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Time periodic traveling wave solutions for periodic advection–reaction–diffusion systems

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Abstract

We study the existence, uniqueness, and asymptotic stability of time periodic traveling wave solutions to a class of periodic advection–reaction–diffusion systems. Under certain conditions, we prove that there exists a maximal wave speed c^* such that for each wave speed $c \le c^*$, there is a time periodic traveling wave connecting two periodic solutions of the corresponding kinetic system. It is shown that such a traveling wave is unique modulo translation and is monotone with respect to its co-moving frame coordinate. We also show that the traveling wave solutions with wave speed $c \le c^*$ are asymptotically stable in certain sense. In addition, we establish the nonexistence of time periodic traveling waves with speed $c > c^*$.

MSC: 35B10; 35B35; 35B40; 35C07; 35K40

Keywords: Advection–reaction–diffusion system; Time periodic traveling waves; Asymptotic stability; Maximal wave speed; Lotka–Volterra competition system

Contents

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1. Introduction

Traveling wave solutions of reaction–diffusion systems have been studied intensively over the last four decades since wave phenomena are observed in many time dependent processes described by evolution equations (see Conley and Gardner [8], Dunbar [9], Gardner [12], Gourley and Ruan [15], Hosono [19], Kan-On [20], Lewis et al. [21], Li et al. [22], Sandstede and Scheel [31], Volpert et al. [32], Weinberger [34] and references therein). Moreover, the study of traveling wave solutions has been such an essential part of mathematical analysis of evolving spatial patterns generated by nonlinear parabolic equations because of their importance in governing the long time behavior and stability.

Although the study of traveling wave solutions has a longstanding history, most of the existing studies are devoted to autonomous equations. Recently, an interest in both space and time periodic traveling wave solutions has been stimulated by a vast number of examples of biological and physical systems where relevant parameters are either space periodic (Berestycki and Hamel [4], Berestycki et al. [5–7]) or time periodic (Alikakos et al. [1], Liang et al. [24], Liang and Zhao [25], Nolen and Xin [30], Xin [35], Zhao [36]). For pulsating fronts, Hamel [16] and Hamel and Roques [17] presented a systematic analysis of the qualitative behavior, uniqueness, and stability of monostable pulsating fronts for reaction–diffusion equations in periodic media with KPP nonlinearities. The established results provide a complete classification of all KPP pulsating fronts. Most recently, Zhao and Ruan [37] investigated time periodic traveling wave solutions of a diffusive Lotka–Volterra competition with periodic forcing. The basic existence and uniqueness results for traveling waves connecting two semi-trivial periodic solutions of the corresponding kinetic system were obtained. The asymptotic stability of traveling wave solutions was also established.

On the other hand, advection–reaction–diffusion equations have been used extensively to model some reaction–diffusion processes taking place in moving media such as fluids, for example, combustion, atmospheric chemistry, and plankton distributions in the sea, etc. Berestycki [2], Gilding and Kersner [14], Malaguti and Marcelli [28], and Malaguti et al. [29] investigated the influence of advection on the propagation of traveling wave fronts in some reaction–diffusion systems. See also Liang and Wu [23] and Wang et al. [33].

In this paper, we are interested in studying the existence and other qualitative behaviors of time periodic traveling wave solutions of a periodic advection–reaction–diffusion system of the following form:

$$\begin{cases} u_t = d_1(t)\Delta u + \mathbf{k}(t)\cdot\nabla u + f(t,u,v), \\ v_t = d_2(t)\Delta v + \mathbf{l}(t)\cdot\nabla v + g(t,u,v), \end{cases} \tag{1.1}$$

where $u = u(t,x)$, $v = v(t,x)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ $(n \geq 1)$, $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $\nabla := (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$, $\mathbf{k}(t) = (k_1(t), \cdots, k_n(t))$, $\mathbf{l}(t) = (l_1(t), \cdots, l_n(t))$, d_i $(i = 1, 2)$ and k_i and l_i $(i = 1, \cdots, n)$ are T-periodic and Hölder continuous functions of t, d_i is strictly positive in $[0, T]$, while k_i and l_i

may change sign, and both f and g are T-periodic in t. Nonlinear periodic advection–reaction–diffusion systems like (1.1) arise in many areas of biology, chemistry and physics and may be utilized to model a vast variety of phenomena. In such a system, diffusion and advection play a crucial role in determining its spatio-temporal patterns and dynamics.

Time periodic traveling waves to (1.1) are solutions of the form

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \begin{pmatrix} X(t, x\cdot\nu - ct) \\ Y(t, x\cdot\nu - ct) \end{pmatrix}, \qquad \begin{pmatrix} X(t+T,z) \\ Y(t+T,z) \end{pmatrix} = \begin{pmatrix} X(t,z) \\ Y(t,z) \end{pmatrix}, \quad z = x\cdot\nu - ct \tag{1.2}$$

satisfying

$$\begin{pmatrix} X(t,\pm\infty) \\ Y(t,\pm\infty) \end{pmatrix} = \lim_{z\to\pm\infty} \begin{pmatrix} X(t,z) \\ Y(t,z) \end{pmatrix} = \begin{pmatrix} u^{\pm}(t) \\ v^{\pm}(t) \end{pmatrix},$$

where the given vector $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{R}^n$ with $|\nu| = 1$ denotes the direction of motion of the wave. $\begin{pmatrix} u^+(t) \\ v^+(t) \end{pmatrix}$ and $\begin{pmatrix} u^-(t) \\ v^-(t) \end{pmatrix}$ are the periodic solutions of the corresponding ordinary differential equations

$$\begin{cases} \dfrac{du}{dt} = f(t,u,v), \\ \dfrac{dv}{dt} = g(t,u,v). \end{cases} \tag{1.3}$$

Notice that time periodic traveling waves of the form (1.2) enjoy the property

$$\big(u(t,x), v(t,x)\big) = \big(u(t+T, x + cT\nu), v(t+T, x+cT\nu)\big).$$

In the present work, define

$$\overline{J} = \frac{1}{T}\int_0^T J(t)dt$$

as the average of a function J that is integrable in $[0,T]$. We make the following assumptions:

(H1) $f(t+T,\cdot,\cdot) = f(t,\cdot,\cdot)$, $g(t+T,\cdot,\cdot) = g(t,\cdot,\cdot)$ for all $t\in\mathbb{R}$, f and $g \in C^{\beta,2}(\mathbb{R}\times\mathbb{R}^2,\mathbb{R})$ for some $\beta \in\,]0,1[$, $f(t,0,0) = g(t,0,0) = f(t,1,1) = g(t,1,1) = 0$ for all $t\in\mathbb{R}$, and $f(t,u,v) = uh(t,u,v)$.

(H2) $f_v(t,u,v) \gneq 0$ for all $(t,u,v) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$, $g_u(t,u,v) \gneq 0$ for all $(t,u,v)\in\mathbb{R}\times\mathbb{R}\times(-\infty,1]$.

(H3) $\overline{h(t,0,0)} > 0$ and $h_v(t,u,v) > 0$ for all $(t,u,v)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$.

(H4) $\overline{g_v(t,0,0)} < 0$, $\overline{\theta^{-1}g(t,0,\theta)} < 0$ for any $0<\theta<1$.

(H5) $f(t,u,v) \geq f_u(t,0,0)u + f_v(t,0,0)v - \varpi(|u|+|v|)^{1+\gamma}$ and $g(t,u,v) \geq g_u(t,0,0)u + g_v(t,0,0)v - \varpi(|u|+|v|)^{1+\gamma}$ for all $(t,u,v)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, where ϖ and γ are certain positive constants.

(H6) $f^*(t,s) := f(t,s,s)$, $g^*(t,s) := g(t,s,s)$. Assume that $f_s^*(t,0) \geq g_s^*(t,0)$, $f_s^*(t,0)s \geq f^*(t,s)$, and $g_s^*(t,0)s \geq g^*(t,s)$ for all $(t,s)\in\mathbb{R}\times\mathbb{R}^+$.

(H7) Let

$$\Pi^0 := \left\{ \big(p_i(t), q_i(t)\big) \in [0,1] \times [0,1] \;\middle|\; \begin{pmatrix} p_i(\cdot + T) \\ q_i(\cdot + T) \end{pmatrix} = \begin{pmatrix} p_i(\cdot) \\ q_i(\cdot) \end{pmatrix}, \right.$$
$$\left. \frac{d}{dt}\begin{pmatrix} p_i \\ q_i \end{pmatrix} = \begin{pmatrix} f(t, p_i, q_i) \\ g(t, p_i, q_i) \end{pmatrix} \right\},$$

where $i = 1, \cdots, m$. Assume that m is finite. Let $\Pi^+ = \{(p(t), q(t)) \in \Pi^0 \mid p(t)q(t) > 0$ for all $t \in [0, T]\}$. Assume that $\Pi^+ = \{(1, 1)\}$.

(H8) Let μ^+ be a characteristic exponent of the linear periodic system

$$\frac{d\mathbf{w}}{dt} - A(t)\mathbf{w} = 0,$$

where $A(t) = \begin{pmatrix} f_u(t,1,1) & f_v(t,1,1) \\ g_u(t,1,1) & g_v(t,1,1) \end{pmatrix}$. Let $\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$ be the eigensolution associated with μ^+. Assume that $\mu^+ < 0$, and both ψ_1 and ψ_2 are strictly positive in $[0, T]$.

The paper is organized as follows. In Section 2, under certain conditions we establish the existence of c^* such that there exists, for any $c \le c^*$, a time periodic traveling wave solution to (1.1) which is monotone in z. In Section 3, we study the uniqueness of time periodic traveling wave solutions of (1.1) for $c \le c^*$. Our approach is to obtain the exact exponential decay rate of a traveling wave solution as it tends towards its unstable limiting state. We would like to point out that unlike the diffusive Lotka–Volterra competition system studied in Zhao and Ruan [37] where the diffusion coefficients are independent of time and advection is absent, the time dependence of both diffusion and advection coefficients in system (1.1) cause substantial technical difficulties, and one cannot use the Laplace transform method and spectral theory employed in [37] to obtain the exponential decay rate of a traveling wave solution of (1.1). To obtain a good understanding of the asymptotic properties of travel wave solutions, different techniques have to be utilized to address this issue. We also show that the components of such a solution are monotone with respect to the variable z. With these asymptotic properties, we employ the sliding method (Berestycki and Nirenberg [3]) to establish the uniqueness of the aforementioned solution. We also show that the wave speed c^* obtained in Section 2 is the maximal speed such that (1.1) has no solutions with wave speed $c > c^*$. In Section 4, under the same conditions presented in Section 3, we utilize similar methods as in Hamel and Roques [17] and Zhao and Ruan [37] to study the asymptotic stability of the time periodic traveling wave solution of (1.1).

We would like to mention that the techniques and results in this paper can be used to study some biological and epidemiological models described by advection–reaction–diffusion systems with periodic coefficients. In particular, by applying the results in this paper we can obtain the existence, uniqueness, and stability of time periodic traveling wave solutions for the two species time-periodic Lotka–Volterra advection–reaction–diffusion systems that will generalize the corresponding results in Zhao and Ruan [37].

For future reference, we denote a vector by printing a letter in boldface $\mathbf{u} = (u_1, \cdots, u_i, \cdots, u_n)$, where u_i stands for the i-th component of $\mathbf{u}$. The following notation shall be adopted. Let $I, \Gamma \subseteq \mathbb{R}$ be two (possibly unbounded) intervals and $M \subseteq \mathbb{R}^n$. Denote by $BUC(I \times \Gamma, M)$ the space of uniformly continuous and bounded functions $\mathbf{u} : I \times \Gamma \to M$ and $C_b(I \times \Gamma, M)$ the space of continuous and bounded functions $\mathbf{u} \in C(I \times \Gamma, M)$. Given $\alpha \in\,]0, 1[$, we let

$C^{\alpha/2,\alpha}(I \times \Gamma, M)$, $C^{1+\alpha/2,2+\alpha}(I \times \Gamma, M)$ be the space of functions defined in Lunardi [27] (see page 177 of [27]). Set $[a,b]^2 := [a,b] \times [a,b]$, where $-\infty \le a < b < \infty$. In what follows, $]a,b[$ with $a < b$ stands for an open interval with end points a and b.

2. Existence of time periodic traveling wave solutions

This section is devoted to the existence of periodic traveling wave solutions to system (1.1).

Definition 2.1. (See Fife and Tang [10].) Let D be an open and connected domain of $\mathbb{R} \times \mathbb{R}^n$. A vector valued function $\mathbf{w} \in C^{1,2}(D, \mathbb{R}^m)$ is called a *regular super-solution* of

$$\frac{\partial u_k}{\partial t} = \sum_{i,j=1}^{n} a_{ij}^k(t,x) \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^k(t,x) \frac{\partial u_k}{\partial x_i} + h_k(t, u_1, \cdots, u_m), \quad k = 1, \cdots, m \tag{2.1}$$

in D provided that

$$\sum_{i,j=1}^{n} a_{ij}^k(t,x) \frac{\partial^2 w_k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^k(t,x) \frac{\partial w_k}{\partial x_i} + h_k(t, w_1, \cdots, w_m) - \frac{\partial w_k}{\partial t} \le 0, \quad \text{for } (t,x) \in D,$$

for each $k \in \{1, \cdots, m\}$. It is called a *regular sub-solution* of (2.1) if the above inequality is reversed. Here $a_{i,j}^k, b_i^k \in C_b^{\theta/2,\theta}(D)$, $\theta \in]0,1[$. Moreover, there exists $\omega > 0$ such that $a_{i,j}^k(t,x)\xi_i\xi_j \ge \omega \sum_{i=1}^n \xi_i^2$ for any n-tripe of real numbers $(\xi_1, \cdots, \xi_n)$ and for any $(t,x) \in D$. In addition, $h_i \in C^{0,1}(I \times \mathbb{R}^m, \mathbb{R})$.

Remark 2.2. Let $I \times \Gamma \subseteq \mathbb{R} \times \mathbb{R}$ be an open connected domain, where $I \subseteq \mathbb{R}$ and $\Gamma \subseteq \mathbb{R}$ are both open intervals (possibly unbounded). If $\mathbf{w}(t,z) = (w_1(t,z), w_2(t,z)) \in C^{1,2}(I \times \Gamma)$ is a regular super-solution (sub-solution) of

$$\begin{cases} u_t = d_1(t)u_{zz} + \big[c + k(t)\big]u_z + f(t,u,v), \\ v_t = d_2(t)v_{zz} + \big[c + l(t)\big]v_z + g(t,u,v) \end{cases} \tag{2.2}$$

in $(t,z) \in I \times \Gamma$, where

$$k(t) = \sum_{i=1}^{n} \nu_i k_i(t), \qquad l(t) = \sum_{i=1}^{n} \nu_i l_i(t), \tag{2.3}$$

then $\widehat{\mathbf{w}}(t,x) := (w_1(t, x\cdot\nu - ct), w_2(t, x\cdot\nu - ct)) \in C^{1,2}(D)$ with $z = x\cdot\nu - ct$ is a regular super-solution (sub-solution) of (1.1), where $D = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid t \in I,\ x\cdot\nu - ct \in \Gamma\}$.

Definition 2.3. A vector function $\mathbf{w}(t,x) = (u(t, x\cdot\nu - ct), v(t, x\cdot\nu - ct)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ is said to be a *nonnegative time periodic traveling wave* of (1.1) connecting $(0,0)$ and $(1,1)$ if $(u(t,z), v(t,z)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve

$$\begin{cases} u_t = d_1(t)u_{zz} + \big[c + k(t)\big]u_z + f(t, u, v), \\ v_t = d_2(t)v_{zz} + \big[c + l(t)\big]v_z + g(t, u, v), \\ \big(u(t, z), v(t, z)\big) = \big(u(t + T, z), v(t + T, z)\big), \quad \big(u(t, z), v(t, z)\big) \ge (0, 0), \\ \lim\limits_{z\to-\infty} \big(u(t, z), v(t, z)\big) = (0, 0), \quad \lim\limits_{z\to\infty} \big(u(t, z), v(t, z)\big) = (1, 1). \end{cases} \tag{2.4}$$

Here $z = x \cdot \nu - ct$, and $k(t)$ and $l(t)$ are given by (2.3).

Remark 2.4. Suppose that $\{k_i(t)\}$ and $\{l_i(t)\}$ are both linearly dependent, i.e., $\mathbb{A} := \{\omega \in \mathbb{R}^n \mid \sum_{i=1}^n \omega_i k_i(t) \equiv 0,\ \prod_{i=1}^n \omega_i \neq 0\} \neq \varnothing$ and $\mathbb{B} := \{\omega \in \mathbb{R}^n \mid \sum_{i=1}^n \omega_i l_i(t) \equiv 0,\ \prod_{i=1}^n \omega_i \neq 0\} \neq \varnothing$, where $\omega = (\omega_1, \cdots, \omega_n)$. In particular, if $\mathbb{A} \cap \mathbb{B} \neq \varnothing$, then it is easy to see that $(U(t, x \cdot \widehat{\nu} - ct), W(t, x \cdot \widehat{\nu} - ct))$ is also a time periodic traveling wave of (1.1) with speed c provided that $(U(t, x \cdot \nu - ct), W(t, x \cdot \nu - ct))$ and c solve (1.1) and (1.2) and $(\mathbb{A} \cap \mathbb{B}) \setminus \operatorname{span}\{\nu\} \neq \varnothing$, and $\widehat{\nu} - \nu \in (\mathbb{A} \cap \mathbb{B}) \setminus \operatorname{span}\{\nu\}$.

Definition 2.5. (See Fife and Tang [10].) If $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, the relation $\mathbf{u} < \mathbf{v}$ ($\mathbf{u} \le \mathbf{v}$ respectively) is to be understood componentwise: $u_i < v_i$ ($u_i \le v_i$) for each i. The other relations, such as "max", "min", "sup", and "inf", are similarly to be understood componentwise.

Definition 2.6. (See Fife and Tang [10].) A vector valued function $\mathbf{w}$ is said to be an *irregular super-solution* of (2.1) if there exist regular super-solutions $\mathbf{w}^1, \cdots, \mathbf{w}^k$ of (2.1) such that $\mathbf{v} = \min\{\mathbf{w}^1, \cdots, \mathbf{w}^k\}$. It is called an *irregular sub-solution* of (2.1) if there exist regular sub-solutions $\mathbf{v}^1, \cdots, \mathbf{v}^k$ of (2.1) with $\mathbf{v} = \max\{\mathbf{v}^1, \cdots, \mathbf{v}^k\}$.

Lemma 2.7. *Suppose that there exist $\underline{\mathbf{w}}(t, z) \in C_b^\theta(\mathbb{R} \times \mathbb{R})$ and $\overline{\mathbf{w}}(t, z) \in C_b^\theta(\mathbb{R} \times \mathbb{R})$ such that $\underline{\mathbf{w}}(t, z)$ and $\overline{\mathbf{w}}(t, z)$ are the irregular super- and sub-solutions of (2.2) in $\mathbb{R} \times (-\infty, z_0)$ and $\mathbb{R} \times \mathbb{R}$, respectively. Here $\theta \in]0, 1[$, $z_0 \in \mathbb{R}$. Assume further that $\underline{\mathbf{w}} \le \overline{\mathbf{w}}$ for all $(t, z) \in \mathbb{R} \times (-\infty, z^0]$ and $\mathbf{0} \le \overline{\mathbf{w}} \le (1, 1)$, $\overline{\mathbf{w}}(0, z) \ge \overline{\mathbf{w}}(T, z)$, $\underline{\mathbf{w}}(T, z) \ge \underline{\mathbf{w}}(0, z)$, and $\underline{\mathbf{w}}(t, z^0) \le \mathbf{0}$ for all $t \in \mathbb{R}$. Then there exists a positive solution $\mathbf{w}^* \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ to (2.2) such that $\mathbf{w}^*(\cdot + T, z) = \mathbf{w}^*(\cdot, z)$, $\mathbf{w}^* \le \overline{\mathbf{w}}$ for all $(t, z) \in [0, T] \times \mathbb{R}$, and $\mathbf{w}^* \ge \underline{\mathbf{w}}$ for all $(t, z) \in [0, T] \times (-\infty, z^0]$. In addition, if $\overline{\mathbf{w}}$ is nondecreasing with respect to z, then either $(w_i^*)_z > 0$ or $(w_i^*)_z \equiv 0$.*

Proof. The proof is essentially the same as that of Lemma 2.4 in [37]. We give a sketch for the sake of clarity. Let the operator $\mathcal{F} : C_b^{\theta/2,\theta}([0, T] \times \mathbb{R}, \mathbb{R}^2) \to C_b^{\theta/2,\theta}([0, T] \times \mathbb{R}, \mathbb{R}^2)$ be defined by

$$(\mathcal{F}\mathbf{w})(t) = G(t, 0)\mathbf{w}(T) + \int_0^t G(t, \tau)\big[K\mathbf{w}(\tau) + H\big(\tau, \mathbf{w}(\tau)\big)\big]d\tau, \quad t \in (0, T].$$

Here $\theta \in]0, 1[$,

$$H(t, \mathbf{w}) = \begin{pmatrix} f(t, w_1, w_2) \\ g(t, w_1, w_2) \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$K \geq \sup_{\mathbb{R}\times[-2,2]^2}[|f(t,w_1,w_2)|+|g(t,w_1,w_2)|]$, and $G(t,s)_{s\leq t}$ is the evolution operator associated with the family $\mathbf{A}(t): D(\mathbf{A}(t)) \subset X_0 \to X_0$, $D(\mathbf{A}(t)) = X_1$, defined by

$$\mathbf{A}(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_1(t)\frac{\partial^2 u}{\partial z^2} + [c+k(t)]\frac{\partial u}{\partial z} - Ku & 0 \\ 0 & d_2(t)\frac{\partial^2 v}{\partial z^2} + [c+l(t)]\frac{\partial v}{\partial z} - Kv \end{pmatrix},$$

where $X_0 = BUC(\mathbb{R},\mathbb{R}^2)$ and $X_1 = \{\binom{u}{v} \in \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2),\ \binom{u}{v}, \mathbf{A}(t)\binom{u}{v} \in X_0\}$. In terms of Corollary 6.1.8 of Lunardi [27], $\mathcal{F}\mathbf{w} \in C^{\theta/2,\theta}([0,T]\times\mathbb{R})$ provided $\mathbf{w} \in C^{\theta/2,\theta}([0,T]\times\mathbb{R})$ for some $\theta \in]0,1[$. Note that $\mathcal{F}\mathbf{w}$ is the mild solution of

$$\begin{cases} u_t = d_1(t)\dfrac{\partial^2 u}{\partial z^2} + \big[k(t)+c\big]\dfrac{\partial u}{\partial z} - Ku + Kw_1 + f(t,w_1,w_2), \\ v_t = d_2(t)\dfrac{\partial^2 v}{\partial z^2} + \big[l(t)+c\big]\dfrac{\partial v}{\partial z} - Kv + Kw_2 + g(t,w_1,w_2), \\ \big(u(0),v(0)\big) = \big(w_1(T),w_2(T)\big). \end{cases}$$

Now set $\mathbf{w}^0 = \overline{\mathbf{w}}$ and $\mathbf{w}^{n+1} = \mathcal{F}\mathbf{w}^n$, $n = 0,1,\cdots$. Using the techniques given in [37] (see the proof of Lemma 2.4 [37] for details), we can show that $\underline{\mathbf{w}} \leq \mathbf{w}^n$ in $[0,T]\times(-\infty,z_0]$ and $(0,0) \lneqq \mathbf{w}^n \leq \overline{\mathbf{w}}$ for all $(t,z) \in [0,T]\times\mathbb{R}$. This implies that $\|\mathbf{w}^n\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R})}$ are uniformly bounded. Moreover, there exists $\beta \in]0,1[$ for which $\|\mathbf{w}^n(T)\|_{C^{2+\beta}(\mathbb{R})}$ are uniformly bounded for all $n \geq 1$. Thus, for $n \geq 2$, Corollary 6.1.8 of Lunardi [27] implies that $\|\mathbf{w}^n\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R})} \leq C$ for some $\alpha \in]0,1[$ and a positive constant C depending only upon d_i, k, l, c, and $\|\overline{\mathbf{w}}\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R})}$. Notice that $\mathbf{w}^{n+1}$ is in fact a strict solution with $\mathbf{w}^{n+1}(0) = \mathbf{w}^n(T)$ if $n \geq 1$ (see page 123 of Lunardi [27] for the definition of a strict solution). In particular, we have $\mathbf{w}^{n+1} \leq \mathbf{w}^n$. Therefore, the sequence $\{\mathbf{w}^n\}$ converges in $C^{1,2}_{\mathrm{loc}}([0,T]\times\mathbb{R})$ to a function $\mathbf{w}^*$, which solves (2.2). With the same arguments as those of Lemma 2.4 of [37], we can finally show that $\mathbf{w}^*(0) = \mathbf{w}^*(T)$ and either $(w_i^*)_z > 0$ or $(w_i^*)_z \equiv 0$ provided that $\overline{\mathbf{w}}$ is nondecreasing with respect to z. The proof is completed. $\square$

In the following, we set

$$\Lambda_c(\lambda) = \overline{d_1(t)}\lambda^2 + \overline{c+k(t)}\lambda + \overline{f_u(t,0,0)}, \quad c \in \mathbb{R},\ \lambda \in \mathbb{R},$$

and

$$\Phi^\lambda(t) = \exp\left(\int_0^t \big[d_1(s)\lambda^2 + \big(c+k(s)\big)\lambda + f_u(s,0,0) - \Lambda_c(\lambda)\big]ds\right),$$

that is,

$$\Lambda_c(\lambda)\Phi^\lambda(t) = \big\{f_u(t,0,0) + \big[c+k(t)\big]\lambda + d_1(t)\lambda^2\big\}\Phi^\lambda(t) - \frac{\partial\Phi^\lambda(t)}{\partial t}. \tag{2.5}$$

We also let

$$\kappa = \overline{d_1(t)}\,\overline{f_u(t,0,0)}, \qquad \lambda_c = \frac{-c - \overline{k(t)} - \sqrt{(c+\overline{k(t)})^2 - 4\kappa}}{2\overline{d_1(t)}} \quad \text{if } c \le c^* =: -2\sqrt{\kappa} - \overline{k(t)}. \tag{2.6}$$

Clearly, $\Lambda(\lambda)$ has positive zeros if and only if $c \le c^*$ since $\overline{f_u(t,0,0)} > 0$.

In case that $c \le c^*$, we write, for convenience, that

$$\varphi_1(t) = \exp\left(\int_0^t \left[d_1(\tau)\lambda_c^2 + c\lambda_c + k(\tau)\lambda_c + f_u(\tau,0,0)\right]d\tau\right), \quad c \le c^*. \tag{2.7}$$

Let $\Theta : \mathbb{R} \to \mathbb{R}^+$ be defined by

$$\Theta(s) = \begin{cases} s & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases} \tag{2.8}$$

Throughout the paper, we will assume that

$$k(t) - l(t) \ge \frac{\sqrt{\kappa}\,\Theta(d_2(t) - d_1(t))}{\overline{d_1(t)}} \quad \text{for all } t \in [0,T]. \tag{2.9}$$

If (2.9) holds, then we set

$$\begin{cases} \varphi_2(t) = \varphi_2(0)\exp\left(\int_0^t \varrho(s)ds\right) + \int_0^t \exp\left(\int_s^t \varrho(\tau)d\tau\right) g_u(s,0,0)\varphi_1(s)ds, \quad c \le c^*, \\ \varphi_2(0) = \left[1 - \exp\left(\int_0^T \varrho(s)ds\right)\right]^{-1} \int_0^T \exp\left(\int_s^T \varrho(\tau)d\tau\right) g_u(s,0,0)\varphi_1(s)ds, \end{cases} \tag{2.10}$$

where $\varrho(t) = d_2(t)\lambda_c^2 + (c + l(t))\lambda_c + g_v(t,0,0)$. Note that $\varphi_2(t)$ is well defined and is the unique positive periodic solution of

$$g_u(t,0,0)\varphi_1 + \left[d_2(t)\lambda_c^2 + \big(c + l(t)\big)\lambda_c + g_v(t,0,0)\right]v - \frac{dv}{dt} = 0, \quad c \le c^*$$

since

$$\begin{aligned} \overline{\varrho(t)} &= \overline{d_2(t)\lambda_c^2 + \big(c + l(t)\big)\lambda_c + g_v(t,0,0)} \le \overline{d_1(t)\lambda_c^2 + \big(c + k(t)\big)\lambda_c + g_v(t,0,0)} \\ &= \overline{g_v(t,0,0) - f_u(t,0,0)} < 0. \end{aligned}$$

In case that $c < c^*$, we fix

$$\epsilon \in \left(0, \min\left\{\frac{\gamma\lambda_c}{2}, \frac{\sqrt{(c+\overline{k(t)})^2-4\kappa}}{2\overline{d_1(t)}}, \frac{\overline{d_1(t)}[\overline{f_u(t,0,0)-g_v(t,0,0)}]}{2\sqrt{\kappa}[\overline{d_1(t)+d_2(t)}]}\right\}\right] \tag{2.11}$$

and let

$$\Lambda^\epsilon = \Lambda_c(\lambda_c+\epsilon) = \overline{(\lambda_c+\epsilon)^2 d_1(t) + (\lambda_c+\epsilon)\big(c+k(t)\big) + f_u(t,0,0)}, \tag{2.12}$$

$$\phi_1(t) = \exp\left(\int_0^t \big[d_1(\tau)(\lambda_c+\epsilon)^2 + (\lambda_c+\epsilon)\big(c+k(\tau)\big) + f_u(\tau,0,0) - \Lambda^\epsilon\big]d\tau\right). \tag{2.13}$$

Clearly, $\Lambda^\epsilon < 0$ and $\phi_1(t)$ is T-periodic and satisfies

$$\big[d_1(t)(\lambda_c+\epsilon)^2 + (\lambda_c+\epsilon)\big(c+k(t)\big) + f_u(t,0,0) - \Lambda^\epsilon\big]\phi_1(t) - \frac{d\phi_1}{dt} = 0.$$

We also set

$$\begin{cases} \phi_2(t) = \phi_2(0)\exp\left(\int_0^t \varrho_\epsilon(s)ds\right) + \int_0^t \exp\left(\int_s^t \varrho_\epsilon(\tau)d\tau\right) g_u(s,0,0)\phi_1(s)ds, \quad c<c^*, \\ \phi_2(0) = \left[1-\exp\left(\int_0^T \varrho_\epsilon(s)ds\right)\right]^{-1} \int_0^T \exp\left(\int_s^T \varrho_\epsilon(\tau)d\tau\right) g_u(s,0,0)\phi_1(s)ds, \end{cases} \tag{2.14}$$

where $\varrho_\epsilon(t) = d_2(t)(\lambda_c+\epsilon)^2 + [c+l(t)](\lambda_c+\epsilon) - \Lambda^\epsilon + g_v(t,0,0)$. Since

$$\begin{aligned} \overline{\varrho_\epsilon(t)} &= \overline{d_2(t)(\lambda_c+\epsilon)^2 + \big(c+l(t)\big)(\lambda_c+\epsilon) - \Lambda^\epsilon + g_v(t,0,0)} \\ &= \overline{\lambda_c\big[l(t)-k(t)+\lambda_c\big(d_2(t)-d_1(t)\big)\big] + \epsilon\big[l(t)-k(t)+\epsilon\big(d_2(t)-d_1(t)\big)\big]} \\ &\quad + \overline{2\epsilon\lambda_c\big[d_2(t)-d_1(t)\big] - f_u(t,0,0) + g_v(t,0,0)} < 0, \end{aligned}$$

it is easy to see that $\phi_2(t)$ is the unique positive periodic solution to

$$g_u(t,0,0)\phi_1 + \big[d_2(t)(\lambda_c+\epsilon)^2 + \big(c+l(t)\big)(\lambda_c+\epsilon) - \Lambda^\epsilon + g_v(t,0,0)\big]v - \frac{dv}{dt} = 0, \quad c<c^*.$$

We now construct a regular sub-solution for system (2.2).

Proposition 2.8. *Suppose that (H1)–(H5) are satisfied. Let $\nu \in \mathbb{R}^n$ with $|\nu| = 1$. Let $k(t)$ and $l(t)$ be given by (2.3). Assume that $k(t) - l(t) \geq \frac{\sqrt{\kappa}\Theta(d_2(t)-d_1(t))}{\overline{d_1(t)}}$ for any $t \in [0,T]$. For each $c < c^* = -2\sqrt{\overline{d_1(t)}\,\overline{f_u(t,0,0)}} - \overline{k(t)}$, set*

$$\big(\underline{U}(t,z), \underline{W}(t,z)\big) = \left(\delta_1\varphi_1 e^{\lambda_c z}\left[1 - \frac{n_0\phi_1}{\varphi_1}e^{\epsilon z}\right], \delta_2\varphi_2 e^{\lambda_c z}\left[1 - \frac{\delta_1 n_0\phi_2}{\delta_2\varphi_2}e^{\epsilon z}\right]\right),$$
$$\forall (t,z) \in \mathbb{R} \times (-\infty, z_0].$$

Here λ_c is specified by (2.6), ϵ is given by (2.11), and

$$z_0 \le z_* := \frac{-1}{(\gamma\lambda_c - \epsilon)} \ln\left(\frac{\varpi C_\gamma[2 + |\frac{\max_t \frac{\phi_1}{\varphi_1}}{\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}}|^{1+\gamma} + |\frac{\max_t \frac{\phi_2}{\varphi_2}}{\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}}|^{1+\gamma}]}{|\Lambda^\epsilon| \min\{\min_t \phi_1, \min_t \phi_2\}}\right), \tag{2.15}$$

$$n_0 = \frac{e^{-\epsilon z_0}}{\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}}, \qquad 0 < \delta_2 \le \delta_1 \le \min\left\{\min\left\{\frac{e^{-\lambda_c z_0}}{\max_t \varphi_1}, \frac{e^{-\lambda_c z_0}}{\max_t \varphi_2}\right\}, n_0^{\frac{1}{\gamma}}\right\}, \tag{2.16}$$

where $C_\gamma > 0$ is the least constant such that $|a+b|^{1+\gamma} \le C_\gamma(|a|^{1+\gamma} + |b|^{1+\gamma})$, $a, b \in \mathbb{R}$. Then $(\underline{U}, \underline{W})$ is a regular sub-solution of (2.2) for $(t, z) \in \mathbb{R} \times]-\infty, z_0[$.

