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Existence of periodic solutions in abstract semilinear equations and applications to biological models ☆

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Abstract

In this paper, we study the existence of mild periodic solutions of abstract semilinear equations in a setting that includes several other types of equations such as delay differential equations, first-order hyperbolic partial differential equations, and reaction-diffusion equations. Under different assumptions on the linear operator and the nonhomogeneous function, sufficient conditions are derived to ensure the existence of mild periodic solutions in the abstract semilinear equations. When the semigroup generated by the linear operator is not compact, Banach fixed point theorem is used whereas when the semigroup generated by the linear operator is compact, Schauder fixed point theorem is employed. In applications, we apply the main results to establish the existence of periodic solutions in delayed red-blood cell models, age-structured models with periodic harvesting, and the diffusive logistic equation with periodic coefficients.

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1. Introduction

The existence of periodic solutions is a fundamental property in all types of differential equations. One basic and important result on this topic is Massera Theorem. In 1950, Massera [29] studied the existence of periodic solutions for the following ordinary differential equation

$$\frac{du}{dt} = f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and periodic in t. He proved that the existence of periodic solutions of equation (1.1) is equivalent to the existence of a bounded solution on $\mathbb{R}_+$ of equation (1.1). Important facts, results and references on periodic solutions of ordinary differential equations can be found in Yoshizawa [44] and Farkas [14].

The problem on the existence of periodic solutions for differential equations in infinite dimensional spaces has been investigated in various directions. One of them is to generalize Massera Theorem to infinite dimensional systems. In fact, Massera and Schäffer [30] studied the relationship between the periodic solutions and bounded solutions for the linear equation

$$\frac{du}{dt} = A(t)u + f(t), \quad t \geq 0 \tag{1.2}$$

in infinite dimensional spaces. Chow [7] and Chow and Hale [8] established the existence of periodic solutions under the existence of bounded solutions for the nonhomogeneous linear functional differential equation

$$\frac{du}{dt} = L(t, u_t) + f(t), \tag{1.3}$$

where $u_t \in C_r := C([-r, 0], \mathbb{R})$, $L : (-\infty, +\infty) \times C_r \to \mathbb{R}$ is continuous, linear with respect to the second argument and T-periodic in t, $T \geq r$, and f is continuous and T-periodic. They proved that Massera Theorem holds for equation (1.3) by showing that the Poincaré map defined by $P : \varphi \to u_T(\cdot, \varphi, f)$, where $u_T(\cdot, \varphi, f)$ is the unique mild solution of equation (1.3) initiated at φ, has a fixed point. Further results on periodic solutions of functional differential equations can be found in Hale and Verduyn Lunel [16], Diekmann et al. [10], and Burton [4].

Prüss [36] investigated the following abstract semilinear equation

$$\frac{du}{dt} = Au(t) + F(t, u(t)), \quad t \geq 0 \tag{1.4}$$

in a Banach space X, under the condition that A generates a C_0-semigroup $\{U(t)\}_{t\geq 0}$ of type (M, ω), the domain $D(A)$ is closed, bounded and convex, and F is continuous and T-periodic in t. By constructing a Poincaré map and using Schauder's fixed point theorem and k-set contraction argument, he proved the existence of mild T-periodic solutions of (1.4) when $U(t)$ is compact for $t > 0$ or $\omega < 0$ and F is compact. By applying Horn's fixed point theorem to the Poincaré map, Liu [23] and Ezzinbi and Liu [13] established the existence of bounded and ultimate bounded solutions of evolution equations with or without delay, implying the existence of periodic solutions. Benkhalti and Ezzinbi [3] and Kpoumiè et al. [20] proved that under some conditions, the existence of a bounded solution for some nondensely defined nonautonomous partial functional differential equations implies the existence of periodic solutions. The approach was to construct a map on the space of T-periodic functions from the corresponding nonhomogeneous linear equation and use a fixed-point theorem concerning set-valued maps to prove the existence of a fixed point for this map. Li et al. [22] proved several Massera-type criteria for linear periodic evolution equations with delay and applied the results to nonlinear evolution equations, functional and partial differential equations.

For the abstract semilinear evolution equation

$$\frac{du}{dt} = A(t)u(t) + F(t, u(t)), \quad t \geq 0 \tag{1.5}$$

in a Banach space X, Nguyen and Ngo [32,33] assumed that $A(t)$ is T-periodic, F is T-periodic in t and satisfies the φ-Lipschitz condition $\|F(t, x_1) - F(t, x_2)\| \leq \varphi(t)\,\|x_1 - x_2\|$ for $\varphi(t)$ being a real and positive function belonging to an admissible function space. They proved the existence of periodic solutions to (1.5) in the case that the family $\{A(t)\}_{t\geq 0}$ generates a strongly continuous, exponentially bounded evolution family. They started with the linear equation (1.2) and used the Cesàro limit to prove the existence of periodic solutions. Then they constructed a map from periodic solutions of (1.2) and used the admissibility of function spaces combined with the Banach fixed point argument to prove the existence of a unique fixed point of the constructed map. The existence and uniqueness of a periodic solution of (1.5) follows from the existence and uniqueness of the fixed point. Naito et al. [31] developed a decomposition technique to prove the existence of periodic solutions to periodic evolution equations in the form of (1.2). Vrabie [40] studied the existence of periodic mild solutions to nonlinear evolution inclusions that include equation (1.4).

In this paper, we study the existence of mild periodic solutions of the abstract semilinear equation (1.4) and abstract semilinear evolution equation (1.5) in a setting that includes several types of equations such as delay differential equations, first-order hyperbolic partial differential equations, and reaction-diffusion equations. We consider the general cases where A is a linear operator on X (not necessarily densely defined) satisfying the Hille-Yoshida condition and $A(t)$ is a T-periodic linear operators on X satisfying the hyperbolic conditions $(A1)$-$(A3)$ introduced by Tanaka [37,38], which will be specified later. In section 2, we recall some preliminary results on semigroups generated by a Hille-Yoshida operator, the evolution family and the existence theorem of solutions of nonhomogeneous linear equations (2.1) and (2.6). In section 3, we start with the linear equations (2.1) and (2.6) to show the existence of mild periodic solutions, whose initial value is controlled by the norm of the input function $f(t)$. Using this result and the fixed point argument, we prove the existence of mild periodic solutions of (1.4) and (1.5) under some assumptions on F. At the end of section 3, we also discuss the case where the semigroup $\{U(t)\}_{t\geq 0}$ generated by A in (1.4) is compact for $t > 0$ and give existence theorem of mild periodic solutions of (1.4). The approach is also to start with the linear equation (2.1) to show the existence of mild periodic solutions of it and use this result combined with the Schauder's fixed point theorem to prove the existence of mild periodic solutions of (1.4). In section 4 we use the main results of this paper to discuss the existence of periodic solutions in three types of biological models, delayed periodic red-blood cell models, age-structured models with periodic harvesting, and diffusive logistic models with periodic coefficients.

2. Preliminary results

In this section, we consider the nonhomogeneous linear Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = Au(t) + f(t), \quad t \geq 0, \\ u(0) = x \in \overline{D(A)}, \end{cases} \tag{2.1}$$

where the linear operator A is densely or non-densely defined in a Banach space X, the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and T-periodic.

First we make the following assumptions.

Assumption 2.1.

(a) $A : D(A) \subset X \to X$ is a linear operator and there exist real constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $\left\| (\lambda I - A)^{-n} \right\| \leq \frac{M}{(\lambda - \omega)^n}$ for $n \geq 1$ and $\lambda > \omega$;

(b) $x \in X_0 = \overline{D(A)}$;

(c) $f : [0, \infty) \to X$ is continuous.

A linear operator $A : D(A) \subset X \to X$ satisfying Assumption 2.1 (a) is called *a Hille-Yosida operator*.

Remark 2.2. Note that the renorming lemma (Lemma 5.1 in Pazy [35]) holds. By exactly the same argument as in [35], we see that if $\left\| (\lambda I - A)^{-n} \right\| \leq \frac{M}{(\lambda - \omega)^n}$ for $n \geq 1$ and $\lambda > \omega$, then there exists a norm $|.|$ on X which is equivalent to the original norm $\|.\|$ on X and satisfies $\|x\| \leq |x| \leq M \|x\|$ for $x \in X$ and $\left| (\lambda I - A)^{-n} \right| \leq \frac{1}{(\lambda - \omega)^n}$ for $n \geq 1$ and $\lambda > \omega$. That is, without loss of generality, M can be chosen to be 1.

Definition 2.3. A continuous function $u : [0, \infty) \to X$ is called a *mild (or an integrated) solution* to (2.1) if $\int_0^t u(s)ds \in D(A)$ and

$$u(t) = x + A \int_0^t u(s)ds + \int_0^t f(s)ds \tag{2.2}$$

for all $t \geq 0$ (see Thieme [39]).

The existence theorem for (2.1) is as follows:

Theorem 2.4 *(Da Prato and Sinestrari [9]). Under Assumption 2.1, there exists a unique mild solution to (2.1) with value in $X_0 = \overline{D(A)}$. Moreover, u satisfies the estimate*

$$\|u(t)\| \leq Me^{\omega t} \|x\| + \int_0^t Me^{\omega t} \|f(s)\| ds \tag{2.3}$$

for all $t \geq 0$.

If $\overline{D(A)} \neq X$; that is, A is nondensely defined, let $X_0 = \overline{D(A)}$. If $f(t) = 0$, then the family of operators $\{U_A(t)\}_{t \geq 0}$ with $U_A(t) : X_0 \to X_0$, $t \geq 0$, defined by $U_A(t)x = u(t)$ for all $t \geq 0$ is the C_0-semigroup generated by A_0, the part of A in X_0. For the rest of the article, we denote by $\{U_A(t)\}_{t \geq 0}$ the semigroup generated by A_0. Moreover, if u is a solution of (2.2) we have the approximation formula (see Magal and Ruan [26,27])

$$u(t) = U_A(t)x + \lim_{\lambda \to +\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} f(s)ds. \tag{2.4}$$

Kato [19] initiated a study on the evolution family of solutions of the hyperbolic linear evolution Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t), & t \geq s \\ u(s) = x \in X \end{cases} \tag{2.5}$$

in a Banach space X. To recall some results about the linear evolution Cauchy problem (2.5), we make the following assumptions.

Assumption 2.5.

(A1) $D(A(t)) := D$ is independent of t and not necessarily densely defined;
(A2) The family $\{A(t)\}_{t \geq 0}$ is stable in the sense that there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t))$ for $t \in [0, \infty)$ and

$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^k}$$

for $\lambda > \omega$ and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \leq t_1 \leq t_2 \leq ... \leq t_k$ and $k = 1, 2, \ldots$;
(A3) The mapping $t \to A(t)x$ is continuously differentiable in X for each $x \in D$.

Note that the linear evolution Cauchy problem (2.5) is called *hyperbolic* if the linear time-dependent operator $A(t)$ satisfies the *hyperbolic conditions* $(A1)$-$(A3)$ in Assumption 2.5. Now we recall some classical results due to Kato [19].

Theorem 2.6 *(Kato [19]). Let $\{A(t), D(A(t))\}_{t \geq 0}$ be a family of linear operators on a Banach space X satisfying Assumption 2.5 such that D is dense in X. Then the Cauchy problem (2.5) is well-posed and the family of operators $\{A(t)\}_{t \geq 0}$ generates an evolution family $\{U(t, s)\}_{t \geq s \geq 0}$. Moreover, for $x \in D$ the map $t \to U(t, s)x$ is the unique continuous function which solves the Cauchy problem (2.5).*

For $\lambda > 0$, $0 \leq s \leq t$ and $x \in \overline{D}$, set

$$U_\lambda(t, s)x = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x.$$

Theorem 2.7 *(Tanaka [38], Kpoumiè et al. [20]). Let $\{A(t)\}_{t\geq 0}$ be a family of linear operators on a Banach space X satisfying Assumption 2.5. If $x \in D$ satisfies the condition that $A(s)x \in \overline{D}$, then there exists an evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ defined on $\overline{D}$ by $U(t,s)x = \lim_{\lambda\to 0^+} U_\lambda(t,s)x$ uniformly for $x \in \overline{D}$ and satisfying:*

(i) $U(t,s)D(s) \subset D(t)$ for all $0 \leq s \leq t$, where $D(t) := \{x \in D : A(t)x \in \overline{D}\}$;
(ii) for all $x \in D(s)$ and $t \geq s$, the mapping $t \mapsto U(t,s)x$ is continuous in D;
(iii) for all $x \in D(s)$ and $t \geq s$, the mapping $t \mapsto U(t,s)x$ is continuously differentiable with

$$\partial_t U(t,s)x = A(t)U(t,s)x$$

and

$$\partial_s^+ U(t,s)x = -U(t,s)A(s)x.$$

Theorem 2.8 *(Oka and Tanaka [34], Tanaka [38], Kpoumiè et al. [20]). Assume that $\{A(t)\}_{t\geq 0}$ satisfies Assumption 2.5. Then the limit*

$$U(t,s)x = \lim_{\lambda\to 0^+} U_\lambda(t,s)x$$

exists for $x \in \overline{D}$, $0 \leq s \leq t$, where the convergence is uniform on $\Gamma := \{(t,s) : 0 \leq s \leq t\}$. Moreover, the family $\{U(t,s) : (t,s) \in \Gamma\}$ satisfies the following properties:

(i) For $x \in \overline{D}$, $\lambda > 0$ and $0 \leq s \leq r \leq t$, one has

$$U_\lambda(t,t)x = x$$

and

$$U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x;$$

(ii) $U(t,s) : \overline{D} \to \overline{D}$ for $(t,s) \in \Gamma$;
(iii) $U(t,t)x = x$ and $U(t,s)x = U(t,r)U(r,s)x$ for $x \in \overline{D}$ and $0 \leq s \leq r \leq t$;
(iv) the mapping $(t,s) \to U(t,s)x$ is continuous on Γ for any $x \in \overline{D}$;
(v) $\|U(t,s)x\| \leq Me^{\omega(t-s)}\|x\|$ for $x \in \overline{D}$ and $(t,s) \in \Gamma$.

