Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 258 (2015) 2435–2470

Journal of Differential Equations

www.elsevier.com/locate/jde

Entire solutions for nonlocal dispersal equations with spatio-temporal delay: Monostable case

Shi-Liang Wu [a,*,1], Shigui Ruan [b,2]

[a] *School of Mathematics and Statistics, Xidian University, Xi'an, Shaanxi 710071, PR China*
[b] *Department of Mathematics, University of Miami, Coral Gables, FL 33124-4250, USA*

Received 7 August 2014; revised 3 November 2014

Available online 31 December 2014

Abstract

This paper deals with entire solutions for a general nonlocal dispersal monostable equation with spatio-temporal delay, i.e., solutions that are defined in the whole space and for all time $t \in \mathbb{R}$. We first derive a particular model for a single species and show how such systems arise from population biology. Then we construct some new types of entire solutions other than traveling wave solutions and equilibrium solutions of the equation under consideration with quasi-monotone and non-quasi-monotone nonlinearities. Various qualitative properties of the entire solutions are also investigated. In particular, the relationship between the entire solutions and the traveling wave fronts which they originated is considered. Our main arguments are based on the comparison principle, the method of super- and sub-solutions, and the construction of auxiliary control systems.

MSC: 34K25; 35R10; 92D25

Keywords: Entire solution; Traveling wave solution; Nonlocal dispersal equation; Spatio-temporal delay; Monostable nonlinearity

* Corresponding author.
E-mail addresses: slwu@xidian.edu.cn (S.-L. Wu), ruan@math.miami.edu (S. Ruan).

[1] Research was partially supported by the NNSF of China (11301407), NSF of Shaanxi Province (2013JQ1012) and Fundamental Research Funds for the Central Universities (K5051370002).

[2] Research was partially supported by the NSF (DMS-1022728 and DMS-1412454).

http://dx.doi.org/10.1016/j.jde.2014.12.013
0022-0396/

1. Introduction

In recent years, many reaction–diffusion equations with *spatio-temporal delay or nonlocal delay* have been proposed and studied to model the interactions of time lag of feedback and spatial diffusion of biological species. See the survey papers of Gourley and Wu [13] and Ruan [32]. Two typical and important examples which have been extensively studied are the diffusive Nicholson's blowflies equation with spatio-temporal delay [10,11,20]:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \delta u(x,t) + p(G*u)(x,t)\exp\big\{-a(G*u)(x,t)\big\}, \tag{1.1}$$

and the following equation describing the evolution of matured population of a single species [1,12,33]:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - du(x,t) + \int_0^\infty \int_{-\infty}^{+\infty} G(x-y,s)b\big(u(y,t-s)\big)dyds, \tag{1.2}$$

where $u(x,t)$ denotes the density of the population at location $x \in \mathbb{R}$ and time $t \geq 0$, $D, \delta, p, a, d > 0$ are constants, $G(\cdot,\cdot)$ is the kernel function, and

$$(G*u)(x,t) = \int_0^\infty \int_{-\infty}^{+\infty} G(x-y,s)u(y,t-s)dyds. \tag{1.3}$$

Note that a basic assumption behind (1.1) and (1.2) is that the internal interaction of species is random and local, i.e. individuals move randomly between the adjacent spatial locations. In reality, the movements and interactions of many species in ecology and biology can occur between non-adjacent spatial locations, see e.g. Lee et al. [18] and Murray [29]. Taking this fact into account, (1.1) and (1.2) can be extended to the following *nonlocal dispersal equations with spatio-temporal delay*:

$$\frac{\partial u}{\partial t} = D(J*u-u)(x,t) - \delta u + p(G*u)(x,t)\exp\big\{-a(G*u)(x,t)\big\}, \tag{1.4}$$

and

$$\frac{\partial u}{\partial t} = D(J*u-u)(x,t) - du + \int_0^\infty \int_{-\infty}^{+\infty} G(x-y,s)b\big(u(y,t-s)\big)dyds, \tag{1.5}$$

respectively, where $(G*u)(x,t)$ is defined in (1.3), $J*u-u$ is a nonlocal dispersal operator and $J*u$ is a spatial convolution defined by

$$(J*u)(x,t) = \int_{-\infty}^{+\infty} J(x-y)u(y,t)dy. \tag{1.6}$$

In the next section, we shall derive model (1.5) and show how such systems arise from population biology. To the best of our knowledge, it is the first work to derive the nonlocal dispersal model with spatio-temporal delay.

Due to their significant applications, front propagation dynamics are one of the most important dynamical issues about biological and epidemiological models. In particular, there have been many results on the traveling wave solutions for nonlocal dispersal equations, see [2,3,5,6,30,31,45]. For example, Coville [6] studied the existence and uniqueness of traveling waves of a nonlocal dispersal diffusion equation with bistable and ignition nonlinearity. Pan [30] and Pan et al. [31] considered the existence of traveling waves for monostable nonlocal dispersal systems with delayed reaction terms satisfying quasi-monotonicity and exponential quasi-monotonicity, respectively. Results in [30,31] were well applied to the Nicholson's blowflies equation with nonlocal diffusion (1.4). Yu and Yuan [45] studied the existence, asymptotic behavior and uniqueness of traveling wave fronts for a general monostable nonlocal dispersal equation with delay.

Although the traveling wave solution is a key concept characterizing the dynamics of reaction–diffusion equations, the dynamics of reaction–diffusion equations are so rich that there might be other interesting patterns, see [7,8,17]. More recently, front-like *entire solutions* that are defined for all space and time and behave like a combination of traveling fronts as $t \to -\infty$ have been observed in various diffusion problems. These solutions can not only describe the interaction of traveling waves but also characterize new dynamics of diffusion equations. For the study of such entire solutions, we refer to [4,14–16,19,23,27] for reaction–diffusion equations without delay, [22,35,37,42] for reaction–diffusion equations with nonlocal delay, [38,39] for delayed lattice differential equations with nonlocal interaction, [21,34] for nonlocal dispersal equations without delay ((1.9) below), [28,40,44] for reaction–diffusion systems, and [43] for periodic lattice dynamical systems. However, to the best of our knowledge, the issues on entire solutions for nonlocal dispersal equations with spatio-temporal delay have not been addressed, especially for infinite delay equations. This is the motivation of the current study.

More precisely, in this paper, we consider the entire solutions of the following general nonlocal dispersal equation with spatio-temporal delay:

$$u_t = D(J * u - u)(x,t) + f\big(u(x,t), \big(G * S(u)\big)(x,t)\big), \tag{1.7}$$

where $(J * u)(x,t)$ is defined in (1.6), and

$$\big(G * S(u)\big)(x,t) = \int_0^{\infty} \int_{-\infty}^{+\infty} G(x-y,s) S\big(u(y,t-s)\big) dy ds. \tag{1.8}$$

It is clear that (1.7) is a generalized version of the models (1.4) and (1.5). In particular, if $G(x,t) = \delta(x)\delta(t)$, then (1.7) reduces to the nonlocal dispersal equation without delay:

$$u_t = D(J * u - u)(x,t) + f\big(u(x,t)\big), \tag{1.9}$$

which has been studied by many researchers, see e.g. [5,6,21,34].

For the kernel functions J and G, we impose the following conditions:

(G$_1$) $J(-x) = J(x) \geq 0$, $G(x,t) = G(-x,t) \geq 0$,

$$\int_{-\infty}^{+\infty} J(y)dy = 1, \qquad \int_0^{\infty}\int_{-\infty}^{+\infty} G(y,s)dyds = 1 \quad \text{(normalized)},$$

and for any $c, \lambda \geq 0$,

$$\int_{-\infty}^{+\infty} e^{-\lambda y} J(y)dy < +\infty \quad \text{and} \quad \int_0^{\infty}\int_{-\infty}^{+\infty} e^{-\lambda(y+cs)} G(y,s)dyds < +\infty.$$

We also make the following basic conditions for the reaction functions f and S:

(C$_1$) $f \in C^2(\bar{I}, \mathbb{R})$, $S \in C^2([0,K], \mathbb{R})$, $f(K, S(K)) = 0$, $f(u, S(u)) > 0$ for $u \in (0, K)$, $f(u,v) \leq \partial_1 f(0,0)u + \partial_2 f(0,0)v$ for $(u,v) \in \bar{I}$ and $0 \leq S(u) \leq S'(0)u$ for $u \in [0,K]$, where $K > 0$ is a constant and $\bar{I} = [0,K] \times [0, S(K)]$.

Note that (C$_1$) implies that $f(0,0) = S(0) = 0$ and that the nonlinearity of (1.7) is monostable. As usual, if the following condition holds:

(C$_2$) $S'(u) \geq 0$ for $u \in [0,K]$ and $\partial_2 f(u,v) \geq 0$ for $(u,v) \in \bar{I}$,

then (1.7) is called a quasi-monotone system; otherwise, it is called a non-quasi-monotone system. In the monostable case, the existence of traveling waves of (1.7) can be obtained by the methods used in [24,25,31,41].

The aim of this paper is to construct some new types of entire solutions for (1.7) with quasi-monotone or non-quasi-monotone nonlinearity. In the quasi-monotone case (i.e. (C$_2$) holds), we first establish a series of comparison principles for (1.7) and a related linear equation (see Lemmas 3.6–3.8). Then, we prove the existence and qualitative features of entire solutions by using these comparison principles. The relationship between the entire solutions and the traveling fronts which they originated is also considered. In particular, we find some interesting phenomenon for the nonlocal dispersal equation, which is similar to those obtained in [16] for the Fisher–KPP equation. The method used for the quasi-monotone case is inspired by the works of Hamel and Nadirashvili [15,16].

More precisely, the idea is to study the solutions $u^k(x,t)$ of a sequence of Cauchy problems for (1.7) starting at times $-k$ ($k \in \mathbb{N}$), where the combinations of any finite traveling fronts with speeds $c \geq c_*$ (c_* is the minimal wave speed) and a spatially independent solution (SIS for short) of (1.7) are taken as the initial values. In this case, the sequence of functions $\{u^k(x,t)\}_{k\in\mathbb{N}}$ is monotone with respect to k. Then, by constructing subsolutions and appropriate upper estimates and using the comparison principles, some desired entire solutions are obtained by passing the limit $k \to \infty$. We mention that when $c > c_*$, the upper estimates are constructed via the exponential decay of the traveling wave fronts and SISs at $-\infty$. However, the decay of the traveling fronts with $c = c_*$ at $-\infty$ is not *exponential* (see Lemma 3.4). To overcome this deficiency, we propose a new concavity assumption on the functions S and f (see the assumption (C_3)) and then obtain an appropriate upper estimate by establishing a corresponding comparison principle.

On the other hand, it is well known that the comparison principle is not applicable for non-quasi-monotone systems. To overcome this difficulty, we make the following assumptions:

(C_2)$'$ There exist $K^{\pm}$ and K with $0 < K^- \le K \le K^+$ and twice piecewise continuously differentiable functions $S^{\pm} : [0, K^+] \to \mathbb{R}$ and $f^{\pm} : \bar{I}^+ \to \mathbb{R}$ such that
- (i) $f \in C^2(\bar{I}^+, \mathbb{R})$ and $S \in C^2([0, K^+], \mathbb{R})$, $f(K, S(K)) = 0$, $f(u, S(u)) > 0$ for $u \in (0, K)$;
- (ii) $\partial_1 f^{\pm} \in C(\bar{I}^+, \mathbb{R})$, $f^{\pm}(K^{\pm}, S^{\pm}(K^{\pm})) = 0$, $f^{\pm}(u, S^{\pm}(u)) > 0$ for $u \in (0, K^{\pm})$, $(S^{\pm})'(0) = S'(0)$ and $\partial_i f^{\pm}(0, 0) = \partial_i f(0, 0)$, $i = 1, 2$;
- (iii) $S^{\pm}(u)$ are non-decreasing on $[0, K^{\pm}]$ and $f^{\pm}(u, v)$ are non-decreasing with respect to the second variable v on $\bar{I}^+$;
- (iv) $0 \le S^-(u) \le S(u) \le S^+(u) \le S'(0)u$ for $u \in [0, K^+]$, and $f^-(u, v) \le f(u, v) \le f^+(u, v) \le \partial_1 f(0, 0)u + \partial_2 f(0, 0)v$ for $(u, v) \in \bar{I}^+$;
- (v) there exists positive constants $L_f, L_S > 0$ such that

$$f^{\pm}(u, v_1) - f^{\pm}(u, v_2) \le L_f \max\{0, v_1 - v_2\}, \quad \forall (u, v_1), (u, v_2) \in \bar{I}^+,$$
$$S^{\pm}(u_1) - S^{\pm}(u_2) \le L_S \max\{0, u_1 - u_2\}, \quad \forall u_1, u_2 \in \big[0, K^+\big].$$

Here and in the following, we denote $\bar{I}^+ := [0, K^+] \times [0, S^+(K^+)]$. It is clear that $f^{\pm} = f$, $S^{\pm} = S$, and $K^{\pm} = K$ if $\partial_2 f(u, v) \ge 0$ for $(u, v) \in \bar{I}$ and $S'(u) \ge 0$ for $u \in [0, K]$. We shall give some examples on the constructions of $f^{\pm}$ and $S^{\pm}$, see Section 6 for applications.

Based on (C_2)$'$, we introduce two auxiliary quasi-monotone nonlocal dispersal equations with spatio-temporal delay to "trap" the original equation, and establish a comparison theorem for the Cauchy problems of the three systems (see Lemma 5.2). In this case, we consider the solutions $U^k(x, t)$ of a sequence of initial value problems of (1.7), where the combinations of traveling fronts and SISs of *the lower system* (i.e. the auxiliary system with smaller reaction term) are taken as the initial values. Due to the non-quasi-monotone nonlinearity, the sequence of functions $\{U^k(x, t)\}_{k\in\mathbb{N}}$ may not be monotone. Thus, we cannot prove the convergence of $\{U^k(x, t)\}_{k\in\mathbb{N}}$ as $k \to -\infty$. An alternative is to prove that there is a convergent subsequence of $\{U^k(x, t)\}_{k\in\mathbb{N}}$. Unfortunately, the solution functions $\{U^k(x, t)\}$ of the nonlocal dispersal equation (1.7) are not smooth enough with respect to x. To obtain a convergent subsequence, we have to make $\{U^k(x, t)\}$ possess a property which is similar to a global Lipschitz condition with respect to x (Lemma 5.3). For this, we impose the following assumption:

(G_2) There exists $\bar{L} > 0$ such that for any $\eta > 0$,

$$\int_{-\infty}^{+\infty} \big|J(y + \eta) - J(y)\big| dy \le \bar{L}\eta \quad \text{and} \quad \int_0^{\infty} \int_{-\infty}^{+\infty} \big|G(y + \eta, s) - G(y, s)\big| dy ds \le \bar{L}\eta.$$

It should be point out that in [21,34], the authors proved a similar property for solutions of (1.9) by making the following assumption:

(G_2)* $J(\cdot) \in C^1(\mathbb{R})$ and $J(\cdot)$ is compactly supported.

Clearly, if $J(\cdot)$ satisfies $(G_2)^*$ and $G(x,t)=\delta(x)\delta(t)$, then (G_2) holds. Moreover, it is easy to verify that the functions $J(x)=\frac{1}{\sqrt{4\pi\varrho}}e^{-\frac{x^2}{4\varrho}}$ and $G(x,t)=\delta(x)\delta(t)$ satisfy (G_2), which implies that (G_2) is weaker than $(G_2)^*$. Here $\varrho>0$ is the nonlocal dispersal constant.