Proof. We assume without loss of generality that $\|\varphi_i\| \le 1$ and $\|\phi_i\| \le 1$ $(i = 1, 2)$. It is easy to see that $(\underline{U}(t,z), \underline{W}(t,z)) \le (1, 1)$ for all $(t, z) \in \mathbb{R} \times (-\infty, z_0]$ and $(\underline{U}(t, z_0), \underline{W}(t, z_0)) \le (0, 0)$ for all $t \in \mathbb{R}$. Moreover, when $(t, z) \in \mathbb{R} \times]-\infty, z_0[$, we have

$$\begin{aligned}
& f(t, \underline{U}, \underline{W}) + d_1(t)\underline{U}_{zz} + \big(c + k(t)\big)\underline{U}_z - \underline{U}_t \\
&\quad \ge \delta_1 e^{\lambda_c z} f_u(t, 0, 0)\varphi_1(t) - \varpi\big(|\underline{U}|^{1+\gamma} + |\underline{W}|^{1+\gamma}\big) \\
&\qquad + \delta_1 e^{\lambda_c z}\big[\big(d_1(t)\lambda_c^2 + \big(c + k(t)\big)\lambda_c\big)\varphi_1(t) - \varphi_1'(t)\big] \\
&\qquad - \delta_1 n_0 e^{(\lambda_c+\epsilon)z}\big\{\big[d_1(t)(\lambda_c + \epsilon)^2 + (\lambda_c + \epsilon)\big(c + k(t)\big) + f_u(t, 0, 0)\big]\phi_1(t) - \phi_1'(t)\big\} \\
&\quad = e^{\lambda_c z}\left\{\delta_1 n_0|\Lambda^\epsilon|\phi_1 e^{\epsilon z} - \big(\varpi\delta_1^{1+\gamma}\varphi_1^{1+\gamma} e^{\gamma\lambda_c z}\big)\left|1 - n_0\frac{\phi_1}{\varphi_1}e^{\epsilon z}\right|^{1+\gamma}\right. \\
&\qquad \left. - \big(\varpi\varphi_2^{1+\gamma} e^{\gamma\lambda_c z}\big)\left|\delta_2 - \delta_1 n_0\frac{\phi_2}{\varphi_2}e^{\epsilon z}\right|^{1+\gamma}\right\} \\
&\quad \ge e^{\lambda_c z}\left\{\delta_1 n_0|\Lambda^\epsilon|\phi_1 e^{\epsilon z} - C_\gamma\varpi\delta_1^{1+\gamma}\varphi_1^{1+\gamma}e^{\gamma\lambda_c z}\left[1 + \left(n_0\frac{\phi_1}{\varphi_1}e^{\epsilon z}\right)^{1+\gamma}\right]\right. \\
&\qquad \left. - C_\gamma\varpi\delta_1^{1+\gamma}\varphi_2^{1+\gamma}e^{\gamma\lambda_c z}\left[\left(\frac{\delta_2}{\delta_1}\right)^{1+\gamma} + \left(n_0\frac{\phi_2}{\varphi_2}e^{\epsilon z}\right)^{1+\gamma}\right]\right\} \\
&\quad \ge \delta_1 n_0 e^{(\lambda_c+\epsilon)z}\left\{|\Lambda^\epsilon|\phi_1 - C_\gamma\varpi\frac{\delta_1^\gamma}{n_0}e^{(\gamma\lambda_c-\epsilon)z}\left[2 + \left|\frac{\max_t \frac{\phi_1}{\varphi_1}}{\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}}\right|^{1+\gamma}\right.\right. \\
&\qquad \left.\left. + \left|\frac{\max_t \frac{\phi_2}{\varphi_2}}{\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}}\right|^{1+\gamma}\right]\right\} \ge 0
\end{aligned}$$

and

$$\begin{aligned}
& g(t, \underline{U}, \underline{W}) + d_2(t)\underline{W}_{zz} + \big(c + l(t)\big)\underline{W}_z - \underline{W}_t \\
&\quad \ge g_u(t, 0, 0)\big[\delta_1\varphi_1 e^{\lambda_c z} - \delta_1 n_0\phi_1 e^{(\lambda_c+\epsilon)z}\big] + g_v(t, 0, 0)\big[\delta_2\varphi_2 e^{\lambda_c z} - \delta_1 n_0\phi_2 e^{(\lambda_c+\epsilon)z}\big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \varpi\left(|\underline{U}|^{1+\gamma} + |\underline{W}|^{1+\gamma}\right) + \delta_2 e^{\lambda_c z}\left[\left(d_2(t)\lambda_c^2 + \left(c + l(t)\right)\lambda_c\right)\varphi_2 - \varphi_2'\right] \\
&\quad - \delta_1 n_0 e^{(\lambda_c+\epsilon)z}\left\{\left[d_2(t)(\lambda_c+\epsilon)^2 + (\lambda_c+\epsilon)\left(c + l(t)\right)\right]\phi_2(t) - \phi_2'(t)\right\} \\
&\geq \delta_2 e^{\lambda_c z}\left\{g_u(t,0,0)\varphi_1 + \left[d_2(t)\lambda_c^2 + \left(c + l(t)\right)\lambda_c + g_v(t,0,0)\right]\varphi_2 - \varphi_2'\right\} \\
&\quad - \varpi\left(|\underline{U}|^{1+\gamma} + |\underline{W}|^{1+\gamma}\right) - \delta_1 n_0 e^{(\lambda_c+\epsilon)z}\left\{g_u(t,0,0)\phi_1\right. \\
&\quad \left. + \left[d_2(t)(\lambda_c+\epsilon)^2 + (\lambda_c+\epsilon)\left(c + l(t)\right) + g_v(t,0,0)\right]\phi_2 - \phi_2'\right\} \\
&\geq e^{\lambda_c z}\left\{\delta_1 n_0 \left|\Lambda^\epsilon\right|\phi_2 e^{\epsilon z} - \left(\varpi[\delta_1\varphi_1]^{1+\gamma} e^{\gamma\lambda_c z}\right)\left|1 - n_0\frac{\phi_1}{\varphi_1}e^{\epsilon z}\right|^{1+\gamma}\right. \\
&\quad \left. - \left(\varpi\varphi_2^{1+\gamma} e^{\gamma\lambda_c z}\right)\left|\delta_2 - \delta_1 n_0\frac{\phi_2}{\varphi_2}e^{\epsilon z}\right|^{1+\gamma}\right\} \\
&\geq 0.
\end{aligned}
$$

The proof is completed. □

We are in a position to state and prove the existence of time periodic traveling wave solutions for system (1.1) when $c < c^*$.

Theorem 2.9. *Suppose all the assumptions given in Proposition 2.8 are satisfied. In addition, assume that (H6) and (H7) hold. Then, for any $c < c^*$, there exists $\mathbf{w}(t,x) = (u(t, x\cdot\nu - ct)$, $v(t, x\cdot\nu - ct)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ such that $\mathbf{w}$ is a nonnegative time periodic traveling wave of (1.1) connecting $(0,0)$ and $(1,1)$. Moreover, $(u_z(t,z), v_z(t,z)) > (0,0)$ for all $(t,z) \in \mathbb{R}\times\mathbb{R}$, where $z = x\cdot\nu - ct$.*

Proof. We utilize Lemma 2.7 and Proposition 2.8 to establish the existence of a periodic traveling wave solution satisfying (1.1). We will first establish the existence of a periodic solution to (2.2). To this end, a pair of ordered (irregular) super- and sub-solutions is needed. Let $w(t,z) = m\varphi_1(t)e^{\lambda_c z}$ and $(\overline{U}, \overline{W}) = \min\{(w,w), (1,1)\}$, where $m > 0$ is an arbitrary constant. We now show that $(\overline{U}, \overline{W})$ is an irregular super-solution of (2.2). Since $(1,1)$ is a solution of (2.2), it suffices to show that (w,w) is a super-solution of (2.2). Note that

$$
\left[d_1(t)\lambda_c^2 + \left(c + k(t)\right)\lambda_c + f_u(t,0,0)\right]\varphi_1 - \frac{d\varphi_1}{dt} = 0.
$$

It follows from (H1) and (H2) that $f_s^*(t,0) = h(t,0,0) = f_u(t,0,0)$ and $f(t,s,s) \leq f_s^*(t,0)$ and $g(t,s,s) \leq g_s^*(t,0)$ for all $(t,s) \in \mathbb{R}\times\mathbb{R}^+$. In addition, $f_s^*(t,0) \geq g_s^*(t,0)$ for all $t \in \mathbb{R}$. Hence, we have

$$
\begin{aligned}
& d_1(t)w_{zz} + \left[c + k(t)\right]w_z + f(t,w,w) - w_t \\
&\quad \leq d_1(t)w_{zz} + \left[c + k(t)\right]w_z + f_s^*(t,0)w - w_t \\
&\quad = m e^{\lambda_c z}\left\{\left[d_1(t)\lambda_c^2 + \left(c + k(t)\right)\lambda_c + f_u(t,0,0)\right]\varphi_1 - \varphi_1'\right\} \\
&\quad = 0,
\end{aligned}
$$

$$
\begin{aligned}
& d_2(t)w_{zz} + \big(c + l(t)\big)w_z + g(t, w, w) - w_t \\
& \quad \le d_2(t)w_{zz} + \big(c + l(t)\big)w_z + g_s^*(t, 0)w - w_t \\
& \quad \le d_1(t)w_{zz} + \big(c + k(t)\big)w_z + f_s^*(t, 0)w - w_t + \big[d_2(t) - d_1(t)\big]w_{zz} + \big[l(t) - k(t)\big]w_z \\
& \quad = m\varphi_1\lambda_c e^{\lambda_c z}\big[l(t) - k(t) + \lambda_c\big(d_2(t) - d_1(t)\big)\big] \\
& \quad \le m\varphi_1\lambda_c e^{\lambda_c z}\left[l(t) - k(t) + \frac{\sqrt{\kappa}}{d_1(t)}\Theta\big(d_2(t) - d_1(t)\big)\right] \\
& \quad \le 0.
\end{aligned}
$$

Now in view of Lemma 2.7 and Proposition 2.8, we can conclude that for each $c < c^*$, there exists $\mathbf{w}(t, z) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ such that $\mathbf{w}$ and c solve (2.2). Clearly, $\mathbf{w}(t + T, z) = \mathbf{w}(t, z)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. Moreover, by arguing with the same way as in the proof of Theorem 2.5 of [37], we have $\lim_{z\to-\infty} \mathbf{w}(t, z) = (0, 0)$ uniformly in $t \in \mathbb{R}$ and $\lim_{z\to\infty} \mathbf{w}(t, z) = (1, 1)$ uniformly in $t \in \mathbb{R}$. The proof is completed. □

The following result is about the existence of time periodic traveling wave solutions for system (1.1) when $c = c^*$.

Theorem 2.10. *Suppose all the assumptions given in Theorem 2.9 are satisfied. Then, for $c = c^*$, there exists $\mathbf{w}^*(t, x) = (u^*(t, x \cdot \nu - c^*t), v^*(t, x \cdot \nu - c^*t)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ such that $\mathbf{w}^*$ is a nonnegative time periodic traveling wave of (1.1) connecting $(0, 0)$ and $(1, 1)$. In addition, $(u_z^*(t, z), v_z^*(t, z)) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$, where $z = x \cdot \nu - ct$.*

Proof. Let $\mathbf{w}^c(t, x) = (u^c(t, x \cdot \nu - ct), v^c(t, x \cdot \nu - ct)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ be nonnegative time periodic traveling waves of (1.1) connecting $(0, 0)$ and $(1, 1)$ with $c \in [c^* - 1, c^*)$. Clearly, $(u^c(t, z), v^c(t, z))$ solves (2.2) with $z = x \cdot \nu - ct$. Since $|u^c|$ and $|v^c|$ are uniformly bounded, it follows from parabolic estimates that

$$
\|u^c\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}(\mathbb{R}\times\mathbb{R})} + \|v^c\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}(\mathbb{R}\times\mathbb{R})} < \infty
$$

for some $\alpha \in]0, 1[$. Let $\{c_n\}$ be a sequence with $c_n \in [c^* - 1, c^*)$ such that $c_n \to c^*$ as $n \to \infty$. Note that $(u^c(t, z + s), v^c(t, z + s))$ is still a solution of (2.4) for fixed $s \in \mathbb{R}$. Let $(p_i(t), q_i(t)) \in \Pi^0$, where Π^0 is specified by (H7). Now we fix $\eta \in]0, 1[$ such that $\eta < p_i(0)$, $i = 1, \cdots, m$. Since $(0, 0) < (u^{c_n}, v^{c_n}) < (1, 1)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$, by translation, we may assume without loss of generality that $u^{c_n}(0, 0) = \eta$ for all n. By taking a subsequence of $\{(u^{c_n}, v^{c_n})\}$ if necessary, we conclude that $\{(u^{c_n}, v^{c_n})\}$ converges in $C_{\mathrm{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R}) \times C_{\mathrm{loc}}^{1,2}(\mathbb{R}^2, \mathbb{R})$ to a function denoted by (u^*, v^*). Since $(0, 0) < (u^{c_n}, v^{c_n}) < (1, 1)$ and $(u_z^{c_n}, v_z^{c_n}) > (0, 0)$, we have $(0, 0) \le (u^*, v^*) \le (1, 1)$ and $(u_z^*, v_z^*) \ge (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. In particular, $(u^*(t + T, \cdot), v^*(t + T, \cdot)) = (u^*(t, \cdot), v^*(t, \cdot))$ for all $t \in \mathbb{R}$, and $u^*(0, 0) = \eta$ since $u^{c_n}(0, 0) \equiv \eta$. By taking the limits in (2.2), we find that (u^*, v^*) solves (2.2) in $(t, z) \in \mathbb{R} \times \mathbb{R}$ with $c = c^*$. Moreover, it follows from the (strong) maximum principle that either $u_z^* > 0$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$ or $u_z^* \equiv 0$. Likewise, the same holds for v_z^*. Next we show by contradiction that $u_z^* > 0$ and $v_z^* > 0$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.

First, assume that $u_z^* \equiv 0$ and $v_z^* \equiv 0$, that is, $(u^*, v^*) = (u^*(t), v^*(t))$. Thus, $(u^*, v^*) \in \Pi^0$. However, this contradicts the fact that $u^*(0) = \eta \neq p_i(0)$. Consequently, either $u_z^* > 0$ or $v_z^* > 0$. Suppose that $u_z^* \equiv 0$ while $v_z^* > 0$. Then, we have

$$\frac{du^*(t)}{dt} = f\big(t, u^*(t), v^*(t,z)\big) = u^*(t)h\big(t, u^*(t), v^*(t,z)\big).$$

By virtue of (H3), this is impossible. Hence we are led to consider the case that $u_z^* > 0$ and $v_z^* \equiv 0$. If this is true, then u^* and $c = c^*$ solve

$$\frac{du(t)}{dt} = \frac{\partial^2 u(t,z)}{\partial z^2} + \big[c + k(t)\big]\frac{\partial u(t,z)}{\partial z} + u(t,z)h\big(t, u(t,z), v^*(t)\big). \tag{2.17}$$

Recall that $v^*(t) \geq 0$. Due to (H2), $v^*(t) \gneq 0$. Define $\widehat{c} =: -2\sqrt{\overline{d_1(t)}\,\overline{h(t,0,v^*(t))}} - \overline{k(t)}$. In view of (H1) and (H3), it is easy to see that

$$\widehat{c} < c^* = -2\sqrt{\overline{d_1(t)}\,\overline{f_u(t,0,0)}} - \overline{k(t)} = -2\sqrt{\overline{d_1(t)}\,\overline{h(t,0,0)}} - \overline{k(t)}.$$

Meanwhile, from the same reasoning as that of Proposition 3.2, it follows that (2.17) has no bounded positive solutions with $u_z \neq 0$ provided that $c > \widehat{c}$, and we reached a contradiction since $\widehat{c} < c^*$. This contradiction excludes the possibility that $u_z^* > 0$ and $v_z^* \equiv 0$, and we readily conclude that $u_z^* > 0$ and $v_z^* > 0$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$. Next we show that $\lim_{z\to-\infty}(u^*, v^*) = (0,0)$ and $\lim_{z\to\infty}(u^*, v^*) = (1,1)$. To this end, let $(u^*(t,\pm\infty), v^*(t,\pm\infty)) := \lim_{z\to\pm\infty}(u^*, v^*)$. Thanks to the periodicity of (u^*, v^*) with respect to t and the regularity of (u^*, v^*) with respect to (t,z), we see that $(u^*(t,z), v^*(t,z))$ converges to $(u^*(t,\pm\infty), v^*(t,\pm\infty))$ uniformly in t as $z \to \pm\infty$, and both $(u^*(t,\pm\infty), v^*(t,\pm\infty))$ are the periodic solutions to (1.3). Clearly, $(u^*(t,-\infty), v^*(t,-\infty)) < (u^*(t,z), v^*(t,z)) < (u^*(t,+\infty), v^*(t,+\infty))$ for any finite $(t,z) \in \mathbb{R} \times \mathbb{R}$. In view of (H7), we readily infer that $(u^*(t,+\infty), v^*(t,+\infty)) \equiv (1,1)$. In particular, as $u^*(0,0) < p_i(0)$, where $p_i(t)$ are given in (H7), it is easy to see that $u^*(t,-\infty) \equiv 0$. Thanks to (H4), the kinetic system (1.3) has no semi-trivial periodic solutions of the form $(0, q(t))$ with $0 < q(t) < 1$. Hence, it follows that $v^*(t,-\infty) \equiv 0$. The proof is completed. □

Remark 2.11. In the proof of Theorem 2.10, it is shown that (u^{c_n}, v^{c_n}) converges locally to (u^*, v^*). As a matter of fact, we can show that $\lim_{n\to\infty}(\|u^{c_n} - u^*\|_{C(\mathbb{R}\times\mathbb{R})} + \|v^{c_n} - u^*\|_{C(\mathbb{R}\times\mathbb{R})}) = 0$. Indeed, let $\varepsilon > 0$ be given, since $(u^*(t,-\infty), v^*(t,-\infty)) = (0,0)$ and $(u^*(t,\infty), v^*(t,\infty)) = (1,1)$, there exists $M > 0$ such that $|u^*(t,z)| + |v^*(t,z)| \leq \frac{\varepsilon}{4}$ for all $(t,z) \in \mathbb{R} \times (-\infty, -M]$, and $|1 - u^*(t,z)| + |1 - v^*(t,z)| \leq \frac{\varepsilon}{4}$ for all $(t,z) \in \mathbb{R} \times [M, \infty)$. As $\{(u^{c_n}, v^{c_n})\}$ converges in $C^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}) \times C^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ to (u^*, v^*), there exists $N > 0$ such that $|u^{c_n}(t,-M)| + |v^{c_n}(t,-M)| \leq \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ whenever $n > N$ and $|1 - u^{c_n}(t,M)| + |1 - v^{c_n}(t,M)| \leq \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ as $n > N$. As a result, for any $(t,z) \in \mathbb{R} \times (-\infty, M]$, we find that

$$\begin{aligned} &\big|u^{c_n}(t,z) - u^*(t,z)\big| + \big|v^{c_n}(t,z) - v^*(t,z)\big| \\ &\quad \leq \big(u^{c_n}(t,-M) + u^*(t,-M)\big) + \big(v^{c_n}(t,-M) + v^*(t,-M)\big) \leq \varepsilon \end{aligned}$$

provided that $n > N$. While, for any $(t,z) \in \mathbb{R} \times [M, \infty)$, we have that

$$\begin{aligned}
&\left|u^{c_n}(t,z)-u^*(t,z)\right|+\left|v^{c_n}(t,z)-v^*(t,z)\right| \\
&\quad\le \left(\left|1-u^{c_n}(t,z)\right|+\left|1-u^*(t,z)\right|\right)+\left(\left|1-v^{c_n}(t,z)\right|+\left|1-v^*(t,z)\right|\right) \\
&\quad\le \left[\left(1-u^{c_n}(t,M)\right)+\left(1-u^*(t,M)\right)\right]+\left[\left(1-v^{c_n}(t,M)\right)+\left(1-v^*(t,M)\right)\right] \\
&\quad\le \varepsilon
\end{aligned}$$

if $n>N$. Here we used the fact that both (u^{c_n},v^{c_n}) and (u^*,v^*) are monotonically increasing with respect to z.

3. Uniqueness of time periodic traveling wave solutions

In this section we study the uniqueness and asymptotic behavior of periodic traveling waves of (1.1). We hereafter consider the following system

$$\begin{cases} u_t=d_1(t)u_{zz}+\left[c+k(t)\right]u_z+f(t,u,v), \\ v_t=d_2(t)v_{zz}+\left[c+l(t)\right]v_z+g(t,u,v), \\ \left(u(t+T,z),v(t+T,z)\right)=\left(u(t,z),v(t,z)\right), \quad (u,v)\ge(0,0), \\ \lim\limits_{z\to-\infty}(u,v)=(0,0), \quad \lim\limits_{z\to\infty}(u,v)=(1,1), \end{cases} \tag{3.1}$$

where $f(t+T,u,v)=f(t,u,v)$, $g(t+T,u,v)=g(t,u,v)$ for any $(t,u,v)\in\mathbb{R}\times\mathbb{R}^2$, and $f,g\in C^{\theta,1}(\mathbb{R}\times\mathbb{R}^2,\mathbb{R})$ for some $\theta\in\,]0,1[$. Throughout this section, the notations specified in Section 2 will be maintained. For the sake of convenience, a few technical lemmas and propositions used in this section are placed in Appendix A. The following result is Lemma 3.6 in Földes and Poláčik [11].

Lemma 3.1. *Let the differential operators $L_k:=\sum_{i,j=1}^n a_{i,j}^k(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i^k\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t}$ $(k=1,2,\cdots,l)$ be uniformly parabolic in an open domain $]\tau,M[\,\times\Omega$ of $(t,x)\in\mathbb{R}\times\mathbb{R}^n$; that is, there is $\alpha_0>0$ such that $a_{i,j}^k(t,x)\xi_i\xi_j\ge\alpha_0\sum_{i=1}^n\xi_i^2$ for any n-tuples of real numbers $(\xi_1,\xi_2,\cdots,\xi_n)$, where $-\infty<\tau<M\le\infty$ and $\Omega\subset\mathbb{R}^n$ is an open bounded region. Suppose that $a_{i,j}^k,b^k\in C([\tau,M)\times\overline{\Omega},\mathbb{R})$ and $|b^k(t,x)|+|a_{i,j}^k(t,x)|\le\beta_0$ for some β_0 and all $(t,x)\in[\tau,M)\times\overline{\Omega}$. Assume that $\mathbf{w}=(w_1,w_2,\cdots,w_l)\in C([\tau,M)\times\overline{\Omega})\cap C^{1,2}(]\tau,M[\,\times\Omega,\mathbb{R}^l)$ satisfies*

$$\sum_{s=1}^l c^{k,s}(t,x)w_s+L_kw^k\le 0, \quad (t,x)\in\,]\tau,M[\,\times\Omega,\ k=1,2,\cdots,l \tag{3.2}$$

where $c^{k,s}\in C((\tau,M)\times\Omega,\mathbb{R})$ and $c^{k,s}\ge 0$ if $k\ne s$, and $|c^{k,s}|\le\beta_0$ $(k,s=1,2,\cdots,l)$. Let D and U be domains in Ω such that $D\subset\subset U$, $\operatorname{dist}(\overline{D},\partial U)>\varrho$, and $|D|>\varepsilon$ for some positive constants ϱ and ε. Let θ be a positive constant with $\tau+4\theta\le M$. Then there exist positive constants p, ω and ω_1, determined only by α_0, β_0, ϱ, ε, n, $\operatorname{diam}\Omega$ and θ, such that

$$\inf_{]\tau+3\theta,\tau+4\theta[\times D} w_k\ge\omega\left\|w_k^+\right\|_{L^p(]\tau+\theta,\tau+2\theta[\times D)}-\omega_1\max_{j=1,\cdots,k}\sup_{\partial_P(]\tau,\tau+4\theta[\times U)} w_j^-.$$

Here $w_k^+ = \max\{w_k, 0\}$, $w_k^- = \max\{-w_k, 0\}$, *and* $\partial_P((\tau, \tau+4\theta)\times U) = \{\tau\}\times U \cup [\tau, \tau+4\theta]\times \partial U$. *Moreover, if all inequalities in (3.2) are replaced by equalities, then the conclusion holds with* $p=\infty$ *and* ω, ω_1 *independent of* ε.

Proposition 3.2. *Suppose that (H1)–(H3) are satisfied. Assume that* $(u, v) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ *and* $c \in \mathbb{R}$ *solve (2.2) with* $(0,0) \le (u, v) \le (1,1)$, $\lim_{z\to-\infty} u(t,z) = \lim_{z\to-\infty} v(t,z) = 0$, *and* $u_z v_z \ne 0$. *Then*

$$0 < \underline{\lambda} := \liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}} \frac{u_z(t,z)}{u(t,z)}\right\} \le \bar{\lambda} := \limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}} \frac{u_z(t,z)}{u(t,z)}\right\} < \infty. \tag{3.3}$$

Here $\underline{\lambda}$ *and* $\bar{\lambda}$ *satisfy the equation*

$$\overline{d_1(t)}\lambda^2 + \overline{c+k(t)}\lambda + \overline{f_u(t,0,0)} = 0. \tag{3.4}$$

Moreover, (2.2) has no solutions satisfying $(0,0)\le(u,v)\le(1,1)$, $\lim_{z\to-\infty}(u(t,z), v(t,z)) = 0$, *and* $u_z v_z \ne 0$ *provided that* $c > c^* := -2\sqrt{\kappa} - \overline{k(t)}$, *where* $\kappa = \overline{d_1(t)}\,\overline{f_u(t,0,0)}$.

Proof. As $(0,0)$ is a solution of (2.2), it follows that

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1(t)\dfrac{\partial^2 u}{\partial z^2} + \big[c+k(t)\big]\dfrac{\partial u}{\partial z} + \left[\displaystyle\int_0^1 f_u(t, su, sv)ds\right]u + \left[\displaystyle\int_0^1 f_v(t, su, sv)ds\right]v \\ \dfrac{\partial v}{\partial t} = d_2(t)\dfrac{\partial^2 v}{\partial z^2} + \big[c+l(t)\big]\dfrac{\partial v}{\partial z} + \left[\displaystyle\int_0^1 g_u(t, su, sv)ds\right]u + \left[\displaystyle\int_0^1 g_v(t, su, sv)ds\right]v. \end{cases} \tag{3.5}$$

Let $l>0$ be a fixed constant. Let $D =]z-\frac{l}{4}, z+\frac{l}{4}[$, $U =]z-\frac{l}{2}, z+\frac{l}{2}[$, and $\Omega =]z-l, z+l[$, $\tau = 0$, and $\theta = T$. In light of the periodicity and positivity of u and v, by applying Lemma 3.1 to (3.5), we obtain that

$$\sup_{s\in I_{l/4}(z)} u(t', s) \le N_l \inf_{s\in I_{l/4}(z)} u(t,s), \qquad \sup_{s\in I_{l/4}(z)} v(t', s) \le N_l \inf_{s\in I_{l/4}(z)} v(t,s), \quad \forall z, t, t' \in \mathbb{R}, \tag{3.6}$$

where $I_{l/4}(z) =]z-\frac{l}{4}, z+\frac{l}{4}[$, $N_l > 0$ is a constant independent of u and v. Note that

$$u_t = d_1(t)u_{zz} + \big[c+k(t)\big]u_z + \left[\frac{f(t,u,v)}{u}\right]u.$$

Due to (H1), $\frac{f(t,u,v)}{u} = h(t,u,v)$ is uniformly bounded for $(t,z)\in\mathbb{R}\times\mathbb{R}$. From the (interior) parabolic L^p estimates, it follows that

$$\left(\int_T^{2T}\int_{z-\frac{l}{8}}^{z+\frac{l}{8}} \big|u_\tau(\tau,s)\big|^p + \big|u_s(\tau,s)\big|^p + \big|u_{ss}(\tau,s)\big|^p ds d\tau\right)^{\frac{1}{p}} \le C\left(\int_0^{2T}\int_{z-\frac{l}{4}}^{z+\frac{l}{4}} \big|u(\tau,s)\big|^p\right)^{\frac{1}{p}}$$

for some positive constant C independent of t and z. Thus

$$\left(\int_T^{2T}\int_{-\frac{l}{8}}^{\frac{l}{8}}\left|u_\tau(\tau,s+z)\right|^p+\left|u_s(\tau,s+z)\right|^p+\left|u_{ss}(\tau,s+z)\right|^p ds\,d\tau\right)^{\frac{1}{p}}$$
$$\le C\sup_{(t,s)\in[0,2T]\times I_{l/4}(z)}\left|u(\tau,s)\right|.$$

By Sobolev imbedding theorem and (3.6), we readily conclude that there exists a constant $C_T>0$ such that

$$\frac{|u_z(t,z)|}{|u(t,z)|}\le C_T\quad \text{for all } (t,z)\in\mathbb{R}\times\mathbb{R}.$$

Consequently

$$-\infty<\underline{\lambda}:=\liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}}\frac{u_z(t,z)}{u(t,z)}\right\}\le\bar{\lambda}:=\limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}}\frac{u_z(t,z)}{u(t,z)}\right\}<\infty.$$

Now we proceed to show (3.4). To this end, we adopt a technique presented in Nolen and Xin [30] (see also Hamel [16]). Let $\underline{\lambda}$ be defined by (3.4) and let $\{(t_n,z_n)\}$ be the sequence such that

$$\lim_{n\to\infty}\frac{u_z(t_n,z_n)}{u(t_n,z_n)}=\underline{\lambda}.$$

Since both u_z and u are T-periodic in t, we may without loss of generality assume that $t_n\in[0,T]$. Define $u^n(t,z)=\frac{u(t,z+z_n)}{u(t_n,z_n)}$. Then

$$u_t^n=d_1(t)u_{zz}^n+\left[c+k(t)\right]u_z^n+\frac{f(t,u(t,z+z_n),v(t,z+z_n))}{u(t,z+z_n)}u^n.$$

Owing to Lemma 3.1, u^n are locally uniformly bounded. In view of the parabolic estimates and the fact that $f(t,u,v)=[\int_0^1 f_u(t,su,v)ds]u$, there exists a subsequence of $\{u^n\}$, still labeled by $\{u^n\}$, such that $\{u^n\}$ converges in $C_{\mathrm{loc}}^{1,2}(\mathbb{R}\times\mathbb{R})$ to a function w which satisfies

$$w_t=d_1(t)w_{zz}+\left[c+k(t)\right]w_z+f_u(t,0,0)w. \tag{3.7}$$

Note that $u^n\ge0$. Assume that $\lim_{n\to\infty}t_n=t^*$. Then $w(t^*,0)=1$, and hence $w>0$ in terms of the strong maximal principle. In addition, observe that $\frac{u_z^n(t,z)}{u^n(t,z)}=\frac{u_z(t,z+z_n)}{u(t,z+z_n)}$. Therefore $\frac{w_z(t,z)}{w(t,z)}\ge\underline{\lambda}$ and $\frac{w_z(t^*,0)}{w(t^*,0)}=\underline{\lambda}$. A direct calculation shows that $\frac{w_z(t,z)}{w(t,z)}$ solves

$$0=d_1(t)\zeta_{zz}+2\left[c+k(t)\right]\frac{w_z}{w}\zeta_z-\zeta_t\quad \text{for all } (t,z)\in\mathbb{R}\times\mathbb{R}.$$

It then follows from the strong maximum principle that $\frac{w_z(t,z)}{w(t,z)}\equiv\underline{\lambda}$. This further implies that $\partial_z(w(t,z)e^{-\underline{\lambda}z})=0$. Thus, $w(t,z)$ must be of the form $w(t,z)=e^{\underline{\lambda}z}\phi(t)$. Since w is strictly positive and is T-periodic in t, we have $\phi(t) > 0$ and $\phi(t+T) = \phi(t)$ for all $t \in \mathbb{R}$. Substituting $w(t,z) = e^{\underline{\lambda} z}\phi(t)$ into (3.7), we find that

$$0 = \left[d_1(t)\underline{\lambda}^2 + \left[c + k(t)\right]\underline{\lambda} + f_u(t,0,0)\right]\phi(t) - \frac{d\phi(t)}{dt}.$$

Hence $\underline{\lambda}$ has to be a real zero of $\Lambda_c(\lambda) := \overline{d_1(t)}\lambda^2 + \overline{c + k(t)}\lambda + \overline{f_u(t,0,0)}$. Similarly, we can show that $\bar{\lambda}$ is also a real zero of $\Lambda_c(\lambda)$.

Note that $\Lambda_c(\lambda)$ has no real zeros if $|c + \overline{k(t)}| < 2\sqrt{\overline{d_1(t)} \times \overline{f_u(t,0,0)}} := 2\sqrt{\kappa}$. This immediately implies that (2.4) has no solutions when $-2\sqrt{\kappa} - \overline{k(t)} < c < 2\sqrt{\kappa} - \overline{k(t)}$. On the other hand, $\Lambda_c(\lambda)$ has two real zeros with the same sign provided $|c + \overline{k(t)}| \geq 2\sqrt{\kappa}$ (counting the multiplicity). In particular, it has two positive zeros if $c \leq -2\sqrt{\kappa} - \overline{k(t)}$ and two negative zeros provided that $c \geq 2\sqrt{\kappa} - \overline{k(t)}$. Since $u(t,z) > 0$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$ and $\lim_{z\to-\infty} u(t,z) = 0$, we must have $0 < \underline{\lambda} \leq \bar{\lambda}$. As a result, (2.4) has solutions only if $c \leq c^* := -2\sqrt{\kappa} - \overline{k(t)}$. $\square$

Theorem 3.3. *Suppose that (H1)–(H4) and (H8) are satisfied. Let $(u,v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (2.4) with $c \leq c^* := -2\sqrt{\kappa} - \overline{k(t)}$. Then $(u_z, v_z) > (0,0)$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$.*

Proof. The proof will be divided into two steps.

Step 1. We first show that there exists $\bar{s}$ such that $(u(t,z+s), v(t,z+s)) \geq (u(t,z), v(t,z))$ for all $(t,z,s) \in \mathbb{R} \times \mathbb{R} \times [\bar{s},\infty)$, or equivalently, $(u(t,z), v(t,z)) \geq (u(t,z-s), v(t,z-s))$ for all $(t,z,s) \in \mathbb{R} \times \mathbb{R} \times [\bar{s},\infty)$.

To this end, we denote $\overline{g_v(t,0,0)}$ by μ^- and set

$$\eta^* := \frac{\sup\{\eta \mid |f_u(t,\cdot,\cdot) - f_u(t,1,1)| + |f_v(t,\cdot,\cdot) - f_v(t,1,1)| \leq \frac{\theta^+|\mu^+|}{2},\ \forall(t,\cdot,\cdot) \in \mathbb{R} \times [1-\eta, 1+\eta]^2\}}{\|\psi_1\| + \|\psi_2\|}$$

$$\eta_* := \frac{\sup\{\eta \mid |g_u(t,\cdot,\cdot) - g_u(t,1,1)| + |g_v(t,\cdot,\cdot) - g_v(t,1,1)| \leq \frac{\theta^+|\mu^+|}{2},\ \forall(t,\cdot,\cdot) \in \mathbb{R} \times [1-\eta, 1+\eta]^2\}}{\|\psi_1\| + \|\psi_2\|},$$

where $\eta \geq 0$ and $\theta^+ = \frac{\min\{\min_t \psi_1, \min_t \psi_1\}}{\|\psi_1\| + \|\psi_2\|}$.

Let

$$\eta^0 = \min\{\eta^*, \eta_*\} \min\Big\{\min_t \psi_1, \min_t \psi_2\Big\} \tag{3.8}$$

and

$$\hat{\varphi}(t) = \exp\left(\int_0^t \left[g_v(s,0,0) - \mu^-\right] ds\right), \quad \mu^- = \overline{g_v(t,0,0)}. \tag{3.9}$$

We also set

$$\eta_0 = \sup\left\{\eta \in \mathbb{R}^+ \;\middle|\; \left|g_v(t,\cdot,\cdot) - g_v(t,0,0)\right| \leq \theta^- \left|\mu^-\right|,\ \forall(t,\cdot,\cdot) \times \mathbb{R} \times [-\eta,\eta]^2\right\}, \tag{3.10}$$

where $\theta^- = \frac{\min_t \hat{\varphi}}{\|\hat{\varphi}\|}$.