In the following, we give some results on the existence of solutions for the following non-densely defined nonhomogeneous linear evolution Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t), \quad t \in [0,a] \\ u(0) = x, \end{cases} \tag{2.6}$$

where $f : [0,a] \to X$ is a function. The following theorem gives a generalized variation of constant formula for equation (2.6).

Theorem 2.9 *(Tanaka [37]). Let $x \in \overline{D}$ and $f \in L^1([0,a],X)$. Then the limit*

$$u(t) := U(t,0)x + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r) f(r) dr \tag{2.7}$$

exists uniformly for $t \in [0,a]$, and u is a continuous function on $[0,a]$.

As in Tanaka [37,38], for $x \in \overline{D}$ a continuous function $u : [0,a] \to X$ is called a *mild (or an integrated) solution* of equation (2.6) if it satisfies (2.7). Furthermore, we have the following estimate.

Lemma 2.10 *(Kpoumiè et al. [20]). Assume that $f \in L^1([0,a],X)$. If u is a mild solution of (2.6), then*

$$\|u(t)\| \le Me^{\omega t} \|x\| + \int_0^t Me^{\omega(t-s)} \|f(s)\| ds. \tag{2.8}$$

3. Existence of periodic solutions

In this section we will present our main results on the existence of periodic solutions in systems (1.4) and (1.5) under different conditions.

3.1. Time-independent operators

We first assume that the operator is time-independent and consider the nonhomogeneous linear equations

$$\frac{du}{dt} = Au(t) + f(t) \tag{3.1}$$

and the semilinear equation

$$\frac{du}{dt} = Au(t) + F(t,u), \tag{3.2}$$

where $A : D(A) \subset X \to X$ is a linear operator, $f \in C([0,\infty),X)$ and $F \in C([0,\infty) \times \overline{D(A)}, X)$ are both T-periodic in t.

We have the following results for the nonhomogeneous linear equation (3.1).

Theorem 3.1. *Assume that A is a Hille-Yosida operator with $M \ge 1$ and $\omega \in \mathbb{R}$, $f \in C([0,\infty),X)$ is T-periodic, i.e. $f(t+T) = f(t)$ for all $t \ge 0$. Further, suppose that $\omega < 0$. Then the linear equation (3.1) has a unique mild T-periodic solution $u_0(t)$. Moreover, we have*

$$\|u_0(0)\| < N \sup_{s \in [0,T]} \|f(s)\|, \quad N = \frac{T}{1 - e^{\omega T}}.$$

Proof. Since A is a Hille-Yosida operator, the Cauchy problem (2.1) has a unique mild solution $u(t) : [0, \infty) \to \overline{D(A)}$ on $t \in [0, \infty)$ for each $x \in \overline{D(A)}$ by Theorem 2.4. Now by the variation of constant formula, we have

$$u(t) = U_A(t)x + \lim_{\lambda \to +\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} f(s)ds, \tag{3.3}$$

where $\{U_A(t)\}_{t \geq 0}$ is the C_0-semigroup generated by A on $\overline{D(A)}$. Let $P_T : \overline{D(A)} \to \overline{D(A)}$ be the Poincaré map, i.e.,

$$P_T(x) = u(T) = U_A(T)x + \lim_{\lambda \to +\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1} f(s)ds. \tag{3.4}$$

Since by assumption $\omega < 0$, $\|U_A(T)\| \leq Me^{\omega T}$. Without loss of generality (W.L.O.G.), assume that $M = 1$. Then $\|U_A(T)\| \leq e^{\omega T} < 1$. Thus, the operator $I - U_A(T)$ is invertible and $P_T(x) = x$ has a unique solution

$$x_0 = (I - U_A(T))^{-1} \lim_{\lambda \to +\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1} f(s)ds, \tag{3.5}$$

i.e., x_0 is a unique fixed point of P_T.

Now let $u_T(t) = u(t+T)$, where $u(t)$ is the unique solution of (2.1) with initial value x_0. Then

$$\begin{aligned}
u_T(t) &= U_A(t+T)x_0 + \lim_{\lambda \to +\infty} \int_0^{t+T} U_A(t+T-s)\lambda(\lambda I - A)^{-1} f(s)ds \\
&= U_A(t)U_A(T)x_0 + \lim_{\lambda \to +\infty} \int_0^T U_A(t)U_A(T-s)\lambda(\lambda I - A)^{-1} f(s)ds \\
&\quad + \lim_{\lambda \to +\infty} \int_T^{t+T} U_A(t+T-s)\lambda(\lambda I - A)^{-1} f(s)ds \\
&= U_A(t)u(T) + \lim_{\lambda \to +\infty} \int_0^t U_A(t-\theta)\lambda(\lambda I - A)^{-1} f(\theta+T)d\theta \\
&= U_A(t)u_T(0) + \lim_{\lambda \to +\infty} \int_0^t U_A(t-\theta)\lambda(\lambda I - A)^{-1} f(\theta)d\theta.
\end{aligned}$$

Since $u_T(0) = u(T) = x_0$, $u_T(t)$ is also a mild solution of (2.1) with initial value x_0. By the uniqueness of solutions, $u_T(t) = u(t)$. Thus, we have $u(t+T) = u(t)$ for $t \in [0, \infty)$.

Moreover, by (3.5), we have

$$
\begin{aligned}
\|x_0\| &\le \frac{\left\| \lim_{\lambda\to+\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1} f(s)ds \right\|}{\|I - U_A(T)\|} \\
&\le \frac{\lim_{\lambda\to+\infty} \frac{\lambda}{\|\lambda I - A\|} \sup_{s\in[0,T]} \|f(s)\| \int_0^T e^{\omega(T-s)} ds}{|1 - \|U_A(T)\||} \\
&\le \frac{T}{1-e^{\omega T}} \sup_{s\in[0,T]} \|f(s)\|,
\end{aligned}
$$

i.e., $\|u_0(0)\| \le \frac{T}{1-e^{\omega T}} \sup_{s\in[0,T]} \|f(s)\|$. This completes the proof. □

Now we make the following assumptions.

Assumption 3.2.

(H1) A is a Hille-Yosida operator on X; i.e., there exist $M \ge 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $\left\|(\lambda I - A)^{-n}\right\|_{L(X)} \le \frac{M}{(\lambda-\omega)^n}$ for $\lambda > \omega$, $n \ge 1$;

(H2) $F : [0, \infty) \times \overline{D(A)} \to X$ is continuous and Lipschitz on bounded sets; i.e., for each $C > 0$ there exists $K_F(C) \ge 0$ such that $\|F(t,u) - F(t,v)\| \le K_F(C)\|u - v\|$ for $t \in [0, \infty)$ and $\|u\| \le C$ and $\|v\| \le C$;

(H3) $F : [0, \infty) \times \overline{D(A)} \to X$ is continuous and bounded on bounded sets; i.e., there exists $L_F(T, \rho) \ge 0$ such that $\|F(t,u)\| \le L_F(T, \rho)$ for $t \le T$ and $\|u\| \le \rho$.

With these assumptions, we have the following result for the semilinear equation (3.2).

Theorem 3.3. *Let Assumption 3.2 hold with $\omega < 0$, $M = 1$ and F being T-periodic in t. Suppose that there exists $\rho > 0$ such that $(N+T)K_F(\rho) < 1$ and $(N+T)L_F(T,\rho) \le \rho$, where $N = \frac{T}{1-e^{\omega T}}$. Then the semilinear equation (3.2) has a mild T-periodic solution.*

Proof. Denote

$$
B_\rho = \{v \in C(\mathbb{R}_+, \overline{D(A)}), v(t+T) = v(t), \|v\| = \sup_{s\in[0,T]} \|v(s)\| \le \rho\}.
$$

By Theorem 3.1, for each $v \in B_\rho$ let $f(t) = F(t, v(t))$, then (3.1) has a mild T-periodic solution

$$
u(t) = U_A(t)u(0) + \lim_{\lambda\to+\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1} F(l, v(l))dl. \tag{3.6}
$$

Define an operator ϕ on B_ρ by $\phi(v)(t) = u(t)$. Then

$$\|\phi(v)(t)\| \le Me^{\omega t}\|u(0)\| + \lim_{\lambda\to+\infty}\int_0^t Me^{\omega(t-l)}\frac{M\lambda}{\lambda-\omega}\|F(l, v(l))\|\,dl.$$

Let $M = 1$. Since $\|u(0)\| \le \frac{T}{1-e^{\omega T}}\sup_{s\in[0,T]}\|f(s)\|$, we have

$$\|\phi(v)(t)\| \le e^{\omega t}N\sup_{s\in[0,T]}\|F(s, v(s))\| + \lim_{\lambda\to+\infty}T\frac{\lambda}{\lambda-\omega}\sup_{s\in[0,T]}\|F(s, v(s))\|,$$

$$\sup_{s\in[0,T]}\|\phi(v)(t)\| \le (N+T)\sup_{s\in[0,T]}\|F(s, v(s))\| \le (N+T)L_F(T,\rho) \le \rho.$$

So ϕ maps B_ρ to B_ρ. Furthermore, let $v_1, v_2 \in B_\rho$. Then

$$\begin{aligned}\phi(v_1)(t) - \phi(v_2)(t) &= u_1(t) - u_2(t)\\ &= U_A(t)(u_1(0) - u_2(0))\\ &\quad + \lim_{\lambda\to+\infty}\int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1}(F(s, v_1(s)) - F(s, v_2(s))),\end{aligned}$$

$$\begin{aligned}\|\phi(v_1(t)) - \phi(v_2(t))\| &\le Me^{\omega t}\|u_1(0) - u_2(0)\|\\ &\quad + \lim_{\lambda\to+\infty}\int_0^t Me^{\omega(t-s)}\frac{M\lambda}{\lambda-\omega}\|F(s, v_1(s)) - F(s, v_2(s))\|\,ds.\end{aligned}$$

Again let $M = 1$. Since $\|u_1(0) - u_2(0)\| \le N\sup_{s\in[0,T]}\|F(s, v_1(s)) - F(s, v_2(s))\|$, by the result in Theorem 3.1, we have

$$\begin{aligned}\|\phi(v_1)(t) - \phi(v_2)(t)\| &\le e^{\omega t}N\sup_{s\in[0,T]}\|F(s, v_1(s)) - F(s, v_2(s))\|\\ &\quad + \lim_{\lambda\to+\infty}T\frac{\lambda}{\lambda-\omega}\sup_{s\in[0,T]}\|F(s, v_1(s)) - F(s, v_2(s))\|,\end{aligned}$$

$$\sup_{s\in[0,T]}\|\phi(v_1)(t) - \phi(v_2)(t)\| \le (N+T)K_F(\rho)\sup_{s\in[0,T]}\|v_1(s) - v_2(s)\|.$$

So it implies that

$$\|\phi(v_1) - \phi(v_2)\| \le (N+T)K_F(\rho)\sup_{s\in[0,T]}\|v_1(s) - v_2(s)\|.$$

Since $(N+T)K_F(\rho) < 1$, by Banach Fixed Point Theorem, $\phi : B_\rho \to B_\rho$ has a fixed point; i.e., there exists $u \in B_\rho$ such that

$$u(t) = U_A(t)u(0) + \lim_{\lambda \to +\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, u(s))ds,$$

which is a T-periodic solution for (3.2). □

Remark 3.4. For the case $M \neq 1$, it can be transfered to $M = 1$ by re-norming the Banach space X [35].

3.2. Time-dependent operators

Now consider the linear evolution equation

$$\frac{du}{dt} = A(t)u(t) + f(t) \tag{3.7}$$

and the semilinear evolution equation

$$\frac{du}{dt} = A(t)u(t) + F(t, u), \tag{3.8}$$

where $A(t)$ is a T-periodic linear operator on a Banach space X, $f : \mathbb{R}_+ \to X$ is continuous and T-periodic, and $F : \mathbb{R}_+ \times X \to X$ is continuous and T-periodic in t. We make the following assumptions.

Assumption 3.5.