The rest of the paper is organized as follows. In Section 2, we derive model (1.5) along with an explicit formula to calculate $G(\cdot,\cdot)$. In Section 3, we give some preliminaries. First, we state some results on traveling fronts and SISs of (1.7). Some existence and comparison theorems for solutions, supersolutions and subsolutions of (1.7) are then obtained. According to the preliminaries established in Section 3, we prove the existence and qualitative properties of entire solutions of (1.7) with quasi-monotone and non-quasi-monotone nonlinearities in Sections 4 and 5, respectively. Finally, we apply our abstract results to models (1.4) and (1.5) in Section 6.

2. Important particular cases

In this section, we derive model (1.5) and give an explicit formula for the kernel function $G(\cdot,\cdot)$.

Consider a single species population with age structure distributed over $\Omega=\mathbb{R}$. Let $v(x,t,a)$ be the density of individuals at location $x\in\mathbb{R}$, time $t\ge 0$ and age $a\ge 0$. Using $D(a)$ and $d(a)$ to denote the diffusion and death rate of the population at age a, respectively.

In reality, individuals in a population do not necessarily always mature at the same age. Therefore, in this paper, we assume that there is a probability density function $p(a)$ specifying the probability of maturing at each age a. Motivated by ecological considerations, we assume that

$$p(a)\ge 0 \quad \text{and} \quad \int_0^\infty p(a)da=1.$$

Assume that the spatial dispersal of individuals is isotropic and can occur between nonadjacent spatial locations. See Lee et al. [18] and Murray [29]. Let $J(x-y)$ be the probability distribution of individuals moving from location x to location y. Then, we have

$$J(-x)=J(x)\ge 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} J(y)dy=1.$$

Since only the matured population can reproduce, we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}v+\dfrac{\partial}{\partial a}v=D(a)\displaystyle\int_{-\infty}^{+\infty} J(x-y)\big[v(y,t,a)-v(x,t,a)\big]dy-d(a)v(x,t,a), \\ v(x,t,0)=b\big(u(x,t)\big), \end{cases} \tag{2.1}$$

where $b(\cdot)$ is the birth function.

Note that the probability of maturing before the age a is $F(a)=\int_0^a p(s)ds$. Thus, the total number of matured population $u(x,t)$ is

$$u(x,t) = \int_0^\infty F(a)v(x,t,a)da. \tag{2.2}$$

Now, we aim to find a differential equation satisfied by $u(x,t)$. Differentiating (2.2) with respect to t and using (2.1) yields

$$\frac{\partial u}{\partial t} = \int_0^\infty F(a)\left(-\frac{\partial}{\partial a}v + D(a)\int_{-\infty}^{+\infty} J(x-y)\big[v(y,t,a) - v(x,t,a)\big]dy - d(a)v\right)da. \tag{2.3}$$

As the cost of assuming that individuals may mature at different ages, we need to require that the diffusion and death rates of the population are age independent, i.e. $D(a) = D$ and $d(a) = d$ for $a \geq 0$, where D and d are positive constants. Then, using $F(0) = 0$ and the ecologically reasonable assumption $v(x,t,\infty) = 0$, it follows from (2.3) that

$$\frac{\partial u}{\partial t} = D\int_{-\infty}^{+\infty} J(x-y)\big[u(y,t) - u(x,t)\big]dy - du + \int_0^\infty p(a)v(x,t,a)da. \tag{2.4}$$

In order to obtain a closed system for $u(x,t)$, we need to evaluate $v(x,t,a)$. For fixed $s \geq 0$, let

$$V^s(x,t) = v(x,t,t-s) \quad \text{for } s \leq t \leq s+a.$$

Then, $V^s(x,s) = v(x,s,0) = b(u(x,s))$. Moreover, by (2.1), we have

$$\begin{aligned}\frac{\partial}{\partial t}V^s(x,t) &= \frac{\partial}{\partial t}v(x,t,a)\Big|_{a=t-s} + \frac{\partial}{\partial a}v(x,t,a)\Big|_{a=t-s}\\ &= D\int_{-\infty}^{+\infty} J(x-y)\big[V^s(y,t) - V^s(x,t)\big]dy - dV^s(x,t).\end{aligned} \tag{2.5}$$

Note that the function $V^s(x,t)$ can be viewed as the continuous spectral of a function $v^s(t,\lambda)$ by Fourier transform:

$$v^s(t,\lambda) = \int_{-\infty}^{+\infty} e^{-\mathbf{i}\lambda x}V^s(x,t)dx, \tag{2.6}$$

$$V^s(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\mathbf{i}\lambda x}v^s(t,\lambda)d\lambda, \tag{2.7}$$

where $\mathbf{i}$ is the imaginary unit. Applying the Fourier transform (2.6) to (2.5), we obtain

$$\frac{\partial}{\partial t}v^s(t,\lambda)=\left[-2D\int_{-\infty}^{+\infty}J(y)\sin^2\frac{\lambda y}{2}dy-d\right]v^s(t,\lambda).$$

Solving the linear equation, we get

$$v^s(t,\lambda)=\exp\left\{-2D(t-s)\int_{-\infty}^{+\infty}J(y)\sin^2\frac{\lambda y}{2}dy-d(t-s)\right\}v^s(s,\lambda).$$

Using the inverse Fourier transform (2.7), we obtain

$$V^s(x,t)=\frac{1}{2\pi}e^{-d(t-s)}\int_{-\infty}^{+\infty}e^{\mathbf{i}\lambda x}\exp\left\{-2D(t-s)\int_{-\infty}^{+\infty}J(z)\sin^2\frac{\lambda z}{2}dz\right\}v^s(s,\lambda)d\lambda.$$

Since $V^s(x,s)=b(u(x,s))$, we have

$$v^s(s,\lambda)=\int_{-\infty}^{+\infty}e^{-\mathbf{i}\lambda y}b\big(u(y,s)\big)dy.$$

Thus,

$$\begin{aligned}V^s(x,t)=\frac{1}{2\pi}e^{-d(t-s)}&\int_{-\infty}^{+\infty}b\big(u(y,s)\big)\\&\times\int_{-\infty}^{+\infty}e^{\mathbf{i}\lambda(x-y)}\exp\left\{-2D(t-s)\int_{-\infty}^{+\infty}J(z)\sin^2\frac{\lambda z}{2}dz\right\}d\lambda dy.\end{aligned}$$

Letting $t=s+a$, we get

$$\begin{aligned}v(x,t,a)=V^{t-a}(x,t)=\frac{1}{2\pi}e^{-da}&\int_{-\infty}^{+\infty}b\big(u(y,t-a)\big)\\&\times\int_{-\infty}^{+\infty}e^{\mathbf{i}\lambda(x-y)}\exp\left\{-2Da\int_{-\infty}^{+\infty}J(z)\sin^2\frac{\lambda z}{2}dz\right\}d\lambda dy.\end{aligned}$$

Therefore, the model for the matured population finally becomes

$$u_t=D(J*u-u)(x,t)-du+\int_0^{\infty}\int_{-\infty}^{+\infty}e^{-ds}p(s)G_0(x-y,s)b\big(u(y,t-s)\big)dyds, \tag{2.8}$$

where

$$G_0(x,s)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{\mathbf{i}\lambda x}\exp\left\{-2Ds\int_{-\infty}^{+\infty}J(z)\sin^2\frac{\lambda z}{2}dz\right\}d\lambda$$
$$=\frac{1}{2\pi}e^{-Ds}\int_{-\infty}^{+\infty}\cos\lambda x\exp\left\{Ds\int_{-\infty}^{+\infty}J(z)\cos\lambda z dz\right\}d\lambda.$$

Let $G(x,s)=e^{-ds}p(s)G_0(x,s)$. Then (2.8) reduces to model (1.5).

If $p(s)=\delta(s-r_0)$ for some positive constant r_0, that is, all individuals mature at the same age r_0, then (2.8) becomes

$$u_t=D(J*u-u)(x,t)-du+e^{-dr_0}\int_{-\infty}^{+\infty}G_0(x-y,r_0)b\big(u(y,t-r_0)\big)dy. \tag{2.9}$$

Furthermore, if $r_0=0$, then $G_0(x,0)=\delta(x)$, hence we obtain the nonlocal dispersal equation without delay

$$u_t=D(J*u-u)(x,t)-du+b\big(u(x,t)\big). \tag{2.10}$$

We note that it is easy to prove that $G_0(x,t)=G_0(-x,t)$ and $\int_0^\infty\int_{-\infty}^{+\infty}G_0(y,s)dyds=1$. It seems difficult to prove $G_0(x,t)\geq 0$ for general kernel $J(\cdot)$. Nevertheless, in the rest of this paper, we consider (1.7) for a general kernel functions $J(\cdot)$ and $G(\cdot,\cdot)$ satisfying the assumption (G_1) in Section 1.

Remark 2.1. Thanks to the referee, we would like to make some remarks.

(1) In the present paper, we do not assume that all individuals become mature at the same age. As a consequence, we need to require that the diffusion and death rates of the population are age independent. This is a strong assumption which makes such models only appropriate for certain kinds of species, perhaps some mammals, where juveniles stay with their parents and are subject to the same per-capita death rates.

(2) If individuals mature at the same age, say $r_0>0$, then we only need to assume that the diffusion and death rates for the mature population are age independent. In fact, by applying the method used in So et al. [33], we can derive a nonlocal dispersal equation with a discrete delay which is similar to (2.9).

3. Preliminaries

In this section, we first state some results on traveling wave fronts and spatially independent solutions of (1.7). Then we discuss the well-posedness of the initial value problem of (1.7), and establish a series of comparison theorems for supersolutions and subsolutions of (1.7) and a related linear problem, which will play an important role in constructing entire solutions.

3.1. Traveling fronts and spatially independent solutions

A traveling wave of (1.7) connecting 0 and K is a solution of the special form $u(x,t) = \phi_c(\xi)$, $\xi = x + ct$, where the velocity c and the wave profile ϕ satisfy the following functional differential equation

$$c\phi_c'(\xi) = D\big[(J * \phi_c)(\xi) - \phi_c(\xi)\big] + f\big(\phi_c(\xi), \big(G * S(\phi_c)\big)(\xi)\big) \tag{3.1}$$

with asymptotic boundary conditions

$$\phi_c(-\infty) = 0 \quad \text{and} \quad \phi_c(+\infty) = K, \tag{3.2}$$

where $(J * \phi_c)(\xi) = \int_{-\infty}^{+\infty} J(y)\phi_c(\xi - y)dy$ and

$$\big(G * S(\phi_c)\big)(\xi) = \int_0^{\infty}\int_{-\infty}^{+\infty} G(y,s)S\big(\phi_c(\xi - y - cs)\big)dyds.$$

We say ϕ_c is a traveling (wave) front if $\phi_c(\cdot)$ is monotone.

It is clear that the characteristic function for (3.1) with respect to the trivial equilibrium 0 can be represented by

$$\begin{aligned}\triangle(c,\lambda) :=\, & c\lambda - D\left[\int_{-\infty}^{+\infty} e^{-\lambda y}J(y)dy - 1\right] - \partial_1 f(0,0) \\ & - \partial_2 f(0,0)S'(0)\int_0^{\infty}\int_{-\infty}^{+\infty} e^{-\lambda(y+cs)}G(y,s)dyds = 0.\end{aligned} \tag{3.3}$$

From (C_1), we see that $\partial_1 f(0,0) + \partial_2 f(0,0)S'(0) \geq \frac{2}{K} f(\frac{K}{2}, S(\frac{K}{2})) > 0$. Thus, one can easily show that the following result holds, see also [45].

Proposition 3.1. *Let* (G_1) *and* (C_1) *hold. There exist* $\lambda_* > 0$ *and* $c_* > 0$ *such that*

$$\triangle(c_*, \lambda_*) = 0 \quad \textit{and} \quad \frac{\partial}{\partial\lambda}\triangle(c_*, \lambda)\bigg|_{\lambda=\lambda_*} = 0.$$

Furthermore, if $c > c_*$*, then the equation* $\triangle(c,\lambda) = 0$ *has two positive real roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $\lambda_1(c) < \lambda_* < \lambda_2(c)$*,* $\frac{\partial}{\partial c}\lambda_1(c) < 0$*, and* $\frac{\partial}{\partial c}[c\lambda_1(c)] < 0$.

Now, we consider the traveling fronts and SISs of (1.7) with quasi-monotone assumption. Here, the SIS of (1.7) means the solution of the following delayed problem:

$$\Gamma'(t) = f\left(\Gamma(t), \int_0^{\infty} S\big(\Gamma(t-s)\big)\int_{-\infty}^{+\infty} G(y,s)dyds\right), \quad t \in \mathbb{R}. \tag{3.4}$$

Using similar methods as in [24,41,45], we can obtain the following existence result on the traveling fronts and SISs.

Proposition 3.2. *Assume* (G_1), (C_1) *and* (C_2).

(i) *For each* $c \geq c_*$, (1.7) *has a traveling wave front* $\phi_c(x+ct)$ *which satisfies* $\phi_c'(\cdot) > 0$, $\phi_c(-\infty) = 0$, *and* $\phi_c(+\infty) = K$. *Furthermore, for* $c > c_*$,

$$\lim_{\xi \to -\infty} \phi_c(\xi) e^{-\lambda_1(c)\xi} = 1 \quad \textit{and} \quad \phi_c(\xi) \leq e^{\lambda_1(c)\xi}, \quad \xi \in \mathbb{R}. \tag{3.5}$$

(ii) *There exists a solution* $\Gamma(t) : \mathbb{R} \to [0, K]$ *of* (3.4) *which satisfies* $\Gamma(-\infty) = 0$ *and* $\Gamma(+\infty) = K$. *Furthermore,*

$$\Gamma'(t) > 0, \qquad \lim_{t \to -\infty} \Gamma(t) e^{-\lambda^* t} = 1 \quad \textit{and} \quad \Gamma(t) \leq e^{\lambda^* t} \quad \textit{for all } t \in \mathbb{R},$$

where λ^* *is the unique positive root of the equation*

$$\lambda - \partial_1 f(0,0) - \partial_2 f(0,0) S'(0) \int_0^{\infty} \int_{-\infty}^{+\infty} e^{-\lambda s} G(y,s) dy ds = 0.$$

We can further obtain the asymptotic behavior for any traveling wave fronts of (1.7) by applying the following version of Ikehara's theorem, see e.g. Carr and Chmaj [2].

Lemma 3.3. *Let* $u(\xi)$ *be a positive decreasing function and* $J_1(\Lambda) := \int_0^{+\infty} e^{-\Lambda\xi} u(\xi) d\xi$. *If* J_1 *can be written as* $J_1(\Lambda) = J(\Lambda)(\Lambda + \Lambda_0)^{-(k+1)}$, *where* $k > -1$, $\Lambda_0 > 0$ *are two constants and* J *is analytic in the strip* $-\Lambda_0 \leq \operatorname{Re} \Lambda < 0$, *then*

$$\lim_{\xi \to +\infty} \frac{u(\xi)}{\xi^k e^{-\Lambda_0 \xi}} = \frac{J(-\Lambda_0)}{\Gamma(\Lambda_0 + 1)}.$$

Lemma 3.4. *Assume* (G_1), (C_1) *and* (C_2). *Let* $\phi_c(\xi)$ *be any traveling wave front of* (1.7) *with speed* $c \geq c_*$.