In view of Proposition 3.2, we see that $u_z > 0$ for all $(t, z) \in \mathbb{R} \times (-\infty, \underline{z}]$ with some $\underline{z} \in \mathbb{R}$. In addition, as $\lim_{z\to-\infty}(u, v) = (0, 0)$ and $\lim_{z\to\infty}(u, v) = (1, 1)$ uniformly in $t \in \mathbb{R}$, there exists $M > 0$ such that

$$-M \le \underline{z}, \quad \left|u(t, z)\right| + \left|v(t, z)\right| \le \eta_0 \quad \text{for all } (t, z) \in \mathbb{R} \times (-\infty, -M] \tag{3.11}$$

and

$$\left|u(t, z) - 1\right| + \left|v(t, z) - 1\right| \le \eta^0 \quad \text{for all } (t, z) \in \mathbb{R} \times [M, \infty). \tag{3.12}$$

Since $\min\{\inf_{(t,z)\in\mathbb{R}\times[-M,M]} u(t, z), \inf_{(t,z)\in\mathbb{R}\times[-M,M]} v(t, z)\} > 0$, there exists $\bar{s} \in \mathbb{R}^+$ such that

$$\bigl(u(t, z), v(t, z)\bigr) \ge \bigl(u(t, z - s), v(t, z - s)\bigr) \quad \text{for all } (t, z, s) \in \mathbb{R} \times [-M, M] \times [\bar{s}, \infty). \tag{3.13}$$

We now proceed to show that

$$\bigl(u(t, z), v(t, z)\bigr) \ge \bigl(u(t, z - s), v(t, z - s)\bigr) \quad \text{for all } (t, z, s) \in \mathbb{R} \times \mathbb{R} \times [\bar{s}, \infty). \tag{3.14}$$

To achieve this goal, we first show that for each $s \ge \bar{s}$, $(u(0, z), v(0, z)) \ge (u(0, z - s), v(0, z - s))$ for all $z \in [M, \infty)$. Let $s \ge \bar{s}$ be fixed. Assume to the contrary that the above statement is not true, then there exists $z' > M$ such that either $u(0, z') < u(0, z' - s)$ or $v(0, z') < v(0, z' - s)$. Assume without loss of generality that

$$u\bigl(0, z'\bigr) - u\bigl(0, z' - s\bigr) = -\varepsilon \quad \text{for some } \varepsilon > 0. \tag{3.15}$$

As $(u(0, z-s), v(z-s)) < (1, 1)$ for all $z \in \mathbb{R}$, in light of (3.12), there exists $0 < \eta \le \min\{\eta^*, \eta_*\}$ such that

$$\bigl(u(0, z) + \eta\psi_1(0), v(0, z) + \eta\psi_2(0)\bigr) \ge \bigl(u(0, z - s), v(0, z - s)\bigr) \quad \text{for all } z \in [M, \infty).$$

Now write

$$\bigl(u^\eta(t, z), v^\eta(t, z)\bigr) = \bigl(u(t, z) + \eta\psi_1(t)e^{\frac{\mu^+ t}{2}}, v(t, z) + \eta\psi_2(t)e^{\frac{\mu^+ t}{2}}\bigr), \quad (t, z) \in \mathbb{R}^+ \times [M, \infty).$$

We then show that $(u^\eta(t, z), v^\eta(t, z))$ is a regular super-solution of (2.2) in $]0, \infty[\times]M, \infty[$, the argument employed here is similar to that given for Theorem 2.1 in Alikakos at el. [1]. In fact, for any $(t, z) \in]0, \infty[\times]M, \infty[$, we have

$$\begin{aligned} &f\bigl(t, u^\eta, v^\eta\bigr) + d_1(t)u^\eta_{zz} + \bigl[c + k(t)\bigr]u^\eta_z - u^\eta_t \\ &\quad = f\bigl(t, u^\eta, v^\eta\bigr) - f(t, u, v) - \eta e^{\frac{\mu^+ t}{2}}\biggl[f_u(t, 1, 1)\psi_1 + f_v(t, 1, 1)\psi_2 + \frac{\mu^+}{2}\psi_1\biggr] \end{aligned}$$

$$
\begin{aligned}
&= \eta e^{\frac{\mu^+}{2}t}\Bigg\{\int_0^1 \big[f_u\big(t, u + \tau\eta\psi_1(t)e^{\frac{\mu^+ t}{2}}, v + \tau\eta\psi_2(t)e^{\frac{\mu^+ t}{2}}\big)d\tau - f_u(t,1,1)\big]\psi_1 \\
&\quad + \int_0^1 \big[f_v\big(t, u + \tau\eta\psi_1(t)e^{\frac{\mu^+ t}{2}}, v + \tau\eta\psi_2(t)e^{\frac{\mu^+ t}{2}}\big)d\tau - f_v(t,1,1)\big]\psi_2 + \frac{\mu^+}{2}\psi_1\Bigg\} \\
&\le 0
\end{aligned}
$$

and

$$
\begin{aligned}
&g\big(t, u^\eta, v^\eta\big) + d_2(t)v^\eta_{zz} + \big[c + l(t)\big]v^\eta_z - v^\eta_t \\
&\quad = g\big(t, u^\eta, v^\eta\big) - g(t,u,v) - \eta e^{\frac{\mu^+ t}{2}}\left[g_u(t,1,1)\psi_1 + g_v(t,1,1)\psi_2 + \frac{\mu^+}{2}\psi_2\right] \\
&\quad \le 0.
\end{aligned}
$$

In terms of (3.13), $(u^\eta(t,M), v^\eta(t,M)) \ge (u(t,M-s), v(t,M-s))$ for all $t \ge 0$. Thus, Proposition A.3 (see Appendix A for its proof) implies that $(u^\eta(t,z), v^\eta(t,z)) \ge (u(t,z-s), v(t,z-s))$ for all $(t,z) \in \mathbb{R}^+ \times [M,\infty)$. Consequently, we have

$$
u\big(0,z'\big) - u\big(0,z'-s\big) = u\big(n'T,z'\big) - u\big(n'T,z'-s\big) \ge -\eta\psi_1\big(n'T\big)e^{\frac{\mu^+ n'T}{2}} \ge -\frac{\varepsilon}{2},
$$

where $n' \in \mathbb{N}$ and $n' \ge 2\ln\frac{\varepsilon}{2\psi(0)\eta}/\mu^+ T$, ε is given by (3.15). This obviously contradicts (3.15). The contradiction shows that $(u(0,z), v(0,z)) \ge (u(0,z-s), v(0,z-s))$ for all $z \in [M,\infty)$, and hence it follows from Proposition A.3 that $(u(t,z), v(t,z)) \ge (u(t,z-s), v(t,z-s))$ for all $(t,z,s) \in \mathbb{R}^+ \times [M,\infty) \times [\bar{s},\infty)$. Thanks to the periodicity of (u,v) with respect to t, we have $(u(t,z), v(t,z)) \ge (u(t,z-s), v(t,z-s))$ for all $(t,z,s) \in \mathbb{R} \times [M,\infty) \times [\bar{s},\infty)$. Furthermore, note that $u_z > 0$ for all $(t,z) \in \mathbb{R} \times]-\infty,-M[$. This together with (3.13) shows that $u(t,z) \ge u(t,z-s)$ for all $(t,z,s) \in \mathbb{R} \times \mathbb{R} \times [\bar{s},\infty)$. Thus, it remains to show that $v(t,z) \ge v(t,z-s)$ for all $(t,z,s) \in \mathbb{R} \times (-\infty,-M] \times [\bar{s},\infty)$. Again, let $s \ge \bar{s}$ be fixed. Due to (H2), we see that

$$
g\big(t, u(t,z-s), v(t,z)\big) + d_2(t)\frac{\partial^2 v(t,z)}{\partial z^2} + \big[c + l(t)\big]\frac{\partial v(t,z)}{\partial z} - \frac{\partial v(t,z)}{\partial t} \le 0.
$$

Now write $w^s(t,z) = v(t,z) - v(t,z-s)$. Then

$$
\left[\int_0^1 g_v\big(t, u(t,z-s), u + \tau w^s\big)d\tau\right]w^s + d_2(t)w^s_{zz} + \big[c + l(t)\big]w^s_z - w^s_t \le 0, \quad (t,z) \in \mathbb{R} \times \mathbb{R}. \tag{3.16}
$$

As $w^s(\cdot,z)$ is T-periodic, we only need to show that $w^s(t,z) \ge 0$ for all $(t,z) \in [0,2T] \times (-\infty,M]$. First, in terms of (3.10) and (3.11), for any $(t,z) \in \mathbb{R} \times (-\infty,-M]$, we observe that

$$\left[\int_0^1 g_v\big(t, u(t,z-s), u+\tau w^s\big)d\tau\right]\hat{\varphi} + d_2(t)\hat{\varphi}_{zz} + \big[c+l(t)\big]\hat{\varphi}_z - \hat{\varphi}_t$$

$$= \left[\int_0^1 g_v\big(t, u(t,z-s), u+\tau w^s\big)d\tau - g_v(t,0,0) + \mu^-\right]\hat{\varphi} \le 0.$$

Since w^s is bounded in $[0,2T]\times(-\infty,-M]$ and $\hat{\varphi}$ is strictly positive, there exists $\delta_1>0$ such that $w^s+\delta_1\hat{\varphi}\ge 0$ for all $(t,z)\in[0,2T]\times(-\infty,-M]$. Now define

$$\delta^* = \inf\big\{\delta\in\mathbb{R}^+ \bigm| w^s+\delta\hat{\varphi}\ge 0 \text{ for all } (t,z)\in[0,2T]\times(-\infty,-M]\big\}.$$

To complete Step 1, it suffices to prove that $\delta^*=0$. Assume that this is not true, that is, $\delta^*>0$. Then, in view of (3.13), we see that $w^s(t,-M)+\delta^*\hat{\varphi}(t)>0$ for all $t\in[0,2T]$. In addition, $\lim_{z\to-\infty}\{\inf_{t\in[0,2T]} w^s+\delta^*\hat{\varphi}\}\ge\delta^*\min_t\hat{\varphi}>0$, and for any $(t,z)\in\,]0,2T[\,\times(-\infty,M]$, there holds

$$\left[\int_0^1 g_v\big(t, u(t,z-s), u+\tau w^s\big)d\tau\right]\big(w^s+\delta^*\hat{\varphi}\big) + d_2(t)\big[w^s+\delta^*\hat{\varphi}\big]_{zz}$$
$$+\big[c+l(t)\big]\big[w^s+\delta^*\hat{\varphi}\big]_z - \big[w^s+\delta^*\hat{\varphi}\big]_t \le 0.$$

By continuity, we have $\inf_{(t,z)\in[0,2T]\times(-\infty,-M]} w^s+\delta^*\hat{\varphi}=0$. Thus, there exists $(t^*,z^*)\in\,]0,2T[\,\times\,]-\infty,-M[$ such that $(w^s+\delta^*\hat{\varphi})(t^*,z^*)=0$. However, the (strong) maximum principle implies that $w^s+\delta^*\hat{\varphi}\equiv 0$ for all $(t,z)\in[0,t^*]\times(-\infty,-M]$, which contradicts that $w^s(t,-M)+\delta^*\hat{\varphi}(t)>0$ for all $t\in[0,2T]$. Therefore, from this contradiction, we finally deduce that (3.14) holds.

Step 2. Now define

$$s^* = \inf\big\{s\in\mathbb{R} \bigm| \big(u(t,z),v(t,z)\big)\ge\big(u(t,z-\eta),v(t,z-\eta)\big) \text{ for all } (t,z,\eta)\in\mathbb{R}\times\mathbb{R}\times[s,\infty)\big\}.$$

Clearly, (3.14) implies that s^* is bounded from above. Furthermore, since $u_z>0$ for all $(t,z)\in\mathbb{R}\times\,]-\infty,M[$, where $M>0$ is given in Step 1, it is easy to see that $s^*\ge 0$. We next argue by contradiction that $s^*=0$. Assume that this is not true, then we show that there exists $(t',z')\in\mathbb{R}\times\mathbb{R}$ such that either $u(t',z')=u(t',z'-s^*)$, or $v(t',z')=v(t',z'-s^*)$. If not, namely, $(u(t,z),v(t,z))>(u(t,z-s^*),v(t,z-s^*))$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}$. Then, by following the same lines of Step 1 and using the definition of s^*, we can infer that

$$\big(u(t,z),v(t,z)\big)\ge\big(u\big(t,z-s^*+\delta'-\eta\big),v\big(t,z-s^*+\delta'-\eta\big)\big),\quad (t,z,\eta)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+.$$

This however contradicts the definition of s^*. Thus, there exists $(t',z')\in\mathbb{R}\times\mathbb{R}$ such that either $u(t',z')=u(t',z'-s^*)$ or $v(t',z')=v(t',z'-s^*)$. Therefore, the maximum principle yields that $v(t,z)\equiv v(t,z-s^*)$ and $u(t,z)\equiv u(t,z-s^*)$. This is clearly impossible since $u_z>0$ for all $(t,z)\in\mathbb{R}\times(-\infty,\underline{z})$, where $\underline{z}$ is given in Step 1. Hence this contradiction confirms that $s^*=0$. As a consequence, we have $(u(t,z),v(t,z))\ge(u(t,z-\eta),v(t,z-\eta))$ for all $(t,z,\eta)\in$

$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. In particular, in light of the maximum principle, it is clear that $(u(t,z), v(t,z)) > (u(t,z-\eta), v(t,z-\eta))$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$ provided that $\eta > 0$. The proof is completed. □

Lemma 3.4. *Suppose that (H1)–(H3) are satisfied. Let $(u,v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (2.4) with $c \le c^* := -2\sqrt{\kappa} - \overline{k(t)}$. Then there exists a positive constant M_c such that $u(t,z) \le M_c v(t,z)$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Since $\lim_{z\to\infty}(u(t,z), v(t,z)) = (1,1)$ uniformly in t, there exist positive constants C and M such that $u(t,z) \le Cv(t,z)$ whenever $z \ge M$. Since $v(t,z) > 0$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$, v is bounded from below by certain positive constants on any compact subsets of $\mathbb{R} \times \mathbb{R}$. In addition, recall that u and v are both periodic in t. Thus, to complete the proof, it suffices to show that u is bounded by a constant multiple of v whenever $(t,z) \in \mathbb{R} \times (-\infty, -M']$ for some positive constant M'. Assume to the contrary that this is not true. Then there must be a sequence $\{(t_n, z_n)\}$ such that

$$\lim_{n\to\infty} z_n \to -\infty, \qquad \lim_{n\to\infty} \frac{v(t_n,z_n)}{u(t_n,z_n)} = 0.$$

Now define

$$u^n(t,z) = \frac{u(t,z+z_n)}{u(t_n,z_n)}, \qquad v^n(t,z) = \frac{v(t,z+z_n)}{u(t_n,z_n)}.$$

Since both u and v are T-periodic in t, we once again assume that $t_n \in [0,T]$. Note that $v^n(t,z) = \frac{v(t,z+z_n)}{v(t_n,z_n)} \frac{v(t_n,z_n)}{u(t_n,z_n)}$. Given $M > 0$, it follows from (3.6) that u^n and v^n are uniformly bounded for all $(t,z) \in \mathbb{R} \times [-M, M]$. In particular, $\lim_{n\to\infty} v^n(t,z) = 0$ uniformly for all $(t,z) \in \mathbb{R} \times [-M, M]$. We have

$$u_t^n = d_1(t)u_{zz}^n + \big[c + k(t)\big]u_z^n + \frac{f(t, u(t,z+z_n), v(t,z+z_n))}{u(t,z+z_n)} u^n, \quad (t,z) \in \mathbb{R} \times \left]-M, M\right[$$

and

$$\begin{aligned} v_t^n &= d_2(t)v_{zz}^n + \big[c + l(t)\big]v_z^n + g_u(t,0,0)u^n + g_v(t,0,0)v^n \\ &\quad + \left[\frac{g(t, u(t,z+z_n), v(t,z+z_n))}{u(t_n,z_n)} - g_u(t,0,0)v^n - g_v(t,0,0)u^n\right], \\ &\quad (t,z) \in \mathbb{R} \times \left]-M, M\right[. \end{aligned}$$

Note that $|g(t,u,v) - g_u(t,0,0)u - g_v(t,0,0)v| \le C(|u|^2 + |u||v| + |v|^2)$ for some positive constant C independent of t and z. As $u^n(\cdot, z)$ and $v^n(\cdot, z)$ are T-periodic functions, the parabolic (interior) estimates imply that, up to an extraction of a subsequence of $\{(u^n, v^n)\}$, $\{(u^n, v^n)\}$ converges uniformly in $C_b^{1,2}(\mathbb{R} \times [-\frac{M}{2}, \frac{M}{2}])$ to a function denoted by (u^*, v^*) which solves

$$\begin{cases} f_u(t,0,0)u^* + \big[c + k(t)\big]\dfrac{\partial u^*}{\partial z} + d_1(t)\dfrac{\partial u^*}{\partial z} - \dfrac{\partial u^*}{\partial t} = 0, \\ g_u(t,0,0)u^* + g_v(t,0,0)v^* + \big[c + l(t)\big]\dfrac{\partial v^*}{\partial z} + d_2(t)\dfrac{\partial^2 v^*}{\partial z^2} - \dfrac{\partial v^*}{\partial t} = 0. \end{cases}$$

Note that $u^n \geq 0$. Let $t^* = \lim_{n\to\infty} t_n$. Then $u^*(t^*, 0) = 1$, hence the (strong) maximum principle implies that $u^*(t, z) > 0$ for all $(t, z) \in \mathbb{R} \times]-\frac{M}{2}, \frac{M}{2}[$. On the other hand, as $\lim_{n\to\infty} v^n(t, z) = 0$ uniformly in $\mathbb{R} \times [-M, M]$, $v^*(t, z) = 0$ for all $(t, z) \in \mathbb{R} \times [-M, M]$. Consequently, we find that

$$g_u(t, 0, 0)u^*(t, z) = 0, \quad (t, z) \in \mathbb{R} \times \left]-\frac{M}{2}, \frac{M}{2}\right[.$$

This is impossible since $g_u(t, 0, 0) \gneq 0$. Therefore, the desired conclusion follows. □

To establish the uniqueness of time periodic traveling wave solutions we now consider two cases, namely, $c < c^*$ and $c = c^*$.

Case I: $c < c^*$.

Lemma 3.5. *Suppose that (H1)–(H8) are satisfied. Assume that* $k(t) - l(t) \geq \frac{\sqrt{\kappa}\Theta(d_2(t)-d_1(t))}{\overline{d_1(t)}}$. *Let* $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ *and c solve (2.4) with* $c < c^* = -2\sqrt{\kappa} - \overline{k(t)}$. *Then*

$$\limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}} \frac{u(t, z)}{\varphi_1(t)e^{\lambda_c z}}\right\} < \infty, \quad \textit{and} \quad \liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}} \frac{u(t, z)}{\varphi_1(t)e^{\lambda_c z}}\right\} > 0. \tag{3.17}$$

Here

$$\varphi_1(t) = \exp\left(\int_0^t \left[d_1(\tau)\lambda_c^2 + c\lambda_c + k(\tau)\lambda_c + f_u(\tau, 0, 0)\right]d\tau\right)$$

is defined by (2.7) and $\kappa = \overline{d_1(t)}\,\overline{f_u(t, 0, 0)}$.

Proof. The proof will be broken into several steps for the sake of clarity.

Step 1. We show in this step that

$$\limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}} \frac{u(t, z)}{\varphi_1(t)e^{\lambda_c z}}\right\} < \infty. \tag{3.18}$$

Assume this is not true. Then there exists a sequence $\{(t_n, z_n)\}$ such that

$$z_n \to -\infty \quad \text{as } n \to \infty, \quad \text{and} \quad \lim_{n\to\infty} \frac{u(t_n, z_n)}{\varphi_1(t_n)e^{\lambda_c z_n}} = \infty. \tag{3.19}$$

Let $\widehat{z}_0 \in (-\infty, z_*]$ be fixed, where z_* is given by (2.15). Let $\widehat{n}_0$ and $\widehat{\delta}$ be fixed positive constants such that

$$\widehat{n}_0 = \frac{1}{e^{\epsilon\widehat{z}_0}\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}}, \qquad \widehat{\delta} \leq \min\left\{e^{-\lambda_c\widehat{z}_0}\min\left\{\frac{1}{\max_t \varphi_1}, \frac{1}{\max_t \varphi_2}\right\}, \widehat{n}_0^{\frac{1}{\gamma}}\right\}.$$

Now set

$$\underline{u}(t,z)=\widehat{\delta}\varphi_1(t)e^{\lambda_c z}\left(1-\widehat{n}_0\frac{\phi_1(t)}{\varphi_1(t)}e^{\epsilon z}\right).$$

As $\lim_{z\to-\infty}u(t,z)=0$ uniformly in t and $\underline{u}(t,z)>0$ for all $(t,z)\in\mathbb{R}\times(-\infty,z']$ with some $z'\le\widehat{z}_0$, there exist $(t_1,z_1)\in\mathbb{R}\times(-\infty,z']$ and $s_1\le 0$ such that $u(t_1,z_1+s_1)\le\underline{u}(t_1,z_1)$. Without loss of generality, we may assume that $s_1=0$. Now if (3.19) were true, then there would be an n^* such that

$$z_n<z_1 \quad\text{and}\quad u(t_n,z_n)\ge N_l\widehat{\delta}\frac{\max_t\varphi_1}{\min_t\varphi_1}\varphi_1(t_n)e^{\lambda_c z_n}\quad\text{whenever}\quad n\ge n^*,$$

where N_l is the positive constant specified by (3.6). It then follows from (3.6) that

$$u(t,z_{n^*})\ge\widehat{\delta}\varphi_1(t)e^{\lambda_c z_{n^*}}>\underline{u}(t,z_{n^*})\quad\text{for all } t\in\mathbb{R}.$$

Let $\underline{v}(t,z)=\varepsilon\widehat{\delta}\varphi_2(t)e^{\lambda_c z}-\widehat{n}_0\phi_2(t)e^{(\lambda_c+\epsilon)z}$, where $\varepsilon\in\,]0,1[$ is a constant sufficiently small such that $\frac{\min_t\varphi_1}{M_c}\ge\varepsilon\max_t\varphi_2$. Then it follows from Lemma 3.4 that

$$v(t,z_{n^*})\ge\frac{u(t,z_{n^*})}{M_c}\ge\varepsilon\widehat{\delta}\varphi_2(t)e^{\lambda_c z}>\underline{v}(t,z_{n^*})\quad\text{for all } t\in\mathbb{R}.$$

Note that $(\underline{u},\underline{v})<(1,1)$ for all $(t,z)\in\mathbb{R}\times(-\infty,\widehat{z}_0]$ and $(\underline{u}(t,\widehat{z}_0),\underline{v}(t,\widehat{z}_0))\le(0,0)$ for all $t\in\mathbb{R}$. Thanks to Proposition 2.8, $(\underline{u},\underline{v})$ is a (regular) sub-solution in $\mathbb{R}\times(-\infty,\widehat{z}_0]$. On the other hand, Theorem 3.3 shows that $(u_z,v_z)>(0,0)$, that is, $(u(t,z),v(t,z))\ge(u(t,z_{n^*}),v(t,z_{n^*}))$ for all $(t,z)\in\mathbb{R}\times[z_{n^*},\infty)$. Hence Lemma A.2 implies that $u(t,z)>\underline{u}(t,z)$ for all $(t,z)\in\mathbb{R}\times[z_{n^*},\widehat{z}_0]$. However, this contradicts the fact that $u(t_1,z_1)\le\underline{u}(t_1,z_1)$. Therefore (3.18) follows.

Step 2. We show in this step that

$$u(t,z)\le K_c e^{\lambda_c z},\qquad v(t,z)\le K_c e^{\lambda_c z},\quad (t,z)\in\mathbb{R}\times\mathbb{R}\tag{3.20}$$

for some positive constant K_c. Note that the first inequality is an immediate consequence of (3.18). To show the second inequality, recall $\mu^-=\overline{g_v(t,0,0)}$, we let $w=\frac{v}{\hat{\varphi}}$, where $\hat{\varphi}$ is given by (3.9). Then a direct computation yields that

$$\frac{[g(t,u,v)-g_u(t,0,0)u-g_v(t,0,0)v]}{\hat{\varphi}}+g_u(t,0,0)\frac{u}{\hat{\varphi}}$$
$$+\mu^-w+d_2(t)w_{zz}+\big[c+l(t)\big]w_z-w_t=0.$$

Since $u\to 0$ and $v\to 0$ uniformly in t as $z\to-\infty$, for each $\varepsilon\in(0,\frac{|\mu^-|}{2}]$, there exists $M_\varepsilon>0$ such that $|g(t,u,v)-g_u(t,0,0)u-g_v(t,0,0)v|\le\varepsilon(u+v)$ whenever $z\le -M_\varepsilon$. Let $\varepsilon\in(0,\frac{|\mu^-|}{2}]$ be fixed, then, when $(t,z)\in\mathbb{R}\times\,]-\infty,-M_\varepsilon[$, it is easy to see that $w=\frac{v}{\hat{\varphi}}$ satisfies

$$\big[\varepsilon+g_u(t,0,0)\big]\frac{u}{\hat{\varphi}}+\frac{\mu^-}{2}w+d_2(t)w_{zz}+\big[c+l(t)\big]w_z-w_t\ge 0,\quad (t,z)\in\mathbb{R}\times\,]-\infty,M_\varepsilon[.$$

Now let $\varphi_\varepsilon(t)$ be the unique positive periodic solution to

$$\frac{[\varepsilon + g_u(t,0,0)]}{\hat{\varphi}(t)} + \left[d_2(t)\lambda_c^2 + \bigl(c + l(t)\bigr)\lambda_c + \frac{\mu^-}{2}\right]\xi - \frac{d\xi}{dt} = 0.$$

Note that φ_ε exists and is unique and positive since

$$\overline{d_2(t)\lambda_c^2 + \bigl[c + l(t)\bigr]\lambda_c + \frac{\mu^-}{2}} \le \overline{d_1(t)\lambda_c^2 + \bigl[c + k(t)\bigr]\lambda_c + \frac{\mu^-}{2}} = \overline{\frac{\mu^-}{2} - f_u(t,0,0)} < 0.$$

We next let $\overline{w}(t,z) = C_\varepsilon \varphi_\varepsilon(t) e^{\lambda_c z}$, where $C_\varepsilon \ge K_c$. It is straightforward to verify that

$$\begin{aligned}&\bigl[\varepsilon + g_u(t,0,0)\bigr]\frac{u}{\hat{\varphi}} + \frac{\mu^-}{2}\overline{w} + d_2(t)\overline{w}_{zz} + \bigl[c + l(t)\bigr]\overline{w}_z - \overline{w}_t \\ &\quad \le \frac{e^{\lambda_c z}[\varepsilon + g_u(t,0,0)]}{\hat{\varphi}}(K_c - C_\varepsilon) \le 0.\end{aligned}$$

Since v is bounded, we may choose C_ε such that $\overline{w}(t,-M_\varepsilon) \ge v(t,-M_\varepsilon)$ for all $t \in \mathbb{R}$. In addition, we have

$$\begin{aligned}&\frac{\mu^-}{2}(w - \overline{w}) + d_2(t)(w - \overline{w})_{zz} + \bigl[c + l(t)\bigr](w - \overline{w})_z - (w - \overline{w})_t \ge 0, \\ &\quad (t,z) \in \mathbb{R} \times]-\infty, M_\varepsilon[.\end{aligned}$$

Since $\lim_{z\to-\infty}(w - \overline{w}) = 0$, $w(t,-M_\varepsilon) - \overline{w}(t,-M_\varepsilon) \le 0$, and $\frac{\mu^-}{2} < 0$, it follows from the maximum principle that $w - \overline{w} \le 0$ for all $(t,z) \in \mathbb{R} \times (-\infty, M_\varepsilon]$. As v is bounded, there exists $C'_\varepsilon > 0$ for which $v(t,z) \le C'_\varepsilon e^{\lambda_c z}$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}$. Without loss of generality, we assume that $C'_\varepsilon \le K_c$. This confirms the existence of K_c.

Step 3. We now proceed to prove by contradiction that

$$\liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}} \frac{u(t,z)}{\varphi_1(t)e^{\lambda_c z}}\right\} > 0. \tag{3.21}$$

Assume that (3.21) is not true, then there exists a sequence $\{(t_n, z_n)\}$ such that

$$\lim_{n\to\infty} z_n = -\infty, \qquad \lim_{n\to\infty} \frac{u(t_n, z_n)}{\varphi_1(t_n)e^{\lambda_c z_n}} = 0. \tag{3.22}$$

Once again, we assume without loss of generality that $t_n \in [0,T]$ because of the periodicity of u with respect to t. We next show that $\lim_{n\to\infty} \frac{v(t_n,z_n)}{\varphi_1(t_n)e^{\lambda_c z_n}} = 0$ provided that (3.22) is true.

Let

$$u_n(t,z) = \frac{u(t, z + z_n)}{\varphi_1(t)e^{\lambda_c(z+z_n)}}, \qquad v_n(t,z) = \frac{v(t, z + z_n)}{\varphi_1(t)e^{\lambda_c(z+z_n)}}.$$

Write

$$f^n(t,z)=\frac{f(t,u(t,z+z_n),v(t,z+z_n))}{u(t,z+z_n)},$$
$$g^n(t,z)=\frac{[g(t,u(t,z+z_n),v(t,z+z_n))-g_u(t,0,0)u(t,z+z_n)-g_v(t,0,0)v(t,z+z_n)]}{v(t,z+z_n)},$$
$$\omega(t)=g_v(t,0,0)-f_u(t,0,0)+\lambda_c\big[\lambda_c\big(d_2(t)-d_1(t)\big)+l(t)-k(t)\big].$$

Note that $\overline{\omega(t)}<0$. As shown in (3.20), u_n and v_n are uniformly bounded. Furthermore, a straightforward calculation gives that

$$\begin{cases}\big[f^n(t,z)-f_u(t,0,0)\big]u_n+d_1(t)\dfrac{\partial^2u_n}{\partial z^2}+\big[2\lambda_cd_1(t)+c+k(t)\big]\dfrac{\partial u_n}{\partial z}-\dfrac{\partial u_n}{\partial t}=0,\\ \big[g^n(t,z)+\omega(t)\big]v_n+g_u(t,0,0)u_n+d_2(t)\dfrac{\partial^2v^n}{\partial z^2}+\big[2\lambda_cd_2(t)+c+l(t)\big]\dfrac{\partial v_n}{\partial z}-\dfrac{\partial v_n}{\partial t}=0.\end{cases}\tag{3.23}$$

From the parabolic estimates, it follows that there exists a subsequence of $\{(u_n,v_n)\}$ (which we continue to denote by $\{(u_n,v_n)\}$ for convenience) such that $\{(u_n,v_n)\}$ converges in $C^{1,2}_{\text{loc}}(\mathbb{R}\times\mathbb{R})$ to a function denoted by $(u^\diamond,v^\diamond)\in C^{1,2}_b(\mathbb{R}\times\mathbb{R})$.

Since $|g(t,u,v)-g_u(t,0,0)u-g_v(t,0,0)v|\le C(|u|^2+|u||v|+|v|^2)$ for some positive constant C and $\lim_{n\to\infty}u(t,z+z_n)=0$ and $\lim_{n\to\infty}v(t,z+z_n)=0$ uniformly in any compact subset of $\mathbb{R}\times\mathbb{R}$, by taking the limit in (3.23), we find that

$$\begin{cases}d_1(t)\dfrac{\partial^2u^\diamond}{\partial z^2}+\big[2\lambda_cd_1(t)+c+k(t)\big]\dfrac{\partial u^\diamond}{\partial z}-\dfrac{\partial u^\diamond}{\partial t}=0,\\ g_u(t,0,0)u^\diamond+\omega(t)v^\diamond+d_2(t)\dfrac{\partial^2v^\diamond}{\partial z^2}+\big[2\lambda_cd_2(t)+c+l(t)\big]\dfrac{\partial v^\diamond}{\partial z}-\dfrac{\partial v^\diamond}{\partial t}=0.\end{cases}$$

Observe that $u_n\ge 0$ and $v_n\ge 0$, and hence $u^\diamond\ge 0$ and $v^\diamond\ge 0$. Moreover, let $t^*=\lim_{n\to\infty}t_n$, then we have $u^\diamond(t^*,0)=\lim_{n\to\infty}u_n(t_n,0)=0$. It follows from the maximum principle that $u^\diamond(t,z)\equiv 0$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}$. Thus, $v^\diamond$ is a bounded periodic solution to

$$\xi_t=d_2(t)\xi_{zz}+\big[2\lambda_cd_2(t)+c+l(t)\big]\xi_z+\omega(t)\xi,\quad z\in\mathbb{R}.\tag{3.24}$$

Let $G^\diamond(t,s)_{t\ge s}$ be the family of evolution operators associated with (3.24) on $BUC(\mathbb{R})$. As $\omega^-:=\overline{\omega(t)}<0$, it is easy to see $\pm Ke^{\frac{\omega^-t}{2}}e^{\int_0^t[\omega(s)-\omega^-]ds}$ are the sub- and super-solutions of (3.24), respectively, where $K>0$ is arbitrary. Thus, the comparison principle implies that $\|G^\diamond(t,s)\|\le Ce^{\frac{\omega^-(t-s)}{2}}$, $t\ge s$, for some positive constant C. Namely $G^\diamond(t,s)$ enjoys a (trivial) exponential dichotomy, in view of Exercise 4* in Henry [18], we must have $v^\diamond(t,z)\equiv 0$. Thus, (3.22) implies that

$$\lim_{n\to\infty}\frac{v(t_n,z_n)}{\varphi_1(t_n)e^{\lambda_cz_n}}=0.$$

For the sake of contradiction, we let

$$w(t,z)=\frac{N_l \max_t \varphi_1(t)}{\min_t \varphi_1(t)}\varphi_1(t)e^{\lambda_c z}, \qquad \big(\bar u(t,z),\bar v(t,z)\big)=\min\big\{\big(w(t,z),w(t,z)\big),(1,1)\big\},$$

where $l>0$ is a fixed constant and N_l is given by (3.6). In view of the proof of Theorem 2.9, $(w(t,z),w(t,z))$ is a regular super-solution of (2.2). Thus $(\bar u,\bar v)$ is an irregular super-solution of (2.2) in $\mathbb{R}\times\mathbb{R}$. In particular, it is nondecreasing in z. Obviously, there exists $\bar\sigma$ such that $(\bar u(t,z),\bar v(t,z))=(1,1)$ for all $(t,z)\in\mathbb{R}\times[\bar\sigma,\infty)$. As $(\bar u(t,z),\bar v(t,z))\to(0,0)$ uniformly in t as $z\to-\infty$ and $(u(t,z),v(t,z))\to(1,1)$ uniformly in t as $z\to\infty$, there exist (t',z') and $s'>0$ such that $(\bar u(t',z'),\bar v(t',z))\le(u(t',z'+s'),v(t',z'+s'))<(1,1)$. Clearly, $z'<\bar\sigma$. Assume without loss of generality that $s'=0$ (otherwise we may consider $(u^{s'}(t,z),v^{s'}(t,z))=(u(t,z+s'),v(t,z+s'))$, which is also a solution of (2.4)).

Now if (3.22) is true, we then have

$$\varepsilon_n:=\frac{u(t_n,z_n)}{\varphi_1(t_n)e^{\lambda_c z_n}}\to 0, \qquad \epsilon_n:=\frac{v(t_n,z_n)}{\varphi_1(t_n)e^{\lambda_c z_n}}\to 0 \quad \text{as } n\to\infty.$$

Furthermore, it follows from (3.6) that

$$\big(u(t,z_n),v(t,z_n)\big)\le N_l\frac{\max_t \varphi_1(t)}{\min_t \varphi_1(t)}[\varepsilon_n+\epsilon_n]\varphi_1(t)\big(e^{\lambda_c z_n},e^{\lambda_c z_n}\big) \quad \text{for all } t\in\mathbb{R}.$$

Let n' be sufficiently large such that $z_{n'}<z'$ and

$$N_l\frac{\max_t \varphi_1(t)}{\min_t \varphi_1(t)}[\varepsilon_n+\epsilon_n]\varphi_1(t)e^{\lambda_c z_{n'}}<N_l\frac{\max_t \varphi_1(t)}{\min_t \varphi_1(t)}\varphi_1(t)e^{\lambda_c z_{n'}}\le\frac12 \quad \text{for all } t\in\mathbb{R}.$$

Namely

$$\big(u(t,z_{n'}),v(t,z_{n'})\big)<\big(\bar u(t,z_{n'}),\bar v(t,z_{n'})\big)\le\left(\frac12,\frac12\right) \quad \text{for all } t\in\mathbb{R}.$$

Therefore, Lemma A.1 implies that

$$\big(u(t,z),v(t,z)\big)<\big(\bar u(t,z),\bar v(t,z)\big)$$

for all $(t,z)\in\mathbb{R}\times[z_{n'},\infty)$. However this contradicts the fact that $(u(t',z'),v(t',z'))\ge(\bar u(t',z'),\bar v(t',z'))$. Hence, we deduce from this contradiction that (3.21) is true. The proof is completed. $\square$

We now state and prove a result on the exponential decay rate of the time periodic traveling wave solutions when $c<c^*$.

Theorem 3.6. *Suppose all the assumptions given in Lemma 3.5 are satisfied. Let $(u,v)\in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ and c solve (2.4) with $c<c^*=-2\sqrt{\kappa}-\overline{k(t)}$. Then*

$$\lim_{z\to-\infty}\frac{u(t,z)}{\rho e^{\lambda_c z}\varphi_1(t)}=1, \qquad \lim_{z\to-\infty}\frac{v(t,z)}{\rho e^{\lambda_c z}\varphi_2(t)}=1, \quad \textit{uniformly in } t\in\mathbb{R}$$

for some positive constant ρ. *Here* φ_1 *and* φ_2 *are given by* (2.7) *and* (2.10), *respectively, and* $\lambda_c = \frac{-c-\sqrt{c^2-4\kappa}}{2}$.

Proof. By virtue of Lemma 3.5, we have

$$0 < \rho_* := \liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}} \frac{u(t,z)}{e^{\lambda_c z}\varphi_1(t)}\right\} \le \limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}} \frac{u(t,z)}{e^{\lambda_c z}\varphi_1(t)}\right\} = \rho^* < \infty.$$

We next show that

$$\rho_* = \lim_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}} \frac{u(t,z)}{e^{\lambda_c z}\varphi_1(t)}\right\}. \tag{3.25}$$

This idea is motivated by Hamel [16]. Given $\varepsilon > 0$, we claim that there exists $z_\varepsilon \in \mathbb{R}$ such that

$$\left\{\inf_{t\in\mathbb{R}} \frac{u(t,z)}{e^{\lambda_c z}\varphi_1(t)}\right\} < \rho_*(1+2\varepsilon) \quad \text{whenever} \quad z \le z_\varepsilon.$$

Assume to the contrary that this is not true, then there exists a sequence $\{z_n\}$ such that

$$z_n \to -\infty, \quad \inf_{t\in\mathbb{R}}\left\{\frac{u(t,z_n)}{\varphi_1(t)e^{\lambda_c z_n}}\right\} \ge \rho_*(1+2\varepsilon).$$

Let $\widetilde{z}_0 \le z_*$ be chosen so that

$$\rho_*(1+2\varepsilon) \le \min\left\{e^{-\lambda_c\widetilde{z}_0}\min\left\{\frac{1}{\max_t \varphi_1}, \frac{1}{\max_t \varphi_2}\right\}, \widetilde{n}_0^{\frac{1}{\gamma}}\right\},$$
$$\widetilde{n}_0 = \frac{1}{e^{\epsilon\widetilde{z}_0}\min\{\min_t \frac{\phi_1}{\varphi_1}, \min_t \frac{\phi_2}{\varphi_2}\}},$$

where z_* is specified by (2.15), ϕ_1 and ϕ_2 are given by (2.13) and (2.14), respectively.