(A1) $D(A(t)) := D$ is independent of t and not necessarily densely defined;
(A2) The family $\{A(t)\}_{t \geq 0}$ is stable in the sense that there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t))$ for $t \in [0, \infty)$ and

$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^k}$$

for $\lambda > \omega$ and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \leq t_1 \leq t_2 \leq ... \leq t_k$ and $k = 1, 2, \ldots$;
(A3) The mapping $t \to A(t)x$ is continuously differentiable in X for each $x \in D$.

For $\lambda > 0$, $0 \leq s \leq t$, and $x \in \overline{D}$. Set

$$U_\lambda(t, s)x = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x. \tag{3.9}$$

Then the generalized variation of constant formula of (3.7) with initial value $u(0) = x$ is given by

$$u(t) = U(t,0)x + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)f(r)dr. \tag{3.10}$$

Now we state and prove the results for the nonhomogeneous linear evolution equation (3.7).

Theorem 3.6. *Let Assumption 3.5 hold, $Me^{\omega T} < 1$, $f \in C([0,\infty), X)$, $f(t+T) = f(t)$ for $t \in [0,\infty)$, and $\omega < 0$. Then the linear evolution equation (3.7) has a unique mild T-periodic solution $u(t)$. Moreover, $\|u(0)\| \le N \sup_{s\in[0,T]} \|f(s)\|$, where $N = \frac{MT}{1-Me^{\omega T}}$.*

Proof. By assumptions, the variation of constant formula (3.10) holds. Let $P_T : \overline{D} \to \overline{D}$ be the Poincaré map

$$P_T(x) = u(T) = U(T,0)x + \lim_{\lambda \to 0^+} \int_0^T U_\lambda(T,r)f(r)dr. \tag{3.11}$$

Since $\|U(T,0)\| \le Me^{\omega T} < 1$, $I - U(T,0)$ is invertible. P_T has a unique fixed point which is given by $x = (I - U(T,0))^{-1} \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,r)f(r)dr$.

Now let $u(t)$ be the unique solution with initial value $u(0) = x$. Let $u_T(t) = u(t+T)$. Then

$$\begin{aligned}
u_T(t) &= U(t+T,0)x + \lim_{\lambda\to 0^+} \int_0^{T+t} U_\lambda(T+t,r)f(r)dr \\
&= U(t+T,T)U(T,0)x + \lim_{\lambda\to 0^+} \int_0^T U_\lambda(T+t,T)U_\lambda(T,r)f(r)dr \\
&\quad + \lim_{\lambda\to 0^+} \int_T^{T+t} U_\lambda(T+t,r)f(r)dr \\
&= U(t+T,T)U(T,0)x + \lim_{\lambda\to 0^+} U_\lambda(T+t,T) \int_0^T U_\lambda(T,r)f(r)dr \\
&\quad + \lim_{\lambda\to 0^+} \int_T^{t+T} U_\lambda(T+t,r)f(r)dr \\
&= U(t,0)U(T,0)x + U(T+t,T) \lim_{\lambda\to 0^+} \int_0^T U_\lambda(T,r)f(r)dr
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\lambda\to 0^+}\int_T^{t+T} U_\lambda(T+t,r)f(r)dr \\
&=U(t,0)U(T,0)x+U(t,0)\lim_{\lambda\to 0^+}\int_0^T U_\lambda(T,r)f(r)dr \\
&+\lim_{\lambda\to 0^+}\int_T^{t+T} U_\lambda(T+t,r)f(r)dr \\
&=U(t,0)[U(T,0)x+\lim_{\lambda\to 0}\int_0^T U_\lambda(T,r)f(r)dr] \\
&+\lim_{\lambda\to 0^+}\int_0^t U_\lambda(T+t,T+s)f(T+s)ds \\
&=U(t,0)u(T)+\lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s)ds \\
&=U(t,0)x+\lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,s)f(s)ds.
\end{aligned}
$$

So $u_T(t)$ is a solution of (3.7) with initial value $u_T(0)=x$. By uniqueness, $u_T(t)=u(t)$, i.e. $u(t+T)=u(t)$ for $t\in[0,\infty)$. Furthermore,

$$
\begin{aligned}
\|x\| &= \left\|(I-U(T,0))^{-1}\right\| \left\|\lim_{\lambda\to 0^+}\int_0^T U_\lambda(T,r)f(r)dr\right\| \\
&\le \frac{1}{\|I-U(T,0)\|}\lim_{\lambda\to 0^+}\int_0^T \|U_\lambda(T,r)\|\,\|f(r)\|\,dr \\
&\le \frac{1}{\|I-U(T,0)\|}\lim_{\lambda\to 0^+}\int_0^T \left\|\prod_{i=[\frac{r}{\lambda}]+1}^{[\frac{T}{\lambda}]}(I-\lambda A(i\lambda))^{-1}\right\|\,\|f(r)\|\,dr \\
&\le \frac{1}{\|I-U(T,0)\|}\lim_{\lambda\to 0^+}\int_0^T M(\frac{1}{1-\lambda\omega})^{[\frac{T}{\lambda}]-[\frac{r}{\lambda}]}\,\|f(r)\|\,dr \\
&\le \frac{1}{\|I-U(T,0)\|}\lim_{\lambda\to 0^+}\int_0^T M(\frac{1}{1-\lambda\omega})^{\frac{T-r}{\lambda}+1}\,\|f(r)\|\,dr
\end{aligned}
$$

$$\leq \frac{1}{|1-\|U(T,0)\||} \lim_{\lambda\to 0^+} \int_0^T \frac{M}{1-\lambda\omega} e^{-\omega(T-r)\frac{\ln(1-\lambda\omega)}{\lambda\omega}} \|f(r)\|\, dr$$

$$\leq \frac{MT}{1-Me^{\omega T}} \sup_{s\in[0,T]} \|f(s)\|.$$

This completes the proof. □

In order to study the semilinear evolution equation (3.8), we introduce the following definition.

Definition 3.7. A continuous function $u : \mathbb{R}_+ \to X$ is called *a mild (or an integrated) solution* of equation (3.8) if it satisfies the following

$$u(t) = U(t,0)u(0) + \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,\sigma)F(\sigma,u(\sigma))d\sigma, \quad t \geq 0. \tag{3.12}$$

Next we establish the existence of periodic solutions for the semilinear evolution equation (3.8).

Theorem 3.8. *Let Assumption 3.2 (H2) (H3) and Assumption 3.5 hold, $\omega < 0$, $Me^{\omega T} < 1$, $F(t+T,\cdot) = F(t,\cdot)$ for $t \geq 0$. Suppose that there exists $\rho > 0$ such that $M(N+T)K_F(\rho) < 1$ and $M(N+T)L_F(T,\rho) \leq \rho$, where $N = \frac{MT}{1-Me^{\omega T}}$. Then the semilinear evolution equation (3.8) has a mild T-periodic solution.*

Proof. Let $B_\rho = \{v \in C(\mathbb{R}_+, \overline{D}), v(t+T) = v(t), \|v\| = \sup_{s\in[0,T]} \|v(s)\| \leq \rho\}$. By Theorem 3.6, for each $v \in B_\rho$ let $f(t) = F(t,v(t))$, then (3.7) has a unique mild T-periodic solution given by

$$u(t) = U(t,0)u(0) + \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,r)F(r,v(r))dr, t \geq 0. \tag{3.13}$$

Let ϕ be an operator on B_ρ defined by $\phi(v)(t) = u(t)$. Then by the argument in Theorem 3.6, we have

$$\|\phi(v)(t)\| \leq Me^{\omega t} \|u(0)\| + \int_0^t Me^{\omega(t-r)} \|F(r,v(r))\|\, dr,$$

$$\begin{aligned}\sup_{t\in[0,T]} \|\phi(v)(t)\| &\leq MN \sup_{r\in[0,T]} \|F(r,v(r))\| + MT \sup_{r\in[0,T]} \|F(r,v(r))\| \\ &\leq M(N+T)L_F(T,\rho) \\ &\leq \rho.\end{aligned}$$

So $\phi : B_\rho \to B_\rho$. Moreover, let $v_1, v_2 \in B_\rho$, then

$$
\begin{aligned}
\phi(v_1)(t) - \phi(v_2)(t) = u_1(t) - u_2(t) \\
&= U(t,0)(u_1(0) - u_2(0)) \\
&\quad + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)[F(r, v_1(r)) - F(r, v_2(r))]dr,
\end{aligned}
$$

$$
\begin{aligned}
\|\phi(v_1)(t) - \phi(v_2)(t)\| &\le Me^{\omega t} \|u_1(0) - u_2(0)\| \\
&\quad + \int_0^t Me^{\omega(t-r)} \|F(r, v_1(r)) - F(r, v_2(r))\| \, dr,
\end{aligned}
$$

$$
\begin{aligned}
\sup_{t\in[0,T]} \|\phi(v_1)(t) - \phi(v_2)(t)\| &\le MN \sup_{r\in[0,T]} \|F(r, v_1(r)) - F(r, v_2(r))\| \\
&\quad + MT \sup_{r\in[0,T]} \|F(r, v_1(r)) - F(r, v_2(r))\| \\
&\le M(N+T)K_F(\rho) \sup_{t\in[0,T]} \|v_1(t) - v_2(t)\| .
\end{aligned}
$$

Thus, we have

$$
\|\phi(v_1) - \phi(v_2)\| \le M(N+T)K_F(\rho) \|v_1 - v_2\| .
$$

Since $M(N+T)K_F(\rho) < 1$, by Banach Fixed Point Theorem, ϕ has a fixed point $u \in B_\rho$; i.e.,

$$
u(t) = U(t,0)u(0) + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t,r)F(r, u(r))dr,
$$

which is a mild T-periodic solution for (3.8). □

3.3. Time-independent operators - revisited

Now consider (3.1) and (3.2) again when A is time independent. We will investigate the case when A is compact.

Theorem 3.9. *Let Assumption 3.2 (H1) hold, $f \in C([0,\infty), X)$, $f(t+T) = f(t)$. Assume that $U_A(T)$ is compact on $\overline{D(A)}$. If there exists $x \in \overline{D(A)}$ such that the Cauchy problem (2.1) has a unique bounded mild solution $u : [0,\infty) \to \overline{D(A)}$ for $u(0) = x \in \overline{D(A)}$, then the nonhomogeneous linear equation (3.1) has a mild T-periodic solution.*

Proof. It suffices to prove that the Poincaré map P_T has a fixed point x_0, where

$$
P_T(x) = U_A(T)x + \lim_{\lambda \to +\infty} \int_0^T U_A(t-s)\lambda(\lambda I - A)^{-1} f(s)ds.
$$

By the same argument as in the proof of Theorem 3.1, let $u(t)$ be the solution such that $u(t+T) = u(t)$ for $t \geq 0$, which implies that $u(t)$ is a T-periodic solution of (3.1).

Suppose P_T has no fixed point, i.e.,

$$x = U_A(T)x + \lim_{\lambda\to+\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1} f(s)ds$$

has no solution in $\overline{D(A)}$. Let $P = U_A(T) : \overline{D(A)} \to \overline{D(A)}$ and

$$x_0 = \lim_{\lambda\to+\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1} f(s)ds \in \overline{D(A)}.$$

Then $x = Px + x_0$ has no solution in $\overline{D(A)}$. Since P is assumed to be compact on $\overline{D(A)}$, 1 is an eigenvalue of P, $I - P$ is Fredholm, thus its range $\mathcal{R}(I - P)$ is closed in $\overline{D(A)}$. Then there exists $x^* \in \overline{D(A)}'$ such that $x^*((I - P)x') = 0$ for each $x' \in \overline{D(A)}$ and $x^*(x_0) \neq 0$. Let

$$x_n = P_T^n(x) = P^n x + (P^{n-1} + ... + I)x_0,$$

where x is chosen such that (3.1) has a unique bounded solution for $u(0) = x$. Then

$$\begin{aligned} x^*(x_n) &= x^*[P^n x + (P^{n-1} + ... + I)x_0] \\ &= x^*(P^n x) + x^*[(P^{n-1} + ... + I)x_0] \\ &= (P')^n x^*(x) + [(P')^{n-1} + ... + I]x^*(x_0). \end{aligned}$$

Note that $x^*(x) = x^*(Px)$, so $P'x^*(x) = x^*(x)$ for $x \in \overline{D(A)}$. Then we get $x^*(x_n) = x^*(x) + nx^*(x_0)$. Let $n \to \infty$, it follows that $nx^*(x_0) \to \infty$. Then $x^*(x_0) \to \infty$, which contradicts the fact that x_n is bounded, since (3.1) has a unique bounded solution for $x \in \overline{D(A)}$. Therefore, P_T has a fixed point in $\overline{D(A)}$ and (3.1) has a mild T-periodic solution. □

Finally we prove an existence theorem of periodic solutions for the semilinear equation (3.2) when the operator A is compact.

