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Fast propagation for reaction–diffusion cooperative systems

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Abstract

This paper deals with the spatial propagation for reaction–diffusion cooperative systems. It is well-known that the solution of a reaction–diffusion equation with monostable nonlinearity spreads at a finite speed when the initial condition decays to zero exponentially or faster, and propagates fast when the initial condition decays to zero more slowly than any exponentially decaying function. However, in reaction–diffusion cooperative systems, a new possibility happens in which one species propagates fast although its initial condition decays exponentially or faster. The fundamental reason is that the growth sources of one species come from the other species. Simply speaking, we find a new interesting phenomenon that the spatial propagation of one species is accelerated by the other species. This is a unique phenomenon in reaction–diffusion systems. We present a framework of fast propagation for reaction–diffusion cooperative systems.
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1. Introduction

1.1. Spatial propagation of reaction–diffusion equations

One interesting problem in the study of parabolic partial differential equations is the spatial propagation of the following reaction–diffusion equation

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(u(t,x)),\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with Fisher-KPP nonlinearity $f(u)$ and continuous front-like initial condition $u_0(x)$, in the sense that

$$f(0) = f(1) = 0,\ 0 < f(u) \leqslant f'(0)u \text{ for } u \in (0,1),$$

$$\liminf_{x\to-\infty} u_0(x) > 0,\ \lim_{x\to+\infty} u_0(x) = 0 \text{ and } 0 \leqslant u_0(x) < 1 \text{ for } x \in \mathbb{R}.$$

For many front-like initial conditions, the spatial propagation of a solution at large time is related to the traveling wave solution $u(t,x) = \phi_c(x - ct)$ satisfying

$$\begin{cases} \phi_c''(\xi) + c\phi_c'(\xi) + f(\phi_c(\xi)) = 0, \\ \phi_c(-\infty) = 1,\ \phi_c(+\infty) = 0, \end{cases} \tag{1.2}$$

where $\xi = x - ct$. It is well-known that equation (1.2) with Fisher-KPP nonlinearity admits a solution $\phi_c(\xi)$ for speed $c \geqslant c^* = 2\sqrt{f'(0)}$, and does not have any solution when $0 < c < c^*$. Therefore, the constant c^* is called the *minimal speed* (for the existence of traveling wave solutions).

It is widely shown that the decay behavior of $u_0(x)$ to zero determines the spatial propagation of $u(t,x)$ at large time. First, if

$$u_0(x) = 0 \text{ in } [x_0, +\infty) \tag{1.3}$$

for some constant $x_0 \in \mathbb{R}$, then $u(t,x)$ approaches a traveling wave solution as follows

$$\lim_{t\to+\infty} u(t, x + m(t)) = \phi_{c^*}(x), \tag{1.4}$$

where $m(t) \sim c^*t + O(\ln t)$ as $t \to +\infty$. If $u_0(x)$ satisfies that

$$u_0(x) \sim O(e^{-\lambda x}) \text{ as } x \to +\infty, \tag{1.5}$$

where λ is a positive constant, then (1.4) holds in case of $\lambda \geqslant \sqrt{f'(0)}$, and when $\lambda < \sqrt{f'(0)}$, the following approach holds

$$\lim_{t\to+\infty} u(t, x + m(t)) = \phi_c(x), \tag{1.6}$$

where $c = \lambda + f'(0)/\lambda$ and $m(t) \sim ct + O(\ln t)$ as $t \to +\infty$. We refer to [5,13,19,27–29] for more results on the convergence to traveling wave solutions.

It can be found from (1.4) that, for the initial condition $u_0(x)$ satisfying (1.3), if we look in a moving frame at any speed less than c^*, then $u(t,x)$ goes to 1 as $t \to +\infty$, and if we look in a moving frame at any speed larger than c^*, then $u(t,x)$ goes to 0 as $t \to +\infty$. It can be described as

$$\begin{cases} \inf\limits_{x \leqslant (c^*-\epsilon)t} u(t,x) \to 1, \\ \sup\limits_{x \geqslant (c^*+\epsilon)t} u(t,x) \to 0, \end{cases} \text{as } t \to +\infty \text{ for any } \epsilon \in (0, c^*). \tag{1.7}$$

Such property is called *spreading property* and the quantity c^* is called the *spreading speed* associated with the initial condition $u_0(x)$. When $u_0(x)$ satisfies (1.5), from (1.6) the same spreading property as (1.7) can also be obtained and here the spreading speed is c. Generally, if $u_0(x)$ decays to zero exponentially or faster (also called **exponentially bounded**), then $u(t,x)$ has a similar spreading property and the spreading speed is finite. So we call $u_0(x)$ an initial condition **spreading at a finite speed** if it is exponentially bounded. For more results on spreading property, we refer to [3,4,13,18,32].

Let us consider two important and interesting problems: (i) Is it possible to describe the spatial propagation for more general initial conditions? (ii) If the spreading speed associated with some $u_0(x)$ is infinite, since the spreading property cannot provide an exact description on the spatial propagation, then how to describe it? Hamel et al. [14] first consider the **fast propagation** (whose spreading speed goes to infinite as $t \to +\infty$) for initial conditions which decay to zero more slowly than any exponentially decaying function (also called **exponentially unbounded**). This type of initial conditions is fully different from the exponentially bounded ones. For any $\mu \in (0,1)$, denote the level set of value μ by

$$E^{\mu}(t) = \{x \in \mathbb{R},\ u(t,x) = \mu\}.$$

For a continuous function $g(x) : \mathbb{R} \to [0,1]$ and a subset $A \subset [0,1]$, denote

$$g^{-1}\{A\} = \{x \in \mathbb{R},\ g(x) \in A\}$$

as the inverse image of A by $g(x)$. Then under some conditions on $f(u)$ and $u_0(x)$,

$$E^{\mu}(t) \subset u_0^{-1}\left\{[\gamma e^{-(f'(0)+\epsilon)t}, \Gamma e^{-(f'(0)-\epsilon)t}]\right\} \tag{1.8}$$

holds at large time t for any $\mu \in (0,1)$, $\epsilon \in (0, f'(0))$, $\gamma > 0$ and $\Gamma > 0$. It is easy to see that the exponentially unbounded initial condition leads to that the spreading speed goes to infinite as $t \to +\infty$, and (1.8) provides an exact description on the locations of level sets $E^{\mu}(t)$ for all $\mu \in (0,1)$ at large time t. Therefore, we call $u_0(x)$ an initial condition propagating fast if it is exponentially unbounded.

For more related results on fast propagation of reaction–diffusion equations, we refer to [1,15] for the ignition nonlinearity, [16] for the inhomogeneous nonlinearity, [2,11,12] for nonlocal diffusion equations, and [6,8,10] for fractional diffusion equations.

Remark 1.1. Fast propagation vs. transition fronts. A transition front means a solution $u(t,x)$ satisfying that there exists a continuous function $\xi(t):\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} u(t,x+\xi(t))\to 1 \text{ as } x\to-\infty, \\ u(t,x+\xi(t))\to 0 \text{ as } x\to+\infty, \end{cases} \text{uniformly for } t\in\mathbb{R}.$$

If $u(t,x)$ is a transition front, then for any $\epsilon\in(0,\frac{1}{2})$ there exists $C=C(\epsilon)$ such that

$$I_\epsilon(t):=\{x\in\mathbb{R}, \epsilon\leqslant u(t,x)\leqslant 1-\epsilon\}\subset[\xi(t)-C,\xi(t)+C]. \tag{1.9}$$

This is just the opposite to the solution propagating fast. Indeed, if $u(t,x)$ propagates fast, because of the different spreading speeds of level sets $E^\mu(t)$ for different $\mu\in(0,1)$, then $u(t,x)$ becomes uniformly flat as $t\to+\infty$, in the sense that $\|u_x(t,\cdot)\|_{L^\infty(\mathbb{R})}\to 0$ as $t\to+\infty$ (see [14, Theorem 1.5]). Therefore $\operatorname{diam}\{I_\epsilon(t)\}\to+\infty$ as $t\to+\infty$, which contradicts (1.9).

1.2. Problem descriptions

In this paper, we are interested in the spatial propagation of the following reaction–diffusion cooperative system

$$U_t=DU_{xx}+F(U),\ t>0,\ x\in\mathbb{R}, \tag{1.10}$$

with initial condition

$$U(0,x)=U_0(x),\ x\in\mathbb{R}, \tag{1.11}$$

where $U=(u_1,\cdots,u_m)$, $U_0=(u_{0,1},\cdots,u_{0,m})$, $F=(f_1,\cdots,f_m)$, $D=\operatorname{diag}\{d_1,\cdots,d_m\}$ with $d_j>0$ for $j\in\{1,\cdots,m\}$ and $m\geqslant 2$. To the best of our knowledge, there is no result on fast propagation of reaction–diffusion systems.

We first consider system (1.10) with initial condition $U_0(x)=(u_{0,1}(x),\cdots,u_{0,m}(x))$ satisfying that all components decay more slowly than any exponentially decaying function as $x\to+\infty$, in the sense that,

$$\forall\sigma>0, \exists x_\sigma\in\mathbb{R} \text{ s.t. } u_{0,j}(x)\geqslant e^{-\sigma x} \text{ in } [x_\sigma,+\infty) \text{ for } j\in J, \tag{1.12}$$

where $J=\{1,\cdots,m\}$. One example satisfying (1.12) is the function $u_0(x)$ with the properties $u_0(x)\in C^1$ and $u_0'(x)/u_0(x)\to 0$ as $x\to+\infty$. Although one may believe that the fast propagation will happen in system (1.10) when the initial condition $U_0(x)$ satisfies (1.12), there are some difficulties in the proof because it is complex to consider the interactions between all species. Our result of this problem is given in Theorem 2.2.

Further, we consider system (1.10) with initial condition $U_0(x)$ satisfying that there is a nonempty subset I of J such that

$$\forall\ \sigma>0,\ \exists\ x_\sigma\in\mathbb{R} \text{ s.t. } u_{0,j}(x)\geqslant e^{-\sigma x} \text{ in } [x_\sigma,+\infty) \text{ for } j\in I. \tag{1.13}$$

If $I=J$, then (1.13) becomes (1.12), and if there are no other subsets satisfying (1.13) except the empty set, then $U(t,x)$ spreads at a finite speed from Lemma 2.1 below.

When $U_0(x)$ satisfies (1.13) with $I \notin \{\varnothing, J\}$ and $u_{0,j}(x)$ for $j \in J\backslash I$ is an exponentially bounded initial condition, one direct question is: how does the solution $U(t,x)$ spread, at a finite speed or an infinite speed? In a scalar reaction–diffusion equation, $u_{0,j}(x)$ could lead to that $u_j(t,x)$ spreads at a finite speed when $j \in J\backslash I$, and $u_{0,i}(x)$ could lead to that $u_i(t,x)$ propagates fast when $i \in I$. However, there is a big difference between the reaction–diffusion cooperative systems and the scalar reaction–diffusion equations, that is, the cooperation property decides that the growth sources of one species may come from the other $m-1$ species. This makes it possible that all species propagate in the same way, but have different types of initial conditions. We prove that all species propagate fast for initial condition $U_0(x)$ satisfying (1.13) in Theorem 2.3, which means that the spatial propagation of $u_j(t,x)$ with $j \in J\backslash I$ is accelerated by $u_i(t,x)$ with $i \in I$. This is a unique phenomenon in reaction–diffusion systems and has not been observed before.

The remaining part of this article is organized as follows. In section 2, we give our main results on fast propagation and apply them to some examples. The detailed proofs are given in section 3.

