Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 265 (2018) 4016–4051
Journal of Differential Equations
www.elsevier.com/locate/jde

On an advection–reaction–diffusion competition system with double free boundaries modeling invasion and competition of *Aedes Albopictus* and *Aedes Aegypti* mosquitoes

Canrong Tian [a,1], Shigui Ruan [b,*,2]

[a] *School of Mathematics and Physics, Yancheng Institute of Technology, Yancheng, Jiangsu 224003, PR China*
[b] *Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA*

Received 11 July 2017; revised 31 March 2018
Available online 5 June 2018

Abstract

Based on the invasion of the *Aedes albopictus* mosquitoes and the competition between *Ae. albopictus* and *Ae. aegypti* mosquitoes in the United States, we consider an advection–reaction–diffusion competition system with two free boundaries consisting of an invasive species (*Ae. albopictus*) with density u and a local species (*Ae. aegypti*) with density v in which u invades the environment with leftward front $x = g(t)$ and rightward front $x = h(t)$. In the case that the competition between the two species is strong-weak and species v wins over species u, the solution (u, v) converges uniformly to the semi-positive equilibrium $(0, 1)$, while the two fronts satisfy that $\lim_{t\to\infty}(g(t), h(t)) = (g_\infty, h_\infty) \subset \mathbb{R}$. In the case that the competition between the two species is weak, we show that when the advection coefficients are less than fixed thresholds there are two scenarios for the long time behavior of solutions: (i) when the initial habitat $h_0 < \pi\left(\sqrt{4 - \nu_1^2}\right)^{-1}$ and the initial value of u is sufficiently small, the solution (u, v) converges uniformly to the semi-positive equilibrium $(0, 1)$ with the two fronts $(g_\infty, h_\infty) \subset \mathbb{R}$; (ii) when the initial habitat $h_0 \ge \pi\left(\sqrt{4 - \nu_1^2}\right)^{-1}$, the solution (u, v) converges locally uniformly to the interior equilibrium with the two fronts $(g_\infty, h_\infty) = \mathbb{R}$. In addition, we propose an upper bound and a lower bound for the asymptotic

* Corresponding author.
E-mail address: ruan@math.miami.edu (S. Ruan).

1 Research was partially supported by the Jiangsu Province 333 Talent Project.
2 Research was partially supported by NSF grant (DMS-1412454) and CDC Southeastern Regional Center of Excellency in Vector-Borne Diseases – The Gateway Program (1U01-CK000510-01).

https://doi.org/10.1016/j.jde.2018.05.027

spreading speeds of the leftward and rightward fronts. Numerical simulations are also provided to confirm our theoretical results.

MSC: 35B35; 35K60; 92B05

Keywords: Invasion; Competition; Advection–reaction–diffusion model; Free boundary; Traveling wave; Asymptotic spreading speed

1. Introduction

Aedes aegypti and *Aedes albopictus* are two prominent mosquito species which transmit the viruses that cause dengue fever, yellow fever, West Nile fever, chikungunya, and Zika, along with many other diseases. *Ae. aegypti* mosquito is a species with tropical and subtropical worldwide distribution and is an insect closely associated with humans and their dwellings. *Ae. albopictus*, a mosquito native to the tropical and subtropical areas of Southeast Asia and a most invasive species, has spread recently to many countries (including the U.S.) through the transport of goods and international travel. Inter-specific competition among mosquito larvae on larval, adult, and life-table traits exists between *Ae. aegypti* and *Ae. albopictus* and affects primarily larva-to-adult survivorship and the larval development time (Noden et al. [36]).

Before the arrival of *Ae. albopictus, Ae. aegypti* was a common mosquito in artificial containers throughout Florida (Morlan and Tinker [35], Frank [17]). *Ae. albopictus* was found for the first time in northern counties in Florida in 1986 (Peacock et al. [39]). Over the next six years, *Ae. albopictus* spread slowly but steadily southward, and by the summer of 1994 it had spread to all 67 counties of the state (O'Meara et al. [37]). Meanwhile, major declines in *Ae. aegypti* abundance were associated with the invasion and expansion of *Ae. albopictus* populations, not only in Florida, but elsewhere in the southern part of the continental United States (Hobbs et al. [26], O'Meara et al. [38]). By 2008, *Ae. albopictus* had spread to 36 states and continued to expand its range (Enserink [16]). In 2013, Rochlin et al. [42] predicted that North American land favoring the environmental conditions of the *Ae. albopictus* mosquito is expected to more than triple in size in the next 20 years, especially in urban areas. By the estimates of CDC [6] in 2016, *Ae. albopictus* not only has spread to all states where *Ae. aegypti* presents but also has reached habitats beyond *Ae. aegypti*'s boundaries (see Fig. 1).

Taking into account the effect of wind on the movement of mosquitoes, Takahashi et al. [44] proposed an advection–reaction–diffusion equation model to investigate the dispersal dynamics of *A. aegypti* and predicted the existence of stable traveling waves in several situations. Although Takahashi et al. gave an estimation of the speed of traveling waves of *A. aegypti*, it is the asymptotic wave speed that usually gives an approximation of the progressive spreading speed of *A. aegypti*, and it does not really show the spread of *A. aegypti* in the early stage of spatial expanding to larger areas. To describe the spatial spreading of *A. aegypti*, we (Tian and Ruan [46]) generalized the model of Takahashi et al. [44] to an advection–reaction–diffusion equation model with free boundary, where the population of the vector mosquitoes is described by a system for the two life stages: the winged form (mature female mosquitoes) and an aquatic population (eggs, larvae and pupae), the expanding front is expressed by a free boundary which models the spatial expanding of the source area. The female mosquitoes are initially located at a habitat, then spread

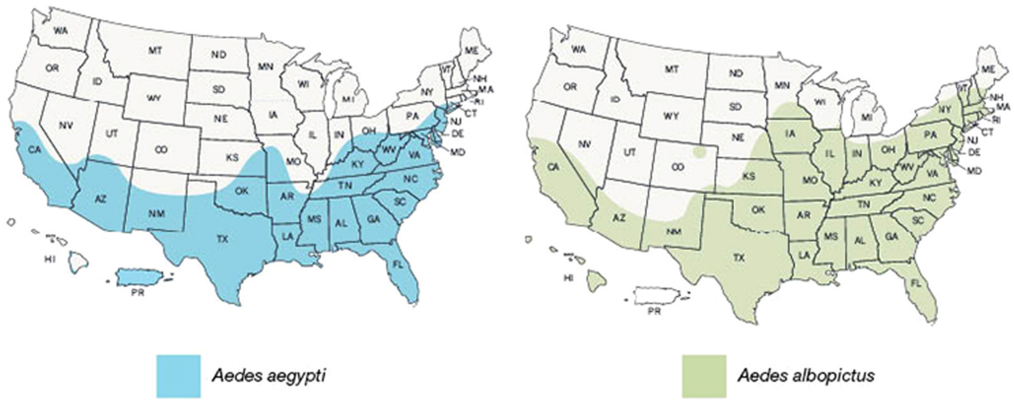

Fig. 1. Estimated range of *Ae. aegypti* and *Ae. albopictus* in the U.S., 2016 (CDC [6]).

to other places owing to their dispersal ability, which includes the long distance dispersal as well as the short distance dispersal. The long distance dispersal of *A. aegypti* is caused by wind and described by the advection, while the short dispersal is due to the random walk of each individual mosquito and described by the classical Laplacian diffusion. The free boundary theory was used to show the existence and uniqueness of the global solution to an advection–reaction–diffusion equation model and the boundedness of moving speed of the free boundary was also estimated.

To model the invasion of *Ae. albopictus* species and the competition between *A. aegypti* and *Ae. albopictus* and take into account the effect of wind, we consider the following advection–reaction–diffusion system with two free boundaries:

$$\begin{cases} U_t = D_1 U_{xx} - \tilde{\nu}_1 U_x + r_1 U(1 - \dfrac{U}{K_1}) - \tilde{a}_1 UV, & t > 0,\ \tilde{g}(t) < x < \tilde{h}(t), \\ U(t,x) \equiv 0, & t > 0,\ x \notin (\tilde{g}(t), \tilde{h}(t)), \\ V_t = D_2 V_{xx} - \tilde{\nu}_2 V_x + r_2 V(1 - \dfrac{V}{K_2}) - \tilde{a}_2 UV, & t > 0,\ x \in \mathbb{R}, \\ U = 0,\ \tilde{g}'(t) = -\tilde{\mu} U_x(t, \tilde{g}(t)), & t \geq 0,\ x = \tilde{g}(t), \\ U = 0,\ \tilde{h}'(t) = -\tilde{\mu} U_x(t, \tilde{h}(t)), & t \geq 0,\ x = \tilde{h}(t), \\ \tilde{g}(0) = -\tilde{h}_0,\ \tilde{h}(0) = \tilde{h}_0, & \\ U(0,x) = U_0(x), & x \in [-\tilde{h}_0, \tilde{h}_0]; \\ V(0,x) = V_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The biological meanings of (1.1) are described as follows: $V(t,x)$ represents the density of the local species (*A. aegypti*) and $U(t,x)$ represents the density of the invasive species (*Ae. albopictus*) at time t and space location x, respectively. These two species have a competition relation. The invasive species, initially limited to a specific part of the domain $[-\tilde{h}_0, \tilde{h}_0]$, spreads over the space with the leftward front $x = \tilde{g}(t)$ and rightward front $x = \tilde{h}(t)$ (i.e. free boundaries). The boundaries are assumed to evolve according to the Stefan boundary conditions $\tilde{g}'(t) = -\tilde{\mu} U_x(t, \tilde{g}(t))$ and $\tilde{h}'(t) = -\tilde{\mu} U_x(t, \tilde{h}(t))$ (Hilhorst et al. [24,25]), which are a kind of free boundary conditions. D_1 and D_2 are the diffusion rates of the two species respectively; $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are the advection speeds of the two species respectively; r_1 and r_2 are the intrinsic growth rates of the two species respectively; K_1 and K_2 are the carrying capacities of the two species respectively; $\tilde{a}_1$ and $\tilde{a}_2$ are the interspecific competition rates. Free boundary problems have been extensively studied in the literature including realistic biological problems with free boundaries, see for example, Bounting and Du [3], Cao et al. [5], Chen and Friedman [7], Du and Lin [8], Du and Lou [10], Du et al. [13,14,11], Lin and Zhu [31], Liu and Lou [32], and the references cited therein.

In order to minimize the number of parameters involved in the model, we introduce the dimensionless variables. Set

$$u = \frac{1}{K_1}U,\ v = \frac{1}{K_2}V,\ \bar{t} = r_1 t,\ \bar{x} = \sqrt{\frac{1}{K_1 D_1}}x. \tag{1.2}$$

Then the free boundaries become $\sqrt{\frac{r_1}{D_1}}\tilde{g}(\frac{\bar{t}}{r_1})$ and $\sqrt{\frac{r_1}{D_1}}\tilde{h}(\frac{\bar{t}}{r_1})$. Denote by $g(\bar{t}) \equiv \sqrt{\frac{r_1}{D_1}}\tilde{g}(\frac{\bar{t}}{r_1})$ and $h(\bar{t}) \equiv \sqrt{\frac{r_1}{D_1}}\tilde{h}(\frac{\bar{t}}{r_1})$, respectively. For the sake of simplicity, we omit the caps of t and x. Rewrite the problem (1.1) as follows:

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1-u-a_1 v), & t>0,\ g(t)<x<h(t), \\ u(t,x) \equiv 0, & t>0,\ x \notin (g(t),h(t)), \\ v_t = D v_{xx} - \nu_2 v_x + v(r - a_2 u - r v), & t>0,\ x\in\mathbb{R}, \\ u=0,\ g'(t) = -\mu u_x(t,g(t)), & t\ge 0,\ x=g(t), \\ u=0,\ h'(t) = -\mu u_x(t,h(t)), & t\ge 0,\ x=h(t), \\ g(0)=-h_0,\ h(0)=h_0, & \\ u(0,x)=u_0(x), & x\in[-h_0,h_0]; \\ v(0,x)=v_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.3}$$

where $\nu_1 = \frac{\tilde{\nu}_1}{r_1}\sqrt{\frac{1}{K_1 D_1}}$, $a_1 = \frac{K_2\tilde{a}_1}{r_1}$, $D = \frac{D_2}{r_1 K_1 D_1}$, $a_2 = \frac{K_1\tilde{a}_2}{r_1}$, $\mu = \frac{K_1\tilde{\mu}}{r_1}\sqrt{\frac{1}{K_1 D_1}}$, $h_0 = \sqrt{\frac{r_1}{D_1}}\tilde{h}_0$, $u_0(x) = \frac{1}{K_1}U_0(\sqrt{K_1 D_1}x)$, and $v_0(x) = \frac{1}{K_2}V_0(\sqrt{K_1 D_1}x)$. Moreover, we assume that u_0 and v_0 satisfy

$$\begin{aligned} &u_0 \in C^2([-h_0,h_0]),\ u_0(\pm h_0)=0,\ u_0(x)>0 \text{ in } (-h_0,h_0), \\ &v_0 \in C(\mathbb{R})\cap L^\infty(\mathbb{R}),\ v_0(x)>0 \text{ in } \mathbb{R}. \end{aligned} \tag{1.4}$$

Two special cases of problem (1.3) have been studied in the literature.

(a) In absence of the local species, named $v\equiv 0$, problem (1.3) reduces to the following single-species advection–reaction–diffusion equation with free boundary:

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1-u), & t>0,\ g(t)<x<h(t), \\ u=0,\ g'(t) = -\mu u_x(t,g(t)), & t\ge 0,\ x=g(t), \\ u=0,\ h'(t) = -\mu u_x(t,h(t)), & t\ge 0,\ x=h(t), \\ g(0)=-h_0,\ h(0)=h_0, & \\ u(0,x)=u_0(x), & x\in[-h_0,h_0]. \end{cases} \tag{1.5}$$

Problem (1.5) was first studied by Gu et al. [19] who showed that if $0 < \nu_1 < 2$, then the behavior is characterized by a spreading-vanishing dichotomy, namely, one of the following alternatives occurs:

- Spreading: $\lim_{t\to\infty}(g(t), h(t)) = \mathbb{R}$, $\lim_{t\to\infty} u(t, x) = 1$ locally uniformly in $\mathbb{R}$.
- Vanishing: $\lim_{t\to\infty}(g(t), h(t)) = (g_\infty, h_\infty)$, $\lim_{t\to\infty} \|u(t, \cdot)\|_{C([g(t),h(t)])} = 0$, here g_∞ and h_∞ are finite constants.

In [20], Gu et al. also showed that the leftward and rightward asymptotic spreading speeds c_l^* and c_r^* exist and satisfy $c_l^* < c^* < c_r^*$, here $c_l^* = \lim_{t\to\infty} \frac{-g(t)}{t}$, $c_r^* = \lim_{t\to\infty} \frac{h(t)}{t}$, c^* is the minimal speed of traveling wave speed connecting 0 and 1. In [21], they further proved that when ν_1 is large the solution of problem (1.5) has an asymptotic behavior more complicated than spreading and vanishing.

