firstly investigated the single-strain model (3.1) and showed that the disease persists if $R_0^i > 1$ and the disease becomes extinct if $R_0^i \le 1$. Consequently, in Section 3.2 we showed the dynamics of the two-strains model, where the potential dynamical outcomes of system (2.5) are listed in Table 1 of Section 4. In particular, when both $R_0^1 > 1$ and $R_0^2 > 1$ hold, the disease infected with Strain 1 and the disease infected with Strain 2 can coexist due to the effects of time heterogeneity (time periodicity) and spatial heterogeneity, which has been proved by Theorem 3.16 and simulated in Section 4. It should be emphasized that such a coexistence phenomenon of (2.5) when $R_0^1 > 1$ and $R_0^2 > 1$ is different from those of the ODE model with constant coefficients corresponding to (2.5). Clearly, the ODE model with constant coefficients corresponding to (2.5) is as follows:

$$\begin{cases} S'(t) = \mu - dS(t) + \delta_1 I_1(t) + \delta_2 I_2(t) - \frac{\beta_1 S(t)I_1(t)}{S(t)+I_1(t)+I_2(t)} - \frac{\beta_2 S(t)I_2(t)}{S(t)+I_1(t)+I_2(t)}, \quad t > 0, \\ I_i'(t) = -\left(d + \kappa_i + \delta_i\right) I_i(t) + e^{-(d+\kappa_i)\tau_i} \frac{\beta_i(t)S(t-\tau_i)I_i(t-\tau_i)}{S(t-\tau_i)+I_1(t-\tau_i)+I_2(t-\tau_i)}, \ t > 0, \ i = 1,2. \end{cases} \tag{5.1}$$

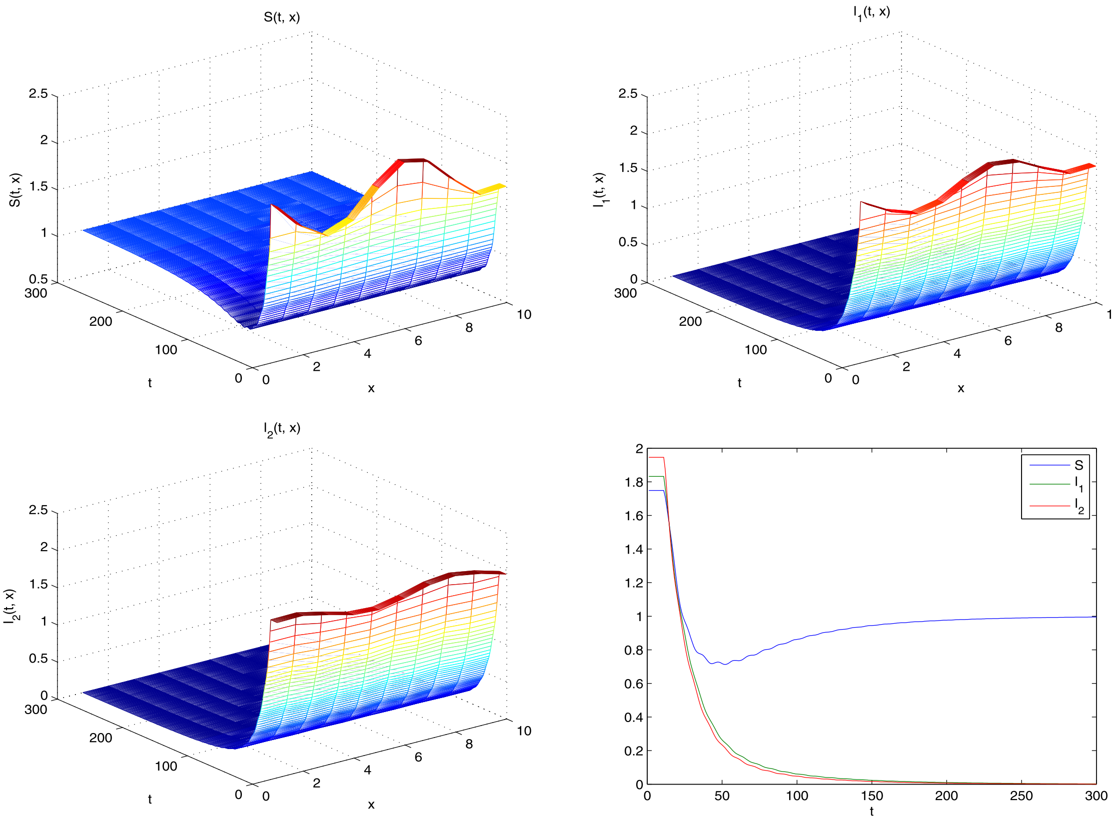

Fig. 5. The dynamic behavior of system (2.5) when $R_0^i = 1, \quad i = 1, 2$.

By similar arguments to those in [5, Section 5], we can show that the solution $(S(t), I_1(t), I_2(t))$ of (5.1) always satisfies that $I_i(t)$ goes to zero as $t \to \infty$ if $R_0^i < R_0^j,\ i \neq j,\ i, j = 1, 2$. Obviously, the competitive exclusion is the unique outcome of the dynamical behaviors of (5.1) when $1 < R_0^i < R_0^j,\ i \neq j,\ i, j$.