Now define

$$\big(\underline{u}(t,z), \underline{v}(t,z)\big) = e^{\lambda_c z}\left(\delta_1^*\varphi_1\left[1 - \frac{\widetilde{n}_0\phi_1}{\varphi_1}e^{\epsilon z}\right], \delta_2^*\varphi_2\left[1 - \frac{\delta_1^*\widetilde{n}_0\phi_2}{\delta_2^*\varphi_1}e^{\epsilon z}\right]\right),$$
$$(t,z) \in \mathbb{R}\times(-\infty, \widetilde{z}_0],$$

where $\delta_1^* = \rho_*(1+\frac{3}{2}\varepsilon)$ and $\delta_2^* = \min\{1, M_c\rho(1+\frac{3}{2}\varepsilon)\}$, and M_c is specified by Lemma 3.4. Therefore, we have

$$\lim_{n\to\infty}\frac{u(t,z_n)}{\underline{u}(t,z_n)} > 1, \qquad \lim_{n\to\infty}\frac{v(t,z_n)}{\underline{v}(t,z_n)} > 1 \quad \text{for all } t\in\mathbb{R}. \tag{3.26}$$

On the other hand, by the definition of ρ_*, there exists a sequence $\{(\tau_n, s_n)\}_{n\in\mathbb{N}}$ such that

$$s_n \to -\infty, \quad \frac{u(\tau_n, s_n)}{e^{\lambda_c z_n}\varphi_1(\tau_n)} = \rho_* \quad \text{as } n\to\infty.$$

Therefore, there exists n^* such that $s_{n^*} < \widetilde{z}_0$ and

$$u(\tau_{n^*}, s_{n^*}) \leq \rho\left(1 + \frac{\varepsilon}{2}\right)\varphi_1(\tau_{n^*})e^{\lambda_c s_{n^*}} \leq \underline{u}(\tau_{n^*}, s_{n^*}). \tag{3.27}$$

In view of (3.26), there exists n' such that $z_{n'} < s_{n^*}$ and

$$\bigl(\underline{u}(t, z_{n'}), \underline{v}(t, z_{n'})\bigr) < \bigl(u(t, z_{n'}), v(t, z_{n'})\bigr) \quad \text{for all } t \in \mathbb{R}.$$

Observe that $(\underline{u}, \underline{v})$ is a (regular) sub-solution of (2.2) in $\mathbb{R} \times (-\infty, \widetilde{z}_0[$ with $(\underline{u}(t, \widetilde{z}_0), \underline{v}(t, \widetilde{z}_0)) \leq (0, 0)$ for all $t \in \mathbb{R}$ and $\sup_{(t,z)\in\mathbb{R}\times(-\infty,\widetilde{z}_0]}(\underline{u}, \underline{v}) < (1, 1)$, hence Lemma A.2 implies that $(\underline{u}(t, z), \underline{v}(t, z)) < (u(t, z), v(t, z))$ for all $(t, z) \in \mathbb{R} \times [z_{n'}, \widetilde{z}_0]$. However, this contradicts (3.27) since $(\tau_{n^*}, s_{n^*}) \in \mathbb{R} \times [z_{n'}, \widetilde{z}_0]$. As $\varepsilon > 0$ is arbitrary, we readily conclude that (3.25) is true.

Now let $\{(t'_n, z'_n)\} \in \mathbb{R} \times \mathbb{R}$ be the sequence such that

$$z'_n \to -\infty, \qquad \lim_{z'_n \to -\infty} \frac{u(t'_n, z'_n)}{e^{\lambda_c z'_n}\varphi_1(t'_n)} = \rho^*, \quad \text{as } n \to -\infty.$$

Since both u and ϕ are T-periodic in t, we may assume that $t_n \in [0, T]$ for all n. Set

$$u^n(t, z) = \frac{u(t, z + z'_n)}{e^{\lambda_c(z+z'_n)}\varphi_1(t)}.$$

Clearly, $\{u^n\}$ is uniformly bounded for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. In particular, u^n satisfies

$$\frac{f(t, u, v)}{u}u^n - f_u(t, 0, 0)u^n + d_1(t)u^n_{zz} + \bigl[2\lambda_c + c + k(t)\bigr]u^n_z - u^n_t = 0.$$

With the same reasoning as that presented in the proof of Lemma 3.5, we may assume, by taking a subsequence if necessary, that $\{u^n\}$ converges in $C^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R})$ to a function $u^\star \geq 0$ that solves

$$d_1(t)u^\star_{zz} + \bigl[2\lambda_c + c + k(t)\bigr]u^\star_z - u^\star_t = 0. \tag{3.28}$$

Thanks to the compactness of $[0, T]$, $t'_n \to t^\star$ for some $t^\star \in [0, T]$. Clearly, $u^\star(t^\star, 0) = \rho^*$. Moreover, owing to the definition of ρ^*, it is easy to see that $u^\star \leq \rho^*$. Note that $u^\star(\cdot, z)$ is also T-periodic. Therefore, the strong maximum principle implies that $u^\star \equiv \rho^*$. Consequently,

$$\lim_{z'_n \to -\infty} \frac{u(t, z'_n)}{e^{\lambda_c z'_n}\varphi_1(t)} = \rho^* \quad \text{uniformly in } t \in [0, T].$$

As u is periodic in t, it is readily seen that $\lim_{z'_n \to -\infty}\{\inf_{t\in\mathbb{R}} \frac{u(t, z'_n)}{e^{\lambda_c z'_n}\varphi_1(t)}\} = \rho^*$. It then follows from (3.25) that $\rho_* = \rho^*$. More precisely, we have

$$\lim_{z \to -\infty} \frac{u(t, z)}{\rho e^{\lambda_c z}\phi(t)} = 1 \quad \text{uniformly in } t \in \mathbb{R} \tag{3.29}$$

for some positive constant ρ.

We now proceed to prove the claimed asymptotic behavior for v. Set

$$\varsigma(t,z)=u(t,z)-\rho\varphi_1(t)e^{\lambda_c z},\qquad r(t,z)=v(t,z)-\rho\varphi_2(t)e^{\lambda_c z}.$$

Then it is easy to see that r and ς are both T-periodic in t and satisfy

$$R(t,z)+g_u(t,0,0)\varsigma+g_v(t,0,0)r+d_2(t)r_{zz}+\big[c+l(t)\big]r_z-r_t=0,\quad (t,z)\in\mathbb{R}\times\mathbb{R}$$

where $R(t,z)=g(t,u(t,z),v(t,z))-g_u(t,0,0)u(t,z)-g_v(t,0,0)v(t,z)$. As shown in Lemma 3.5 (see Step 2), there exists some positive constant C' such that $|\varsigma|\le C'e^{\lambda_c z}$ and $|r|\le C'e^{\lambda_c z}$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}$.

Now let

$$m_*=\liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}}\frac{r(t,z)}{\varphi_2(t)e^{\lambda_c z}}\right\},\qquad m^*=\limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}}\frac{r(t,z)}{\varphi_2(t)e^{\lambda_c z}}\right\}.$$

To complete the proof, it suffices to show that $m_*=m^*=0$. We only show that $m_*=0$ since the arguments for the other are similar. Let $\{(t_n,z_n)\}$ be the sequence such that $z_n\to-\infty$ as $n\to\infty$ and $\lim_{n\to\infty}\frac{r(t_n,z_n)}{\varphi_2(t_n)e^{\lambda_c z_n}}=m_*$. Once again, we assume without loss of generality that $t_n\in[0,T]$ and $\lim_{n\to\infty}t_n=t^*$ for some $t^*\in[0,T]$. Set

$$\varsigma^n(t,z)=\frac{\varsigma(t,z+z_n)}{\varphi_1(t)e^{\lambda_c(z+z_n)}},\qquad r^n(t,z)=\frac{r(t,z+z_n)}{\varphi_1(t)e^{\lambda_c(z+z_n)}}.$$

Note that $\{\varsigma^n\}$ and $\{r^n\}$ are uniformly bounded. In particular, (3.29) implies that $\lim_{n\to\infty}\varsigma^n(t,z)=0$ uniformly in any compact subsets of $\mathbb{R}\times\mathbb{R}$. Moreover, the same calculation as that given for Lemma 3.5 yields that

$$\frac{R(t,z+z_n)}{\varphi_1(t)e^{\lambda_c(z+z_n)}}+g_u(t,0,0)\varsigma^n+\omega(t)r^n+d_2(t)r^n_{zz}+\big[2\lambda_c d_2(t)+c+l(t)\big]r^n_z-r^n_t=0.$$

By following the same reasoning as the above, we see that $\{r^n\}$ converges uniformly to zero in any compact subset of $\mathbb{R}\times\mathbb{R}$. Thus,

$$0=\lim_{n\to\infty}r^n(t_n,0)=\lim_{n\to\infty}\frac{r(t_n,z_n)}{\varphi_1(t_n)e^{\lambda_c z_n}}=m_*\frac{\varphi_2(t^*)}{\varphi_1(t^*)}.$$

As $\frac{\varphi_2(t^*)}{\varphi_1(t^*)}>0$, we must have $m_*=0$. Likewise, we can deduce that $m^*=0$. Therefore, it follows that

$$\lim_{z\to-\infty}\frac{v(t,z)}{\rho\varphi_2(t)e^{\lambda_c z}}=1\quad\text{uniformly in } t\in\mathbb{R}.$$

This completes the proof. $\square$

Case II: $c=c^*$.

We now turn to the case $c=c^*$. In this case, note that $\lambda_{c^*}=\frac{\sqrt{\kappa}}{d_1(t)}$, where λ_c $(c\le c^*)$ is given by (2.6). Here and subsequently, we will write $\lambda_*=\lambda_{c^*}$. In order to derive an a priori estimate similar to (3.17), with slightly abuse of symbols, we introduce the following notations:

$$\varphi_1(t) = \exp\left(\int_0^t \left[d_1(\tau)\lambda_*^2 + c\lambda_* + k(\tau)\lambda_* + f_u(\tau, 0, 0)\right]d\tau\right) \tag{3.30}$$

and

$$\begin{cases} \varphi_2(t) = \varphi_2(0)\exp\left(\int_0^t \varrho_*(s)ds\right) + \int_0^t \exp\left(\int_s^t \varrho_*(\tau)d\tau\right) g_u(s, 0, 0)\varphi_1(s)ds, \\ \varphi_2(0) = \left[1 - \exp\left(\int_0^T \varrho_*(s)ds\right)\right]^{-1} \int_0^T \exp\left(\int_s^T \varrho_*(\tau)d\tau\right) g_u(s, 0, 0)\varphi_1(s)ds, \end{cases} \tag{3.31}$$

where $\varrho_*(t) = d_2(t)\lambda_*^2 + (c^* + l(t))\lambda_* + g_v(t, 0, 0)$.

Let $\epsilon^* > 0$ be fixed such that

$$0 < \epsilon^* \le \min\left\{\gamma\lambda_*, \frac{\overline{\overline{f_u(t,0,0) - g_v(t,0,0)}\,\overline{d_1(t)}}}{2\sqrt{\kappa}[\overline{d_1(t) + d_1(t)}]}\right\} \tag{3.32}$$

and set

$$\Lambda^{\epsilon^*} = \Lambda_{c^*}\left(\lambda_* + \epsilon^*\right) = \overline{d_1(t)\left(\lambda_* + \epsilon^*\right)^2 + \left[c^* + k(t)\right]\left(\lambda_* + \epsilon^*\right) + f_u(t, 0, 0)}. \tag{3.33}$$

Clearly, $\Lambda^{\epsilon^*} > 0$. Accordingly, we set

$$\phi_1(t) = \exp\left(\int_0^t \left[d_1(\tau)\lambda_*^2 + c\lambda_* + k(\tau)\lambda_* + f_u(\tau, 0, 0) - \Lambda^{\epsilon^*}\right]d\tau\right) \tag{3.34}$$

and

$$\begin{cases} \phi_2(t) = \phi_2(0)\exp\left(\int_0^t \varrho_{\epsilon^*}(s)ds\right) + \int_0^t \exp\left(\int_s^t \varrho_{\epsilon^*}(\tau)d\tau\right) g_u(s, 0, 0)\phi_1(s)ds, \\ \phi_2(0) = \left[1 - \exp\left(\int_0^T \varrho_{\epsilon^*}(s)ds\right)\right]^{-1} \int_0^T \exp\left(\int_s^T \varrho_{\epsilon^*}(\tau)d\tau\right) g_u(s, 0, 0)\phi_1(s)ds, \end{cases} \tag{3.35}$$

where $\varrho_{\epsilon^*}(t) = d_2(t)(\lambda_* + \epsilon^*)^2 + [c + l(t)](\lambda_* + \epsilon^*) - \Lambda^{\epsilon^*} + g_v(t, 0, 0)$. Note that

$$\begin{aligned} \overline{\varrho_{\epsilon^*}(t)} &= \overline{d_2(t)\left(\lambda_* + \epsilon^*\right)^2 + \left[c^* + l(t)\right]\left(\lambda_* + \epsilon^*\right) - \Lambda^{\epsilon^*} + g_v(t, 0, 0)} \\ &= \overline{\lambda_*\left[l(t) - k(t) + \lambda_*\left(d_2(t) - d_1(t)\right)\right] + \epsilon^*\left[l(t) - k(t) + \epsilon^*\left(d_2(t) - d_1(t)\right)\right]} \\ &\quad + \overline{2\epsilon^*\lambda_*\left[d_2(t) - d_1(t)\right] - f_u(t, 0, 0) + g_v(t, 0, 0)} < 0. \end{aligned}$$

Thus, $\phi_2(t)$ is well defined. It is easy to see that $\phi_2(t)$ is positive and periodic and solves

$$g_u(t,0,0)\phi_2 + \left\{g_v(t,0,0) + d_2(t)\left(\lambda_* + \epsilon^*\right)^2 + \left[c + l(t)\right](\lambda_* + \epsilon) - \Lambda^{\epsilon^*}\right\}\xi - \frac{d\xi}{dt} = 0.$$

Observe that $\Lambda_{c^*}(\lambda_*) = \frac{d\Lambda_{c^*}}{d\lambda}|_{\lambda=\lambda_*} = 0$. Set $\psi_1^*(t) = -\frac{\partial \Phi^\lambda(t)}{\partial\lambda}|_{\lambda=\lambda_*}$. By differentiating both sides of (2.5) with respect to λ at λ_*, we find that

$$\begin{aligned}\left[c^* + k(t) + 2\lambda_* d_1(t)\right]\varphi_1(t) = {} & \left\{f_u(t,0,0) + \left[c^* + k(t)\right]\lambda_c + d_1(t)\lambda_*^2\right\}\psi_1^*(t)\\ & - \frac{d\psi_1^*(t)}{dt}.\end{aligned} \tag{3.36}$$

Let $\psi_2^*(t)$ be the periodic solution of

$$\begin{aligned}& g_u(t,0,0)\psi_1^* - \left[c^* + l(t) + 2\lambda_* d_2(t)\right]\varphi_2\\ &\quad + \left[g_v(t,0,0) + \left(c^* + l(t)\right)\lambda_* + d_2(t)\lambda_*^2\right]\varsigma - \frac{d\varsigma}{dt} = 0.\end{aligned} \tag{3.37}$$

As shown before, $\overline{g_v(t,0,0) + (c^* + l(t))\lambda_* + d_2(t)\lambda_*^2} < 0$, thus, ψ_2^* exists and is unique. We also let $\check{z} \geq 0$ be the least number such that

$$\ln z \leq (1+\gamma)^{-1}\ln\left(\frac{\Lambda^{\epsilon^*}\min\{\min_{t\in\mathbb{R}}\phi_1, \min_{t\in\mathbb{R}}\phi_2\}}{\varpi 2\cdot 6^{1+\gamma}}\right) + \frac{(\gamma\lambda_* - \epsilon^*)z}{1+\gamma} \quad \text{for all } z \geq \check{z}. \tag{3.38}$$

Set

$$z^* = \min\left\{-\check{z}, -1, -\frac{1}{\lambda_*}, -\max_{t\in\mathbb{R}}\left|\frac{\psi_1^*(t)}{\varphi_1(t)}\right|, -\max_{t\in\mathbb{R}}\left|\frac{\psi_2^*(t)}{\varphi_2(t)}\right|\right\}, \tag{3.39}$$

$$n_0 = \frac{e^{-\epsilon^* z_0}}{\max_{t\in\mathbb{R}}|\frac{\phi_1(t)}{\varphi_1(t)}| + \max_{t\in\mathbb{R}}|\frac{\phi_2(t)}{\varphi_2(t)}|}, \qquad M_0 = 3|z_0|, \quad -\infty < z_0 \leq z^*, \tag{3.40}$$

$$0 < \delta_2 \leq \delta_1 \leq \min\left\{\frac{e^{-\lambda_* z_0}}{6|z_0|[\sup_{t\in\mathbb{R}}(\varphi_1 + \varphi_2)]}, n_0^{\frac{1}{\gamma}}\right\}. \tag{3.41}$$

Proposition 3.7. *Suppose that (H1)–(H5) are satisfied. Assume that* $k(t) - l(t) \geq \frac{\sqrt{\kappa}\Theta(d_2(t) - d_1(t))}{d_1(t)}$. *Let*

$$\begin{aligned}&\left(\underline{U}(t,z), \underline{W}(t,z)\right)\\ &= \left(\delta_1 e^{\lambda_* z}\varphi_1\left[|z| + \frac{\psi_1^*}{\varphi_1} - M_0 + \frac{n_0\phi_1 e^{\epsilon^* z}}{\varphi_1}\right], \delta_2 e^{\lambda_* z}\varphi_2\left[|z| + \frac{\psi_2^*}{\varphi_2} - \frac{\delta_1 M_0}{\delta_2} + \frac{\delta_1 n_0 \phi_2 e^{\epsilon^* z}}{\delta_2\varphi_2}\right]\right)\end{aligned}$$

provided that $c = c^*$. *Then* $(\underline{U}, \underline{W})$ *is a regular sub-solution of (2.2) in* $\mathbb{R} \times (-\infty, z_0[$, *where* z_0, n_0, M_0, δ_1, *and* δ_2 *are given in (3.40) and (3.41), respectively.*

Proof. First observe that $e^{\lambda_* z}|z|$ is nondecreasing in $(-\infty, z_0]$ if $z_0 \le z^*$. Thus, in view of (3.40) and (3.41), we see that $(\underline{U}, \underline{W}) < (1, 1)$ for all $(t, z) \in \mathbb{R} \times (-\infty, z_0]$. In particular, it is easy to see that $(\underline{U}(t, z_0), \underline{W}(t, z_0)) \le (0, 0)$ for all $t \in \mathbb{R}$. Moreover, for any $(t, z) \in \mathbb{R} \times]-\infty, z_0[$, a direct computation shows that

$$\begin{aligned}
&f(t, \underline{U}, \underline{W}) + \big[c^* + k(t)\big]\underline{U}_z + d_1(t)\underline{U}_{zz} - \underline{U}_t \\
&\quad \ge \delta_1 e^{\lambda_* z} f_u(t, 0, 0)\big[-z\varphi_1 + \psi_1^* - M_0\varphi_1\big] \\
&\qquad + \delta_1 e^{\lambda_* z}\big[c + k(t)\big]\big[-\varphi_1 - z\lambda_*\varphi_1 + \lambda_*\psi_1^* - M_0\lambda_*\varphi_1\big] \\
&\qquad + d_1(t)\delta_1 e^{\lambda_* z}\big[-2\lambda_*\varphi_1 - \lambda_*^2 z\varphi_1 + \lambda_*^2\psi_1^* - M_0\lambda_*^2\varphi_1\big] - \delta_1 e^{\lambda_* z}\big[-z\varphi_1' + \psi_1^{*\prime} - M_0\varphi_1'\big] \\
&\qquad + \delta_1 n_0 e^{(\lambda_* + \epsilon^*)z}\big\{\big[f_u(t, 0, 0) + \big(c^* + k(t)\big)\big(\lambda_* + \epsilon^*\big) + d_1(t)\big(\lambda_* + \epsilon^*\big)^2\big]\phi_1 - \phi_1'\big\} \\
&\qquad - \varpi\big(|\underline{U}|^{1+\gamma} + |\underline{W}|^{1+\gamma}\big) \\
&\quad \ge -(M_0 + z)\delta_1 e^{\lambda_* z}\big\{\big[f_u(t, 0, 0) + \big(c^* + k(t)\big)\lambda_* + d_1(t)\lambda_*^2\big]\varphi_1 - \varphi_1'\big\} \\
&\qquad + \delta_1 e^{\lambda_* z}\big\{-\big[c^* + k(t) + 2d_1(t)\lambda_*\big]\varphi_1 + \big[f_u(t, 0, 0) + \big(c^* + k(t)\big)\lambda_* + d_1(t)\lambda_*^2\big]\psi_1^* - \psi_1^{*\prime}\big\} \\
&\qquad + e^{\lambda_* z}\left\{e^{\epsilon^* z}\delta_1 n_0 \Lambda^{\epsilon^*}\phi_1 - \varpi\delta_1^{1+\gamma} e^{\gamma\lambda_* z}\varphi_1\big(6|z|\big)^{1+\gamma} - \varpi\delta_2^{1+\gamma} e^{\gamma\lambda_* z}\varphi_2\left(6\frac{\delta_1}{\delta_2}|z|\right)^{1+\gamma}\right\} \\
&\quad \ge \delta_1 n_0 e^{(\lambda_* + \epsilon^*)z}\left\{\Lambda^{\epsilon^*}\phi_1 - \varpi\frac{\delta_1^\gamma}{n_0} 2\cdot 6^{(1+\gamma)}|z|^{1+\gamma} e^{(\gamma\lambda_* - \epsilon^*)z}\right\} \ge 0
\end{aligned}$$

and

$$\begin{aligned}
&g(t, \underline{U}, \underline{W}) + \big[c^* + l(t)\big]\underline{W}_z + d_2(t)\underline{W}_{zz} - \underline{W}_t \\
&\quad \ge \delta_2 e^{\lambda_* z}\big\{g_u(t, 0, 0)\big[-z\varphi_1 + \psi_1^*\big] + g_v(t, 0, 0)\big[-z\varphi_2 + \psi_2^*\big]\big\} \\
&\qquad + \delta_1 n_0 e^{(\lambda_* + \epsilon^*)z}\big\{g_u(t, 0, 0)\phi_1 + \big[g_v(t, 0, 0) + c^* + l(t) + d_2(t)\lambda_*^2\big]\phi_2 - \phi_2'\big\} \\
&\qquad - M_0\delta_1 e^{\lambda_* z}\big\{g_u(t, 0, 0)\varphi_1 + \big[g_v(t, 0, 0) + c^* + l(t) + d_2(t)\lambda_*^2\big]\varphi_2 - \varphi_2'\big\} \\
&\qquad + \delta_2 e^{\lambda_* z}\big[c^* + l(t)\big]\big[-\varphi_2 - z\lambda_*\varphi_2 + \lambda_*\psi_2^*\big] + \delta_2 e^{\lambda_* z} d_2(t)\big[-2\lambda_*\varphi_2 - z\lambda_*^2\varphi_2 + \lambda_*^2\psi_2^*\big] \\
&\qquad - \delta_2 e^{\lambda_* z}\big[-z\varphi_2' + \psi_2^{*\prime}\big] - \varpi\big(|\underline{U}|^{1+\gamma} + |\underline{W}|^{1+\gamma}\big) \\
&\quad \ge -(M_0\delta_1 + z\delta_2)e^{\lambda_* z}\big\{g_u(t, 0, 0)\varphi_1 + \big[g_v(t, 0, 0) + c^* + l(t) + d_2(t)\lambda_*^2\big]\varphi_2 - \varphi_2'\big\} \\
&\qquad + \delta_2 e^{\lambda_* z}\big\{g_u(t, 0, 0)\psi_1^* - \big[c^* + l(t) + 2\lambda_* d_2(t)\big]\varphi_2 \\
&\qquad + \big[g_v(t, 0, 0) + \big(c^* + l(t)\big)\lambda_* + d_2(t)\lambda_*^2\big]\psi_2^* - \psi_2^{*\prime}\big\} \\
&\qquad + e^{\lambda_* z}\left\{e^{\epsilon^* z}\delta_1 n_0 \Lambda^{\epsilon^*}\phi_2 - \varpi\delta_1^{1+\gamma} e^{\gamma\lambda_* z}\varphi_1\big(6|z|\big)^{1+\gamma} - \varpi\delta_2^{1+\gamma} e^{\gamma\lambda_* z}\varphi_2\left(6\frac{\delta_1}{\delta_2}|z|\right)^{1+\gamma}\right\} \\
&\quad \ge \delta_1 n_0 e^{(\lambda_* + \epsilon^*)z}\left\{\Lambda^{\epsilon^*}\phi_2 - \varpi\frac{\delta_1^\gamma}{n_0} 2\cdot 6^{(1+\gamma)}|z|^{1+\gamma} e^{(\gamma\lambda_* - \epsilon^*)z}\right\} \ge 0.
\end{aligned}$$

Thus, $(\underline{U}, \underline{W})$ is a regular sub-solution of (2.2) in $\mathbb{R} \times]-\infty, z_0[$. The proof is completed. □

Proposition 3.8. *Suppose that (H1)–(H4) and (H6) are satisfied. Assume that* $k(t) - l(t) \geq \frac{\sqrt{\kappa}\Theta(d_2(t)-d_1(t))}{d_1(t)}$. *Let*

$$\big(\overline{U}(t,z), \overline{W}(t,z)\big) = \left(me^{\lambda_* z}\varphi_1(t)\left[n + \frac{\psi_1^*(t)}{\varphi_1(t)} - z\right], me^{\lambda_* z}\varphi_1(t)\left[n + \frac{\psi_1^*(t)}{\varphi_1(t)} - z\right]\right)$$

provided that $c = c^*$, *where* $m, n \in \mathbb{R}^+$. *Then* $(\overline{U}, \overline{W})$ *is a regular super-solution of (2.2) in* $\mathbb{R} \times]-\infty, z^0[$, *where* $z^0 \leq n - \frac{2}{\lambda_*} - \max_{t\in\mathbb{R}}|\frac{\psi_1^*}{\varphi_1}|$, *and* ψ_1^* *is given by (3.36).*

Proof. For the sake of simplicity, we write $w(t,z) = me^{\lambda_* z}\varphi_1(t)[n + \frac{\psi_1^*(t)}{\varphi_1(t)} - z]$. Notice that $w > 0$, $w_z \geq 0$, and $w_{zz} \geq 0$ in $(-\infty, z^0]$. Moreover, when $(t,z) \in \mathbb{R} \times]-\infty, z^0[$, we have

$$\begin{aligned}
&f(t,\overline{U},\overline{W}) + \big[c + k(t)\big]\overline{U}_z + d_1(t)\overline{U}_{zz} - \overline{U}_t \\
&\quad = f(t,w,w) + \big[c + k(t)\big]w_z + d_1(t)w_{zz} - w_t \\
&\quad \leq me^{\lambda_* z}\left\{f_u(t,0,0)\left[n + \frac{\psi_1^*}{\varphi_1} - z\right] + \varphi_1\big[c^* + k(t)\big]\left[\lambda_* n - 1 - z\lambda_* + \lambda_*\frac{\psi_1^*}{\varphi_1}\right]\right. \\
&\qquad \left. + d_1(t)\varphi_1\left[-2\lambda_* - z\lambda_*^2 + \lambda_*^2\frac{\psi_1^*}{\varphi_1} + n\lambda_*^2\right] - \varphi_1'\left[n - z + \frac{\psi_1^{*\prime}}{\varphi_1'}\right]\right\} \\
&\quad = me^{\lambda_* z}(n-z)\big\{\big[f_u(t,0,0) + \big(c^* + k(t)\big)\lambda_* + d_1(t)\lambda_*^2\big]\varphi_1 - \varphi_1'\big\} \\
&\qquad + me^{\lambda_* z}\big\{-\big[c^* + k(t) + 2\lambda_*\big]\varphi_1 + \big[f_u(t,0,0) + \big(c^* + k(t)\big)\lambda_* + d_1(t)\lambda_*^2\big]\psi_1^* - \psi_1^{*\prime}\big\} \\
&\quad = 0
\end{aligned}$$

and

$$\begin{aligned}
&g(t,\overline{U},\overline{W}) + \big[c + l(t)\big]\overline{W}_z + d_2(t)\overline{W}_{zz} - \overline{W}_t \\
&\quad \leq g_s^*(t,0,0)w + \big[c + l(t)\big]w_z + d_2(t)w_{zz} - w_t \\
&\quad \leq f_u(t,0,0)w + \big[c + k(t)\big]w_z + d_1(t)w_{zz} - w_t + \big[l(t) - k(t)\big]w_z + \big[d_2(t) - d_1(t)\big]w_{zz} \\
&\quad \leq m\lambda_* e^{\lambda_* z}\varphi_1\left[n - \frac{2}{\lambda_*} - z + \frac{\psi_1^*}{\varphi_1}\right]\big[l(t) - k(t) + \lambda_*\big(d_2(t) - d_1(t)\big)\big] \\
&\quad \leq 0.
\end{aligned}$$

Hence, $(\overline{U}, \overline{W})$ is a regular super-solution of (2.2) in $\mathbb{R} \times]-\infty, z^0[$. The proof is completed. □

Lemma 3.9. *Suppose that (H1)–(H8) are satisfied. Let* $(u,v) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ *and* c *solve (2.4) with* $c = c^* = -2\sqrt{\kappa} - \overline{k(t)}$. *Then*

$$\limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}}\frac{u(t,z)}{\varphi_1(t)|z|e^{\lambda_* z}}\right\} < \infty \quad \text{and} \quad \liminf_{z\to-\infty}\left\{\inf_{t\in\mathbb{R}}\frac{u(t,z)}{\varphi_1(t)|z|e^{\lambda_* z}}\right\} > 0. \tag{3.42}$$

Proof. Thanks to Proposition 3.7, by following the same reasoning as that in Lemma 3.5 (see Step 1 of the proof for Lemma 3.5), we can infer that

$$\limsup_{z\to-\infty}\left\{\sup_{t\in\mathbb{R}}\frac{u(t,z)}{\varphi_1(t)|z|e^{\lambda_* z}}\right\}<\infty. \tag{3.43}$$

We next show that there exist positive constants K^* and M^* for which

$$u(t,z)\le K^*|z|e^{\lambda_* z},\qquad v(t,z)\le K^*|z|e^{\lambda_* z},\quad \text{for all } (t,z)\in\mathbb{R}\times\left(-\infty,-M^*\right]. \tag{3.44}$$

Notice that the first inequality is an immediate consequence of (3.43). To show the second inequality, we let $\mu^-=\overline{g_v(t,0,0)}$ and $w=\frac{v}{\hat\varphi}$ again, where $\hat\varphi$ is given by (3.9). Then by following the same lines, we arrive

$$\left[\varepsilon+g_u(t,0,0)\right]\frac{u}{\hat\varphi}+\frac{\mu^-}{2}w+d_2(t)w_{zz}+\left[c+l(t)\right]w_z-w_t\ge 0,\quad \forall(t,z)\in\mathbb{R}\times\left]-\infty,-M^+\right[$$

for some positive constant M^+. We will assume without loss of generality that $M^+\ge M^*$. Let $m^*\ge\max_{t\in[0,T]}\frac{|c^*+l(t)+2\lambda_* d_2(t)|\varphi_2(t)\hat\varphi(t)}{[\varepsilon+g_u(t,0,0)]\varphi_1(t)}$, and let $\varphi_\varepsilon(t)$ be the periodic solution to

$$\begin{aligned}&\left[\varepsilon+g_u(t,0,0)\right]\varphi_1-\frac{1}{m^*}\left[c^*+l(t)+2\lambda_* d_2(t)\right]\varphi_2\\&\quad+\left[\frac{\mu^-}{2}+\lambda_*\left(c^*+l(t)\right)+\lambda_*^2 d_2(t)\right]\xi-\frac{d\xi}{dt}=0.\end{aligned}$$

Since $\overline{\frac{\mu^-}{2}+\lambda_*(c^*+l(t))+\lambda_*^2 d_2(t)}<0$ and $\frac{[\varepsilon+g_u(t,0,0)]}{\hat\varphi}\varphi_1-\frac{1}{m^*}[c^*+l(t)+2\lambda_* d_2(t)]\varphi_2\ge 0$, $\varphi_\varepsilon(t)$ is unique and nonnegative.

Now set $\overline{w}(t,z)=C_\varepsilon(-ze^{\lambda_* z}\varphi_2+m^*e^{\lambda_* z}\varphi_\varepsilon)$, where $C_\varepsilon\ge\frac{K^*}{\min_t\varphi_1}$. Observe that $\overline{w}$ satisfies the equation

$$\frac{[\varepsilon+g_u(t,0,0)]}{\hat\varphi}\left[C_\varepsilon\left(-ze^{\lambda_* z}\varphi_1+m^*e^{\lambda_* z}\varphi_1\right)\right]+\frac{\mu^-}{2}w+\left[c^*+l(t)\right]w_z+d_2(t)w_{zz}-w_t=0.$$

As a consequence, there holds

$$\left[\varepsilon+g_u(t,0,0)\right]\frac{u}{\hat\varphi}+\frac{\mu^-}{2}\overline{w}+d_2(t)\overline{w}_{zz}+\left[c+l(t)\right]\overline{w}_z-\overline{w}_t\le 0,\quad \forall(t,z)\in\mathbb{R}\times\left]-\infty,M^*\right[.$$

Since $C_\varepsilon\ge\frac{K^*}{\min_t\varphi_1^*}$ is arbitrary, by the same arguments as those given for Lemma 3.5, we can deduce that there exists $K^*>0$ for which $v(t,z)\le K^*|z|e^{\lambda_* z}$ for all $(t,z)\in\mathbb{R}\times(-\infty,M^*]$.

The proof for the second inequality of (3.42) is analogous to that presented in the proof of Lemma 3.5, we choose to skip it and leave the detailed calculations to interested readers. The proof is completed. □

The exponential decay rate of the time periodic traveling wave solutions when $c=c^*$ is given in the following result.

Theorem 3.10. *Suppose that (H1)–(H8) are satisfied. Let $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (2.4) with $c = c^* = -2\sqrt{\kappa} - \overline{k(t)}$. Then*

$$\lim_{z \to -\infty} \frac{u(t,z)}{\rho \varphi_1(t)|z|e^{\lambda_* z}} = 1, \qquad \lim_{z \to -\infty} \frac{v(t,z)}{\rho \varphi_2(t)|z|e^{\lambda_* z}} = 1 \quad \text{uniformly in } t \in \mathbb{R}$$

for some positive constant ρ, where φ_1 and φ_2 are given by (3.30) and (3.31), respectively.

Proof. In light of Lemma 3.9, there exist positive constants ρ_* and ρ^* such that

$$0 < \rho_* = \liminf_{z \to -\infty} \left\{ \inf_{t \in \mathbb{R}} \frac{u(t,z)}{\varphi_1(t)|z|e^{\lambda_* z}} \right\} \le \limsup_{z \to -\infty} \left\{ \sup_{t \in \mathbb{R}} \frac{u(t,z)}{\varphi_1(t)|z|e^{\lambda_* z}} \right\} = \rho^* < \infty.$$

We next show that for a given $\varepsilon \in (0, 1]$, there exists $z_\varepsilon \in \mathbb{R}$ such that

$$\inf_{t \in \mathbb{R}} \frac{u(t,z)}{\varphi_1(t)|z|e^{\lambda_* z}} \le \rho_*(1 + 2\varepsilon) \quad \text{whenever} \quad z \le z_\varepsilon. \tag{3.45}$$

Assume to the contrary that this is not true. Then, for given $\varepsilon \in (0, 1]$, there exists a sequence $\{z_n\}$ such that

$$z_n \to -\infty, \quad \inf_{t \in \mathbb{R}} \frac{u(t,z_n)}{\varphi_1(t)|z_n|e^{\lambda_* z_n}} \ge \rho_*(1 + 2\varepsilon). \tag{3.46}$$

Pick $\delta > 0$ sufficiently small such that $\frac{\min_t \varphi_1}{M_{c^*}} \ge \delta \max_t \varphi_2$, where M_{c^*} is the positive number specified in Lemma 3.4 with $c = c^*$. Let $z_0 \in (-\infty, z^*]$ be chosen so that

$$\rho\left(\frac{1+3\varepsilon}{2}\right) \le \min\left\{ \frac{e^{-\lambda_* z_0}}{6|z_0|[\sup_{t\in\mathbb{R}}(\varphi_1 + \varphi_2)]}, \frac{e^{-\epsilon^* z_0}}{\max_{t\in\mathbb{R}} |\frac{\phi_1(t)}{\varphi_1(t)}| + \max_{t\in\mathbb{R}} |\frac{\phi_2(t)}{\varphi_2(t)}|} \right\}$$

where z^* is given by (3.39) and ϵ^* is given by (3.32). Define

$$\begin{aligned}
&\big(\underline{u}(t,z), \underline{v}(t,z)\big) \\
&\quad = \rho_*\left(1 + \frac{3\varepsilon}{2}\right)e^{\lambda_* z}\left(\varphi_1\left[|z| + \frac{\psi_1^*}{\varphi_1} - M_0 + \frac{n_0\phi_1 e^{\epsilon^* z}}{\varphi_1}\right], \delta\varphi_2\left[|z| + \frac{\psi_2^*}{\varphi_2} - \frac{M_0}{\delta} + \frac{n_0\phi_2 e^{\epsilon^* z}}{\delta\varphi_2}\right]\right),
\end{aligned}$$

where n_0 and M_0 are specified by (3.40). In view of (3.46) and Lemma 3.4, we have

$$\lim_{n\to\infty} \frac{u(t,z_n)}{\underline{u}(t,z_n)} > 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{v(t,z_n)}{\underline{v}(t,z_n)} > 1 \quad \text{uniformly in } t.$$

Hence, there exists n^* such that $(u(t,z_n), v(t,z_n)) > (\underline{u}(t,z_n), v(t,z_n))$ for all $t \in \mathbb{R}$ and $z_n < z_0$ provided that $n \ge n^*$.