(b) When $\nu_1 = \nu_2 = 0$, that is, there is no advection in the environment, many researchers studied the qualitative properties of problem (1.3) under one or two free boundaries, see for example, Du and Lin [9], Guo and Wu [22], Lin [30], Wang and Zhao [49], Wu [51], and Zhao and Wang [53]. These researchers extended the asymptotic stability and traveling wave solutions to the corresponding Cauchy problem of system (1.3), which can be listed in details as follows:

(i) If $a_1 > 1$ and $a_2 < r$, then the semi-positive equilibrium $(0, 1)$ is globally asymptotically stable, and there exists a traveling wave solution connecting the two semi-positive equilibria $(1, 0)$ and $(0, 1)$ (cf. Kan-On [27]);

(ii) If $a_1 < 1$ and $a_2 < r$, then the interior equilibrium $\left(\frac{r(1-a_1)}{r-a_1a_2}, \frac{r-a_2}{r-a_1a_2}\right)$ is globally asymptotically stable, and there exists a traveling wave solution connecting the trivial equilibrium $(0, 0)$ and the interior equilibrium $\left(\frac{r(1-a_1)}{r-a_1a_2}, \frac{r-a_2}{r-a_1a_2}\right)$ (cf. Tang and Fife [45]);

(iii) If $a_1 < 1$ and $a_2 > r$, then the semi-positive equilibrium $(1, 0)$ is globally asymptotically stable, and there exists a traveling wave solution connecting the two semi-positive equilibria $(0, 1)$ and $(1, 0)$ (cf. Kan-On [27]).

The competition in cases (i) and (iii) is called *weak-strong* since one species wins the competition in the long run, whereas the competition in case (ii) is called *weak* since the two species coexist with no one winning or losing the competition. Recently, Du et al. [15] found the exact spreading speed in the case of spreading.

The current studies show that when the size of the initial habitat is big, the solutions of the free boundary problem (without advection) have the same asymptotic convergence with the corresponding Cauchy problem, and the asymptotic spreading speed of the front is no more than the minimal speed of the traveling wave solution. However, when the initial habitat is small, the free boundary problem (without advection) cannot spread over the whole space, which is different from the Cauchy problem where the traveling wave solution exists.

Our main purpose in this paper is to study the influence of the advection terms on the asymptotic behavior of the competition system (1.3). We will extend the results of asymptotic convergence of Cauchy problem with cases (i) and (ii) to the free boundary problem (with advection). Moreover, we will estimate an upper and lower bounded asymptotic spreading speeds of the front to improve the previous results of competition system (1.3). We would like to mention that the advection coefficients in problem (1.3) cause substantial technical difficulties, studying the asymptotic stability and traveling wave solutions for reaction–diffusion systems to advection–

reaction–diffusion systems is far from trivial, even for regular boundary value problems (Zhao and Ruan [52]).

The rest of the article is organized as follows. In section 2 we prove the global existence and uniqueness of solutions to the free boundary problem (1.3). Moreover, we give a priori estimate of the derivative of boundary. In section 3, we investigate the weak-strong competition case in which u will vanish. In section 4, we investigate the weak competition case that u and v will coexist and the boundary will spread. Section 5 deals with the asymptotic spreading speed when the boundary spreads. In section 6 we carry out numerical simulations to confirm our analytical findings. In section 7 we use our conclusions to interpret the invasion of *Ae. albopictus* mosquitoes and the competition between *A. aegypti* and *Ae. albopictus* as an example of biological applications.

2. Existence and uniqueness

In this section, we first present the following local existence and uniqueness result by using the contraction mapping theorem.

Theorem 2.1. *For any given (u_0, v_0) satisfying (1.4) and any $\alpha \in (0,1)$, there is a $T > 0$ such that problem (1.3) admits a unique bounded solution*

$$(u, v, g, h) \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_T) \times \left(C_{loc}^{1+\frac{\alpha}{2}, 2+\alpha}(D_T^\infty) \cap C(\overline{D}_T^\infty)\right) \times \left[C^{1+\frac{\alpha}{2}}([0,T])\right]^2.$$

Moreover,

$$\|u\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_T)} + \|v\|_{L^\infty(\overline{D}_T^\infty)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C, \tag{2.1}$$

where

$$D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, g(t) < x < h(t)\},$$
$$D_T^\infty = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, x \in \mathbb{R}\},$$

here positive constants C and T depend only on h_0, α, $\|u_0\|_{C^2([-h_0,h_0])}$ and $\|v_0\|_{L^\infty(\mathbb{R})}$.

Proof. We introduce an auxiliary function $\zeta(y)$ satisfying $|\zeta'(y)| < \frac{6}{h_0}$ for $y \in [0,\infty)$, and

$$\zeta(y) = \begin{cases} 1, & \text{if } |y - h_0| < \frac{h_0}{4}, \\ 0, & \text{if } |y - h_0| > \frac{h_0}{2}, \end{cases}$$

it is obvious that such a $C^3\big([0,\infty)\big)$ function $\zeta(y)$ exists. As in Wang and Zhao [50], we perform the following coordinate transformation

$$x = y + \zeta(y)(h(t) - h_0) + \xi(y)(g(t) + h_0), y \in \mathbb{R}, \tag{2.2}$$

here $\xi(y) = -\zeta(-y)$. When t is confined to

$$|h(t) - h_0| + |g(t) + h_0| \le \frac{h_0}{16},$$

the transformation (2.2) implies that $x \to y$ is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, meanwhile, the free boundaries $x = g(t)$ and $x = h(t)$ are transformed to the lines $y = -h_0$ and $y = h_0$, respectively. The rest of the proof is similar to that of Theorem 2.1 of Du and Lin [9]. See also Belgacem and Cosner [1], Friedman [18] and Ladyzenskaja et al. [28]. We omit the details. □

We extend the local solution obtained in Theorem 2.1 to the maximal time. To do so, we need the following priori estimate.

Lemma 2.2. *Let (u, v, g, h) be a solution to problem (1.3) defined for $t \in [0, T)$ for some $T \in (0, \infty)$. Then there exist constants M_1, M_2, and M_2 independent of T such that*

$$0 < u(t,x) \le M_1, \ \textit{for}\ 0 \le t \le T,\ g(t) < x < h(t), \tag{2.3}$$

$$0 < v(t,x) \le M_2, \ \textit{for}\ 0 \le t \le T,\ x \in \mathbb{R}, \tag{2.4}$$

$$-M_3 \le g'(t) < 0,\ 0 < h'(t) \le M_3,\ \textit{for}\ 0 < t \le T. \tag{2.5}$$

Proof. As $g(t), h(t)$ are fixed, we use the strong maximum principle to get

$$u > 0 \text{ for } (t,x) \in [0,T] \times (g(t), h(t)), \text{ and } v > 0 \text{ for } (t,x) \in [0,T] \times \mathbb{R}. \tag{2.6}$$

Consequently, since v satisfies

$$\begin{cases} v_t \le D v_{xx} - \nu_2 v_x + r v(1-v), & 0 < t \le T,\ x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases}$$

we as well obtain $v \le \max\{\|v_0\|_{L^\infty(\mathbb{R})}, 1\} := M_2$. Similarly, u satisfies

$$\begin{cases} u_t \le u_{xx} - \nu_1 u_x + u(1-u), & 0 < t \le T,\ g(t) < x < h(t), \\ u(0,x) = u_0(x), & x \in [-h_0, h_0]. \end{cases}$$

It is obvious that $u \le \max\{\|u_0\|_{C([-h_0,h_0])}, 1\} := M_1$.

It remains to prove (2.5). We only prove the boundedness of $h'(t)$, since $g'(t)$ can be treated in a similar way. Applying the Hopf Lemma (cf. Du and Lin [8]) to problem (1.3), we have $u_x(t, h(t)) < 0$ for $t \in (0, T]$. We thereby obtain $h'(t) > 0$ for $t \in (0, T]$. In order to prove the upper bound of $h'(t)$, we compare u with an auxiliary function. Define

$$\Omega =: \{(t,x):\ 0 < t \le T,\ h(t) - \frac{1}{M} < x < h(t)\}$$

and construct an auxiliary function

$$w(t,x) := M_1[2M(h(t)-x) - M^2(h(t)-x)^2].$$

We choose M so that $w(t,x) \ge u(t,x)$ in Ω.

Owing to $h'(t) > 0$, for $(t,x) \in \Omega$ we calculate

$$w_t = 2M_1 M h'(t)(1 - M(h(t)-x)) > 0.$$

Consequently,

$$w_t - w_{xx} + \nu_1 w_x > 2M_1 M^2 - 2\nu_1 M_1 M \text{ for } (t,x) \in \Omega. \tag{2.7}$$

As $u, v > 0$, we have

$$\begin{aligned} u_t - u_{xx} + \nu_1 u_x &= u(1 - u - a_1 v) \\ &< u(1-u) < M_1 \text{ for } (t,x) \in \Omega. \end{aligned} \tag{2.8}$$

In light of (2.7) and (2.8), setting $z = w - u$, we obtain

$$z_t - z_{xx} + \nu_1 z_x > M_1(2M^2 - 2\nu_1 M - 1) \ge 0, \text{ for } (t,x) \in \Omega, \tag{2.9}$$

provided

$$M \ge \frac{\nu_1 + \sqrt{\nu_1^2 + 2}}{2}. \tag{2.10}$$

On the other hand, we have

$$\begin{aligned} &z(t, h(t) - \frac{1}{M}) = M_1 - u(t, h(t) - \frac{1}{M}) \ge 0 \text{ for } t \in (0,T), \\ &z(t, h(t)) = 0 \text{ for } t \in (0,T). \end{aligned} \tag{2.11}$$

Hence, we expect to find M such that

$$u_0(x) \le w(0,x) \text{ for } x \in [h_0 - \frac{1}{M}, h_0]. \tag{2.12}$$

In view of (2.9), (2.11), and (2.12), we can apply the parabolic maximum principle to z over Ω to deduce that $u(t,x) \le w(t,x)$ for $(t,x) \in \Omega$. Consequently, $u_x(t,h(t)) \ge w_x(t,h(t)) = -2MM_1$, which leads to

$$h'(t) = -\mu v_x(t, h(t)) \le 2MM_1\mu := M_3. \tag{2.13}$$

It remains to seek for some M independent of T such that (2.12) holds. We compute

$$w_x(0,x) = -2M_1 M(1 - M(h_0 - x)) \le -M_1 M \text{ for } x \in [h_0 - \frac{1}{2M}, h_0].$$

In view of (2.10), choosing

$$M := \max\left\{ \frac{\nu_1 + \sqrt{\nu_1^2 + 2}}{2}, \frac{4\|u_0\|_{C^1([-h_0,h_0])}}{3M_1} \right\}, \tag{2.14}$$

we have

$$w_x(0,x) \le -MM_1 \le -\frac{4}{3}\|u_0\|_{C^1([-h_0,h_0])} \le u_0'(x) \text{ for } x \in [h_0-(2M)^{-1}, h_0].$$

As $w(0,h_0) = u_0(h_0) = 0$, the above inequality implies that

$$w(0,x) \ge u_0(x) \text{ for } x \in [h_0 - \frac{1}{2M}, h_0].$$

Moreover, in term of $M \ge \frac{4}{3M_1}\|u_0\|_{C^1([-h_0,h_0])}$ for $x \in [h_0 - \frac{1}{M}, h_0 - \frac{1}{2M}]$, we obtain

$$w(0,x) \ge \frac{3}{4}M_1,\ u_0(x) \le \frac{1}{M}\|u_0\|_{C^1([-h_0,h_0])} \le \frac{3}{4}M_1, \text{ for } x \in [h_0 - \frac{1}{M}, h_0 - \frac{1}{2M}].$$

Therefore, $u_0(x) \le w(0,x)$ for $x \in [h_0 - \frac{1}{M}, h_0]$, which means that (2.13) is valid. The proof is complete. □

Theorem 2.3. *For any given (u_0, v_0) satisfying (1.4), the solution of problem (1.3) exists and is unique for all $t \in [0,\infty)$.*

Proof. It follows from the uniqueness of solutions (Theorem 2.1) that there is a fixed T such that the solution (u,v,g,h) is confined to $[0,T)$. We now fix $\delta \in (0,T)$. By L^p estimates, the Sobolev's embedding theorem, and the Hölder estimates for parabolic equations, we can find $M_4 > 0$ depending on δ, T, $M_i (i = 1,2,3)$ such that

$$\|u(t,\cdot)\|_{C^2([g(t),h(t)])}, \|v(t,\cdot)\|_{C^2(\mathbb{R})} \le M_4 \text{ for } t \in [\delta, T].$$

Again it follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending on $M_i (i = 1,2,3)$ but not on t such that the solution of problem (1.3) with initial time $T - \frac{\tau}{2}$ can be extended uniquely to the time $T + \frac{\tau}{2}$. This means that the solution of (1.3) can be extended as long as u and v remain bounded. Therefore, the solution of problem (1.3) can be extended to the infinite time interval $[0,\infty)$. □

3. Weak-strong competition

In view of Lemma 2.2, we can observe that $x = g(t)$ is monotone decreasing and $x = h(t)$ is monotone increasing. Therefore, there exist $g_\infty \in [-\infty, 0)$ and $h_\infty \in (0,\infty]$ such that $\lim_{t\to\infty} g(t) = g_\infty$ and $\lim_{t\to\infty} h(t) = h_\infty$. In this section, we try to extend the long time behavior to solutions of Cauchy problem corresponding to (1.3) in the case of weak-strong competition and v winning.

Theorem 3.1. *Suppose that*

$$a_1 > 1 \text{ and } a_2 < r. \tag{3.1}$$

If δ is an arbitrary small positive constant such that $v_0(x) \ge \delta$ for $x \in \mathbb{R}$, then

$$\lim_{t\to\infty}(u(t,x), v(t,x)) = (0,1) \text{ uniformly in any compact subset of } \mathbb{R}.$$

Proof. Consider the following ordinary differential equation:

$$\begin{cases} U' = U(1-U), & t > 0, \\ U(0) = \|u_0\|_{L^\infty([-h_0,h_0])}. \end{cases}$$

We solve the equation and find

$$U = e^t\big(e^t - 1 + \frac{1}{U(0)}\big)^{-1}.$$

By the comparison principle, we see that $u(t,x) \le U(t)$ for $t > 0$ and $x \in [g(t), h(t)]$. Since $\lim_{t\to\infty} U(t) = 1$, we have

$$\limsup_{t\to\infty} u(t,x) \le 1 \text{ uniformly in } \mathbb{R}. \tag{3.2}$$

Similarly, we can deduce that

$$\limsup_{t\to\infty} v(t,x) \le 1 \text{ uniformly in } \mathbb{R}. \tag{3.3}$$

Therefore, for $\varepsilon_1 = \frac{1}{2}(\frac{r}{a_2} - 1)$, there exists $t_1 > 0$ such that $u(t,x) \le 1 + \varepsilon_1$ for $t \ge t_1$, $x \in \mathbb{R}$. Thus, v satisfies

$$\begin{cases} v_t - Dv_{xx} + \nu_2 v_x \ge v(\varepsilon_1 - rv), & t > t_1,\ x \in \mathbb{R}, \\ v(t_1, x) > 0, & x \in \mathbb{R}. \end{cases}$$

Let $V(t)$ be the unique solution to

$$\begin{cases} V' = V(\varepsilon_1 - rV), & t > t_1, \\ V(t_1) = \inf_{x\in\mathbb{R}} v(t_1, x). \end{cases}$$

Since $v_0(x) \ge \delta$, we have $V(t_1) > 0$. By using the comparison principle, we have $v(t,x) \ge V(t)$ for $t \ge t_1$ and $x \in \mathbb{R}$. Since $\lim_{t\to\infty} V(t) = \frac{\varepsilon_1}{r}$, for any $L > 0$, there exists $t_L > t_1$ such that

$$v(t,x) \ge V(t) \ge \frac{\varepsilon_1}{2r} \text{ for } t \ge t_L,\ -L \le x \le L.$$