2. Spatial propagation of diffusive cooperative systems

Before giving some spatial propagation properties of system (1.10), we introduce some notations. Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ and $\beta = (\beta_1, \cdots, \beta_m)$ be two vectors in $\mathbb{R}^m$. The inequality $\alpha \leqslant \beta$ ($\alpha \ll \beta$) means that $\alpha_j \leqslant \beta_j$ ($\alpha_j < \beta_j$) for all $j \in J = \{1, \cdots, m\}$. Denote $[\alpha, \beta] = \{u \in \mathbb{R}^m, \alpha \leqslant u \leqslant \beta\}$ for $\alpha \leqslant \beta$. Define

$$\max\{\alpha, \beta\} = \Big(\max\{\alpha_1, \beta_1\}, \cdots, \max\{\alpha_m, \beta_m\}\Big),$$

$$\min\{\alpha, \beta\} = \Big(\min\{\alpha_1, \beta_1\}, \cdots, \min\{\alpha_m, \beta_m\}\Big).$$

Let $X = \mathrm{BUC}(\mathbb{R}, \mathbb{R}^m)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}^m$ with supremum norm $\|\cdot\|_X$. Further, define the set of vector-valued functions

$$C_{[\alpha,\beta]} = \{U = (u_1, \cdots, u_m) \in X,\ u_j : \mathbb{R} \to [\alpha_j, \beta_j] \text{ for all } j \in J\}.$$

If $\mathbf{u}, \mathbf{v} \in C_{[\alpha,\beta]}$, then the inequality $\mathbf{u} \leqslant \mathbf{v}$ ($\mathbf{u} \ll \mathbf{v}$) means that $\mathbf{u}(x) \leqslant \mathbf{v}(x)$ ($\mathbf{u}(x) \ll \mathbf{v}(x)$) for any $x \in \mathbb{R}$.

To recall the known results on the spreading property and traveling wave solutions of system (1.10), we assume that

(A1) (a) There is a strongly positive equilibrium $\mathbf{K} = (K_1, K_2, ...K_m)$ in $\mathbb{R}^m$ such that $F \in C^2([\mathbf{0}, \mathbf{K}], \mathbb{R}^m)$ and $F(\mathbf{0}) = F(\mathbf{K}) = \mathbf{0}$, while there is no other equilibrium α in $\mathbb{R}^m$ such that $F(\alpha) = \mathbf{0}$ and $\mathbf{0} \ll \alpha \leqslant \mathbf{K}$;

(b) $F(U)$ is cooperative in $[\mathbf{0}, \mathbf{K}]$ in the sense that $\partial_j F_i(U) \geqslant 0$ for all $U \in [\mathbf{0}, \mathbf{K}]$ and $i \neq j$;

(c) The matrix $F'(\mathbf{0})$ is irreducible with $\lambda_0 \triangleq s(F'(\mathbf{0})) > 0$, where

$$s(F'(\mathbf{0})) = \max\{\mathrm{Re}\lambda \,|\, \det(\lambda I - F'(\mathbf{0})) = 0\}.$$

Considering a traveling wave solution $U(x,t) = \Phi(x - ct)$ and substituting it into system (1.10), we obtain the following wave form system

$$D\Phi''(\xi)+c\Phi'(\xi)+F(\Phi(\xi))=0.$$

Then the eigenvalue problem of its linearization at the origin can be written as

$$\frac{A(\lambda)V}{\lambda}=cV,$$

where $V\in\mathbb{R}^m$, $\lambda>0$ and $A(\lambda)=\lambda^2 D+F'(\mathbf{0})$. Denote $m(\lambda)=s(A(\lambda))>0$. If (A1) holds, then $A(\lambda)$ is also cooperative and irreducible. Using Perron–Frobenius theorem, by considering $A(\lambda)+\nu I$ with $\nu\in\mathbb{R}$ large enough if necessary, we have that $m(\lambda)$ is the simple principal eigenvalue of $A(\lambda)$ with a strongly positive unit eigenvector $V(\lambda)\in\mathbb{R}^m$. We similarly get that $\lambda_0=s(F'(\mathbf{0}))$ is the simple principal eigenvalue of $F'(\mathbf{0})$ with a strongly positive unit eigenvector $V=(v_1,\cdots,v_m)\in\mathbb{R}^m$. Moreover, by using the same arguments as in [24, Lemma 3.8] and [9, Lemma 2.1], there exist c^*, λ^*, $\lambda_1(c)$ and $\lambda_2(c)$ such that

$$c^*=\frac{m(\lambda^*)}{\lambda^*}=\inf_{\lambda>0}\frac{m(\lambda)}{\lambda}\text{ and }c=\frac{m(\lambda_i(c))}{\lambda_i(c)}\text{ for any }c>c^*$$

with $0<\lambda_1(c)<\lambda^*<\lambda_2(c)<\infty$. We remark that the irreducibility hypothesis of $F'(\mathbf{0})$ is not necessary and it was replaced by a weaker one in [33, Hypotheses 4.1 v] and [31, H2].

Differing from the scalar reaction–diffusion equations, in reaction–diffusion systems different species can spread at different finite spreading speeds (see [20,33]). It was also proved in [22] that the slowest spreading speed can always be characterized as the minimal speed for the existence of traveling wave solutions. However, some sufficient conditions were proposed to guarantee that all species spread at a single finite spreading speed (see [22,33]). We refer to [9,21,23,25, 26] for more results on spreading property of reaction–diffusion cooperative systems. Since this paper focuses on the interaction between spreading at a finite speed and fast propagation, we just consider the case that all species spread at a single finite spreading speed. Therefore, we make the following assumption to guarantee the uniqueness of a finite spreading speed, which means that the nonlinearity $F(U)$ is less than its linearization at the origin along the directions of $V(\lambda)$ and V.

(A2) $F(\min\{\mathbf{K},pV\})\leqslant F'(\mathbf{0})pV$ and $F(\min\{\mathbf{K},pV(\lambda)\})\leqslant F'(\mathbf{0})pV(\lambda)$, for $p>0$ and $0<\lambda\leqslant\lambda^*$.

The following lemma gives the results on the existence of traveling wave solutions and spreading property of system (1.10). Detailed proof can be found in [9, Theorem 3.1], [22, Theorem 4.1, Theorem 4.2], [31, Theorem 2.1] and [33, Theorem 4.2]. For more results on traveling wave solutions of reaction–diffusion systems, we refers to [30].

Lemma 2.1. *If (A1) and (A2) hold, then*

(i) *For any $c\in[c^*,+\infty)$, system (1.10) has a nonincreasing traveling wave solution $U(t,x)=\Phi_c(x-ct)$ satisfying*

$$\Phi_c(+\infty)=\mathbf{0},\ \ \Phi_c(-\infty)=\mathbf{K}\textit{ and }\Phi_c(\xi)\gg\mathbf{0}\textit{ for }\xi\in\mathbb{R},$$

while if $c\in(0,c^*)$, there is no traveling wave solution connecting $\mathbf{0}$ *and* $\mathbf{K}$;

(ii) *Assume that* $U_0(x)$ *satisfies* $U_0(x) = \mathbf{0}$ *for all sufficiently large* x, $\liminf_{x\to-\infty} U_0(x) \gg \mathbf{0}$ *and* $\mathbf{0} \leqslant U_0(x) \ll \mathbf{K}$ *for* $x \in \mathbb{R}$. *Then the solution* $U(t,x)$ *of system* (1.10) *with initial condition* $U_0(x)$ *has the following spreading property*

$$\lim_{t\to\infty} \sup_{x\geqslant(c^*+\epsilon)t} U(t,x) = \mathbf{0}, \quad \lim_{t\to\infty} \inf_{x\leqslant(c^*-\epsilon)t} U(t,x) = \mathbf{K} \textit{ for } \epsilon \in (0, c^*).$$

Next we give our results on fast propagation for initial conditions satisfying (1.12) and (1.13), respectively. Assume that

(A3) There exist positive numbers M, δ and p_0 such that $F(pV) \geqslant F'(\mathbf{0})pV - Mp^{1+\delta}V$ for $0 < p \leqslant p_0$.

For any $\mu \in (0,1)$ and a solution $U(t,x) = (u_1(t,x), \cdots, u_m(t,x))$ of system (1.10), denote the level set of value μK_j by

$$E_j^{\mu}(t) = \{x \in \mathbb{R},\ u_j(t,x) = \mu K_j\} \text{ for } j \in J.$$

Let $U_0(x) \in C_{[\mathbf{0},\mathbf{K}]}$ be the front-like initial condition, namely,

$$\lim_{x\to+\infty} U_0(x) = \mathbf{0},\ \liminf_{x\to-\infty} U_0(x) \gg \mathbf{0},\ u_{0,j}(x) > 0 \text{ for } x \in \mathbb{R},\ j \in J. \tag{2.1}$$

Denote

$$\bar{u}(x) = \max_{j\in J}\left\{\frac{u_{0,j}(x)}{v_j}\right\},\ \underline{u}_J(x) = \min_{j\in J}\left\{\frac{u_{0,j}(x)}{v_j}\right\}.$$

Therefore, the initial condition $U_0(x)$ is trapped between $\bar{u}(x)V$ and $\underline{u}_J(x)V$, which are both in the direction of $V = (v_1, \cdots, v_m)$.

Theorem 2.2. *Assume that (A1)–(A3) hold and* $U_0(x)$ *satisfies* (1.12) *and* (2.1). *Let* $U(t,x)$ *be the solution of system* (1.10) *with initial condition* $U_0(x)$.

(i) *We have*

$$\lim_{x\to+\infty} U(t,x) = \mathbf{0} \textit{ for any } t \geqslant 0 \textit{ and } \lim_{t\to\infty} \liminf_{x\to-\infty} U(t,x) \to \mathbf{K}. \tag{2.2}$$

Therefore, for any given $\mu \in (0,1)$, *one can choose* $t_\mu \geqslant 0$ *such that* $E_j^{\mu}(t)$ *is compact and non-empty for any* $t \geqslant t_\mu$ *and* $j \in J$.

(ii) *Moreover, assume that there is* $\xi_0 \in \mathbb{R}$ *such that* $\bar{u}(x)$ *and* $\underline{u}_J(x)$ *are in* $C^2[\xi_0, +\infty)$ *and nonincreasing in* $[\xi_0, +\infty)$, *and* $\bar{u}''(x) = o(\bar{u}(x))$, $\underline{u}_J''(x) = o(\underline{u}_J(x))$ *as* $x \to +\infty$. *Then, for any* $\mu \in (0,1)$, $\epsilon \in (0, \lambda_0)$, $\gamma > 0$ *and* $\Gamma > 0$, *there exists* $T = T_{\mu,\epsilon,\gamma,\Gamma,\lambda_0,V,\mathbf{K}} \geqslant t_\mu$ *satisfying*

$$\sup\{E_j^{\mu}(t)\} < \inf\left\{\bar{u}^{-1}(\gamma e^{-(\lambda_0+\epsilon)t}) \cap (\xi_0, +\infty)\right\}, \tag{2.3}$$

$$\inf\{E_j^{\mu}(t)\} > \sup\left\{\underline{u}_J^{-1}(\Gamma e^{-(\lambda_0-\epsilon)t}) \cap (\xi_0, +\infty)\right\}, \tag{2.4}$$

for any $t \geqslant T$ *and* $j \in J$.

Since $\lim\limits_{x\to-\infty} \bar{u}(x) > 0$ and $\bar{u}(x) > 0$ for $x \in \mathbb{R}$, then $\inf\limits_{(-\infty,\xi_0]} \bar{u}(x) > \gamma e^{-(\lambda_0+\epsilon)t}$ for large time t. Therefore, we have $\bar{u}^{-1}(\gamma e^{-(\lambda_0+\epsilon)t}) \subseteq (\xi_0, +\infty)$ for large time t. If $\bar{u}(x)$ is nonincreasing in $[\xi_0, +\infty)$, then $\bar{u}^{-1}(\gamma e^{-(\lambda_0+\epsilon)t})$ is either a closed interval or a singleton. The same result on $\underline{u}_J^{-1}(\Gamma e^{-(\lambda_0-\epsilon)t})$ also holds. Moreover, because we do not assume that $U_0(x)$ is monotone decreasing in $\mathbb{R}$, $E_j^{\mu}(t)$ may contain several closed intervals and singletons.