Theorem 3.10. *Let Assumption 3.2 (H1)(H3) hold with $M = 1$ and $F(t + T, x) = F(t, x)$ for $t \geq 0$, $x \in \overline{D(A)}$. Let $U_A(t)$ be compact on $\overline{D(A)}$ for $t > 0$. Suppose that there exists $\rho > 0$ such that $(N + T)L_F(T, \rho) \leq \rho$, where $N = \frac{T}{1-e^{\omega T}}$ for $\omega < 0$, and $(N + T)e^{\omega T} L_F(T, \rho) \leq \rho$, where $N = \frac{Te^{\omega T}}{\|I-U_A(T)\|}$ for $\omega \geq 0$. If for each T-periodic $f \in C([0, \infty), X)$, there exists $x \in \overline{D(A)}$ such that the Cauchy problem (2.1) has a unique bounded mild solution for $u(0) = x \in \overline{D(A)}$, then the semilinear equation (3.2) has a T-periodic solution.*

Proof. Define

$$B_\rho = \{v \in C(\mathbb{R}^+, \overline{D(A)}), v(t + T) = v(t), \|v\| = \sup_{s\in[0,T]} \|v(s)\| \leq \rho\}.$$

By Theorem 3.9, for each $v \in B_\rho$, let $f(t) = F(t, v(t))$. Then equation (3.1) has a unique mild T-periodic solution given by

$$u(t) = U_A(t)u(0) + \lim_{\lambda\to+\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1} F(l, v(l))dl. \tag{3.14}$$

Moreover,

$$u(0) = (I - U_A(T))^{-1} \lim_{\lambda\to+\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds, \tag{3.15}$$

$$\|u(0)\| \le \begin{cases} \frac{e^{\omega T} T}{\|I - U_A(T)\|} \sup_{s\in[0,T]} \|F(s, v(s))\|, & \omega \ge 0, \\ \frac{T}{\|I - U_A(T)\|} \sup_{s\in[0,T]} \|F(s, v(s))\|, & \omega < 0. \end{cases} \tag{3.16}$$

Since $\frac{T}{\|I-U_A(T)\|} \le \frac{T}{1-e^{\omega T}}$ for $\omega < 0$, let

$$N = \begin{cases} \frac{T}{1-e^{\omega T}}, & \omega < 0, \\ \frac{e^{\omega T} T}{\|I-U_A(T)\|}, & \omega \ge 0. \end{cases}$$

Then we have $\|u(0)\| \le N \sup_{s\in[0,T]} \|F(s, v(s))\|$. Define an operator ϕ on B_ρ as follows:

$$\phi(v)(t) = u(t) = U_A(t)u(0) + \lim_{\lambda\to+\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1} F(l, v(l))dl.$$

Then

$$\begin{aligned} \|\phi(v)(t)\| &\le Me^{\omega t} \|u(0)\| + \int_0^t Me^{\omega(t-l)} \|F(l, v(l))\| dl \\ &\overset{M=1}{=\!=\!=} e^{\omega t} \|u(0)\| + \int_0^t e^{\omega(t-l)} \|F(l, v(l))\| dl. \end{aligned}$$

It follows that if $\omega < 0$,

$$\begin{aligned} \sup_{t\in[0,T]} \|\phi(v)(t)\| &\le \|u(0)\| + T \sup_{t\in[0,T]} \|F(t, v(t))\| \\ &\le (N + T)L_F(T, \rho) \\ &\le \rho. \end{aligned}$$

If $\omega \ge 0$,

$$\begin{aligned}\sup_{t\in[0,T]}\|\phi(v)(t)\| &\le \|u(0)\| + Te^{\omega T}\sup_{t\in[0,T]}\|F(t,v(t))\|\\ &\le e^{\omega T}(N+T)L_F(T,\rho)\\ &\le \rho.\end{aligned}$$

So $\phi : B_\rho \to B_\rho$.

Next, we show that ϕ is compact. Let $t > 0$, $u \in \phi(B_\rho)$. Then there exists $v \in B_\rho$ such that

$$u(t) = U_A(t)u(0) + \lim_{\lambda\to+\infty}\int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1}F(l, v(l))dl.$$

Let $0 < \varepsilon < t$, then

$$\begin{aligned}u(t) &= U_A(t)u(0) + \lim_{\lambda\to+\infty}\int_0^{t-\varepsilon} U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds\\ &\quad + \lim_{\lambda\to+\infty}\int_{t-\varepsilon}^{t} U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds\\ &= U_A(t)u(0) + U_A(\varepsilon)\lim_{\lambda\to+\infty}\int_0^{t-\varepsilon} U_A(t-\varepsilon-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds\\ &\quad + \lim_{\lambda\to+\infty}\int_{t-\varepsilon}^{t} U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds.\end{aligned}$$

Since

$$\|F(s, v(s))\| \le L_F(t,\rho) \ \text{ and } \ \left\|\lambda(\lambda I - A)^{-1}F(s, v(s))\right\| \le \frac{\lambda}{\lambda-\omega}L_F(t,\rho),$$

it then follows that

$$\lim_{\lambda\to+\infty}\int_0^{t-\varepsilon} U_A(t-\varepsilon-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds$$

is bounded. By the compactness of $U_A(\varepsilon)$, it follows that

$$\{U_A(\varepsilon)\lim_{\lambda\to+\infty}\int_0^{t-\varepsilon} U_A(t-\varepsilon-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds, v \in B_\rho\}$$

is relatively compact in $\overline{D(A)}$. Moreover, there exists some $b > 0$ such that

$$\left\| \lim_{\lambda\to+\infty} \int_{t-\varepsilon}^{t} U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \right\| \leq b\varepsilon$$

for $v \in B_\rho$. Hence, $\{u(t), v \in \phi(B_\rho)\}$ is relatively compact in $\overline{D(A)}$ for each $t > 0$. By the periodicity, $\{u(0) : u \in \phi(B_\rho)\}$ is relatively compact in $\overline{D(A)}$.

Now we show the equi-continuity of $\{u(t), v \in \phi(B_\rho)\}$. For $T + \varepsilon \geq t > \tau > 0$, we have

$$\begin{aligned}
u(t) - u(\tau) &= (U_A(t) - U_A(\tau))u(0) + \lim_{\lambda\to+\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&\quad - \lim_{\lambda\to+\infty} \int_0^\tau U_A(\tau-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&= (U_A(t) - U_A(\tau))u(0) + \lim_{\lambda\to+\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&\quad - \lim_{\lambda\to+\infty} \int_0^\tau U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&\quad + \lim_{\lambda\to+\infty} \int_0^\tau U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&\quad - \lim_{\lambda\to+\infty} \int_0^\tau U_A(\tau-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&= (U_A(t) - U_A(\tau))u(0) + \lim_{\lambda\to+\infty} \int_\tau^t U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \\
&\quad + \lim_{\lambda\to+\infty} \int_0^\tau (U_A(t-\tau) - I)U_A(\tau-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds,
\end{aligned}$$

$$\begin{aligned}
\|u(t) - u(\tau)\| &\leq \|U_A(t) - U_A(\tau)\|\rho + \left\| \lim_{\lambda\to+\infty} \int_\tau^t U_A(t-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \right\| \\
&\quad + \left\| (U_A(t-\tau) - I) \lim_{\lambda\to+\infty} \int_0^\tau U_A(\tau-s)\lambda(\lambda I - A)^{-1} F(s, v(s))ds \right\|.
\end{aligned}$$

Since $\{U_A(t)\}_{t>0}$ is compact on $\overline{D(A)}$, it is continuous in uniform topology. Then $\lim_{t\to\tau}\|U_A(t)-U_A(\tau)\|=0$. Since $\|F(s,v(s))\|\le L_F(T+\varepsilon,\rho)$ for $v\in B_\rho$, $0<s<T+\varepsilon$, there exists $C>0$ such that

$$\left\|\lim_{\lambda\to+\infty}\int_\tau^t U_A(t-s)\lambda(\lambda I-A)^{-1}F(s,v(s))ds\right\|\le C(t-\tau)\ \text{ for }\ v\in B_\rho.$$

Then

$$\lim_{t\to+\tau}\left\|\lim_{\lambda\to+\infty}\int_\tau^t U_A(t-s)\lambda(\lambda I-A)^{-1}F(s,v(s))ds\right\|\le\lim_{t\to+\tau}C(t-\tau)=0$$

uniformly for $v\in B_\rho$. Since $\{u(t):v\in\phi(B_\rho)\}$ is relatively compact in $\overline{D(A)}$ for each $t\ge0$ as shown above, $\{u(t)-U_A(t)u(0):v\in B_\rho\}$ is also relatively compact in $\overline{D(A)}$ for each $t\ge 0$, which implies that $\{\lim_{\lambda\to+\infty}\int_0^\tau U_A(\tau-s)\lambda(\lambda I-A)^{-1}F(s,v(s))ds, v\in B_\rho\}$ is relatively compact in $\overline{D(A)}$ for each $\tau>0$. So there exists a compact set $K\subset\overline{D(A)}$ such that

$$\lim_{\lambda\to+\infty}\int_0^\tau U_A(\tau-s)\lambda(\lambda I-A)^{-1}F(s,v(s))ds\in K$$

for all $v\in B_\rho$.

Since $\lim_{h\to0}\sup_{\alpha\in K}\|(U_A(h)-I)\alpha\|=0$ for compact K, it follows that

$$\lim_{t\to\tau}\sup_{v\in B_\rho}\left\|(U_A(t-\tau)-I)\lim_{\lambda\to+\infty}\int_0^\tau U_A(\tau-s)\lambda(\lambda I-A)^{-1}F(s,v(s))ds\right\|=0.$$

Summarizing the above analysis, we have

$$\lim_{t\to\tau,t>\tau>0}\sup_{v\in B_\rho}\|u(t)-u(\tau)\|=0.$$

Similarly,

$$\lim_{t\to\tau,\tau>t>0}\sup_{v\in B_\rho}\|u(t)-u(\tau)\|=0.$$

By periodicity, $u(t)$ is also equi-continuous at $t=0$. Now by Arzelà-Ascoli theorem, $\phi(B_\rho)$ is relatively compact in $C=\{\varphi|\varphi\in C(\mathbb{R}_+,\overline{D(A)}),\varphi(t+T)=\varphi(t)\}$. So ϕ has a fixed point in B_ρ; i.e., there exists $u\in B_\rho$ such that

$$u(t)=U_A(t)u(0)+\lim_{\lambda\to+\infty}\int_0^t U_A(t-s)\lambda(\lambda I-A)^{-1}F(s,u(s))ds,$$

which is a mild T-periodic solution for (3.2). □

Remark 3.11. Note that if F is bounded, i.e., $\|F(t,x)\| \le B$ for each $t \in [0,\infty)$ and $x \in \overline{D(A)}$, it is a special case of Theorem 3.10. In this case, we choose $\rho \ge (N+T)B$, then $\|\phi(v)\| \le (N+T)B \le \rho$, which implies that $\phi : B_\rho \to B_\rho$. By the argument in Theorem 3.10, ϕ has a fixed point in B_ρ, which is a T-periodic solution for (3.2). For the case $M \ne 1$, it can be transfered to $M = 1$ by re-norming the Banach space X [35].

4. Applications

The results obtained in last section can be applied to study the existence of periodic solutions in several types of equations including delay differential equations, first-order hyperbolic partial differential equations, and reaction-diffusion equations, in particular some biological and physical models described by these equations. In this section we consider retarded periodic functional differential equations with application to a delayed red-blood cell mode, age-structured population models with periodic harvesting, and the diffusive logistic equation with periodic coefficients.

4.1. Retarded functional differential equations and delayed red-blood cell models

The existence of periodic solutions in periodic functional differential equations has been studied by many researchers (see, for example, Chow [7] and Chow and Hale [8]), we refer to the monographs of Hale and Verduyn Lunel [16] and Burton [4], and the references cited therein. In this subsection, we will apply the results in section 3 to obtain existence of periodic solutions in periodic functional differential equations. Namely, we will first consider a general class of retarded periodic functional differential equations, then we will consider a delayed red-blood cell model with periodic coefficients.

(i) Retarded periodic functional differential equations. For $r \ge 0$, let $\mathcal{C} = C([-r,0],\mathbb{R}^n)$ be the Banach space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ endowed with the supremum norm

$$\|\varphi\| = \sup_{\theta\in[-r,0]} |\varphi(\theta)|_{\mathbb{R}^n}$$

Consider the retarded functional differential equations (RFDE) of the form

$$\begin{cases} \frac{dx(t)}{dt} = Bx(t) + \hat{L}(x_t) + f(t,x_t), \forall t \ge 0, \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \tag{4.1}$$

where $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$, $B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, $\hat{L} : \mathcal{C} \to \mathbb{R}^n$ is a bounded linear operator given by

$$\hat{L}(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta),$$

here $\eta : [-r,0] \to M_n(\mathbb{R})$ is a map of bounded variation, i.e. $V(\eta,[-r,0]) = \sup \sum_{i=1}^{n} \|\eta(\theta_{i+1}) - \eta(\theta_i)\| < +\infty$ in which the supremum is taken over all subdivisions $-r = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} = 0$, and $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is a continuous map.