We obtain that (u, v) satisfies

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1 - u - a_1 v), & t > t_L,\ g(t) < x < h(t), \\ v_t = Dv_{xx} - \nu_2 v_x + v(r - a_2 u - rv), & t > t_L,\ x \in \mathbb{R}, \\ u(t,x) \le 1 + \varepsilon_1,\ v(t,x) \ge \frac{\varepsilon_1}{2r}, & t \ge t_L,\ x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Consider $(\overline{u}, \underline{v})$ as a unique solution to the following equations

$$\begin{cases} \overline{u}_t = \overline{u}_{xx} - \nu_1 \overline{u}_x + \overline{u}(1 - \overline{u} - a_1 \underline{v}), & t > t_L,\ -L < x < L, \\ \underline{v}_t = D\underline{v}_{xx} - \nu_2 \underline{v}_x + \underline{v}(r - a_2 \overline{u} - r\underline{v}), & t > t_L,\ -L < x < L, \\ \overline{u}(t, \pm L) = 1 + \varepsilon_1,\ \underline{v}(t, \pm L) = \frac{\varepsilon_1}{2r}, & t > t_L, \\ \overline{u}(t_L, x) = 1 + \varepsilon_1,\ \underline{v}(t_L, x) = \frac{\varepsilon_1}{2r}, & -L < x < L. \end{cases} \tag{3.5}$$

Since $u \equiv 0$ for $t > t_L$ and $x \notin (g(t), h(t))$, no matter whether or not $(g(t), h(t)) \subset (-L, L)$, the comparison principle implies that $u \le \overline{u}$ and $v \ge \underline{v}$ in $[t_L, \infty) \times [-L, L]$. Thanks to the fact that system (3.5) is quasimonotone nonincreasing, the theory of monotone dynamical systems (see, e.g. Smith [43, Corollary 7.3.6]) yields that

$$\lim_{t\to\infty} \overline{u}(t,x) = \overline{u}_L(x), \quad \lim_{t\to\infty} \underline{v}(t,x) = \underline{v}_L(x) \text{ uniformly in } [-L, L], \tag{3.6}$$

where $(\overline{u}_L(x), \underline{v}_L(x))$ is the solution to

$$\begin{cases} \overline{u}_{Lxx} - \nu_1 \overline{u}_{Lx} + \overline{u}_L(1 - \overline{u}_L - a_1 \underline{v}_L) = 0, & -L < x < L, \\ D\underline{v}_{Lxx} - \nu_2 \underline{v}_{Lx} + \underline{v}_L(r - a_2 \overline{u}_L - r\underline{v}_L) = 0, & -L < x < L, \\ \overline{u}_L(\pm L) = 1 + \varepsilon_1, \ \underline{v}_L(\pm L) = \frac{\varepsilon_1}{2r}. \end{cases} \tag{3.7}$$

Note that if $0 < L_1 < L_2$, by employing the comparison principle to system (3.5) with the corresponding boundary conditions, we discover that $\overline{u}_{L_1}(x) \ge \overline{u}_{L_2}(x)$ and $\underline{v}_{L_1}(x) \ge \underline{v}_{L_2}(x)$ in $[-L_1, L_1]$. Letting $L \to \infty$, by classical elliptic regularity theory and a diagonal procedure, we obtain that $(\overline{u}_L(x), \underline{v}_L(x))$ converges uniformly on any compact subset of $\mathbb{R}$ to $(\overline{u}_\infty(x), \underline{v}_\infty(x))$, which satisfies

$$\begin{cases} \overline{u}_{\infty xx} - \nu_1 \overline{u}_{\infty x} + \overline{u}_\infty(1 - \overline{u}_\infty - a_1 \underline{v}_\infty) = 0, & x \in \mathbb{R}, \\ D\underline{v}_{\infty xx} - \nu_2 \underline{v}_{\infty x} + \underline{v}_\infty(r - a_2 \overline{u}_\infty - r\underline{v}_\infty) = 0, & x \in \mathbb{R}, \\ \overline{u}_\infty(x) \le 1 + \varepsilon_1, \ \underline{v}_\infty(x) \ge \frac{\varepsilon_1}{2r}, & x \in \mathbb{R}. \end{cases} \tag{3.8}$$

Recalling (3.1), we shall illustrate that $(\overline{u}_\infty, \underline{v}_\infty) = (0, 1)$. Let us consider the following ordinary differential equations:

$$\begin{cases} z' = z(1 - z - a_1 w), & t > 0, \\ w' = w(r - a_2 z - rw), & t > 0, \\ z(0) = 1 + \varepsilon_1, \ w(0) = \frac{\varepsilon_1}{2r}. \end{cases}$$

By (3.1), it follows that $(z, w) \to (0, 1)$ as $t \to \infty$ (see, e.g. Morita and Tachibana [34]). If (Z, W) is the solution to

$$\begin{cases} Z_t = Z_{xx} - \nu_1 Z_x + Z(1 - Z - a_1 W), & t > 0, \ x \in \mathbb{R}, \\ W_t = DW_{xx} - \nu_2 W_x + W(r - a_2 Z - rW), & t > 0, \ x \in \mathbb{R}, \\ Z(0,x) = 1 + \varepsilon_1, \ W(0,x) = \frac{\varepsilon_1}{2r}, & x \in \mathbb{R}, \end{cases} \tag{3.9}$$

then $(Z, W) \to (0, 1)$ as $t \to \infty$ uniformly in $\mathbb{R}$. By utilizing the comparison principle to (3.8) and (3.9), we have $\overline{u}_\infty(x) \le Z(t,x)$ and $\underline{v}_\infty(x) \ge W(t,x)$ for $t > 0$, which immediately gives that $\overline{u}_\infty(x) = 0$ and $\underline{v}_\infty(x) = 1$.

Thanks to (3.6), $u(t,x) \le \overline{u}(t,x) \to \overline{u}_L(x)$ and $v(t,x) \ge \underline{v}(t,x) \to \underline{v}_L(x)$. Letting $L \to \infty$, we obtain that $\limsup_{t\to\infty} u(t,x) \le 0$ and $\liminf_{t\to\infty} v(t,x) \ge 1$ uniformly in any compact subset of $\mathbb{R}$. Taking into account (3.3) and $\liminf_{t\to\infty} u(t,x) \ge 0$, we can deduce that $\lim_{t\to\infty}(u(t,x), v(t,x)) = (0, 1)$ uniformly in any compact subset of $\mathbb{R}$. □

Remark 3.2. Consider the Cauchy problem corresponding to (1.3):

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1 - u - a_1 v), & t > 0,\ x \in \mathbb{R}, \\ v_t = Dv_{xx} - \nu_2 v_x + v(r - a_2 u - rv), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x),\ v(0, x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.10}$$

where $u_0(x)$ and $v_0(x)$ are nonnegative and bounded. Note that the proof of Theorem 3.1 is valid. In addition, the solution (u, v) of (3.10) converges to $(0, 1)$ locally uniformly in $\mathbb{R}$ as $t \to \infty$.

In the above theorem, we only give the long time behavior of $(u(t, x), v(t, x))$. The next theorem addresses the long time behavior of the double boundaries $x = g(t)$ and $x = h(t)$.

Theorem 3.3. *Suppose that (3.1) holds. If δ is an arbitrary small positive constant such that $v_0(x) \geq \delta$ for $x \in \mathbb{R}$, then $-g_\infty$, $h_\infty < \infty$ and*

$$\lim_{t\to\infty} u(t, x) = 0 \text{ and } \lim_{t\to\infty} v(t, x) = 1 \text{ uniformly in } \mathbb{R}.$$

Proof. In view of (3.2), for any given $\varepsilon_1 = \frac{1}{2}(\frac{r}{a_2} - 1)$, there exists $t_1 > 0$ such that $u(t, x) \leq 1 + \varepsilon_1$ for $t \geq t_1$, $x \in \mathbb{R}$. In addition, it follows from Lemma 2.2 that $0 < u(t, x) < M_1 := \max\{1, \|u_0\|_{C([-h_0,h_0])}\}$ and $0 < v(t, x) < M_2 := \max\{1, \|v_0\|_{L^\infty(\mathbb{R})}\}$ for $t \geq 0$ and $x \in \mathbb{R}$. Hence, v satisfies

$$\begin{cases} v_t - Dv_{xx} + \nu_2 v_x \geq v(r - a_2 M_1 - rM_2), & t > 0,\ x \in \mathbb{R}, \\ v(0, x) \geq \delta, & x \in \mathbb{R}. \end{cases}$$

Therefore, we obtain that $v(t, x) \geq \delta e^{(-a_2 M_1 - rM_2)t}$ for $t > 0$ and $x \in \mathbb{R}$.

We now consider the following problem:

$$\begin{cases} z' = z(1 - z - a_1 w), & t > t_1, \\ w' = w(r - a_2 z - rw), & t > t_1, \\ z(t_1) = 1 + \varepsilon_1,\ w(t_1) = \delta e^{(-a_2 M_1 - rM_2)t_1}. \end{cases}$$

By employing the comparison principle, we obtain that $u(t, x) \leq z(t)$ and $v(t, x) \geq w(t)$ for $t \geq t_1$ and $x \in \mathbb{R}$. Owing to the assumption (3.1) that $a_1 > 1$ and $a_2 < r$, it follows that $(z, w) \to (0, 1)$ as $t \to \infty$. Thus we have

$$\lim_{t\to\infty} u(t, x) = 0 \text{ uniformly in } \mathbb{R}.$$

Next we claim that $\lim_{t\to\infty} v(t, x) = 1$ uniformly in $\mathbb{R}$. Since $\lim_{t\to\infty} u(t, x) = 0$ uniformly in $\mathbb{R}$, for any $\varepsilon > 0$ there exists $t_2 > 0$ such that $0 < u(t, x) \leq \varepsilon$ for $t \geq t_2$ and $x \in \mathbb{R}$. Recalling that $v(t, x) \geq \delta e^{(-a_2 M_1 - rM_2)t_2}$, we thereby obtain

$$\begin{cases} v_t - Dv_{xx} + \nu_2 v_x \geq v(r - a_2\varepsilon - rv), & t > t_2,\ x \in \mathbb{R}, \\ v(t_2, x) \geq \delta e^{(-a_2 M_1 - rM_2)t_2}, & x \in \mathbb{R}. \end{cases}$$

Let us consider the following ordinary differential equation:

$$\begin{cases} \tilde{v}' = \tilde{v}(r - a_2\varepsilon - r\tilde{v}), & t > 0, \\ \tilde{v}(t_2) = \delta e^{(-a_2 M_1 - r M_2)t_2}. \end{cases}$$

By the comparison principle, we have $v(t,x) \geq \tilde{v}(t)$ for $t > t_2$ and $x \in \mathbb{R}$. Since $\lim_{t\to\infty} \tilde{v}(t) = 1 - \frac{a_2\varepsilon}{r}$, we deduce that $\liminf_{t\to\infty} v(t,x) \geq 1 - \frac{a_2\varepsilon}{r}$ uniformly in $\mathbb{R}$. Due to the arbitrariness of ε, it follows that $\lim_{t\to\infty} v(t,x) \geq 1$ uniformly in $\mathbb{R}$.

It remains to show that $-g_\infty$ and $h_\infty < \infty$. Since $(u(t,x), v(t,x)) \to (0,1)$ as $t \to \infty$ uniformly in $\mathbb{R}$, for any $\varepsilon > 0$, there exists $T > 0$ such that $u(t,x) < \varepsilon$ and $v(t,x) > 1 - \varepsilon$ for $t > T$ and $x \in \mathbb{R}$. By setting $\varepsilon = 1 - \frac{1}{a_1}$, we have

$$u(1 - u - a_1 v) \leq u(1 - a_1 + a_1\varepsilon) \leq 0 \text{ for } t > T \text{ and } x \in \mathbb{R}. \tag{3.11}$$

Owing to (3.11), it follows from the boundary condition of (1.3) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{g(t)}^{h(t)} u(t,x)\mathrm{d}x &= \int_{g(t)}^{h(t)} u_t(t,x)\mathrm{d}x + h'(t)u(t,h(t)) - g'(t)u(t,g(t)) \\ &= \int_{g(t)}^{h(t)} (u_{xx} - \nu_1 u_x)\mathrm{d}x + \int_{g(t)}^{h(t)} u(1 - u - a_1 v)\mathrm{d}x \\ &\leq \int_{g(t)}^{h(t)} (u_{xx} - \nu_1 u_x)\mathrm{d}x \\ &\leq u_x(h(t)) - u_x(g(t)) - \nu_1 u(h(t)) + \nu_1 u(g(t)) \\ &= u_x(h(t)) - u_x(g(t)) \\ &= \frac{-h'(t) + g(t)}{\mu}. \end{aligned} \tag{3.12}$$

Integrating (3.12) from T to t gives

$$\int_{g(t)}^{h(t)} u(t,x)\mathrm{d}x - \int_{g(T)}^{h(T)} u(T,x)\mathrm{d}x \leq -\frac{h(t) - g(t) - h(T) + g(T)}{\mu}. \tag{3.13}$$

It follows from (3.13) that

$$\begin{aligned} h(t) - g(t) &\leq h(T) - g(T) + \mu\int_{g(T)}^{h(T)} u(T,x)\mathrm{d}x - \mu\int_{g(t)}^{h(t)} u(t,x)\mathrm{d}x \\ &\leq h(T) - g(T) + \mu\int_{g(T)}^{h(T)} u(T,x)\mathrm{d}x. \end{aligned} \tag{3.14}$$

By letting $t \to \infty$ in (3.14), we have $h_\infty - g_\infty \leq h(T) - g(T) + \mu\int_{g(T)}^{h(T)} u(T,x)\mathrm{d}x < \infty$, which implies that h_∞ and $-g_\infty < \infty$. $\square$

4. Weak competition

In the section, we try to extend the long time behavior to solutions of Cauchy problem corresponding to (1.3) in the case of weak competition. In order to do that, we give the following two lemmas. These lemmas without the advection terms were proved in Propositions 2.1 and 2.2 of Wang and Zhao [49] or in Propositions B1 and B2 of Wang and Zhao [50].