For any $\mu \in (0, 1)$, Theorem 2.2 (ii) implies that $E_j^{\mu}(t)$ is included in the moving sets

$$\left[\sup\left\{\underline{u}_J^{-1}(\Gamma e^{-(\lambda_0-\epsilon)t})\right\},\ \inf\left\{\bar{u}^{-1}(\gamma e^{-(\lambda_0+\epsilon)t})\right\}\right] \tag{2.5}$$

when t is large enough. That means

$$\inf_{x\leqslant \sup\left\{\underline{u}_J^{-1}(\Gamma e^{-(\lambda_0-\epsilon)t})\right\}} U(t,x) \to \mathbf{K} \text{ and } \sup_{x\geqslant \inf\left\{\bar{u}^{-1}(\gamma e^{-(\lambda_0+\epsilon)t})\right\}} U(t,x) \to \mathbf{0} \tag{2.6}$$

as $t \to +\infty$, which can be considered as the propagating property of cooperative systems with exponentially unbounded initial condition.

Now we consider the spatial propagation of system (1.10) with the initial condition satisfying (1.13) which includes the initial conditions of both the exponentially bounded type and the exponentially unbounded type.

Denote $f_{j,i} = \frac{\partial f_j}{\partial u_i}(\mathbf{0})$ for $i, j \in J$. Let $n = \dim\{I\}$, and redefine the order of $\{1, \cdots, m\}$ so that $I = \{1, \cdots, n\}$. Define $\tilde{F} = \{\tilde{f}_j\}_{j\in I} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tilde{f}_j(\{u_i\}_{i\in I}) = f_j\left(\{u_i\}_{i\in I},\ \{u_i = 0\}_{i\in J\setminus I}\right) \text{ for } j \in I.$$

(A3*) (a) Assume $n \geqslant 2$. The matrix $\tilde{F}'(\mathbf{0})$ is irreducible with $\tilde{\lambda}_0 \triangleq s(\tilde{F}'(\mathbf{0})) > 0$. There are positive numbers M, δ and p_0 such that

$$\tilde{F}(p\tilde{V}) \geqslant \tilde{F}'(\mathbf{0})p\tilde{V} - Mp^{1+\delta}\tilde{V} \text{ for } 0 \leqslant p \leqslant p_0,$$

where $\tilde{V} = (\tilde{v}_1, \cdots, \tilde{v}_n) \in \mathbb{R}^n$ is the strongly positive unit eigenvector of $\tilde{F}'(\mathbf{0})$ corresponding to the simple principal eigenvalue $\tilde{\lambda}_0$;

(b) There are positive numbers M, δ and p_0 such that

$$f_j(U) \geqslant \sum_{i=1}^{m}\left(f_{j,i}u_i - Mu_i^{1+\delta}\right)$$

for $j \in J\setminus I$ and $U = (u_i) \in \mathbb{R}^m$ with $|u_i| \leqslant p_0$ for $i \in J$.

By choosing the larger M, the larger δ, the smaller p_0 between those in (A3*-a) and (A3*-b), we can simply denote them also by M, δ and p_0 for convenience, respectively. Denote

$$\underline{u}_I(x) = \min_{j\in I}\left\{\frac{u_{0,j}(x)}{r\tilde{v}_j}\right\},$$

where the constant r satisfies that $\min_{j\in I}\{\tilde{v}_j\}r \geqslant \delta^{\frac{1}{1+\delta}}$. Then $\underline{u}_I(x)$ is in $C_{[\mathbf{0},\mathbf{K}]}$ and exponentially unbounded.

Theorem 2.3. *Assume that (A1) and (A2) hold and $U(t,x)$ is the solution of system (1.10) with initial condition $U_0(x)$ satisfying (2.1). Assume that there exists a non-empty subset $I \subseteq J$ such that (A3*) and (1.13) hold. Then Theorem 2.2 (i) holds.*

Moreover, assume that there is $\xi_0 \in \mathbb{R}$ such that $\bar{u}(x)$ and $\underline{u}_I(x)$ are in $C^2(\xi_0,+\infty)$ and nonincreasing in $[\xi_0,+\infty)$, and $\bar{u}''_I(x) = o(\bar{u}(x))$, $\underline{u}''_I(x) = o(\underline{u}_I(x))$ as $x \to +\infty$. Then for any $\mu \in (0,1)$, $\epsilon \in (0,\tilde{\lambda}_0)$, $\gamma > 0$ and $\Gamma > 0$, there exists $T = T_{\mu,\epsilon,\gamma,\Gamma,\lambda_0,V,\mathbf{K}} \geqslant t_\mu$ so that (2.3) and

$$\inf\{E_j^\mu(t)\} > \sup\left\{\underline{u}_I^{-1}(\Gamma e^{-(\tilde{\lambda}_0-\epsilon)t}) \cap (\xi_0,+\infty)\right\} \text{ for any } t \geqslant T,\ j\in J, \tag{2.7}$$

hold.

Although $u_{0,j}(x)$ with $j \in J\backslash I$ may be exponentially bounded, we have that $u_j(t,x)$ propagates fast by (2.7). The fundamental reason is that the growth sources of $u_j(t,x)$ with $j \in J\backslash I$ come from $u_i(t,x)$ with $i \in I$. Roughly speaking, the jth equation with $j \in J\backslash I$ of system (1.10) can be considered as

$$\begin{cases} \partial_t u_j = \triangle u_j - |f_{j,j}|u_j + \sum_{i\in I} f_{j,i}\, C\underline{u}_I(x)e^{\tilde{\lambda}_0 t},\ t>0,\ x\in\mathbb{R}, \\ u_j(0,x) = u_{0,j}(x),\ x\in\mathbb{R}. \end{cases} \tag{2.8}$$

To get a lower bound of $E_j^\mu(t)$, we make a lower solution of equation (2.8) in the following form

$$\underline{u}_j(t,x) = \eta_j(t)\underline{u}_I(x) - \zeta_j(t)[\underline{u}_I(x)]^{1+\delta}$$

where $\eta_j(0) = \zeta_j(0) = 0$ and $\eta_j(t) \to e^{\tilde{\lambda}_0 t}$, $\zeta_j(t) \to e^{\tilde{\lambda}_0(1+\delta)t}$ as $t \to +\infty$. Then we can prove that the solution $u_j(t,x)$ of equation (2.8) with front-like initial condition propagates fast, even when $u_{0,j}(x) = 0$, $x \in [x_0,+\infty)$ for some $x_0 \in \mathbb{R}$.

Remark 2.4. We thank the reviewer for pointing out that the results in Theorems 2.2 and 2.3 imply that the fast propagation phenomenon does not depend on the diffusion coefficient D.

Remark 2.5. When the initial condition $U_0(x)$ satisfies (1.12), if $\tilde{\lambda}_0 \geqslant \lambda_0$, then (2.7) can be regarded as a more accurate calculation of the lower bound than (2.4), which means that $\underline{u}_I^{-1}(\Gamma e^{-(\tilde{\lambda}_0-\epsilon)t}) \geqslant \underline{u}_J^{-1}(\Gamma e^{-(\lambda_0-\epsilon)t})$. Further, reselect a subset of I and denote it also by I, which satisfies (A3*) and $\underline{u}_I(x) \in C^2$, $\underline{u}''_I(x) = o(\underline{u}_I(x))$ as $x \to +\infty$. Then by the same method we obtain a more accurate calculation of the level set. Repeat this process until I is the smallest, and the most accurate calculation of the lower bound is obtained.

Lastly, we apply Theorems 2.2 and 2.3 to two examples.

Example 2.6. Consider the following epidemic model

$$\begin{cases} \frac{\partial}{\partial t}u_1(t,x) = d_1\Delta u_1(t,x) - \alpha u_1(t,x) + f(u_2(t,x)), & x \in \mathbb{R}, t > 0, \\ \frac{\partial}{\partial t}u_2(t,x) = d_2\Delta u_2(t,x) - \beta u_2(t,x) + g(u_1(t,x)), & x \in \mathbb{R}, t > 0, \\ u_1(0,x) = u_{0,1}(x),\ u_2(0,x) = u_{0,2}(x), & x \in \mathbb{R}, \end{cases} \tag{2.9}$$

where α, β, d_1 and d_2 are some positive constants. This system describes the spread of the infectious diseases by oral-faecal transmission such as typhoid fever, cholera, hepatitis A, poliomyelitis, etc (see [7,17,34,35]). Here $u_1(t,x)$ and $u_2(t,x)$ biologically stand for the spatial concentration of the bacteria and the spatial density of the infective human population, respectively. Assume that

$$f(0) = g(0) = 0,\ \alpha K_1 = f(K_2),\ \beta K_2 = g(K_1) \ \text{ for some } K_1, K_2 \in \mathbb{R}^+, \tag{2.10}$$

and $f \in C^2[0, K_2]$, $g \in C^2[0, K_1]$,

$$f'(u) > 0,\ f''(u) < 0 \text{ in } [0, K_2] \ \text{ and } \ g'(u) > 0,\ g''(u) < 0 \text{ in } [0, K_1]. \tag{2.11}$$

Then it is easy to see that $\alpha\beta < f'(0)g'(0)$ and our assumptions (A1), (A2) and (A3) in Theorem 2.2 hold.

If $u_{0,1}(x) \sim \exp\{-x^{a_1}\}$ and $u_{0,2}(x) \sim \exp\{-x^{a_2}\}$ with $a_1, a_2 \in (0,1)$, then there exist C_1 and C_2 such that for all $j \in \{1,2\}$,

$$C_1 t^a \leqslant x_j^\mu(t) \leqslant C_2 t^b \quad \text{for all } \ x_j^\mu(t) \in W_j^\mu(t),\ \mu \in (0,1) \text{ and large } t \tag{2.12}$$

and

$$\inf_{x \leqslant C_1 t^a} u_j(t,x) \to K_j, \quad \sup_{x \geqslant C_2 t^b} u_j(t,x) \to 0 \text{ as } t \to +\infty, \tag{2.13}$$

where $a = \min\{1/a_1, 1/a_2\}$ and $b = \max\{1/a_1, 1/a_2\}$.

If $u_{0,1}(x) \sim x^{-a_1}$ and $u_{0,2}(x) \sim x^{-a_2}$ with $a_1, a_2 \in (0, +\infty)$, then there exist C_1 and C_2 such that for all $j \in \{1,2\}$,

$$C_1 e^{(\lambda_0-\epsilon)t/b} \leqslant x_j^\mu(t) \leqslant C_2 e^{(\lambda_0+\epsilon)t/a} \text{ for all } x_j^\mu(t) \in W_j^\mu(t),\ \mu \in (0,1) \text{ and large } t \tag{2.14}$$

and

$$\inf_{x \leqslant C_1 e^{(\lambda_0-\epsilon)t/b}} u_j(t,x) \to K_j, \quad \sup_{x \geqslant C_2 e^{(\lambda_0+\epsilon)t/a}} u_j(t,x) \to 0 \text{ as } t \to +\infty, \tag{2.15}$$

where $a = \min\{a_1, a_2\}$, $b = \max\{a_1, a_2\}$ and λ_0 is the unique positive root of equation $(\lambda + \alpha)(\lambda + \beta) = g'(0)h'(0)$.