Now following Liu et al. [24] we rewrite (4.1) as an abstract non-densely defined Cauchy problem so that our theorems can be applied. First, we write it as a PDE. Define $u \in C([0,\infty) \times [-r,0], \mathbb{R}^n)$ by

$$u(t,\theta) = x(t+\theta), \ \forall t \geq 0, \forall \theta \in [-r,0].$$

If $x \in C^1([-r,+\infty), \mathbb{R}^n)$, then

$$\frac{\partial u(t,\theta)}{\partial t} = x'(t+\theta) = \frac{\partial u(t,\theta)}{\partial \theta}.$$

So we have

$$\frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0, \forall t \geq 0, \forall \theta \in [-r,0].$$

Moreover, for $\theta = 0$, we have

$$\begin{aligned} \frac{\partial u(t,0)}{\partial \theta} &= x'(t) = Bx(t) + \hat{L}(x_t) + f(t,x_t) \\ &= Bu(t,0) + \hat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0. \end{aligned}$$

Thus, u satisfies the PDE

$$\begin{cases} \frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \frac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \hat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0, \\ u(0,.) = \varphi \in \mathcal{C}. \end{cases} \tag{4.2}$$

To rewrite (4.2) as an abstract non-densely defined Cauchy problem, let $X = \mathbb{R}^n \times \mathcal{C}$ with the usual product norm

$$\left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\| = |x|_{\mathbb{R}^n} + \|\varphi\|.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A), \tag{4.3}$$

with $D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-r,0], \mathbb{R}^n)$. Then $\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times \mathcal{C} \neq X$. Define $L : \overline{D(A)} \to X$ by

$$L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \hat{L}(\varphi) \\ 0_{\mathcal{C}} \end{pmatrix}$$

and $F : \mathbb{R} \times \overline{D(A)} \to X$ by

$$F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} f(t,\varphi) \\ 0_{\mathcal{C}} \end{pmatrix}.$$

Set

$$v(t)=\begin{pmatrix}0_{\mathbb{R}^n}\\ u(t)\end{pmatrix}.$$

Then the PDE (4.2) can be written as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt}=Av(t)+L(v(t))+F(t,v(t)),\ t\geq 0;\ \ v(0)=\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi\end{pmatrix}\in\overline{D(A)}. \tag{4.4}$$

Now we give an existence theorem of periodic solutions for equation (4.1).

Assumption 4.1.

(B1) $f:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^n$ is Lipschitz on bounded sets; i.e., for each $C>0$ there exists $K_f(C)\geq 0$ such that $\|f(t,u)-f(t,v)\|\leq K_f(C)\,\|u-v\|$ for $t\in[0,\infty)$ and $\|u\|\leq C$ and $\|v\|\leq C$;
(B2) $f:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^n$ is bounded on bounded sets; i.e., there exists $L_f(T,\rho)\geq 0$ such that $\|f(t,u)\|\leq L_f(T,\rho)$ for $t\leq T$ and $\|u\|\leq\rho$.

With these assumptions and the notation $\omega_0(B):=\sup_{\lambda\in\sigma(B)}\mathrm{Re}(\lambda)$, we have the following result for equation (4.1).

Theorem 4.2. *Let Assumption 4.1 hold with $\omega_0(B)<0$ and f being T-periodic in t. Suppose that there exists $\rho>0$ such that $(N+T)(K_f(\rho)+V(\eta,[-r,0]))<1$ and $(N+T)(L_f(T,\rho)+V(\eta,[-r,0]))\leq\rho$, where $N=\frac{T}{1-e^{\omega_0(B)T}}$, then equation (4.1) has a T-periodic solution.*

Proof. Since (4.1) can be written as (4.4), denote $G(t,v(t))=L(v(t))+F(t,v(t))$, it suffices to prove that

(a) A satisfies Assumption 3.2 (H1) with $\omega<0$;
(b) $G:[0,\infty)\times 0_{\mathbb{R}^n}\times\mathcal{C}\to\mathbb{R}^n\times\mathcal{C}$ satisfies Assumption 3.2(H1) (H2);
(c) There exists $\rho>0$ such that $(N+T)K_G(\rho)<1$ and $(N+T)L_G(T,\rho)\leq\rho$, where $N=\frac{T}{1-e^{\omega T}}$.

Then it follows from Theorem 3.3 that equation (4.4) has a T-periodic mild solution, which implies that equation (4.2) has a T-periodic mild solution with initial $u(0,.)=\varphi_0\in\mathcal{C}$. Meanwhile, by Theorem 2.1 in [16], equation (4.1) has a unique solution $x_0(t)\in C^1([0,\infty),\mathbb{R}^n)$ with initial $x_0(\theta)=\varphi_0(\theta)$ for $\theta\in[-r,0]$. Therefore, $x_0(t)$ is a T-periodic solution for (4.1).

From Lemma 7.1 in [28], we know that A as defined in (4.3) is a Hille-Yoshida operator with $\omega=\omega_0(B)<0$ and $M=1$, which proves (a).

For $\varphi_1,\varphi_2\in\mathcal{C}$ such that $\|\varphi_1\|\leq C$ and $\|\varphi_2\|\leq C$, we have

$$\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi_1\end{pmatrix},\begin{pmatrix}0_{\mathbb{R}^n}\\ \varphi_2\end{pmatrix}\in 0_{\mathbb{R}^n}\times\mathcal{C}=\overline{D(A)}$$

and

$$\left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix} \right\| = \|\varphi_1\| \le C, \left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \right\| = \|\varphi_2\| \le C.$$

Then

$$\begin{aligned}
&\left\| G(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}) - G(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}) \right\| \\
&= \left\| L(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}) - L(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}) + F(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}) - F(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}) \right\| \\
&\le \left\| L(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}) - L(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}) \right\| + \left\| F(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}) - F(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}) \right\| \\
&= \left\| \begin{pmatrix} \int_{-r}^0 d\eta(\theta)(\varphi_1(\theta) - \varphi_2(\theta)) \\ 0_{\mathcal{C}} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi_1) - f(t, \varphi_2) \\ 0_{\mathcal{C}} \end{pmatrix} \right\| \\
&= \left| \int_{-r}^{0} d\eta(\theta)(\varphi_1(\theta) - \varphi_2(\theta)) \right|_{\mathbb{R}^n} + |f(t, \varphi_1) - f(t, \varphi_2)|_{\mathbb{R}^n} \\
&\le K_f(C) \|\varphi_1 - \varphi_2\| + V(\eta, [-r, 0]) \|\varphi_1 - \varphi_2\| \\
&= (K_f(C) + V(\eta, [-r, 0])) \|\varphi_1 - \varphi_2\| \\
&= (K_f(C) + V(\eta, [-r, 0])) \left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \right\|.
\end{aligned}$$

So there exists $K_G(C) = K_f(C) + V(\eta, [-r, 0])$ such that

$$\left\| G(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}) - G(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}) \right\| \le K_G(C) \left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \right\|.$$

Furthermore, for $t \le T$ and $\left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \right\| \le \rho$, we have

$$\begin{aligned}
\left\| G(t, \begin{pmatrix} 0_{\mathbb{R}_+{}^n} \\ \varphi \end{pmatrix}) \right\| &= \left\| L\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + F(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}) \right\| \\
&\le \left\| L\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \right\| + \left\| F(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}) \right\| \\
&= \left\| \begin{pmatrix} \int_{-r}^0 d\eta(\theta)\varphi(\theta) \\ 0_{\mathcal{C}} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi) \\ 0_{\mathcal{C}} \end{pmatrix} \right\| \\
&= \left| \int_{-r}^{0} d\eta(\theta)\varphi(\theta) \right|_{\mathbb{R}^n} + |f(t, \varphi)|_{\mathbb{R}^n} \\
&\le V(\eta, [-r, 0])\rho + L_f(T, \rho).
\end{aligned}$$

So there exists $L_G(T,\rho) = V(\eta, [-r, 0])\rho + L_f(T,\rho)$ such that $\left\| G(t, \begin{pmatrix} 0_{\mathbb{R}_+{}^n} \\ \varphi \end{pmatrix}) \right\| \le L_G(T,\rho)$, which completes the proof of (b).

With $K_G(C)$ and $L_G(T,\rho)$ given as above, (c) follows directly from the assumptions. □

(ii) A delayed periodic red-blood cell model. Now as an example, we consider a delayed red-blood cell model with periodic coefficients which is a modification of the model of Wazewska-Czyzewska and Lasota [41] (see also Arino and Kimmel [2], Gopalsamy [15], Kuang [21], called *Lasota-Wazewska model* in the literature):

$$N'(t) = -\mu N(t) + p(t)e^{-\gamma(t)N(t-r)}, \tag{4.5}$$

where $N(t)$ denotes the number of red-blood cells at time t, $\mu \in (0,\infty)$ is the probability of death of a red-blood cell, $p(t)$ and $\gamma(t)$ are positive and T-periodic continuous functions related to the production of red-blood cells per unit time and r is the time required to produce a red-blood cell.

Proposition 4.3. *Assume that*

(i) $p \in C([0,\infty), \mathbb{R}^+)$, $p(t+T) = p(t)$ *for* $t \ge 0$ *and* $p(t) \le p_+$ *for* $t \ge 0$*;*
(ii) $\gamma \in C([0,\infty), \mathbb{R}^+)$, $\gamma(t+T) = \gamma(t)$ *for* $t \ge 0$ *and* $\gamma(t) \le \gamma_+$ *for* $t \ge 0$*;*
(iii) There exists $\rho > 0$ *such that* $(\frac{T}{1-e^{-\mu T}} + T)p_+\gamma_+e^{\gamma_+\rho} < 1$ *and* $(\frac{T}{1-e^{-\mu T}} + T)p_+e^{\gamma_+\rho} \le \rho$.

Then equation (4.5) *has a* T*-periodic solution.*

Proof. Equation (4.5) can be written as equation (4.1), where $B = -\mu$, $\hat{L} = 0$ and $f(t,\varphi) = p(t)e^{-\gamma(t)\varphi(-r)}$. Then it suffices to check assumptions of Theorem 4.2. First note that $\omega_0(B) = -\mu < 0$. Since $\hat{L} = 0$, $V(\eta, [-r, 0]) = 0$. For $\varphi_1, \varphi_2 \in C([-r, 0], \mathbb{R})$ and $\|\varphi_1\| \le \rho$, $\|\varphi_2\| \le \rho$, we have

$$\begin{aligned} |f(t,\varphi_1) - f(t,\varphi_2)| &= \left| p(t)(e^{-\gamma(t)\varphi(-r)} - e^{-\gamma(t)\varphi_2(-r)}) \right| \\ &\le p(t)\gamma(t)e^{\gamma(t)\rho} \|\varphi_1 - \varphi_2\| \\ &\le p_+\gamma_+e^{\gamma_+\rho} \|\varphi_1 - \varphi_2\| . \end{aligned}$$

So we can pick $K_f(\rho) = p_+\gamma_+e^{\gamma_+\rho}$. Moreover, for $\varphi \in C([-r, 0], \mathbb{R})$, $\|\varphi\| \le \rho$ and $0 \le t \le T$,

$$|f(t,\varphi)| = \left| p(t)e^{-\gamma(t)\varphi(-r)} \right| \le p_+e^{\gamma_+\rho}.$$

So we get $L_f(T,\rho) = p_+e^{\gamma_+\rho}$. Then Assumption (iii) implies $(N+T)(K_f(\rho)+V(\eta, [-r, 0])) < 1$ and $(N+T)(L_f(T,\rho)+V(\eta, [-r, 0])) \le \rho$ in the assumption of Theorem 4.2. The conclusion follows from Theorem 4.2. □

Now we choose parameters for equation (4.5) such that assumptions in Proposition 4.3 are satisfied and perform numerical simulations to show the existence of a T-periodic solution. Let $T = 1$, $r = 1$, $\mu = 10$, $p(t) = 0.3 + 0.2\sin(2\pi t)$ and $\gamma(t) = 0.15 + 0.05\cos(2\pi t)$. It can be

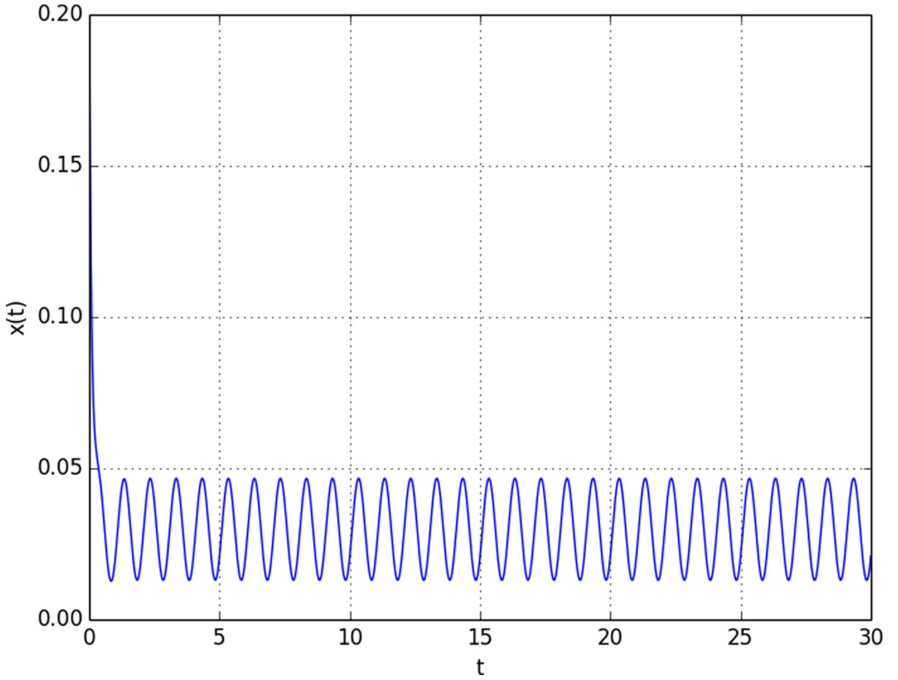

Fig. 1. A T-periodic solution of the delayed periodic red-blood cell model (4.5) with $r = 1$ starting at $\varphi(\theta) = 0.2$, $\theta \in [-1, 0]$, where $p(t) = 0.3 + 0.2\sin(2\pi t)$, $T = 1$ and $\gamma(t) = 0.15 + 0.05\cos(2\pi t)$.

easily checked we have all the assumptions in Proposition 4.3, then there exists a 1-periodic solution, which can be seen from Fig. 1.