On the other hand, by the definition of ρ_*, there exists a sequence $\{(t_n, s_n)\}$ such that $\lim_{n\to\infty} \frac{u(t_n,s_n)}{\varphi_1(t_n)|s_n|e^{\lambda_* s_n}} = \rho_*$. Note that

$$\lim_{n\to\infty} \frac{\underline{u}(t_n,s_n)}{\rho_*(1+\frac{\varepsilon}{2})\varphi_1(t_n)|s_n|e^{\lambda_* s_n}} > 1.$$

Thus, there exists n_* such that

$$u(t_n, s_n) \leq \rho_* \left(1 + \frac{\varepsilon}{2}\right) \varphi_1(t_n)|s_n| e^{\lambda_* s_n} \leq \underline{u}(t_n, s_n) \quad \text{and} \quad s_n < z_0 \quad \text{whenever} \quad n \geq n_*.$$

Without loss of generality, we assume that $z_{n^*+1} < s_{n_*+1}$. By virtue of Proposition 3.7, $(\underline{u}, \underline{v})$ is a sub-solution of (2.2) for $(t, z) \in \mathbb{R} \times]-\infty, z_0[$, and $(\underline{u}(t, z_0), \underline{v}(t, z_0)) \leq (0, 0)$ for all $t \in \mathbb{R}$. It then follows from Lemma A.1 in Appendix A that $(u(t, z), v(t, z)) > (\underline{u}(t, z), \underline{v}(t, z))$ for all $(t, z) \in \mathbb{R} \times [z_{n^*+1}, z_0]$. Consequently,

$$u(t_{n_*+1}, s_{n_*+1}) \leq \underline{u}(t_{n_*+1}, s_{n_*+1}) < u(t_{n_*+1}, s_{n_*+1}).$$

This is a contradiction and it shows that (3.45) holds. As $\varepsilon > 0$ is arbitrary, we readily conclude that

$$\lim_{z \to -\infty} \left\{ \inf_{t \in \mathbb{R}} \frac{u(t, z)}{\varphi_1(t)|z| e^{\lambda_* z}} \right\} = \rho_*. \tag{3.47}$$

Again by following the same lines as those given in Theorem 3.6, we deduce that $\rho_* = \rho^*$. Namely,

$$\lim_{z \to -\infty} \frac{u(t, z)}{\rho \varphi_1(t)|z| e^{\lambda_* z}} = 1 \quad \text{uniformly in } t \in \mathbb{R}$$

for some positive constant ρ.

Now let

$$r(t, z) = v(t, z) - \rho \varphi_2(t) e^{\lambda_* z} \left[|z| + \frac{\psi_2^*(t)}{\varphi_2(t)} \right], \qquad \varsigma(t, z) = u(t, z) - \rho \varphi_1(t) e^{\lambda_* z} \left[|z| + \frac{\psi_1^*(t)}{\varphi_1(t)} \right].$$

Then by using the same arguments as those given in the proof of Theorem 3.6, we infer that

$$\liminf_{z \to -\infty} \left\{ \inf_{t \in \mathbb{R}} \frac{r(t, z)}{\varphi_2(t)|z| e^{\lambda_* z}} \right\} = \limsup_{z \to -\infty} \left\{ \sup_{t \in \mathbb{R}} \frac{r(t, z)}{\varphi_2(t)|z| e^{\lambda_* z}} \right\} = 0.$$

Therefore, it follows that

$$\lim_{z \to -\infty} \frac{v(t, z)}{\rho \varphi_2(t)|z| e^{\lambda_* z}} = 1 \quad \text{uniformly in } t \in \mathbb{R}.$$

The proof is completed. $\square$

Corollary 3.11. *Suppose that (H1)–(H8) are satisfied. Assume that* $k(t) - l(t) \geq \frac{\sqrt{\kappa}\Theta(d_2(t) - d_1(t))}{\overline{d_1(t)}}$. *Let* $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ *and* c *solve* (2.4) *with* $c \leq c^* := -2\sqrt{\kappa} - \overline{k(t)}$. *Then there exists some positive constant* ρ *such that*

$$\lim_{z \to -\infty} \frac{u_z(t, z)}{\rho \varphi_1(t) e^{\lambda_c z}} = \lambda_c, \qquad \lim_{z \to -\infty} \frac{v_z(t, z)}{\rho \varphi_2(t) e^{\lambda_c z}} = \lambda_c \quad \textit{uniformly in } t \in \mathbb{R}, \textit{ if } c < c^*, \tag{3.48}$$

where φ_1 and φ_2 are given by (2.7) and (2.10), respectively, and

$$\lim_{z\to-\infty}\frac{u_z(t,z)}{\rho\varphi_1(t)|z|e^{\lambda_* z}}=\lambda_*,$$
$$\lim_{z\to-\infty}\frac{v_z(t,z)}{\rho\varphi_2(t)|z|e^{\lambda_* z}}=\lambda_*\quad \textit{uniformly in } t\in\mathbb{R},\ \textit{if } c=c^* \tag{3.49}$$

where φ_1 and φ_2 are given by (3.30) and (3.31), respectively.

Proof. We will only give a proof for (3.49) since (3.48) can be proved similarly. Using (3.5) and parabolic estimates, we see that

$$\left[\int_T^{2T}\int_{-\frac{l}{8}}^{\frac{l}{8}}\left(\left|u_\tau(\tau,s+z)\right|^p+\left|u_s(\tau,s+z)\right|^p+\left|u_{ss}(\tau,s+z)\right|^p\right)ds\,d\tau\right]^{\frac{1}{p}}$$
$$\le C\sup_{[0,2T]\times[z-\frac{l}{4},z+\frac{l}{4}]}\left(|u|+|v|\right),$$
$$\left[\int_T^{2T}\int_{-\frac{l}{8}}^{\frac{l}{8}}\left(\left|v_\tau(\tau,s+z)\right|^p+\left|v_s(\tau,s+z)\right|^p+\left|v_{ss}(\tau,s+z)\right|^p\right)ds\,d\tau\right]^{\frac{1}{p}}$$
$$\le C\sup_{[0,2T]\times[z-\frac{l}{4},z+\frac{l}{4}]}\left(|u|+|v|\right)$$

for some positive constant C, where $l>0$ is some constant and $p\ge 3$.

Therefore, Sobolev embedding theorem implies that

$$\frac{|u_z(t,z)-u_z(\tau,z)|}{|t-\tau|^\alpha}\le C'\left|u(t,z)\right|,\qquad \frac{|v_z(t,z)-v_z(\tau,z)|}{|t-\tau|^\alpha}\le C'\left|v(t,z)\right|,\quad \forall t,\tau,z\in\mathbb{R},\ t\ne\tau$$

for some positive constant C'. This yields that, whenever $z\le -M^*$, there holds

$$\frac{|u_z(t,z)-u_z(\tau,z)|}{|t-\tau|^\alpha|z|e^{\lambda_* z}}+\frac{|v_z(t,z)-v_z(\tau,z)|}{|t-\tau|^\alpha|z|e^{\lambda_* z}}\le K\quad \text{for all } t,\tau\in\mathbb{R} \text{ with } t\ne\tau. \tag{3.50}$$

On the other hand, for each fixed $t\in\mathbb{R}$, l'Hôpital's rule gives that

$$\lim_{z\to-\infty}\frac{u_z(t,z)}{\lambda_*\rho\varphi_1(t)|z|e^{\lambda_* z}}=1,\qquad \lim_{z\to-\infty}\frac{v_z(t,z)}{\lambda_*\rho\varphi_2(t)|z|e^{\lambda_* z}}=1. \tag{3.51}$$

Since both u_z and v_z are periodic with t and $[0,T]$ is compact, (3.49) follows from (3.50) and (3.51). The proof is completed. □

Summarizing the above results on the exponential decay rate, we finally can state and prove the uniqueness of the time periodic traveling wave solutions for system (1.1).

Theorem 3.12. *Assume that (H1)–(H8) are satisfied. Suppose that* $\mathbf{w}_i(t,x) = (u_i(t, x\cdot\nu - ct), v_i(t, x\cdot\nu - ct)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ *with* $c \le c^* := -2\sqrt{\kappa} - \overline{k(t)}$ *are two time periodic traveling waves of (1.1)* $(i = 1, 2)$*. Then there exists* $s_0 \in \mathbb{R}$ *such that* $(u_2(t,z), v_2(t,z)) = (u_1(t, z+s_0), v_1(t, z+s_0))$ *for all* $(t,z) \in \mathbb{R}\times\mathbb{R}$*, where* $z = x\cdot\nu - ct$*.*

Proof. The proof is similar to that of Theorem 3.11 in [37], we will give a sketch. Thanks to Theorems 3.6 and 3.10, there exist two positive constants ρ_1 and ρ_1 such that

$$\lim_{z\to-\infty} \frac{u_i(t,z)}{\rho_i\varphi_1(t)e^{\lambda_c z}} = 1, \qquad \lim_{z\to-\infty} \frac{v_i(t,z)}{\rho_i\varphi_2(t)e^{\lambda_c z}} = 1, \quad c < c^*\ (i = 1, 2)$$

and

$$\lim_{z\to-\infty} \frac{u_i(t,z)}{\rho_i\varphi_1(t)|z|e^{\lambda_* z}} = 1, \qquad \lim_{z\to-\infty} \frac{v_i(t,z)}{\rho_i\varphi_2(t)|z|e^{\lambda_* z}} = 1, \quad c = c^*\ (i = 1, 2)$$

From the same reasoning as shown in Theorem 3.11 of [37], it follows that there exists $\bar{s}$ such that $(u_1(t, z+s), v_1(t, z+s)) \ge (u_2(t,z), v_2(t,z))$ for all $(t,z) \in \mathbb{R}\times\mathbb{R}$ whenever $s \ge \bar{s}$. Now define

$$s^* = \inf\bigl\{s \in \mathbb{R} \bigm| \bigl(u_1(t, z+s), v_1(t, z+s)\bigr) \ge \bigl(u_2(t,z), v_2(t,z)\bigr),\ \forall (t,z) \in \mathbb{R}\times\mathbb{R}\bigr\}.$$

Clearly, s^* is bounded. In addition, with the same arguments as that given for Theorem 3.11 in [37] together with Proposition A.5 in Appendix A, we can show that $\rho_1 e^{\lambda_c s^*} = \rho_2$.

Next define

$$s_* = \sup\bigl\{s \in \mathbb{R} \bigm| \bigl(u_1(t, z+s), v_1(t, z+s)\bigr) \le \bigl(u_2(t,z), v_2(t,z)\bigr),\ \forall (t,z) \in \mathbb{R}\times\mathbb{R}\bigr\}.$$

Clearly, s_* is bounded. Indeed, note that

$$-s_* = \inf\bigl\{-s \in \mathbb{R} \bigm| \bigl(u_2(t, z-s), v_2(t, z-s)\bigr) \ge \bigl(u_1(t,z), v_1(t,z)\bigr),\ \forall (t,z) \in \mathbb{R}\times\mathbb{R}\bigr\}.$$

By following the same lines, we can conclude that $\rho_2 e^{-\lambda_c s_*} = \rho_1$. It immediately follows that $s^* = s_*$. Therefore, by the definitions of s^* and s_*, we have

$$\bigl(u_1\bigl(t, z+s^*\bigr), v_1\bigl(t, z+s^*\bigr)\bigr) = \bigl(u_2(t,z), v_2(t,z)\bigr)$$

for all $(t,z) \in \mathbb{R}\times\mathbb{R}$. This completes the proof. □

4. Asymptotic stability of time periodic traveling wave solutions

In this section, we concentrate on the asymptotic stability of time periodic traveling wave solutions discussed in the previous sections. We thereafter consider

$$\begin{cases} u_t = d_1(t)\Delta u + \mathbf{k}(t)\cdot\nabla u + f(t,u,v), \\ v_t = d_2(t)\Delta v + \mathbf{l}(t)\cdot\nabla v + g(t,u,v), \\ \bigl(u(0,x,u_0), v(0,x,v_0)\bigr) = \bigl(u_0(x), v_0(x)\bigr) \end{cases} \tag{4.1}$$

where $(u_0(x), v_0(x)) \in C^\theta(\mathbb{R}^n, \mathbb{R}^2)$, $0 < \theta \le 1$, and $(0,0) \lneqq (u_0, v_0) \lneqq (1,1)$. Throughout this section, the assumptions (H1)–(H8) given in Section 2 will remain true. We often denote by $(u(t,x,u_0), v(t,x,v_0))$ the solution of (4.1) with positive initial data $(u_0(x), v_0(x))$. Assume that $(0,0) \lneqq (u_0, v_0) \lneqq (1,1)$. Since $(1,1)$ is a solution of (4.1), it is easy to show that $(0,0) \lneqq (u(t,x,u_0), v(t,x,v_0)) \lneqq (1,1)$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. In what follows, given $\nu \in \mathbb{R}^n$ with $|\nu| = 1$, a time periodic traveling wave solution to (4.1) will be always denoted by $(U(t, x\cdot\nu - ct), W(t, x\cdot\nu - ct))$ or (U, W) in short. We shall always assume that (2.9) holds. We will use the same type of methods as those given in Zhao and Ruan [37] to establish the asymptotic stability of (U, W) (see also Hamel and Roques [17]).

As before, our main results will be completed through a series of lemmas and propositions. We will divide our discussion into two cases.

Case I: $c < c^*$.

In this case, we let $\chi(s)$ be a smooth function such that $\chi(s) = 1$ for $s \le \underline{s}$; $\chi(s) = 0$ for $s \ge \bar{s}$, and $0 \le \chi'(s)$ and $|\chi'| + |\chi''| \le 1$, where $\underline{s}$ and $\bar{s}$ are fixed constants with $\underline{s} < \bar{s}$.

Fix

$$\epsilon \in \left(0, \min\left\{\frac{\gamma\lambda_c}{2}, \frac{\sqrt{(c+\overline{k(t)})^2 - 4\kappa}}{2\overline{d_1(t)}}, \frac{\overline{d_1(t)}[\overline{f_u(t,0,0) - g_v(t,0,0)}]}{2\sqrt{\kappa}[\overline{d_1(t) + d_2(t)}]}\right\}\right] \tag{4.2}$$

such that

$$\beta := -\frac{\overline{(\lambda_c+\epsilon)^2 d_1(t) + (\lambda_c+\epsilon)(c+k(t)) + f_u(t,0,0)}}{2} \le \frac{|\mu^+|}{2}. \tag{4.3}$$

Set

$$\begin{cases} \xi_c(t,s) = \chi(s)e^{(\lambda_c+\epsilon)s}\phi_1(t) + \big(1-\chi(s)\big)\psi_1(t) \\ \varsigma_c(t,s) = \chi(s)e^{(\lambda_c+\epsilon)s}\phi_2(t) + \big(1-\chi(s)\big)\psi_2(t), \end{cases} \tag{4.4}$$

where ϕ_i $(i = 1,2)$ are given by (2.13) and (2.14), respectively.

We also set

$$\ell^+ := \min\left\{\min_{t\in[0,T]} \frac{1}{\psi_1(t)}, \min_{t\in[0,T]} \frac{1}{\psi_2(t)}\right\}. \tag{4.5}$$

Proposition 4.1. *Assume that (H1)–(H8) are satisfied. Let $(U(t, x\cdot\nu - ct), W(t, x\cdot\nu - ct))$ be a traveling wave of (4.1) with $c < c^*$ such that $(U(t,z), W(t,z))$ and c solve (2.4). Then*

$$\limsup_{s\to\infty} \sup_{(t,z)\in\mathbb{R}\times\mathbb{R},\ \ell\in(0,\ell^+]} \frac{U(t,z) - \ell\xi_c(t,z+s) - 1}{\ell\psi_1(t)} \le -1,$$

$$\limsup_{s\to\infty} \sup_{(t,z)\in\mathbb{R}\times\mathbb{R},\ \ell\in(0,\ell^+]} \frac{W(t,z) - \ell\varsigma_c(t,z+s) - 1}{\ell\psi_2(t)} \le -1.$$

Proof. The proof is similar to that of Proposition 4.2 in [37] and is omitted here. □

In the following, we fix $s_0 \in \mathbb{R}$ such that

$$\sup_{(t,s)\in\mathbb{R}\times\mathbb{R}} \frac{U(t,s)-\ell\xi_c(t,s+s_0)-1}{\psi_1(t)} \le -\frac{\ell}{2} \quad \text{for all } \ell \in \left(0,\ell^+\right] \tag{4.6}$$

and

$$\sup_{(t,s)\in\mathbb{R}\times\mathbb{R}} \frac{W(t,s)-\ell\varsigma_c(t,s+s_0)-1}{\psi_2(t)} \le -\frac{\ell}{2} \quad \text{for all } \ell \in \left(0,\ell^+\right]. \tag{4.7}$$

Lemma 4.2. *Suppose that (H1)–(H8) are satisfied. Let $(U(t,x\cdot\nu-ct), W(t,x\cdot\nu-ct))$ be a traveling wave of (4.1) with $c<c^*$ such that $(U(t,z),W(t,z))$ and c solve (2.4). Let ϵ, β, ℓ^+, and s_0 be given by (4.2), (4.3), (4.5), and (4.6), respectively. Then there exists $\delta_c \in (0,\ell^+]$ such that for each $z_0 \in \mathbb{R}$ and each $\sigma \ge 1/\beta$, $(u^\pm(t,x), v^\pm(t,x))$ are respectively the super- and sub-solutions of (4.1) in $\mathbb{R}^+\times\mathbb{R}^n$ whenever $\delta\in(0,\delta_c]$. Here*

$$\begin{aligned} u^\pm(t,x) &= U\left(t, x\cdot\nu-ct+z_0\pm\sigma\left(1-e^{-\beta t}\right)\right) \\ &\quad \pm\delta\xi_c\left(t,x\cdot\nu-ct+z_0+s_0\pm\sigma\left(1-e^{-\beta t}\right)\right)e^{-\beta t}, \\ v^\pm(t,x) &= W\left(t, x\cdot\nu-ct+z_0\pm\sigma\left(1-e^{-\beta t}\right)\right) \\ &\quad \pm\delta\varsigma_c\left(t,x\cdot\nu-ct+z_0+s_0\pm\sigma\left(1-e^{-\beta t}\right)\right)e^{-\beta t}. \end{aligned}$$

Proof. We will only show that u^+ is a super-solution of (4.1), the other cases can be proved similarly. Throughout the proof, we always let $z' = x\cdot\nu-ct+z_0+s_0+\sigma(1-e^{-\beta t})$ and $z = x\cdot\nu-ct+z_0+\sigma(1-e^{-\beta t})$. A straightforward calculation shows that

$$\begin{aligned} &f\left(t,u^+,v^+\right)+\mathbf{k}(t)\cdot\nabla u^+ + d_1(t)\Delta u^+ - u_t^+ \\ &\quad = f\left(t, U(t,z)+e^{-\beta t}\delta\xi_c\left(t,z'\right), W(t,z)+e^{-\beta t}\delta\varsigma_c\left(t,z'\right)\right) \\ &\qquad - f\left(t,U(t,z),W(t,z)\right)+\delta e^{-\beta t}\beta\xi_c\left(t,z'\right) \\ &\qquad + e^{-\beta t}\delta\left\{\frac{-\sigma\beta}{\delta}U_z+\chi e^{(\lambda_c+\epsilon)z'}\left[\left(d_1(t)(\lambda_c+\epsilon)^2+\left(c+k(t)\right)(\lambda_c+\epsilon)\right)\phi_1-\phi_1'\right]\right. \\ &\qquad \left. -(1-\chi)\psi_1'+r_1\left(t,z'\right)\right\} \\ &\quad = e^{-\beta t}\delta\left[f_1(t,z)\xi_c(t,z)+f_2(t,z)\varsigma_c\left(t,z'\right)+\beta\xi_c(t,z)\right] \\ &\qquad + e^{-\beta t}\delta\left\{\frac{-\sigma\beta}{\delta}U_z-\chi e^{(\lambda_c+\epsilon)z'}\left[f_u(t,0,0)\phi_1+f_v(t,0,0)\phi_2+2\beta\phi_1\right]\right. \\ &\qquad \left. -(1-\chi)\psi_1'+r_1\left(t,z'\right)\right\} \\ &\quad = e^{-\beta t}\delta\left\{\frac{-\sigma\beta}{\delta}U_z+e^{(\lambda_c+\epsilon)z'}\chi\left[\left(f_1-f_u(t,0,0)\right)\phi_1+\left(f_2-f_v(t,0,0)\right)\phi_2-\beta\phi_1\right]\right. \\ &\qquad \left. +r_1\left(t,z'\right)+(1-\chi)\left[\left(f_1-f_u(t,1,1)\right)\psi_1+\left(f_2-f_v(t,1,1)\right)\psi_2+\mu^+\psi_1+\beta\psi_1\right]\right\}, \end{aligned}$$

where

$$r_1(t,z') = e^{(\lambda_c+\epsilon)z'}\phi_1(t)\{k(t)\chi' + d_1(t)[\chi'' + 2\chi'(\lambda_c+\epsilon)] + \chi'[c - \sigma\beta e^{-\beta t}]\}$$
$$+ \psi_1(t)[\chi'(\sigma\beta e^{-\beta t} - c) - k(t)\chi' - d_1(t)\chi''] - e^{(\lambda_c+\epsilon)z'}\phi_1(t)(\lambda_c+\epsilon)\chi\sigma\beta e^{-\beta t},$$

$$f_1 = f_1(t,z,z') = \int_0^1 [f_u(t, U(t,z) + \tau\delta\xi_c(t,z')e^{-\beta t}, W(t,z) + \tau\delta\varsigma_c(t,z')e^{-\beta t})]d\tau,$$

$$f_2 = f_2(t,z,z') = \int_0^1 [f_v(t, U(t,z) + \tau\delta\xi_c(t,z')e^{-\beta t}, W(t,z) + \tau\delta\varsigma_c(t,z')e^{-\beta t})]d\tau.$$

Likewise, we have

$$\begin{aligned}
&g(t,u^+,v^+) + \mathbf{l}(t)\cdot\nabla v^+ + d_2(t)\Delta v^+ - v_t^+ \\
&\quad = g(t, U(t,z) + e^{-\beta t}\delta\xi_c(t,z'), W(t,z) + e^{-\beta t}\delta\varsigma_c(t,z')) \\
&\qquad - g(t, U(t,z), W(t,z)) + \delta e^{-\beta t}\beta\varsigma_c(t,z') + e^{-\beta t}\delta\biggl\{\frac{-\sigma\beta}{\delta}W_z \\
&\qquad + \chi e^{(\lambda_c+\epsilon)z'}[(d_2(t)(\lambda_c+\epsilon)^2 + (c + l(t))(\lambda_c+\epsilon))\phi_2 - \phi_2'] - (1-\chi)\psi_2' + r_2(t,z')\biggr\} \\
&\quad = e^{-\beta t}\delta[g_1(t,z)\xi_c(t,z) + g_2(t,z)\varsigma_c(t,z') + \beta\varsigma_c(t,z)] + e^{-\beta t}\delta\biggl\{\frac{-\sigma\beta}{\delta}W_z \\
&\qquad - \chi e^{(\lambda_c+\epsilon)z'}[g_u(t,0,0)\phi_1 + g_v(t,0,0)\phi_2 + 2\beta\phi_2] - (1-\chi)\psi_1' + r_2(t,z')\biggr\} \\
&\quad = e^{-\beta t}\delta\biggl\{\frac{-\sigma\beta}{\delta}W_z + e^{(\lambda_c+\epsilon)z'}\chi[(g_1 - g_u(t,0,0))\phi_1 + (g_2 - g_v(t,0,0))\phi_2 - \beta\phi_2] \\
&\qquad + r_2(t,z') + (1-\chi)[(g_1 - g_u(t,1,1))\psi_1 + (g_2 - g_v(t,1,1))\psi_2 + \mu^+\psi_2 + \beta\psi_2]\biggr\},
\end{aligned}$$

where

$$r_2(t,z') = e^{(\lambda_c+\epsilon)z'}\phi_2(t)\{l(t)\chi' + d_2(t)[\chi'' + 2\chi'(\lambda_c+\epsilon)] + \chi'[c - \sigma\beta e^{-\beta t}]\}$$
$$+ \psi_2(t)[\chi'(\sigma\beta e^{-\beta t} - c) - l(t)\chi' - d_2(t)\chi''] - e^{(\lambda_c+\epsilon)z'}\phi_2(t)(\lambda_c+\epsilon)\chi\sigma\beta e^{-\beta t},$$

$$g_1 = g_1(t,z,z') = \int_0^1 [g_u(t, U(t,z) + \tau\delta\xi_c(t,z')e^{-\beta t}, W(t,z) + \tau\delta\varsigma_c(t,z')e^{-\beta t})]d\tau,$$

$$g_2 = g_2(t,z,z') = \int_0^1 [g_v(t, U(t,z) + \tau\delta\xi_c(t,z')e^{-\beta t}, W(t,z) + \tau\delta\varsigma_c(t,z')e^{-\beta t})]d\tau.$$

Now denote

$$\Delta_L = \left|f_1 - f_u(t,0,0)\right| + \left|f_2 - f_v(t,0,0)\right| + \left|g_1 - g_u(t,0,0)\right| + \left|g_2 - g_v(t,0,0)\right|,$$
$$\Delta_R = \left|f_1 - f_u(t,1,1)\right| + \left|f_2 - f_v(t,1,1)\right| + \left|g_1 - g_u(t,1,1)\right| + \left|g_2 - g_v(t,1,1)\right|.$$

Notice that

$$\lim_{z\to-\infty} \Delta_L = 0 \quad \text{uniformly in } t, \qquad \Delta_R \le K\left[(U-1) + (W-1) + \delta\left(\|\psi_1\| + \|\psi_2\|\right)\right],$$

where $K > 0$ depends only upon $4\max_{(t,u,v)\in\mathbb{R}\times[-2,2]^2}\{|f_{uu}|, |f_{vv}|, |f_{uv}|, |g_{uu}|, |g_{vv}|, |g_{uv}|\}$.

Consequently, there exists $M > 0$ sufficiently large and $\delta^0 > 0$ such that $-M < \underline{s}$ and $M > \bar{s}$, $\delta^0 \le \ell^+$, and

$$\Delta_L\left(\|\phi_1\| + \|\phi_2\|\right) \le \frac{\beta}{2}\min\left\{\min_t \phi_1, \min_t \phi_2\right\} \quad \text{for all } (t,z) \in \mathbb{R}\times(-\infty,-M],$$
$$\Delta_R\left(\|\psi_1\| + \|\psi_2\|\right) \le \frac{\beta}{2}\min\left\{\min_t \psi_1, \min_t \psi_2\right\} \quad \text{for all } (t,z) \in \mathbb{R}\times[M,-\infty).$$

As a result, with $0 < \delta \le \delta^0$ and $\sigma > 0$, we have

$$f\left(t,u^+,v^+\right) + \mathbf{k}(t)\cdot\nabla u^+ + d_1(t)\Delta u^+ - u_t^+ \le e^{-\beta t}\delta\sigma\beta\left\{-\frac{U_z}{\delta} - e^{(\lambda_c+\epsilon)z'}\phi_1(\lambda_c+\epsilon)\chi\right\} < 0,$$
$$g\left(t,u^+,v^+\right) + \mathbf{l}(t)\cdot\nabla v^+ + d_2(t)\Delta v^+ - v_t^+ \le e^{-\beta t}\delta\sigma\beta\left\{-\frac{W_z}{\delta} - e^{(\lambda_c+\epsilon)z'}\phi_2(\lambda_c+\epsilon)\chi\right\} < 0$$

for all $(t,z) \in \mathbb{R}\times(-\infty,M]$. We have

$$f\left(t,u^+,v^+\right) + \mathbf{k}(t)\cdot\nabla u^+ + d_1(t)\Delta u^+ - u_t^+ \le -e^{-\beta t}\sigma\beta U_z < 0,$$
$$g\left(t,u^+,v^+\right) + \mathbf{l}(t)\cdot\nabla v^+ + d_2(t)\Delta v^+ - v_t^+ \le -e^{-\beta t}\sigma\beta W_z < 0$$

for all $(t,z) \in \mathbb{R}\times[M,\infty)$.

Let

$$\Delta_C = \frac{1}{\sigma\beta}\left\{e^{(\lambda_c+\epsilon)z'}\left[\Delta_L\left(\|\phi_1\| + \|\phi_2\|\right) + \Delta_R\left(\|\psi_1\| + \|\psi_2\|\right)\right] + \left|r_1\left(t,z'\right)\right| + \left|r_2\left(t,z'\right)\right|\right\}.$$

Choose $\sigma \ge \frac{1}{\beta}$, then it is easy to see that

$$\begin{aligned}\Delta_C \le e^{(\lambda_c+\epsilon)z'}\Big\{&\left[\Delta_L\left(\|\phi_1\| + \|\phi_2\|\right) + \Delta_R\left(\|\psi_1\| + \|\psi_2\|\right)\right] + \left(\|\phi_1\| + \|\phi_2\|\right)\left[\left(\|k\| + \|l\|\right)\left|\chi'\right|\right.\\ &\left. + \left(\|d_2\| + \|d_2\|\right)\left(\left|\chi''\right| + \left|\chi'\right|(2\lambda_c + 2\epsilon + c + 1)\right) + |\chi|(\lambda_c+\epsilon)\right]\\ &+ \left(\|\psi_1\| + \|\psi_2\|\right)\left[\left|\chi'\right|(c+1) + \left(\|k\| + \|l\|\right)\left|\chi'\right| + \left(\|d_1\| + \|d_2\|\right)\left|\chi''\right|\right]\Big\}.\end{aligned}$$

Then, for all $(t,z) \in \mathbb{R}\times[-M,M]$, it follows that

$$f\big(t,u^+,v^+\big)+\mathbf{k}(t)\cdot\nabla u^+ + d_1(t)\Delta u^+ - u_t^+ \le e^{-\beta t}\delta\sigma\beta\left[-\frac{U_z}{\delta}+\Delta_C\right]$$
$$g\big(t,u^+,v^+\big)+\mathbf{l}(t)\cdot\nabla v^+ + d_2(t)\Delta v^+ - v_t^+ \le e^{-\beta t}\delta\sigma\beta\left[-\frac{W_z}{\delta}+\Delta_C\right].$$

Since $(U_z, W_z) > (0,0)$, there exists $\gamma > 0$ for which $\gamma \le \min\{\inf_{(t,z)\in\mathbb{R}\times[-M,M]} U_z, \inf_{(t,z)\in\mathbb{R}\times[-M,M]} W_z\}$. Notice that $z' = z + s_0$ and it is easy to see that $\sup_{(t,z)\in\mathbb{R}\times[-M,M]} \Delta_C$ is finite. Now set $\delta_c = \min\{\delta^0, \ell^+, \frac{\gamma}{\sup_{(t,z)\in\mathbb{R}\times[-M,M]} \Delta_C}\}$. Then, as long as $\delta \in (0, \delta_c]$, we have

$$f\big(t,u^+,v^+\big)+\mathbf{k}(t)\cdot\nabla u^+ + d_1(t)\Delta u^+ - u_t^+ \le 0,$$
$$g\big(t,u^+,v^+\big)+\mathbf{l}(t)\cdot\nabla v^+ + d_2(t)\Delta v^+ - v_t^+ \le 0$$

for all $(t,z)\in\mathbb{R}\times[-M,M]$. This completes the proof. □

In what follows, if $c < c^*$, we set

$$\begin{aligned}
u_\sigma^\pm(t,x,z_0) &= U\big(t, x\cdot\nu - ct + z_0 + \pm\sigma\big(1-e^{-\beta t}\big)\big)\\
&\quad \pm \delta_c\xi_c\big(t, x\cdot\nu - ct + z_0 + s_0 \pm \sigma\big(1-e^{-\beta t}\big)\big)e^{-\beta t},\\
v_\sigma^\pm(t,x,z_0) &= W\big(t, x\cdot\nu - ct + z_0 \pm \sigma\big(1-e^{-\beta t}\big)\big)\\
&\quad \pm \delta_c\varsigma_c\big(t, x\cdot\nu - ct + z_0 + s_0 \pm \sigma\big(1-e^{-\beta t}\big)\big)e^{-\beta t}.
\end{aligned}$$

Lemma 4.3. *Suppose that (H1)–(H8) are satisfied. Assume that*

$$\lim_{x\cdot\nu\to-\infty}\frac{u_0(x)}{k\varphi_1(0)e^{\lambda_c(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty}\frac{v_0(x)}{k\varphi_2(0)e^{\lambda_c(x\cdot\nu)}} = 1$$

for some positive constant k. Furthermore, assume that

$$\liminf_{x\cdot\nu\to\infty}\big(u_0(x)-1\big) \ge -\varepsilon_0, \qquad \liminf_{x\cdot\nu\to\infty}\big(v_0(x)-1\big) \ge -\varepsilon_0$$

for some $\varepsilon_0 \in [0, \frac{\delta^c}{2\ell^+})$. Then there exist $z_0\in\mathbb{R}$, $\sigma_c \ge 1$, and $t_c > 0$ such that

$$\big(u_\sigma^-(t,x,z_0), v^-(t,x,z_0)\big) \le \big(u(t,x,u_0), v(t,x,v_0)\big) \le \big(u_\sigma^+(t,x,z_0), v^+(t,x,z_0)\big)$$

for all $(t,x)\in[t_c,\infty)\times\mathbb{R}^n$ and $\sigma \ge \sigma_c$.

Proof. The proof is the same as that of Lemma 4.7 given below. We omit it and refer to Lemma 4.7 for details. □

Lemma 4.4. *Suppose that all the assumptions given in Lemma 4.3 are satisfied. Let $(U,W) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ and c solve (2.4) with $c < c^*$. Let ϵ be given by (4.2). Let z_0 be the number for which*

$$\lim_{x\cdot\nu\to-\infty}\frac{U(0, x\cdot\nu+z_0)}{k\varphi_1(0)e^{\lambda_*(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty}\frac{W(0, x\cdot\nu+z_0)}{k\varphi_2(0)e^{\lambda_*(x\cdot\nu)}} = 1.$$

Then for each $\eta > 0$, there exist $\theta_\eta \in \mathbb{R}$ and $D_\eta > 0$ such that

$$U(t, x\cdot\nu - ct - \eta) - D_\eta\varphi_1(t)e^{(\lambda_c+\epsilon)(x\cdot\nu-ct)} \le u(t,x,u_0),$$
$$W(t, x\cdot\nu - ct - \eta) - D_\eta\varphi_2(t)e^{(\lambda_c+\epsilon)(x\cdot\nu-ct)} \le v(t,x,v_0),$$

and

$$u(t,x,u_0) \le U(t, x\cdot\nu - ct + \eta) + D_\eta\varphi_1(t)e^{(\lambda_c+\epsilon)(x\cdot\nu-ct)},$$
$$v(t,x,v_0) \le W(t, x\cdot\nu - ct + \eta) + D_\eta\varphi_2(t)e^{(\lambda_c+\epsilon)(x\cdot\nu-ct)}$$

for all $(t,x) \in \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \mid x\cdot\nu - ct \le \theta_\eta\}$.

Proof. The proof is similar to that of Lemma 4.8 given later and is omitted here. □

Case II: $c = c^*$.

