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A GENERALIZATION OF THE BUTLER-McGEHEE LEMMA AND ITS APPLICATIONS IN PERSISTENCE THEORY

Feng Yang
Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Shigui Ruan*
Department of Mathematics, Statistics and Computing Science, Dalhousie University
Halifax, Nova Scotia, Canada B3H 3J5

(Submitted by: Klaus Schmitt)

Abstract. The so-called Butler-McGehee lemma was first stated and proposed by Freedman and Waltman [11] to study persistence in three interacting predator-prey population models. Roughly speaking, the lemma says that if a trajectory, not on the stable manifold of a given isolated hyperbolic equilibrium P, has that equilibrium in its ω-limit set, then its ω-limit set also contains points on the stable and unstable manifolds of the equilibrium different from P. The lemma has been extended to different forms. The main purpose of this paper is to generalize one of the various formats of the Butler-McGehee lemma (Butler and Waltman [4]) in such a way as to encompass orbits from a set G rather than from a single point. An application to the uniform persistence of a class of dynamical systems which are not necessarily point dissipative is given.

1. Introduction. Recently, there have appeared in the literature several papers dealing with persistence theory in dynamical systems; see Butler, Freedman and Waltman [3], Butler and Waltman [4], Freedman and Moson [7], Freedman, Ruan and Tang [9], Garay [12], Hofbauer and So [18], Tang [23] and Teng and Duan [24]; in semi-dynamical systems and related systems, see Dunbar, Rybakowski and Schmitt [5], Fonda [6], Freedman and Ruan [8], Freedman and So [10], Hale and Waltman [16], Hallam and Ma [17] and Thieme [25], and their applications to ecological models, see Burton and Hutson [2], Freedman and Waltman [11], Gard [13], Gopalsamy [14] and Kirlinger [20], Ruan [21, 22], etc. The connections of various types of persistence for dynamical systems have been discussed by Freedman and Moson [7]. For more details and more references on persistence theory, we refer to the recent survey paper by Hutson and Schmitt [19].

When applying persistence theory to the question of survival versus extinction in models of interacting populations, a theorem known as the Butler-McGehee lemma is usually required. This lemma may take on various formats, depending on the nature of the dynamical or semi-dynamical systems.

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The original version of this lemma appeared in Freedman and Waltman [11], the setting for which was a hyperbolic restpoint of an autonomous ordinary differential equation. Since then, it has been extended to a compact isolated invariant set, instead of just a restpoint, to a continuous flow on a locally compact metric space by Butler and Waltman [4] and to continuous semi-flows by Dunbar, Rybakowsky, and Schmitt [5]. Later, Freedman and So [10] have developed this lemma to a form utilizable for discrete semi-dynamical systems. The latest form for locally compact metric space has been given a new proof by Hofbauer and So [18]. Hale and Waltman [16] also have obtained results for a complete metric space, not necessarily locally compact. Their results, which are in the setting of an asymptotically smooth C^0-semigroup, are useful in studying the persistence of population models whose dynamics involve such concepts as delays or diffusions in functional differential equations or partial differential equations, respectively.

All the above theorems deal with a point x and its limit set $\omega(x)$ or $\alpha(x)$ in phase space. The object of this paper is to generalize the Butler-McGehee lemma in such a way as to encompass orbits from a set G rather than from a single point and to consider the ω-limit (α-limit) set of the set G. Obviously, if we take $G = \{x\}$, our result will reduce to one of the various known forms of the Butler-McGehee lemma. With this generalization, we can establish a uniform persistence theorem for certain dynamical systems which are not necessarily point dissipative (point dissipativity is needed only on a subset of the boundary). For similar results we refer to Freedman, Ruan and Tang [9] and Thieme [25]. Thieme has established some very interesting persistence criteria under relaxed point dissipativity and has applied the so-called persistence theorems for lazy-bones to an endemic model.

This paper is organized as follows: in Section 2, basic notation and terminologies are introduced. In Section 3, the main result is given together with a variation for discrete dynamical systems; a corollary is also given. Section 4 considers some applications to uniformly persistent systems. Finally, an example is given to illustrate the result obtained.