Now we change the parameters so that assumptions in Proposition 4.3 are not satisfied. Let $T = 1$, $r = 1$, $\mu = 10$, $p(t) = 3 + 2\sin(2\pi t)$ and $\gamma(t) = 10 + 5\cos(2\pi t)$. Fig. 2 shows a solution in this scenario.

Remark 4.4. Similar techniques can be used to discuss the existence of periodic solutions in other delayed biological models such as the delayed periodic logistic equation (Chen [5]) and delayed periodic Nicholson's blowflies equation (Chen [6]).

Remark 4.5. Following the settings in Wu [43] and Ducrot et al. [12], we can also use the results in section 3 to study the existence of periodic solutions in abstract evolution equations with delay (Liu [23], Ezzinbi and Liu [13], Benkhalti and Ezzinbi [3], Kpoumiè et al. [20]) and partial functional differential equations with periodicity (Li et al. [22]).

4.2. Age-structured population models with periodic harvesting

Consider the following age-structured population model with periodic harvesting and global population dependent boundary condition (Anita et al. [1]):

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) + \mu(a)u(t,a) = f(t,a) - v(t,a)u(t,a), (a,t) \in [0,a^+] \times \mathbb{R}_+, \\ u(t,0) = \int_0^{a^+} \gamma(t,a)u(t,a)da, \\ u(t,a) = u(t+T,a), \end{cases} \tag{4.6}$$

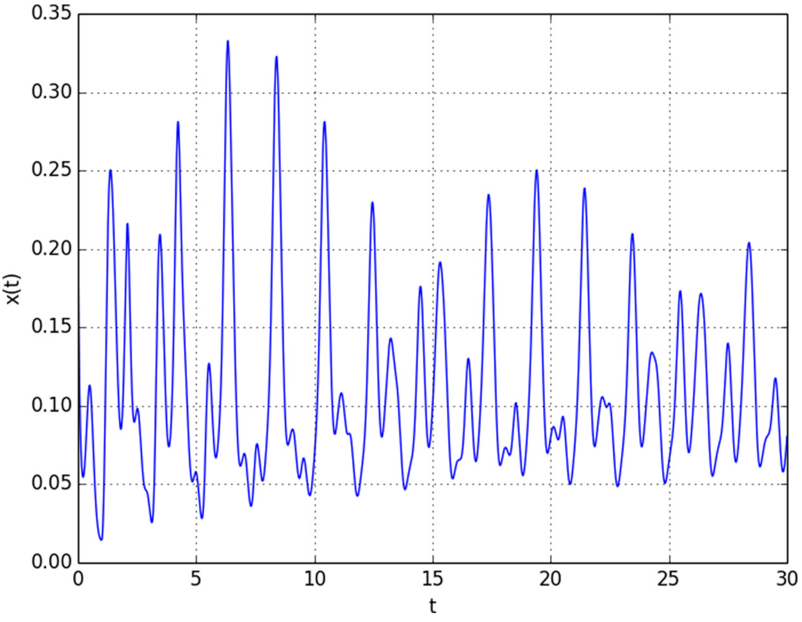

Fig. 2. An irregular solution of the delayed periodic red-blood cell model (4.5) with $r = 1$ starting at $\varphi(\theta) = 0.2, \theta \in [-1, 0]$, where $p(t) = 3 + 2\sin(2\pi t)$, $T = 1$ and $\gamma(t) = 10 + 5\cos(2\pi t)$.

where t is the time variable, a is the age variable, and $u(t, a)$ is the density of the population at time t with age a. This is a linear model for an age-structured population (see for instance Iannelli [18] and Webb [42]), where $\mu(a)$ is the age-specific death rate. Moreover, the population is subject to a T-periodic external flow $f(t, a)$ and a T-periodic age-specific harvesting effort $v(t, a)$ (see for instance Anita et al. [1]).

We have the following results.

Proposition 4.6. *Assume that*

(i) $f \in C([0,\infty), L^1[0,a^+))$, $f(t,a) = f(t+T,a)$ *for* $t \geq 0$, $a \in [0,a^+)$ *and* $\sup_{t\in[0,T]} \int_0^{a^+} |f(t,a)|\, da \leq f_+(T)$;

(ii) $\mu(a) \in L^1[0,a^+)$ *and there exists* $\mu_- > 0$ *such that* $\mu(a) \geq \mu_-$ *for* $a \in [0,a^+)$;

(iii) $\gamma(t,a) \in C([0,\infty), L^1[0,a^+))$, $\gamma(t,a) = \gamma(t+T,a)$ *and there exists* $\gamma_+ > 0$ *such that* $0 \leq \gamma(t,a) \leq \gamma_+$ *for* $t \geq 0$, $a \in [0,a^+)$;

(iv) $v(t,a) \in C^1([0,\infty), L^1[0,a^+))$ *and* $v(t,a) = v(t+T,a)$ *for* $t \geq 0$, $a \in [0,a^+)$;

(v) $(\frac{T}{1-e^{-(\mu_-+v_-)T}} + T)\gamma_+ < 1$ *and the inequality* $(\frac{T}{1-e^{-(\mu_-+v_-)T}} + T)(\gamma_+\rho + f_+(T)) \leq \rho$ *has solution.*

Then problem (4.6) *has a mild* T*-periodic solution* $u(t,a) \in C([0,\infty), L^1[0,a^+))$.

Proof. Consider the space $X := \mathbb{R} \times L^1(0,a^+)$ endowed with the product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0,a^+)} .$$

Define the time-dependent linear operator $A(t) : D(A(t)) \subset X \to X$ by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi - v(t)\varphi \end{pmatrix}$$

with $D(A(t)) = D = \{0\} \times W^{1,1}(0, a^+)$ and $\overline{D(A(t))} = \overline{D} = \{0\} \times L^1(0, a^+) \neq X$. Define $F : \mathbb{R}_+ \times \overline{D} \to X$ by

$$F\left(t, \begin{pmatrix} 0 \\ \phi \end{pmatrix}\right) = \begin{pmatrix} \int_0^{a^+} \gamma(t,a)\phi(a)da \\ f(t,a) \end{pmatrix}.$$

Then the partial differential equation (4.6) can be written as the evolution equation (3.8). For some $t \in \mathbb{R}_+$, let

$$(\lambda I - A(t))\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix},$$

$$(\lambda I - A(t))^{-1}\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}.$$

Since

$$(\lambda I - A(t))\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi(0) \\ \phi' + (\lambda + \mu + v(t))\phi \end{pmatrix},$$

we have

$$\phi(0) = \theta,$$

$$\phi'(a) + (\lambda + \mu(a) + v(t,a))\phi(a) = \varphi(a).$$

Then

$$\begin{aligned}\phi(a) &= \theta e^{-\int_0^a (\lambda+\mu(s)+v(t,s))ds} + e^{-\int_0^a (\lambda+\mu(s)+v(t,s))ds} \int_0^a e^{\int_0^s (\lambda+\mu(\tau)+v(t,\tau))d\tau}\varphi(s)ds \\ &= \theta e^{-\lambda a - \int_0^a \mu(s)ds - \int_0^a v(t,s)ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau}\varphi(s)ds.\end{aligned}$$

So

$$(\lambda I - A(t))^{-1}\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \theta e^{-\lambda a - \int_0^a \mu(s)ds - \int_0^a v(t,s)ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau}\varphi(s)ds \end{pmatrix}$$

and

$$
\begin{aligned}
&\left\| (\lambda I - A(t))^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \right\|_{L^1} \\
&= \left\| \theta e^{-\lambda a - \int_0^a \mu(s)ds - \int_0^a v(t,s)ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau} \varphi(s)ds \right\|_{L^1} \\
&\le |\theta| \left\| e^{-\lambda a - \int_0^a \mu(\tau)d\tau - \int_0^a v(t,\tau)d\tau} \right\|_{L^1} \\
&\quad + \left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau} \varphi(s)ds \right\|_{L^1} .
\end{aligned}
$$

W.L.O.G. assume $\varphi(a) \equiv 0$ for $a > a^+$ and extend $\varphi(a)$ to the whole $\mathbb{R}_+$. Since $\mu(a) \ge \mu_-$ and $v(t,a) \ge v_-$, we have

$$
\begin{aligned}
&\left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau} \varphi(s)ds \right\|_{L^1} \\
&= \int_0^{a^+} \left| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau} \varphi(s)ds \right| da \\
&\le \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t,\tau)d\tau} |\varphi(s)|\, dsda \\
&\le \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)|\, dsda \\
&\le \int_0^{\infty} \int_0^a e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)|\, dsda \\
&= \int_0^{\infty} \int_s^{\infty} e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)|\, dads \\
&= \int_0^{\infty} \Big(\int_s^{\infty} e^{-(\lambda+\mu_-+v_-)a} da \Big) e^{(\lambda+\mu_-+v_-)s} |\varphi(s)|\, ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \int_0^{\infty} |\varphi(s)|\, ds
\end{aligned}
$$

$$= \frac{1}{\lambda + \mu_- + v_-} \int_0^{a^+} |\varphi(s)|\, ds$$
$$= \frac{1}{\lambda + \mu_- + v_-} \|\varphi\|_{L^1}.$$

Moreover,

$$\left\| e^{-\lambda a - \int_0^a \mu(\tau)d\tau - \int_0^a v(t,\tau)d\tau} \right\|_{L^1}$$
$$= \int_0^{a^+} e^{-\lambda a - \int_0^a \mu(\tau)d\tau - \int_0^a v(t,\tau)d\tau} da$$
$$\le \int_0^{a^+} e^{-(\lambda+\mu_-+v_-)a} da$$
$$\le \int_0^{\infty} e^{-(\lambda+\mu_-+v_-)a} da$$
$$= \frac{1}{\lambda + \mu_- + v_-}.$$

So we obtain

$$\left\| (\lambda I - A(t))^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \right\|_{L^1} \le \frac{1}{\lambda + \mu_- + v_-} (|\theta| + \|\varphi\|_{L^1})$$

for all $t \in \mathbb{R}_+$ and $\lambda > -(\mu_- + v_-)$. It then follows that

$$\left\| (\lambda I - A(t))^{-1} \right\| \le \frac{1}{\lambda + \mu_- + v_-}$$

for all $t \in \mathbb{R}_+$ and $\lambda > -(\mu_- + v_-)$ so that

$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \le \frac{1}{(\lambda + \mu_- + v_-)^k}$$

for $\lambda > -(\mu_- + v_-)$ and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \le t_1 \le t_2 \le ... \le t_k$ and $k = 1, 2, \ldots$. Hence, Assumption 3.5 holds for $\{A(t)\}_{t\ge 0}$.