Example 2.7. Consider the following epidemic model

$$\begin{cases} \dfrac{\partial}{\partial t} u_1(t,x) = d_1 \Delta u_1(t,x) - \alpha u_1(t,x) + f(u_2(t,x)), & x \in \mathbb{R}, t > 0, \\ \dfrac{\partial}{\partial t} u_2(t,x) = d_2 \Delta u_2(t,x) - \beta u_2(t,x) + g(u_1(t,x)), & x \in \mathbb{R}, t > 0, \\ \dfrac{\partial}{\partial t} u_3(t,x) = d_3 \Delta u_3(t,x) - \gamma u_3(t,x) + h(u_1(t,x)), & x \in \mathbb{R}, t > 0, \\ u_1(0,x) = u_{0,1}(x),\ u_2(0,x) = u_{0,2}(x),\ u_3(0,x) = u_{0,3}(x) & x \in \mathbb{R}, \end{cases}$$

where α, β, γ, d_1, d_2 and d_3 are some positive constants. This model can be regarded as an extension of model (2.9), where the epidemic can infect two species, such as bird flu and rabies. Here $u_1(t,x)$ stands for the spatial concentration of the bacteria, $u_2(t,x)$ and $u_3(t,x)$ stand for the spatial densities of the infective population in the two species, respectively. Assume that (2.10) and (2.11) hold. Assume further that

$$h(0) = 0,\ h(u) \in C^{1+\delta}([0, K_3]),\ 0 < h'(u) \leqslant h'(0) \text{ in } [0, K_3],$$

for some $\delta \in (0,1)$ and $K_3 = h(K_1)/\gamma$. Then our assumptions (A1), (A2) and (A3*) in Theorem 2.3 holds, and here $I = \{1, 2\}$.

If $u_{0,1}(x) \sim \exp\{-x^{a_1}\}$, $u_{0,2}(x) \sim \exp\{-x^{a_2}\}$ and $u_{0,3}(x) = 0$ for large x, where $a_1, a_2 \in (0,1)$, then there exist C_1, C_2 such that (2.12) and (2.13) hold for all $j \in \{1,2,3\}$. If $u_{0,1}(x) \sim x^{-a_1}$, $u_{0,2}(x) \sim x^{-a_2}$ and $u_{0,3}(x) = 0$ for large x, where $a_1, a_2 \in (0, +\infty)$, then there exist C_1 and C_2 such that (2.14) and (2.15) hold for $j \in \{1,2,3\}$. It can be found that the spatial propagation of $u_3(t,x)$ is accelerated by $u_1(t,x)$.

3. Proofs of Theorems

We first state the comparison principle of reaction–diffusion cooperative systems, which is used in the proofs of our results. By cooperation property, we can choose η large enough such that $Q^\eta U = \eta U + F(U) : \mathbb{R}^m \to \mathbb{R}^m$ is nondecreasing, in the sense that $Q^\eta U_1 \geqslant Q^\eta U_2$ holds for $U_1 \geqslant U_2$. We further define a family of linear operators by

$$T^\eta(t) = \operatorname{diag}(T_1^\eta(t), \cdots, T_m^\eta(t)) : X \to X, \quad t \geqslant 0$$

satisfying $T^\eta(0)\Psi = \Psi$ and

$$(T_j^\eta(t)\psi_j)(x) = e^{-\eta t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_j t}} e^{-\frac{y^2}{4 d_j t}} \psi_j(x-y)dy,\ t > 0,$$

with $\Psi = (\psi_j) \in X$ and $j \in J$.

Definition 3.1. A continuous function $U(t,x) : \mathbb{R}^+ \times \mathbb{R} \to [\mathbf{0}, \mathbf{K}]$ is called *an upper (a lower) solution* of system (1.10) if it satisfies

$$U(t,x) \geqslant \ (\leqslant) \ T^\eta(t)U(0,x) + \int_0^t T^\eta(t-s)Q^\eta(U(s,x))ds \ \text{ for } t>0,\ x \in \mathbb{R}.$$

It is easy to see that if the continuous function $U(t,x)$ is differentiable for $t \geqslant 0$ and twice differentiable for almost every $x \in \mathbb{R}$, with the property

$$U_t \geqslant \ (\leqslant) \ DU_{xx} + F(U) \quad \text{for } t>0 \text{ and almost every } \ x \in \mathbb{R},$$

then $U(t,x)$ is an upper (a lower) solution of system (1.10).

Lemma 3.2 *(Comparison Principle). Assume that $F(U)$ is cooperative and continuous. Assume also that $\bar{U}(t,x)$ and $\underline{U}(t,x)$ are a pair of upper and lower solutions of system* (1.10)*. If $\bar{U}(0,x) \geqslant \underline{U}(0,x)$ for $x \in \mathbb{R}$, then $\bar{U}(t,x) \geqslant \underline{U}(t,x)$ holds for $t \geqslant 0$, $x \in \mathbb{R}$.*

The proof of Lemma 3.2 is similar to [9, Lemma 4.1].

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. We first prove part (i). If (2.2) holds, then for any given $\mu \in (0,1)$, there is $t_\mu \geqslant 0$ such that

$$\lim_{x\to+\infty} u_j(t,x) = 0 < \mu K_j, \ \liminf_{x\to-\infty} u_j(t,x) > \mu K_j \text{ for any } t \geqslant t_\mu \text{ and } j \in J.$$

By the continuity of $u_j(t,x)$, we have that $E_j^\mu(t)$ is compact and non-empty for any $t \geqslant t_\mu$ and $j \in J$. So it is sufficient to just prove (2.2).

For any sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying $\lim\limits_{n\to+\infty} x_n = +\infty$, denote $U_n(t,x) = U(t,x+x_n)$. Then by Arzela–Ascoli theorem, $\{U_n(t,x)\}$ converges locally uniformly in $[0,+\infty) \times \mathbb{R}$, up to extraction of a subsequence, to a classical solution $U_\infty(t,x)$ of system (1.10) with initial condition $U_\infty(0,x) = \mathbf{0}$. Therefore, $U_\infty(t,x) \equiv \mathbf{0}$ for any $t \geqslant 0$, $x \in \mathbb{R}$. By the uniqueness of limit, we have $U(t,x) \to \mathbf{0}$ as $x \to +\infty$ for any $t \geqslant 0$.

Since $U_0(x)$ satisfies (1.12) and (2.1), we can find a uniformly continuous function $\tilde{U}_0(x) : \mathbb{R} \to [\mathbf{0}, \mathbf{K}]$ with the properties: (i) $\mathbf{0} \leqslant \tilde{U}_0(x) \leqslant U_0(x)$ for all $x \in \mathbb{R}$; (ii) $\tilde{U}_0(x)$ vanishes at x large enough; (iii) $\liminf\limits_{x\to-\infty} U_0(x) \gg \mathbf{0}$ and $\mathbf{0} \leqslant U_0(x) \ll \mathbf{K}$ for $x \in \mathbb{R}$. Let $\tilde{U}(t,x)$ be the solution of system (1.10) with initial condition $\tilde{U}_0(x)$. Then Lemma 3.2 implies that

$$\mathbf{0} \leqslant \tilde{U}(t,x) \leqslant U(t,x) \leqslant \mathbf{K} \text{ for } t \geqslant 0,\ x \in \mathbb{R}.$$

By Lemma 2.1 (ii), for any $0 < \epsilon' < c^*$ we have that

$$\lim_{t\to\infty} \inf_{x \leqslant (c^*-\epsilon')t} \tilde{U}(t,x) = \mathbf{K}.$$

Then for any $\mathbf{k} \in \mathbb{R}^m$ satisfying $\mathbf{0} \ll \mathbf{k} \ll \mathbf{K}$, there is $t_{\mathbf{k}} \geqslant 0$ such that

$$\mathbf{K}-\mathbf{k} \leqslant \inf_{x \leqslant (c^*-\epsilon')t} \tilde{U}(t,x) \leqslant \inf_{x \leqslant (c^*-\epsilon')t} U(t,x) \leqslant \mathbf{K} \text{ for } t \geqslant t_{\mathbf{k}}. \tag{3.1}$$

So

$$\mathbf{K}-\mathbf{k} \leqslant \liminf_{x\to-\infty} U(t,x) \leqslant \mathbf{K} \text{ for } t \geqslant t_{\mathbf{k}},$$

which means that $\liminf_{x\to-\infty} U(t,x) \to \mathbf{K}$ as $t \to \infty$.

Next we prove part (ii). By the definitions of $\bar{u}(x)$ and $\underline{u}_J(x)$, we have (1.12) and (2.1) also hold for $\bar{u}(x)$ and $\underline{u}_J(x)$, namely,

$$\lim_{x\to+\infty} \bar{u}(x) = 0, \quad \lim_{x\to+\infty} \underline{u}_J(x) = 0, \tag{3.2}$$

and

$$\forall\, \sigma > 0, \;\; \exists\, x_\sigma \in \mathbb{R} \;\text{ s.t. }\; \bar{u}(x) \geqslant e^{-\sigma x}, \;\; \underline{u}_J(x) \geqslant e^{-\sigma x} \;\text{ in }\; [x_\sigma, +\infty). \tag{3.3}$$

Now we begin to prove (2.3). For any fixed $\mu \in (0,1)$, let $\mathbf{k} \ll (1-\mu)\mathbf{K}$. From (3.1) there is $T_1 = \max\{t_{\mathbf{k}}, t_\mu, \xi_0/(c^*-\epsilon')\}$ such that $U(t,x) \gg \mu\mathbf{K}$ in $[T_1,+\infty)\times(-\infty,\xi_0]$. Then every element in $E_j^\mu(t)$ is larger than ξ_0 for $t \geqslant T_1$ and $j \in J$. By the monotone property of $\bar{u}(x)$ in $[\xi_0,+\infty)$, we have

$$\bar{u}(y_j^\mu(t)) \leqslant \bar{u}(\xi_0) \text{ for any } y_j^\mu(t) \in E_j^\mu(t), \;\; j \in J \text{ and } t \geqslant T_1. \tag{3.4}$$

So we just need to prove that there is $T \geqslant T_1$ such that

$$\bar{u}(\xi_0)) \geqslant \gamma e^{-(\lambda_0+\epsilon)t} \text{ and } E_j^\mu(t) \subseteq \bar{u}^{-1}\left\{(\gamma e^{-(\lambda_0+\epsilon)t}, \bar{u}(\xi_0))\right\} \text{ for } t \geqslant T.$$

For any fixed $\epsilon > 0$, by (3.2) and $\bar{u}''(x) = o(\bar{u}(x))$ as $x \to +\infty$, we choose $\xi_\epsilon \geqslant \xi_0$ satisfying

$$\bar{u}(\xi_\epsilon) < \min\left\{\inf_{x\leqslant\xi_0} \bar{u}(x),\;\; h\right\}, \;\; |\bar{u}''(x)| \leqslant \frac{\epsilon}{2d}\bar{u}(x) \text{ for any } x \geqslant \xi_\epsilon,$$

where $d = \max_{j\in J}\{d_j\} > 0$ and the positive number h satisfies $hV \geqslant \mathbf{K}$. Let $\rho = \lambda_0 + \epsilon/2$ and define

$$\bar{U}(t,x) = \left(\bar{u}_j(t,x)\right) = \min\left\{\mathbf{K},\; h\frac{\bar{u}(x)e^{\rho t}}{\bar{u}(\xi_\epsilon)}V\right\} \text{ for } t \geqslant 0, x \in \mathbb{R}.$$

Notice that

$$\bar{U}(0,x) \geqslant \min\{\mathbf{K},\; \bar{u}(x)V\} \geqslant U_0(x) \text{ for } x \in \mathbb{R}, \tag{3.5}$$

$$\mathbf{K} \geqslant \bar{U}(t,x) \geqslant \min\{\mathbf{K},\; hV\} = \mathbf{K} \text{ for } x \leqslant \xi_\epsilon, \;\; t \geqslant 0. \tag{3.6}$$