2. Preliminary results. Let X be a locally compact metric space with metric d and $\mathcal{F} = (X, \mathbb{R}, \pi)$ be a continuous flow on X, where $\pi : X \times \mathbb{R} \to X$ is a continuous map such that $\pi(x, 0) = x$ for all $x \in X$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in X$, $t, s \in \mathbb{R}$. For the basic definitions and results on dynamical systems, we refer to Bhatia and Szegö [1] and Butler and Waltman [4].

Let $\gamma(x), \gamma^+(x)$, and $\gamma^-(x)$ be the orbit, positive semi-orbit, and negative semi-orbit of $\mathcal{F}$ through $x \in X$, respectively. The ω-limit set is defined as

$$\omega(x) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \{\pi(x, t)\}}.$$

This is equivalent to saying that $y \in \omega(x)$ if and only if there is a sequence $t_n \to \infty$ as $n \to \infty$ such that $\pi(x, t_n) \to y$ as $n \to \infty$. If M is a subset of X, we define the ω-limit set of M as

$$\omega(M) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \{\pi(M, t)\}},$$

where $\pi(M,t) = \cup_{x\in M}\{\pi(x,t)\}$. The α-limit sets $\alpha(x)$ and $\alpha(M)$ can be defined similarly. Usually the set

$$L^+(M) = \bigcup_{x\in M} \omega(x)$$

is considered as a candidate for the limiting behavior of the set M since it contains the ω-limit set of each point. Generally $L^+(M)$ is smaller than $\omega(M)$. In fact, ω-limit sets of points in M could be disconnected even when $\omega(M)$ is connected. By Lemma 3.1.1 of Hale [15], if $\omega(M)$ is compact and attracts M (i.e., $d(\pi(M,t),\omega(M)) \to 0$ as $t \to \infty$), then $\omega(M)$ is invariant. In addition, if M is connected, then $\omega(M)$ is connected. As pointed out by Hale and Waltman [16], from the point of view of the qualitative behavior of the dynamics generated by a semigroup, it is necessary to consider the set $\omega(M)$.

Let $M \subset X$ be any set. Then $\overline{M}$ and $\overset{\circ}{M}$ will denote the closure and the interior, respectively, of M. Finally, let $W^+(M) = \{x \in X : \omega(x) \neq \emptyset,\ \omega(x) \subset M\}$ be the stable set of $M, W^-(M) = \{x \in X : \alpha(x) \neq \emptyset,\ \alpha(x) \subset M\}$ be the unstable set of M, $W_w^+(M) = \{x \in X : \omega(x) \cap M \neq \emptyset\}$ be the weakly stable set of M and $W_w^-(M) = \{x \in X : \alpha(x) \cap M \neq \emptyset\}$ be the weakly unstable set of M. Note that if M is compact, $x \in W^+(M)$ is equivalent to $\lim_{t\to\infty} d(\pi(x,t),M) = 0$ and a similar statement holds for $W^-(M)$.

3. Main results. First we give a lemma which is useful to the proof of the main theorem.

Definition 3.1. A nonempty subset M of X, invariant for $\mathcal{F}$, is called an *isolated invariant set* if it is the maximal invariant set in some neighborhood of itself. The neighborhood is called an *isolating neighborhood*.

Lemma 3.2. *Let $M \subset X$ be a compact isolated invariant set for $\mathcal{F}$. For any set $G \subset X\backslash W^+(M)$, if $\omega(G) \cap M \neq \emptyset$, then there exist a compact isolating neighborhood V of M and a sequence $\{y_n\} \subset \overset{\circ}{V}$ satisfying the following*:

(i) $y_n \to M$ *as* $n \to \infty$;

(ii) *there exists* $\{t_n\} \subset \mathbb{R}_+$ *such that for all* $n, \pi(y_{n,} - t_n) \in \partial V$ *and* $\pi(y_n, -t) \in \overset{\circ}{V}$ *if* $t \in [0, t_n)$;

(iii) $\{\pi(y_{n,} - t_n)\}$ *converges to a point* $p \in \omega(G)$.