(i) *For* $c > c_*$,

$$\lim_{\xi \to -\infty} \phi_c(\xi) e^{-\lambda_1(c)\xi} = a(c) \quad \textit{and} \quad \lim_{\xi \to -\infty} \phi_c'(\xi) e^{-\lambda_1(c)\xi} = a(c)\lambda_1(c); \tag{3.6}$$

(ii) *for* $c = c_*$,

$$\lim_{\xi \to -\infty} \phi_c(\xi) \xi^{-1} e^{-\lambda_1(c)\xi} = -a(c) \quad \textit{and} \quad \lim_{\xi \to -\infty} \phi_c'(\xi) \xi^{-1} e^{-\lambda_1(c)\xi} = -a(c)\lambda_1(c), \tag{3.7}$$

where $a(c)$ *is a positive constant.*

Proof. The proof is similar to that of [36, Theorem 4.8] and [2, Theorem 1]. Here, we only sketch the outline. The proof is divided into three steps.

Step 1. We show that $\phi_c(\xi)$ is integrable on $(-\infty, \xi']$ for some $\xi' \in \mathbb{R}$.

Step 2. We prove that $\phi_c(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$ for some $\gamma > 0$. To get the assertion, we first show that $W(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$, where $W(\xi) := \int_{-\infty}^{\xi} \phi_c(s)ds$.

Step 3. For $0 < \mathrm{Re}\,\lambda < \gamma$, define a two-sided Laplace transform of ϕ by $\mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} \phi_c(\xi) \times e^{-\lambda\xi} d\xi$. Using Lemma 3.3 and a property of Laplace transforms, one can show that for $c > c_*$, $\lim_{\xi\to-\infty} \phi_c(\xi)e^{-\lambda_1(c)\xi} = a(c)$ and for $c = c_*$, $\lim_{\xi\to-\infty} \phi_c(\xi)\xi^{-1}e^{-\lambda_1(c)\xi} = -a(c)$.

Furthermore, using (3.1), we can verify that for $c > c_*$, $\lim_{\xi\to-\infty} \phi_c'(\xi)e^{-\lambda_1(c)\xi} = a(c)\lambda_1(c)$ and for $c = c_*$, $\lim_{\xi\to-\infty} \phi_c'(\xi)\xi^{-1}e^{-\lambda_1(c)\xi} = -a(c)\lambda_1(c)$. This completes the proof. □

3.2. *Initial value problem*

In this subsection, we consider the initial value problem of (1.7) with the following initial data:

$$u(x, s) = \varphi(x, s), \quad x \in \mathbb{R},\ s \in (-\infty, 0]. \tag{3.8}$$

We shall study the well-posedness of the IVP (1.7) and (3.8), and establish a series of comparison theorems for supersolutions and subsolutions of (1.7) and a related linear problem.

Let $X = \mathrm{BUC}(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the supremum norm $\|\cdot\|_X$. Let

$$X_{[0,K]} := \{\varphi \in X\colon\ \varphi(x) \in [0, K],\ \forall x \in \mathbb{R}\}, \tag{3.9}$$

$$\mathcal{C}_{[0,K]} := C\big((-\infty, 0], X_{[0,K]}\big). \tag{3.10}$$

For any $\varphi \in \mathcal{C}_{[0,K]}$, define $\|\varphi\|_{\mathcal{C}} = \sup_{\theta\in(-\infty,0]} \|\varphi(\theta)\|_X$.

As usual, we identify an element $\varphi \in \mathcal{C}$ as a function from $\mathbb{R} \times (-\infty, 0]$ into $\mathbb{R}$ defined by $\varphi(x, s) = \varphi(s)(x)$. For any continuous function $u : (-\infty, b) \to X$, $b > 0$, we define $u_t \in \mathcal{C}$, $t \in [0, b)$ by $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$. Define $F : \mathcal{C}_{[0,K]} \to X$ by

$$F(\varphi)(x) = (J * \varphi)(x, 0) + \bar{L}_1\varphi(x, 0) + f\big(\varphi(x, 0), \big(G * S(\varphi)\big)(x, 0)\big),$$

where $\bar{L}_1 := \max_{(u,v)\in\bar{I}} |\partial_1 f(u, v)|$. It is easy to see that $F : \mathcal{C}_{[0,K]} \to X$ is globally Lipschitz continuous.

Let $T(t) = e^{-(D+\bar{L}_1)t}$. Clearly, $T(t)$ is a linear semigroup on X. The definitions of supersolution and subsolution are given as follows.

Definition 3.5. A continuous function $u : (-\infty, b) \to X_{[0,K]}$, $b > 0$, is called a supersolution (or a subsolution) of (1.7) on $[0, b)$ if

$$u(t) \geq \text{ (or } \leq) \ T(t-s)\big[u(s)\big] + \int_s^t T(t-r)\big[F(u_r)\big]dr$$

for any $0 \leq s < t < b$. If u is both a supersolution and a subsolution on $[0, b)$, then it is said to be a mild solution of (1.7).

According to Definition 3.5, we have the following result.

Lemma 3.6. *Assume* (G_1)*,* (C_1) *and* (C_2)*.*

(i) *For any* $\varphi \in \mathcal{C}_{[0,K]}$*, (1.7) has a unique mild solution* $u(x,t;\varphi)$ *on* $[0,\infty)$ *with* $0 \le u(x,t;\varphi) \le K$ *for* $x \in \mathbb{R}$*,* $t \ge 0$*. Moreover,* $u(x,t;\varphi)$ *is a classical solution of (1.7) for* $(x,t) \in \mathbb{R} \times (0,\infty)$*.*

(ii) *For any pair of supersolution* u^+ *and subsolution* u^- *of (1.7) on* $[0,\infty)$ *with* $u^+(x,s) \ge u^-(x,s)$ *for* $x \in \mathbb{R}$ *and* $s \in (-\infty,0]$*, one has* $0 \le u^-(x,t) \le u^+(x,t) \le K$ *for* $(x,t) \in \mathbb{R} \times [0,\infty)$*.*

Proof. (i) The first part of this assertion can be proved by using the contracting mapping theorem, see e.g. Fang et al. [9, Lemma 2.8]. It can also be proved by applying the theory of abstract functional differential equations (Martin and Smith [26, Corollary 5]). Since the process is standard, we omit the details here. Suppose that $u(x,t;\varphi)$ is the unique solution of (1.7) with initial value $\varphi \in \mathcal{C}_{[0,K]}$. For simplicity, we denote $u(x,t;\varphi)$ by $u(x,t)$. From Definition 3.5, $u(x,t)$ satisfies

$$u(t)(x) = T(t)[\varphi](x) + \int_0^t T(t-r)\big[F(u_r)\big](x)dr.$$

Differentiating both sides of the above equation, we obtain

$$\begin{aligned}
\partial_t u(x,t) &= -(D+\bar{L}_1)e^{-(D+\bar{L}_1)t}\varphi(x) + F(u_t)(x) \\
&\quad - (D+\bar{L}_1)\int_0^t T(t-r)\big[F(u_r)\big](x)dr \\
&= -(D+\bar{L}_1)e^{-(D+\bar{L}_1)t}\varphi(x) + F(u_t)(x) \\
&\quad - (D+\bar{L}_1)u(x,t) + (D+\bar{L}_1)T(t)[\varphi](x) \\
&= D(J*u-u)(x,t) + f\big(u(x,t),\big(G*S(u)\big)(x,t)\big).
\end{aligned}$$

Therefore, $u(x,t;\varphi)$ is a classical solution of (1.7) for $(x,t) \in \mathbb{R} \times (0,\infty)$.

The assertion (ii) follows from [26, Corollary 5]. This completes the proof. □

The following comparison theorem plays an important role in constructing upper estimates for solutions of (1.7).

Lemma 3.7. *Let* (G_1) *and* (C_1) *hold. Assume further that* $\partial_2 f(0,0)S'(0) \ge 0$*. Let* $u^+ \in C(\mathbb{R}^2,[0,+\infty))$ *and* $u^- \in C(\mathbb{R}^2,(-\infty,K])$ *be such that* $u^+(x,s) \ge u^-(x,s)$ *for* $x \in \mathbb{R}$*,* $s \in (-\infty,0]$*. If*

$$\begin{aligned}
u_t^+ &\ge D\big[\big(J*u^+\big)(x,t) - u^+(x,t)\big] \\
&\quad + \partial_1 f(0,0)u^+(x,t) + \partial_2 f(0,0)S'(0)\big(G*u^+\big)(x,t),
\end{aligned} \tag{3.11}$$

$$u_t^- \le D\big[\big(J*u^-\big)(x,t)-u^-(x,t)\big] \\ +\partial_1 f(0,0)u^-(x,t)+\partial_2 f(0,0)S'(0)\big(G*u^-\big)(x,t) \tag{3.12}$$

for $x\in\mathbb{R}$ and $t>0$, then $u^+(x,t)\ge u^-(x,t)$ for $x\in\mathbb{R}$ and $t\ge 0$.

Proof. Set $w(x,t)=u^-(x,t)-u^+(x,t)$ for $(x,t)\in\mathbb{R}^2$. Then $w(x,t)\le K$ for $(x,t)\in\mathbb{R}^2$. From (3.11) and (3.12), we have

$$w_t\le D(J*w)(x,t)+\big(\partial_1 f(0,0)-D\big)w(x,t)+\partial_2 f(0,0)S'(0)(G*w)(x,t) \tag{3.13}$$

for $x\in\mathbb{R}$ and $t>0$. Note that $w(x,s)\le 0$ for $x\in\mathbb{R}$ and $s\in(-\infty,0]$. Take $\mu=\partial_1 f(0,0)-D$. Since $\partial_2 f(0,0)S'(0)\ge 0$, it follows from (3.13) that

$$\begin{aligned} w(x,t)&\le w(x,0)e^{\mu t}+\int_0^t e^{\mu(t-s)}\big[D(J*w)(x,s)+\partial_2 f(0,0)S'(0)(G*w)(x,s)\big]ds \\ &\le \int_0^t e^{\mu(t-s)}\Bigg[D\int_{-\infty}^{+\infty}J(y)\max\big\{w(x-y,s),0\big\}dy \\ &\quad+\partial_2 f(0,0)S'(0)\int_0^{\infty}\int_{-\infty}^{+\infty}G(y,r)\max\big\{w(x-y,s-r),0\big\}dydr\Bigg]ds. \end{aligned} \tag{3.14}$$

Denote $[B]_+:=\max\{B,0\}$ for any $B\in\mathbb{R}$. Then, from (3.14), we have

$$\begin{aligned} \big[w(x,t)\big]_+ \le \int_0^t e^{\mu(t-s)}\Bigg[&D\int_{-\infty}^{+\infty}J(y)\big[w(x-y,s)\big]_+dy \\ &+\partial_2 f(0,0)S'(0)\int_0^{\infty}\int_{-\infty}^{+\infty}G(y,r)\big[w(x-y,s-r)\big]_+dydr\Bigg]ds \end{aligned}$$

for $x\in\mathbb{R}$ and $t\in[0,+\infty)$. Moreover, set

$$w_\lambda(t):=\sup_{x\in\mathbb{R}}\big[w(x,t)\big]_+e^{-\lambda t}\quad\text{and}\quad w_\lambda:=\sup_{t\in\mathbb{R}}w_\lambda(t)\quad\text{for } \lambda>\max\{\mu,0\}.$$

Then, we have

$$w_\lambda(t)\le\int_0^t e^{-(\lambda-\mu)(t-s)}\Bigg[Dw_\lambda(s)+\partial_2 f(0,0)S'(0)\int_0^{\infty}\int_{-\infty}^{+\infty}G(y,r)e^{-\lambda r}w_\lambda(s-r)dydr\Bigg]ds,$$

which yields that

$$
\begin{aligned}
w_\lambda &\le w_\lambda \int_0^t e^{-(\lambda-\mu)(t-s)}\left[D + \partial_2 f(0,0)S'(0)\int_0^{\infty}\int_{-\infty}^{+\infty} G(y,r)e^{-\lambda r}\,dy\,dr\right]ds \\
&\le w_\lambda \int_0^t e^{-(\lambda-\mu)(t-s)}\big[D + \partial_2 f(0,0)S'(0)\big]ds \\
&\le w_\lambda\big[D + \partial_2 f(0,0)S'(0)\big]/(\lambda-\mu).
\end{aligned}
$$

Hence, $w_\lambda \le 0$ for sufficiently large λ. Therefore, $u^-(x,t) \le u^+(x,t)$ for $(x,t) \in \mathbb{R} \times [0,+\infty)$. This completes the proof. $\square$

To obtain another comparison theorem which will be used to construct upper estimates, we need the concavity assumption of the functions S and f:

(C$_3$) $S(u)$ is concave on $[0,K]$ and for any $m \in \mathbb{Z}^+$, $a_i \in [0,K]$, $b_i \in [0,S(K)]$, $i = 1,\dots,m$,

$$
\begin{aligned}
&\bar{L}_1 \min\left\{K, \sum_{i=1}^m a_i\right\} + f\left(\min\left\{K, \sum_{i=1}^m a_i\right\}, \min\left\{S(K), \sum_{i=1}^m b_i\right\}\right) \\
&\le \sum_{i=1}^m \big[\bar{L}_1 a_i + f(a_i,b_i)\big].
\end{aligned}
$$

We would like to point out that assumption (C$_3$) is not a more restrictive condition. Indeed, in general monostable nonlinearities satisfy such a concave condition, see Section 6 for applications.