We claim that $\bar{U}(t,x)$ is an upper solution of system (1.10) in $(0,+\infty)\times\mathbb{R}$. Indeed, for any $j \in J$, if $\bar{u}_j(t,x) = K_j$, it is obviously that

$$\partial_t \bar{u}_j - d_j \triangle \bar{u}_j - f_j(\bar{U}) \geqslant 0,$$

and if $\bar{u}_j(t,x) = h\frac{\bar{u}(x)e^{\rho t}}{\bar{u}(\xi_\epsilon)}v_j < K_j$, then $x \geqslant \xi_\epsilon$ holds from (3.6) and we have

$$\begin{aligned}\partial_t \bar{u}_j - d_j \triangle \bar{u}_j - f_j(\bar{U}) &\geqslant \frac{he^{\rho t}}{\bar{u}(\xi_\epsilon)}(\rho \bar{u}(x)v_j - d_j \bar{u}''(x)v_j - f_j'(\mathbf{0})\bar{u}(x)V) \\ &\geqslant \frac{he^{\rho t}}{\bar{u}(\xi_\epsilon)}[\rho \bar{u}(x) - d|\bar{u}''(x)| - \lambda_0 \bar{u}(x)]v_j \\ &\geqslant 0.\end{aligned}$$

Therefore, combining with (3.5), Lemma 3.2 implies that

$$U(t,x) \leqslant \bar{U}(t,x) \text{ for } t \geqslant 0, x \in \mathbb{R}.$$

For any $t \geqslant T_1$, $j \in J$, and $y_j^\mu(t) \in E_j^\mu(t)$, we have

$$\mu K_j = u_j(t, y_j^\mu(t)) \leqslant \bar{u}_j(t, y_j^\mu(t)) \leqslant h\frac{\bar{u}(y_j^\mu(t))e^{\rho t}}{\bar{u}(\xi_\epsilon)}v_j,$$

which implies that

$$\bar{u}(y_j^\mu(t)) \geqslant \frac{\mu K_j \bar{u}(\xi_\epsilon)}{hv_j}e^{-\rho t}. \tag{3.7}$$

Redefine $T_1 = T_{\mu,\epsilon,\gamma,\lambda_0,V,\mathbf{K}}$ large enough such that

$$e^{\frac{\epsilon}{2}T_1} > \max_{j\in J}\left\{\frac{\gamma h v_j}{\mu K_j \bar{u}(\xi_\epsilon)}\right\}.$$

By (3.4), (3.7) and $y_j^\mu(t) \geqslant \xi_0$, we have

$$\gamma e^{-(\lambda_0+\epsilon)t} < \bar{u}(y_j^\mu(t)) \leqslant \bar{u}(\xi_0) \text{ for } t \geqslant T_1,\ \ j \in J,$$

which implies that (2.3) holds.

Lastly, we prove (2.4). Recall that we have proved there is $T_2 = \max\{t_{\mathbf{k}}, t_\mu, \xi_0/(c^* - \epsilon')\}$ such that every element in $E_j^\mu(t)$ is larger than ξ_0 when $t \geqslant T_2$. So we just need to prove that for fixed $\mu \in (0,1)$, $\epsilon \in (0, \lambda_0)$ and $\Gamma > 0$ there exists $T \geqslant T_2$ such that

$$E_j^\mu(t) \subseteq \underline{u}_J^{-1}\left\{(0, \Gamma e^{-(\lambda_0-\epsilon)t})\right\} \text{ for } t \geqslant T.$$

Because (3.3) holds and $\underline{u}_J''(x) = o(\underline{u}_J(x))$ as $x \to +\infty$, from the first note in [14], we have $|\underline{u}_J'(x)| = o(\underline{u}_J(x))$ as $x \to +\infty$. For fixed $\epsilon \in (0, \lambda_0)$, choose ρ satisfying

$$\max\{\lambda_0 - \epsilon,\ \frac{\lambda_0}{1+\delta}\} < \rho < \lambda_0. \tag{3.8}$$

Let $\xi_\rho \geqslant \xi_0$ satisfy

$$\underline{u}_J(\xi_\rho) \leqslant \inf_{x \leqslant \xi_0} \{u_J(x)\}, \ |\underline{u}_J''(x)| \leqslant l_1 \underline{u}_J(x), \ |\underline{u}_J'(x)|^2 \leqslant l_2 |\underline{u}_J(x)|^2 \text{ for } x \geqslant \xi_\rho, \tag{3.9}$$

where

$$0 < l_1 \leqslant \min\left\{\frac{\lambda_0 - \rho}{d}, \ \frac{\rho(1+\delta) - \lambda_0}{4d(1+\delta)}\right\},$$
$$0 < l_2 \leqslant \frac{\rho(1+\delta) - \lambda_0}{4d\delta(1+\delta)}.$$

By (2.1) and the definition of $\underline{u}_J(x)$, we have $\kappa = \underline{u}_J(\xi_\rho) = \inf_{(-\infty,\xi_\rho]} \{\underline{u}_J(x)\} > 0$. Denote

$$p_1 = \min\{p_0, \kappa\} \text{ and } L = \max\left\{p_1^{-\delta}, \ \frac{2M}{\rho(1+\delta) - \lambda_0}\right\}. \tag{3.10}$$

Introduce the auxiliary function $g(s) : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$g(s) = s - Ls^{1+\delta}. \tag{3.11}$$

By some simple calculations, we have that $g(s) \leqslant 0$ for $s \geqslant p_1$ and $g(s) \leqslant p_1 \leqslant \kappa$ for $s > 0$.

Define a lower solution by

$$\underline{U}(t,x) = (\underline{u}_j(t,x)) = \max\left\{\mathbf{0}, g\left(p_1 \frac{\underline{u}_J(x)}{\underline{u}_J(\xi_\rho)} e^{\rho t}\right) V\right\} \text{ for } t \geqslant 0, x \in \mathbb{R}.$$

Then

$$\underline{U}(0,x) \leqslant \max\left\{\mathbf{0}, \ \min\left\{\kappa, p_1 \frac{\underline{u}_J(x)}{\underline{u}_J(\xi_\rho)}\right\} V\right\} \leqslant \min\left\{\kappa, \ \underline{u}_J(x)\right\} V \text{ for } x \in \mathbb{R} \tag{3.12}$$

and

$$\underline{U}(t,\xi_\rho) = \max\left\{\mathbf{0}, g\left(p_1 e^{\rho t}\right) V\right\} = \mathbf{0} \text{ for } t \geqslant 0.$$

Since $\underline{u}_J(x) \geqslant \underline{u}_J(\xi_\rho)$ for $x \leqslant \xi_\rho$, then

$$\underline{U}(t,x) = \mathbf{0} \text{ for } x \leqslant \xi_\rho, \ t \geqslant 0. \tag{3.13}$$

We now prove that $\underline{U}(t,x)$ is a lower solution in $(0,+\infty) \times \mathbb{R}$. Indeed, when $\underline{U}(t,x) = \mathbf{0}$, it can be easily checked by $F(\mathbf{0}) = \mathbf{0}$. If $\underline{U}(t,x) = g\left(p_1 \frac{\underline{u}_J(x)}{\underline{u}_J(\xi_\rho)} e^{\rho t}\right) V > \mathbf{0}$, then $x \geqslant \xi_\rho$ holds from (3.13). By (A3) and $0 \leqslant g(s) \leqslant p_1 \leqslant p_0$ for $s > 0$, omitting some simple calculations, we have

$$
\begin{aligned}
&\underline{U}_t(t,x) - D\underline{U}_{xx}(t,x) - F(\underline{U}(t,x)) \\
&\leqslant \underline{U}_t(t,x) + d|\underline{U}_{xx}(t,x)| - F'(\mathbf{0})\underline{U}(t,x) + M\left[g\left(p_1\frac{\underline{u}_J(x)}{\underline{u}_J(\xi_\rho)}e^{\rho t}\right)\right]^{1+\delta} V \\
&\leqslant \frac{p_1 e^{\rho t}}{\underline{u}_J(\xi_\rho)} H_1(x) V + L\frac{p_1^{1+\delta}e^{\rho(1+\delta)t}}{\left[\underline{u}_J(\xi_\rho)\right]^{1+\delta}} H_2(x) V
\end{aligned}
$$

where

$$
\begin{aligned}
&H_1(x) = (\rho - \lambda_0)\underline{u}_J(x) + d|\underline{u}_J''(x)| \\
&H_2(x) = \left[\underline{u}_J(x)\right]^{\delta-1}\left[q_1|\underline{u}_J(x)|^2 + q_2|\underline{u}_J'(x)|^2 + q_3|\underline{u}_J(x)\underline{u}_J''(x)|\right] \\
&q_1 = -\rho(1+\delta) + \lambda_0 + \frac{M}{L},\ q_2 = d\delta(1+\delta),\ q_3 = d(1+\delta).
\end{aligned}
$$

By (3.8)–(3.10), we have $H_1(x) \leqslant 0$ and $H_2(x) \leqslant 0$ for $x \geqslant \xi_\rho$, which imply that

$$
\underline{U}_t(t,x) - D\underline{U}_{xx}(t,x) - F(\underline{U}(t,x)) \leqslant 0 \text{ for } t > 0, x \geqslant \xi_\rho. \tag{3.14}
$$

Therefore, the function $\underline{U}(t,x)$ is a lower solution in $(0,+\infty)\times\mathbb{R}$.

Let $\underline{U}^*(t,x)$ be the solution of system (1.10) with initial condition $\underline{u}^*(x)V$, where

$$
\underline{u}^*(x) = \begin{cases} \underline{u}_J(x), & x \geqslant \xi_\rho, \\ \kappa = \underline{u}_J(\xi_\rho), x < \xi_\rho. \end{cases}
$$

Since $\underline{u}^*(x)$ is nonincreasing in $\mathbb{R}$, it follows that $\underline{U}^*(t,x)$ is also nonincreasing in $x \in \mathbb{R}$ for any $t \geqslant 0$. From $\underline{u}_J(x) \geqslant \underline{u}_J(\xi_\rho)$ for $x \in (-\infty, \xi_\rho)$, combining with (3.12), we have

$$
\underline{U}(0,x) \leqslant \underline{u}^*(x)V \leqslant \underline{u}_J(x)V \leqslant U_0(x),\ \text{ for } x \in \mathbb{R}. \tag{3.15}
$$

By Lemma 3.2, from (3.13)–(3.15), we have

$$
\underline{U}(t,x) \leqslant \underline{U}^*(t,x) \leqslant U(t,x) \text{ for } t \geqslant 0, x \in \mathbb{R}. \tag{3.16}
$$

Choose some constant $\omega \in (0, L^{-\frac{1}{\delta}})$, and denote $\theta = g(\omega) > 0$. Since $\underline{u}_J(x)$ is nonincreasing in $[\xi_0, +\infty)$ and (3.2) holds, there is $T_2' = (\ln\omega - \ln p_1)/\rho$ such that the set

$$
O_\omega(t) = \left\{x \in [\xi_\rho, +\infty),\ p_1\frac{\underline{u}_J(x)}{\underline{u}_J(\xi_\rho)}e^{\rho t} = \omega\right\},
$$

is non-empty when $t \geqslant T_2'$. Denote $z_\omega(t) = \sup\{O_\omega(t)\} \geqslant \xi_\rho$. Then the function $z_\omega(t) : [T_2', +\infty) \to [\xi_\rho, +\infty)$ is nondecreasing and right-continuous. Moreover, the discontinuous points appear (only) at time t satisfying that there is a closed interval of x, which is not a single point and where $\underline{u}_J(x) = \frac{\omega e^{-\rho t}}{p_1}\underline{u}_J(\xi_\rho)$ holds. Here this closed interval can be written as $[z_\omega(t^-), z_\omega(t^+)]$.