Proof. With the assumption of $\omega(G) \cap M \neq \emptyset$, we may pick a compact isolating neighborhood V_1 of M and a sequence $\{z_n\} \subset \omega(G) \cap \overset{\circ}{V}_1$ so that $z_n \to M$ as $n \to \infty$. If $\gamma^-(z_n)\backslash\overset{\circ}{V}_1 \neq \emptyset$ for all n, then define $y_n = z_n$ and $V = V_1$. It is easy to see that $\{y_n\}$ and V satisfy (i) and (ii). In the other case, if $\gamma^-(z_n) \subset \overset{\circ}{V}_1$ for some n, then $\alpha(z_n) \neq \emptyset$ and $\alpha(z_n) \subset M$. It follows that there exists $x \in G \subset X\backslash W^+(M)$ such that $\omega(x) \cap M \neq \emptyset$ since $z_n \in \omega(G)$. A compact isolating neighborhood V of M can be found so that $x \notin V$. Since $x \in W_w^+(M)\backslash W^+(M)$, there exists a sequence $\{s_n\} \subset \mathbb{R}_+$ satisfying: (a) $s_n \to \infty$ as $n \to \infty$ while $s_n < s_{n+1}$; (b) $\pi(x, [s_n, s_{n+1}])\backslash V \neq \emptyset$; and (c)

$\pi(x, s_n) \to M$ as $n \to \infty$ while $\pi(x, s_n) \in \overset{\circ}{V}$ for all n. Now, we define $y_n = \pi(x, s_n)$. By property (b), there exists $\{t_n\} \subset \mathbb{R}_+$ such that $\pi(x, s_n - t_n) \in \partial V$ while $\pi(x, s_n - t) \in \overset{\circ}{V}$ for all $t \in (0, t_n)$. The V and $\{y_n\}$ thus defined also satisfy (i) and (ii).

Since ∂V is compact, there exists a convergent subsequence $\{\pi(y_n, -t_n)\}$. Corresponding to this subsequence, we obtain subsequences of $\{y_n\}$ and $\{t_n\}$, denoted by $\{y_n\}$ and $\{t_n\}$ again. Since $x \in G$, $\pi(y_n, -t_n) = \pi(\pi(x, s_n), -t_n) = \pi(x, s_n - t_n)$ and $\omega(G)$ is invariant, there is a point $p \in \omega(G)$, such that $\pi(y_n, -t_n) \to p$ as $n \to \infty$.

Theorem 3.3. *Let $M \subset X$ be a compact isolated set for $\mathcal{F}$. For any $G \subset X \backslash W^+(M)$, if $\omega(G) \cap M \neq \emptyset$, then $\omega(G) \cap (W^+(M) \backslash M) \neq \emptyset$ and $\omega(G) \cap (W^-(M) \backslash M) \neq \emptyset$. (A similar result holds for $\alpha(G)$, and for $G \subset X \backslash W^-(M)$.)*

Proof. By the above lemma, to show Theorem 3.3, it suffices to show that $t_n \to \infty$ as $n \to \infty$ in Lemma 3.2. In fact, if it is true, then for any $t > 0$, there exists N such that $t_n > t$ for all $n \geq N$. It follows that for all $t > 0$,

$$\pi(p, t) = \lim_{n\to\infty} \pi\big(\pi(y_{n,} - t_n), t\big) = \lim_{n\to\infty} \pi\big(y_{n,} - (t_n - t)\big) \in V, \tag{3.1}$$

since $\pi\big(y_n, -(t_n - t)\big) \in \overset{\circ}{V}$ for large n. (3.1) shows that $\gamma^+(p) \subset V$ and hence that $\omega(p) \neq \emptyset$ and $\omega(p) \subset M$ since V is a compact isolated neighborhood of M. It follows that $p \in \omega(G) \cap (W^+(M) \backslash M)$, and so $\omega(G) \cap (W^+(M) \backslash M) \neq \emptyset$.

However, the boundedness of any subsequence of $\{t_n\}$ violates the continuous dependence of orbits on initial points since M is compact and invariant. It follows that $t_n \to \infty$ as $n \to \infty$. This completes the proof of the theorem. $\square$

The following corollary is one of the various forms of the Butler-McGehee lemma (Theorem 4.1 in Butler and Waltman [4]).