Lemma 3.8. *Assume* (G$_1$) *and* (C$_1$)–(C$_3$). *Let* $m \in \mathbb{Z}^+$ *and* $u_i^0, u^0 \in \mathcal{C}_{[0,K]}$, $i = 1,\dots,m$, *be* $m+1$ *given functions with*

$$
u^0(x,s) \le \min\left\{K, \sum_{i=1}^m u_i^0(x,s)\right\} \quad \textit{for } x \in \mathbb{R},\ s \in (-\infty,0].
$$

Let u_i *and* u *be the solutions of the Cauchy problems of* (1.7) *with the initial values:*

$$
u_i(x,s) = u_i^0(x,s) \quad \textit{and} \quad u(x,s) = u^0(x,s), \qquad x \in \mathbb{R},\ s \in (-\infty,0], \tag{3.15}
$$

respectively. Then

$$
0 \le u(x,t) \le \min\left\{K, \sum_{i=1}^m u_i(x,t)\right\} \quad \textit{for all } x \in \mathbb{R} \textit{ and } t \ge 0.
$$

Proof. Set $Z(x,t) := \min\{K, \sum_{i=1}^m u_i(x,t)\}$, then $u(x,s) \le Z(x,s)$ for $x \in \mathbb{R}$ and $s \in (-\infty,0]$. By the second part of Lemma 3.6, it suffices to show that $Z(t)(\cdot) = Z(\cdot,t) \in C((-\infty,+\infty), X_{[0,K]})$ is a supersolution of (1.7), i.e.

$$T(t-s)\big[Z(s)\big](x)+\int_s^t T(t-r)\big[F(Z_r)\big](x)dr \le Z(t)(x) \quad \text{for } 0 \le s < t < +\infty. \tag{3.16}$$

Since $S'(u) \ge 0$ for $u \in [0, K]$ and $\partial_2 f(u, v) \ge 0$ for $(u, v) \in [0, K] \times [0, S(K)]$, we have

$$\begin{aligned} &T(t-s)\big[Z(s)\big](x)+\int_s^t T(t-r)\big[F(Z_r)\big](x)dr \\ &\quad \le e^{-(D+\bar{L}_1)(t-s)}K + (D+\bar{L}_1)K\int_s^t e^{-(D+\bar{L}_1)(t-r)}dr = K \end{aligned} \tag{3.17}$$

for $0 \le s < t < +\infty$. Using the concave condition of $S(u)$ on $[0, K]$ and mathematical induction, we can show that for any $d_i \in (0, K]$, $i = 1, \dots, m$,

$$S\big(\min\{K, d_1+\cdots+d_m\}\big) \le S(d_1)+\cdots+S(d_m).$$

Using the assumption $S'(u) \ge 0$ for $u \in [0, K]$ again, we have

$$S\big(\min\{K, d_1+\cdots+d_m\}\big) \le \min\big\{S(K), S(d_1)+\cdots+S(d_m)\big\}. \tag{3.18}$$

Then, by (3.18) and (C_3), we have

$$\begin{aligned} F(Z_r)(x) &= (J * Z)(x,r) + \bar{L}_1 Z(x,r) + f\big(Z(x,r), \big(G * S(Z)\big)(x,r)\big) \\ &\le D\sum_{i=1}^m (J*u_i)(x,r) + \bar{L}_1 \min\left\{K, \sum_{i=1}^m u_i(x,r)\right\} \\ &\quad + f\left(\min\left\{K, \sum_{i=1}^m u_i(x,r)\right\}, \min\left\{S(K), \sum_{i=1}^m \big(G * S(u_i)\big)(x,r)\right\}\right) \\ &\le \sum_{i=1}^m \big[D(J*u_i)(x,r) + \bar{L}_1 u_i(x,r) + f\big(u_i(x,r), \big(G*S(u_i)\big)(x,r)\big)\big] \\ &= \sum_{i=1}^m F\big((u_i)_r\big)(x). \end{aligned}$$

Noting that

$$T(t-s)\big[u_i(s)\big](x)+\int_s^t T(t-r)\big[F\big((u_i)_r\big)\big](x)dr = u_i(t)(x), \quad i = 1, \dots, m,$$

for $0 \le s < t < +\infty$, we obtain

$$
\begin{aligned}
&T(t-s)\big[Z(s)\big](x)+\int_s^t T(t-r)\big[F(Z_r)\big](x)dr \\
&\quad\le \sum_{i=1}^m \left\{T(t-s)\big[u_i(s)\big](x)+\int_s^t T(t-r)\big[F\big((u_i)_r\big)\big](x)dr\right\} \\
&\quad= \sum_{i=1}^m u_i(t)(x) \quad \text{for } 0\le s<t<+\infty.
\end{aligned} \tag{3.19}
$$

Therefore, (3.16) follows from (3.17) and (3.19), i.e., $Z(x,t)$ is a supersolution of (1.7) and the assertion of this lemma follows from Lemma 3.6. □

4. Entire solutions: quasi-monotone case

In this section, we consider entire solutions of (1.7) in the quasi-monotone case. First, we use the conclusions established in the previous section to obtain some appropriate upper estimates for solutions of (1.7). Then, we prove the existence and various qualitative properties of entire solutions.

For any $k\in\mathbb{N}$, $m,n\in\mathbb{N}\cup\{0\}$, $h_1,\dots,h_m,\vartheta_1,\dots,\vartheta_n,h\in\mathbb{R}$, $c_1,\dots,c_m,\bar{c}_1,\dots,\bar{c}_n\ge c_*$, and $\chi\in\{0,1\}$ with $m+n+\chi\ge 2$, we denote

$$
\begin{aligned}
\varphi^k(x,s) &:= \max\Big\{\max_{i=1,\dots,m}\phi_{c_i}(x+c_is+h_i),\ \max_{j=1,\dots,n}\phi_{\bar{c}_j}(-x+\bar{c}_js+\vartheta_j),\ \chi\varGamma(s+h)\Big\},\\
\underline{u}(x,t) &:= \max\Big\{\max_{i=1,\dots,m}\phi_{c_i}(x+c_it+h_i),\ \max_{j=1,\dots,n}\phi_{\bar{c}_j}(-x+\bar{c}_jt+\vartheta_j),\ \chi\varGamma(t+h)\Big\},
\end{aligned}
$$

where $x\in\mathbb{R}$, $s\in(-\infty,-k]$ and $t>-k$. Let $u^k(x,t)$ be the unique solution of the following initial value problem

$$
\begin{cases}
u_t=D(J*u-u)(x,t)+f\big(u(x,t),\big(G*S(u)\big)(x,t)\big),\\
u(x,s)=\varphi^k(x,s),
\end{cases} \tag{4.1}
$$

for $x\in\mathbb{R}$, $s\in(-\infty,-k]$ and $t>-k$. From Lemma 3.6, we see that

$$
\underline{u}(x,t)\le u^k(x,t)\le K \quad \text{for } (x,t)\in\mathbb{R}^2.
$$

4.1. Existence of entire solutions

Using the comparison theorems established in Section 3, we can obtain some appropriate upper estimates of $u^k(x,t)$. For simplicity, denote

$$
\varPi(x,t):=\sum_{i=1}^m\phi_{c_i}(x+c_it+h_i)+\sum_{j=1}^n\phi_{\bar{c}_j}(-x+\bar{c}_jt+\vartheta_j)+\chi\varGamma(t+h),
$$

$$\varPi_0(x,t) := \sum_{1\le i\le m} e^{\lambda_1(c_i)(x+c_it+h_i)} + \sum_{1\le j\le n} e^{\lambda_1(\bar c_j)(-x+\bar c_jt+\vartheta_j)} + \chi e^{\lambda^*(t+h)},$$

$$\varPi_1(x,t) := \min_{1\le i_0\le m}\Big\{\phi_{c_{i_0}}(x+c_{i_0}t+h_{i_0}) + \sum_{1\le i\le m, i\ne i_0} e^{\lambda_1(c_i)(x+c_it+h_i)} + \sum_{1\le j\le n} e^{\lambda_1(\bar c_j)(-x+\bar c_jt+\vartheta_j)} + \chi e^{\lambda^*(t+h)}\Big\},$$

$$\varPi_2(x,t) := \min_{1\le j_0\le n}\Big\{\phi_{\bar c_{j_0}}(-x+\bar c_{j_0}t+\vartheta_{j_0}) + \sum_{1\le j\le n, j\ne j_0} e^{\lambda_1(\bar c_j)(-x+\bar c_jt+\vartheta_j)} + \sum_{1\le i\le m} e^{\lambda_1(c_i)(x+c_it+h_i)} + \chi e^{\lambda^*(t+h)}\Big\},$$

$$\varPi_3(x,t) := \sum_{1\le i\le m} e^{\lambda_1(c_i)(x+c_it+h_i)} + \sum_{1\le j\le n} e^{\lambda_1(\bar c_j)(-x+\bar c_jt+\vartheta_j)} + \chi\varGamma(t+h).$$

Lemma 4.1. *Assume* (G_1) *and* (C_1)–(C_3). *The unique solution* $u^k(x,t)$ *of* (4.1) *satisfies*

$$u^k(x,t) \le u^+(x,t) := \min\{K, \varPi(x,t)\} \quad \textit{for } (x,t)\in\mathbb{R}^2.$$

Proof. The assertion of this lemma follows directly from Lemma 3.8. So, we omit it here. □

Lemma 4.2. *Assume* (G_1) *and* (C_1)–(C_2). *If* $c_1,\dots,c_m,\bar c_1,\dots,\bar c_n > c_*$, *then the unique solution* $u^k(x,t)$ *of* (4.1) *satisfies*

$$u^k(x,t) \le \tilde u(x,t) := \min\{K, \varPi_0(x,t)\} \tag{4.2}$$

for $(x,t)\in\mathbb{R}^2$. *If, in addition, the following condition*

(C_4) $\partial_i f(u,v) \le \partial_i f(0,0)$ *and* $S'(u)\le S'(0)$ *for any* $(u,v)\in\bar I$, $i=1,2$,

holds, then

$$u^k(x,t) \le \bar u(x,t) := \min\{K, \varPi_1(x,t), \varPi_2(x,t), \varPi_3(x,t)\} \tag{4.3}$$

for $(x,t)\in\mathbb{R}^2$.

Proof. We first prove (4.3). Since $u^k(x,t)\le K$ for $(x,t)\in\mathbb{R}^2$, it suffices to show that $u^k(x,t)\le \varPi_i(x,t)$, $i=1,2,3$, for $(x,t)\in\mathbb{R}^2$. We only prove $u^k(x,t)\le\varPi_1(x,t)$ for $(x,t)\in\mathbb{R}^2$. The other case can be proved similarly. Given any $i_0\in\{1,\dots,m\}$. Take

$$W^k(x,t) := u^k(x,t) - \phi_{c_{i_0}}(x+c_{i_0}t+h_{i_0}) \quad \text{for } (x,t)\in\mathbb{R}^2.$$

Then $0\le W^k(x,t)\le K$ for $(x,t)\in\mathbb{R}^2$. Using $\partial_i f(u,v)\le\partial_i f(0,0)$ and $S'(u)\le S'(0)$ for any $(u,v)\in\bar I$, $i=1,2$, we obtain

$$\begin{cases} \dfrac{\partial W^k}{\partial t} \le D\big[\big(J * W^k\big)(x,t) - W^k(x,t)\big] \\ \quad + \partial_1 f(0,0) W^k(x,t) + \partial_2 f(0,0) S'(0)\big(G * W^k\big)(x,t), \\ W^k(x,s) := u^k(x,s) - \phi_{c_{i_0}}(x + c_{i_0} s + h_{i_0}), \end{cases} \tag{4.4}$$

where $x \in \mathbb{R}$, $t > -k$, $s \in (-\infty, -k]$. Taking

$$V(x,t) := \sum_{1 \le i \le m, i \ne i_0} e^{\lambda_1(c_i)(x + c_i t + h_i)} + \sum_{1 \le j \le n} e^{\lambda_1(\bar{c}_j)(-x + \bar{c}_j t + \vartheta_j)} + \chi e^{\lambda^*(t+h)}$$

for $(x,t) \in \mathbb{R}^2$. It is easy to verify that

$$\frac{\partial V}{\partial t} = D\big[(J * V)(x,t) - V(x,t)\big] + \partial_1 f(0,0) V(x,t) + \partial_2 f(0,0) S'(0)(G * V)(x,t)$$

where $x \in \mathbb{R}$, $t > -k$. According to Proposition 3.2, we have

$$\begin{aligned} W^k(x,s) &= \varphi^k(x,s) - \phi_{c_{i_0}}(x + c_{i_0} s + h_{i_0}) \\ &\le \sum_{1 \le i \le m, i \ne i_0} \phi_{c_i}(x + c_i s + h_i) + \sum_{1 \le j \le n} \phi_{\bar{c}_j}(-x + \bar{c}_j s + \vartheta_j) + \chi \Gamma(s+h) \\ &\le \sum_{1 \le i \le m, i \ne i_0} e^{\lambda_1(c_i)(x + c_i s + h_i)} + \sum_{1 \le j \le n} e^{\lambda_1(\bar{c}_j)(-x + \bar{c}_j s + \vartheta_j)} + \chi e^{\lambda^*(s+h)} \\ &= V(x,s) \quad \text{for } x \in \mathbb{R},\ s \in (-\infty, -k]. \end{aligned}$$

Then, it follows from Lemma 3.7 that $W^k(x,t) \le V(x,t)$ for $(x,t) \in \mathbb{R}^2$, that is,

$$\begin{aligned} u^k(x,t) \le \phi_{c_{i_0}}(x + c_{i_0} t + h_{i_0}) &+ \sum_{1 \le i \le m, i \ne i_0} e^{\lambda_1(c_i)(x + c_i t + h_i)} \\ &+ \sum_{1 \le j \le n} e^{\lambda_1(\bar{c}_j)(-x + \bar{c}_j t + \vartheta_j)} + \chi e^{\lambda^*(t+h)}. \end{aligned}$$

Since $i_0 \in \{1, \ldots, m\}$ is arbitrary, we have $u^k(x,t) \le \Pi_1(x,t)$ for $(x,t) \in \mathbb{R}^2$. Therefore, (4.3) holds. The proof of (4.2) is similar and is omitted. This completes the proof. □

By using upper estimates of Lemmas 4.1 and 4.2, we can obtain the following result.

Theorem 4.3. *Assume* (G$_1$) *and* (C$_1$)–(C$_2$). *For any $m, n \in \mathbb{N} \cup \{0\}$, $h_1, \ldots, h_m, \vartheta_1, \ldots, \vartheta_n$, $h \in \mathbb{R}$, $c_1, \ldots, c_m, \bar{c}_1, \ldots, \bar{c}_n \ge c_*$, and $\chi \in \{0, 1\}$ with $m + n + \chi \ge 2$, there exists an entire solution $\Phi_p(x,t)$ of (1.7) such that*

$$\underline{u}(x,t) \le \Phi_p(x,t) \le K \quad \text{for } (x,t) \in \mathbb{R}^2, \tag{4.5}$$

where $p := p_{m,n,\chi} = (c_1, h_1, \ldots, c_m, h_l, \bar{c}_1, \vartheta_1, \ldots, \bar{c}_n, \vartheta_n, \chi h)$. Furthermore, the following results hold.

(i) *If* (C_3) *holds, then* $\Phi_p(x,t) \le u^+(x,t)$ *for* $(x,t) \in \mathbb{R}^2$.
(ii) *If* $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n > c_*$, *then* $\Phi_p(x,t) \le \tilde{u}(x,t)$ *for* $(x,t) \in \mathbb{R}^2$. *If, in addition,* (C_4) *holds, then* $\Phi_p(x,t) \le \bar{u}(x,t)$ *for* $(x,t) \in \mathbb{R}^2$.

Proof. Recall that $u^k(x,t)$ is the unique solution of the initial value problem (4.1). By Lemma 3.6, it is easy to see that

$$\underline{u}(x,t) \le u^k(x,t) \le u^{k+1}(x,t) \le K \tag{4.6}$$

for all $x \in \mathbb{R}$ and $t \ge -k$. Then there exists a function $\Phi_p(x,t)$ satisfying $0 \le \Phi_p(x,t) \le K$ such that for any $(x,t) \in \mathbb{R}^2$, there is $\lim_{k\to\infty} u^k(x,t) = \Phi_p(x,t)$. For any given $t_0 \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $t_0 > -k$ and $u^k(x,t)$ satisfies

$$u^k(t)(x) = T(t-t_0)\big[u^k(t_0)\big](x) + \int_{t_0}^{t} T(t-r)\big[F\big(u_r^k\big)\big](x)dr,$$

where T and F are defined as in Section 3. By Lebesgue's dominated convergence theorem, we get

$$\Phi_p(t)(x) = T(t-t_0)\big[\Phi_p(t_0)\big](x) + \int_{t_0}^{t} T(t-r)\big[F\big((\Phi_p)_r\big)\big](x)dr.$$

It is clear that $\Phi_p(x,t)$ is continuous and differentiable about t. Differentiating two sides of the above equation, it is easy to verify that

$$\partial_t \Phi_p(x,t) = D\big[(J * \Phi_p)(x,t) - \Phi_p(x,t)\big] + f\big(\Phi_p(x,t), \big(G * S(\Phi_p)\big)(x,t)\big).$$

Therefore, $\Phi_p(x,t)$ is an entire solution of (1.7). Moreover, the assertions of (i) and (ii) follow from Lemmas 4.1–4.2. The proof is complete. □

4.2. *Qualitative properties of entire solutions*

In the previous subsection, some new types of entire solutions of (1.7) were constructed by considering a combination of any finite number of traveling wave fronts with speeds $c \ge c_*$ and a spatial independent solution. In this subsection we continue to investigate the qualitative properties of the entire solutions, such as the monotonicity and limit of $\Phi_p(x,t)$ with respect to the variables x and t, and the shift parameters h_i, ϑ_j and h.