By (3.16), we have $\underline{U}^*(t, z_\omega(t)) \geqslant \underline{U}(t, z_\omega(t)) = g(p_1 \frac{\underline{u}_J(z_\omega(t))}{\underline{u}_J(\xi_\rho)} e^{\rho t})V = \theta V$ for $t \geqslant T_2'$. Since $\underline{U}^*(t, x)$ is nonincreasing in $x \in \mathbb{R}$ for any $t \geqslant 0$, we have

$$\underline{U}^*(t, x) \geqslant \theta V \quad \text{for } x \in (-\infty, z_\omega(t)],\ t \geqslant T_2'.$$

Denote $\mu^* = \min\limits_{j \in J}\{\frac{v_j}{K_j}\}\theta$, then (3.16) implies that

$$U(t, x) \geqslant \theta V \geqslant \mu^* \mathbf{K} \quad \text{for } x \in (-\infty, z_\omega(t)],\ t \geqslant T_2'. \tag{3.17}$$

If $0 < \mu < \mu^*$, then for any $t \geqslant \max\{T_2, T_2'\}$, $j \in J$ and $y_j^\mu(t) \in E_j^\mu(t)$ we have

$$y_j^\mu(t) \geqslant z_\omega(t) \geqslant \xi_\rho \geqslant \xi_0.$$

Since $\underline{u}_J(x)$ is nonincreasing in $[\xi_0, +\infty)$, we have

$$\underline{u}_J(y_j^\mu(t)) \leqslant \underline{u}_J(z_\omega(t)) = \frac{\underline{u}_J(\xi_\rho)\omega}{p_1} e^{-\rho t}.$$

For fixed $\Gamma > 0$ and $\epsilon \in (0, \lambda_0)$, redefine $T_2 = T_{\mu,\epsilon,\Gamma,\lambda_0} \geqslant T_2'$ such that

$$e^{(\rho-\lambda_0+\epsilon)T_2} > \frac{\underline{u}_J(\xi_\rho)\omega}{p_1\Gamma}.$$

Therefore, $\underline{u}_J(y_j^\mu(t)) < \Gamma e^{-(\lambda_0-\epsilon)t}$ for any $t \geqslant T_2$.

If $\mu \geqslant \mu^*$, let $U_{\mu^*}(t, x)$ be the solution of system (1.10) with initial condition

$$U_{\mu^*}(0, x) = \begin{cases} \mu^* \mathbf{K}, & x \in (-\infty, -1], \\ -x\mu^* \mathbf{K}, & x \in (-1, 0), \\ \mathbf{0}, & x \in [0, +\infty). \end{cases}$$

By (3.17), we have $U(\tau, x + z_\omega(\tau)) \geqslant U_{\mu^*}(0, x)$ for any $\tau \geqslant T_2'$, $x \in \mathbb{R}$. Lemma 3.2 implies

$$U(t + \tau, x + z_\omega(\tau)) \geqslant U_{\mu^*}(t, x) \text{ for } t \geqslant 0,\ \tau \geqslant T_2',\ x \in \mathbb{R}. \tag{3.18}$$

Let $\mathbf{0} \ll \mathbf{k} \ll (1-\mu)\mathbf{K}$, then by Lemma 2.1 (ii) there exists $t_{\mathbf{k}} \geqslant 0$ so that

$$\mu \mathbf{K} \ll \mathbf{K} - \mathbf{k} \leqslant \inf_{x \leqslant (c^*-\epsilon')t} U_{\mu^*}(t, x) \leqslant \mathbf{K} \text{ for } t \geqslant t_{\mathbf{k}}.$$

Combining with (3.18), by the arbitrariness of τ, we have

$$\mu \mathbf{K} \ll \inf_{x \leqslant (c^*-\epsilon')t_{\mathbf{k}}} U_{\mu^*}(t_{\mathbf{k}}, x) \leqslant \inf_{x \leqslant z_\omega(t-t_{\mathbf{k}})+(c^*-\epsilon')t_{\mathbf{k}}} U(t, x) \text{ for } t \geqslant t_{\mathbf{k}} + T_2'. \tag{3.19}$$

Then (3.19) implies that for $t \geqslant t_{\mathbf{k}} + T_2'$, $j \in J$ and $y_j^\mu(t) \in E_j^\mu(t)$,

$$y_j^{\mu}(t) \geqslant z_\omega(t-t_{\mathbf{k}}) + (c^* - \epsilon')t_{\mathbf{k}} \geqslant z_\omega(t-t_{\mathbf{k}}) \geqslant \xi_\rho \geqslant \xi_0,$$

which implies that

$$\underline{u}_J(y_j^{\mu}(t)) \leqslant \underline{u}_J(z_\omega(t-t_{\mathbf{k}})) \leqslant \frac{\underline{u}_J(\xi_\rho)\omega}{p_1} e^{-\rho(t-t_{\mathbf{k}})}.$$

For fixed $\Gamma > 0$ and $\epsilon \in (0, \lambda_0)$, redefine $T_2 = T_{\mu,\epsilon,\Gamma,\lambda_0} \geqslant t_{\mathbf{k}} + T_2'$ large enough so that

$$e^{(\rho+\epsilon-\lambda_0)T_2} > \frac{\underline{u}_J(\xi_\rho)\omega}{p_1\Gamma} e^{\rho t_{\mathbf{k}}}.$$

Therefore, $\underline{u}_J(y_j^{\mu}(t)) < \Gamma e^{-(\lambda_0-\epsilon)t}$ for any $t \geqslant T_2$. This completes the proof of (2.4).

The proof of Theorem 2.2 is complete. □

Remark 3.3. Since $U(t,x)$ is not monotonous, we introduce a monotone function $\underline{U}^*(t,x)$ to prove $U(t,x) \geqslant \theta V$ in $[T_2', +\infty) \times (-\infty, z_\omega(t)]$. This makes it easier than the method in [14], where the parabolic maximum principle was used.

Before proving Theorem 2.3, we introduce the following auxiliary condition.

(A4) For any $j \in J \backslash I$, u_j is connected to $\{u_i | i \in I\}$ with respect to $F'(\mathbf{0})$, in the sense that there is $i \in I$ such that $f_{j,i} > 0$.

We first give the proof under condition (A4), where u_j with $J\backslash I$ is accelerated directly by u_i with $i \in I$.

Proof of Theorem 2.3 Under (A4). The proofs of property (i) and (2.3) are similar to Theorem 2.2, so we just need to prove (2.7). For any fixed $\epsilon \in (0, \tilde{\lambda}_0)$, choose $\tilde{\rho} > 0$ such that

$$\max\left\{\tilde{\lambda}_0 - \epsilon,\ \frac{2+\delta}{2+2\delta}\tilde{\lambda}_0\right\} < \tilde{\rho} < \min\left\{\tilde{\lambda}_0,\ \frac{2+\delta}{2+2\delta}\tilde{\lambda}_0 + M\tilde{p}_1^{\delta}\right\}.$$

Define $\underline{U}(t,x) = (\underline{u}_j(t,x))$ for $j \in J$ such that

$$(\underline{u}_i(t,x))_{i\in I} = \max\left\{\mathbf{0},\ \tilde{g}\left(\tilde{p}_1 \frac{\underline{u}_I(x)}{\underline{u}_I(\xi_{\tilde{\rho}})} e^{\tilde{\rho}t}\right) r\tilde{V}\right\}, \tag{3.20}$$

where

$$\tilde{g}(s) = s - \tilde{L}s^{1+\delta}, \quad r\min_{i\in I}\{\tilde{v}_i\} \geqslant \delta^{\frac{1}{1+\delta}},$$

$$\tilde{L} = \frac{2M}{(1-k)[\tilde{\rho}(1+\delta)-\tilde{\lambda}_0]} > \frac{\tilde{p}_1^{-\delta}}{1+\delta},$$

$$k = \frac{(\tilde{\lambda}_0-\tilde{\rho})(1+\delta)}{\tilde{\rho}(1+\delta)-\tilde{\lambda}_0} < 1,$$

$$\tilde{p}_1 = \min\left\{\frac{p_0}{r}, \inf_{(-\infty,\xi_{\tilde{\rho}}]}\{\underline{u}_I(x)\}\right\} > 0,$$

and $\xi_{\tilde{\rho}} \geqslant \xi_0$ satisfying

$$\underline{u}_I(\xi_{\tilde{\rho}}) < \inf_{x\leqslant\xi_0}\{\underline{u}_I(x)\},$$

$$|\underline{u}_I''(x)| \leqslant \tilde{l}_1\underline{u}_I(x), \ |\underline{u}_I'(x)|^2 \leqslant \tilde{l}_2|\underline{u}_I(x)|^2, \ x\in[\xi_{\tilde{\rho}}, +\infty),$$

$$\tilde{l}_1 = \frac{\tilde{\lambda}_0-\tilde{\rho}}{d} = k\cdot\frac{\tilde{\rho}(1+\delta)-\tilde{\lambda}_0}{d(1+\delta)},$$

$$\tilde{l}_2 = \frac{1-k}{2}\cdot\frac{\tilde{\rho}(1+\delta)-\tilde{\lambda}_0}{d(1+\delta)\delta}, \ d = \max_{j\in J}\{d_j\}.$$

For fixed $j \in J\backslash I$, there exists $i \in I$ such that $f_{j,i} > 0$ by (A4). Let a_j satisfy

$$0 < a_j \leqslant \min\left\{ f_{j,i}, \ |f_{j,j}| + \tilde{\lambda}_0\right\}.$$

Define

$$\underline{u}_j(t,x) = \max\left\{0, G_j(t, \underline{u}_I(x))\right\} \text{ for } j \in J\backslash I, \tag{3.21}$$

where

$$G_j(t,u) = A_j u\left(e^{\tilde{\rho}t} - e^{-R_j t}\right) - B_j u^{1+\delta}\left(e^{\tilde{\rho}(1+\delta)t} - e^{-S_j t}\right),$$

$$A_j = \frac{a_j\tilde{p}_1 r\tilde{v}_i}{\underline{u}_I(\xi_{\tilde{\rho}})(R_j+\tilde{\rho})}, \quad B_j = \frac{C_j}{S_j+\tilde{\rho}(1+\delta)},$$

$$C_j = \tilde{L}\frac{a_j(\tilde{p}_1)^{1+\delta}r\tilde{v}_i}{[\underline{u}_I(\xi_{\tilde{\rho}})]^{1+\delta}} + M\frac{(\tilde{p}_1 r\tilde{v}_i)^{1+\delta}}{[\underline{u}_I(\xi_{\tilde{\rho}})]^{1+\delta}} + M(A_j)^{1+\delta},$$

$$R_j = |f_{j,j}| + d\tilde{l}_1, \quad S_j = |f_{j,j}| - d(1+\delta)(\tilde{l}_1+\delta\tilde{l}_2).$$