Moreover, we have

$$F\left(t, \begin{pmatrix} 0 \\ \phi \end{pmatrix}\right) = \begin{pmatrix} \int_0^{a^+} \gamma(t,a)\phi(a)da \\ f(t,a) \end{pmatrix},$$

$$F\left(t,\begin{pmatrix}0\\ \phi_1\end{pmatrix}\right)-F\left(t,\begin{pmatrix}0\\ \phi_2\end{pmatrix}\right)=\begin{pmatrix}\int_0^{a^+}\gamma(t,a)\phi_1(a)da\\ f(t,a)\end{pmatrix}-\begin{pmatrix}\int_0^{a^+}\gamma(t,a)\phi_2(a)da\\ f(t,a)\end{pmatrix}$$
$$=\begin{pmatrix}\int_0^{a^+}\gamma(t,a)(\phi_1(a)-\phi_2(a))da\\ 0\end{pmatrix}.$$

From the discussion of the case $v(t,a)\equiv 0$, we obtain

$$\left\|F\left(t,\begin{pmatrix}0\\ \phi_1\end{pmatrix}\right)-F\left(t,\begin{pmatrix}0\\ \phi_2\end{pmatrix}\right)\right\|\le\gamma_+\left\|\begin{pmatrix}0\\ \phi_1\end{pmatrix}-\begin{pmatrix}0\\ \phi_2\end{pmatrix}\right\|,$$

where $\gamma(t,a)\le\gamma_+$. So $K_F(\rho)=\gamma_+$. Assume $\sup_{t\in[0,T]}\int_0^{a^+}|f(t,a)|\le f_+(T)$, then from the discussion in the case $v(t,a)\equiv 0$, for $\|\phi\|_{L^1}\le\rho$,

$$\left\|F\left(t,\begin{pmatrix}0\\ \phi\end{pmatrix}\right)\right\|\le\gamma_+\rho+f_+(T).$$

Thus, we have $L_F(T,\rho)=\gamma_+\rho+f_+(T)$. So we have checked Assumption 3.2 (H2)(H3) and Assumption 3.5 (A1)(A2), and (iv) implies Assumption 3.5 (A3). Moreover, we have

$$M(N+T)K_F(\rho)=(\frac{T}{1-e^{-(\mu_-+v_-)T}}+T)\gamma_+<1$$

and there exists $\rho>0$ such that

$$M(N+T)L_F(T,\rho)=(\frac{T}{1-e^{-(\mu_-+v_-)T}}+T)(\gamma_+\rho+f_+(T))\le\rho.$$

So all assumptions in Theorem 3.8 are satisfied which ensures that there is a mild T-periodic solution. □

As an example, now we choose some specific functions and coefficients for problem (4.6) such that they satisfy conditions in Proposition 4.6. Let $T=1$, $v(t,a)=0.5+0.4a(1-a)\sin(2\pi t)$ and $\mu(a)=\frac{e^{-4a}}{1.0-a}$, then $\omega=-\mu_--v_-=-0.299<0$ and $N=\frac{MT}{1-Me^{\omega T}}=\frac{1}{1-e^{-0.299}}\approx 3.87$. Let $a^+=1$, $f(t,a)=1+2\sin(2\pi t)$ and $\gamma(t,a)=0.2a^2(1-a)(1+\sin(2\pi t))$. Then $K_F(\rho)\approx 0.059$, $(N+T)K_F(\rho)\approx 4.87\times 0.059\approx 0.28733<1$ for all $\rho>0$. In addition, $L_F(T,\rho)=0.059\rho+3$, then $(N+T)L_F(T,\rho)\le\rho\Leftrightarrow 4.87\times(0.059\rho+3)\le\rho\Leftrightarrow 0.28733\rho+14.61\le\rho$, which means that $\rho\ge 20.5$. Then equation (4.6) has a mild 1-periodic solution by Proposition 4.6. A solution of equation (4.6) is shown in Fig. 3.

Again, we change the parameters a little bit such that the assumptions of Proposition 4.6 are NOT satisfied. Let $\gamma(t,a)=4a^2(1-a)(1+\sin(2\pi t))$, then $\gamma_+=1.18$ and $(N+T)K_F(\rho)=4.87\times 1.18\approx 5.7466>1$. Then assumptions of Proposition 4.6 are not satisfied. The graph below shows a solution with the same initial value as the previous one, which is not periodic (see Fig. 4).

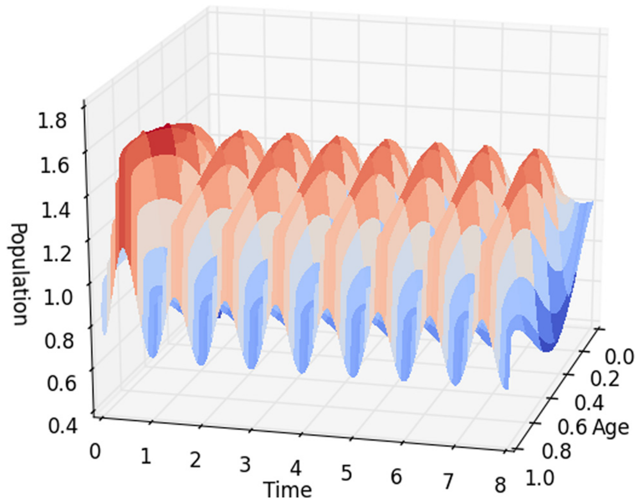

Fig. 3. A T-periodic solution of (4.6) starting at $u(0,a)=1$ and with global boundary condition $u(t,0)=\int_0^1\gamma(t,a)u(t,a)da$, where $\mu(a)=\frac{e^{-4a}}{1.0-a}$, $T=1$, $v(t,a)=0.5+0.4a(1-a)\sin(2\pi t)$, $\gamma(t,a)=0.2a^2(1-a)(1+\sin(2\pi t))$ and $f(t,a)=1+2\sin(2\pi t)$.

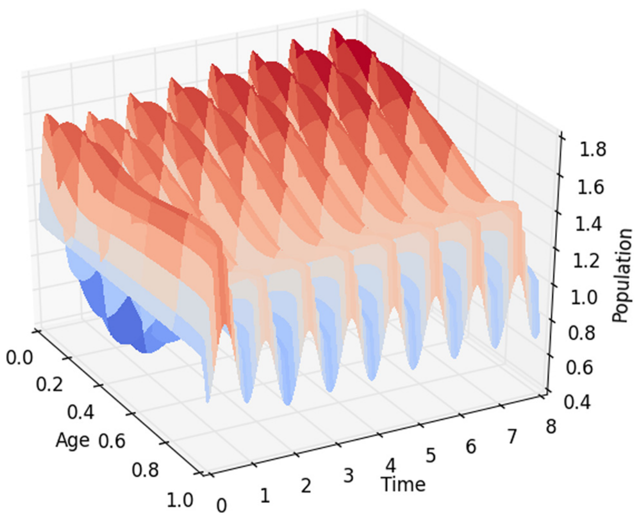

Fig. 4. A solution of (4.6) starting at $u(0,a)=1$ and with boundary condition $u(t,0)=\int_0^1\gamma(t,a)u(t,a)da$, where $\mu(a)=\frac{e^{-4a}}{1.0-a}$, $T=1$, $v(t,a)=0.5+0.4a(1-a)\sin(2\pi t)$, $\gamma(t,a)=4a^2(1-a)(1+\sin(2\pi t))$ and $f(t,a)=1+2\sin(2\pi t)$.

4.3. The diffusive logistic model with periodic coefficients

This subsection is concerned with a diffusive logistic model in T-periodic environment. Consider the following problem (Hess [17])

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x) + r(t)u(t,x)[1 - \frac{u(t,x)}{K(t)}],\ t \in \mathbb{R}_+,\ x \in [0,1], \\ u(t,0) = u(t,1) = 1, \\ u(t,x) = u(t+T,x), \end{cases} \tag{4.7}$$

where t is the time variable, x is the space variable, and $u(t,x)$ is the density of a population at time t and location x. In the logistic term, we have a T-periodic intrinsic growth rate $r(t)$ and a T-periodic carrying capacity $K(t)$. Moreover, we give constant boundary values. We assume that the population reaches certain steady state on the boundary. Without loss of generality, it is assumed that $u(t,0) = u(t,1) = 1$.

To study the existence of periodic solutions around the steady state $u^* = 1$, let $v(t,a) = u(t,a) - 1$, then

$$\begin{cases} \partial_t v(t,x) = \partial_x^2 v(t,x) + r(t)[v(t,x)+1][1 - \frac{v(t,x)+1}{K(t)}],\ t \in \mathbb{R}_+,\ x \in [0,1], \\ v(t,0) = v(t,1) = 0, \\ v(t,x) = v(t+T,x), \end{cases} \tag{4.8}$$

where $r(t)$ and $K(t)$ are T-periodic. The existence of solutions for (4.7) and that for (4.8) are equivalent. From now on, we consider (4.8).

Let $X = C[0,1]$. Define

$$Au = u''.$$

Then $D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$, $\overline{D(A)} = C_0[0,1] = \{u \in C[0,1] : u(0) = u(1) = 0\} \neq C[0,1] = X$. $D(A_0) = \{u \in D(A) : Au \in \overline{D(A)}\} = \{u \in C^2[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$, where A_0 is the part of A in $\overline{D(A)}$. It follows that A_0 generates a semigroup $\{U_A(t)\}_{t\geq 0}$ on $\overline{D(A)}$ given by

$$U_A(t)f(x) = \sum_{n=1}^{\infty}(2\int_0^1 f(\xi)\sin(n\pi\xi)d\xi)\sin(n\pi x)e^{-(n\pi)^2 t}.$$

Define $F : \mathbb{R}_+ \times \overline{D(A)} \to X$ by

$$F(t,\varphi) = r(t)(\varphi+1)(1 - \frac{\varphi+1}{K(t)}).$$

Then as before, we can rewrite (4.8) as abstract Cauchy problem (3.2).

Proposition 4.7. *Assume that*

(i) $r(t) \in C[0,\infty)$, there exists $r_+ > 0$ such that $0 \leq r(t) \leq r_+$ for $t \geq 0$, $r(t) = r(t+T)$;
(ii) $K(t) \in C[0,\infty)$, there exists $k_- > 0$ such that $K(t) \geq k_-$ for $t \geq 0$, $K(t) = K(t+T)$;
(iii) There exists $\rho > 0$ such that $(\frac{T}{\|I-U_A(T)\|} + T)r_+(\rho+1)(1+\frac{1+\rho}{k_-}) \leq \rho$.

Then problem (4.7) has a mild T-periodic solution.

Proof. It suffices to prove the following

(a) A is Hille-Yoshida operator with $M = 1$ and $\omega = 0$;
(b) There exists $L_F(T, \rho) \geq 0$ such that $\|F(t, u)\| \leq L_F(T, \rho)$ for $t \leq T$ and $\|u\| \leq \rho$;
(c) $U_A(t)$ is compact on $\overline{D(A)}$ for $t > 0$;
(d) There exists $\rho > 0$ such that $(\frac{T}{\|I-U_A(T)\|} + T)r_+(\rho + 1)(1 + \frac{1+\rho}{k_-}) \leq \rho$;
(e) The Cauchy problem (2.1) has a unique mild solution for each $x \in \overline{D(A)}$ and $f \in C([0, \infty), X)$, $f(t + T) = f(t)$. Moreover there exits $x \in \overline{D(A)}$ such that the solution $u(t)$ with $u(0) = x$ is bounded.

Note that if we rewrite (4.8) as abstract Cauchy problem (3.2), (a)-(e) cover all assumptions in Theorem 3.10. Then the existence of a mild T-periodic solution to problem (4.8) is guaranteed by Theorem 3.10. Thus, we get existence of a mild T-periodic solution to problem (4.7).

Now we prove (a)-(e).

(a) Let $\psi \in X$. Let $\lambda > 0$. Then

$$(\lambda I - A)\varphi = \psi \Leftrightarrow \lambda\varphi - \varphi'' = \psi.$$

Set $\hat{\varphi} = \varphi'$. Then

$$(\lambda I - A)\varphi = \psi \Leftrightarrow \begin{cases} \varphi' = \hat{\varphi} \\ \hat{\varphi}' = \lambda\varphi - \psi \end{cases}$$
$$\Leftrightarrow \begin{cases} \sqrt{\lambda}\varphi' + \hat{\varphi}' = \sqrt{\lambda}(\sqrt{\lambda}\varphi + \hat{\varphi}) - \psi \\ \sqrt{\lambda}\varphi' - \hat{\varphi}' = -\sqrt{\lambda}(\sqrt{\lambda}\varphi - \hat{\varphi}) + \psi. \end{cases}$$

Define

$$w = (\sqrt{\lambda}\varphi + \hat{\varphi}),$$
$$\hat{w} = (\sqrt{\lambda}\varphi - \hat{\varphi}).$$

Then we have

$$(\lambda I - A)\varphi = \psi \Leftrightarrow \begin{cases} w' = \sqrt{\lambda}w - \psi, \\ \hat{w}' = -\sqrt{\lambda}\hat{w} + \psi. \end{cases} \tag{4.9}$$

The first equation of (4.9) is equivalent to

$$e^{-\sqrt{\lambda}x}w(x) = e^{-\sqrt{\lambda}y}w(y) - \int_y^x e^{-\sqrt{\lambda}l}\psi(l)dl,\ \forall x \geq y. \tag{4.10}$$

In (4.10) let $y = 0$, then we obtain

$$w(x) = e^{\sqrt{\lambda}x}w(0) - e^{\sqrt{\lambda}x}\int_0^x e^{-\sqrt{\lambda}l}\psi(l)dl, \tag{4.11}$$

where $w(0) = \sqrt{\lambda}\varphi(0) + \hat{\varphi}(0) = \hat{\varphi}(0)$. In (4.10) let $x = 1$, we have

$$w(y) = e^{\sqrt{\lambda}y-\sqrt{\lambda}}w(1) + e^{\sqrt{\lambda}y}\int_y^1 e^{-\sqrt{\lambda}l}\psi(l)dl, \tag{4.12}$$

where $w(1) = \sqrt{\lambda}\varphi(1) + \hat{\varphi}(1) = \hat{\varphi}(1)$.