Theorem 4.4. *Assume* (G_1) *and* (C_1)–(C_2). *Let* $\Phi_p(x,t)$ *be the entire solution of* (1.7) *as stated in Theorem* 4.3, *then the following properties hold.*

(i) $0 < \Phi_p(x,t) < K$ *and* $\frac{\partial}{\partial t}\Phi_p(x,t) > 0$ *for any* $(x,t) \in \mathbb{R}^2$.
(ii) $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |\Phi_{p_{m,n,1}}(x,t) - K| = 0$, $\lim_{t\to+\infty} \sup_{|x|\le A} |\Phi_{p_{m,n,0}}(x,t) - K| = 0$ *for any* $A \in \mathbb{R}_+$.

(iii) *If* $m \geq 1$, *then* $\lim_{x\to+\infty}\sup_{t\geq a}|\Phi_p(x,t)-K|=0$ *for every* $a\in\mathbb{R}$; *if* $n\geq 1$, *then* $\lim_{x\to-\infty}\sup_{t\geq a}|\Phi_p(x,t)-K|=0$ *for every* $a\in\mathbb{R}$.

(iv) *If* (C_3) *holds or* $c_1,\dots,c_m,\bar{c}_1,\dots,\bar{c}_n > c_*$, *then* $\lim_{t\to-\infty}\sup_{x\in[A_1,A_2]}\Phi_p(x,t)=0$ *for any* $A_1<A_2$.

(v) *For any* $(x,t)\in\mathbb{R}^2$, $\Phi_p(x,t)$ *is increasing with respect to* h_i, ϑ_j *and* h, *respectively,* $\forall i=1,\dots,m$ *and* $j=1,\dots,n$.

(vi) *For any* $A,\gamma\in\mathbb{R}$, $\Phi_p(x,t)$ *converges to* K
  (a) *as* $h_i\to+\infty$ *uniformly on* $(x,t)\in[A,+\infty)\times[\gamma,+\infty)$, $i=1,\dots,m$;
  (b) *as* $\vartheta_j\to+\infty$ *uniformly on* $(x,t)\in(-\infty,A]\times[\gamma,+\infty)$, $j=1,\dots,n$;
  (c) *as* $h\to+\infty$ *uniformly on* $(x,t)\in\mathbb{R}\times[\gamma,+\infty)$.

(vii) *If* $c_1,\dots,c_m,\bar{c}_1,\dots,\bar{c}_n>c_*$, *then for every* $x\in\mathbb{R}$,

$$\Phi_{p_{m,n,1}}(x,t)\sim\Gamma(t+h)\sim e^{\lambda^*(t+h)}\quad\text{and}\quad\Phi_{p_{m,n,0}}(x,t)=O\big(e^{c_{\max}\lambda_1(c_{\max})}\big),$$

as $t\to-\infty$, *where* $c_{\max}:=\max\{\max_{i=1,\dots,m}\{c_i\},\max_{j=1,\dots,n}\{\bar{c}_j\}\}$.

(viii) *Let* (C_4) *hold.*
  (a) *For any* $\gamma\in\mathbb{R}$, $\Phi_{p_{m,n,1}}(x,t)$ *converges to* $\Phi_{p_{m,n,0}}(x,t)$ *as* $h\to-\infty$ *uniformly on* $(x,t)\in\widetilde{T}_\gamma:=\mathbb{R}\times(-\infty,\gamma]$.
  (b) *If* $c_{i_0}>c_*$ *for some* $i_0\in\{1,\dots,m\}$, *then for any* $A,\gamma\in\mathbb{R}$,

$$\Phi_p(x,t)\text{ converges to }\Phi_{(c_i,h_i;i\in\{1,\dots,m\}\setminus\{i_0\},\bar{c}_j,\vartheta_j;j\in\{1,\dots,n\},h)}(x,t)$$

  as $h_{i_0}\to-\infty$ *uniformly on* $(x,t)\in(-\infty,A]\times(-\infty,\gamma]$.
  (c) *If* $\bar{c}_{j_0}>c_*$ *for some* $j_0\in\{1,\dots,n\}$, *then for any* $A,\gamma\in\mathbb{R}$,

$$\Phi_p(x,t)\text{ converges to }\Phi_{(c_i,h_i;i\in\{1,\dots,m\},\bar{c}_j,\vartheta_j;j\in\{1,\dots,n\}\setminus\{j_0\},h)}(x,t)$$

  as $\vartheta_{j_0}\to-\infty$ *uniformly on* $(x,t)\in[A,+\infty)\times(-\infty,\gamma]$.

Proof. The proofs of parts (ii)–(vi) are straightforward and omitted.

(i) Clearly, $0<\Phi_p(x,t)\leq K$ for all $(x,t)\in\mathbb{R}^2$. Since

$$u^k(x,t)\geq\underline{u}(x,t)\geq\underline{u}(x,s)=\varphi(x,s)=u^k(x,s)$$

for $x\in\mathbb{R}$, $s\in(-\infty,-k]$ and $t>-k$, by Lemma 3.6, we have $\frac{\partial}{\partial t}u^k(x,t)\geq 0$ for $(x,t)\in\mathbb{R}\times(-k,+\infty)$. This yields $\frac{\partial}{\partial t}\Phi_p(x,t)\geq 0$ for all $(x,t)\in\mathbb{R}^2$. Now, we show that $\frac{\partial}{\partial t}\Phi_p(x,t)>0$ for $(x,t)\in\mathbb{R}^2$. By direct computations, we have

$$\begin{aligned}\partial_{tt}\Phi_p&=D\big[(J*\partial_t\Phi_p)(x,t)-\partial_t\Phi_p(x,t)\big]\\&\quad+\partial_1 f\big(u(x,t),\big(G*S(u)\big)(x,t)\big)\partial_t\Phi_p(x,t)\\&\quad+\partial_2 f\big(u(x,t),\big(G*S(u)\big)(x,t)\big)\big(G*\big(S'(\Phi_p)\partial_t\Phi_p\big)\big)(x,t)\\&\geq-(D+\bar{L}_1)\partial_t\Phi_p(x,t),\end{aligned}$$

where $\bar{L}_1=\max_{(u,v)\in[0,K]\times[0,S(K)]}|\partial_1 f(u,v)|$. For any $r\in\mathbb{R}$, we can obtain

$$(\Phi_p)_t(x,t) \ge (\Phi_p)_t(x,r)e^{-(D+\bar{L}_1)(t-r)} \ge 0, \quad \forall x \in \mathbb{R},\ t > r. \tag{4.7}$$

Suppose for the contrary that there exists $(x_0, t_0) \in \mathbb{R}^2$ such that $(\Phi_p)_t(x_0, t_0) = 0$. By (4.7), we see that $(\Phi_p)_t(x_0, r) = 0$ for all $r \le t_0$. Hence $\Phi_p(x_0, t) = \Phi_p(x_0, t_0)$ for all $t \le t_0$, which implies that $\lim_{t\to-\infty} \Phi_p(x_0, t) = \Phi_p(x_0, t_0)$. But following from (i) and (ii) of Theorem 4.3, we see that $\Phi_p(x_0, t_0) > 0$ and $\lim_{t\to-\infty} \Phi_p(x_0, t) = 0$. This contradiction yields that $\frac{\partial}{\partial t}\Phi_p(x, t) > 0$ for all $(x, t) \in \mathbb{R}^2$.

If there exists $(x_0, t_0) \in \mathbb{R}^2$ such that $\Phi_p(x_0, t_0) = K$, then

$$\begin{aligned} 0 < \partial_t \Phi_p(x_0, t_0) &= D\big[(J * \Phi_p)(x_0, t_0) - K\big] + f\big(K, \big(G * S(\Phi_p)\big)(x_0, t_0)\big) \\ &\le f\big(K, S(K)\big) = 0, \end{aligned}$$

which is impossible. Therefore, $\Phi_p(x, t) < K$ for $(x, t) \in \mathbb{R}^2$.

(vii) When $c_1, \ldots, c_m, \bar{c}_1, \ldots, \bar{c}_n > c_*$, the assertion (ii) of Theorem 4.3 implies that

$$\begin{aligned} \chi\Gamma(t+h) \le \Phi_{p_{m,n,\chi}}(x,t) \le &\sum_{1\le j\le m} e^{\lambda_1(c_j)(x+c_jt+h_j)} \\ &+ \sum_{1\le j\le n} e^{\lambda_1(\bar{c}_j)(-x+\bar{c}_jt+\vartheta_j)} + \chi e^{\lambda^*(t+h)}. \end{aligned} \tag{4.8}$$

The second part of this statement follows from (4.8) with $\chi = 0$ and the fact that $\frac{\partial}{\partial c}[c\lambda_1(c)] < 0$ for any $c > c_*$.

Since $\lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = 1$, to prove the first part of this statement, it suffices to prove that $c\lambda_1(c) > \lambda^*$ for any $c > c_*$. Suppose for the contrary that there exists $c_0 > c_*$ such that $c_0\lambda_1(c_0) \le \lambda^*$. Then, from Propositions 3.1–3.2, we have

$$\begin{aligned} 0 &\ge c_0\lambda_1(c_0) - \lambda^* \\ &= D\left[\int_{-\infty}^{+\infty} e^{-\lambda_1(c_0)y} J(y)dy - 1\right] \\ &\quad + \partial_2 f(0,0)S'(0)\left[\int_0^{+\infty}\int_{-\infty}^{+\infty} \big[e^{-\lambda_1(c_0)(y+cs)} - e^{-\lambda_* s}\big]G(y,s)dyds\right] \\ &> \partial_2 f(0,0)S'(0)\int_0^{+\infty}\int_{-\infty}^{+\infty} e^{-\lambda_1(c_0)cs}\big[e^{-\lambda_1(c_0)y} - 1\big]G(y,s)dyds \ge 0. \end{aligned}$$

This contradiction shows that $c\lambda_1(c) > \lambda^*$ for any $c > c_*$, and the first part of this statement follows.

(viii) We only prove part (a) of this assertion, since the other cases can be considered similarly. Recall that $u^k(x, t)$ is the unique solution of the initial value problem (4.1). For $\chi = 1$, we denote $\varphi^k(x, s)$ by $\varphi^k_{p_{m,n,1}}(x, s)$ and $u^k(x, t)$ by $u^k_{p_{m,n,1}}(x, t)$, respectively. Similarly, when $\chi = 0$, we denote $\varphi^k(x, s)$ and $u^k(x, t)$ by $\varphi^k_{p_{m,n,0}}(x, s)$ and $u^k_{p_{m,n,0}}(x, t)$, respectively. Take

$$Z^k(x,t) := u^k_{p_{m,n,1}}(x,t) - u^k_{p_{m,n,0}}(x,t), \quad \text{for } (x,t) \in \mathbb{R} \times (-k,+\infty).$$

Then, $0 \le Z^k(x,t) \le K$ for all $(x,t) \in \mathbb{R} \times (-k,+\infty)$. By assumption ($C_4$), we see that

$$\begin{aligned}\frac{\partial Z^k}{\partial t} &= D\big[\big(J * Z^k\big)(x,t) - Z^k(x,t)\big] \\ &\quad + f\big(u^k_{p_{m,n,1}}(x,t), \big(G * S\big(u^k_{p_{m,n,1}}\big)\big)(x,t)\big) - f\big(u^k_{p_{m,n,0}}(x,t), \big(G * S\big(u^k_{p_{m,n,0}}\big)\big)(x,t)\big) \\ &\le D\big[\big(J * Z^k\big)(x,t) - Z^k(x,t)\big] + \partial_1 f(0,0) Z^k(x,t) + \partial_2 f(0,0) S'(0)\big(G * Z^k\big)(x,t).\end{aligned}$$

Define the function $\widehat{Z}(x,t) := e^{\lambda^*(t+h)}$, $(x,t) \in \mathbb{R}^2$. By Proposition 3.2, we have

$$Z^k(x,s) = u^k_{p_{m,n,1}}(x,s) - u^k_{p_{m,n,0}}(x,s) \le \Gamma(s+h) \le e^{\lambda^*(s+h)} = \widehat{Z}(x,s)$$

for $x \in \mathbb{R}$ and $s \in (-\infty,-k]$. Moreover, it is easy to see that $\widehat{Z}(x,t)$ satisfies the linear equation:

$$\frac{\partial \widehat{Z}}{\partial t} = D\big[(J * \widehat{Z})(x,t) - \widehat{Z}(x,t)\big] + \partial_1 f(0,0)\widehat{Z}(x,t) + \partial_2 f(0,0) S'(0)(G * \widehat{Z})(x,t).$$

It then follows from Lemma 3.7 that

$$0 \le Z^k(x,t) = u^k_{p_{m,n,1}}(x,t) - u^k_{p_{m,n,0}}(x,t) \le \widehat{Z}(x,t) = e^{\lambda^*(t+h)}$$

for all $(x,t) \in \mathbb{R} \times [-k,+\infty)$. Since $\lim_{k\to\infty} u^k_{p_{m,n,\chi}}(x,t) = \Phi_{p_{m,n,\chi}}(x,t)$, we have

$$0 \le \Phi_{p_{m,n,1}}(x,t) - \Phi_{p_{m,n,0}}(x,t) \le e^{\lambda^*(t+h)} \quad \text{for all } (x,t) \in \mathbb{R}^2,$$

which implies that $\Phi_{p_{m,n,1}}(x,t)$ converges to $\Phi_{p_{m,n,0}}(x,t)$ as $h \to -\infty$ uniformly on $(x,t) \in \widetilde{T}_\gamma$ for any $\gamma \in \mathbb{R}$, and the assertion of this part follows. This completes the proof. □

In the following theorem, we establish the relationship between the entire solution $\Phi_p(x,t)$ and the traveling wave fronts which they originated.