It is clear that $R_j - S_j = d\tilde{l}_1 + d(1+\delta)(\tilde{l}_1+\tilde{l}_2\delta) = \tilde{\rho}\delta - M\tilde{L}^{-1}$. For $t > 0$, denote

$$W_i(t) = \left\{x \;\middle|\; \tilde{g}\left(\tilde{p}_1\frac{\underline{u}_I(x)}{\underline{u}_I(\xi_{\tilde{\rho}})}e^{\tilde{\rho}t}\right) = 0\right\},$$

$$W_j(t) = \left\{x \mid G_j(t, \underline{u}_I(x)) = 0\right\}.$$

We claim that $\xi_{\tilde{\rho}} < x_i(t) \leqslant x_j(t)$ for any $x_i(t) \in W_i(t)$, $x_j(t) \in W_j(t)$ and $t > 0$. Indeed, from the definitions of $\underline{u}_i(t,x)$ and $\underline{u}_j(t,x)$, we have

$$\underline{u}_I(x_i(t))^\delta e^{\tilde{\rho}\delta t} = \frac{[\underline{u}_I(\xi_{\tilde{\rho}})]^\delta}{\tilde{L}\tilde{p}_1^\delta}, \quad \underline{u}_I(x_i(t)) < \underline{u}_I(\xi_{\tilde{\rho}})$$

and

$$\begin{aligned}
&\underline{u}_I(x_j(t))^\delta e^{\tilde{\rho}\delta t} \\
&= \frac{A_j}{B_j} \cdot \frac{e^{\tilde{\rho}(1+\delta)t} - e^{(\tilde{\rho}\delta - R_j)t}}{e^{\tilde{\rho}(1+\delta)t} - e^{-S_j t}} \\
&\leqslant \frac{A_j}{B_j} \leqslant \frac{[\underline{u}_I(\xi_{\tilde{\rho}})]^\delta}{\tilde{L}\tilde{p}_1^\delta} \cdot \frac{S_j + \tilde{\rho}(1+\delta)}{R_j + \tilde{\rho}} \cdot \frac{1}{1 + r^\delta \tilde{v}_i^\delta M \tilde{L}^{-1}\left(\frac{1}{a_j} + \frac{a_j^\delta}{(R_j+\tilde{\rho})^{1+\delta}}\right)} \\
&\leqslant \frac{[\underline{u}_I(\xi_{\tilde{\rho}})]^\delta}{L\tilde{p}_1^\delta} \cdot \frac{S_j + \tilde{\rho}(1+\delta)}{R_j + \tilde{\rho}} \cdot \frac{1}{1 + r^\delta \tilde{v}_i^\delta M \tilde{L}^{-1} \delta^{-\frac{\delta}{1+\delta}} (R_j + \tilde{\rho})^{-1}} \\
&\leqslant \frac{[\underline{u}_I(\xi_{\tilde{\rho}})]^\delta}{L\tilde{p}_1^\delta} \cdot \frac{S_j + \tilde{\rho}(1+\delta)}{R_j + \tilde{\rho} + M\tilde{L}^{-1}} \\
&\leqslant \frac{[\underline{u}_I(\xi_{\tilde{\rho}})]^\delta}{\tilde{L}\tilde{p}_1^\delta}.
\end{aligned}$$

Then

$$\underline{u}_I(x_j(t)) \leqslant \underline{u}_I(x_i(t)) < \underline{u}_I(\xi_{\tilde{\rho}}) < \inf_{x \leqslant \xi_0} \{\underline{u}_I(x)\} \text{ for } t > 0.$$

It implies that

$$\xi_0 \leqslant \xi_{\tilde{\rho}} \leqslant x_i(t) \leqslant x_j(t). \tag{3.22}$$

Moreover, it is easy to see that for $t > 0$

$$\underline{u}_i(t,x) = \begin{cases} \tilde{g}\left(\tilde{p}_1 \dfrac{\underline{u}_I(x)}{\underline{u}_I(\xi_{\tilde{\rho}})} e^{\tilde{\rho}t}\right) r\tilde{v}_i, & x \in [x_i(t), +\infty), \\ 0, & x \in (-\infty, x_i(t)) \end{cases} \tag{3.23}$$

and

$$\underline{u}_j(t,x) = \begin{cases} G_j(t, \underline{u}_I(x)), & x \in [x_j(t), +\infty), \\ 0, & x \in (-\infty, x_j(t)). \end{cases} \tag{3.24}$$

Next, we prove that $\underline{u}_j(t,x) \leqslant p_0$ for any $t \geqslant 0$, $x \in \mathbb{R}$ and $j \in J \backslash I$. When $\underline{u}_j(t,x) = G_j(t, \underline{u}_I(x)) \geqslant 0$, we have

$$\begin{aligned}
\underline{u}_j(t,x) &\leqslant \max\{0,\ A_j\underline{u}_I(x)(e^{\tilde{\rho}t}-e^{-R_jt})-B_j[\underline{u}_I(x)]^{1+\delta}(e^{\tilde{\rho}t}-e^{-R_jt})e^{\tilde{\rho}\delta t}\}\\
&\leqslant \max\{0,\ A_j\underline{u}_I(x)e^{\tilde{\rho}t}-B_j[\underline{u}_I(x)]^{1+\delta}e^{\tilde{\rho}(1+\delta)t}\}\\
&\leqslant A_j(\frac{A_j}{B_j})^{\frac{1}{\delta}}\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}}\\
&\leqslant \frac{a_j\tilde{p}_1r\tilde{v}_i}{\underline{u}_I(\xi_{\tilde{\rho}})(R_j+\tilde{\rho})}\cdot\frac{\underline{u}_I(\xi_{\tilde{\rho}})}{\tilde{L}^{\frac{1}{\delta}}\tilde{p}_1}\cdot\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}}\\
&\leqslant \tilde{p}_1r\tilde{v}_i\\
&\leqslant p_0.
\end{aligned}$$

Now, we prove $\underline{U}(t,x)=(\underline{u}_j(t,x))$ defined by (3.20) and (3.21) is a lower solution. When $j\in I$, by using the same method in the proof of Theorem 2.2, we have

$$\partial_t\underline{u}_j-d_j\partial_{xx}\underline{u}_j-f_j(\underline{U})\leqslant\partial_t\underline{u}_j-d_j\partial_{xx}\underline{u}_j-\tilde{f}_j(\{\underline{u}_i\}_{i\in I})\leqslant 0 \text{ for } x\in\mathbb{R}, t>0.$$

When $j\in J\backslash I$ and

$$\underline{u}_j(t,x)=G_j(t,\underline{u}_I(x))\geqslant 0,$$

omitting some calculations, by (3.22)–(3.24) we obtain that for $x\geqslant x_j(t)$ and $t>0$,

$$\begin{aligned}
&\partial_t\underline{u}_j-d_j\partial_{xx}\underline{u}_j-f_j(\underline{U})\\
\leqslant\ &\partial_t\underline{u}_j+d|\partial_{xx}\underline{u}_j|-f_j(\underline{u}_i,\underline{u}_j,\mathbf{0})\\
\leqslant\ &\partial_t\underline{u}_j+d|\partial_{xx}\underline{u}_j|-a_j\underline{u}_i+|f_{j,j}|\underline{u}_j+M(\underline{u}_i^{1+\delta}+\underline{u}_j^{1+\delta})\\
\leqslant\ &H_1e^{\tilde{\rho}t}\underline{u}_I(x)+H_2e^{-R_jt}\underline{u}_I(x)+H_3e^{\tilde{\rho}(1+\delta)t}[\underline{u}_I(x)]^{1+\delta}+H_4e^{-S_jt}[\underline{u}_I(x)]^{1+\delta},
\end{aligned}$$

where

$$\begin{aligned}
H_1&=A_j\tilde{\rho}+A_jd\tilde{l}_1+A_j|f_{j,j}|-a_j\frac{\tilde{p}_1r\tilde{v}_i}{\underline{u}_I(\xi_{\tilde{\rho}})}=0,\\
H_2&=A_jR_j-A_jd\tilde{l}_1-A_j|f_{j,j}|=0,\\
H_3&=B_j\Big[-\tilde{\rho}(1+\delta)+d(1+\delta)(\tilde{l}_1+\delta\tilde{l}_2)-|f_{j,j}|\Big]\\
&\qquad+\tilde{L}\frac{a_j(\tilde{p}_1)^{1+\delta}r\tilde{v}_i}{[\underline{u}_I(\xi_{\tilde{\rho}})]^{1+\delta}}+M\frac{(\tilde{p}_1r\tilde{v}_i)^{1+\delta}}{[\underline{u}_I(\xi_{\tilde{\rho}})]^{1+\delta}}+M(A_j)^{1+\delta}\\
&=-B_j(S_j+\tilde{\rho}(1+\delta))+C_j=0,\\
H_4&=B_j\Big[-S_j-d(1+\delta)(\tilde{l}_1+\delta\tilde{l}_2)+|f_{j,j}|\Big]=0.
\end{aligned}$$

Therefore,

$$\underline{U}_t(t,x)-D\underline{U}_{xx}(t,x)-F(\underline{U}(t,x))\leqslant\mathbf{0} \text{ for } x\in\mathbb{R},\ t>0.$$

Combining with

$$\underline{u}_i(0,x) \leqslant r\underline{u}_I(x)\tilde{v}_i \leqslant u_{0,i}(x) \text{ for } i \in I,\ x \in \mathbb{R},$$
$$\underline{u}_j(0,x) = 0 \leqslant u_{0,j}(x) \text{ for } j \in J\backslash I,\ x \in \mathbb{R},$$

by Lemma 3.2, we have

$$\underline{U}(t,x) \leqslant U(t,x) \text{ for } x \in \mathbb{R},\ t \geqslant 0.$$

By the definition of $\underline{U}(t,x)$, there exists $T > 0$ such that for $j \in J\backslash I$, $t \geqslant T$, $x \in \mathbb{R}$,

$$\max\left\{0, \frac{A_j}{2}\underline{u}_I(x)e^{\tilde{\rho}t} - B_j[\underline{u}_I(x)]^{1+\delta}e^{\tilde{\rho}(1+\delta)t}\right\} \leqslant \underline{u}_j(t,x) \leqslant u_j(t,x).$$

Then (2.7) can be proved by the same method as in the proof of Theorem 2.2. □

Proof of Theorem 2.3. For all $k \in J\backslash I$, now we structure a lower solution for the kth equation in system (1.10) under the assumption that $u_k(t,x)$ is connected to $u_j(t,x)$ with $j \in J\backslash I$, in the sense that $f_{k,j} > 0$. We state that all the notations in the proof with (A4) are valid.

We assume that there is a lower solution $\underline{u}_j(t,x)$ of the jth equation in system (1.10) satisfying

$$\underline{u}_j(t,x) = \max\left\{0,\ \underline{u}_I(x)\eta_j(t) - [\underline{u}_I(x)]^{1+\delta}\zeta_j(t)\right\}, \tag{3.25}$$

where

$$\eta_j(t) = A_j\left(e^{\tilde{\rho}t} - e^{-R_jt}\right),\ \ \zeta_j(t) = B_j\left(e^{\tilde{\rho}(1+\delta)t} - e^{-S_jt}\right),$$
$$R_j = |f_{j,j}| + d\tilde{l}_1,\ \ S_j = |f_{j,j}| - d(1+\delta)(\tilde{l}_1 + \delta\tilde{l}_2),$$

and A_j and B_j are two positive constants. Further, we assume that

$$A_j(\frac{A_j}{B_j})^{\frac{1}{\delta}}\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \leqslant p_0 \text{ and } x_j(t) \geqslant \xi_{\tilde{\rho}} \text{ for any } x_j(t) \in W_j(t),$$

where

$$W_j(t) = \left\{x\ \middle|\ \underline{u}_I(x)\eta_j(t) - [\underline{u}_I(x)]^{1+\delta}\zeta_j(t) = 0\right\} \text{ for } t > 0.$$

Then

$$\begin{aligned}\underline{u}_j(t,x) &\leqslant \max\left\{0,\ A_j\underline{u}_I(x)(e^{\tilde{\rho}t} - e^{-R_jt}) - B_j[\underline{u}_I(x)]^{1+\delta}(e^{\tilde{\rho}t} - e^{-R_jt})e^{\tilde{\rho}\delta t}\right\}\\ &\leqslant \max\left\{0,\ A_j\underline{u}_I(x)e^{\tilde{\rho}t} - B_j[\underline{u}_I(x)]^{1+\delta}e^{\tilde{\rho}(1+\delta)t}\right\}\\ &\leqslant A_j(\frac{A_j}{B_j})^{\frac{1}{\delta}}\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \leqslant p_0.\end{aligned}$$

Let a_k satisfy

$$0 < a_k \leqslant \min\left\{ f_{k,j},\ |f_{k,k}| + \tilde{\lambda}_0,\ \frac{b_j(\tilde{\rho} + R_k)M}{(1-b_j)(\tilde{\rho}+R_k) + M\tilde{L}^{-1}} \cdot \frac{A_j^{1+\delta}}{B_j} \right\}, \tag{3.26}$$

where

$$0 < b_j = \inf_{t>0} \left\{ \frac{e^{|f_{j,j}|t} - e^{-(\tilde{\rho}+d\tilde{l}_1)t}}{e^{|f_{j,j}|t} - e^{-(\tilde{\rho}+d\tilde{l}_1+M\tilde{L}^{-1})t}} \right\} < 1.$$