Fix

$$\epsilon^* \in \left(0, \min\left\{\frac{\lambda_*}{4}, \frac{\overline{f_u(t,0,0) - g_v(t,0,0)}\,\overline{d_1(t)}}{2\sqrt{\kappa}[\overline{d_1(t)} + \overline{d_1(t)}]}\right\}\right] \tag{4.8}$$

such that

$$\beta := \Lambda_{c^*}(\lambda_* + \epsilon^*) = \overline{d_1(t)\big(\lambda_* + \epsilon^*\big)^2 + \big[c^* + k(t)\big]\big(\lambda_* + \epsilon^*\big) + f_u(t,0,0)} \le \frac{|\mu^+|}{2}, \tag{4.9}$$

where $\Lambda_{c^*}(\lambda_* + \epsilon^*) > 0$ is given by (3.33). Let

$$\bar{s} = \frac{1}{\epsilon^*}\ln\left(\min\left\{\min_t \frac{\varphi_1}{\phi_1}, \min_t \frac{\varphi_2}{\phi_2}\right\}\right),$$

where φ_i $(i = 1,2)$ are given by (3.30) and (3.31), and ϕ_i $(i = 1,2)$ are given by (3.34) and (3.35). Clearly,

$$1 - \frac{\phi_1}{\varphi_1}e^{\epsilon^* s} \ge 0, \qquad 1 - \frac{\phi_2}{\varphi_2}e^{\epsilon^* s} \ge 0 \quad \text{for all } (t,s) \in \mathbb{R} \times (-\infty, \bar{s}].$$

Now let $\chi(s)$ be a smooth function such that $\chi(s) = 1$ for $s \le \underline{s}$; $\chi(s) = 0$ for $s \ge \bar{s}$, and $0 \le \chi'(s)$ and $|\chi'| + |\chi''| \le 1$, where $\underline{s}$ is a fixed constant with $\underline{s} < \bar{s}$. Set

$$\begin{cases} \xi_*(t,s) = \chi(s)e^{\lambda_* s}\varphi_1(t)\left[1 - \dfrac{\phi_1(t)}{\varphi_1(t)}e^{\epsilon^* s}\right] + \big(1 - \chi(s)\big)\psi_1(t), \\ \varsigma_*(t,s) = \chi(s)e^{\lambda_* s}\varphi_2(t)\left[1 - \dfrac{\phi_2(t)}{\varphi_2(t)}e^{\epsilon^* s}\right] + \big(1 - \chi(s)\big)\psi_2(t). \end{cases} \tag{4.10}$$

Proposition 4.5. *Assume that (H1)–(H8) are satisfied. Let $(U(t, x\cdot\nu - ct), W(t, x\cdot\nu - ct))$ be a traveling wave of (4.1) with $c = c^*$ such that $(U(t,z), W(t,z))$ and c solve (2.4). Then*

$$\limsup_{s\to\infty} \sup_{(t,z)\in\mathbb{R}\times\mathbb{R},\ \ell\in(0,\ell^+]} \frac{U(t,z) - \ell\xi_*(t,z+s) - 1}{\ell\psi_1(t)} \le -1, \tag{4.11}$$

$$\limsup_{s\to\infty} \sup_{(t,z)\in\mathbb{R}\times\mathbb{R},\ \ell\in(0,\ell^+]} \frac{W(t,z) - \ell\varsigma_*(t,z+s) - 1}{\ell\psi_2(t)} \le -1. \tag{4.12}$$

Proof. Notice that both ξ_* and ς_* are nonnegative. Hence by arguing in a manner similar to that of Proposition 4.1 or Proposition 4.2 in [37], we obtained the estimates. □

In what follows, we fix $s_0 \in \mathbb{R}$ such that

$$\sup_{(t,s)\in\mathbb{R}\times\mathbb{R}} \frac{U(t,s) - \ell\xi_*(t,s+s_0) - 1}{\psi_1(t)} \le -\frac{\ell}{2} \quad \text{for all } \ell \in \big(0, \ell^+\big] \tag{4.13}$$

and

$$\sup_{(t,s)\in\mathbb{R}\times\mathbb{R}} \frac{W(t,s) - \ell\varsigma_*(t,s+s_0) - 1}{\psi_2(t)} \le -\frac{\ell}{2} \quad \text{for all } \ell \in \big(0, \ell^+\big]. \tag{4.14}$$

Lemma 4.6. *Suppose that (H1)–(H8) are satisfied. Let $(U(t, x\cdot\nu - ct), W(t, x\cdot\nu - ct))$ be a traveling wave of (4.1) with $c = c^*$ such that $(U(t,z), W(t,z))$ and c^* solve (2.4). Let ϵ^*, β, ℓ^+, and s_0 be given by (4.8), (4.9), (4.5), and (4.13), respectively. Then there exists $\delta^* \in (0, \ell^+]$ such that for each $z_0 \in \mathbb{R}$ and each $\sigma \ge 1/\beta$, $(u^\pm(t,x), v^\pm(t,x))$ are respectively the super- and sub-solutions of (4.1) in $\mathbb{R}^+ \times \mathbb{R}^n$ whenever $\delta \in (0, \delta^*]$. Here*

$$\begin{aligned} u^\pm(t,x) &= U\big(t, x\cdot\nu - c^*t + z_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big) \\ &\quad \pm \delta\xi_*\big(t, x\cdot\nu - c^*t + z_0 + s_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big)e^{-\beta t}, \\ v^\pm(t,x) &= W\big(t, x\cdot\nu - c^*t + z_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big) \\ &\quad \pm \delta\varsigma_*\big(t, x\cdot\nu - c^*t + z_0 + s_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big)e^{-\beta t}. \end{aligned}$$

Proof. The proof is similar to the proof of Lemma 4.2. We will give a sketch. Once again we only show that u^+ is a super-solution of (4.1), the other case can be proved similarly. Set $z' = x\cdot\nu - c^*t + z_0 + s_0 + \sigma(1 - e^{-\beta t})$ and $z = x\cdot\nu - c^*t + z_0 + \sigma(1 - e^{-\beta t})$. A straightforward calculation yields that

$$\begin{aligned} & f\big(t, u^+, v^+\big) + \mathbf{k}(t)\cdot\nabla u^+ + d_1(t)\Delta u^+ - u_t^+ \\ &\quad = f\big(t, U(t,z) + e^{-\beta t}\delta\xi_*\big(t,z'\big), W(t,z) + e^{-\beta t}\delta\varsigma_*\big(t,z'\big)\big) - f\big(t, U(t,z), W(t,z)\big) \\ &\qquad + \delta e^{-\beta t}\beta\xi_*\big(t,z'\big) + e^{-\beta t}\delta\bigg\{\frac{-\sigma\beta}{\delta}U_z + \chi e^{\lambda_* z'}\big[\big(d_1(t)\lambda_*^2 + \big(c^* + k(t)\big)\lambda_*\big)\varphi_1 - \varphi_1'\big] \\ &\qquad - \chi e^{(\lambda_*+\epsilon^*)z'}\big[\big(d_1(t)\big(\lambda_* + \epsilon^*\big)^2 + \big(c^* + k(t)\big)\big(\lambda_* + \epsilon^*\big)\big)\phi_1 - \phi_1'\big] - (1-\chi)\psi_1' + r_1\big(t,z'\big)\bigg\} \end{aligned}$$

$$
\begin{aligned}
&= e^{-\beta t}\delta\big[f_1(t,z)\xi_*(t,z) + f_2(t,z)\varsigma_*\big(t,z'\big) + \beta\xi_*(t,z)\big] \\
&\quad + e^{-\beta t}\delta\bigg\{\frac{-\sigma\beta}{\delta}U_z + \chi e^{\lambda_* z'}\big[-f_u(t,0,0)\varphi_1 - f_v(t,0,0)\varphi_2\big] \\
&\quad - \chi e^{(\lambda_*+\epsilon)z'}\big[-f_u(t,0,0)\phi_1 - f_v(t,0,0)\phi_2 + \beta\phi_1\big] - (1-\chi)\psi_1' + r_1\big(t,z'\big)\bigg\} \\
&= e^{-\beta t}\delta\bigg\{\frac{-\sigma\beta}{\delta}U_z + e^{\lambda_* z'}\chi\big[\big(f_1 - f_u(t,0,0)\big)\varphi_1 + \big(f_2 - f_v(t,0,0)\big)\varphi_2 + \beta\varphi_1\big] \\
&\quad - e^{(\lambda_*+\epsilon^*)z'}\chi\big[\big(f_1 - f_u(t,0,0)\big)\phi_1 + \big(f_2 - f_v(t,0,0)\big)\phi_2 + 2\beta\phi_1\big] + r_1\big(t,z'\big) \\
&\quad + (1-\chi)\big[\big(f_1 - f_u(t,1,1)\big)\psi_1 + \big(f_2 - f_v(t,1,1)\big)\psi_2 + \mu^+\psi_1 + \beta\psi_1\big]\bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
r_1\big(t,z'\big) &= e^{\lambda_* z'}\varphi_1(t)\big\{k(t)\chi' + d_1(t)\big[\chi'' + 2\chi'\lambda_*\big] + \chi'\big[c^* - \sigma\beta e^{-\beta t}\big]\big\} \\
&\quad - e^{(\lambda_*+\epsilon^*)z'}\phi_1(t)\big\{k(t)\chi' + d_1(t)\big[\chi'' + 2\chi'(\lambda_*+\epsilon)\big] + \chi'\big[c^* - \sigma\beta e^{-\beta t}\big]\big\} \\
&\quad + \psi_1(t)\big[\chi'\big(\sigma\beta e^{-\beta t} - c^*\big) - k(t)\chi' - d_1(t)\chi''\big] \\
&\quad - \chi\sigma\beta e^{-\beta t}e^{\lambda_* z'}\big[\varphi_1(t)\lambda_* - e^{\epsilon^* z'}\phi_1(t)\big(\lambda_* + \epsilon^*\big)\big], \\
f_1 = f_1\big(t,z,z'\big) &= \int_0^1 \big[f_u\big(t, U(t,z) + \tau\delta\xi_*\big(t,z'\big)e^{-\beta t}, W(t,z) + \tau\delta\varsigma_*\big(t,z'\big)e^{-\beta t}\big)\big]d\tau, \\
f_2 = f_2\big(t,z,z'\big) &= \int_0^1 \big[f_v\big(t, U(t,z) + \tau\delta\xi_*\big(t,z'\big)e^{-\beta t}, W(t,z) + \tau\delta\varsigma_*\big(t,z'\big)e^{-\beta t}\big)\big]d\tau.
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
&g\big(t,u^+,v^+\big) + \mathbf{l}(t)\cdot\nabla v^+ + d_2(t)\Delta v^+ - v_t^+ \\
&= g\big(t, U(t,z) + e^{-\beta t}\delta\xi_*\big(t,z'\big), W(t,z) + e^{-\beta t}\delta\varsigma_*\big(t,z'\big)\big) - g\big(t, U(t,z), W(t,z)\big) \\
&\quad + \delta e^{-\beta t}\beta\varsigma_*\big(t,z'\big) + e^{-\beta t}\delta\bigg\{\frac{-\sigma\beta}{\delta}W_z + \chi e^{\lambda_* z'}\big[\big(d_2(t)\lambda_*^2 + \big(c^* + l(t)\big)\lambda_*\big)\varphi_2 - \varphi_2'\big] \\
&\quad - \chi e^{(\lambda_*+\epsilon^*)z'}\big[\big(d_2(t)\big(\lambda_*+\epsilon^*\big)^2 + \big(c^*+l(t)\big)\big(\lambda_*+\epsilon^*\big)\big)\phi_2 - \phi_2'\big] - (1-\chi)\psi_1' + r_2\big(t,z'\big)\bigg\} \\
&= e^{-\beta t}\delta\big[g_1(t,z)\xi_*(t,z) + g_2(t,z)\varsigma_*\big(t,z'\big) + \beta\varsigma_*(t,z)\big] \\
&\quad + e^{-\beta t}\delta\bigg\{\frac{-\sigma\beta}{\delta}W_z + \chi e^{\lambda_* z'}\big[-g_u(t,0,0)\varphi_1 - g_v(t,0,0)\varphi_2\big] \\
&\quad - \chi e^{(\lambda_*+\epsilon)z'}\big[-g_u(t,0,0)\phi_1 - g_v(t,0,0)\phi_2 + \beta\phi_2\big] - (1-\chi)\psi_2' + r_2\big(t,z'\big)\bigg\}
\end{aligned}
$$

$$= e^{-\beta t}\delta\Big\{\frac{-\sigma\beta}{\delta}W_z + e^{\lambda_* z'}\chi\big[\big(g_1 - g_u(t,0,0)\big)\varphi_1 + \big(g_2 - f_v(t,0,0)\big)\varphi_2 + \beta\varphi_2\big]$$
$$- e^{(\lambda_*+\epsilon^*)z'}\chi\big[\big(g_1 - g_u(t,0,0)\big)\phi_1 + \big(g_2 - g_v(t,0,0)\big)\phi_2 + 2\beta\phi_2\big] + r_2(t,z')$$
$$+ (1-\chi)\big[\big(g_1 - g_u(t,1,1)\big)\psi_1 + \big(g_2 - g_v(t,1,1)\big)\psi_2 + \mu^+\psi_2 + \beta\psi_2\big]\Big\},$$

where

$$r_2(t,z') = e^{\lambda_* z'}\varphi_2(t)\big\{l(t)\chi' + d_2(t)\big[\chi'' + 2\chi'\lambda_*\big] + \chi'\big[c^* - \sigma\beta e^{-\beta t}\big]\big\}$$
$$- e^{(\lambda_*+\epsilon^*)z'}\phi_2(t)\big\{l(t)\chi' + d_1(t)\big[\chi'' + 2\chi'(\lambda_* + \epsilon^*)\big] + \chi'\big[c^* - \sigma\beta e^{-\beta t}\big]\big\}$$
$$+ \psi_2(t)\big[\chi'\big(\sigma\beta e^{-\beta t} - c^*\big) - l(t)\chi' - d_2(t)\chi''\big]$$
$$- \chi\sigma\beta e^{-\beta t}e^{\lambda_* z'}\big[\varphi_2(t)\lambda_* - e^{\epsilon^* z'}\phi_2(t)(\lambda_* + \epsilon^*)\big],$$

$$g_1 = g_1(t,z,z') = \int_0^1 \big[g_u\big(t, U(t,z) + \tau\delta\xi_*(t,z')e^{-\beta t}, W(t,z) + \tau\delta\varsigma_*(t,z')e^{-\beta t}\big)\big]d\tau,$$

$$g_2 = g_2(t,z,z') = \int_0^1 \big[g_v\big(t, U(t,z) + \tau\delta\xi_*(t,z')e^{-\beta t}, W(t,z) + \tau\delta\varsigma_*(t,z')e^{-\beta t}\big)\big]d\tau.$$

In terms of Corollary 3.11, if $z < 0$ and $|z|$ is sufficiently large, then $U_z \ge \frac{\lambda_*\rho}{2}|z|e^{\lambda_* z}$ and $W_z \ge \frac{\lambda_*\rho}{2}|z|e^{\lambda_* z}$ for some $\rho > 0$. Recall $z' = z + s_0$. The rest of the proof follows by using the same arguments as that in the proof of Lemma 4.2. The proof is completed. □

In what follows, we set

$$u_\sigma^\pm(t,x,z_0) = U\big(t, x\cdot\nu - c^*t + z_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big)$$
$$\pm \delta^*\xi_*\big(t, x\cdot\nu - c^*t + z_0 + s_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big)e^{-\beta t},$$
$$v_\sigma^\pm(t,x,z_0) = W\big(t, x\cdot\nu - c^*t + z_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big)$$
$$\pm \delta^*\varsigma_*\big(t, x\cdot\nu - c^*t + z_0 + s_0 \pm \sigma\big(1 - e^{-\beta t}\big)\big)e^{-\beta t}.$$

Lemma 4.7. *Suppose that (H1)–(H8) are satisfied. Assume that*

$$\lim_{x\cdot\nu\to-\infty}\frac{u_0(x)}{k\varphi_1(0)|x\cdot\nu|e^{\lambda_*(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty}\frac{v_0(x)}{k\varphi_2(0)|x\cdot\nu|e^{\lambda_*(x\cdot\nu)}} = 1$$

for some positive constant k. Furthermore, assume that

$$\liminf_{x\cdot\nu\to\infty}\big(u_0(x) - 1\big) \ge -\varepsilon_0, \qquad \liminf_{x\cdot\nu\to\infty}\big(v_0(x) - 1\big) \ge -\varepsilon_0$$

for some $\varepsilon_0 \in [0, \frac{\delta^*}{2\ell^+})$. Then there exist $z_0 \in \mathbb{R}$, $\sigma^* \ge 1$, and $t_* > 0$ such that

$$\big(u_\sigma^-(t,x,z_0), v_\sigma^-(t,x,z_0)\big) \le \big(u(t,x,u_0), v(t,x,v_0)\big) \le \big(u_\sigma^+(t,x,z_0), v_\sigma^+(t,x,z_0)\big)$$

for all $(t,x) \in [t_*,\infty) \times \mathbb{R}^n$ *and* $\sigma \ge \sigma^*$.

Proof. Let the operator $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset X_0 \to X_0$ with $D(\mathcal{A}(t)) = X_1$ be defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_1(t)\Delta u + \mathbf{k}(t)\cdot\nabla u & 0 \\ 0 & d_2(t)\Delta v + \mathbf{l}(t)\cdot\nabla v \end{pmatrix},$$

where $X_0 = BUC(\mathbb{R}^n, \mathbb{R}^2)$ and $X_1 = \{\binom{u}{v} \in \bigcap_{p\ge 1} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^2),\ \binom{u}{v}, \mathcal{A}(t)\binom{u}{v} \in X_0\}$. Let $G(t,s)_{s\le t}$ be the evolution operator for the family $\mathcal{A}(t)$, then we have

$$\begin{pmatrix} u(t,x,u_0) \\ v(t,x,v_0) \end{pmatrix} = G(t,0)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(x) + \int_0^t G(t,s)\begin{pmatrix} f(s,u,v) \\ g(s,u,v) \end{pmatrix} ds.$$

As $(u(t,x,u_0), v(t,x,v_0))$ is bounded, it follows that

$$\lim_{t\to 0}\left\| \begin{pmatrix} u(t,\cdot,u_0) - u_0 \\ v(t,\cdot,v_0) - v_0 \end{pmatrix} \right\|_{X_0} = 0.$$

In view of assumption, there exists $\alpha > 0$ for which $\alpha\varepsilon_0 \le \frac{\delta^*}{2\ell^+}$. Since

$$\liminf_{x\cdot\nu\to\infty}\big(u(t,x,u_0) - 1\big) \ge \liminf_{x\cdot\nu\to\infty}\big(u(t,x,u_0) - u_0(x)\big) + \liminf_{x\cdot\nu\to\infty}\big(u_0(x) - 1\big)$$

and

$$\liminf_{x\cdot\nu\to\infty}\big(v(t,x,v_0) - 1\big) \ge \liminf_{x\cdot\nu\to\infty}\big(v(t,x,v_0) - v_0(x)\big) + \liminf_{x\cdot\nu\to\infty}\big(v_0(x) - 1\big),$$

there exists t_* such that

$$\liminf_{x\cdot\nu\to\infty}\left\{ \inf_{t\in\mathbb{R}} \frac{u(t_*,x,u_0) - 1}{\psi_1(t)} \right\} > -\ell^+\alpha\varepsilon_0 e^{-\beta t_*},$$
$$\liminf_{x\cdot\nu\to\infty}\left\{ \inf_{t\in\mathbb{R}} \frac{v(t_*,x,v_0) - 1}{\psi_2(t)} \right\} > -\ell^+\alpha\varepsilon_0 e^{-\beta t_*}.$$

Consequently, (4.13) and (4.14) imply that

$$\sup_{(t,s)\in\mathbb{R}\times\mathbb{R}} \frac{U(t,s) - \delta^*\xi_*(t,s+s_0)e^{-\beta t_*} - 1}{\psi_1(t)} < \liminf_{x\cdot\nu\to\infty}\left\{ \inf_{t\in\mathbb{R}} \frac{u(t_*,x,u_0) - 1}{\psi_1(t)} \right\} \tag{4.15}$$

and

$$\sup_{(t,s)\in\mathbb{R}\times\mathbb{R}} \frac{W(t,s) - \delta^*\varsigma_*(t,s+s_0)e^{-\beta t_*} - 1}{\psi_2(t)} < \liminf_{x\cdot\nu\to\infty}\left\{ \inf_{t\in\mathbb{R}} \frac{v(t_*,x,v_0) - 1}{\psi_2(t)} \right\}. \tag{4.16}$$

Now in view of Theorem 3.10, we can fix $z_0 \in \mathbb{R}$ such that

$$\lim_{x\cdot\nu\to-\infty}\frac{U(0,x\cdot\nu+z_0)}{k\varphi_1(0)|x\cdot\nu|e^{\lambda_*(x\cdot\nu)}}=1,\qquad \lim_{x\cdot\nu\to-\infty}\frac{W(0,x\cdot\nu+z_0)}{k\varphi_2(0)|x\cdot\nu|e^{\lambda_*(x\cdot\nu)}}=1.$$

Notice that such a z_0 is uniquely determined by k. We next show that there exists $\sigma^* \geq 1$ for which

$$u_\sigma^-(t_*,x,z_0)\leq u(t_*,x,u_0)\quad \text{whenever}\quad \sigma\geq\sigma^*.$$

Assume to the contrary that this is not true, then there exist two sequences $\{x_n\}$ and $\{\sigma_n\}$ such that

$$\sigma_n\to\infty\quad \text{as } n\to\infty,\quad \text{and}\quad u_{\sigma_n}^-(t_*,x_n,z_0)>u(t_*,x_n,u_0). \tag{4.17}$$

Let $z_n = x_n\cdot\nu - c^*t_* + z_0 - \sigma_n(1-e^{-\beta t_*})$. Note that $u^-(t_*,x_n,z_0) = U(t_*,z_n) - \delta^*\xi_*(t_*,z_n + s_0)e^{-\beta t_*}$. Up to extraction of a subsequence of $\{z_n\}$, two cases may occur: either $\lim_{n\to\infty} z_n = -\infty$ or $\{z_n\}$ is bounded from below. If $\{z_n\}$ is bounded from below, then $x_n\cdot\nu\to\infty$ as $n\to\infty$. In case that $\lim_{n\to\infty} z_n=-\infty$, we need to consider two possibilities: either $\{x_n\cdot\nu\}$ is bounded or $\{x_n\cdot\nu\}$ is unbounded. We shall focus on the possibility that $\{x_n\cdot\nu\}$ is bounded since by utilizing (4.15) and (4.16) we can follow the same lines as those of Lemma 4.4 of [37] to reach a contradiction provided that $\{x_n\cdot\nu\}$ is unbounded.

Suppose that $\{x_n\cdot\nu\}$ is bounded, then either $\{x_n\}$ is unbounded or bounded. If $\{x_n\}$ is unbounded, let $y_n=(x_n\cdot\nu)\nu$ and $\mathbf{s}_n=\frac{x_n-y_n}{T}$. Note that $\mathbf{s}_nT\cdot\nu=0$. Now set $(u_n(t,x),v_n(t,x)) = (u(t,x+\mathbf{s}_nT,u_0), v(t,x+\mathbf{s}_nT,v_0))$. Clearly, for each n, $(u_n(t,x),v_n(t,x))$ is also a solution of (4.1) with $(u_n(0,x),v_n(0,x))=(u_0(x+\mathbf{s}_nT),v_0(x+\mathbf{s}_nT))$.

Recall y_n is bounded, there exists $R>0$ sufficiently large so that $|y_n|\leq\frac{R}{4}$. In particular, by virtue of assumption, we may choose $R>0$ such that $(u_0(x'+\mathbf{s}_nT),v_0(x'+\mathbf{s}_nT))\geq\frac{1-\varepsilon_0}{2}(1,1)$ for some $x'\in\{x\in\mathbb{R}^n:|x|\leq\frac{R}{4}\}$. Due to the regularity of $(u_n(t,x),v_n(t,x))$, up to extraction of a subsequence, $(u_n(t,x),v_n(t,x))$ converges uniformly on $[0,t_*+1]\times B_R(x)$ to a function $(u_\infty(t,x),v_\infty(t,x))$, where $B_R(x)=\{x\in\mathbb{R}^n:|x|\leq R\}$. By passing the limits in (4.1), we find that $(u_\infty(t,x),v_\infty(t,x))$ satisfies (4.1) in $]0,t_*+1[\times\dot{B}_{\frac{R}{2}}(x)$. Here $\dot{B}_{\frac{R}{2}}(x)=\{|x|<\frac{R}{2}\}$. Moreover, it is easy to see that $(u_\infty(t,x),v_\infty(t,x))\geq(0,0)$ for all $(t,x)\in[0,t_*+1]\times\{|x|=\frac{R}{2}\}$ and $(u_\infty(0,x),v_\infty(0,x))\gneq(0,0)$ for $x\in B_{\frac{R}{2}}(x)$. Hence, the comparison principle implies that

$$\big(u_\infty(t_*,x),v_\infty(t_*,x)\big)>(0,0)\quad \text{for all } x\in B_{\frac{R}{4}}(x).$$

On the other hand, if (4.17) is true, then

$$0=\lim_{n\to\infty}u_{\sigma_n}^-(t_*,x_n,z_0)\geq\lim_{n\to\infty}u(t_*,x_n,u_0)=\lim_{n\to\infty}u_n(t_*,y_n)=u_\infty(t_*,y_\infty)\geq0,$$

where $y_\infty=\lim_{n\to\infty}y_n$. This forces that $u_\infty(t_*,y_\infty)=0$, which is a contradiction since $y_\infty\in B_{\frac{R}{4}}(x)$ and $u_\infty(t_*,y_\infty)>0$. If $\{x_n\}$ is bounded, the continuity of $u(t,x,u_0)$ yields

$$0=\lim_{n\to\infty}u_{\sigma_n}^-(t_*,x_n,z_0)\geq\lim_{n\to\infty}u(t_*,x_n,u_0)=u(t_*,x_\infty,u_0)\geq0,$$

where $x_\infty = \lim_{n\to\infty} x_n$. Thus, $u(t_*, x_\infty, u_0) = 0$, which contradicts the fact that $u(t_*, x, u_0) > 0$. Therefore, we readily conclude that $u_\sigma^-(t_*, x, z_0) \le u(t_*, x, u_0)$ if $\sigma \ge \sigma_1$ for some $\sigma_1 \ge 1$. Likewise, we can show that $v_\sigma^-(t_*, x, z_0) \le v(t_*, x, u_0)$ if $\sigma \ge \sigma_2$ for some $\sigma_2 \ge 1$. Furthermore, by using the same arguments as those given in the proof of Lemma 4.4 of [37], we can show that there exist $\sigma_3 \ge 1$ and $\sigma_4 \ge 1$ such that

$$u_\sigma^+(t_*, x, z_0) \ge u(t_*, x, u_0) \quad \text{if } \sigma \ge \sigma_3, \qquad v_\sigma^+(t_*, x, z_0) \ge v(t_*, x, v_0) \quad \text{provided } \sigma \ge \sigma_4.$$

Now choose $\sigma^* = \max_{1\le i\le 4}\{\sigma_i\}$. Clearly

$$\big(u_\sigma^-(t_*, x, z_0), v_\sigma^-(t_*, x, z_0)\big) \le \big(u(t_*, x, u_0), v(t_*, x, u_0)\big) \le \big(u_\sigma^+(t_*, x, z_0), v_\sigma^+(t_*, x, z_0)\big)$$

for all $\sigma \ge \sigma^*$. Notice that $(u_\sigma^-(t_*, x, z_0), v_\sigma^-(t_*, x, z_0)) \le (1,1)$ and $(u_\sigma^+(t_*, x, z_0), v_\sigma^+(t_*, x, z_0)) > (0,0)$. The conclusion follows from Proposition A.3 in Appendix A. The proof is completed. $\square$

Lemma 4.8. *Suppose that all the assumptions of Lemma 4.7 are satisfied. Let $(U, W) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ and c solve (2.4) with $c = c^*$. Let ϵ^* be given by (4.8). Let z_0 be the number for which*

$$\lim_{x\cdot\nu\to-\infty} \frac{U(0, x\cdot\nu + z_0)}{k\varphi_1(0)|z|e^{\lambda_*(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty} \frac{W(0, x\cdot\nu + z_0)}{k\varphi_2(0)|z|e^{\lambda_*(x\cdot\nu)}} = 1.$$

Then for each $\eta > 0$, there exist $\theta_\eta \le \frac{1}{\epsilon^*}\ln(\min\{\min_t \frac{\varphi_1}{\phi_1}, \min_t \frac{\varphi_2}{\phi_2}\})$ and $D_\eta > 0$ such that

$$U\big(t, x\cdot\nu - c^*t - \eta\big) - D_\eta e^{\lambda_*(x\cdot\nu - c^*t)}\varphi_1(t)\left(1 - \frac{\phi_1(t)}{\varphi_1(t)}e^{\epsilon^*(x\cdot\nu - c^*t)}\right) \le u(t, x, u_0), \tag{4.18}$$

$$W\big(t, x\cdot\nu - c^*t - \eta\big) - D_\eta e^{\lambda_*(x\cdot\nu - c^*t)}\varphi_2(t)\left(1 - \frac{\phi_2(t)}{\varphi_2(t)}e^{\epsilon^*(x\cdot\nu - c^*t)}\right) \le v(t, x, v_0), \tag{4.19}$$

and

$$u(t, x, u_0) \le U\big(t, x\cdot\nu - c^*t + \eta\big) + D_\eta e^{\lambda_*(x\cdot\nu - c^*t)}\varphi_1(t)\left(1 - \frac{\phi_1(t)}{\varphi_1(t)}e^{\epsilon^*(x\cdot\nu - c^*t)}\right), \tag{4.20}$$

$$v(t, x, v_0) \le W\big(t, x\cdot\nu - c^*t + \eta\big) + D_\eta e^{\lambda_*(x\cdot\nu - c^*t)}\varphi_2(t)\left(1 - \frac{\phi_2(t)}{\varphi_2(t)}e^{\epsilon^*(x\cdot\nu - c^*t)}\right) \tag{4.21}$$

for all $(t, x) \in \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \mid x\cdot\nu - c^*t \le \theta_\eta\}$.

Proof. We first chose $z_* \le 0$ such that

$$1 - \max_t \frac{\phi_1(t)}{\varphi_1(t)}e^{\epsilon^* z} \ge \frac{1}{2}, \qquad 1 - \max_t \frac{\phi_2(t)}{\varphi_2(t)}e^{\epsilon^* z} \ge \frac{1}{2} \quad \text{for all } z \in (-\infty, z_*].$$

Again assume without loss of generality that $z_0 = 0$. It follows from the monotonicity of $(U(0,\cdot), W(0,\cdot))$ and the assumptions that

$$\lim_{x\cdot\nu\to-\infty}\frac{U(0,x\cdot\nu-\eta)}{u_0(x)}<1,\qquad \lim_{x\cdot\nu\to-\infty}\frac{W(0,x\cdot\nu-\eta)}{v_0(x)}<1.$$

Thus, $(U(0,x\cdot\nu-\eta),W(0,x\cdot\nu-\eta))\le(u_0(x),v_0(x))$ for $x\cdot\nu\le -M$, where $M>0$ is sufficiently large. Since $\min\{\inf_{x\cdot\nu\ge -M}u_0(x),\inf_{x\cdot\nu\ge -M}v_0(x)\}\ge 0$, there exists $\hat{D}_0(\eta)>1$ such that

$$U(0,x\cdot\nu-\eta)-\hat{D}_0(\eta)e^{\lambda_*(x\cdot\nu)}\varphi_1(0)\left(1-\frac{\phi_1(0)}{\varphi_1(0)}e^{\epsilon^*(x\cdot\nu)}\right)\le u_0(x),$$
$$W(0,x\cdot\nu-\eta)-\hat{D}_0(\eta)e^{\lambda_*(x\cdot\nu)}\varphi_2(0)\left(1-\frac{\phi_2(0)}{\varphi_2(0)}e^{\epsilon^*(x\cdot\nu)}\right)\le v_0(x)$$

as long as $x\cdot\nu\le z_*$. By virtue of Theorem 3.10, there exist $\rho>0$ and $\underline{z}\le z_*$ for which

$$\left|U(t,z)\right|\le\frac{3\rho}{2}|z|e^{\lambda_* z}\varphi_1(t),\qquad \left|W(t,z)\right|\le\frac{3\rho}{2}|z|e^{\lambda_* z}\varphi_2(t).$$

Set $\hat{m}=\frac{\min\{\min_t\phi_1,\min_t\phi_2\}}{\max_t(\phi_1+\phi_2)}$ and $m_*=\min\{\min_t\phi_1,\min_t\phi_2\}$. As $f,g\in C^{0,2}$, there exists $\varepsilon\in\,]0,1[$ such that

$$\left|f_u(t,u,v)-f_u(t,0,0)\right|+\left|f_v(t,u,v)-f_v(t,0,0)\right|\le\frac{\hat{m}\beta}{2},$$
$$\left|g_u(t,u,v)-g_u(t,0,0)\right|+\left|g_v(t,u,v)-g_v(t,0,0)\right|\le\frac{\hat{m}\beta}{2}$$

whenever $|u|+|v|\le\varepsilon$, and there exists $K>0$ such that

$$\left|f_u(t,u,v)-f_u(t,0,0)\right|+\left|f_v(t,u,v)-f_v(t,0,0)\right|\le K|u|+|v|,$$
$$\left|g_u(t,u,v)-g_u(t,0,0)\right|+\left|g_v(t,u,v)-g_v(t,0,0)\right|\le K|u|+|v|$$

whenever $|u|+|v|\le 1$. Let $m^*=\max_t(\varphi_1+\varphi_2)$, and chose $z_\eta\le\underline{z}$ with $|z_\eta|$ sufficiently large such that $|z_\eta|\ge\hat{D}_0(\eta)$, and

$$2|z|m^*\left(\frac{3\rho}{2}+2\left(\frac{3\rho}{2}+1\right)\right)e^{\lambda_* z}\le\varepsilon,$$
$$2K|z|\left(m^*\right)^2\left(\frac{3\rho}{2}+2\left(\frac{3\rho}{2}+1\right)\right)e^{(\lambda_*-\epsilon^*)z}\le\frac{m_*\beta}{2}\quad\text{for all } z\le z_\eta.$$

Let $\hat{D}_\eta=2(3\frac{\rho}{2}+1)z_\eta$ and define

$$u_\eta(t,x)=U\left(t,x\cdot\nu-c^*t-\eta\right)-\hat{D}_\eta e^{\lambda_*(x\cdot\nu-c^*t)}\varphi_1(t)\left(1-\frac{\phi_1(t)}{\varphi_1(t)}e^{\epsilon^*(x\cdot\nu-c^*t)}\right),$$
$$v_\eta(t,x)=W\left(t,x\cdot\nu-c^*t-\eta\right)-\hat{D}_\eta e^{\lambda_*(x\cdot\nu-c^*t)}\varphi_2(t)\left(1-\frac{\phi_2(t)}{\varphi_2(t)}e^{\epsilon^*(x\cdot\nu-c^*t)}\right)$$

It is clear that $(u_\eta(t,x),v_\eta(t,x))\le(0,0)$ for all $(t,x)\in\{x\cdot\nu-c^*t=z_\eta\}$, and $(u_\eta(t,x),v_\eta(t,x))\le(1,1)$ for all $(t,x)\in\{x\cdot\nu-c^*t\le z_\eta\}$. As $(u(t,x,u_0),v(t,x,v_0))\ge(0,0)$ for all

$(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$, and $\hat{D}_\eta \ge \hat{D}_0(\eta)$, it follows that $(u_\eta(t,x), v_\eta(t,x)) \le (u(t,x,u_0), v(t,x,v_0))$ for all $(t,x)\in\{t=0,\ x\cdot\nu\le z_\eta\}\cup\{t\ge 0,\ x\cdot\nu - c^*t = z_\eta\}$. In addition, whenever $(t,x)\in\{x\cdot\nu - c^*t < z_\eta\}$, we find

$$\begin{aligned}
& f(u_\eta, v_\eta) + \mathbf{k}(t)\cdot\nabla u_\eta + d_1(t)\Delta u_\eta - (u_\eta)_t \\
&= \hat{D}_\eta e^{\lambda_* z}\left\{\int_0^1 \left[f_u(t,0,0) - f_u\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big)ds\right]\varphi_1 \right.\\
&\quad \left. + \int_0^1 \left[f_v(t,0,0) - f_v\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big)ds\right]\varphi_2\right\} \\
&\quad + \hat{D}_\eta e^{(\lambda_*+\epsilon^*)z}\left\{\int_0^1 \left[f_u\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big) - f_u(t,0,0)ds\right]\phi_1 \right.\\
&\quad \left. + \int_0^1 \left[f_v\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big) - f_v(t,0,0)ds\right]\phi_2 + \beta\phi_1\right\} \\
&\ge \hat{D}_\eta\left[-2K|z|\big(m^*\big)^2\left(\frac{3\rho}{2} + 2\left(\frac{3\rho}{2}+1\right)\right)e^{2\lambda_* z} + e^{(\lambda_*+\epsilon^*)z}\frac{m_*\beta}{2}\right] \\
&= \hat{D}_\eta e^{(\lambda_*+\epsilon^*)z}\left[\frac{m_*\beta}{2} - 2K|z|\big(m^*\big)^2\left(\frac{3\rho}{2} + 2\left(\frac{3\rho}{2}+1\right)\right)e^{(\lambda_*-\epsilon^*)z}\right] \\
&\ge 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& g(u_\eta, v_\eta) + \mathbf{l}(t)\cdot\nabla v_\eta + d_2(t)\Delta v_\eta - (v_\eta)_t \\
&= \hat{D}_\eta e^{\lambda_* z}\left\{\int_0^1 \left[g_u(t,0,0) - g_u\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big)ds\right]\varphi_1 \right.\\
&\quad \left. + \int_0^1 \left[g_v(t,0,0) - g_v\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big)ds\right]\varphi_2\right\} \\
&\quad + \hat{D}_\eta e^{(\lambda_*+\epsilon^*)z}\left\{\int_0^1 \left[g_u\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big) - g_u(t,0,0)ds\right]\phi_1 \right.\\
&\quad \left. + \int_0^1 \left[g_v\big(t, su_\eta + (1-s)U, sv_\eta + (1-s)W\big) - g_v(t,0,0)ds\right][\phi_2 + \beta\phi_2]\right\} \\
&\ge 0.
\end{aligned}$$

Now let $\Gamma = \{(t,x) \mid t > 0,\ x \cdot \nu - c^* t < z_\eta\}$. Obviously Γ is an open connected subset of $\mathbb{R}^+ \times \mathbb{R}^n$, and $\partial\Gamma = \{(t,x) \mid t = 0,\ x \cdot \nu < z_\eta\} \cup \{(t,x) \mid t \geq 0,\ x \cdot \nu - c^* t = z_\eta\}$. Thus, it follows from Proposition A.3 that

$$\big(u_\eta(t,x), v_\eta(t,x)\big) \leq \big(u(t,x,u_0), v(t,x,v_0)\big) \quad \text{for all } (t,x) \in \big\{(t,x) \bigm| t \geq 0,\ x \cdot \nu - c^* t \leq z_\eta\big\}.$$

Now what is left is to establish (4.20) and (4.21). Analogously, it can be shown that there exist $\check{D}_\eta > 0$ and $z^\eta \in \mathbb{R}$ for which $(u(t,x,u_0), v(t,x,v_0)) \leq (u^\eta(t,x), v^\eta(t,x))$ for all $(t,x) \in \{(t,x) \mid t \geq 0,\ x \cdot \nu - c^* t \leq z^\eta\}$. Here

$$u^\eta(t,x) = U\big(t, x \cdot \nu - c^* t + \eta\big) + \check{D}_\eta e^{\lambda_*(x\cdot\nu - c^* t)} \varphi_1(t)\left(1 - \frac{\phi_1(t)}{\varphi_1(t)} e^{\epsilon^*(x\cdot\nu - c^* t)}\right),$$

$$v^\eta(t,x) = W\big(t, x \cdot \nu - c^* t + \eta\big) + \check{D}_\eta e^{\lambda_*(x\cdot\nu - c^* t)} \varphi_2(t)\left(1 - \frac{\phi_2(t)}{\varphi_2(t)} e^{\epsilon^*(x\cdot\nu - c^* t)}\right).$$

Set $D_\eta = \max\{\hat{D}_\eta, \check{D}_\eta\}$ and $\theta_\eta = \min\{z_\eta, z^\eta\}$. Then (4.18), (4.19); (4.20), and (4.21) hold for all $(t,x) \in \{(t,x) \mid t \geq 0,\ x \cdot \nu - c^* t \leq \theta_\eta\}$. The proof is completed. □

Lemma 4.9. *Suppose that (H1)–(H8) are satisfied. Assume that $(U,W) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (2.4) with $c \leq c^*$. Assume that $(u(t,x), v(t,x)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ solves the first and second equations of (4.1) for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and satisfies that*

$$\begin{aligned}&\big(U(t, z + z_0 + \underline{\omega}), W(t, z + z_0 + \underline{\omega})\big)\\ &\quad \leq \big(u(t,x), v(t,x)\big) \leq \big(U(t, z + z_0 + \overline{\omega}), W(t, z + z_0 + \overline{\omega})\big)\end{aligned}$$

for certain constants $\underline{\omega}$, $\overline{\omega}$ and z_0 with $\underline{\omega} \leq 0 \leq \overline{\omega}$ and $z_0 \in \mathbb{R}$, where $z = x \cdot \nu - ct$. In addition, assume that for each $\eta > 0$, there exist $\theta_\eta \in \mathbb{R}$ and $D_\eta \in \mathbb{R}^+$ such that

$$\begin{aligned}&U(t, x\cdot\nu - ct + z_0 - \eta) - D_\eta \varphi_1 e^{(\lambda_c+\epsilon)(x\cdot\nu - ct)}\\ &\quad \leq u(t,x) \leq U(t, x\cdot\nu - ct + z_0 + \eta) + D_\eta \varphi_1 e^{(\lambda_c+\epsilon)(x\cdot\nu - ct)}\\ &W(t, x\cdot\nu - ct + z_0 - \eta) - D_\eta \varphi_2 e^{(\lambda_c+\epsilon)(x\cdot\nu - ct)}\\ &\quad \leq v(t,x) \leq W(t, x\cdot\nu - ct + z_0 + \eta) + D_\eta \varphi_2 e^{(\lambda_c+\epsilon)(x\cdot\nu - ct)}\end{aligned}$$

for all $(t,x) \in \{(t,x) \mid x \cdot \nu - ct \leq \theta_\eta\}$ provided that $c < c^*$. In case that $c = c^*$, suppose that

$$\begin{aligned}&U(t, z + z_0 - \eta) - D_\eta \varphi_1 e^{\lambda_* z}\left(1 - \frac{\phi_1}{\varphi_1} e^{\epsilon_* z}\right)\\ &\quad \leq u(t,x) \leq U(t, z + z_0 + \eta) + D_\eta \varphi_1 e^{\lambda_* z}\left(1 - \frac{\phi_1}{\varphi_1} e^{\epsilon_* z}\right)\\ &W(t, z + z_0 - \eta) - D_\eta \varphi_2 e^{\lambda_* z}\left(1 - \frac{\phi_2}{\varphi_2} e^{\epsilon_* z}\right)\\ &\quad \leq v(t,x) \leq W(t, z + z_0 + \eta) + D_\eta \varphi_2 e^{\lambda_* z}\left(1 - \frac{\phi_2}{\varphi_2} e^{\epsilon_* z}\right)\end{aligned}$$

for all $(t,x) \in \{(t,x) \mid x \cdot \nu - c^* t \le \theta_\eta\}$ *provided that* $c = c^*$, *where* $z = x \cdot \nu - c^* t$. *Then*

$$\big(u(t,x), v(t,x)\big) = \big(U(t, x \cdot \nu - ct + z_0), W(t, x \cdot \nu - ct + z_0)\big)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$.