Corollary 3.4. *Let M be a compact isolated invariant set for $\mathcal{F}$. Then for any $x \in W_w^+(M) \backslash W^+(M)$, it follows that $\omega(x) \cap (W^+(M) \backslash M) \neq \emptyset$, $\omega(x) \cap (W^-(M) \backslash M) \neq \emptyset$. (A similar result holds for $\alpha(x)$.)*

Proof. The corollary follows by applying Theorem 3.3 to $G = \{x\}$. In this case, $\omega(G) = \omega(x)$. $\square$

We now give a variant of Theorem 3.3 which is valid for discrete dynamical systems. All notation is adopted from Freedman and So [10] except that $f : X \to X$ is a bijection which defines a discrete dynamical system on X. Further, $\omega(G)$ is defined similarly as in Section 2 except in the sense of discrete systems (see, cf. Hale [15]).

Theorem 3.5. *Let M be a compact isolated invariant set in X. For any $G \subset X \backslash W^+(M)$, if $\omega(G) \cap M \neq \emptyset$, then $\omega(G) \cap (W^+(M) \backslash M) \neq \emptyset$ and $\omega(G) \cap (W^-(M) \backslash M) \neq \emptyset$.*

Proof. The proof follows analogously to the proof of Theorem 3.3, noting that f maps a compact set into a compact set.

Remark 3.6. Here, we can demonstrate by an example that Theorem 3.3 is not a trivial extension of the Butler-McGehee lemma (i.e., the conditions in Theorem 3.3

imply those in the Butler-McGehee lemma). This example is given by the Lotka-Volterra equations

$$\begin{cases} \dot{x} = x(\alpha - \beta y), \\ \dot{y} = y(-\gamma + \delta x), \quad x(0),\ y(0) \geq 0, \quad \alpha, \beta, \gamma, \delta > 0. \end{cases} \tag{3.2}$$

Consider $X = \mathbb{R}^2_+,\ M = \{(0,0)\}$. Then $W^+(M) = \{(0,y) \,:\, y \geq 0\}$. Let $G = \overset{\circ}{\mathbb{R}}{}^2_+$. Then $\omega(G) = X$ while $W^+_w(M) = W^+(M)$.

4. Applications to uniform persistence. In this section, we consider only continuous flows. Let $E \subset X$ be a positively invariant set for $\mathcal{F}$ and let D be a closed subset in E.

Definition 4.1. $\mathcal{F}$ will be called *uniformly persistent* in E relative to D if there exists $\eta > 0$ such that for all $x \in E \backslash D$,

$$\liminf_{t \to \infty} d\big(\pi(x,t), D\big) \geq \eta.$$

Suppose B is a maximal invariant set in D for $\mathcal{F}$ such that $D \cap \omega(D) \subset B$. Denote by $\mathcal{F}_B$ the flow $\mathcal{F}$ restricted to B.

Definition 4.2. Let $\mathcal{M} = \{M_i\}_{i=1}^k = \{M_1, M_2, \dots, M_k\}$ be a class of finite pairwise disjoint nonempty sets. $\mathcal{M}$ is called a *covering* of $\omega(\mathcal{F}_B)$ if for each i, $M_i \subset D$, $M_i \cap \omega(\mathcal{F}_B) \neq \emptyset$ and $\omega(\mathcal{F}_B) \subset \bigcup_{i=1}^k M_i$.

It should be pointed out that generally the covering $\mathcal{M} = \{M_i\}_{i=1}^k$ and the union set $\bigcup_{i=1}^k M_i$ are different. However, if $M_1, M_2, \dots, M_k$ are isolated, then they are the same as stated in the current literature (see Butler and Waltman [4] and Hale and Waltman [16]).

Definition 4.3. $\mathcal{F}_B$ is *isolated* if there exists a covering $\mathcal{M} = \{M_i\}_{i=1}^k$ of $\omega(\mathcal{F}_B)$ by pairwise disjoint, compact, isolated invariant sets $M_1, \dots, M_k$ for $\mathcal{F}_B$ such that for each i, M_i is also isolated invariant for $\mathcal{F}$. $\mathcal{M}$ is called an *isolated covering*.