Lemma 4.1. *Assume that ν, D, α and β are fixed positive constants. Suppose that*

$$\nu < 2\sqrt{D\alpha}. \tag{4.1}$$

For any given $\varepsilon > 0$ and $L > 0$, there exist $l_\varepsilon > \max\left\{L, 2D\pi\left(\sqrt{4D\alpha - \nu^2}\right)^{-1}\right\}$ and $T_\varepsilon > 0$, such that when the continuous and non-negative function $w(t,x)$ satisfies

$$\begin{cases} w_t - Dw_{xx} + \nu w_x \geq\ (\leq) w(\alpha - \beta w), & t > 0,\ -l_\varepsilon < x < l_\varepsilon, \\ w(t, \pm l_\varepsilon) \geq\ (=) 0, & t \geq 0, \\ w(0,x) > 0, & x \in (-l_\varepsilon, l_\varepsilon), \end{cases} \tag{4.2}$$

then

$$w(t,x) > \frac{\alpha}{\beta} - \varepsilon\ (w(t,x) < \frac{\alpha}{\beta} + \varepsilon),\ \textit{for}\ t \geq T_\varepsilon,\ x \in [-L, L]. \tag{4.3}$$

Moreover, the above inequality implies that

$$\liminf_{t\to\infty} w(t,x) > \frac{\alpha}{\beta} - \varepsilon\ (\limsup_{t\to\infty} w(t,x) < \frac{\alpha}{\beta} + \varepsilon)\ \textit{uniformly in}\ [-L, L]. \tag{4.4}$$

Proof. Consider the following eigenvalue problem with advection:

$$-D\Phi'' + \nu\Phi' = \lambda\Phi \text{ in}(-l, l), \qquad \Phi(\pm l) = 0. \tag{4.5}$$

The first eigenvalue λ_1 and eigenfunction Φ_1 of (4.5) are

$$\lambda_1 = \frac{\nu^2}{4D} + D\frac{\pi^2}{l^2},\ \Phi_1(x) = e^{\frac{\nu}{2D}x} \sin\left(\sqrt{\frac{\lambda_1}{D} - \frac{\nu^2}{4D^2}}\,x\right).$$

Let $l > 2D\pi\left(\sqrt{4D\alpha - \nu^2}\right)^{-1}$, we obtain $\lambda_1 < \alpha$. Thus, the problem

$$\begin{cases} -Dw_{xx} + \nu w_x = w(\alpha - \beta w), & -l < x < l, \\ w(\pm l) = 0 \end{cases} \tag{4.6}$$

admits a unique positive solution $0 < w = w_l < \frac{\alpha}{\beta}$ (Proposition 3.3 in Cantrell and Cosner [4]). By the comparison principle, $w_l(x)$ is monotone decreasing with respect to l. Letting $l \to \infty$, by using the classical elliptic regularity theory and a diagonal procedure, the limit $\lim_{l\to\infty} w_l(x) = w^*(x)$ exists and satisfies the following Cauchy equation:

$$-Dw_{xx} + \nu w_x = w(\alpha - \beta w), \quad x \in \mathbb{R}. \tag{4.7}$$

In view of Liouville theorem, we have

$$\lim_{l \to \infty} w_l(x) = \frac{\alpha}{\beta} \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.8}$$

Therefore, for any given $L > 0$ and $\varepsilon > 0$, there exists $l_\varepsilon > 2D\pi\left(\sqrt{4D\alpha - \nu^2}\right)^{-1}$ such that

$$\frac{\alpha}{\beta} - \frac{\varepsilon}{2} < w_l(x) < \frac{\alpha}{\beta} + \frac{\varepsilon}{2}, \text{ for } x \in [-L, L], \; l \geq l_\varepsilon. \tag{4.9}$$

Since the parabolic equation

$$\begin{cases} w_t - Dw_{xx} + \nu w_x = w(\alpha - \beta w), & t > 0, \; -l_\varepsilon < x < l_\varepsilon, \\ w(t, \pm l_\varepsilon) = 0, & t \geq 0, \\ w(0, x) = w_0(x), & x \in [-l_\varepsilon, l_\varepsilon]. \end{cases} \tag{4.10}$$

is a gradient system, the solution $w(t, x)$ of (4.10) converges to a solution $w_l(x)$ of the corresponding stationary problem (4.6) as $t \to \infty$ (see Brunovsky and Chow [2], Hale and Massatt [23]). By (4.9), there exists a $T_\varepsilon > 0$ such that

$$\frac{\alpha}{\beta} - \varepsilon < w(t, x) < \frac{\alpha}{\beta} + \varepsilon, \text{ for } t \geq T_\varepsilon, \; x \in [-L, L]. \tag{4.11}$$

Combining (4.2) and (4.10), the comparison principle implies that (4.3) and (4.4) hold immediately. □

Lemma 4.2. *Assume that ν, D, α and β are fixed positive constants and (4.1) holds. For any given $\varepsilon > 0$ and $L > 0$, there exist $l_\varepsilon > \max\left\{L, 2D\pi\left(\sqrt{4D\alpha - \nu^2}\right)^{-1}\right\}$ and $T_\varepsilon > 0$, such that when the continuous and non-negative function $z(t, x)$ satisfies*

$$\begin{cases} z_t - Dz_{xx} + \nu z_x \geq \; (\leq) z(\alpha - \beta z), & t > 0, \; -l_\varepsilon < x < l_\varepsilon, \\ z(t, \pm l_\varepsilon) \geq \; (\leq) k, & t \geq 0, \\ z(0, x) > 0, & x \in (-l_\varepsilon, l_\varepsilon), \end{cases} \tag{4.12}$$

then

$$z(t, x) > \frac{\alpha}{\beta} - \varepsilon \; \left(z(t, x) < \frac{\alpha}{\beta} + \varepsilon\right), \; \textit{for } t \geq T_\varepsilon, \; x \in [-L, L]. \tag{4.13}$$

Moreover, the above inequality implies that

$$\liminf_{t \to \infty} z(t, x) > \frac{\alpha}{\beta} - \varepsilon \; \left(\limsup_{t \to \infty} z(t, x) < \frac{\alpha}{\beta} + \varepsilon\right) \textit{ uniformly in } [-L, L]. \tag{4.14}$$

Proof. Let $l > 2D\pi\left(\sqrt{4D\alpha-\nu^2}\right)^{-1}$. By using a similar argument as in the proof of Lemma 4.1, we can show that the problem

$$\begin{cases} -Dz_{xx}+\nu z_x = z(\alpha-\beta z), & -l<x<l, \\ z(\pm l)=k \end{cases} \tag{4.15}$$

admits a unique positive solution $z = z_l$. We claim that

$$\lim_{l\to\infty} z_l(x) = \frac{\alpha}{\beta} \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.16}$$

For the case $k > \frac{\alpha}{\beta}$, the strong maximum principle implies that $\frac{\alpha}{\beta} \le z_l(x) \le k$ for $x \in [-l, l]$. Since $z_l(x) \le k$, the comparison principle yields that $z_l(x)$ is monotone decreasing with respect to l. Hence, the limit $\lim_{l\to\infty} z_l(x) = z^*(x)$ exists and satisfies the following Cauchy equation:

$$-Dz_{xx}+\nu z_x = z(\alpha-\beta z),\, x\in\mathbb{R}. \tag{4.17}$$

By Liouville theorem, $\lim_{l\to\infty} z_l(x) = \frac{\alpha}{\beta}$ uniformly in any compact subset of $\mathbb{R}$. Thus (4.16) holds.

For the case $k \le \frac{\alpha}{\beta}$, choose $k_0 > \frac{\alpha}{\beta}$ and let $z_l^0(x)$ be the unique solution of (4.15) with $k = k_0$. By the comparison principle, $w_l(x) \le z_l(x) \le z_l^0(x)$ for $x \in [-l, l]$, where $w_l(x)$ is the solution of (4.6). Since $w_l(x)$ and $z_l^0(x)$ converge to $\frac{\alpha}{\beta}$ as $l \to \infty$, we obtain (4.16) immediately.

By (4.16), for any given $L > 0$ and $\varepsilon > 0$, there exists $l_\varepsilon > \max\left\{L, 2D\pi\left(\sqrt{4D\alpha-\nu^2}\right)^{-1}\right\}$ such that

$$\frac{\alpha}{\beta}-\frac{\varepsilon}{2} < z_l(x) < \frac{\alpha}{\beta}+\frac{\varepsilon}{2} \text{ for } x\in[-L,L],\ l\ge l_\varepsilon. \tag{4.18}$$

Let $z_0(x) \in C([-l_\varepsilon, l_\varepsilon])$ be a positive function and $z(t, x)$ be the solution of

$$\begin{cases} z_t - Dz_{xx}+\nu z_x = z(\alpha-\beta z), & t>0,\ -l_\varepsilon<x<l_\varepsilon, \\ z(t,\pm l_\varepsilon)=k, & t\ge 0, \\ z(0,x)=z_0(x), & x\in[-l_\varepsilon,l_\varepsilon]. \end{cases} \tag{4.19}$$

In view of $l_\varepsilon > \max\left\{L, 2D\pi\left(\sqrt{4D\alpha-\nu^2}\right)^{-1}\right\}$, we show that

$$\lim_{t\to\infty} z(t,x) = z_l(x) \text{ uniformly in any compact subset of } (-l_\varepsilon, l_\varepsilon). \tag{4.20}$$

Fix a positive constant q and let $\psi_q(t, x)$ be the unique solution of

$$\begin{cases} \psi_t - D\psi_{xx}+\nu\psi_x = \psi(\alpha-\beta\psi), & t>0,\ -l_\varepsilon<x<l_\varepsilon, \\ \psi(t,\pm l_\varepsilon)=k, & t\ge 0, \\ \psi(0,x)=q, & x\in[-l_\varepsilon,l_\varepsilon]. \end{cases} \tag{4.21}$$

We choose $M >> 1$ and $0 < m << 1$ such that M and m are the upper and lower solutions of (4.15). Then $\psi_M(t, x)$ is monotone decreasing and $\psi_m(t, x)$ is monotone increasing with respect to t. Hence, the limits $\lim_{t\to\infty}\psi_M(t,x)=\psi_M(x)$ and $\lim_{t\to\infty}\psi_m(t,x)=\psi_m(x)$ exist. $\psi_M(x)$ and $\psi_m(x)$ are both positive solutions of (4.15) with $l=l_\varepsilon$. Therefore,

$$\psi_M(x)=\psi_m(x)=z_l(x), \text{ where } z_l(x) \text{ is the solution of (4.15) with } l=l_\varepsilon. \tag{4.22}$$

By comparing the equations (4.19) and (4.21), the comparison principle leads to $\psi_m(t,x)\le z(t,x)\le\psi_M(t,x)$. Since (4.22) holds, $\lim_{t\to\infty}z(t,x)=z_l(x)$. By using the interior estimate, the limit $\lim_{t\to\infty}z(t,x)=z_l(x)$ is uniformly in any compact subset of $(-l_\varepsilon,l_\varepsilon)$.

Owing to (4.14) and (4.20), there exists $T_\varepsilon >> 1$ such that

$$\frac{\alpha}{\beta}-\varepsilon<z(t,x)<\frac{\alpha}{\beta}+\varepsilon, \text{ for } t\ge T_\varepsilon,\ x\in[-L,L].$$

By comparing the two systems (4.12) and (4.19), we obtain that (4.13) and (4.14) hold immediately. $\square$

4.1. Long term behavior of boundaries and convergence of solutions

In fact, whether the size of (g_∞,h_∞) is finite determines the asymptotic behavior of u and v. In the next two theorems, we give the relationship between the long term behavior of boundaries and the convergence of u and v.

Theorem 4.3. *Assume that $h_\infty=\infty$ and $g_\infty=-\infty$. If*

$$a_1<1,\ a_2<r,\ \nu_1<2\sqrt{1-a_1},\ \nu_2<2\sqrt{D(r-a_2)}, \tag{4.23}$$

then the solution (u,v,g,h) to (1.3) satisfies

$$\lim_{t\to\infty}(u(t,x),v(t,x)) = \Big(\frac{r(1-a_1)}{r-a_1a_2},\frac{r-a_2}{r-a_1a_2}\Big) \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.24}$$

Proof. By Lemma 2.2, $0<u(t,x)\le M_1$ for $t>0$ and $x\in(g(t),h(t))$, $0<v(t,x)\le M_2$ for $t>0$ and $x\in\mathbb{R}$. Then we find that v satisfies

$$\begin{cases} v_t-Dv_{xx}-\nu_2v_x\le v(r-rv), & t>0,\ x\in\mathbb{R},\\ v(0,x)=v_0(x), & x\in\mathbb{R}.\end{cases}$$

The comparison principle yields that

$$\limsup_{t\to\infty}v(t,x)\le 1:=\overline{v}_1 \text{ uniformly in } \mathbb{R}.$$

For any $0<\varepsilon_1<<1$, there exists $T_1>0$ such that

$$v(t,x)\le\overline{v}_1+\varepsilon_1 \text{ for } t\ge T_1,\ x\in\mathbb{R}. \tag{4.25}$$

(i) For any given $L > 0$, and $0 < \varepsilon << 1$, let l_ε be given by Lemma 4.1. In view of $h_\infty = \infty$ and $g_\infty = -\infty$, there exists $T_2 \geq T_1$ such that

$$g(t) < -l_\varepsilon, \ h(t) > l_\varepsilon, \ \text{for } t \geq T_2.$$

Recalling (4.25), we see that u satisfies

$$\begin{cases} u_t - u_{xx} + \nu_1 u_x \geq u(1 - u - a_1(\overline{v}_1 + \varepsilon_1)), & t > T_2, \ x \in (-l_\varepsilon, l_\varepsilon), \\ u(t, \pm l_\varepsilon) \geq 0, & t \geq T_2, \\ u(T_2, x) > 0, & x \in (-l_\varepsilon, l_\varepsilon). \end{cases}$$

In order to use Lemma 4.1, we need to verify condition (4.1), here it corresponds to $\nu_1 < 2\sqrt{1 - a_1(\overline{v}_1 + \varepsilon_1)}$. Thus, we should choose $\varepsilon_1 < \frac{4(1-a_1\overline{v}_1)-\nu_1^2}{4a_1}$. Due to $\nu_1 < 2\sqrt{1-a_1}$ of (4.23), we induce that $\frac{4(1-a_1\overline{v}_1)-\nu_1^2}{4a_1} = \frac{4(1-a_1)-\nu_1^2}{4a_1} > 0$. Hence ε_1 satisfying Lemma 4.1 exists. Applying Lemma 4.1, we have

$$\liminf_{t\to\infty} u(t, x) > 1 - a_1\overline{v}_1 - a_1\varepsilon_1 - \varepsilon \ \text{ uniformly in } [-L, L].$$

Since $1 - a_1 > 0$, by the arbitrariness of L, ε and ε_1, we have

$$\liminf_{t\to\infty} u(t, x) \geq 1 - a_1\overline{v}_1 := \underline{u}_1 > 0 \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.26}$$

(ii) For any given $L > 0$ and $0 < \varepsilon << 1$, let l_ε be given by Lemma 4.2 with $k = M_2$. Taking into account (4.26), there exists $T_3 \geq T_2$ such that

$$u(t, x) \geq \underline{u}_1 + \varepsilon_2 \text{ for } t \geq T_3, \ x \in [-l_\varepsilon, l_\varepsilon].$$

Then v satisfies

$$\begin{cases} v_t - Dv_{xx} - \nu_2 v_x \leq v(r - a_2(\underline{u}_1 + \varepsilon_2) - rv), & t > T_3, \ x \in (-l_\varepsilon, l_\varepsilon), \\ v(t, \pm l_\varepsilon) \leq M_2, & t \geq T_3, \\ v(T_3, x) > 0, & x \in (-l_\varepsilon, l_\varepsilon). \end{cases}$$

In order to use Lemma 4.2, we need to verify condition (4.1), here it corresponds to $\nu_2 < 2\sqrt{D(r - a_2(\underline{u}_1 + \varepsilon_2))}$. Thus we should choose $\varepsilon_2 < \frac{4D(r-a_2\underline{u}_1)-\nu_2^2}{4Da_2}$. Since $\underline{u}_1 < 1$, $\frac{4D(r-a_2\underline{u}_1)-\nu_2^2}{4Da_2} > \frac{4D(r-a_2)-\nu_2^2}{4Da_2}$. Consequently, in order to ensure ε_2 existing, we only need to verify $\frac{4D(r-a_2)-\nu_2^2}{4Da_2} > 0$, which can be induced by the condition $\nu_2 < 2\sqrt{D(r-a_2)}$ of (4.23). Applying Lemma 4.2, we have

$$\limsup_{t\to\infty} v(t, x) < 1 - \frac{a_2}{r}(\underline{u}_1 + \varepsilon_2) + \varepsilon \ \text{ uniformly in } [-L, L].$$

By the arbitrariness of L, ε and ε_2, we have

$$\limsup_{t\to\infty} v(t, x) \leq 1 - \frac{a_2}{r}\underline{u}_1 := \overline{v}_2 \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.27}$$

(iii) For any given $L > 0$ and $0 < \varepsilon << 1$, let l_ε be given by Lemma 4.1. Taking into account (4.27), there exists $T_4 \geq T_3$ such that

$$g(t) < -l_\varepsilon,\ h(t) > l_\varepsilon,\ v(t,x) \leq \overline{v}_2 + \varepsilon_1,\ \text{for } t \geq T_4,\ x \in [-l_\varepsilon, l_\varepsilon].$$

We see that u satisfies

$$\begin{cases} u_t - u_{xx} + \nu_1 u_x \geq u(1 - u - a_1(\overline{v}_2 + \varepsilon_1)), & t > T_4,\ x \in (-l_\varepsilon, l_\varepsilon), \\ u(t, \pm l_\varepsilon) \geq 0, & t \geq T_4, \\ u(T_4, x) > 0, & x \in (-l_\varepsilon, l_\varepsilon). \end{cases}$$