Proof. We only give a proof for the case of $c = c^*$ since the other can be proved similarly. The proof will be divided into a few steps. Once again we assume without loss of generality that $z_0 = 0$.

Step 1. Define

$$\bar{\eta} := \inf\left\{\eta \in [0,\infty) \;\middle|\; \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \le \begin{pmatrix} U(t, x \cdot \nu - c^* t + \eta) \\ W(t, x \cdot \nu - c^* t + \eta) \end{pmatrix},\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n\right\}.$$

Clearly, $\bar{\eta}$ is bounded and satisfies $0 \le \bar{\eta} \le \varpi$ since $(U(t,\cdot), W(t,\cdot))$ is monotonically increasing. To complete the proof, we need to show that $\bar{\eta} = 0$. Assume to the contrary that $\bar{\eta} > 0$. Then, we first claim that there exists $\underline{\theta} \in (-\infty, \theta_{\frac{\bar{\eta}}{2}}]$ such that

$$\big(u(t,x), v(t,x)\big) \le \left(U\left(t, x \cdot \nu - c^* t + \frac{\bar{\eta}}{2}\right), W\left(t, x \cdot \nu - c^* t + \frac{\bar{\eta}}{2}\right)\right) \tag{4.22}$$

for all $(t,x) \in \{x \cdot \nu - c^* t \le \underline{\theta}\}$. Assume this is not true, then there exist two sequences such that

$$\lim_{k\to\infty} x_k \cdot \nu - c^* t_k = -\infty \quad \text{and}$$
$$\big(u(t_k,x_k), v(t_k,x_k)\big) > \left(U\left(t_k, x_k \cdot \nu - c^* t_k + \frac{\bar{\eta}}{2}\right), W\left(t_k, x_k \cdot \nu - c^* t_k + \frac{\bar{\eta}}{2}\right)\right).$$

On the other hand, Theorem 3.10 shows that

$$\lim_{k\to\infty} \frac{U(t_k, z_k + \frac{\bar{\eta}}{4}) + D_{\frac{\bar{\eta}}{4}} \varphi_1(t_k) e^{\lambda_* z_k}[1 - \frac{\phi_1(t_k)}{\varphi_1(t_k)} e^{(\lambda_* + \epsilon_*) z_k}]}{U(t_k, z_k + \frac{\bar{\eta}}{2})} < 1$$

and

$$\lim_{k\to\infty} \frac{W(t_k, z_k + \frac{\bar{\eta}}{4}) + D_{\frac{\bar{\eta}}{4}} \varphi_2(t_k) e^{\lambda_* z_k}[1 - \frac{\phi_2(t_k)}{\varphi_2(t_k)} e^{(\lambda_* + \epsilon_*) z_k}]}{W(t_k, z_k + \frac{\bar{\eta}}{2})} < 1,$$

where $z_k = x_k \cdot \nu - c^* t_k$. It then follows from the assumption that

$$\big(u(t_k,x_k), v(t_k,x_k)\big) \le \left(U\left(t_k, x_k \cdot \nu - c^* t_k + \frac{\bar{\eta}}{2}\right), W\left(t_k, x_k \cdot \nu - c^* t_k + \frac{\bar{\eta}}{2}\right)\right)$$

whenever $x_k \cdot \nu - c^* t_k \le \theta'$ for some $\theta' \le \theta_{\frac{\bar{\eta}}{4}}$. This is a contradiction. Thus, (4.22) holds.

Step 2. In this step, we show that

$$\inf_{\underline{\theta}\le x\cdot\nu-c^*t\le\theta} U\big(t,x\cdot\nu-c^*t+\bar\eta\big)-u(t,x)>0,$$
$$\inf_{\underline{\theta}\le x\cdot\nu-c^*t\le\theta} W\big(t,x\cdot\nu-c^*t+\bar\eta\big)-v(t,x)>0 \tag{4.23}$$

for any $\theta\ge\underline{\theta}$. We only prove the first inequality of (4.23) since the second can be proved in exactly the same way. Assume to the contrary that $\inf_{\underline{\theta}\le x\cdot\nu-c^*t\le\theta} U(t,x\cdot\nu-c^*t+\bar\eta)-u(t,x)=0$. Then there exist two sequences $\{t_k\}$ and $\{x_k\}$ such that

$$\underline{\theta}\le x_k\cdot\nu-c^*t_k\le\theta\quad\text{and}\quad\lim_{k\to\infty}\big[U\big(t,x_k\cdot\nu-c^*t_k+\bar\eta\big)-u(t_k,x_k)\big]=0.$$

To reach a contradiction, we need to consider several scenarios, i.e., (a) $\{t_k\}$ is unbounded; (b) $\{t_k\}$ is bounded while $\{x_k\}$ is unbounded, (c) both $\{t_k\}$ and $\{x_k\}$ are bounded. We only deal with the case (a), the others can be treated similarly. If $\{t_k\}$ is unbounded, upon an extraction of a subsequence, we may assume that $t_k\to\infty$ as $k\to\infty$. Hence, there exists a sequence $\{j_k\}$ with $j_k\in\mathbb{N}^+$ such that $\lim_{k\to\infty} j_k=\infty$ and $t_k\in[j_kT,(j_k+1)T]$. Now let $\tau_k=t_k-j_kT$. Clearly, $\tau_k\in[0,T]$. We also write $z_k=x_k\cdot\nu-c^*j_kT-c^*\tau_k$. Since z_k and τ_k are bounded, $x_k\cdot\nu-c^*j_kT$ must be bounded. We then set

$$y_k=\big(x_k\cdot\nu-c^*j_kT\big)\nu,\qquad \mathbf{s}_k=\frac{x_k-y_k}{T}.$$

Notice that

$$T\mathbf{s}_k\cdot\nu=(x_k-y_k)\cdot\nu=x_k\cdot\nu-\big(x_k\cdot\nu-c^*j_kT\big)\nu\cdot\nu=c^*j_kT.$$

Thus,

$$z_k=x_k\cdot\nu-c^*t_k=y_k\cdot\nu-c^*\tau_k+T\big(\mathbf{s}_k\cdot\nu-c^*j_k\big)=y_k\cdot\nu-c^*\tau_k.$$

Note that $\underline{\theta}\le z_k\le\theta$, and both $y_k\in\mathbb{R}^n$ and $\tau_k\in[0,T]$ are bounded. Thus, up to an extraction of subsequence, we may assume that there exist constants $z_\infty\in[\underline{\theta},\theta]$, $y_\infty\in\mathbb{R}^n$, and $\tau_\infty\in[0,T]$ for which

$$\lim_{k\to\infty} z_k=z_\infty,\qquad\lim_{k\to\infty} y_k=y_\infty,\qquad\lim_{k\to\infty}\tau_k=\tau_\infty.$$

Now we set

$$\big(u_k(t,x),v_k(t,x)\big)=\big(u(t+j_kT,x+\mathbf{s}_kT),v(t+j_kT,x+\mathbf{s}_kT)\big).$$

Since both f and g are periodic in t with the period T, $(u_k(t,x),v_k(t,x))$ are the solutions of (4.1) as well. Thanks to the regularities of $\{(u_k,v_k)\}$ with respect to t and x, up to an extraction of a subsequence, $\{(u_k,v_k)\}$ converges uniformly in any compact subset of $\mathbb{R}\times\mathbb{R}^n$ to a solution of (4.1), denoted by $(u_\infty(t,x),v_\infty(t,x))$. Note that $(0,0)\le(u_\infty(t,x),v_\infty(t,x))\le(1,1)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^n$. Moreover, it is easy to see that

$$\begin{aligned}&\big(u_k(t,x),v_k(t,x)\big)\\&\quad=\big(u(t+j_kT,x+\mathbf{s}_kT),v(t+j_kT,x+\mathbf{s}_kT)\big)\\&\quad\le\big(U\big(t,x\cdot\nu-c^*t+T\big(\mathbf{s}_k\cdot\nu-c^*j_k\big)+\bar\eta\big),W\big(t,x\cdot\nu-c^*t+T\big(\mathbf{s}_k\cdot\nu-c^*j_k\big)+\bar\eta\big)\big)\\&\quad=\big(U\big(t,x\cdot\nu-c^*t+\bar\eta\big),W\big(t,x\cdot\nu-c^*t+\bar\eta\big)\big).\end{aligned}$$

By passing the limits in the above inequality, we find

$$\big(u_\infty(t,x),v_\infty(t,x)\big)\le\big(U\big(t,x\cdot\nu-c^*t+\bar\eta\big),W\big(t,x\cdot\nu-c^*t+\bar\eta\big)\big),\quad\forall(t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

In particular, we have

$$\begin{aligned}&U\big(\tau_\infty,y_\infty\cdot\nu-c^*\tau_\infty+\bar\eta\big)-u_\infty(\tau_\infty,y_\infty)\\&\quad=\lim_{k\to\infty}\big[U\big(\tau_k,y_k\cdot\nu-c^*\tau_k+T\big(\mathbf{s}_k\cdot\nu-c^*j_k\big)+\bar\eta\big)-u_k(\tau_k,y_k)\big]\\&\quad=\lim_{k\to\infty}\big[U\big(\tau_k,x_k\cdot\nu-c^*t_k+\bar\eta\big)-u(\tau_k+j_kT,y_k+\mathbf{s}_kT)\big]\\&\quad=\lim_{k\to\infty}\big[U\big(\tau_k+j_kT,x_k\cdot\nu-c^*t_k+\bar\eta\big)-u(\tau_k+j_kT,y_k+\mathbf{s}_kT)\big]\\&\quad=\lim_{k\to\infty}\big[U\big(t_k,x_k\cdot\nu-c^*t_k+\bar\eta\big)-u(t_k,x_k)\big]=0.\end{aligned}$$

In other words, $U(\tau_\infty,y_\infty\cdot\nu-c^*\tau_\infty+\bar\eta)=u_\infty(\tau_\infty,y_\infty)$. Set

$$\big(U^{\bar\eta}\big(t,x\cdot\nu-c^*t\big),W^{\bar\eta}\big(t,x\cdot\nu-c^*t\big)\big)=\big(U\big(t,x\cdot\nu-c^*t+\bar\eta\big),W\big(t,x\cdot\nu-c^*t+\bar\eta\big)\big).$$

Since

$$\begin{aligned}&\left[\int_0^1 f_u\big(t,sU^{\bar\eta}+(1-s)u_\infty,sW^{\bar\eta}+(1-s)v_\infty\big)ds\right]\big(U^{\bar\eta}-u_\infty\big)+\mathbf{k}(t)\cdot\nabla\big(U^{\bar\eta}-u_\infty\big)\\&\quad+d_1(t)\Delta\big(U^{\bar\eta}-u_\infty\big)-\big(U^{\bar\eta}-u_\infty\big)_t\le 0,\end{aligned}$$

it follows from the maximum principle that $U^{\bar\eta}(t,x\cdot\nu-c^*t)=u_\infty(t,x)$ for all $(t,x)\in(-\infty,\tau_\infty]\times\mathbb{R}^n$. On the other hand, by (4.22), we have

$$\big(u_k(t,x),v_k(t,x)\big)\le\left(U\left(t,x\cdot\nu-c^*t+\frac{\bar\eta}{2}\right),W\left(t,x\cdot\nu-c^*t+\frac{\bar\eta}{2}\right)\right)$$

as long as $x\cdot\nu-c^*t\le\underline{\theta}$. By taking the limit, we find that

$$\big(u_\infty(t,x),v_\infty(t,x)\big)\le\left(U\left(t,x\cdot\nu-c^*t+\frac{\bar\eta}{2}\right),W\left(t,x\cdot\nu-c^*t+\frac{\bar\eta}{2}\right)\right)$$

for all $(t,x) \in \{(t,x) \mid x \cdot \nu - c^* t \le \underline{\theta}\}$. This is a contradiction because

$$U\left(t, x \cdot \nu - c^* t + \frac{\bar{\eta}}{2}\right) < U\big(t, x \cdot \nu - c^* t + \bar{\eta}\big)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. The contradiction shows that (4.23) is true if $\bar{\eta} > 0$.

Step 3. In terms of the assumptions, we have

$$\lim_{z\to\infty} \sup_{x\cdot\nu - c^* t \ge z} \big|u(t,x) - 1\big| = 0, \qquad \lim_{z\to\infty} \sup_{x\cdot\nu - c^* t \ge z} \big|v(t,x) - 1\big| = 0.$$

Thus, there exists $\bar{\theta} > \underline{\theta}$ such that

$$\begin{aligned} &\big(u(t,x), v(t,x)\big) \in \big[1-\omega^0, 1\big)^2, \\ &\big(U\big(t, x\cdot\nu - c^* t\big), W\big(t, x\cdot\nu - c^* t\big)\big) \in \big[1-\omega^0, 1\big)^2 \end{aligned} \tag{4.24}$$

whenever $x \cdot \nu - ct \ge \bar{\theta}$, where ω^0 is specified by Proposition A.5 in Appendix A. Since $(U(t,\cdot), W(t,\cdot))$ is uniformly continuous and $(U(\cdot,z), W(\cdot,z))$ is periodic, in view of (4.23), there exists $\hat{\eta} \in [\frac{\bar{\eta}}{2}, \bar{\eta})$ for which

$$\begin{aligned} &\inf_{\underline{\theta} \le x\cdot\nu - c^* t \le \bar{\theta}} \big[U\big(t, x\cdot\nu - c^* t + \hat{\eta}\big) - u(t,x)\big] \ge 0, \\ &\inf_{\underline{\theta} \le x\cdot\nu - c^* t \le \bar{\theta}} \big[W\big(t, x\cdot\nu - c^* t + \hat{\eta}\big) - v(t,x)\big] \ge 0. \end{aligned} \tag{4.25}$$

We next show that

$$\begin{aligned} &\inf_{x\cdot\nu - c^* t \ge \bar{\theta}} \big[U\big(t, x\cdot\nu - c^* t + \hat{\eta}\big) - u(t,x)\big] \ge 0, \\ &\inf_{x\cdot\nu - c^* t \ge \bar{\theta}} \big[W\big(t, x\cdot\nu - c^* t + \hat{\eta}\big) - v(t,x)\big] \ge 0. \end{aligned} \tag{4.26}$$

To this end, set

$$\begin{aligned} u^\delta(t,x) &= U\big(t, x\cdot\nu - c^* t + \hat{\eta}\big) + \delta\psi_1(t) - u(t,x); \\ v^\delta(t,x) &= W\big(t, x\cdot\nu - c^* t + \hat{\eta}\big) + \delta\psi_2(t) - v(t,x), \end{aligned}$$

where ψ_i $(i = 1,2)$ are specified by (H8). We now define

$$\bar{\delta} := \big\{\delta \in [0,\infty) \,\big|\, \big(u^\delta(t,x), v^\delta(t,x)\big) \ge (0,0),\ \forall (t,x) \in \big\{x\cdot\nu - c^* t \ge \bar{\theta}\big\}\big\}.$$

To prove (4.26), it is sufficient to show that $\bar{\delta} = 0$. Assume to the contrary that $\bar{\delta} > 0$. Then we must have either $\inf_{(t,x)\in\{x\cdot\nu - c^* t \ge \bar{\theta}\}} u^{\bar{\delta}} = 0$ or $\inf_{(t,x)\in\{x\cdot\nu - c^* t \ge \bar{\theta}\}} v^{\bar{\delta}} = 0$. To see this, recall that $(u^\delta, v^\delta) \ge (0,0)$ for any $\delta \in [0,\bar{\delta}]$ as long as $x\cdot\nu - c^* t \le \bar{\theta}$. In addition, for any $\delta \in (0,\bar{\delta}]$, we have

$$\lim_{z\to\infty}\inf_{x\cdot\nu-c^*t\ge z}u^\delta(t,x)\ge\delta\min_{t\in\mathbb{R}}\psi_1(t)>0,$$
$$\lim_{z\to\infty}\inf_{x\cdot\nu-c^*t\ge z}v^\delta(t,x)\ge\delta\min_{t\in\mathbb{R}}\psi_2(t)>0. \tag{4.27}$$

Thus, if both $\inf_{(t,x)\in\{x\cdot\nu-c^*t\ge\bar\theta\}}u^{\bar\delta}$ and $\inf_{(t,x)\in\{x\cdot\nu-c^*t\ge\bar\theta\}}v^{\bar\delta}$ are strictly positive, then there is $0\le\delta'<\bar\delta$ for which $(u^{\delta'},v^{\delta'})\ge(0,0)$, which apparently contradicts the definition of $\bar\delta$. Therefore, we must have that $(\inf_{(t,x)\in\{x\cdot\nu-c^*t\ge\bar\theta\}}u^{\bar\delta})(\inf_{(t,x)\in\{x\cdot\nu-c^*t\ge\bar\theta\}}v^{\bar\delta})=0$.

Assume without loss of generality that $\inf_{(t,x)\in\{x\cdot\nu-c^*t\ge\bar\theta\}}v^{\bar\delta}=0$. Then there exist two sequences $\{t_k\}$ and $\{x_k\}$ such that $x_k\cdot\nu-c^*t_k\ge\bar\theta$ and $\lim_{k\to\infty}v^{\bar\delta}(t_k,x_k)=0$. Notice that (4.27) implies that $\{x_k\cdot\nu-c^*t_k\}$ is bounded. We again reencounter three scenarios as shown in Step 2, we retain the same notations used in Step 2 and consider only the case that $\{t_k\}$ is unbounded. Set

$$u_k^{\bar\delta}(t,x)=u^{\bar\delta}(t+j_kT,x+\mathbf{s}_kT)=U\big(t,x\cdot\nu-c^*t+\hat\eta\big)+\bar\delta\psi_1(t)-u(t+j_kT,x+\mathbf{s}_kT),$$
$$v_k^{\bar\delta}(t,x)=v^{\bar\delta}(t+j_kT,x+\mathbf{s}_kT)=W\big(t,x\cdot\nu-c^*t+\hat\eta\big)+\bar\delta\psi_2(t)-v(t+j_kT,x+\mathbf{s}_kT).$$

Note that $(u_k^{\bar\delta},v_k^{\bar\delta})$ are uniformly bounded and nonnegative in $\mathbb{R}\times\mathbb{R}^n$. Thanks to the regularities of $(u_k^{\bar\delta},v_k^{\bar\delta})$ with respect to t and x, without loss of generality, we may assume that $\{u_k^{\bar\delta},v_k^{\bar\delta}\}$ converges uniformly in any compact subset of $\mathbb{R}\times\mathbb{R}^n$ to a function denoted by (u^∞,v^∞). With a slightly abuse of notation, we still denote by (u_∞,v_∞) the limit function of $\{(u(t+j_kT,x+\mathbf{s}_kT),v(t+j_kT,x+\mathbf{s}_kT))\}$. Therefore,

$$u^\infty(t,x)=U\big(t,x\cdot\nu-c^*t+\hat\eta\big)+\bar\delta\psi_1(t)-u_\infty(t,x)$$
$$v^\infty(t,x)=W\big(t,x\cdot\nu-c^*t+\hat\eta\big)+\bar\delta\psi_2(t)-v_\infty(t,x).$$

Note that $(u_k^{\bar\delta}(t,x),v_k^{\bar\delta}(t,x))\ge(\bar\delta\min_t\psi_1(t),\bar\delta\min_t\psi_2(t))$ if $x\cdot\nu-c^*t\le\bar\theta$. This implies that $(u^\infty(t,x),v^\infty(t,x))>(0,0)$ for all $(t,x)\in\{x\cdot\nu-c^*t\le\bar\theta\}$. Since

$$z_\infty=\lim_{k\to\infty}x_k\cdot\nu-c^*t_k=\lim_{k\to\infty}y_k\cdot\nu-c^*\tau_k=y_\infty\cdot\nu-c^*\tau_\infty,$$

it is easy to see that $z_\infty>\bar\theta$ and $v^\infty(\tau_\infty,y_\infty)=0$. Furthermore, by virtue of (4.24), we have

$$\big(u_\infty(t,x),v_\infty(t,x)\big)\in\big[1-\omega^0,1\big]^2,$$
$$\big(U\big(t,x\cdot\nu-c^*t+\hat\eta\big),W\big(t,x\cdot\nu-c^*t+\hat\eta\big)\big)\in\big[1-\omega^0,1\big]^2$$

whenever $x\cdot\nu-c^*t\ge\bar\theta$. Let $(U^{\hat\eta},W^{\hat\eta})=(U(t,x\cdot\nu-c^*t+\hat\eta),W(t,x\cdot\nu-c^*t+\hat\eta))$. In view of the proof of Proposition A.5 in Appendix A, we find that

$$\left[\int_0^1 g_v\big(t,sU^{\hat\eta}+(1-s)u_\infty,sW^{\hat\eta}+(1-s)v_\infty\big)ds\right]\psi_2+\mathbf{l}(t)\nabla\psi_2+d_2(t)\Delta\psi_2-\psi_2'\le 0$$

provided that $x \cdot \nu - c^* t \geq \bar{\theta}$. It then follows that

$$\left[\int_0^1 g_v\big(t, sU^{\hat{\eta}} + (1-s)u_\infty, sW^{\hat{\eta}} + (1-s)v_\infty\big)ds\right]v^\infty + \mathbf{l}(t)\nabla v^\infty + d_2(t)\Delta v^\infty - v_t^\infty \leq 0$$

for all $(t,x) \in \{x \cdot \nu - c^* t > \bar{\theta}\}$. Since $y_\infty \cdot \nu - c^* \tau_\infty > \bar{\theta}$ and $v^\infty(\tau_\infty, y_\infty) = 0$, the maximum principle implies that $v^\infty(t,x) = 0$ for all $(t,x) \in \{t \leq \tau_\infty,\ x \cdot \nu - c^* t \geq \bar{\theta}\}$. This however contradicts the fact that $v^\infty(t,x) > 0$ for any $(t,x) \in \{x \cdot \nu - c^* t = \bar{\theta}\}$. The contradiction shows that $\bar{\delta} = 0$. Namely, (4.26) is true. As $(U(\cdot, \cdot + \frac{\bar{\eta}}{2}), W(\cdot, \cdot + \frac{\bar{\eta}}{2})) \leq (U(\cdot, \cdot + \hat{\eta}), W(\cdot, \cdot + \hat{\eta}))$, from (4.22), (4.25), and (4.26), it follows that

$$\big(u(t,x), v(t,x)\big) \leq \big(U\big(t, x \cdot \nu - c^* t + \hat{\eta}\big), W\big(t, x \cdot \nu - c^* t + \hat{\eta}\big)\big) \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

It obviously contradicts the definition of $\bar{\eta}$. Therefore, we must have $\bar{\eta} = 0$. Consequently,

$$\big(u(t,x), v(t,x)\big) \leq \big(U\big(t, x \cdot \nu - c^* t\big), W\big(t, x \cdot \nu - c^* t\big)\big) \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

Step 4. Define

$$\underline{\eta} := \inf\left\{\eta \in [0,\infty) \;\middle|\; \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \geq \begin{pmatrix} U(t, x \cdot \nu - c^* t - \eta) \\ W(t, x \cdot \nu - c^* t - \eta) \end{pmatrix},\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n\right\}.$$

Clearly, $0 \leq \underline{\eta} \leq -\underline{\omega}$. Moreover, arguing in a similar manner, it can be shown that $\underline{\eta} = 0$, that is,

$$\big(u(t,x), v(t,x)\big) \geq \big(U\big(t, x \cdot \nu - c^* t\big), W\big(t, x \cdot \nu - c^* t\big)\big) \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

Therefore, $(u(t,x), v(t,x)) = (U(t, x \cdot \nu - c^* t), W(t, x \cdot \nu - c^* t))$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. $\square$

We now state our main result in this section.

Theorem 4.10. *Suppose that (H1)–(H8) are satisfied. Let $(u(t,x,u_0), v(t,x,v_0))$ be a solution of (4.1) with initial data (u_0, v_0) such that $(0,0) \lneq (u_0(x), v_0(x)) \lneq (1,1)$. Let $(U,W) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (2.4) with $c \leq c^*$. If $c < c^*$, assume further that all the assumptions of Lemma 4.3 are satisfied. If $c = c^*$, assume that all the assumptions of Lemma 4.7 are satisfied. Then*

$$\lim_{t\to\infty} \big|u(t,x,u_0) - U(t, x \cdot \nu - ct + z_0)\big| + \big|v(t,x,v_0) - W(t, x \cdot \nu - ct + z_0)\big| = 0 \tag{4.28}$$

for some $z_0 \in \mathbb{R}$. In particular, z_0 is the unique number such that

$$\lim_{x\cdot\nu\to-\infty} \frac{U(0, x \cdot \nu + z_0)}{k\varphi_1(0)e^{\lambda_c(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty} \frac{W(0, x \cdot \nu + z_0)}{k\varphi_2(0)e^{\lambda_c(x\cdot\nu)}} = 1, \quad \textit{if } c < c^*,$$

and

$$\lim_{x\cdot\nu\to-\infty} \frac{U(0, x \cdot \nu + z_0)}{k\varphi_1(0)|x \cdot \nu|e^{\lambda_*(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty} \frac{W(0, x \cdot \nu + z_0)}{k\varphi_2(0)|x \cdot \nu|e^{\lambda_*(x\cdot\nu)}} = 1, \quad \textit{if } c = c^*.$$

Proof. We will give a proof for the case that $c = c^*$. We again assume that $z_0 = 0$. Assume to the contrary that (4.28) is not true. Then there exist $\varepsilon > 0$ and a sequence $\{(t_k, x_k)\}$ such that $\lim_{k\to\infty} t_k = \infty$, and

$$\lim_{k\to\infty} \left|u(t_k, x_k, u_0) - U\big(t_k, x_k \cdot \nu - c^* t_k\big)\right| + \left|v(t_k, x_k, v_0) - W\big(t, x_k \cdot \nu - c^* t_k\big)\right| \geq \varepsilon. \quad (4.29)$$

If $z_k = x_k \cdot \nu - c^* t_k$ are bounded, then we revisit the scenario (a) presented in the proof of Lemma 4.9. To derive a contradiction, we proceed with the same notations used before and set

$$\big(u_k(t,x), v_k(t,x)\big) = \big(u(t + j_k T, x + \mathbf{s}_k T, u_0), v(t + j_k T, x + \mathbf{s}_k T, v_0)\big).$$

Clearly, for each k, $(u_k(t,x), v_k(t,x))$ is a solution of (4.1) in $]-j_k T, \infty[\, \times \mathbb{R}^n$ satisfying $(u_k(-j_k T, x), v_k(-j_k T, x)) = (u_0(x + \mathbf{s}_k T), v_0(x + \mathbf{s}_k T))$. Denote again by $(u_\infty(t,x), v_\infty(t,x))$ the solution of (4.1) to which $\{(u_k, v_k)\}$ converges uniformly in any compact set of $\mathbb{R} \times \mathbb{R}^n$. Due to Lemma 4.7, we have

$$U\big(t, x\cdot\nu - c^* t - \sigma^*\big) - \delta^* \Lambda e^{-\beta(t + j_k T)} \leq u_k(t,x) \leq U\big(t, x\cdot\nu - c^* t + \sigma^*\big) + \delta^* \Lambda e^{-\beta(t + j_k T)},$$
$$W\big(t, x\cdot\nu - c^* t - \sigma^*\big) - \delta^* \Lambda e^{-\beta(t + j_k T)} \leq v_k(t,x) \leq W\big(t, x\cdot\nu - c^* t + \sigma^*\big) + \delta^* \Lambda e^{-\beta(t + j_k T)}$$

for all $[t_* - j_k T, \infty) \times \mathbb{R}^n$, where $\Lambda = \max\{\sup_{(t,s)\in\mathbb{R}^2} \xi_*, \sup_{(t,s)\in\mathbb{R}^2} \varsigma_*\}$. It then follows that

$$U\big(t, x\cdot\nu - c^* t - \sigma^*\big) \leq u_\infty(t,x) \leq U\big(t, x\cdot\nu - c^* t + \sigma^*\big),$$
$$W\big(t, x\cdot\nu - c^* t - \sigma^*\big) \leq v_\infty(t,x) \leq W\big(t, x\cdot\nu - c^* t + \sigma^*\big)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$.

Moreover, Lemma 4.8 shows that for each $\eta > 0$, there exist $D_\eta > 0$ and $\theta_\eta \in \mathbb{R}$ such that

$$U\big(t, x\cdot\nu - c^* t - \eta\big) - D_\eta \varphi_1 e^{\lambda_*(x\cdot\nu - c^* t)}\left(1 - \frac{\phi_1}{\varphi_1} e^{\epsilon^*(x\cdot\nu - c^* t)}\right) \leq u_k(t,x),$$
$$W\big(t, x\cdot\nu - c^* t - \eta\big) - D_\eta \varphi_2 e^{\lambda_*(x\cdot\nu - c^* t)}\left(1 - \frac{\phi_2}{\varphi_2} e^{\epsilon^*(x\cdot\nu - c^* t)}\right) \leq v_k(t,x)$$

for all $(t,x) \in \{t \geq -j_k T,\ x\cdot\nu - c^* t \leq \theta_\eta\}$, and

$$U\big(t, x\cdot\nu - c^* t + \eta\big) + D_\eta \varphi_1 e^{\lambda_*(x\cdot\nu - c^* t)}\left(1 - \frac{\phi_1}{\varphi_1} e^{\epsilon^*(x\cdot\nu - c^* t)}\right) \geq u_k(t,x),$$
$$W\big(t, x\cdot\nu - c^* t + \eta\big) + D_\eta \varphi_2 e^{\lambda_*(x\cdot\nu - c^* t)}\left(1 - \frac{\phi_2}{\varphi_2} e^{\epsilon^*(x\cdot\nu - c^* t)}\right) \geq v_k(t,x)$$

for all $(t,x) \in \{t \geq -j_k T,\ x\cdot\nu - c^* t \leq \theta_\eta\}$. By taking the limits in the above inequalities, we obtain that

$$\big(U^-(t, z-\eta), W^-(t, z-\eta)\big) \leq \big(u_\infty(t,x), v_\infty(t,x)\big) \leq \big(U^+(t, z+\eta), W^+(t, z+\eta)\big)$$

for all $z = x\cdot\nu - c^* t \leq \theta_\eta$, where

$$\begin{aligned}&\big(U^{\pm}(t,z\pm\eta),W^{\pm}(t,z\pm\eta)\big)\\&\quad=\bigg(U(t,z\pm\eta)\pm D_\eta\varphi_1e^{\lambda_*z}\bigg(1-\frac{\phi_1}{\varphi_1}e^{\epsilon^*z}\bigg),W(t,z\pm\eta)\pm D_\eta\varphi_2e^{\lambda_*z}\bigg(1-\frac{\phi_2}{\varphi_2}e^{\epsilon^*z}\bigg)\bigg)\end{aligned}$$

and $z=x\cdot\nu-c^*t$. Consequently, it follows from Lemma 4.9 that

$$\big(u_\infty(t,x),v_\infty(t,x)\big)=\big(U\big(t,x\cdot\nu-c^*t\big),W\big(t,x\cdot\nu-c^*t\big)\big)\quad\text{for all }(t,x)\in\mathbb{R}\times\mathbb{R}^n.\tag{4.30}$$

On the other hand, we have

$$\big(u_\infty(\tau_\infty,y_\infty),v_\infty(\tau_\infty,y_\infty)\big)=\lim_{k\to\infty}\big(u(t_k,x_k,u_0),v(t_k,x_k,v_0)\big)$$

and

$$\big(U\big(\tau_\infty,y_\infty\cdot\nu-c^*\tau_\infty\big),W\big(\tau_k,y_\infty\cdot\nu-c^*\tau_\infty\big)\big)=\lim_{k\to\infty}\big(U\big(t_k,x_k-c^*t_k\big),W\big(t_k,x_k-c^*t_k\big)\big).$$

Hence, it follows from (4.29) that

$$\big|u_\infty(\tau_\infty,y_\infty)-U\big(\tau_\infty,y_\infty\cdot\nu-c^*\tau_\infty\big)\big|+\big|v_\infty(\tau_\infty,y_\infty)-W\big(\tau_\infty,y_\infty\cdot\nu-c^*\tau_\infty\big)\big|\geq\varepsilon,$$

which contradicts (4.30). Hence $\{z_k\}$ has to be unbounded. Recall $z_k=x_k\cdot\nu-c^*t_k$. If $\lim_{k\to\infty}z_k=-\infty$, then it follows from Lemma 4.7 that

$$\lim_{k\to\infty}\big(u(t_k,x_k,u_0),v(t_k,x_k,v_0)\big)=\lim_{k\to\infty}\big(U\big(t_k,x_k\cdot\nu-c^*t_k\big),W\big(t,x_k\cdot\nu-c^*t_k\big)\big)=(0,0),$$

while, if $\lim_{k\to\infty}z_k=\infty$, then Lemma 4.7 yields that

$$\lim_{k\to\infty}\big(u(t_k,x_k,u_0),v(t_k,x_k,v_0)\big)=\lim_{k\to\infty}\big(U\big(t_k,x_k\cdot\nu-c^*t_k\big),W\big(t,x_k\cdot\nu-c^*t_k\big)\big)=(1,1).$$

Both of them contradict (4.29). Therefore (4.28) follows. The proof is completed. □

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Appendix A

In this appendix we first present a few lemmas and propositions used in Section 3.