Definition 4.4. Let M, N be isolated invariant sets. We shall say that M *is chained* to N, written $M \to N$, if there exists $x \notin M \cup N$ such that $x \in W^-(M) \cap W^+(N)$. A finite sequence $M_1, \dots, M_k$ of isolated invariant sets will be called a *chain* if $M_1 \to \cdots \to M_k$. The chain will be called a *cycle* if $M_k = M_1$.

Definition 4.5. $\mathcal{F}_B$ will be called *acyclic* if there exists an isolated covering $\mathcal{M} = \{M_i\}_{i=1}^k$ of $\mathcal{F}_B$ such that no subset of $\mathcal{M}$ forms a cycle. The covering will also be called acyclic. (Otherwise, $\mathcal{F}_B$ will be called *cyclic*.)

Definition 4.6. The flow $\mathcal{F}$ is said to be *point dissipative* over a nonempty set $M \subset X$ if there exists a compact set $N \subset X$ such that for any $x \in M$ there exists $t(x) > 0$ such that $\pi(x,t) \in \overset{\circ}{N}$ for all $t \geq t(x)$.

Now we are in the position to state and prove the main theorem of this section.

Theorem 4.7. *Let E, D and $B \neq \emptyset$ be defined as above. Further assume that $E\backslash D$ is positively invariant and $\mathcal{F}_B$ is point dissipative, isolated and acyclic with acyclic covering $\mathcal{M} = \{M_i\}_{i=1}^k$. Denote $M = \bigcup_{M_i \in \mathcal{M}} M_i$. If there exists a compact neighborhood N of M in D and $\alpha_0 > 0$ such that for all $x \in E\backslash D$,*

$$\liminf_{t\to\infty} d\big(\pi(x,t), D\backslash N\big) \geq \alpha_0, \tag{4.1}$$

then $\mathcal{F}$ is uniformly persistent relative to D if and only if

(H) *for each $M_i \in \mathcal{M}$, $W^+(M_i) \cap (E\backslash D) = \emptyset$.*

Proof. We prove the theorem by following the arguments of Butler and Waltman [4]. The necessity of (H) for the uniform persistence of $\mathcal{F}$ is obvious. Now suppose (H) holds. We divide $E\backslash D$ into two sets:

$$G = \big\{x \in E\backslash D \mid \omega(x) \neq \emptyset\big\}, \quad Q = \big\{x \in E\backslash D \mid \omega(x) = \emptyset\big\} = (E\backslash D)\backslash G.$$

Since X is locally compact, there exists a compact neighborhood N_1 of N with $d(N, \partial N_1) = \alpha_1 > 0$. If $Q \neq \emptyset$, then we claim that $\liminf_{t\to\infty} d(\pi(x,t), N) \geq \alpha_1$ for any $x \in Q$. In fact, $\liminf_{t\to\infty} d(\pi(x,t), N) < \alpha_1$ implies that $\omega(x) \neq \emptyset$, since N_1 is compact, which contradicts the definition of Q. Therefore, it follows from (4.1) that $\liminf_{t\to\infty} d(\pi(x,t), D) \geq \min\{\alpha_0, \alpha_1\} > 0$ for each $x \in Q$.

With the above conclusion on Q, if $G = \emptyset$, we are done. Otherwise, if $G \neq \emptyset$, then $\omega(G) \neq \emptyset$ and $\omega(G) \subset E$ by the positive invariance of $E\backslash D$. Moreover, by (4.1), we have