Furthermore, we take into account the special case of (2.5) without latent period and the effects of temporal heterogeneity. Namely, we consider the following model

$$\begin{cases} \frac{\partial}{\partial t} S = D_S \Delta S + \mu(x,t) - d(x,t)S(x,t) + \delta_1(x,t)I_1(x,t) + \delta_2(x,t)I_2(x,t) \\ \qquad\qquad - \beta_1(x,t)\frac{S(x,t)I_1(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)} - \beta_2(x,t)\frac{S(x,t)I_2(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)},\ x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial t} I_i = D_i \Delta I_i - r_i(x,t)I_i(x,t) + \beta_i(x,t)\frac{S(x,t)I_i(x,t)}{S(x,t)+I_1(x,t)+I_2(x,t)}, \\ \qquad\qquad x \in \Omega,\ t > 0,\ i = 1, 2, \\ \frac{\partial}{\partial n} S = \frac{\partial}{\partial n} I_i = 0, \quad x \in \partial\Omega,\ t > 0,\ i = 1, 2. \end{cases} \tag{5.2}$$

Clearly, the results established in Sections 2–4 still hold for system (5.2). In contrast to model (1.1) studied by Tuncer and Martcheva [14], model (5.2) (and (2.5)) studied in this paper takes into account the demographic structure (the recruitment/birth term $\mu(t, x)$ and the natural death rate $d(t, x)$). Nevertheless, here we would like to emphasize that the results of this paper cannot cover those established by Tuncer and Martcheva [14] for model (1.1). In fact, for any number $N > 0$, $(N/|\Omega|, 0, 0)$ is always a disease-free equilibrium of (1.1), but the model (5.2) (and (2.5)) studied in this paper always admits a unique disease-free steady state. Therefore, in [14] the authors always suppose that $S(x, 0) + I_1(x, 0) + I_2(x, 0) \equiv N > 0$ on $x \in \overline{\Omega}$ holds. Of course, under such an assumption, it seems that the threshold dynamics similar to those in this paper can be established for model (1.1), see Jiang et al. [42, Theorem 2.4].

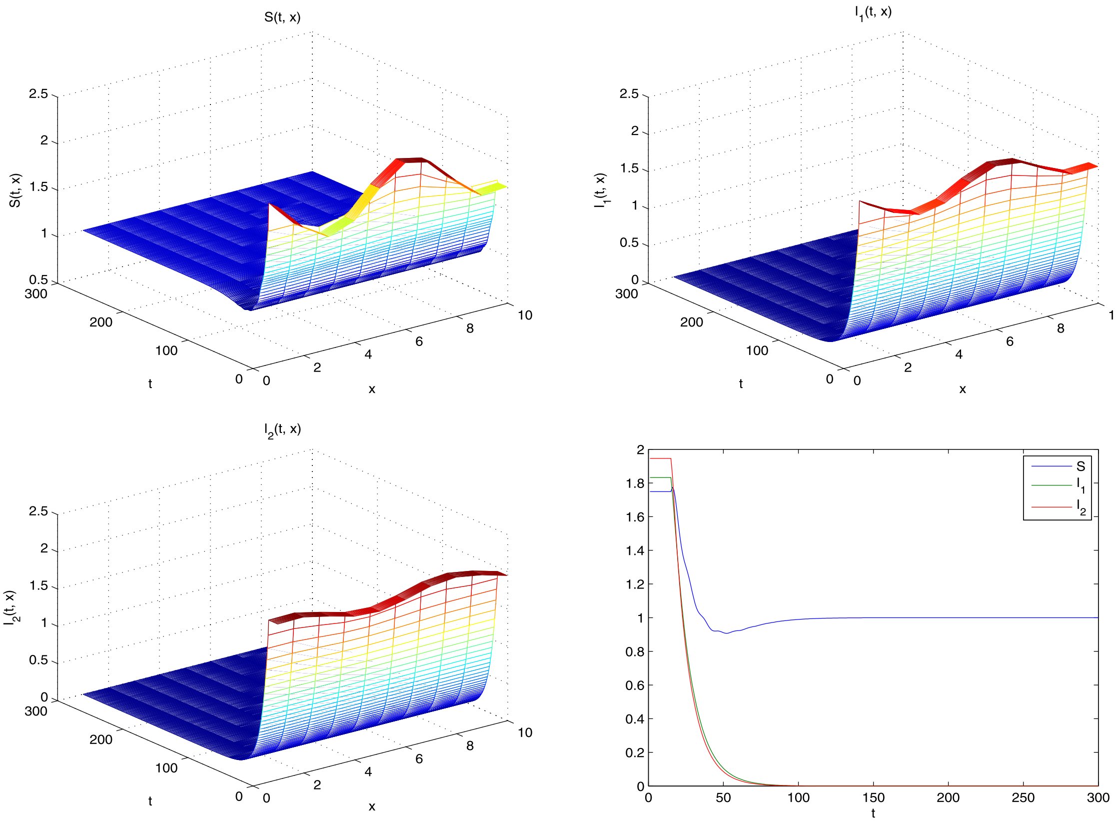

Fig. 6. The dynamic behavior of system (2.5) when $R_0^i < 1, \quad i = 1, 2$.

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