Since $\overline{v}_2 < \overline{v}_1$, the above $\frac{4(1-a_1\overline{v}_1)-\nu_1^2}{4a_1} > 0$ implies $\frac{4(1-a_1\overline{v}_2)-\nu_1^2}{4a_1} > 0$. Thus, our choice ε_1 satisfies the condition (4.1). Applying Lemma 4.1, we have

$$\liminf_{t\to\infty} u(t,x) > 1 - a_1\overline{v}_2 - a_1\varepsilon_1 - \varepsilon \ \text{ uniformly in } [-L, L].$$

By the arbitrariness of L, ε and ε_1, we have

$$\liminf_{t\to\infty} u(t,x) \geq 1 - a_1\overline{v}_2 := \underline{u}_2 \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.28}$$

(iv) For any given $L > 0$ and $0 < \varepsilon << 1$, let l_ε be given by Lemma 4.2 with $k = M_2$. Taking into account (4.28), there exists $T_5 \geq T_4$ such that

$$u(t,x) \geq \underline{u}_2 + \varepsilon_2 \text{ for } t \geq T_5,\ x \in [-l_\varepsilon, l_\varepsilon].$$

Then v satisfies

$$\begin{cases} v_t - Dv_{xx} - \nu_2 v_x \leq v(r - a_2(\underline{u}_2 + \varepsilon_2) - rv), & t > T_5,\ x \in (-l_\varepsilon, l_\varepsilon), \\ v(t, \pm l_\varepsilon) \leq M_2, & t \geq T_5, \\ v(T_5, x) > 0, & x \in (-l_\varepsilon, l_\varepsilon). \end{cases}$$

Since $\underline{u}_2 < 1$ still holds, the above $\frac{4D(r-a_2)-\nu_2^2}{4Da_2} > 0$ implies $\frac{4D(r-a_2\underline{u}_2)-\nu_2^2}{4Da_2} > 0$. Thus the chosen ε_2 satisfies condition (4.1). Applying Lemma 4.2, we have

$$\limsup_{t\to\infty} v(t,x) < 1 - \frac{a_2}{r}(\underline{u}_2 + \varepsilon_2) + \varepsilon \ \text{ uniformly in } [-L, L].$$

By the arbitrariness of L, ε and ε_2, we have

$$\limsup_{t\to\infty} v(t,x) \leq 1 - \frac{a_2}{r}\underline{u}_2 := \overline{v}_3 \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.29}$$

Therefore, as long as the sequence $\{\underline{u}_n\}$ is monotone increasing and the sequence $\{\overline{v}_n\}$ is monotone decreasing, the condition (4.1) is naturally satisfied. We can apply Lemmas 4.1 and 4.2 again. Repeating the above procedure such as (4.26), (4.27), (4.28) and (4.29), we obtain two sequences $\{\underline{u}_n\}$ and $\{\overline{v}_n\}$, which satisfy

$$\underline{u}_1 = 1 - a_1,\ \overline{v}_1 = 1,\ \text{and}\ \underline{u}_n = 1 - a_1\overline{v}_n,\ \overline{v}_{n+1} = 1 - \frac{a_2}{r}\underline{u}_n,\ \text{for}\ n = 1, 2\cdots \tag{4.30}$$

We now claim that $\{\underline{u}_n\}$ is monotone increasing and $\{\overline{v}_n\}$ is monotone decreasing. We prove it by using an induction method. For the case $n = 1$, since $a_1 < 1$, it is easy to see that

$$\overline{v}_2 - \overline{v}_1 = -\frac{a_2}{r}(1 - a_1) < 0,\ \underline{u}_2 - \underline{u}_1 = -a_1(\overline{v}_2 - \overline{v}_1) > 0.$$

Suppose that $\overline{v}_n - \overline{v}_{n-1} < 0,\ \underline{u}_n - \underline{u}_{n-1} > 0$. By (4.30), we compute

$$\begin{aligned}&\overline{v}_{n+1} - \overline{v}_n = 1 - \frac{a_2}{r}\underline{u}_n - (1 - \frac{a_2}{r}\underline{u}_{n-1}) = -\frac{a_2}{r}(\underline{u}_n - \underline{u}_{n-1}) < 0,\\&\underline{u}_{n+1} - \underline{u}_n = -a_1(\overline{v}_{n+1} - \overline{v}_n) > 0.\end{aligned}$$

Thus the induction principle implies the claim.

Since the sequences $\{\underline{u}_n\}$ is monotone increasing and $\{\overline{v}_n\}$ is monotone decreasing, the limits $\lim_{n\to\infty}\underline{u}_n$ and $\lim_{n\to\infty}\overline{v}_n$ exist, which satisfy

$$\begin{aligned}&\liminf_{t\to\infty} u(t,x) \ge \underline{u} = \frac{r(1-a_1)}{r - a_1a_2} \text{ uniformly in any compact subset of } \mathbb{R}.\\&\limsup_{t\to\infty} v(t,x) \le \overline{v} = \frac{r - a_2}{r - a_1a_2} \text{ uniformly in any compact subset of } \mathbb{R}.\end{aligned} \tag{4.31}$$

By using a similar argument, we can show

$$\begin{aligned}&\limsup_{t\to\infty} u(t,x) \le \frac{r(1-a_1)}{r - a_1a_2} \text{ uniformly in any compact subset of } \mathbb{R}.\\&\liminf_{t\to\infty} v(t,x) \ge \frac{r - a_2}{r - a_1a_2} \text{ uniformly in any compact subset of } \mathbb{R}.\end{aligned} \tag{4.32}$$

Combining (4.31) and (4.32), we conclude that (4.24) holds. □

Remark 4.4. For the Cauchy problem (3.10), the proof of Theorem 4.3 is valid. When (4.23) holds, the solution (u, v) of (3.10) converges to the coexistence state $(\frac{r(1-a_1)}{r-a_1a_2}, \frac{r-a_2}{r-a_1a_2})$ locally uniformly in $\mathbb{R}$ as $t \to \infty$.

Theorem 4.5. *Let (u, v, g, h) be a solution of (1.3). If $h_\infty - g_\infty < \infty$, then*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = 0, \tag{4.33}$$

$$\lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0, \tag{4.34}$$

$$\lim_{t\to\infty} v(t,x) = 1 \textit{ uniformly in any compact subset of } \mathbb{R}. \tag{4.35}$$

Proof. We first show that

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}([1,\infty)\times[g(t),h(t)])} + \|g\|_{C^{1+\alpha/2}([1,\infty))} + \|h\|_{C^{1+\alpha/2}([1,\infty))} \le C \tag{4.36}$$

for any $\alpha \in (0,1)$, where the constant C depends on $\alpha, h_0, \|u_0\|_{C^2([-h_0,h_0])}, g_\infty$ and h_∞. We straighten the free boundaries by the following transformation

$$y = \frac{2x}{h(t)-g(t)} - \frac{h(t)+g(t)}{h(t)-g(t)}$$

such that $x = h(t)$ and $x = g(t)$ change to $y = \pm 1$. Some calculations show that

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{2}{h(t)-g(t)} := \sqrt{A(g(t),h(t),y)},\\
\frac{\partial^2 y}{\partial x^2} &= \frac{4}{(h(t)-g(t))^2} := A(g(t),h(t),y),\\
\frac{\partial y}{\partial t} &= -\frac{2x(h'(t)-g'(t)) + 2(g(t)h'(t)+h(t)g'(t))}{(h(t)-g(t))^2} := B(g(t),g'(t),h(t),h'(t),y).
\end{aligned}$$

Our proof is motivated by Theorem 2.1 in Wang [47,48] (see also Theorem 4.1 in Lei et al. [29] and Theorem 4.1 in Cao et al. [5]). Let $w(t,y) = u(t,x)$ and $z(t,y) = v(t,x)$, then we obtain that $w(t,y)$ satisfies

$$\begin{cases} w_t - Aw_{yy} - (A - B - \nu_1\sqrt{A})w_y = f_1(w,z), & t>0, -1<y<1\\ w(t,\pm 1) = 0, & t>0,\\ w(0,y) = u_0(\frac{y}{h_0}) \ge 0, & -1 \le y \le 1. \end{cases}$$

For any integer $n \ge 0$, define

$$w^n(t,y) = w(t+n,y),\ z^n(t,y) = z(t+n,y),$$

then $w^n(t,y)$ satisfies

$$\begin{cases} w^n_t - A^n w^n_{yy} - (A^n - B^n - \nu_1\sqrt{A^n})w^n_y = f_1(w^n,z^n), & t \in (0,3], -1<y<1,\\ w^n(t,\pm 1) = 0, & t \in (0,3],\\ w^n(0,y) = u(n, \frac{y(h(n)-g(n))+h(n)+g(n)}{2}) \ge 0, & -1 \le y \le 1, \end{cases}$$

where $A^n = A(t+n)$ and $B^n = B(t+n)$. In view of Lemma 2.2, it follows that w^n, z^n, A^n and B^n are bounded uniformly on n, and

$$\max_{0\le t_1<t_2\le 3, |t_1-t_2|\le\tau} |A^n(t_1) - A^n(t_2)| \le \frac{8(h^n(t)-g^n(t))'}{(h^n(t)-g^n(t))^3} \le \frac{2M_3\tau}{h_0^3} \to 0 \text{ as } \tau \to 0$$

with $g^n(t) = g(t+n),\ h^n(t) = h(t+n)$. Moreover, we have $A^n \ge \frac{4}{(h_\infty - g_\infty)^2}$ for all $n \ge 0$ and $0 < t \le 3$ as $h_\infty - g_\infty < \infty$.

If we choose the integer p sufficiently large, applying the interior L^p estimate yields that there exists a positive constant C independent of n such that $\|u^n\|_{W_p^{1,2}([1,3]\times[-1,1])} \le C_1$ for all $n > 0$. By Sobolev's imbedding theorem, we therefore have $\|u^n\|_{C^{(1+\alpha)/2,1+\alpha}([1,3]\times[-1,1])} \le C_1$ for all $n > 0$, which implies that $\|u\|_{C^{(1+\alpha)/2,1+\alpha}([n+1,n+3]\times[-1,1])} \le C_1$. Since

$$g'(t) = -\mu u_x(t, g(t)),\ u_x(t, g(t)) = \frac{2}{h(t) - g(t)} u_y(-1, t),$$
$$h'(t) = -\mu u_x(t, h(t)),\ u_x(t, h(t)) = \frac{2}{h(t) - g(t)} u_y(1, t),$$

hold in $[n+1, n+3] \times [-1, 1]$ and $0 < -g'(t), h(t) \le M_3$, we have

$$\|g\|_{C^{1+\alpha/2}([n+1,n+3])} + \|h\|_{C^{1+\alpha/2}([n+1,n+3])} \le C_2.$$

Since the intervals $[n+1, n+3] \times [-1, 1]$ overlap and the above constants C_1 and C_2 are independent of n, it follows that (4.36) holds. Noting that $h(t)$ is bounded and $\|h\|_{C^{1+\alpha/2}([0,\infty))} \le M$, we have $h'(t) \to 0$ as $t \to +\infty$. Analogously, $g'(t) \to 0$ as $t \to +\infty$. Thus (4.34) is proved.

We then show (4.33). Assume that

$$\limsup_{t \to +\infty} \|u(t, \cdot)\|_{C([g(t),h(t)])} = \delta > 0$$

by contradiction. Then there exists a sequence $\{(t_k, x_k)\}$ in $(-\infty, \infty) \times (g(t), h(t))$ such that $u(t_k, x_k) \ge \delta/2$ for all $k \in \mathbb{N}$, and $t_k \to \infty$ as $k \to \infty$.

It follows from Theorem 2.1 that there exists a constant $\tilde{C}$ depending on α, h_0, (u_0, v_0) and h_∞ such that

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(G)} + \|v\|_{C^{(1+\alpha)/2,1+\alpha}(G)} + \|h\|_{C^{1+\alpha/2}([0,\infty))} + \|g\|_{C^{1+\alpha/2}([0,\infty))} \le \tilde{C}, \quad (4.37)$$

where $G = \{(t, x) \in \mathbb{R} :\ t \ge 0, x \in [g(t), h(t)]\}$. Combining $u(t, h(t)) = u(t, g(t)) = 0$ and (4.37), we have $|u_x(t, h(t))|$ and $|u_x(t, g(t))|$ are uniformly bounded for $t \in [0, \infty)$. There exists $\sigma > 0$ such that $x_k \le h(t_k) - \sigma$ for all $k \ge 1$. Therefore there exists a subsequence of $\{x_k\}$ converging to $x_0 \in (g_\infty, h_\infty - \sigma]$. Without loss of generality, we assume $x_k \to x_0$ as $k \to \infty$.

Define

$$U_k(t, x) = u(t_k + t, x) \text{ and } V_k(t, x) = v(t_k + t, x) \text{ for } (t, x) \in G_k$$

with

$$G_k = \{(t, x) \in \mathbb{R} :\ t \in (-t_k, \infty), x \in [g(t_k + t), h(t_k + t)]\}.$$

It follows from (4.37) that $\{(U_k, V_k)\}$ has a subsequence $\{(U_{k_n}, V_{k_n})\}$ such that

$$\|(U_{k_n}, V_{k_n}) - (\hat{U}, \hat{V})\|_{C^{1,2}(G_{k_n}) \times C^{1,2}(G_{k_n})} \to 0 \text{ as } n \to \infty,$$

and $(\hat{U}, \hat{V})$ satisfies

$$\begin{cases} \hat{U}_t = \hat{U}_{xx} - \nu_1 \hat{U}_x + \hat{U}(1 - \hat{U} - a_1 \hat{V}), & -\infty < t < \infty,\ g_\infty < x < h_\infty, \\ \hat{V}_t = D\hat{V}_{xx} - \nu_2 \hat{V}_x + \hat{V}(r - a_2 \hat{U} - r\hat{V}), & -\infty < t < \infty,\ g_\infty < x < h_\infty, \end{cases}$$

with

$$\hat{U}(t, h_\infty) = 0 \text{ for } t \in (-\infty, \infty).$$

Since $\hat{U}(0, x_0) \geq \delta/2$, we have $\widehat{U} > 0$ in $(-\infty, \infty) \times (g_\infty, h_\infty)$ by the strong comparison principle. Let $M = \|1 - \hat{U} - a_1\hat{V}\|_{L^\infty([0,\infty))}$, then $\hat{U}_t - \hat{U}_{xx} + \nu_1\hat{U}_x + M\hat{U} \geq 0$. Using the Hopf Lemma at the point $(0, h_\infty)$, we get $\widehat{U}_x(0, h_\infty) := \sigma_0 < 0$. It follows that

$$u_x(t_{k_n}, h(t_{k_n})) = \partial u_{k_n}(0, h(t_{k_n})) \leq \sigma_0/2 < 0 \tag{4.38}$$

for all large n. Hence, $h'(t_{k_n}) \geq -\mu\sigma_0/2 > 0$ for all large n.

On the other hand, since $\|h\|_{C^{1+\alpha/2}([0,\infty))} \leq \tilde{C}$, $h'(t) > 0$ and $h(t) \leq h_\infty$, we have $h'(t) \to 0$ as $t \to \infty$. We obtain a contradiction, thus (4.33) is true.