Define

$$\underline{u}_k(t,x) = \max\left\{0,\ \underline{u}_I(x)\eta_k(t) - [\underline{u}_I(x)]^{1+\delta}\zeta_k(t)\right\}, \tag{3.27}$$

where

$$\begin{aligned}
&\eta_k(t) = A_k\left(e^{\tilde{\rho}t} - e^{-R_k t}\right),\quad \zeta_k(t) = B_k\left(e^{\tilde{\rho}(1+\delta)t} - e^{-S_k t}\right),\\
&R_k = |f_{k,k}| + d\tilde{l}_1,\quad S_k = |f_{k,k}| - d(1+\delta)(\tilde{l}_1 + \delta\tilde{l}_2),
\end{aligned}$$

and

$$A_k = \frac{a_k A_j}{\tilde{\rho} + R_k} \leqslant A_j,\quad B_k = \frac{a_k B_j + M A_j^{1+\delta} + M A_k^{1+\delta}}{\tilde{\rho}(1+\delta) + S_k}. \tag{3.28}$$

For $t > 0$, denote

$$W_k(t) = \left\{ x \;\middle|\; \underline{u}_I(x)\eta_k(t) - [\underline{u}_I(x)]^{1+\delta}\zeta_k(t) = 0 \right\}.$$

We claim that $x_k(t) \geqslant x_j(t)$ for any $x_j(t) \in W_j(t)$, $x_k(t) \in W_k(t)$ and $t > 0$. Indeed, (3.25) and (3.27) imply that

$$\begin{aligned}
\frac{\underline{u}_I(x_k(t))^\delta}{\underline{u}_I(x_j(t))^\delta} &= \frac{\eta_k(t)\zeta_j(t)}{\eta_j(t)\zeta_k(t)}\\
&= \frac{A_k B_j}{B_k A_j} \cdot \frac{(e^{\tilde{\rho}t} - e^{-R_k t})(e^{\tilde{\rho}(1+\delta)t} - e^{-S_j t})}{(e^{\tilde{\rho}(1+\delta)t} - e^{-S_k t})(e^{\tilde{\rho}t} - e^{-R_j t})}\\
&= \frac{A_k B_j}{B_k A_j} \cdot \frac{e^{|f_{k,k}|t} - e^{-(\tilde{\rho}+d\tilde{l}_1)t}}{e^{|f_{k,k}|t} - e^{-(\tilde{\rho}+d\tilde{l}_1+M\tilde{L}^{-1})t}} \cdot \frac{e^{|f_{j,j}|t} - e^{-(\tilde{\rho}+d\tilde{l}_1+M\tilde{L}^{-1})t}}{e^{|f_{j,j}|t} - e^{-(\tilde{\rho}+d\tilde{l}_1)t}}\\
&\leqslant \frac{1}{b_j} \cdot \frac{A_k B_j}{B_k A_j}.
\end{aligned}$$

By (3.26) and (3.28), we have

$$\begin{aligned}
\frac{A_k B_j}{B_k A_j} &= \frac{a_k}{\tilde{\rho}+R_k}\cdot\frac{B_j\big[\tilde{\rho}(1+\delta)+S_k\big]}{a_k B_j + M A_j^{1+\delta} + M A_k^{1+\delta}} \\
&= \frac{\tilde{\rho}(1+\delta)+S_k}{\tilde{\rho}+R_k}\cdot\frac{1}{1+M\frac{A_j^{1+\delta}}{a_k B_j}+M\frac{a_k^{\delta}A_j^{1+\delta}}{B_j(\tilde{\rho}+R_k)^{1+\delta}}} \\
&\leqslant \frac{\tilde{\rho}+R_k+M\tilde{L}^{-1}}{\tilde{\rho}+R_k+(\tilde{\rho}+R_k)M\frac{A_j^{1+\delta}}{a_k B_j}} \\
&\leqslant b_j < 1.
\end{aligned}$$

Then $\underline{u}_I(x_k(t)) \leqslant \underline{u}_I(x_j(t))$, which implies that

$$\xi_{\tilde{\rho}} \leqslant x_j(t) \leqslant x_k(t) \text{ for all } t>0. \tag{3.29}$$

Moreover, we have

$$\underline{u}_k(t,x) \leqslant A_k\Big(\frac{A_k}{B_k}\Big)^{\frac{1}{\delta}}\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \leqslant A_j\Big(\frac{A_j}{B_j}\Big)^{\frac{1}{\delta}}\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \leqslant p_0.$$

By some simple calculations, it is easy to see that

$$\begin{aligned}
&\partial_t \underline{u}_k - d_k\partial_{xx}\underline{u}_k - f_k(\underline{U}) \\
&\leqslant \partial_t \underline{u}_k + d|\partial_{xx}\underline{u}_k| - f_k(\underline{u}_j, \underline{u}_k, \mathbf{0}) \\
&\leqslant \partial_t \underline{u}_k + d|\partial_{xx}\underline{u}_k| - a_k\underline{u}_j + |f_{k,k}|\underline{u}_k + M(\underline{u}_j^{1+\delta}+\underline{u}_k^{1+\delta}) \\
&\leqslant 0.
\end{aligned}$$

Therefore, $\underline{u}_k(t,x)$ is a lower solution of the kth equation in system (1.10) and it holds that

$$A_k\Big(\frac{A_k}{B_k}\Big)^{\frac{1}{\delta}}\frac{\delta}{(1+\delta)^{1+\frac{1}{\delta}}} \leqslant p_0 \text{ and } x_k(t) \geqslant \xi_{\tilde{\rho}} \text{ for any } x_k(t)\in W_k(t).$$

Repeating this process until there is a lower solution satisfying (3.27) of the kth equation in system (1.10) for all $k \in J\backslash I$. The repeated process can been finished because the matrix $F'(\mathbf{0})$ is irreducible.

By Lemma 3.2, there exists $T>0$ such that for all $k \in J\backslash I$, $t \geqslant T$, $x \in \mathbb{R}$,

$$\max\left\{0, \frac{A_k}{2}\underline{u}_I(x)e^{\tilde{\rho}t} - B_k[\underline{u}_I(x)]^{1+\delta}e^{\tilde{\rho}(1+\delta)t}\right\} \leqslant \underline{u}_k(t,x) \leqslant u_k(t,x).$$

Then the proof is finished by the same method as in the proof of Theorem 2.2. □

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References

[1] M. Alfaro, Slowing Allee effect versus accelerating heavy tails in monostable reaction diffusion equations, Nonlinearity 30 (2017) 687–702.
[2] M. Alfaro, J. Coville, Propagation phenomena in monostable integro-differential equations: acceleration or not?, J. Differential Equations 263 (2017) 5727–5758.
[3] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. Math. 39 (1978) 33–76.
[4] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: J.A. Goldstein (Ed.), Partial Differential Equations and Related Topics, in: Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5–49.
[5] M.R. Booty, R. Haberman, A.A. Minzoni, The accommodation of traveling waves of Fisher's type to the dynamics of the leading tail, SIAM J. Appl. Math. 53 (1993) 1009–1025.
[6] X. Cabré, J.M. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equations, Comm. Math. Phys. 320 (2013) 679–722.
[7] V. Capasso, S.L. Paveri-Fontana, A mathematical model for the 1973 cholera epidemic in the European Mediterranean region, Rev. épidémiol. Santé Publique 27 (1979) 32–121.
[8] A.C. Coulon, J.M. Roquejoffre, Transition between linear and exponential propagation in Fisher-KPP type reaction–diffusion equations, Comm. Partial Differential Equations 37 (2012) 2029–2049.
[9] J. Fang, X.-Q. Zhao, Monotone wavefronts for partially degenerate reaction–diffusion systems, J. Dynam. Differential Equations 21 (2009) 663–680.
[10] P. Felmer, M. Yangari, Fast propagation for fractional KPP equations with slowly decaying initial conditions, SIAM J. Math. Anal. 45 (2013) 662–678.
[11] D. Finkelshtein, Y. Kondratiev, P. Tkachov, Accelerated front propagation for monostable equations with nonlocal diffusion, arXiv:1611.09329.
[12] J. Garnier, Accelerating solutions in integro-differential equations, SIAM J. Math. Anal. 43 (2011) 1955–1974.
[13] F. Hamel, G. Nadin, Spreading properties and complex dynamics for monostable reaction–diffusion equations, Comm. Partial Differential Equations 37 (2012) 511–537.
[14] F. Hamel, L. Roques, Fast propagation for KPP equations with slowly decaying initial conditions, J. Differential Equations 249 (2010) 1726–1745.
[15] J. He, Y. Wu, Y. Wu, Large time behavior of solutions for degenerate p-degree Fisher equation with algebraic decaying initial data, J. Math. Anal. Appl. 448 (2017) 1–21.
[16] C. Henderson, Propagation of solutions to the Fisher-KPP equation with slowly decaying initial data, Nonlinearity 29 (2016) 3215–3240.
[17] C.-H. Hsu, T.-S. Yang, Existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for epidemic models, Nonlinearity 26 (2013) 121–139.
[18] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bull. Univ. État Moscou, Sér. Internat. A 1 (1937) 1–26.
[19] K.-S. Lau, On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov, J. Differential Equations 59 (1985) 44–70.
[20] M.A. Lewis, B. Li, H.F. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219–233.
[21] B. Li, M.A. Lewis, H.F. Weinberger, Existence of traveling waves for integral recursions with nonmonotone growth functions, J. Math. Biol. 58 (2009) 323–338.
[22] B. Li, H.F. Weinberger, M.A. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, Math. Biosci. 196 (2005) 82–98.

[23] X. Liang, Y. Yi, X.-Q. Zhao, Spreading speeds and traveling waves for periodic evolution systems, J. Differential Equations 231 (2006) 57–77.
[24] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, Comm. Pure Appl. Math. 60 (2007) 1–40, Comm. Pure Appl. Math. 61 (2008) 137–138, Erratum.
[25] X. Liang, X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Funct. Anal. 259 (2010) 857–903.
[26] R. Lui, Biological growth and spread modeled by systems of recursions. I. Mathematical theory, Math. Biosci. 93 (1989) 269–295.
[27] J.F. Mallordy, J.M. Roquejoffre, A parabolic equation of the KPP type in higher dimensions, SIAM J. Math. Anal. 26 (1995) 1–20.
[28] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov, Comm. Pure Appl. Math. 28 (1975) 323–331.
[29] D.H. Sattinger, Stability of waves of nonlinear parabolic systems, Adv. Math. 22 (1976) 312–355.
[30] A.I. Volpert, V.A. Volpert, V.A. Volpert, Traveling Wave Solutions of Parabolic Systems, Transl. Math. Monogr., vol. 140, Amer. Math. Soc., Providence, 1994.
[31] H. Wang, Spreading speeds and traveling waves for non-cooperative reaction–diffusion systems, J. Nonlinear Sci. 21 (2011) 747–783.
[32] H.F. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal. 13 (1982) 353–396.
[33] H.F. Weinberger, M.A. Lewis, B. Li, Analysis of linear determinacy for spread in cooperative models, J. Math. Biol. 45 (2002) 183–218.
[34] S.-L. Wu, C.-H. Hsu, Existence of entire solutions for delayed monostable epidemic models, Trans. Amer. Math. Soc. 368 (2016) 6033–6062.
[35] X.-Q. Zhao, W. Wang, Fisher waves in an epidemic model, Discrete Contin. Dyn. Syst. Ser. B 4 (2004) 1117–1128.