$$d\big(\omega(G), D\backslash N\big) \geq \alpha_0 > 0. \tag{4.2}$$

It follows that to complete the proof, it suffices to show that $d(\omega(G), N) > 0$. Suppose that $d(\omega(G), N) = 0$. Then there exists $y \in N \cap \omega(G)$ since N is compact. The invariance of $\omega(G)$ implies that $\gamma(y) \subset \omega(G)$. Since $E\backslash D$ is positively invariant, we have $\gamma^-(y) \subset N \subset D$. Hence, $\alpha(y)$ is a nonempty, compact and connected set contained in N. Since B is maximal in D, $\alpha(y) \subset B$. It follows from the invariance of $\alpha(y)$ that $\alpha(y) \cap M_i \neq \emptyset$ for some $M_i \in \mathcal{M}$. We relabel $\{M_i\}_{i=1}^k$ so that M_i becomes M_1. We have that $\alpha(y) \subset \omega(G)$ and hence $\omega(G) \cap M_1 \neq \emptyset$. By Theorem 3.3, $\omega(G) \cap (W^+(M_1)\backslash M_1) \neq \emptyset$. Since $\mathcal{M}$ is pairwise disjoint, we may choose $y_1 \in \omega(G) \cap (W^+(M_1)\backslash M)$, where M is defined in the theorem. Since all arguments utilized to y are applicable to y_1, we conclude that $\alpha(y_1) \cap M_j \neq \emptyset$ for some $M_j \in \mathcal{M}$. There are two cases: (i) $\alpha(y_1)\backslash M_\ell \neq \emptyset$ for each $M_\ell \in \mathcal{M}$; or (ii) $\alpha(y_1) \subset M_\ell$ for some $M_\ell \in \mathcal{M}$. Actually, in case (ii), $\ell = j$ since $\mathcal{M}$ is pairwise disjoint.

Consider case (i) first. Since $y_1 \in W_w^-(M_j)\backslash W^-(M_j)$, applying Theorem 3.3 to $\{y_1\}$, we can find $z \in \alpha(y_1) \cap (W^-(M_j)\backslash M_j)$, where z can also be chosen so that $z \notin M$. As we mentioned above, all results we obtained for y are true for y_1 too. It follows that $z \in \alpha(y_1) \subset B \cap N$ and hence there exists $M_p \in \mathcal{M}$ such that $\omega(z) \subset M_p$, which implies that $M_j \to M_p$. If $M_j = M_p$, we obtain a cycle in $\mathcal{M}$, a contradiction of the fact that $\mathcal{M}$ is acyclic. Therefore $M_j \neq M_p$. Then we have $\alpha(y_1) \cap M_p \neq \emptyset$ since

$z \in \alpha(y_1)$ implies that $\omega(z) \subset \alpha(y)$. By the assumption of this case, $\alpha(y_1)\backslash M_p \neq \emptyset$, which implies $y_1 \in W^+_w(M_p)\backslash W^+(M_p)$. Applying the above argument to M_p, we can find an $M_q \in \mathcal{M}$ such that $M_p \to M_q$. Repeating this procedure, we shall end up with a cycle in $\mathcal{M}$ since $\mathcal{M}$ is finite, which contradicts the assumption on $\mathcal{M}$, completing the proof in this case.

In case (ii), $M_j \to M_1$ in B. If $M_j = M_1$, we are done. Otherwise, we relabel M_j as M_2. Since $\omega(G)$ is invariant, $\omega(G) \cap M_2 \neq \emptyset$. Repeating the above argument from the very beginning on M_1, we shall end up with a cycle in $\mathcal{M}$ by the same reason as in case (i), completing the proof.

Remark 4.8. The above theorem assumes that B is nonempty. In the case that $B = \emptyset$, if condition (4.1) is changed to that there exists a nonempty compact neighborhood N in D such that (4.1) holds for all $x \in E\backslash D$, then $\mathcal{F}$ is uniformly persistent. In fact, $Q = \omega(E\backslash D) \cap D \subset N$ by (4.1). If there exists a point $q \in Q$, then the positive invariance of $E\backslash D$ and the compactness of N imply that $\gamma^-(q) \cap (E\backslash D) = \emptyset$ and $\gamma^-(q)\backslash N \neq \emptyset$. It follows that $\omega(E\backslash D) \cap [(E\backslash D) \cup (D\backslash N)] \neq \emptyset$. This is a contradiction either to the positive invariance of E or to (4.1). Therefore we must have $d(N, \omega(E\backslash D)) > 0$ which combined with (4.1) implies the proof of our assertion.

Remark 4.9. If $B = D$, condition (4.1) is automatically satisfied; then Theorem 4.7 reduces to Theorem 3.1 of Butler and Waltman [4].