It remains to show (4.35). Let $\overline{v}(t)$ solve the following ordinary differential equation:

$$\begin{cases} \overline{v}' = r\overline{v}(1 - \overline{v}), & t > 0, \\ \overline{v}(0) = \|v_0\|_{L^\infty(\mathbb{R})}, \end{cases}$$

where

$$\overline{v} = e^{rt}\big(e^{rt} - 1 + \frac{1}{\overline{v}(0)}\big)^{-1}.$$

Applying the comparison principle yields that $v(t, x) \leq \overline{v}(t)$ for $(t, x) \in [0, T) \times \mathbb{R}$. Since $\lim_{t\to\infty} \overline{v}(t) = 1$, we have

$$\limsup_{t\to\infty} v(t, x) \leq 1 \text{ uniformly for } x \in \mathbb{R}. \tag{4.39}$$

On the other hand, we have (4.33) and $u(t, x) \equiv 0$ for $x \notin (g(t), h(t))$. Hence, for any given positive σ, there exists $T_\sigma > 0$ such that $u(t, x) < \sigma$ for $(t, x) \in [T_\sigma, \infty) \times \mathbb{R}$. Consequently, v satisfies

$$\begin{cases} v_t - Dv_{xx} + \nu_2 v_x \geq v(r - a_2\sigma - rv), & t > T_\sigma,\ x \in \mathbb{R}, \\ v(T_\sigma, x) > 0, & x \in \mathbb{R}. \end{cases} \tag{4.40}$$

Considering the corresponding Cauchy problem of (4.40),

$$\begin{cases} w_t - Dw_{xx} + \nu_2 w_x = w(r - a_2\sigma - rw), & t > T_\sigma,\ x \in \mathbb{R}, \\ w(0, x) > 0, & x \in \mathbb{R}, \end{cases} \tag{4.41}$$

invoking Theorem 5.4 of Du and Ma [12] shows that

$$\lim_{t\to\infty} w(t, x) = \frac{r - a_2\sigma}{r}$$

uniformly in any compact subset of $\mathbb{R}$. By applying the comparison principle to u and w, we deduce that

$$\liminf_{t\to\infty} u(t, x) \geq \frac{r - a_2\sigma}{r} \text{ uniformly in any compact subset of } \mathbb{R}. \tag{4.42}$$

Letting $\sigma \to 0$, (4.42) implies $\liminf_{t\to\infty} u(t,x) \geq 1$ uniformly in any compact subset of $\mathbb{R}$. Recalling (4.24), (4.33) is proved. The proof is complete. □

4.2. Sufficient conditions for spreading and vanishing

By Theorems 4.3 and 4.5, the boundaries determine the asymptotic behavior of u and v. Hence, we only need to study the asymptotic behavior of the free boundaries h_∞ and g_∞. In order to do that, we give the spreading-vanishing alternative for the following single free boundary problem:

$$\begin{cases} u_t = u_{xx} - \nu u_x + u(\alpha - \beta u), & t > 0,\ g(t) < x < h(t), \\ u = 0,\ g'(t) = -\mu u_x(t, g(t)), & t \geq 0,\ x = g(t), \\ u = 0,\ h'(t) = -\mu u_x(t, h(t)), & t \geq 0,\ x = h(t), \\ g(0) = -h_0,\ h(0) = h_0, & \\ u(0,x) = u_0(x) = \sigma\phi(x), & x \in [-h_0, h_0], \end{cases} \tag{4.43}$$

where $\phi(x) \in C^2([-h_0, h_0])$, $\phi(\pm h_0) = 0$, $\phi'(-h_0) > 0$, $\phi'(h_0) < 0$, $\phi_0(x) > 0$ in $(-h_0, h_0)$. Note that the global existence of solutions and boundedness to problem (4.43) are valid by a similar argument in Theorem 2.3 and Lemma 2.2. Gu et al. [19,21] have used the zero number argument method to show that when $\nu < 2\sqrt{\alpha}$ the problem (4.43) possesses a spreading-vanishing dichotomy; that is, the solution converges either to 1 locally uniformly in $\mathbb{R}$ or to 0 uniformly into a finite domain.

Lemma 4.6 *(Theorem 1.1 in [19] or Theorem 2.1 in [21]). Assume that $\nu < 2\sqrt{\alpha}$ and (u, g, h) is a global solution of (4.43). Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}$ and $\lim_{t\to\infty} u(t,x) = \frac{\alpha}{\beta}$ uniformly in any compact subset of $\mathbb{R}$;

or

(ii) Vanishing: (g_∞, h_∞) is a finite interval with $h_\infty - g_\infty \leq 2\pi\left(\sqrt{4\alpha - \nu^2}\right)^{-1}$ and $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = 0$.

Moreover, in the case $h_0 \geq \pi\left(\sqrt{4\alpha - \nu^2}\right)^{-1}$, spreading happens; In the case $h_0 < \pi\left(\sqrt{4\alpha - \nu^2}\right)^{-1}$, there exists $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ such that vanishing happens when $0 < \sigma \leq \sigma^*$, and spreading happens when $\sigma > \sigma^*$.

Next we give the comparison principle for the single free boundary problem (4.43).

Lemma 4.7. *Let $T \in (0, \infty)$, $\overline{g}, \overline{h} \in C^1([0,T])$. Assume that $D_T^* := \{(t,x) \in \mathbb{R}^2 :\ 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t)\}$ and*

$$\overline{u} \in C(\overline{D_T^*}) \cap C^{1,2}(D_T^*),\ \text{and}\ \overline{u}(t,x) > 0 \text{ in } D_T^*,$$

and $(\overline{u}, \overline{g}, \overline{h})$ satisfies

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} + \nu \overline{u}_x \geq \overline{u}(\alpha - \beta \overline{u}), & 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u}(t, \overline{g}(t)) = 0,\ \overline{g}'(t) \leq -\mu \overline{u}_x(t, \overline{g}(t)), & 0 < t \leq T, \\ \overline{u}(t, \overline{h}(t)) = 0,\ \overline{h}'(t) \geq -\mu \overline{u}_x(t, \overline{h}(t)), & 0 < t \leq T, \\ \overline{g}(0) \leq -h_0,\ \overline{h}(0) \geq h_0, & \\ \overline{u}(0, x) \geq u_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{4.44}$$

Then $(\overline{u}, \overline{g}, \overline{h})$ is called an upper solution of (4.43). Moreover, the solution (u, g, h) of the free boundary problem (4.43) satisfies

$$g(t) \geq \overline{g}(t),\ h(t) \leq \overline{h}(t) \ \text{for}\ t \in [0, T], \tag{4.45}$$

$$u(t, x) \leq \overline{u}(t, x) \ \text{for}\ (t, x) \in [0, T] \times [g(t), h(t)]. \tag{4.46}$$

By reversing all the inequalities in (4.44), we can define a lower solution $(\underline{u}, \underline{g}, \underline{h})$, which satisfies

$$g(t) \leq \underline{g}(t),\ h(t) \geq \underline{h}(t) \ \text{for}\ t \in [0, T], \tag{4.47}$$

$$u(t, x) \geq \underline{u}(t, x) \ \text{for}\ (t, x) \in [0, T] \times [\underline{g}(t), \underline{h}(t)]. \tag{4.48}$$

Proof. We first prove the case of $\overline{g}(0) < -h_0$ and $\overline{h}(0) > h_0$. We claim that $\overline{g}(t) < g(t)$ and $\overline{h}(t) > h(t)$ on $(0, T]$. Clearly, this is true for small $t > 0$. If our claim is not true, without loss of generality, there exists a $T_0 > 0$ such that $\overline{h}(T_0) = h(T_0)$, and $\overline{g}(t) < g(t)$, $\overline{h}(t) > h(t)$ for all $t \in (0, T_0)$. Thus,

$$h'(T_0) \geq \overline{h}'(T_0). \tag{4.49}$$

We now show that $u \leq \overline{u}$ in $[0, T_0] \times [g(t), h(t)]$. Letting $U = (\overline{u} - u)e^{-Kt}$, taking the difference of the equations for (4.43) and (4.44) implies that

$$\begin{cases} U_t - U_{xx} + \nu U_x \geq -KU + \alpha U - \beta U(\overline{u} + u), & 0 < t \leq T_0,\ g(t) < x < h(t), \\ U(t, g(t)) \geq 0,\ U(t, h(t)) \geq 0, & 0 < t \leq T_0, \\ U(0, x) \geq 0, & x \in [-h_0, h_0]. \end{cases} \tag{4.50}$$

Since $\overline{u} \in C(\overline{D_T^*})$, by setting $M = \|\overline{u}\|_{L^\infty(\overline{D_T^*})} + \|u\|_{L^\infty(\overline{D_T^*})}$, the first inequality of (4.50) implies that

$$U_t - U_{xx} + \nu U_x \geq -KU + \alpha U - \beta M U,\ 0 < t \leq T_0,\ g(t) < x < h(t).$$

Upon choosing

$$K \geq \alpha - \beta M \tag{4.51}$$

and applying the maximum principle directly to (4.50), we obtain $\overline{U} \geq 0$ in $[0, T_0] \times [g(t), h(t)]$. Therefore, $u \leq \overline{u}$ in $[0, T_0] \times [g(t), h(t)]$. Since $\overline{u}(T_0, h(T_0)) = u(T_0, h(T_0))$, utilizing Hopf Lemma to U shows $U_x(T_0, h(T_0)) < 0$; that is, $\overline{u}_x(T_0, h(T_0)) < u_x(T_0, h(T_0))$. Thus, we obtain that $h'(T_0) < \overline{h}'(T_0)$, which is a contradiction to (4.49). Hence, we proved our claim that $\overline{h}(t) > h(t)$ and $\overline{g}(t) < g(t)$ on all $[0, T]$. We can obtain the equation (4.50) for $t \in (0, T]$. By applying the maximum principle again, we conclude that $\overline{u} \geq u$ in $[0, T] \times [g(t), h(t)]$. Hence, (4.45) and (4.46) hold in the case of $\overline{g}(0) < -h_0$ and $\overline{h}(0) > h_0$.

It remains to prove the general case that $\overline{g}(0) = -h_0$ and $\overline{h}(0) = h_0$. We set $h_0^\varepsilon = (1-\varepsilon)h_0$, $\mu_\varepsilon = (1-\varepsilon)\mu$ for small $\varepsilon > 0$, and $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ solves the following equations:

$$\begin{cases} u_t = u_{xx} - \nu u_x + u(\alpha - \beta u), & t > 0,\ g(t) < x < h(t), \\ u = 0,\ g'(t) = -\mu_\varepsilon u_x(t, g(t)), & t \geq 0,\ x = g(t), \\ u = 0,\ h'(t) = -\mu_\varepsilon u_x(t, h(t)), & t \geq 0,\ x = h(t), \\ g(0) = -h_0^\varepsilon,\ h(0) = h_0^\varepsilon, & \\ u(0, x) = u_0^\varepsilon(x), & x \in [-h_0^\varepsilon, h_0^\varepsilon]. \end{cases} \tag{4.52}$$

here $u_0^\varepsilon(x) \in C^2([-h_0^\varepsilon, h_0^\varepsilon])$, which also satisfies

$$0 < u_0^\varepsilon(x) \leq u_0(x) \text{ for } x \in [-h_0^\varepsilon, h_0^\varepsilon].$$

Moreover, as $\varepsilon \to 0$,

$$u_0^\varepsilon\Big(\frac{h_0}{h_0^\varepsilon} x\Big) \to u_0(x) \text{ in } C^2([-h_0, h_0]).$$

Thus, $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ satisfies the first case. Consequently we have

$$\begin{aligned} & g_\varepsilon(t) \geq \overline{g}(t),\ h_\varepsilon(t) \leq \overline{h}(t) \ \text{ for } t \in [0, T], \\ & u_\varepsilon(t, x) \leq \overline{u}(t, x) \ \text{ for } t \in [0, T],\ x \in [-h_0^\varepsilon, h_0^\varepsilon]. \end{aligned} \tag{4.53}$$

Since the unique solution of (4.43) depends continuously on the parameters in problem (4.52), as $\varepsilon \to 0$, $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ converges to (u, g, h) the unique solution of (4.43). Then (4.45) and (4.46) are true by taking $\varepsilon \to 0$. $\square$

We now give a sufficient condition for spreading in (1.3).

Theorem 4.8. *Assume that* (4.23) *holds and* (u, v, g, h) *is a global solution of* (1.3).

$$\text{If } h_\infty - g_\infty < \infty, \text{ then } h_\infty - g_\infty \leq 2\pi\Big(\sqrt{4 - \nu_1^2}\Big)^{-1}. \tag{4.54}$$

$$\text{If } h_0 \geq \pi\Big(\sqrt{4 - \nu_1^2}\Big)^{-1}, \text{ then } (g_\infty, h_\infty) = \mathbb{R}. \tag{4.55}$$

Moreover, when $(g_\infty, h_\infty) = \mathbb{R}$, *spreading happens, and the solution* (u, v, g, h) *to* (1.3) *satisfies*

$$\lim_{t\to\infty} (u(t, x), v(t, x)) = \Big(\frac{r(1-a_1)}{r - a_1 a_2}, \frac{r - a_2}{r - a_1 a_2}\Big) \text{ uniformly in any compact subset of } \mathbb{R}.$$

Proof. In order to prove (4.54), we assume $h_\infty - g_\infty > 2\pi\Big(\sqrt{4 - \nu_1^2}\Big)^{-1}$ to get a contradiction. By Theorem 4.5, if $h_\infty - g_\infty < \infty$, then $\|u(t, \cdot)\|_{C([g(t), h(t)])} = 0$, $\lim_{t\to\infty} v(t, x) = 1$ uniformly in any compact subset of $\mathbb{R}$. For any small $0 < \varepsilon_1 << 1$ and any $\varepsilon_2 > 0$, there exists $T_1 > 0$ such that

$$v(t,x) < 1+\varepsilon_1,\ \text{for } t > T_1, x \in [g_\infty, h_\infty].$$
$$h(T_1) - g(T_1) > \max\{2h_0, 2\pi\left(\sqrt{4-\nu_1^2-\varepsilon_1}\right)^{-1}\}.$$

Set $l_1 = g(T_1)$ and $l_2 = h(T_1)$, then $l_2 - l_1 > 2\pi\left(\sqrt{4-\nu_1^2-\varepsilon_1}\right)^{-1}$. Consider the following initial boundary value problem:

$$\begin{cases} w_t - w_{xx} + \nu_1 w_x = w(1 - w - a_1(1+\varepsilon_2)), & t > T_1,\ l_1 < x < l_2, \\ w(t,l_1) = w(t,l_2) = 0, & t > T_1, \\ w(T_1,x) = u(T_1,x) > 0, & x \in (l_1,l_2). \end{cases} \tag{4.56}$$

By the comparison principle, we have

$$w(t,x) \le u(t,x) \text{ for } t \ge T_1, x \in [l_1,l_2]. \tag{4.57}$$

By choosing $\varepsilon_2 < \frac{\varepsilon_1}{4a_1}$, we verify that the first eigenvalue of (4.56) $\lambda_1 = \frac{\nu_1^2}{4} + \frac{\pi^2}{l^2} < 1 - a_1 - a_1\varepsilon_2$. Since (4.56) is a gradient system (see [2,23]), as $t \to \infty$, the solution $w(t,x)$ converges to a solution $\theta(x)$ of the following stationary problem:

$$\begin{cases} \theta_{xx} + \nu_1\theta_x = \theta(1-\theta - a_1(1+\varepsilon_2)), & l_1 < x < l_2, \\ \theta(l_1) = \theta(l_2) = 0. \end{cases}$$

Thus, it follows from (4.57) that $\liminf_{t\to\infty} u(t,x) \ge \lim_{t\to} w(t,x) = \theta(x) > 0$ in (l_1,l_2), which is a contradiction to $\|u(t,\cdot)\|_{C([g(t),h(t)])} = 0$. Hence (4.54) is true.

In addition, if $h_0 \ge \pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$, the monotone properties of $g(t)$ and $h(t)$ yield that $h_\infty - g_\infty > 2\pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$, which leads to (4.55) immediately. Since (4.23) holds, applying Theorem 4.3, the proof is completed. □

In the next, we discuss the case of $h_0 < \pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$. We give a sufficient condition for vanishing in (1.3).