Theorem 4.5. *Assume* (G_1) *and* (C_1)–(C_3). *Let $\Phi_p(x,t)$ be the entire solution of* (1.7) *stated in Theorem 4.3. Then for any $c \ge c_*$, the following properties hold.*

- (i) (a) *If there exists $i_0 \in \{1,\dots,m\}$ such that $c = c_{i_0}$ and $c < c_i$ for any $i \ne i_0$, then $\Phi_p(-ct + x,t) \to \phi_{c_{i_0}}(x + h_{i_0})$; if there exists $j_0 \in \{1,\dots,n\}$ such that $c = \bar{c}_{j_0}$ and $c < \bar{c}_j$ for any $j \ne j_0$, then $\Phi_p(ct + x,t) \to \phi_{\bar{c}_{j_0}}(x + \vartheta_{j_0})$ as $t \to -\infty$;*
  - (b) *if $c < c_i$ for all $i \in \{1,\dots,m\}$, then $\Phi_p(-ct + x,t) \to 0$ as $t \to -\infty$; if $c < \bar{c}_j$ for all $j \in \{1,\dots,n\}$, then $\Phi_p(ct + x,t) \to 0$ as $t \to -\infty$;*
  - (c) *if there exists $i_0 \in \{1,\dots,m\}$ such that $c > c_{i_0}$, then $\Phi_p(-ct + x,t) \to K$ as $t \to -\infty$; if there exists $j_0 \in \{1,\dots,n\}$ such that $c > \bar{c}_{j_0}$, then $\Phi_p(ct + x,t) \to K$ as $t \to -\infty$.*
- (ii) *If there exists $i_0 \in \{1,\dots,m\}$ such that $c < c_{i_0}$, then $\Phi_p(-ct + x,t) \to K$ as $t \to +\infty$; if there exists $j_0 \in \{1,\dots,n\}$ such that $c < \bar{c}_{j_0}$, then $\Phi_p(ct + x,t) \to K$ as $t \to +\infty$.*

All these limits are uniform in x in any compact subset of $\mathbb{R}$.

Proof. (i) We only prove the statement (a), since the others can be proved similarly. From Theorem 4.3, we have

$$
\begin{aligned}
0 &\le \Phi_p(-ct+x,t) - \phi_{c_{i_0}}\big((c_{i_0}-c)t+x+h_{i_0}\big) \\
&\le \sum_{1\le i\le m, i\ne i_0} \phi_{c_i}\big(x+(c_i-c)t+h_i\big) + \sum_{1\le j\le n} \phi_{\bar{c}_j}\big(-x+(\bar{c}_j+c)t+\vartheta_j\big) + \chi\Gamma(t+h)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\le \Phi_p(ct+x,t) - \phi_{\bar{c}_{j_0}}\big((\bar{c}_{j_0}-c)t-x+\vartheta_{j_0}\big) \\
&\le \sum_{1\le i\le m} \phi_{c_i}\big(x+(c_i+c)t+h_i\big) + \sum_{1\le j\le n, j\ne j_0} \phi_{\bar{c}_j}\big(-x+(\bar{c}_j-c)t+\vartheta_j\big) + \chi\Gamma(t+h)
\end{aligned}
$$

for all $(x,t)\in\mathbb{R}^2$, hence the assertion of part (i) follows. Similarly, we can prove the assertion of part (ii). This completes the proof. □

Remark 4.6. Roughly speaking, the convergences in the statements of part (i) of Theorem 4.5 mean that only some fronts, i.e. those with small speeds, can be "viewed" as $t\to-\infty$, the others are being "hidden". But, it seems impossible to view any fronts as $t\to+\infty$. Similar phenomenon has been observed by Hamel and Nadirashvili [16] for the Fisher–KPP equation.

5. Entire solutions: non-quasi-monotone case

In this section, we consider the entire solutions of (1.7) in the non-quasi-monotone case. It is well known that the comparison principle is not applicable for such non-quasi-monotone systems. In addition to $(\mathrm{C}_2)'$ and (G_1), we further make the following assumptions:

$(\mathrm{C}_3)'$ $S^+(u)$ is concave on $[0,K^+]$ and for any $m\in\mathbb{Z}^+$, $(a_i,b_i)\in\bar{I}^+$, $i=1,\dots,m$,

$$
\begin{aligned}
& L_f^+ \min\left\{K^+, \sum_{i=1}^m a_i\right\} + f^+\left(\min\left\{K^+, \sum_{i=1}^m a_i\right\}, \min\left\{S^+(K^+), \sum_{i=1}^m b_i\right\}\right) \\
& \le \sum_{i=1}^m \big[L_f^+ a_i + f^+(a_i,b_i)\big],
\end{aligned}
$$

where $\bar{I}^+ = [0,K^+]\times[0,S^+(K^+)]$ and $L_f^+ := \max_{(u,v)\in\bar{I}^+} |\partial_1 f^+(u,v)|$.

Similar to Lemma 3.6, it is easy to verify that for any $\varphi\in\mathcal{C}_{[0,K^+]}$, (1.7) has a unique solution $u(x,t;\varphi)$ on $[0,\infty)$ with $0\le u(x,t;\varphi)\le K^+$ for $x\in\mathbb{R}$, $t\ge 0$. Moreover, $u(x,t;\varphi)$ is classical on $(0,+\infty)$. Here and in what follows, $X_{[0,K^\pm]}$ and $\mathcal{C}_{[0,K^\pm]}$ are defined as (3.9) and (3.10) by replacing $[0,K]$ with $[0,K^\pm]$.

According to the assumption $(\mathrm{C}_2)'$, we consider the following two auxiliary quasi-monotone nonlocal dispersal equations with spatio-temporal delay:

$$u_t = D(J * u - u)(x,t) + f^+\big(u(x,t), \big(G * S^+(u)\big)(x,t)\big), \tag{5.1}$$

$$u_t = D(J * u - u)(x,t) + f^-\big(u(x,t), \big(G * S^-(u)\big)(x,t)\big), \tag{5.2}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$.

It is clear that $\triangle(c,\lambda) = 0$ is also the characteristic equation of (5.1) and (5.2) with respect to the trivial equilibrium. By Proposition 3.2, we have the following result.

Proposition 5.1. *Assume* (G_1) *and* $(\mathrm{C}_2)'$.

(i) *For any $c \geq c_*$, (5.1) and (5.2) have traveling wave fronts $\phi_c^+(\xi)$, $\phi_c^-(\xi)$, $\xi = x + ct$, respectively, which satisfy $(\phi_c^\pm)'(\cdot) > 0$, $\phi_c^\pm(-\infty) = 0$ and $\phi_c^\pm(+\infty) = K^\pm$. Moreover, if $c > c_*$, then*

$$\lim_{\xi \to -\infty} \phi_c^\pm(\xi) e^{-\lambda_1(c)\xi} = 1, \qquad \phi_c^\pm(\xi) \leq e^{\lambda_1(c)\xi} \quad \textit{for all } \xi \in \mathbb{R}.$$

(ii) *There exist solutions $\Gamma^\pm(t) : \mathbb{R} \to [0, K^+]$ of the following delayed equations:*

$$\Gamma'(t) = f^\pm\left(\Gamma(t), \int_0^\infty S^\pm\big(\Gamma(t-s)\big) \int_{-\infty}^{+\infty} G(y,s)\,dy\,ds\right), \quad t \in \mathbb{R}, \tag{5.3}$$

which satisfy $\Gamma^\pm(-\infty) = 0$, $\Gamma^\pm(+\infty) = K^\pm$ and

$$\frac{d}{dt}\Gamma^\pm(t) > 0, \qquad \lim_{t \to -\infty} \Gamma^\pm(t) e^{-\lambda^* t} = 1 \quad \textit{and} \quad \Gamma^-(t) \leq e^{\lambda^* t} \quad \textit{for all } t \in \mathbb{R}.$$

Define $\tilde{F}, F^\pm : \mathcal{C}_{[0,K^+]} \to X$ by

$$F^\pm(\varphi)(x) := D(J * \varphi)(x,0) + L\varphi(x,0) + f^\pm\big(\varphi(x,0), \big(G * S^\pm(\varphi)\big)(x,0)\big),$$
$$\tilde{F}(\varphi)(x) := D(J * \varphi)(x,0) + L\varphi(x,0) + f\big(\varphi(x,0), \big(G * S(\varphi)\big)(x,0)\big),$$

where

$$L := \max_{(u,v) \in \bar{I}^+} \max\big\{\big|\partial_1 f^+(u,v)\big|, \big|\partial_1 f^-(u,v)\big|, \big|\partial_1 f(u,v)\big|\big\}.$$

It is clear that $F^\pm(\cdot)$ are non-decreasing in $\mathcal{C}_{[0,K^+]}$ and

$$F^-(\varphi) \leq \tilde{F}(\varphi) \leq F^+(\varphi) \quad \text{for } \varphi \in \mathcal{C}_{[0,K^+]}.$$

Moreover, we define $\tilde{T}(t) = e^{-(D+L)t}$.

The following two lemmas play an important role in the proof of our main result for the non-quasi-monotone nonlocal dispersal system with spatio-temporal delay.

Lemma 5.2. *Assume* (G_1) *and* $(C_2)'$. *Given* $r \in \mathbb{R}$. *Let* $u, u^{\pm} \in C(\mathbb{R}, X_{[0,K^+]})$ *be such that*

$$u^-(t)(x) \le \tilde{T}(t-r)\big[u^-(r)\big](x) + \int_r^t \tilde{T}(t-s)\big[F^-\big(u_s^-\big)\big](x)ds, \tag{5.4}$$

$$u(t)(x) = \tilde{T}(t-r)\big[u(r)\big](x) + \int_r^t \tilde{T}(t-s)\big[\tilde{F}(u_s)\big](x)ds, \tag{5.5}$$

$$u^+(t)(x) \ge \tilde{T}(t-r)\big[u^+(r)\big](x) + \int_r^t \tilde{T}(t-s)\big[F^+\big(u_s^+\big)\big](x)ds \tag{5.6}$$

for all $x \in \mathbb{R}$, $t > r$ *and* $u^-(x,s) \le u(x,s) \le u^+(x,s)$ *for* $x \in \mathbb{R}$ *and* $s \in (-\infty, r]$. *Then,*

$$u^-(x,t) \le u(x,t) \le u^+(x,t) \quad \textit{for all } x \in \mathbb{R} \textit{ and } t \ge r.$$

Proof. We only prove $u(x,t) \le u^+(x,t)$ for all $x \in \mathbb{R}$ and $t \ge r$, since the other case can be proved similarly. Let

$$z(x,t) := u(x,t) - u^+(x,t) \quad \text{for } (x,t) \in \mathbb{R}^2.$$

Note that $u(x,s) \le u^+(x,s)$ for $x \in \mathbb{R}$, $s \in (-\infty, r]$. Using the assumption $(C_2)'$, we have for any $x \in \mathbb{R}$ and $s \ge r$ that

$$\begin{aligned}
&\tilde{F}(u_s)(x) - F^+\big(u_s^+\big)(x) \\
&\quad \le F^+(u_s)(x) - F^+\big(u_s^+\big)(x) \\
&\quad = D(J*z)(x,s) + Lz(x,s) \\
&\qquad + f^+\big(u(x,s), \big(G*S^+(u)\big)(x,s)\big) - f^+\big(u^+(x,s), \big(G*S^+(u)\big)(x,s)\big) \\
&\qquad + f^+\big(u^+(x,s), \big(G*S^+(u)\big)(x,s)\big) - f^+\big(u^+(x,s), \big(G*S^+\big(u^+\big)\big)(x,s)\big) \\
&\quad \le D(J*z)(x,s) + \big[L + \partial_1 f^+\big(\eta(x,s), \big(G*S^+(u)\big)(x,s)\big)\big]z(x,s) \\
&\qquad + L_f L_S \int_0^\infty \int_{-\infty}^{+\infty} G(x-y,r)\max\big\{z(y,s-r),0\big\}dydr \\
&\quad \le D\int_{-\infty}^{+\infty} J(y)\max\big\{0, z(x-y,s)\big\}dy + 2L\max\big\{0, z(x,s)\big\} \\
&\qquad + L_f L_S \int_0^\infty \int_{-\infty}^{+\infty} G(x-y,r)\max\big\{z(y,s-r),0\big\}dydr,
\end{aligned} \tag{5.7}$$

where $\eta(x,s) = \theta u(x,s) + (1-\theta)u^+(x,s)$, $\theta \in (0,1)$.

Denote $[B]_+ := \max\{B, 0\}$ for any $B \in \mathbb{R}$. Then, it follows from (5.5)–(5.7) that

$$\begin{aligned} z(x,t) &\le \tilde{T}(t-r)\big[z(r)\big](x) + \int_r^t \tilde{T}(t-s)\big[\tilde{F}(u_s) - F^+\big(u_s^+\big)\big](x)ds \\ &\le \int_r^t \tilde{T}(t-s)\big[\tilde{F}(u_s) - F^+\big(u_s^+\big)\big](x)ds \\ &\le \int_r^t \tilde{T}(t-s)\Bigg\{D\int_{-\infty}^{+\infty} J(y)\big[z(x-y,s)\big]_+dy + 2L\big[z(x,s)\big]_+ \\ &\quad + L_f L_S \int_0^{\infty}\int_{-\infty}^{+\infty} G(y,r)\big[z(x-y,s-r)\big]_+dydr\Bigg\}ds, \end{aligned}$$

which implies that

$$\begin{aligned} \big[z(x,t)\big]_+ &\le \int_r^t \tilde{T}(t-s)\Bigg\{D\int_{-\infty}^{+\infty} J(y)\big[z(x-y,s)\big]_+dy + 2L\big[z(x,s)\big]_+ \\ &\quad + L_f L_S \int_0^{\infty}\int_{-\infty}^{+\infty} G(y,r)\big[z(x-y,s-r)\big]_+dydr\Bigg\}ds. \end{aligned}$$

Using the similar method as in the proof of Lemma 3.7, we can show that $z(x,t) \le 0$ for $x \in \mathbb{R}$ and $t \ge r$. Therefore, $u(x,t) \le u^+(x,t)$ for $x \in \mathbb{R}$ and $t \ge r$. This completes the proof. □

Lemma 5.3. *Assume* (G_1) *and* $(\mathrm{C}_2)'$. *Let* $u(x,t;\varphi)$ *be the solution of* (1.7) *with the initial value* $\varphi \in \mathcal{C}_{[0,K^+]}$.

(i) *There exists a positive constant* M_1, *independent of* φ, *such that for any* $x \in \mathbb{R}$ *and* $t > 0$, $|u_t(x,t;\varphi)| \le M_1$.
(ii) *If, in addition,* (G_2) *holds,* $L_1 := \max_{(u,v)\in \bar{I}^+} \partial_1 f(u,v) < D$ *and there exists a constant* $M > 0$ *such that for any* $\eta > 0$, $\sup_{x\in\mathbb{R}} |\varphi(x+\eta) - \varphi(x)| \le M\eta$, *then for any* $\eta > 0$,

$$\sup_{x\in\mathbb{R},t\ge 0} \big|u(x+\eta,t;\varphi) - u(x,t;\varphi)\big| \le M'\eta,$$

where $M' > 0$ *is a constant which is independent of* φ *and* η.

Proof. Since $0 \le u(x,t) \le K^+$ for $(x,t) \in \mathbb{R}^2$, it is easy to see that the first statement (i) holds.