Lemma A.1. *Assume that (H1) and (H2) are satisfied. Let* $(\underline{u},\underline{v})\in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ *be a regular sub-solution of* (2.2) *such that* $(0,0)\leq(\underline{u},\underline{v})<(1,1)$. *Assume that* $(\underline{u}(\cdot,z),\underline{v}(\cdot,z))$ *is* T*-periodic. Let* $(\bar{u},\bar{v})\in C^\theta(\mathbb{R}\times\mathbb{R})$ $(\theta\in\,]0,1[)$ *be an irregular super-solution of* (2.2) *such that* $(\bar{u},\bar{v})=\min\{(\overline{w}_1,\overline{w}_2),(1,1)\}$, *where* $(\overline{w}_1,\overline{w}_2)$ *is a regular super-solution of* (2.2) *in* $\mathbb{R}\times\,]-\infty,\widehat{z}[$

with $\widehat{z} \le \infty$; $(\overline{w}_1(\cdot, z), \overline{w}_2(\cdot, z))$ *is* T*-periodic, and* $(\overline{w}_1(t,\cdot), \overline{w}_2(t,\cdot))$ *is nondecreasing. In addition there exists* $\bar{\sigma} < \widehat{z}$ *such that* $(\bar{u}(t,z), \bar{v}(t,z)) = (1,1)$ *for any* $(t,z) \in \mathbb{R} \times [\bar{\sigma}, +\infty)$. *Here* $(\bar{u}(t,z), \bar{v}(t,z)) := (1,1)$ *for all* $(t,z) \in \mathbb{R} \times [\widehat{z}, \infty)$ *provided that* $\widehat{z} < \infty$. *If there exists* $\sigma < \bar{\sigma}$ *such that* $(\underline{u}(t,\sigma), \underline{v}(t,\sigma)) < (\bar{u}(t,\sigma), \bar{v}(t,\sigma))$ *for any* $t \in \mathbb{R}$, *then* $(\underline{u}, \underline{v}) < (\bar{u}, \bar{v})$ *for any* $(t,z) \in \mathbb{R} \times [\sigma, +\infty)$.

Proof. The proof is very similar to that of Lemma 3.1 of [37], we thus omit it and refer the readers to [37] for details. □

Lemma A.2. *Assume that (H1) and (H2) are satisfied. Assume that* $(\bar{u}, \bar{v})$ *is a regular super-solution of (2.2) in* $\mathbb{R} \times \mathbb{R}$ *and* $(\bar{u}(\cdot, z), \bar{v}(\cdot, z))$ *is* T*-periodic and satisfies that*

$$\liminf_{z\to\infty}\Big\{\inf_{t\in\mathbb{R}} \bar{u}(t,z)\Big\} \ge 1 \quad \textit{and} \quad \liminf_{z\to\infty}\Big\{\inf_{t\in\mathbb{R}} \bar{v}(t,z)\Big\} \ge 1.$$

Let $(\underline{u}(t,z), \underline{v}(t,z)) \in C_b^{1,2}(\mathbb{R} \times (-\infty, z_0])$ *be a regular sub-solution of (2.2) in* $\mathbb{R} \times (-\infty, z_0)$. *Moreover, assume that* $\sup_{(t,z)\in\mathbb{R}\times(-\infty,z_0]}(\underline{u}, \underline{v}) < (1,1)$. *In particular,* $(\underline{u}, \underline{v})$ *is* T*-periodic in* t *and* $(\underline{u}(t,z_0), \underline{v}(t,z_0)) \le (0,0)$ *for all* $t \in \mathbb{R}$, *and for each* $t \in \mathbb{R}$, $\underline{u}(t,z) > 0$ *for all* $z \in (-\infty, z')$ *provided that* $u(t,z') \ge 0$. *If there exists* $\sigma \in]-\infty, z_0[$ *such that* $(\underline{u}(t,\sigma), \underline{v}(t,\sigma)) < (\bar{u}(t,\sigma), \bar{v}(t,\sigma))$ *for all* t *and* $(\bar{u}(t,\sigma), \bar{v}(t,\sigma)) \le (\bar{u}(t,z), \bar{v}(t,z))$ *for all* $(t,z) \in \mathbb{R} \times [\sigma, \infty)$, *then* $(\underline{u}, \underline{v}) < (\bar{u}, \bar{v})$ *for all* $(t,z) \in \mathbb{R} \times [\sigma, z_0]$.

Proof. We argue by contradiction. Define

$$\vartheta_1 = \inf\big\{\vartheta > 0 \,\big|\, \underline{u}(t,z) \le \bar{u}(t, z+\vartheta) \text{ for all } (t,z) \in \mathbb{R} \times [\sigma, z_0]\big\}.$$
$$\vartheta_2 = \inf\big\{\vartheta > 0 \,\big|\, \underline{v}(t,z) \le \bar{v}(t, z+\vartheta) \text{ for all } (t,z) \in \mathbb{R} \times [\sigma, z_0]\big\}.$$

Since $(\liminf_{z\to\infty}\{\inf_{t\in[0,T]} \bar{u}\}, \liminf_{z\to\infty}\{\inf_{t\in[0,T]} \bar{v}\}) \ge (1,1)$ and $\sup_{(t,z)\in\mathbb{R}\times(-\infty,z_0]}(\underline{u}, \underline{v}) < 1$, both ϑ_1 and ϑ_2 are bounded. Let $\vartheta^* = \max\{\vartheta_1, \vartheta_2\}$. Assume without loss of generality that $\vartheta^* = \vartheta_1$. We next show that $\vartheta^* = 0$. Suppose that this is not true, then there exists a point $(t^*, z^*) \in \mathbb{R} \times [\sigma, z_0)$ such that $\underline{u}(t^*, z^*) = \bar{u}(t^*, z^* + \vartheta^*)$ and $\underline{v}(t^*, z^*) \le \bar{v}(t^*, z^* + \vartheta^*)$. By virtue of assumption, we see that

$$\underline{u}(\cdot, \sigma) < \bar{u}(\cdot, \sigma) \le \bar{u}\big(\cdot, \sigma + \vartheta^*\big). \tag{A.1}$$

Hence, $z^* > \sigma$. In addition, it follows from the assumption that $\underline{u}(t^*, z) > 0$ for all $z \in [\sigma, z^*]$. Due to the continuity of u with respect to (t,z), there exists $\varepsilon > 0$ with $\varepsilon \le z_0 - z^*$ such that $\underline{u}(t,z) \ge 0$ for all $(t,z) \in [t^* - \varepsilon, t^* + \varepsilon] \times [\sigma, z^* + \varepsilon]$. Let $w^*(t,z) = \bar{u}(t, z+\vartheta^*) - \underline{u}(t,z)$. Notice that $w^* \ge 0$ for all $(t,z) \in \mathbb{R} \times [\sigma, z_0]$ and $f_v(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) \ge 0$ for all $(s,t,z) \in [0,1] \times [t^* - \varepsilon, t^* + \varepsilon] \times [\sigma, z^* + \varepsilon]$ in terms of (H4). Then

$$\beta(t,z)w^* + \big[c + k(t)\big]w_z^* + d_1(t)w_{zz}^* - w_t^* \le 0 \quad \text{for all } (t,z) \in \big(t^* - \varepsilon, t^* + \varepsilon\big) \times \big(\sigma, z^* + \varepsilon\big),$$

where $\beta(t,z) = [\int_0^1 f_u(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v})ds]$. Therefore, the strong maximum principle implies that $\underline{u}(t^*, z) = \bar{u}(t^*, z+\vartheta^*)$ for any $z \in [\sigma, z^*]$, which contradicts (A.1). Thus, $\vartheta^* = 0$. The proof is completed. □

Proposition A.3. *Let D be an open connected domain of $\mathbb{R}^+ \times \mathbb{R}^n$ such that $\overline{D} \subseteq \{(t,x) \mid t \ge t^*,\ x \in \mathbb{R}^n\}$, where $t^* \ge 0$, and $\overline{D} \cap \{(t,x) \mid t = t^*,\ x \in \mathbb{R}^n\} \neq \varnothing$. $\overline{D} \cap H_s^\tau \neq \varnothing$ whenever $s \ge s_*$ for certain $s_* > 0$, where $H_s^\tau = \{(t,x) \mid t \ge t^*,\ |x| \le s\}$. Let $(\underline{w}_1, \underline{w}_2) \in C^{1,2}(D) \cap C_b(\overline{D})$ and $(\overline{w}_1, \overline{w}_2) \in C^{1,2}(D) \cap C_b(\overline{D})$ be respectively the sub-solution and super-solution of*

$$\begin{cases} \dfrac{\partial w_1}{\partial t} = \displaystyle\sum_{i,j=1}^n a_{i,j}^1(t,x)\frac{\partial^2 w_1}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^1(t,x)\frac{\partial w_1}{\partial x_i} + f(t,w_1,w_2), \\ \dfrac{\partial w_2}{\partial t} = \displaystyle\sum_{i,j=1}^n a_{i,j}^2(t,x)\frac{\partial^2 w_2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^2(t,x)\frac{\partial w_2}{\partial x_i} + g(t,w_1,w_2) \end{cases} \tag{A.2}$$

in D, where $a_{i,j}^k$ and $b_i^k \in C_b(\overline{D})$ $(k = 1,2)$, and there is $\alpha_0 > 0$ such that $a_{i,j}^k(t,x)\xi_i\xi_j \ge \alpha_0 \sum_{i=1}^n \xi_i^2$ for any n-tuples of real numbers $(\xi_1, \xi_2, \cdots, \xi_n)$. Moreover, for each closed and bounded interval $I \subset \mathbb{R}$, $f, g \in C_b^{0,1}(\mathbb{R} \times I^2)$. In particular, $f_{w_2} \ge 0$ in $\mathbb{R} \times [0,1] \times \mathbb{R}$ and $g_{w_1} \ge 0$ in $\mathbb{R} \times \mathbb{R} \times [0,1]$. Suppose that $(\underline{w}_1(t,x), \underline{w}_2(t,x)) \le (1,1)$ and $(\overline{w}_1(t,x), \overline{w}_2(t,x)) \ge (0,0)$ for all $(t,x) \in \overline{D}$. Let $(w_1, w_2) \in C^{1,2}(D) \cap C_b(\overline{D})$ be a solution of (A.2) such that $(0,0) \le (w_1(t,x), w_2(t,x)) \le (1,1)$ for all $(t,x) \in \overline{D}$. Assume that $(\underline{w}_1, \underline{w}_2) \le (w_1, w_2) \le (\overline{w}_1, \overline{w}_2)$ for all $(t,x) \in \partial D$. Then

$$\big(\underline{w}_1(t,x), \underline{w}_2(t,x)\big) \le \big(w_1(t,x), w_2(t,x)\big) \le \big(\overline{w}_1(t,x), \overline{w}_2(t,x)\big) \quad \textit{for all } (t,x) \in \overline{D}. \tag{A.3}$$

Proof. We present a sketch as the proof is similar to that of Lemma 2.4 of [37]. This lemma will be used in several places. We only prove the last inequality of (A.3) while the other case can be proved similarly. Set

$$\iota = \max_{k=1,2}\left(\sum_{i=1}^n \big|b_i^k\big|_\infty^2\right)^{\frac{1}{2}}, \qquad m^+ = \big|w_1^* - \overline{w}_1\big|_\infty + \big|w_2^* - \overline{w}_2\big|_\infty,$$

$$M^+ = \big|w_1^*\big|_\infty + \big|w_2^*\big|_\infty + |\overline{w}_1|_\infty + |\overline{w}_2|_\infty,$$

$$\vartheta = \max_{k=1,2} 2\sum_{i,j=1}^n \big|a_{ij}^k\big|_\infty, \qquad \omega = \max_{(t,u,v)\in\Delta} 2\big(1 + |f_u| + |f_v| + |g_u| + |g_v|\big),$$

$$\zeta(t,x,s) = \frac{m^+ e^{\omega t}}{\iota^2 + s^2}\big(|x|^2 + \iota^2 + \vartheta t\big),$$

where $\Delta = \mathbb{R} \times [-M^+, M^+] \times [-M^+, M^+]$. Let $s \ge s_*$ and write

$$w_1^s(t,x) = w_1^*(t,x) - \overline{w}_1(t,x) - \zeta(t,x,s), \qquad w_2^s(t,x) = w_2^*(t,x) - \overline{w}_2(t,x) - \zeta(t,x,s).$$

A straightforward computation yields that

$$L_1 w_1^s = \sum_{i,j=1}^n a_{i,j}^1(t,x)\frac{\partial^2 w_1^s}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^1(t,x)\frac{\partial w_1^s}{\partial x_i} - \frac{\partial w_1^s}{\partial t} > -f_1 w_1^s - f_2 w_2^s,$$

where

$$f_1 = \int_0^1 f_u\big(t, \tau\overline{w}_1 + (1-\tau)w_1^*, \tau\overline{w}_2 + (1-\tau)w_2^*\big)d\tau,$$

$$f_2 = \int_0^1 f_v\big(t, \tau\overline{w}_1 + (1-\tau)w_1^*, \tau\overline{w}_2 + (1-\tau)w_2^*\big)d\tau.$$

Likewise, we have

$$L_2 w_2^s = \sum_{i,j=1}^n a_{i,j}^2(t,x)\frac{\partial^2 w_2^s}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^2(t,x)\frac{\partial w_2^s}{\partial x_i} - \frac{\partial w_2^s}{\partial t} > -g_1 w_1^s - g_2 w_2^s.$$

Now by employing the argument similar to that of Lemma 2.1 of Lieberman [26], we can show that $(w_1^s, w_2^s) < (0,0)$ for all $(t,x) \in \overline{D} \cap \{(t,x) \mid t \geq t^*,\ |x| \leq s\}$. Since (w_1^s, w_2^s) converges uniformly to $(w_1^* - \overline{w}_1, w_2^* - \overline{w}_2)$ in any compact sets of $\mathbb{R}^+ \times \mathbb{R}$ contained in $\overline{D}$ as $s \to \infty$, we infer that $(w_1^*, w_2^*) \leq (\overline{w}_1, \overline{w}_2)$ for all (t,x). The proof is completed. □

Next we prove a result that was used in the proof of Lemma 4.7.

Proposition A.4. *Suppose that (H1)–(H8) are satisfied. Assume that*

$$\lim_{x\cdot\nu\to-\infty}\frac{u_0(x)}{k\varphi_1(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty}\frac{v_0(x)}{k\varphi_2(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}} = 1$$

for some positive constant k. Here $\iota = 0$ if $c < c^*$, and $\iota = 1$ provided that $c = c^*$. Let $(u(t,x,u_0), v(t,x,v_0))$ be the solution of (4.1) with $(u(0,x,u_0), v(0,x,v_0)) = (u_0, v_0)$, which satisfies $(0,0) \lneqq (u_0, v_0) \lneqq (1,1)$. Let $I \subset [0,+\infty)$ be any compact subinterval. Then there exists $z_0 \in \mathbb{R}$ such that

$$\lim_{x\cdot\nu\to-\infty}\frac{|u(t,x,u_0) - U(t, x\cdot\nu - ct + z_0)|}{U(t, x\cdot\nu - ct + z_0)} = 0,$$
$$\lim_{x\cdot\nu\to-\infty}\frac{|v(t,x,v_0) - W(t, x\cdot\nu - ct + z_0)|}{W(t, x\cdot\nu - ct + z_0)} = 0$$

uniformly in $t \in I$, where $z_0 \in \mathbb{R}$ is the unique number such that

$$\lim_{x\cdot\nu\to-\infty}\frac{U(0, x\cdot\nu + z_0)}{k\varphi_1(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty}\frac{W(0, x\cdot\nu + z_0)}{k\varphi_2(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}} = 1.$$

Proof. In light of Theorems 3.6 and 3.10, we can fix $z_0 \in \mathbb{R}$ such that

$$\lim_{x\cdot\nu\to-\infty}\frac{U(0, x\cdot\nu + z_0)}{k\varphi_1(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}} = 1, \qquad \lim_{x\cdot\nu\to-\infty}\frac{W(0, x\cdot\nu + z_0)}{k\varphi_2(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}} = 1.$$

It is easy to see that z_0 is uniquely determined by k. Once again, we will assume without loss of generality that $z_0 = 0$ throughout the proof. Let $\rho(r) \in C^3(\mathbb{R})$ be a real positive function with the following properties: (i) $|\rho(r)| + |\rho'(r)| \le C_1 e^{-\delta|r|}$ for certain positive constants C_1 and δ; (ii) $|\frac{\rho'''(r)}{\rho(r)}| + |\frac{\rho''(r)}{\rho(r)}| + |\frac{\rho'(r)}{\rho(r)}| \le C_2$ for some positive constant C_2. By rescaling, we may assume that $\rho(0) = 1$ and $\delta > 2\lambda_*$. Such a function can be easily constructed, for instance, $\rho(r) = \frac{1}{\cosh(\delta r)}$ has the desired properties.

Now we write $\rho^x(y) = \rho(y \cdot \nu - x \cdot \nu)$ and set

$$\begin{aligned}\widehat{u}(t, y) &= \rho^x(y)\big[u(t, y, u_0) - U(t, y \cdot \nu - ct)\big],\\ \widehat{v}(t, y) &= \rho^x(y)\big[v(t, y, v_0) - W(t, y \cdot \nu - ct)\big].\end{aligned}$$

Then

$$\begin{aligned}&\partial_t\widehat{u} - d_1(t)\Delta_y\widehat{u} - \mathbf{k}(t) \cdot \nabla_y\widehat{u}\\ &\quad= \rho^x(y)\big\{\partial_t[u - U] - d_1(t)\Delta_y[u - U] - \mathbf{k}(t) \cdot \nabla_y[u - U]\big\} - d_1(t)\Delta_y\rho^x(y)[u - U]\\ &\qquad - 2d_1(t)\nabla_y\rho^x(y) \cdot \nabla_y[u - U] - \mathbf{k}(t) \cdot \nabla_y\rho^x(y)[u - U]\\ &\quad= \rho^x(y)\big[f(t, u, v) - f(t, U, W)\big] - \frac{d_1(t)\Delta_y\rho^x(y)}{\rho^x(y)}\widehat{u} - \frac{2d_1(t)}{\rho^x(y)}\nabla_y\rho^x(y) \cdot \nabla_y\big[\rho^x(y)(u - U)\big]\\ &\qquad + \frac{2d_1(t)\nabla_y\rho^x(y) \cdot \nabla_y\rho^x(y)}{\rho^x(y)}\big[\rho^x(y)(u - U)\big] - \frac{\mathbf{k}(t) \cdot \nabla_y\rho^x(y)}{\rho^x(y)}\big[\rho^x(y)(u - U)\big],\end{aligned}$$

where $\Delta_y := \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$, and $\nabla_y := (\frac{\partial}{\partial y_i}, \cdots, \frac{\partial}{\partial y_n})$. Define

$$\begin{aligned}(L_1 w)(t, y) := \partial_t w - d_1(t)\Delta_y w &- \left[\mathbf{k}(t) - \frac{2d_1(t)}{\rho^x(y)}\nabla_y\rho^x(y)\right] \cdot \nabla_y w\\ &- \left[\frac{2d_1(t)\nabla_y\rho^x(y) \cdot \nabla_y\rho^x(y) - \mathbf{k}(t) \cdot \nabla_y\rho^x(y) - d_1(t)\Delta_y\rho^x(y)}{\rho^x(y)}\right]w.\end{aligned}$$

Then, we find that

$$\begin{aligned}L_1\widehat{u} = &\left\{\int_0^1 f_u\big(t, su + (1 - s)U, sv + (1 - s)W\big)ds\right\}\widehat{u}\\ &+ \left\{\int_0^1 f_v\big(t, su + (1 - s)U, sv + (1 - s)W\big)ds\right\}\widehat{v}.\end{aligned}$$

Likewise, we have

$$\begin{aligned}L_2\widehat{v} = &\left\{\int_0^1 g_u\big(t, su + (1 - s)U, sv + (1 - s)W\big)ds\right\}\widehat{u}\\ &+ \left\{\int_0^1 g_v\big(t, su + (1 - s)U, sv + (1 - s)W\big)ds\right\}\widehat{v},\end{aligned}$$

where

$$(L_2 w)(t,y) := \partial_t w - d_2(t)\Delta_y w - \left[\mathbf{l}(t) - \frac{2d_2(t)}{\rho^x(y)}\nabla_y \rho^x(y)\right]\cdot \nabla_y w \\ - \left[\frac{2d_2(t)\nabla_y\rho^x(y)\cdot\nabla_y\rho^x(y) - \mathbf{l}(t)\cdot\nabla_y\rho^x(y) - d_2(t)\Delta_y\rho^x(y)}{\rho^x(y)}\right]w.$$

By the variation of constants formula and Gronwall's inequality, we obtain that

$$\left|\widehat{u}(t,y)\right|_{L_\infty(\mathbb{R}^n)} \le Ce^{Kt}\left[\left|\widehat{u}(0,y)\right|_{L_\infty(\mathbb{R}^n)} + \left|\widehat{v}(0,y)\right|_{L_\infty(\mathbb{R}^n)}\right], \quad t\in I$$

for certain positive constants C and K, which depend only upon $d_i(t)$ $(i=1,2)$, $\mathbf{k}(t)$; $\mathbf{l}(t)$, C_1, C_2, and n (see Theorem 3.5 of Garroni and Menaldi [13] and Lunardi [27]). Without loss of generality, we may assume that $I\subseteq[0,T]$. Thus,

$$\begin{aligned}&\left|u(t,x,u_0) - U(t,x\cdot\nu - ct)\right| \\ &\quad\le Ce^{KT}\sup_{|(y-x)\cdot\nu|\le\frac{|x\cdot\nu|}{2}}\left\{\left|\rho^x(y)\left[u_0(y) - U(0,y\cdot\nu)\right]\right| + \left|\rho^x(y)\left[v_0(y) - W(0,y\cdot\nu)\right]\right|\right\} \\ &\qquad + Ce^{KT}\sup_{|(y-x)\cdot\nu|\ge\frac{|x\cdot\nu|}{2}}\left\{\left|\rho^x(y)\left[u_0(y) - U(0,y\cdot\nu)\right]\right| + \left|\rho^x(y)\left[v_0(y) - W(0,y\cdot\nu)\right]\right|\right\}.\end{aligned}$$

Notice that

$$\left|\rho^x(y)\left[u_0(y) - U(0,y\cdot\nu)\right]\right| + \left|\rho^x(y)\left[v_0(y) - W(0,y\cdot\nu)\right]\right| \le 4C_1 e^{-\frac{\delta|x\cdot\nu|}{2}} \quad \text{whenever} \\ \left|(y-x)\cdot\nu\right| \ge \frac{|x\cdot\nu|}{2}.$$

Moreover, if $y\in\{s\in\mathbb{R}^n : |(s-x)\cdot\nu| \le \frac{|x\cdot\nu|}{2}\}$, then

$$\begin{aligned}\left|\rho^x(y)\left[u_0(y) - U(0,y\cdot\nu)\right]\right| &\le C_1 e^{-\delta|(y-x)\cdot\nu|}\left|U(0,y\cdot\nu)\left(1 - \frac{u_0(y)}{U(0,y\cdot\nu)}\right)\right| \\ &\le C_1 C' e^{-\delta|(y-x)\cdot\nu|}|y\cdot\nu|^\iota e^{\lambda_c(y\cdot\nu)}\left|1 - \frac{u_0(y)}{U(0,y\cdot\nu)}\right| \\ &\le \frac{3}{2}C_1 C' e^{-\delta|(y-x)\cdot\nu|}e^{\lambda_c|(y-x)\cdot\nu|}|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}\left|1 - \frac{u_0(y)}{U(0,y\cdot\nu)}\right| \\ &\le \frac{3}{2}C_1 C'|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}\left|1 - \frac{u_0(y)}{U(0,y\cdot\nu)}\right|.\end{aligned}$$

Here $C'>0$ is a constant and we used the fact that $U(0,x\cdot\nu)\sim\varphi_1(0)|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}$ as $x\cdot\nu\to-\infty$.

Similarly, we have

$$\left|\rho^x(y)\left[v_0(y) - W(0,y\cdot\nu)\right]\right| \le \frac{3}{2}C_1 C'|x\cdot\nu|^\iota e^{\lambda_c(x\cdot\nu)}\left|1 - \frac{v_0(y)}{W(0,y\cdot\nu)}\right|.$$

Consequently, for each $t \in I$, if $x \cdot \nu < 0$ and $|x \cdot \nu|$ is sufficiently large, it follows that

$$\begin{aligned}
&\left|u(t,x,u_0) - U(t, x\cdot\nu - ct)\right| \\
&\quad \le \widehat{C}|x\cdot\nu|^{\iota} e^{\lambda_c(x\cdot\nu)} \sup_{|(y-x)\cdot\nu| \le \frac{|x\cdot\nu|}{2}} \left\{\left|1 - \frac{u_0(y)}{U(0, y\cdot\nu)}\right| + \left|1 - \frac{v_0(y)}{W(0, y\cdot\nu)}\right|\right\} + \widehat{C}e^{-\frac{\delta|x\cdot\nu|}{2}},
\end{aligned}$$

where $\widehat{C}$ is a positive constant that depends on C, C_1, K, and T. As $y\cdot\nu \le \frac{|x\cdot\nu|}{2} + x\cdot\nu \le \frac{x\cdot\nu}{2}$

$$\lim_{y\cdot\nu\to-\infty} \left\{\left|1 - \frac{u_0(y)}{U(0, y\cdot\nu)}\right| + \left|1 - \frac{v_0(y)}{W(0, y\cdot\nu)}\right|\right\} = 0,$$

we readily infer that

$$\lim_{x\cdot\nu\to-\infty} \frac{|u(t,x,u_0) - U(t, x\cdot\nu - ct)|}{U(t, x\cdot\nu - ct)} = 0 \quad \text{uniformly in } t \in I.$$

Likewise, we have

$$\lim_{x\cdot\nu\to-\infty} \frac{|v(t,x,v_0) - W(t, x\cdot\nu - ct)|}{W(t, x\cdot\nu - ct)} = 0 \quad \text{uniformly in } t \in I.$$

The proof is completed. $\square$

Finally we prove a result that was used in the proof of Lemma 4.9.

Proposition A.5. *Suppose that (H1), (H2) and (H8) are satisfied. Let $(\bar u, \bar v)$ and $(\underline{u}, \underline{v}) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R})$ be respectively the regular super-solution and sub-solution of (2.2). In particular, both $(\bar u, \bar v)$ and $(\underline{u}, \underline{v})$ are T-periodic in t, and $\liminf_{z\to\infty}\{\inf_{t\in[0,T]}(\bar u - \underline{u})\} \ge 0$, $\liminf_{z\to\infty}\{\inf_{t\in[0,T]}(\bar v - \underline{v})\} \ge 0$. Let*

$$\begin{aligned}
\omega^* := \sup\Big\{\omega \,\Big|\, &\left|f_u(t,\cdot,\cdot) - f_u(t,1,1)\right| + \left|f_v(t,\cdot,\cdot) - f_v(t,1,1)\right| \le \frac{\theta^+|\mu^+|}{2}, \\
&\forall (t,\cdot,\cdot) \in \mathbb{R}\times[1-\omega, 1+\omega]^2\Big\} \\
\omega_* := \sup\Big\{\omega \,\Big|\, &\left|g_u(t,\cdot,\cdot) - g_u(t,1,1)\right| + \left|g_v(t,\cdot,\cdot) - g_v(t,1,1)\right| \le \frac{\theta^+|\mu^+|}{2}, \\
&\forall (t,\cdot,\cdot) \in \mathbb{R}\times[1-\omega, 1+\omega]^2\Big\},
\end{aligned}$$

where $\omega \ge 0$ and $\theta^+ = \frac{\min\{\min_t \psi_1, \min_t \psi_1\}}{\|\psi_1\| + \|\psi_2\|}$. If there exists $z' \in \mathbb{R}$ such that

$$\big(\bar u(t,z), \bar v(t,z)\big) \in \left[1-\omega^0, 1\right]^2 \quad \textit{and} \quad \big(\underline{u}(t,z), \underline{v}(t,z)\big) \in \left[1-\omega^0, 1\right]^2$$

for all $(t,z) \in \mathbb{R}\times[z',\infty)$, and $(\bar u(t,z'), \bar v(t,z')) \ge (\underline{u}(t,z'), \underline{v}(t,z'))$ for all $t \in \mathbb{R}$, where $\omega^0 = \min\{\omega^*, \omega_*\}$, then $(\bar u(t,z), \bar v(t,z)) \ge (\underline{u}(t,z), \underline{v}(t,z))$ for all $(t,z) \in \mathbb{R}\times[z', +\infty)$.

Proof. The proof is similar to that of Proposition 3.9 of [37]. Since it was used in several places, we give a detailed proof. As both $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are T-periodic in t, it suffices to prove that

$$\inf_{(t,z)\in[0,2T]\times[z',+\infty)} \{\bar{u} - \underline{u}\} \geq 0 \quad \text{and} \quad \inf_{(t,z)\in[0,2T]\times[z',+\infty)} \{\bar{v} - \underline{v}\} \geq 0. \tag{A.4}$$

Let

$$u^\tau(t,z) = \bar{u}(t,z) - \underline{u}(t,z) + \tau\psi_1(t), \qquad v^\tau(t,z) = \bar{v}(t,z) - \underline{v}(t,z) + \tau\psi_2(t).$$

Since both $\bar{u} - \underline{u}$ and $\bar{v} - \underline{v}$ are bounded, there exists $M > 0$ such that $(u^\tau(t,z), v^\tau(t,z)) \geq (0,0)$ for all $(t,z) \in [0,2T] \times [z', +\infty)$ as long as $\tau \geq M$. Now define

$$\tau^* = \inf\{\tau \in [0,\infty) \mid \big(u^\tau(t,z), v^\tau(t,z)\big) \geq (0,0) \text{ for all } (t,z) \in [0,2T] \times \big[z', +\infty\big)\}.$$

Notice that τ^* is bounded. To complete the proof, it suffices to show that $\tau^* = 0$.

Assume to the contrary that this is not true. Then it is easy to see that

$$\text{either} \inf_{(t,z)\in[0,2T]\times[z',+\infty)} u^{\tau^*}(t,z) = 0 \quad \text{or} \inf_{(t,z)\in[0,2T]\times[z',+\infty)} v^{\tau^*}(t,z) = 0.$$

Assume without loss of generality that $\inf_{(t,z)\in[0,2T]\times[z',+\infty)} v^{\tau^*} = 0$. Due to that fact that $\liminf_{z\to\infty}\{\inf_{t\in[0,2T]} v^{\tau^*}\} \geq \tau^* \min_t \varphi_2 > 0$, there exists $(t^*, z^*) \in (0,2T) \times (z', \infty)$ such that $v^{\tau^*}(t^*, z^*) = 0$. On the other hand, since

$$\begin{aligned}
&\tau^*\Bigg\{\big[l(t)+c\big](\psi_2)_z + d_2(t)(\psi_2)_{zz} - (\psi_2)_t + \Bigg[\int_0^1 g_u\big(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}\big)ds\Bigg]\psi_1 \\
&\quad + \Bigg[\int_0^1 g_v\big(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}\big)ds\Bigg]\psi_2\Bigg\} \\
&= \tau^*\Bigg\{\mu^+\psi_2 + \Bigg[\int_0^1 g_u\big(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}\big) - g_u(t,1,1)ds\Bigg]\psi_1 \\
&\quad + \Bigg[\int_0^1 g_v\big(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}\big) - g_v(t,1,1)ds\Bigg]\psi_2\Bigg\} \leq 0
\end{aligned}$$

for all $(t,z) \in \mathbb{R} \times [z', \infty)$, we have

$$\begin{aligned}
&\Bigg[\int_0^1 g_v\big(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}\big)ds\Bigg]v^{\tau^*} + \big[k(t)+c\big]v_z^{\tau^*} + d_2(t)v_{zz}^{\tau^*} - v_t^{\tau^*} \\
&\quad \leq -\Bigg[\int_0^1 g_u\big(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}\big)ds\Bigg]u^{\tau^*} \leq 0
\end{aligned}$$

for all $(t, z) \in \mathbb{R} \times [z', \infty)$. Therefore, the strong maximum principle implies that $v^{\tau^*}(t, z) \equiv 0$ for all $(t, z) \in [0, t^*] \times [z', \infty)$. This is impossible since $v^{\tau^*}(t, z') > 0$. Hence we must have $\tau^* = 0$. The proof is completed. $\square$

References

[1] N.D. Alikakos, P.W. Bates, X. Chen, Periodic traveling waves and locating oscillating patterns in multidimensions domains, Trans. Amer. Math. Soc. 351 (1999) 2777–2805.

[2] H. Berestycki, The influence of advection on the propagation of fronts in reaction–diffusion equations, in: H. Berestycki, Y. Pomeau (Eds.), Nonlinear PDEs in Condensed Matter and Reactive Flows, in: NATO Sci. Ser. C, vol. 569, Kluwer, Dordrecht, 2003, pp. 1–45.

[3] H. Berestycki, L. Nirenberg, Traveling fronts in cylinders, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992) 497–572.

[4] H. Berestycki, F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math. 55 (2002) 949–1032.

[5] H. Berestycki, F. Hamel, G. Nadin, Asymptotic spreading in heterogeneous diffusive media, J. Funct. Anal. 255 (2008) 2146–2189.

[6] H. Berestycki, F. Hamel, N. Nadirashvili, The speed of propagation for KPP type problems. I – Periodic framework, J. Eur. Math. Soc. (JEMS) 7 (2005) 173–213.

[7] H. Berestycki, F. Hamel, L. Roques, Analysis of the periodically fragmented environment model: II – Biological invasions and pulsating travelling fronts, J. Math. Pures Appl. 84 (2005) 1101–1146.

[8] C. Conley, R. Gardner, An application of the generalized Morse index to travelling wave solutions of a competitive reaction–diffusion model, Indiana Univ. Math. J. 33 (1984) 319–343.

[9] S.R. Dunbar, Travelling wave solutions of diffusive Lotka–Volterra equations, J. Math. Biol. 17 (1983) 11–32.

[10] P.C. Fife, M. Tang, Comparison principles for reaction–diffusion systems: irregular comparison functions and applications to questions of stability and speed of propagation of disturbances, J. Differential Equations 40 (1981) 168–185.

[11] J. Földes, P. Poláčik, On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry, Discrete Contin. Dyn. Syst. 25 (2009) 133–157.

[12] R. Gardner, Existence and stability of travelling wave solutions of competition models: a degree theoretic approach, J. Differential Equations 44 (1982) 343–364.

[13] M.G. Garroni, J.L. Menaldi, Green Functions for Second Order Parabolic Integro-Differential Problems, Pitman Res. Notes Math. Ser., vol. 275, Longman Scientific & Technical, Harlow, Essex, 1992.

[14] B.H. Gilding, R. Kersner, Travelling Waves in Nonlinear Diffusion–Convection Reaction, Birkhäuser, Boston, 2004.

[15] S. Gourley, S. Ruan, Convergence and travelling fronts in functional differential equations with nonlocal terms: a competition model, SIAM J. Math. Anal. 35 (2003) 806–822.

[16] F. Hamel, Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity, J. Math. Pures Appl. 89 (2008) 355–399.

[17] F. Hamel, L. Roques, Uniqueness and stability properties of monostable pulsating fronts, J. Eur. Math. Soc. (JEMS) 13 (2011) 345–390.

[18] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin, 1981.

[19] Y. Hosono, The minimal speed of traveling fronts for a diffusive Lotka–Volterra competition model, Bull. Math. Biol. 60 (1998) 435–448.

[20] Y. Kan-On, Fisher wave fronts for the Lotka–Volterra competition model with diffusion, Nonlinear Anal. 28 (1997) 145–164.

[21] M.A. Lewis, B. Li, H.F. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219–233.

[22] W.-T. Li, G. Lin, S. Ruan, Existence of travelling wave solutions in delayed reaction–diffusion systems with applications to diffusion–competition systems, Nonlinearity 19 (2006) 1253–1273.

[23] D. Liang, J. Wu, Travelling waves and numerical approximations in a reaction advection diffusion equation with nonlocal delayed effects, J. Nonlinear Sci. 13 (2003) 289–310.

[24] X. Liang, Y. Yi, X. Zhao, Spreading speeds and traveling waves for periodic evolution systems, J. Differential Equations 231 (2006) 57–77.

[25] X. Liang, X. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with application, Comm. Pure Appl. Math. 60 (2007) 1–40.

[26] G.M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, River Edge, NJ, 1998.

[27] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Boston, 1995.

[28] L. Malaguti, C. Marcelli, Travelling wavefronts in reaction–diffusion equations with convection effects and non-regular terms, Math. Nachr. 242 (2002) 148–164.

[29] L. Malaguti, C. Marcelli, S. Matucci, Front propagation in bistable reaction–diffusion–advection equations, Adv. Differential Equations 9 (2004) 1143–1166.

[30] J. Nolen, J. Xin, Existence of KKP type fronts in space–time periodic shear flows and a study of minimal speeds based on variational principle, Discrete Contin. Dyn. Syst. 13 (2005) 1217–1234.

[31] B. Sandstede, A. Scheel, On the structure of spectra of modulated travelling waves, Math. Nachr. 232 (2001) 39–93.

[32] A.I. Volpert, Vi.A. Volpert, Vl.A. Volpert, Traveling Wave Solutions of Parabolic Systems, Transl. Math. Monogr., vol. 140, Amer. Math. Soc., Providence, 1994, translated by J.F. Heyda.

[33] Z.-C. Wang, W.-T. Li, S. Ruan, Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, J. Differential Equations 238 (2007) 153–200.

[34] H.F. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal. 13 (1982) 353–396.

[35] J. Xin, Front propagation in heterogeneous media, SIAM Rev. 42 (2000) 161–230.

[36] G. Zhao, Multidimensional periodic traveling waves in infinite cylinders, Discrete Contin. Dyn. Syst. 24 (2009) 1025–1045.

[37] G. Zhao, S. Ruan, Existence, uniqueness and asymptotic stability of time periodic traveling solutions for a periodic Lotka–Volterra competition system with diffusion, J. Math. Pures Appl. 95 (2011) 627–671.