Theorem 4.9. *Assume that* (4.23) *holds and $h_0 < \pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$. The initial condition of* (1.3) *satisfies $u(0,x) = u_0(x) = \sigma\phi(x)$ for $x \in [-h_0,h_0]$, where $\phi(x) \in C^2([-h_0,h_0])$, $\phi(\pm h_0) = 0$, $\phi'(-h_0) > 0$, $\phi'(h_0) < 0$, $\phi_0(x) > 0$ in $(-h_0,h_0)$. Then there exists $\sigma^* = \sigma^*(h_0,\phi) \in (0,\infty)$ such that when $0 < \sigma \le \sigma^*$, $h_\infty - g_\infty < \infty$. Moreover, when $h_\infty - g_\infty < \infty$, vanishing happens, and the solution (u,v,g,h) to* (1.3) *satisfies*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = 0,$$
$$\lim_{t\to\infty} v(t,x) = 1 \text{ uniformly in any compact subset of } \mathbb{R}.$$

Proof. Since $v(t,x)>0$ in $(0,\infty)\times\mathbb{R}$, the problem (1.3) implies that

$$\begin{cases} u_t \le u_{xx} - \nu_1 u_x + u(1-u), & t>0,\ g(t)<x<h(t),\\ u=0,\ g'(t)=-\mu u_x(t,g(t)), & t\ge 0,\ x=g(t),\\ u=0,\ h'(t)=-\mu u_x(t,h(t)), & t\ge 0,\ x=h(t),\\ g(0)=-h_0,\ h(0)=h_0, & \\ u(0,x)=u_0(x)=\sigma\phi(x), & x\in[-h_0,h_0]. \end{cases} \tag{4.58}$$

Applying Lemma 4.7 directly, we know that (u,g,h) is a lower solution of

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1-u), & t>0,\ \tilde g(t)<x<\tilde h(t),\\ u=0,\ \tilde g'(t)=-\mu u_x(t,\tilde g(t)), & t\ge 0,\ x=\tilde g(t),\\ u=0,\ \tilde h'(t)=-\mu u_x(t,\tilde h(t)), & t\ge 0,\ x=\tilde h(t),\\ \tilde g(0)=-h_0,\ \tilde h(0)=h_0, & \\ u(0,x)=u_0(x)=\sigma\phi(x), & x\in[-h_0,h_0]. \end{cases} \tag{4.59}$$

Thus,

$$g(t)\ge \tilde g(t),\ \ h(t)\le \tilde h(t),\ \ \text{for } t>0. \tag{4.60}$$

By using Lemma 4.6 to (4.59) with $\alpha=1$ and $\nu=\nu_1$, there exists $\sigma^*=\sigma^*(h_0,\phi)\in(0,\infty]$ such that when $0<\sigma\le\sigma^*$, $\tilde h_\infty-\tilde g_\infty<\infty$. In view of (4.60), by choosing $\sigma^*=\sigma^*(h_0,\phi)$, we obtain that when $0<\sigma\le\sigma^*$, $h_\infty-g_\infty<\infty$. Since (4.23) holds, applying Theorem 4.5, the proof is completed. □

5. Asymptotic spreading speed

In term of Theorem 4.8, we obtain sufficient conditions for spreading. Our aim is to show that when spreading happens, the leftward front $g(t)$ and the rightward front $h(t)$ move at different speeds for large time. To do so, we recall the asymptotic spreading speed for the scalar free boundary problem (4.43). Gu et al. [20] used the phase plane analysis to show that when $\nu<2\sqrt{\alpha}$ the problem (4.43) possesses the fixed leftward and rightward asymptotic spreading speeds.

Lemma 5.1 *(Theorems 1.1 and 1.2 in [20]). Assume that $\nu<2\sqrt{\alpha}$ and (u,g,h) is a global solution of (4.43). Then there exist positive constants c_l^* and c_r^* such that*

$$0<c_l^*:=\lim_{t\to\infty}\frac{-g(t)}{t}<c^*<c_r^*:=\lim_{t\to\infty}\frac{h(t)}{t},$$

where c_l^* and c_r^* depend on μ, ν, α and β; c^* is the asymptotic spreading speed of the logistic free boundary problem without the advection, which is given as k_0 in Proposition 4.1 of Du and Lin [8] or c^* in Theorem 1.10 of Du and Lou [10]. If μ, ν, and β are fixed, then c_l^*, c^* and c_r^* are strictly increasing in α. Moreover,

(i) If $\nu\in(0,2\sqrt{\alpha})$ is fixed, then c_l^*, c^* and c_r^* are strictly increasing in μ, and

$$\lim_{\mu\to 0}c_l^*=0,\ \ \lim_{\mu\to\infty}c_l^*=2\sqrt{\alpha}-\nu,$$

$$\lim_{\mu\to 0} c^* = 0, \quad \lim_{\mu\to\infty} c^* = 2\sqrt{\alpha},$$

$$\lim_{\mu\to 0} c_r^* = 0, \quad \lim_{\mu\to\infty} c_r^* = 2\sqrt{\alpha} + \nu.$$

(ii) If μ is fixed, then $-c_l^*$ and c_r^* are strictly increasing in ν, and

$$\lim_{\nu\to 0} c_l^* = \lim_{\nu\to 0} c_r^* = c^*, \quad \lim_{\nu\to 2\sqrt{\alpha}} c_l^* = 0.$$

Theorem 5.2. *Assume that (4.23) and $h_0 \geq \pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$. (u, v, g, h) is a global solution of (1.3). Then there exist positive constants c_l^*, c_r^*, c_l^{**}, and c_r^{**} such that*

$$\begin{aligned} 0 < c_r^{**} \leq \liminf_{t\to\infty} \frac{h(t)}{t} \leq \limsup_{t\to\infty} \frac{h(t)}{t} \leq c_r^*, \\ 0 < c_l^{**} \leq \liminf_{t\to\infty} \frac{-g(t)}{t} \leq \limsup_{t\to\infty} \frac{-g(t)}{t} \leq c_l^*. \end{aligned} \tag{5.1}$$

where c_l^*, c_r^*, c_l^{**}, and c_r^{**} depend on μ, ν_1 and a_1. Moreover, c_l^* and c_r^* are the leftward and rightward asymptotic spreading speeds of the scalar free boundary problem (5.2); c_l^{**} and c_r^{**} are the leftward and rightward asymptotic spreading speeds of the scalar free boundary problem (5.6).

Proof. We first estimate the upper bound of the asymptotic spreading speed. Similar as in Theorem 4.9, we have

$$\begin{cases} u_t \leq u_{xx} - \nu_1 u_x + u(1-u), & t > 0,\ g(t) < x < h(t), \\ u = 0,\ g'(t) = -\mu u_x(t, g(t)), & t \geq 0,\ x = g(t), \\ u = 0,\ h'(t) = -\mu u_x(t, h(t)), & t \geq 0,\ x = h(t), \\ g(0) = -h_0,\ h(0) = h_0, & \\ u(0, x) = u_0(x) > 0, & x \in [-h_0, h_0]. \end{cases}$$

Let $(u, \tilde{g}, \tilde{h})$ be a solution of

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1-u), & t > 0,\ \tilde{g}(t) < x < \tilde{h}(t), \\ u = 0,\ \tilde{g}'(t) = -\mu u_x(t, \tilde{g}(t)), & t \geq 0,\ x = \tilde{g}(t), \\ u = 0,\ \tilde{h}'(t) = -\mu u_x(t, \tilde{h}(t)), & t \geq 0,\ x = \tilde{h}(t), \\ \tilde{g}(0) = -h_0,\ \tilde{h}(0) = h_0, & \\ u(0, x) = u_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{5.2}$$

By Lemma 4.7, we obtain

$$g(t) \geq \tilde{g}(t),\ h(t) \leq \tilde{h}(t),\ \text{for } t > 0.$$

Due to (4.23), $\nu_1 < 2\sqrt{1-a_1}$ implies that $\nu_1 < 2$. Applying Lemma 5.1 to (5.2) yields that there exist positive constants c_l^* and c_r^* such that

$$0 < c_l^* = \lim_{t\to\infty} \frac{-\tilde{g}(t)}{t} < c_r^* = \lim_{t\to\infty} \frac{\tilde{h}(t)}{t}.$$

Hence, we deduce that

$$\limsup_{t\to\infty} \frac{-g(t)}{t} \le c_l^*, \ \limsup_{t\to\infty} \frac{h(t)}{t} \le c_r^*. \tag{5.3}$$

We next deal with the estimate of the lower bound of the asymptotic spreading speed. Applying the comparison principle directly to (1.3), we have $\limsup_{t\to\infty} v(t,x) \le 1$ in $\mathbb{R}$. Thus for any given $0 < \varepsilon << 1$, there exists $T_\varepsilon > 0$ such that

$$v(t,x) \le 1+\varepsilon, \ \text{for } t \ge T_\varepsilon, \ x \in \mathbb{R}.$$

We find that u satisfies

$$\begin{cases} u_t \ge u_{xx} - \nu_1 u_x + u(1-u-a_1-a_1\varepsilon), & t > T_\varepsilon, \ g(t) < x < h(t), \\ u = 0, \ g'(t) = -\mu u_x(t, g(t)), & t \ge T_\varepsilon, \ x = g(t), \\ u = 0, \ h'(t) = -\mu u_x(t, h(t)), & t \ge T_\varepsilon, \ x = h(t), \\ g(0) := g(T_\varepsilon) < -h_0, \ h(0) := h(T_\varepsilon) > h_0, & \\ u(T_\varepsilon, x) > 0, & x \in [g(T_\varepsilon), h(T_\varepsilon)]. \end{cases}$$

Due to (4.23), $\nu_1 < 2\sqrt{1-a_1}$ leads to $\frac{4(1-a_1)-\nu_1^2}{a_1} > 0$. By choosing $\varepsilon < \frac{4(1-a_1)-\nu_1^2}{a_1}$, we can verify that $\nu_1 < 2\sqrt{1-a_1-a_1\varepsilon}$. Applying Lemma 5.1 to

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1-u-a_1-a_1\varepsilon), & t > 0, \ \hat{g}(t) < x < \hat{h}(t), \\ u = 0, \ \hat{g}'(t) = -\mu u_x(t, \hat{g}(t)), & t \ge 0, \ x = \hat{g}(t), \\ u = 0, \ \hat{h}'(t) = -\mu u_x(t, \hat{h}(t)), & t \ge 0, \ x = \hat{h}(t), \\ \hat{g}(0) = -h_0, \ \hat{h}(0) = h_0, & \\ u(0,x) = u_0(x), & x \in [-h_0, h_0], \end{cases} \tag{5.4}$$

we know that there exist positive constants c_l^ε and c_r^ε such that

$$0 < c_l^\varepsilon := \lim_{t\to\infty} \frac{-\hat{g}(t)}{t} < c_r^\varepsilon := \lim_{t\to\infty} \frac{\hat{h}(t)}{t}.$$

In order to ensure that (5.4) is a lower solution, we need to choose the initial condition $u_0(x) \le u(T_\varepsilon, x)$. Hence, using Lemma 4.7, we have

$$g(t) \le \hat{g}(t), \ h(t) \ge \hat{h}(t), \ \text{for } t > 0.$$

Hence, we deduce that

$$\liminf_{t\to\infty} \frac{-g(t)}{t} \ge c_l^\varepsilon > 0, \ \liminf_{t\to\infty} \frac{h(t)}{t} \ge c_r^\varepsilon > 0. \tag{5.5}$$

In term of Lemma 5.1, c_l^ε and c_r^ε are monotonic decreasing in ε. By letting $\varepsilon \to 0$, we have $c_l^\varepsilon \to c_l^{**}$ and $c_r^\varepsilon \to c_r^{**}$, here c_l^{**} and c_r^{**} are the leftward and rightward asymptotic spreading speeds of the following free boundary problem:

$$\begin{cases} u_t = u_{xx} - \nu_1 u_x + u(1 - u - a_1), & t > 0,\ \hat{g}(t) < x < \hat{h}(t), \\ u = 0,\ \hat{g}'(t) = -\mu u_x(t, \hat{g}(t)), & t \geq 0,\ x = \hat{g}(t), \\ u = 0,\ \hat{h}'(t) = -\mu u_x(t, \hat{h}(t)), & t \geq 0,\ x = \hat{h}(t), \\ g(0) = -h_0,\ h(0) = h_0, & \\ u(0,x) = u_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{5.6}$$

Therefore, we obtain

$$\liminf_{t\to\infty} \frac{-g(t)}{t} \geq c_l^{**} > 0,\ \ \liminf_{t\to\infty} \frac{h(t)}{t} \geq c_r^{**} > 0. \tag{5.7}$$

Combining (5.3) and (5.7), we get (5.1), which completes the proof. □

6. Numerical simulations

In this section, we provide numerical computations of problem (1.3) by means of an implicit finite difference scheme [40], using Crank–Nicholson method for time integration and Adams–Bashforth scheme for the nonlinear operator. We take the advection coefficients as $\nu_1 = \nu_2 = 0.08164$, which means that both two mosquito species move 0.08164 km per day (see more biological interpretations in [44]). Consider initial functions as follows:

$$u_0(x) = \begin{cases} 0.1 \cos \frac{\pi x}{2h_0}, & x \in [-h_0, h_0], \\ 0, & x \notin [-h_0, h_0], \end{cases} \qquad v_0(x) = 0.1 + 0.05 \cos \frac{\pi x}{2h_0},\ x \in \mathbb{R}. \tag{6.1}$$

We give three examples to explain the three possible competitive interactions between the two mosquito species.

(i) Weak-strong competition. Theorem 3.3 indicates that in the case of weak-strong competition the free boundaries will vanish. We take the following parameter values of model (1.3):

$$r = 1, a_1 = 1.2, a_2 = 0.05, \mu = 2. \tag{6.2}$$

It is easy to see from Fig. 2 that the density u of the invasive *Ae. albopictus* mosquitoes disappears gradually and the density v of the local *Ae. aegypti* mosquitoes converges to a homogeneous steady state. Moreover, the two free boundaries of u are limited in a bounded interval when the time increases.