(ii) For any given $\eta > 0$, let $w(x,t) = u(x+\eta,t;\varphi) - u(x,t;\varphi)$. For simplicity, we denote $u(x+\eta,t;\varphi)$ and $u(x,t;\varphi)$ by $u(x+\eta,t)$ and $u(x,t)$, respectively. Clearly, $|w(x,0)| = |\varphi(x+\eta) - \varphi(x)| \le M\eta$ for all $x \in \mathbb{R}$. It follows from (G_2) that

$$\begin{aligned}\frac{\partial w}{\partial t} &= D\int_{-\infty}^{+\infty}\big[J(x+\eta-y)-J(x-y)\big]u(y,t)dy - Dw(x,t)\\ &\quad + f\big(u(x+\eta,t),\big(G*S(u)\big)(x+\eta,t)\big) - f\big(u(x,t),\big(G*S(u)\big)(x,t)\big)\\ &\le DK^+\bar{L}\eta + \big[h(x,t)-D\big]w(x,t)\\ &\quad + L_2\int_0^{\infty}\int_{-\infty}^{+\infty}\big|G(x+\eta-y,s)-G(x-y,s)\big|S\big(u(y,t-s)\big)dyds\\ &\le \hat{L}\eta + \big[h(x,t)-D\big]w(x,t)\end{aligned}$$

for $x \in \mathbb{R}$ and $t \ge 0$, where $L_2 = \max_{(u,v)\in\bar{I}^+}|\partial_2 f(u,v)|$, $\hat{L} = DK^+\bar{L} + \bar{L}L_2S^+(K^+)$, and

$$h(x,t) = \partial_1 f\big(\theta u(x+\eta,t)+(1-\theta)u(x,t),\big(G*S(u)\big)(x+\eta,t)\big), \quad \theta \in (0,1).$$

Simple calculations show that, for $x \in \mathbb{R}$ and $t \ge 0$,

$$\begin{aligned} w(x,t) &\le w(x,0)e^{\int_0^t[h(x,s)-D]ds} + \int_0^t \hat{L}\eta e^{\int_s^t[h(x,r)-D]dr}ds\\ &\le M\eta e^{\int_0^t[L_1-D]ds} + \int_0^t \hat{L}\eta e^{\int_s^t[L_1-D]dr}ds\\ &= M\eta e^{-(D-L_1)t} + \int_0^t \hat{L}\eta e^{-(D-L_1)(t-s)}ds\\ &= M\eta e^{-(D-L_1)t} + \hat{L}\eta\big(1-e^{-(D-L_1)t}\big)/(D-L_1) := \overline{w}(t).\end{aligned}$$

Now, let $\tilde{w}(x,t) := -w(x,t) = u(x,t) - u(x+\eta,t)$. Then we have

$$\begin{aligned}\frac{\partial \tilde{w}}{\partial t} &= D\int_{-\infty}^{+\infty}\big[J(x-y)-J(x+\eta-y)\big]u(y,t)dy - D\tilde{w}(x,t)\\ &\quad + f\big(u(x,t),\big(G*S(u)\big)(x,t)\big) - f\big(u(x+\eta,t),\big(G*S(u)\big)(x+\eta,t)\big)\\ &\le DK^+\bar{L}\eta + \big[h(x,t)-D\big]\tilde{w}(x,t)\\ &\quad + L_2\int_0^{\infty}\int_{-\infty}^{+\infty}\big|G(x+\eta-y,s)-G(x-y,s)\big|S\big(u(y,t-s)\big)dyds\\ &\le \hat{L}\eta + \big[h(x,t)-D\big]\tilde{w}(x,t).\end{aligned}$$

Similarly, we can show that $\tilde{w}(x,t) = -w(x,t) \le \overline{w}(t)$ for $x \in \mathbb{R}$ and $t \ge 0$. Therefore, since $L_1 < D$, we get

$$\left|w(x,t)\right| \le \overline{w}(t) \le M'\eta := \left[M + \hat{L}/(D - L_1)\right]\eta \quad \text{for all } x \in \mathbb{R},\ t \ge 0.$$

The proof is complete. $\square$

Applying Lemmas 5.2 and 5.3, we have the following results for the nonlocal dispersal equation (1.7) with monostable and non-quasi-monotone nonlinearity. For the sake of convenience, we denote

$$\underline{U}(x,t) := \max\Big\{\max_{i=1,\dots,m} \phi^-_{c_i}(x + c_i t + h_i),\ \max_{j=1,\dots,n} \phi^-_{\bar{c}_j}(-x + \bar{c}_j t + \vartheta_j),\ \chi\Gamma^-(t+h)\Big\},$$

$$\Pi^+(x,t) := \sum_{i=1}^{m} \phi^+_{c_i}(x + c_i t + h_i) + \sum_{j=1}^{n} \phi^+_{\bar{c}_j}(-x + \bar{c}_j t + \vartheta_j) + \chi\Gamma^+(t+h),$$

$$\overline{\Pi}(x,t) := \sum_{i=1}^{m} e^{\lambda_1(c_i)(x + c_i t + h_i)} + \sum_{j=1}^{n} e^{\lambda_1(\bar{c}_j)(-x + \bar{c}_j t + \vartheta_j)} + \chi e^{\lambda^*(t+h)}.$$

Theorem 5.4. *Assume* (G_1), (G_2), $(C_2)'$ *and* $L_1 < D$. *For any* $m, n \in \mathbb{N} \cup \{0\}$, $h_1, \dots, h_m$, $\vartheta_1, \dots, \vartheta_n, h \in \mathbb{R}$, $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n \ge c_*$, *and* $\chi \in \{0, 1\}$ *with* $m + n + \chi \ge 1$, *there exists an entire solution* $U_p(x,t)$ *of* (1.7) *such that*

$$U_p(x,t) > 0 \quad \text{and} \quad \underline{U}(x,t) \le U_p(x,t) \le K^+ \quad \text{for } (x,t) \in \mathbb{R}^2,$$

where $p := p_{m,n,\chi} = (c_1, h_1, \dots, c_m, h_l, \bar{c}_1, \vartheta_1, \dots, \bar{c}_n, \vartheta_n, \chi h)$.

Furthermore, the following results hold.

(i) *If* $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n > c_*$, *then*

$$U_p(x,t) \le \overline{U}(x,t) := \min\big\{K^+, \overline{\Pi}(x,t)\big\} \quad \text{for } (x,t) \in \mathbb{R}^2. \tag{5.8}$$

(ii) *If* $(C_3)'$ *holds, then*

$$U_p(x,t) \le U^+(x,t) := \min\big\{K^+, \Pi^+(x,t)\big\} \quad \text{for } (x,t) \in \mathbb{R}^2. \tag{5.9}$$

(iii) *If* $(C_3)'$ *holds or* $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n > c_*$, *then*

$$\lim_{t \to -\infty} \sup_{|x| \le A} \left|U_p(x,t)\right| = 0 \quad \text{for any } A \in \mathbb{R}_+.$$

(iv) $\liminf_{t \to +\infty} \inf_{x \in \mathbb{R}} U_{p_{m,n,1}}(x,t) \ge K^-$ *and* $\liminf_{t \to +\infty} \inf_{|x| \le A} U_{p_{m,n,0}}(x,t) \ge K^-$ *for any* $A \in \mathbb{R}_+$.

(v) *If* $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n > c_*$, *then for every* $x \in \mathbb{R}$, *as* $t \to -\infty$,

$$U_{p_{m,n,1}}(x,t) \sim e^{\lambda^*(t+h)} \quad \textit{and} \quad U_{p_{m,n,0}}(x,t) = O\big(e^{c_{\max}\lambda_1(c_{\max})}\big),$$

where $c_{\max} := \max\{\max_{i=1,\dots,m}\{c_i\}, \max_{j=1,\dots,n}\{\bar{c}_j\}\}$.

Proof. For $n \in \mathbb{Z}$, we denote

$$\varphi^{n,-}(x,s) := \max\Big\{\max_{i=1,\dots,m} \phi^-_{c_i}(x + c_i s + h_i), \max_{j=1,\dots,n} \phi^-_{\bar{c}_j}(-x + \bar{c}_j s + \vartheta_j), \chi\Gamma^-(s+h)\Big\},$$

where $x \in \mathbb{R}$, $s \in (-\infty, -k]$. Let $U^k(x,t)$ be the unique mild solution of the initial value problem of (1.7) with initial condition:

$$U^k(x,s) = \varphi^{n,-}(x,s), \quad x \in \mathbb{R} \text{ and } s \in (-\infty, -k].$$

It is clear that $\underline{U}(x,s) = \varphi^{n,-}(x,s) = U^k(x,s) \le K^+$ for $x \in \mathbb{R}$ and $s \in (-\infty, -k]$. Since $F^-(\cdot)$ is non-decreasing in $\mathcal{C}_{[0,K^+]}$, one can easily verify that

$$\underline{U}(t)(x) \le \tilde{T}(t+k)\big[\underline{U}(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^-(\underline{U}_s)\big](x)ds,$$

$$U^k(t)(x) = \tilde{T}(t+k)\big[U^k(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[\tilde{F}\big(U^k_s\big)\big](x)ds,$$

$$K^+ = \tilde{T}(t+k)K^+ + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+\big(K^+\big)\big](x)ds$$

for any $x \in \mathbb{R}$ and $t > -k$. It follows from Lemma 5.2 that

$$\underline{U}(x,t) \le U^k(x,t) \le K^+ \quad \text{for } x \in \mathbb{R},\ t > -k.$$

Now, we prove the following claim.

Claim. *If* $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n > c_*$, *then*

$$U^k(x,t) \le \overline{U}(x,t) \quad \textit{for } x \in \mathbb{R},\ t > -k, \tag{5.10}$$

and if $(\mathrm{C}_3)'$ *holds, then*

$$U^k(x,t) \le U^+(x,t) \quad \textit{for } x \in \mathbb{R},\ t > -k. \tag{5.11}$$

We first prove (5.10). According to Proposition 5.1, if $c_1, \dots, c_m, \bar{c}_1, \dots, \bar{c}_n > c_*$, then $\varphi^{n,-}(x,s) = U^k(x,s) \le \overline{U}(x,s)$ for $x \in \mathbb{R}$ and $s \in (-\infty, -k]$. By Lemma 5.2, it suffices to show that

$$\tilde{T}(t+k)\big[\overline{U}(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+(\overline{U}_s)\big](x)ds \le \overline{U}(t)(x) \tag{5.12}$$

for $x \in \mathbb{R}$ and $t > -k$. Since $F^+(\cdot)$ is non-decreasing in $\mathcal{C}_{[0,K^+]}$, one can easily verify that

$$\tilde{T}(t+k)\big[\overline{U}(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+(\overline{U}_s)\big](x)ds \le K^+ \tag{5.13}$$

for $x \in \mathbb{R}$ and $t > -k$. For any $\varphi \in C((-\infty,0], X)$, define Q by

$$Q(\varphi)(x) = D(J * \varphi)(x,0) + \big[L + \partial_1 f(0,0)\big]\varphi(x,0) + \partial_2 f(0,0)S'(0)(G * \varphi)(x,0).$$

Then, direct computations show that $\overline{\Pi}(t)(\cdot) = \overline{\Pi}(\cdot,t)$ satisfies the integral equation:

$$\overline{\Pi}(t)(x) = \tilde{T}(t+k)\big[\overline{\Pi}(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[Q(\overline{\Pi}_s)\big](x)ds. \tag{5.14}$$

By the assumption $(\mathrm{C}_2)'$, we obtain

$$\begin{aligned}
F^+(\overline{U}_s)(x) &= D(J * \overline{U})(x,s) + L\overline{U}(x,s) + f^+\big(\overline{U}(x,s), \big(G * S^+(\overline{U})\big)(x,s)\big) \\
&\le D(J * \overline{U})(x,s) + \big[L + \partial_1 f(0,0)\big]\overline{U}(x,s) + \partial_2 f(0,0)S'(0)(G * \overline{U})(x,s) \\
&\le D(J * \overline{\Pi})(x,s) + \big[L + \partial_1 f(0,0)\big]\overline{\Pi}(x,s) + \partial_2 f(0,0)S'(0)(G * \overline{\Pi})(x,s) \\
&= Q(\overline{\Pi}_s)(x).
\end{aligned}$$

Then it follows from (5.14) that

$$\begin{aligned}
&\tilde{T}(t+k)\big[\overline{U}(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+(\overline{U}_s)\big](x)ds \\
&\quad \le \tilde{T}(t+k)\big[\overline{\Pi}(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[Q(\overline{\Pi}_s)\big](x)ds = \overline{\Pi}(t)(x).
\end{aligned} \tag{5.15}$$

Combining (5.13) and (5.15), (5.12) holds and (5.10) follows from Lemma 5.2.

Now, we prove (5.11). Clearly, $\varphi^{n,-}(x,s) = U^k(x,s) \le U^+(x,s)$ for $x \in \mathbb{R}$ and $s \in (-\infty, -k]$. Similar to (5.13), we have

$$\tilde{T}(t+k)\big[U^+(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+\big(U_s^+\big)\big](x)ds \le K^+ \tag{5.16}$$

for $x \in \mathbb{R}$ and $t > -k$. For simplicity, denote

$$v_i(x,t) := \phi_{c_i}^+(x + c_i t + h_i), \quad i = 1, \ldots, m,$$
$$v_{m+j}(x,t) := \phi_{\bar{c}_j}^+(-x + \bar{c}_j t + \vartheta_j), \quad j = 1, \ldots, n \quad \text{and} \quad v_{m+n+1}(x,t) = \chi \Gamma^+(t+h).$$

Note that v_i $(i = 1, \ldots, m+n+1)$ satisfy the following equation:

$$\tilde{T}(t+k)\big[v_i(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+\big((v_i)_s\big)\big](x)ds = v_i(t)(x), \quad t > -k. \tag{5.17}$$

Since $S^+(u)$ is concave and non-decreasing on $[0, K^+]$, we have

$$S^+\big(U^+\big) \le \min\big\{S^+\big(K^+\big), S^+(v_1) + \cdots + S^+(v_{m+n+1})\big\}.$$

Furthermore, using assumption $(C_3)'$, we obtain

$$\begin{aligned}
F^+\big(U_s^+\big)(x) &= D\big(J * U^+\big)(x,s) + LU^+(x,s) + f^+\big(U^+(x,s), \big(G * S^+\big(U^+\big)\big)(x,s)\big) \\
&\le D \sum_{i=1}^{m+n+1} (J * v_i)(x,s) + \big(L - L_f^+\big)U^+(x,s) + L_f^+ U^+(x,s) \\
&\quad + f^+\left(U^+(x,s), \min\left\{S^+\big(K^+\big), \sum_{i=1}^{m+n+1} \big(G * S^+(v_i)\big)(x,s)\right\}\right) \\
&\le \sum_{i=1}^{m+n+1} \big[D(J * v_i)(x,s) + \big(L - L_f^+\big)v_i(x,s) \\
&\quad + L_f^+ v_i(x,s) + f^+\big(v_i(x,s), \big(G * S^+(v_i)\big)(x,s)\big)\big] \\
&= \sum_{i=1}^{m+n+1} F^+\big((v_i)_s\big)(x).
\end{aligned}$$

Then it follows from (5.17) that

$$\begin{aligned}
&\tilde{T}(t+k)\big[U^+(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+\big(U_s^+\big)\big](x)ds \\
&\quad \le \sum_{i=1}^{m+n+1} \left\{\tilde{T}(t+k)\big[v_i(-k)\big](x) + \int_{-k}^{t} \tilde{T}(t-s)\big[F^+\big((v_i)_s\big)\big](x)ds\right\}
\end{aligned}$$

$$= \sum_{i=1}^{m+n+1} v_i(t)(x) = \Pi^+(x,t) \tag{5.18}$$

for $x \in \mathbb{R}$ and $t > -k$. Then, (5.11) follows from (5.16), (5.18) and Lemma 5.2.