Theorem 4.7 can be applied to systems which are not point dissipative (the point dissipativity is needed only on a subset of the "boundary" D) while most of these kinds of theorems in current papers deal with point dissipative systems only. The following example fits Theorem 4.7 but not any other theorems which have appeared to the best of our knowledge.

Example 4.10. Consider the following differential equations in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y), \end{cases} \tag{4.3}$$

where the functions f and g are defined as

$$f(x,y) = \begin{cases} 0, & \text{if} \quad x = -1 \quad \text{or} \quad x \geq 1 \\ -\frac{(1+x)(2+x)y}{\ln(-1-x)}, & \text{if} \quad x < -1 \quad \text{and} \quad x \neq -2 \\ -y, & \text{if} \quad x = -2 \\ -\frac{y(1-x^2)}{(1+y^2)(1-p(x)q(y))}, & \text{if} \quad |x| < 1 \end{cases} \tag{4.4}$$

and

$$g(x,y) = \begin{cases} 1, & \text{if} \quad x \geq 1 \\ -(2+x), & \text{if} \quad x \leq -1 \\ x, & \text{if} \quad |x| < 1, \end{cases} \tag{4.5}$$

where $p(x)$ and $q(y)$ are any continuous differentiable functions satisfying

$$
\begin{aligned}
p(x) &= 0, \quad \text{if} \quad x \leq 0 \\
0 < p(x) &< \frac{1}{2}, \quad \text{if} \quad 0 < x < 1 \\
p(x) &= \frac{1}{2}, \quad \text{if} \quad x \geq 1
\end{aligned}
$$

and

$$
\begin{aligned}
0 < q(y) &< \frac{1}{2}, \quad \text{if} \quad y < 0 \\
q(y) &= 0, \quad \text{if} \quad y \geq 0.
\end{aligned}
$$

It is easy to verify that such a system describes a dynamical system as shown in the following figure.

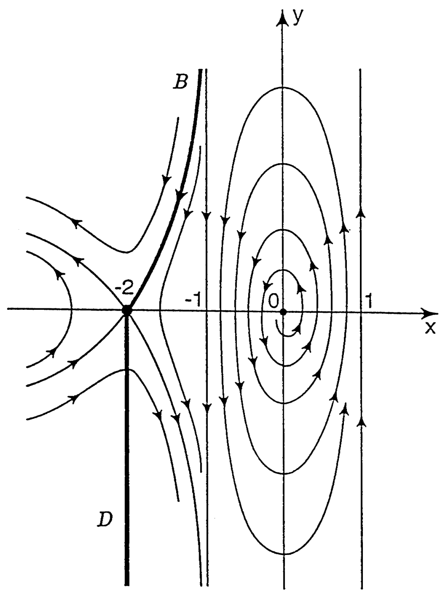

Figure. Example 4.10

We take

$$
\begin{aligned}
E &= \{(x,y) : -2 \leq x < -1, y \leq \ln(-1-x)\} \cup \{(x,y) : x \geq -1\}, \\
D &= \{(x,y) : y = \ln(-1-x), -2 \leq x < -1\} \cup \{(x,y) : x = -2, y \leq 0\}.
\end{aligned}
$$

Then we have $B = \{(x, \ln(-1-x)) : -2 \leq x < -1\}$ and $M = \{(-2,0)\}$. The linear variational matrix of (4.3) at M has the form

$$
\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
$$

It follows that M is a saddle point. From (4.5), we can see that in the area between D and $x = -1$, $\dot{y}$ is always negative. It follows that all the orbits are apart from M

with a certain distance as $t \to \infty$. And the stable set $W^+(M)$ of M in E is the curve $y = \ln(-1-x)$ with $-2 \le x < -1$, a part of D. By Theorem 4.7, the flow defined by (4.3) is uniformly persistent in E relative to D. On the other hand, since it is not a point dissipative system as we can see, the result cannot be obtained by any other theorems requiring point dissipativeness.

As pointed out by the referee, for uniform persistence we only need to consider the behavior of system (4.3) on $(-\infty, -1]$. Finally, for other examples of ecological and epidemiological models which exhibit uniform persistence under relaxed point dissipativity, we refer to Freedman, Ruan and Tang [9], Teng and Duan [24], and Thieme [25].

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