(ii) Weak competition with big initial habitat. Theorem 4.8 implies that in the case of weak competition the free boundaries will spread if the initial habitat is big. We take the following parameter values of model (1.3):

$$r = 1, a_1 = 0.08, a_2 = 0.05, \mu = 2. \tag{6.3}$$

In view of Theorem 4.8, we compute the threshold size of the initial habitat as $h_0 = 2\pi \left(\sqrt{4 - \nu_1^2}\right)^{-1} = 0.7867$. Take $h_0 = 100 > 0.7867$ such that the initial habitat is so big that

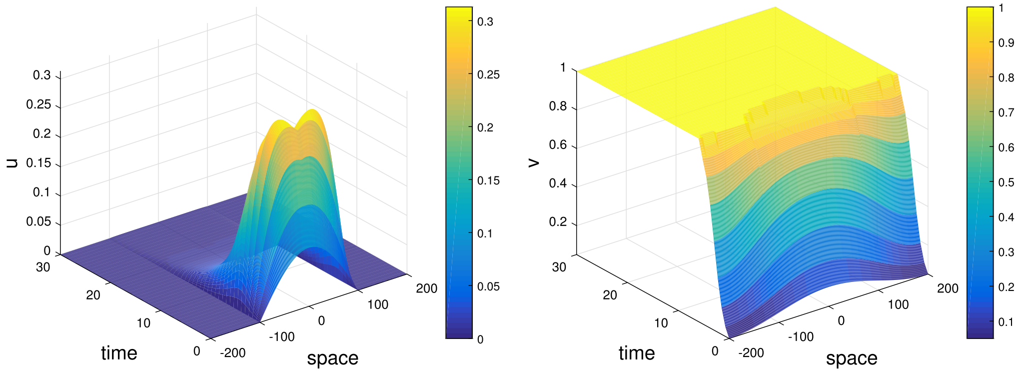

Fig. 2. The long time behaviors of u and v for the weak-strong competition case. Here $h_0 = 100$, and other parameters are given in (6.2). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

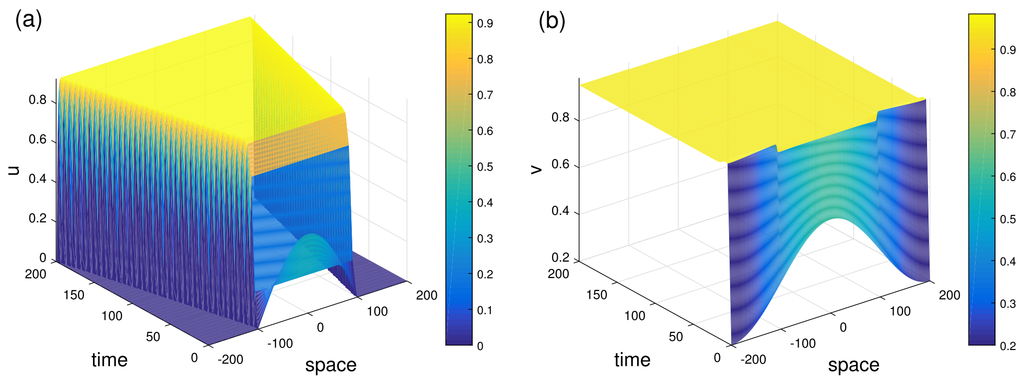

Fig. 3. The long time behaviors of u and v for the weak competition with big initial habitat case. Here $h_0 = 100$, and other parameters are given in (6.3).

the free boundaries spread while (u, v) converges locally uniformly to $(\frac{r(1-a_1)}{r-a_1a_2}, \frac{r-a_2}{r-a_1a_2}) = (0.9237, 0.9538)$. We can see from Fig. 3 that the density u of the invasive *Ae. albopictus* species converges locally uniformly to a homogeneous steady state 0.9237 and the density v of the local *Ae. aegypti* species converges to a homogeneous steady state 0.9538. Moreover, the two free boundaries of u increase slowly and tend to some finite values in a long run.

(iii) Weak competition with small initial habitat. Theorem 4.9 shows that in the case of weak competition the free boundaries will vanish if the initial habitat is small. We take the same parameters as in (6.3). According to Theorem 4.9, if $h_0 < 0.7867$ the free boundaries vanish while the long time behavior of (u, v) is local uniform convergence to $(0, 1)$. Take $h_0 = 0.75$. We can see from Fig. 4 that the density u of the invasive *Ae. albopictus* mosquitoes disappears gradually and the density v of the local *Ae. aegypti* mosquitoes converges to a homogeneous steady state 1. Moreover, the two free boundaries of u are limited in a bounded interval with the time increasing.

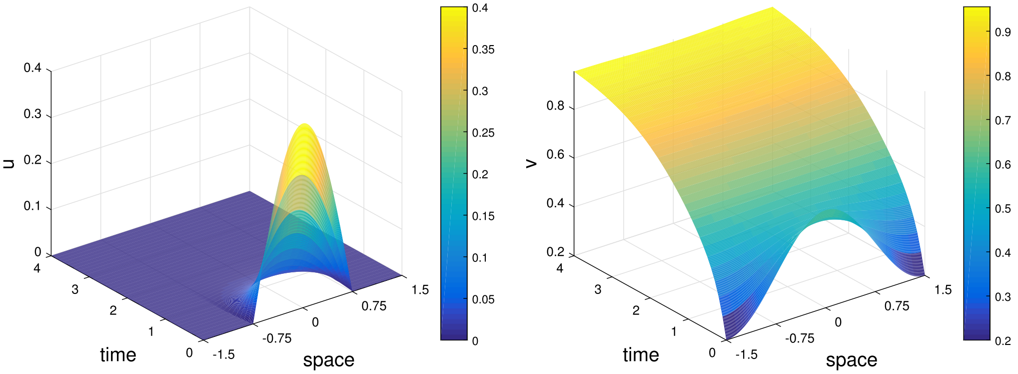

Fig. 4. The long time behaviors of u and v for the weak competition with small initial habitat case. Here $h_0 = 0.75$, and other parameters are given in (6.3).

7. Discussions

Invasions by insect vectors of human diseases such as mosquitoes have profound effects on global public health (Lounibos [33]). *Ae. aegypti* and *Ae. albopictus* mosquitoes are two prominent transmitters of dengue fever virus, chikungunya virus, yellow fever virus, Zika virus, etc. Understanding the dispersal and invasive behavior of *Aedes* mosquitoes is essential in implementing vector control strategies and preventing and controlling mosquito-borne diseases. *Ae. aegypti* mosquito is an invasive domestic species with tropical and subtropical worldwide distribution and *Ae. albopictus* is a most recent invasive species that has spread recently to many countries. After arriving the U.S. in 1983 (Reiter and Darsie [41]), *Ae. albopictus* mosquitoes have been competing with *Ae. aegypti* mosquitoes, coexisting with *Ae. aegypti* where *Ae. aegypti* present, and spreading beyond the boundaries of *Ae. aegypti*'s habitats (see Fig. 1). Our competition model with free boundary (1.3) can be applied to model the invasion of *Ae. albopictus* and the competition between *Ae. aegypti* and *Ae. albopictus*.

In view of Theorem 3.3, in the case of weak-strong competition when the local *Ae. aegypti* wins, the invasive *Ae. albopictus* will eventually vanish and the habitat of *Ae. albopictus* is confined to a finite region. By Theorems 4.8 and 4.9, we see that in the case of weak competition, when the two advection coefficients are less than some fixed constants (satisfying (4.23)), the invasive *Ae. albopictus* may spread over the whole space or vanish. When the size of initial habitat is larger than a fixed constant (satisfying $h_0 \geq \pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$), the invasive *Ae. albopictus* will spread over the whole space, and the two subspecies of mosquitoes will locally uniformly converge to the interior equilibrium. When the size of initial habitat is less than a fixed constant (satisfying $h_0 < \pi\left(\sqrt{4-\nu_1^2}\right)^{-1}$) and if the initial value is small, the invasive *Ae. albopictus* will vanish and the local *Ae. aegypti* will uniformly converge to 1. In term of Theorem 5.2, in the case of spread occurring with weak competition, we find that the asymptotic spreading speed of the leftward front is confined to a finite interval and as well the rightward front.

It is known that temperature, humidity and rainfall impact adult *Aedes* mosquito survival and availability of oviposition sites. It will be interesting to study the effect of climate change on the dynamics of advection–reaction–diffusion competition models with free boundary.

Acknowledgments

The authors would like to thank the anonymous referee for the helpful comments and suggestions.

References

[1] F. Belgacem, C. Cosner, The effect of dispersal along environmental gradients on the dynamics of populations in heterogeneous environment, Can. Appl. Math. Q. 3 (1995) 379–397.
[2] P. Brunovsky, S.N. Chow, Generic properties of stationary state solutions of reaction–diffusion equations, J. Differential Equations 53 (1984) 1–23.
[3] G. Bunting, Y. Du, K. Krakowski, Spreading speed revisited: analysis of a free boundary model, Netw. Heterog. Media 7 (2012) 583–603.
[4] R.S. Cantrell, C. Cosner, Spatial Ecology via Reaction–Diffusion Equations, John Wiley & Sons Ltd., Chichester, UK, 2003.
[5] J.-F. Cao, W.-T. Li, F.-Y. Yang, Dynamics of a nonlocal SIS epidemic model with free boundary, Discrete Contin. Dyn. Syst. Ser. B 22 (2017) 247–266.
[6] Centers for Disease Control and Prevention (CDC), Potential Range in US, https://www.cdc.gov/zika/vector/range.html, 2017, Last updated: January 19.
[7] X. Chen, A. Friedman, A free boundary problem arising in a model of wound healing, SIAM J. Math. Anal. 32 (2000) 778–800.
[8] Y. Du, Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal. 42 (2010) 377–405.
[9] Y. Du, Z. Lin, The diffusive competition model with a free boundary: invasion of a superior or inferior competitor, Discrete Contin. Dyn. Syst. Ser. B 19 (2014) 3015–3132.
[10] Y. Du, B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, J. Eur. Math. Soc. 17 (10) (2015) 2673–2724.
[11] Y. Du, B. Lou, M. Zhou, Nonlinear diffusion problems with free boundaries: convergence, transition speed and zero number arguments, SIAM J. Math. Anal. 47 (5) (2015) 3555–3584.
[12] Y. Du, L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions, J. Lond. Math. Soc. 64 (2001) 107–124.
[13] Y. Du, H. Matano, K. Wang, Regularity and asymptotic behavior of nonlinear Stefan problems, Arch. Ration. Mech. Anal. 212 (2014) 957–1010.
[14] Y. Du, H. Matsuzawa, M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, SIAM J. Math. Anal. 46 (2014) 375–396.
[15] Y. Du, M. Wang, M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, J. Math. Pures Appl. 107 (2017) 253–287.
[16] M. Enserink, Entomology: a mosquito goes global, Science 320 (2008) 864–866.
[17] J.H. Frank, Recycling of discarded tires for control of *Aedes aegypti*, J. Fla. Anti-Mosq. Assoc. 52 (1981) 44–48.
[18] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, N.J., 1964.
[19] H. Gu, Z. Lin, B. Lou, Long time behavior of solutions of a diffusion–advection logistic model with free boundaries, Appl. Math. Lett. 37 (2014) 49–53.
[20] H. Gu, Z. Lin, B. Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries, Proc. Amer. Math. Soc. 143 (2015) 1109–1117.
[21] H. Gu, B. Lou, M. Zhou, Long time behavior of solutions of Fisher-KPP equation with advection and free boundaries, J. Funct. Anal. 269 (2015) 1714–1768.
[22] J.-S. Guo, C.-H. Wu, On a free boundary problem for a two-species weak competition system, J. Dynam. Differential Equations (2012) 1–23.
[23] J.K. Hale, R. Massatt, Asymptotic Behavior of Gradient-Like Systems, Univ. Fla. Symp. Dyn. Sys., vol. II, Academic Press, New York, 1982.
[24] D. Hilhorst, M. Iida, M. Mimura, H. Ninomiya, A competition–diffusion system approximation to the classical two-phase Stefanproblem, Jpn. J. Ind. Appl. Math. 18 (2001) 161–180.
[25] D. Hilhorst, M. Mimura, R. Schätzle, Vanishing latent heat limit in a Stefan-like problem arising in biology, Nonlinear Anal. Real World Appl. 4 (2003) 261–285.

[26] J.H. Hohbs, E.A. Hughes, B.H. Eichold II, Replacement of *Aedes aegypti* by *Aedes alhopictus* in Mobile, Alabama, J. Am. Mosq. Control Assoc. 7 (1991) 488–489.

[27] Y. Kan-On, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, Nonlinear Anal. 28 (1997) 145–164.

[28] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, RI, 1968.

[29] C. Lei, Z. Lin, Q. Zhang, The spreading front of invasive species in favorable habitat or unfavorable habitat, J. Differential Equations 257 (2014) 145–166.

[30] Z. Lin, A free boundary problem for a predator-prey model, Nonlinearity 20 (2007) 1883–1892.

[31] Z. Lin, H. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, J. Math. Biol. 75 (2017) 1381–1409.

[32] X. Liu, B. Lou, Long time behavior of solutions of a reaction–diffusion equation with Robin and free boundary conditions, J. Differential Equations 259 (2015) 423–453.

[33] L.P. Lounibos, Invasions by insect vectors of human disease, Annu. Rev. Entomol. 47 (2002) 233–266.

[34] Y. Morita, K. Tachibana, An entire solution to the Lotka-Volterra competition–diffusion equations, SIAM J. Math. Anal. 40 (2009) 2217–2240.

[35] H.B. Morlan, M.E. Tinker, Distribution of *Aedes aegijpti* infestations in the United States, Am. J. Trop. Med. Hyg. 14 (1965) 892–899.

[36] B.H. Noden, P.A. O'Neal, J.E. Fader, S.A. Juliano, Impact of inter- and intra-specific competition among larvae on larval, adult, and life-table traits of *Aedes aegypti* and *Aedes albopictus* females, Ecol. Entomol. 41 (2016) 192–200.

[37] G.F. O'Meara, L.F. Evans Jr., A.D. Gettman, J.P. Cuda, Spread of *Aedes albopictus* and decline of *Ae. aegypti* (Diptera: Culicidae) in Florida, J. Med. Entomol. 32 (1995) 554–562.

[38] G.F. O'Meara, A.D. Gettman, L.F. Evans Jr., F.D. Scheel, Invasion of cemeteries in Florida by *Aedes albopictus*, J. Am. Mosq. Control Assoc. 8 (1992) 1–10.

[39] B.E. Peacock, J.P. Smith, P.G. Gregory, T.M. Loyless, J.A. Mulrennen Jr., P.R. Simmonds, L. Padgett Jr., E.K. Cook, T.R. Eddins, *Aedes albopictus* in Florida, J. Am. Mosq. Control Assoc. 4 (1988) 362–365.

[40] S. Razvan, D. Gabriel, Numerical approximation of a free boundary problem for a predator-prey model, Numer. Anal. Appl. 5434 (2009) 548–555.

[41] P. Reiter, R.F. Darsie, *Aedes albopictus* in Memphis, Tennessee (USA): an achievement of modern transportation?, Mosq. News 44 (3) (1984) 296–399.

[42] I. Rochlin, D.V. Ninivaggi, M.L. Hutchinson, A. Farajollahi, Climate change and range expansion of the Asian tiger mosquito (*Aedes albopictus*) in Northeastern USA: implications for public health practitioners, PLoS ONE 8 (4) (2013) e60874, https://doi.org/10.1371/journal.pone.0060874.

[43] H.L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Math. Surveys Monogr., vol. 41, Amer. Math. Soc., Providence, RI, 1995.

[44] L.T. Takahashi, N.A. Maidana, W.C. Ferreira Jr, P. Pulino, H.M. Yang, Mathematical models for the *Aedes aegypti* dispersal dynamics: travelling waves by wing and wind, Bull. Math. Biol. 67 (2005) 509–528.

[45] M.M. Tang, P.C. Fife, Propagating front for competing species equations with diffusion, Arch. Ration. Mech. Anal. 73 (1980) 69–77.

[46] C. Tian, S. Ruan, A free boundary problem of *Aedes aegypti* mosquito invasion, Appl. Math. Model. 46 (2017) 203–217.

[47] M. Wang, On some free boundary problems of the prey-predator model, J. Differential Equations 256 (2014) 3365–3394.

[48] M. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment, J. Funct. Anal. 270 (2016) 483–508.

[49] M. Wang, J. Zhao, Free boundary problems for a Lotka-Volterra competition system, J. Dynam. Differential Equations 26 (2014) 655–672.

[50] M. Wang, J. Zhao, A free boundary problem for the predator-prey model with double free boundaries, J. Dynam. Differential Equations (2015) 1–23.

[51] C.-H. Wu, Spreading speed and traveling waves for a two-species weak competition system with free boundary, Discrete Contin. Dyn. Syst. Ser. B 18 (2013) 2441–2455.

[52] G. Zhao, S. Ruan, Time periodic traveling wave solutions for periodic advection–reaction–diffusion systems, J. Differential Equations 257 (2014) 1078–1147.
[53] Y. Zhao, M. Wang, Free boundary problems for the diffusive competition system in higher dimension with sign-changing coefficients, IMA J. Appl. Math. 81 (2015) 255–280.