By the uniform boundedness of $\frac{d}{dz}\phi_c^-(z)$ and $\frac{d}{dz}\Gamma^-(z)$, it is easy to show that there exists a constant $M > 0$ such that for any $\eta > 0$,

$$\sup_{x\in\mathbb{R}} \left|\varphi^{n,-}(x+\eta) - \varphi^{n,-}(x)\right| \le M\eta.$$

Then, it follows from Lemma 5.3 that there exists a subsequence $\{U^{n_k}(x,t)\}$ of $\{U^k(x,t)\}$ and a function $U_p(x,t)$ such that $\lim_{n\to\infty} U^{n_k}(x,t) = U_p(x,t)$. For any given $t_0 \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $t_0 > -k$ and U^k satisfies

$$U^k(t)(x) = T(t-t_0)\big[U^k(t_0)\big](x) + \int_{t_0}^{t} T(t-r)\big[F\big(U_r^k\big)\big](x)dr,$$

where T and F are defined as in Section 3. Using a similar method as in the proof of Theorem 4.3, we can show that $U_p(x,t)$ is an entire solution of (1.7).

The assertions for parts (i) and (ii) follow from (5.10) and (5.11). Note that $c\lambda_1(c) > \lambda^*$ and $\frac{\partial}{\partial c}[c\lambda_1(c)] < 0$ for any $c > c_*$. The assertions for parts (iii)–(v) follow from Proposition 5.1 and (5.8)–(5.9). This completes the proof. $\square$

6. Applications

In previous sections, we study the entire solutions for a class of nonlocal dispersal equations with spatio-temporal delay. In this section, we apply the abstract results to models (1.4) and (1.5).

Example 1. Consider the Nicholson's blowflies model (1.4). Let $f(u,v) = -\delta u + pve^{-av}$ and $S(v) = v$. It is easy to see that (1.4) has two equilibria 0 and $K = \frac{1}{a}\ln\frac{p}{\delta}$ provided that $p > \delta$. We have the following results on entire solutions for (1.4).

Theorem 6.1. *Assume* (G_1).

(i) *If $\delta < p \le \delta e$, then the conclusions of Theorems 4.3–4.5 hold for (1.4).*
(ii) *If $p > \delta e$ and (G_2) holds, then the conclusions of Theorem 5.4 are valid for (1.4).*

If $\delta < p \le \delta e$, then $K = \frac{1}{a}\ln\frac{p}{\delta} \le \frac{1}{a}$, $\partial_1 f(u,v) = -\delta = \partial_1 f(0,0)$, $\partial_2 f(u,v) = p(1-av)e^{-av} \le p = \partial_2 f(0,0)$ and $\partial_2 f(u,v) \ge 0$ for $(u,v) \in [0,K]^2$. Hence (C_1), (C_2) and (C_4) hold. Let $g(v) = pve^{-av}$. It is clear that g is concave on $[0,K]$ when $p \le \delta e$. Note that $\bar{L}_1 := \max_{(u,v)\in[0,K]^2} |\partial_1 f(u,v)| = \delta$. Thus, for any $m \in \mathbb{Z}^+$, $a_i, b_i \in [0,K]$, $i = 1,\dots,m$,

$$\bar{L}_1 \min\left\{K, \sum_{i=1}^m a_i\right\} + f\left(\min\left\{K, \sum_{i=1}^m a_i\right\}, \min\left\{S(K), \sum_{i=1}^m b_i\right\}\right)$$

$$= g\left(\min\left\{K, \sum_{i=1}^{m} b_i\right\}\right) \le \sum_{i=1}^{m} g(b_i) = \sum_{i=1}^{m}\left[\bar{L}_1 a_i + f(a_i, b_i)\right],$$

that is, (C_3) holds.

When $p > \delta e$, (1.4) is non-quasi-monotone. Let $S^{\pm}(u) := u$, $K^+ := \frac{1}{\delta}\max_{v\in[0,K]} pve^{-av}$ and $K^- \in (0, K]$ with $K^- := \frac{1}{\delta} pK^+e^{-aK^+}$. The auxiliary functions $f^{\pm}(u, v)$ are defined as follows:

$$f^+(u, v) := -\delta u + \max_{w\in[0,v]} pwe^{-aw} \quad \text{and} \quad f^-(u, v) := -\delta u + \min_{w\in[v,K^+]} pwe^{-aw},$$

for $(u, v) \in [0, K^+]^2$. Direct computations show that $L_1 = \max_{(u,v)\in[0,K^+]^2} \partial_1 f^+(u, v) = -\delta < D$, $f^+(u, v) \le -\delta u + pv = \partial_1 f(0, 0)u + \partial_2 f(0, 0)v$ for $u, v \ge 0$, and

$$f^{\pm}(u, v_1) - f^{\pm}(u, v_2) \le p\max\{0, v_1 - v_2\}, \quad \forall (u, v_1), (u, v_2) \in \left[0, K^+\right]^2.$$

Hence, $(C_2)'$ holds. Since $h(v) = \max_{w\in[0,v]} pwe^{-aw}$ is concave on $[0, K^+]$, it is easy to show that $(C_3)'$ holds.

Example 2. Consider the population model (1.5). Let $f(u, v) = -du + v$ and $S(u) = b(u)$. We assume that

- (A_1) $b \in C^2[0, +\infty)$, $b(0) = b(K) - dK = 0$, $b'(0) > d$, $b(u) > du$ and $b'(u) \le b'(0)$ for $u \in (0, K)$, where $K > 0$ is a constant.
- (A_2) One of the following holds:
  - (a) $b'(u) \ge 0$ for $u \in [0, +\infty)$ and $b(u)$ is concave on $[0, K]$.
  - (b) There exists a number $u_{\max} > 0$ such that $b(u)$ is increasing for $0 < u \le u_{\max}$ and decreasing for $u \ge u_{\max}$, and $b(u)$ is concave on $[0, u_{\max}]$.

If (A_2)(a) or (A_2)(b) holds with $K \le u_{\max}$, then (1.5) is quasi-monotone on $[0, K]$, and it is easy to verify that (C_1), (C_2) and (C_4) hold. Using the concavity of b, we see that (C_3) holds.

If (A_2)(b) holds with $K > u_{\max}$, then (1.5) is non-quasi-monotone on $[0, K]$. Let $f^{\pm}(u, v) = f(u, v)$, and define $S^{\pm}(u)$ as follows:

$$S^+(u) = \begin{cases} S(u), & u \in [0, u_{\max}], \\ S(u_{\max}), & u > u_{\max}, \end{cases} \quad \text{and} \quad S^-(u) = \begin{cases} S(u), & u \in [0, u_{\min}], \\ S(u_{\min}), & u > u_{\min}, \end{cases}$$

where $u_{\min} \in (0, K)$ satisfies $S(u_{\min}) = S(S(u_{\max})/d)$. Let

$$K^+ = S(u_{\max})/d \quad \text{and} \quad K^- = S(u_{\min})/d.$$

Moreover, it is easy to verify that

$$S^{\pm}(u_1) - S^{\pm}(u_2) \le \max_{u\in[0,u_{\max}]} b'(u)\max\{0, u_1 - u_2\}, \quad \forall u_1, u_2 \in \left[0, K^+\right].$$

Thus, $(C_2)'$ holds. Since $S(u)$ is concave on $[0, u_{\max}]$, we see that $S^+(u)$ is concave on $[0, K^+]$, and hence $(C_3)'$ holds. Moreover, $L_1 = \max_{(u,v)\in[0,K^+]^2} \partial_1 f^+(u, v) = -d < D$. Therefore, the following results hold.

Theorem 6.2. *Assume* (G_1) *and* (A_1).

(i) *If* (A_2)(a) *or* (A_2)(b) *holds with* $K \le u_{\max}$*, then the conclusions of Theorems* 4.3–4.5 *are valid for* (1.5).

(ii) *If* (A_2)(b) *holds with* $K > u_{\max}$ *and* (G_2) *holds, then the conclusions of Theorem* 5.4 *hold for* (1.5).

References

[1] J. Al-Omari, S.A. Gourley, A nonlocal reaction–diffusion model for a single species with stage structure and distributed maturation delay, European J. Appl. Math. 16 (2005) 37–51.
[2] J. Carr, A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations, Proc. Amer. Math. Soc. 132 (2004) 2433–2439.
[3] X. Chen, Existence, uniqueness and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differential Equations 2 (1997) 125–160.
[4] X. Chen, J.-S. Guo, Existence and uniqueness of entire solutions for a reaction–diffusion equation, J. Differential Equations 212 (2005) 62–84.
[5] J. Coville, L. Dupaigne, On a non-local equation arising in population dynamics, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 727–755.
[6] J. Coville, Travelling fronts in asymmetric nonlocal reaction diffusion equations: the bistable and ignition cases, preprint.
[7] S.I. Ei, The motion of weakly interacting pulses in reaction–diffusion systems, J. Dynam. Differential Equations 14 (2002) 85–136.
[8] S.I. Ei, M. Mimura, M. Nagayama, Pulse–pulse interaction in reaction–diffusion systems, Phys. D 165 (2002) 176–198.
[9] J. Fang, J. Wei, X.-Q. Zhao, Spatial dynamics of a nonlocal and time-delayed reaction–diffusion system, J. Differential Equations 245 (2008) 2749–2770.
[10] T. Faria, W. Huang, J. Wu, Travelling waves for delayed reaction–diffusion equations with global response, Proc. R. Soc. Lond. Ser. A 462 (2006) 229–261.
[11] S.A. Gourley, S. Ruan, Dynamics of the diffusive Nicholson's blowflies equation with distributed delays, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 1275–1291.
[12] S.A. Gourley, J.W.-H. So, Extinction and wavefront propagation in a reaction–diffusion model of a structured population with distributed maturation delay, Proc. Roy. Soc. Edinburgh Sect. A 130 (2003) 527–548.
[13] S.A. Gourley, J. Wu, Delayed nonlocal diffusive systems in biological invasion and disease spread, in: Nonlinear Dynamics and Evolution Equations, in: Fields Inst. Commun., vol. 48, Amer. Math. Soc., Providence, RI, 2006, pp. 137–200.
[14] J.-S. Guo, Y. Morita, Entire solutions of reaction–diffusion equations and an application to discrete diffusive equations, Discrete Contin. Dyn. Syst. 12 (2005) 193–212.
[15] F. Hamel, N. Nadirashvili, Entire solutions of the KPP equation, Comm. Pure Appl. Math. 52 (1999) 1255–1276.
[16] F. Hamel, N. Nadirashvili, Travelling fronts and entire solutions of the Fisher–KPP equation in $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 157 (2001) 91–163.
[17] T. Kawahara, M. Tanaka, Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation, Phys. Lett. A 97 (1983) 311–314.
[18] C.T. Lee, et al., Non-local concepts in models in biology, J. Theoret. Biol. 210 (2001) 201–219.
[19] W.-T. Li, N.-W. Liu, Z.-C. Wang, Entire solutions in reaction–advection–diffusion equations in cylinders, J. Math. Pures Appl. 90 (2008) 492–504.
[20] W.-T. Li, S. Ruan, Z.-C. Wang, On the diffusive Nicholson's blowflies equation with nonlocal delay, J. Nonlinear Sci. 17 (2007) 505–525.
[21] W.-T. Li, Y.-J. Sun, Z.-C. Wang, Entire solutions in the Fisher–KPP equation with nonlocal dispersal, Nonlinear Anal. 11 (2010) 2302–2313.
[22] W.-T. Li, Z.-C. Wang, J. Wu, Entire solutions in monostable reaction–diffusion equations with delayed nonlinearity, J. Differential Equations 245 (2008) 102–129.
[23] N.-W. Liu, W.-T. Li, Z.-C. Wang, Entire solutions of reaction–advection–diffusion equations with bistable nonlinearity in cylinders, J. Differential Equations 246 (2009) 4249–4267.

[24] S. Ma, Traveling wavefronts for delayed reaction–diffusion systems via a fixed point theorem, J. Differential Equations 171 (2001) 294–314.
[25] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equation, J. Differential Equations 237 (2007) 259–277.
[26] R.H. Martin, H.L. Smith, Abstract functional differential equations and reaction–diffusion systems, Trans. Amer. Math. Soc. 321 (1990) 1–44.
[27] Y. Morita, H. Ninomiya, Entire solutions with merging fronts to reaction–diffusion equations, J. Dynam. Differential Equations 18 (2006) 841–861.
[28] Y. Morita, K. Tachibana, An entire solution to the Lotka–Volterra competition–diffusion equations, SIAM J. Math. Anal. 40 (2009) 2217–2240.
[29] J.D. Murray, Mathematical Biology, Springer, Berlin, 1989.
[30] S. Pan, Traveling wave fronts of delayed non-local diffusion systems without quasimonotonicity, J. Math. Anal. Appl. 346 (2008) 415–424.
[31] S. Pan, W.-T. Li, G. Lin, Travelling wave fronts in nonlocal delayed reaction–diffusion systems and applications, Z. Angew. Math. Phys. 60 (2009) 377–392.
[32] S. Ruan, Spatial-temporal dynamics in nonlocal epidemiological models, in: Y. Iwasa, K. Sato, Y. Takeuchi (Eds.), Mathematics for Life Science and Medicine, Springer-Verlag, New York, 2007, pp. 99–122.
[33] J.W.-H. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A 457 (2001) 1841–1853.
[34] Y.-J. Sun, W.-T. Li, Z.-C. Wang, Entire solutions in nonlocal dispersal equations with bistable nonlinearity, J. Differential Equations 251 (2011) 551–581.
[35] Z.-C. Wang, W.-T. Li, Dynamics of a nonlocal delayed reaction–diffusion equation without quasi-monotonicity, Proc. Roy. Soc. Edinburgh Sect. A 140 (2010) 1081–1109.
[36] Z.-C. Wang, W.-T. Li, S. Ruan, Travelling fronts in monostable equations with nonlocal delayed effects, J. Dynam. Differential Equations 20 (2008) 563–607.
[37] Z.-C. Wang, W.-T. Li, S. Ruan, Entire solutions in bistable reaction–diffusion equations with nonlocal delayed nonlinearity, Trans. Amer. Math. Soc. 361 (2009) 2047–2084.
[38] Z.-C. Wang, W.-T. Li, J. Wu, Entire solutions in delayed lattice differential equations with monostable nonlinearity, SIAM J. Math. Anal. 40 (2009) 2392–2420.
[39] Z.-C. Wang, W.-T. Li, S. Ruan, Entire solutions in lattice delayed differential equations with nonlocal interaction: bistable case, Math. Model. Nat. Phenom. 8 (2013) 78–103.
[40] M.-X. Wang, G.-Y. Lv, Entire solutions of a diffusive and competitive Lotka–Volterra type system with nonlocal delay, Nonlinearity 23 (2010) 1609–1630.
[41] J. Wu, X. Zou, Traveling wave fronts of reaction–diffusion systems with delay, J. Dynam. Differential Equations 13 (2001) 651–687.
[42] S.-L. Wu, C.-H. Hsu, Entire solutions of non-quasi-monotone delayed reaction–diffusion equations with applications, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014) 1085–1112.
[43] S.-L. Wu, Z.-X. Shi, F.-Y. Yang, Entire solutions in periodic lattice dynamical systems, J. Differential Equations 255 (2013) 3505–3535.
[44] S.-L. Wu, H.-Y. Wang, Front-like entire solutions for monostable reaction–diffusion systems, J. Dynam. Differential Equations 25 (2013) 505–533.
[45] Z.-X. Yu, R. Yuan, Existence, asymptotics and uniqueness of traveling waves for nonlocal diffusion systems with delayed nonlocal response, Taiwanese J. Math. 17 (2013) 2163–2190.