The second equation of (4.9) is equivalent to

$$e^{\sqrt{\lambda}x}\hat{w}(x) = e^{\sqrt{\lambda}y}\hat{w}(y) + \int_y^x e^{\sqrt{\lambda}l}\psi(l)dl,\ \forall x \geq y. \tag{4.13}$$

In (4.13) let $y = 0$, then we have

$$\hat{w}(x) = e^{-\sqrt{\lambda}x}\hat{w}(0) + e^{-\sqrt{\lambda}x}\int_0^x e^{\sqrt{\lambda}l}\psi(l)dl, \tag{4.14}$$

where $\hat{w}(0) = \sqrt{\lambda}\varphi(0) - \hat{\varphi}(0) = -\hat{\varphi}(0)$. In (4.13) let $x = 1$, we have

$$\hat{w}(y) = e^{\sqrt{\lambda}-\sqrt{\lambda}y}\hat{w}(1) - e^{-\sqrt{\lambda}y}\int_y^1 e^{\sqrt{\lambda}l}\psi(l)dl, \tag{4.15}$$

where $\hat{w}(1) = \sqrt{\lambda}\varphi(1) - \hat{\varphi}(1) = -\hat{\varphi}(1)$.

From (4.11) and (4.14), we have

$$e^{2\sqrt{\lambda}x}\hat{w}(x) + w(x) = \int_0^x e^{\sqrt{\lambda}x}(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})\psi(l)dl, \tag{4.16}$$

where $x \in [0, 1]$. Combining (4.12) and (4.15), we obtain

$$e^{2\sqrt{\lambda}(1-x)}w(x) + \hat{w}(x) = \int_x^1 e^{-\sqrt{\lambda}x}(e^{2\sqrt{\lambda}-\sqrt{\lambda}l} - e^{\sqrt{\lambda}l})\psi(l)dl. \tag{4.17}$$

Since $\hat{w} = \sqrt{\lambda}\varphi - \hat{\varphi}$ and $w = \sqrt{\lambda}\varphi + \hat{\varphi}$, (4.16) and (4.17) can be written as

$$\sqrt{\lambda}(e^{2\sqrt{\lambda}x} + 1)\varphi + (1 - e^{2\sqrt{\lambda}x})\hat{\varphi} = \int_0^x e^{\sqrt{\lambda}x}(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})\psi(l)dl \tag{4.18}$$

and

$$(e^{2\sqrt{\lambda}(1-x)} + 1)\sqrt{\lambda}\varphi + (e^{2\sqrt{\lambda}(1-x)} - 1)\hat{\varphi} = \int_x^1 e^{-\sqrt{\lambda}x}(e^{2\sqrt{\lambda}-\sqrt{\lambda}l} - e^{\sqrt{\lambda}l})\psi(l)dl. \tag{4.19}$$

Combining (4.18) and (4.19), we have the following

$$
\begin{aligned}
&\varphi(x)\\
&=\frac{(e^{2\sqrt{\lambda}-\sqrt{\lambda}x}-e^{\sqrt{\lambda}x})\int_0^x(e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l})\psi(l)dl-(e^{-\sqrt{\lambda}x}-e^{\sqrt{\lambda}x})\int_x^1(e^{2\sqrt{\lambda}-\sqrt{\lambda}l}-e^{\sqrt{\lambda}l})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&=\frac{\int_0^x(e^{2\sqrt{\lambda}-\sqrt{\lambda}(x-l)}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}(x-l)})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&\quad-\frac{\int_x^1(e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(l-x)}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(l-x)}+e^{\sqrt{\lambda}(x+l)})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&=\frac{\int_0^x(e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|x-l|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&\quad+\frac{\int_x^1(e^{2\sqrt{\lambda}-\sqrt{\lambda}|l-x|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|l-x|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&=\frac{\int_0^1(e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|x-l|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}.
\end{aligned}
$$

Since $\varphi\in\overline{D(A)}$, it follows that

$$
\begin{aligned}
\|\varphi\|&=\sup_{x\in[0,1]}|\varphi(x)|\\
&=\sup_{x\in[0,1]}\left|\frac{\int_0^1(e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|x-l|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\right|.
\end{aligned}
$$

Since $e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|x-l|}\geq 0$ for $x\in[0,1]$ and $l\in[0,1]$, we have

$$
\begin{aligned}
\|\varphi\|&\leq\sup_{x\in[0,1]}|\psi(x)|\sup_{x\in[0,1]}\frac{\int_0^1\left|e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|x-l|}\right|dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&=\sup_{x\in[0,1]}|\psi(x)|\sup_{x\in[0,1]}\frac{\int_0^1(e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|}-e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)}-e^{\sqrt{\lambda}(x+l)}+e^{\sqrt{\lambda}|x-l|})dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&=\sup_{x\in[0,1]}|\psi(x)|\sup_{x\in[0,1]}\frac{e^{2\sqrt{\lambda}}(\frac{2}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}e^{\sqrt{\lambda}(x-1)}-\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}x}+\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}(x+1)}-\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}x})}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}\\
&\quad+\frac{-\frac{2}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}(x-1)}+\frac{1}{\sqrt{\lambda}}e^{\sqrt{\lambda}x}-\frac{1}{\sqrt{\lambda}}e^{\sqrt{\lambda}(x+1)}+\frac{1}{\sqrt{\lambda}}e^{\sqrt{\lambda}x}}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{(e^{2\sqrt{\lambda}}-1)(\frac{2}{\sqrt{\lambda}}-\frac{1}{\sqrt{\lambda}}e^{\sqrt{\lambda}(x-1)}-\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}x}+\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}(x+1)}-\frac{1}{\sqrt{\lambda}}e^{-\sqrt{\lambda}x})}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)} \\
&\le \sup_{x\in[0,1]} |\psi(x)| \sup_{x\in[0,1]} \frac{(e^{2\sqrt{\lambda}}-1)\frac{2}{\sqrt{\lambda}}}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}}-1)} \\
&= \frac{1}{\lambda} \sup_{x\in[0,1]} |\psi(x)| \\
&= \frac{1}{\lambda} \|\psi\| .
\end{aligned}
$$

Now we have $\left\|(\lambda I - A)^{-1}\psi\right\| \le \frac{1}{\lambda}\|\psi\|$, which implies that $\left\|(\lambda I - A)^{-1}\right\| \le \frac{1}{\lambda}$. So A is Hille-Yoshida with $M = 1$ and $\omega = 0$, which completes the proof of (a).

(b) For $\|\varphi\| \le \rho$ and $t \in [0, 1]$,

$$
\left\| r(t)(\varphi+1)(1-\frac{1+\varphi}{K(t)}) \right\| \le r_+(\rho+1)(1+\frac{1+\rho}{k_-}).
$$

So we have $L_F(1,\rho) = r^+(\rho+1)(1+\frac{1+\rho}{k_-})$, which implies (b).

(c) It is well known and its proof can be found in previous studies like [13].

(d) It follows directly from assumption (iii).

(e) Claim (a) together with Theorem 2.4 implies that the Cauchy problem (2.1) has a unique mild solution for each $x \in \overline{D(A)}$ and $f \in C([0,\infty), X)$ with $f(t) = f(t+T)$, which is the first part of (e).

Now we check that there is a bounded solution. From the variation of constant formula

$$
u(t) = U_A(t)u_0 + \lim_{\lambda\to+\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} f(s)ds,
$$

we first consider the first part

$$
U_A(t)u_0(x) = \sum_{n=1}^{\infty} (2\int_0^1 u_0(\xi)\sin(n\pi\xi)d\xi)\sin(n\pi x)e^{-(n\pi)^2 t}.
$$

Then we have

$$
\begin{aligned}
|U_A(t)u_0(x)| &\le \sup_{x\in[0,1]} \sum_{n=1}^{\infty} \left|(2\int_0^1 u_0(\xi)\sin(n\pi\xi)d\xi)\right| |\sin(n\pi x)|\, e^{-(n\pi)^2 t} \\
&\le 2\sum_{n=1}^{\infty} \sup_{\xi\in[0,1]} |u_0(\xi)| \frac{2}{\pi} e^{-(n\pi)^2 t} \\
&= \sup_{\xi\in[0,1]} |u_0(\xi)| (\sum_{n=1}^{\infty} \frac{4}{\pi} e^{-(n\pi)^2 t}).
\end{aligned}
$$

It follows that

$$\lim_{t\to+\infty}\sup_{x\in[0,1]}|U_A(t)u_0(x)|=0.$$

So there exists an $M>0$ such that $|U_A(t)u_0(x)|\le M$ for $t\in[0,\infty)$ and $x\in[0,1]$.

Now we consider the second part $\lim_{\lambda\to+\infty}\int_0^t U_A(t-s)\lambda(\lambda I-A)^{-1}f(s)ds$ and have

$$\begin{aligned}
&\left|U_A(t-s)\lambda(\lambda I-A)^{-1}f(s)\right|\\
&=\left|\sum_{n=1}^{\infty}(2\int_0^1\lambda(\lambda I-A)^{-1}f(s)(\xi)\sin(n\pi\xi)d\xi)\sin(n\pi x)e^{-(n\pi)^2(t-s)}\right|\\
&\le 2\sum_{n=1}^{\infty}\int_0^1\left\|\lambda(\lambda I-A)^{-1}\right\||f(s)(\xi)||sin(n\pi\xi)|d\xi(e^{-(n\pi)^2(t-s)})\\
&\le 2\sum_{n=1}^{\infty}\sup_{\xi\in[0,1],s\in[0,1]}|f(s)(\xi)|\frac{2}{\pi}e^{-(n\pi)^2(t-s)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\int_0^t U_A(t-s)\lambda(\lambda I-A)^{-1}f(s)ds\right|\\
&\le 2\sup_{\xi\in[0,1],\tau\in[0,1]}|f(\tau)(\xi)|\int_0^t\sum_{n=1}^{\infty}\frac{2}{\pi}e^{-(n\pi)^2(t-s)}ds\\
&=2\sup_{\xi\in[0,1],\tau\in[0,1]}|f(\tau)(\xi)|\sum_{n=1}^{\infty}\frac{2}{\pi}e^{-(n\pi)^2t}\int_0^t e^{(n\pi)^2s}ds\\
&=2\sup_{\xi\in[0,1],\tau\in[0,1]}|f(\tau)(\xi)|\sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1}{(n\pi)^2}(1-e^{-(n\pi)^2t})\\
&<2\sup_{\xi\in[0,1],\tau\in[0,1]}|f(\tau)(\xi)|\sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1}{(n\pi)^2}\\
&\le\frac{4}{\pi^3}\sup_{\xi\in[0,1],\tau\in[0,1]}|f(\tau)(\xi)|\times 2\\
&=\frac{8}{\pi^3}\sup_{\xi\in[0,1],\tau\in[0,1]}|f(\tau)(\xi)|.
\end{aligned}$$

Hence, there exists an $M_0 > 0$ such that

$$\lim_{\lambda\to+\infty}\left|\int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1} f(s)ds\right| \le M_0,\ \forall t \ge 0.$$

Combining the above two parts, we have for each $u_0 \in \overline{D(A)}$, the solution to the Cauchy problem (2.1) is bounded for all $t \ge 0$, which completes the proof of the second part of (e). □

Now we choose specific functions and parameters. Let $T = 1$, $r(t) = 0.15 + 0.1\cos(2\pi t)$ and $K(t) = 15 + \sin(2\pi t)$, then $F(t,\varphi) = (0.15 + 0.1\cos(2\pi t))(\varphi + 1)(1 - \frac{1+\varphi}{15+\sin(2\pi t)})$. $N = \frac{1}{\|I-U_A(1)\|}$, where

$$U_A(t)[f(x)] = \sum_{n=1}^{\infty}(2\int_0^1 f(\xi)\sin(n\pi\xi)d\xi)\sin(n\pi x)e^{-(n\pi)^2 t},$$

$$\begin{aligned}\sup_{x\in[0,1]}|U_A(1)[f(x)]| &= \sup_{x\in[0,1]}\left|\sum_{n=1}^{\infty}(2\int_0^1 f(\xi)\sin(n\pi\xi)d\xi)\sin(n\pi x)e^{-(n\pi)^2}\right| \\ &\le \sup_{\xi\in[0,1]}|f(\xi)|\sum_{n=1}^{\infty}2\times\frac{2}{\pi}e^{-(n\pi)^2},\end{aligned}$$

in which

$$\begin{aligned}e^{-(n\pi)^2} &= \frac{1}{e^{(n\pi)^2}} \\ &= \frac{1}{1 + (n\pi)^2 + \frac{(n\pi)^4}{2} + \dots} \\ &\le \frac{2}{(n\pi)^4}.\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty}2\times\frac{2}{\pi}e^{-(n\pi)^2} \le \frac{4}{\pi}\times\frac{2}{(n\pi)^4} = \frac{8}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^4} < \frac{4}{3}\times\frac{8}{\pi^5} = \frac{32}{3\pi^5} < \frac{6}{\pi^3}.$$

So we derive

$$\sup_{x\in[0,1]}|U_A(1)[f(x)]| < \sup_{\xi\in[0,1]}|f(\xi)|\times\frac{6}{\pi^3},$$

i.e.,

$$\|U_A(1)\| < \frac{6}{\pi^3}.$$

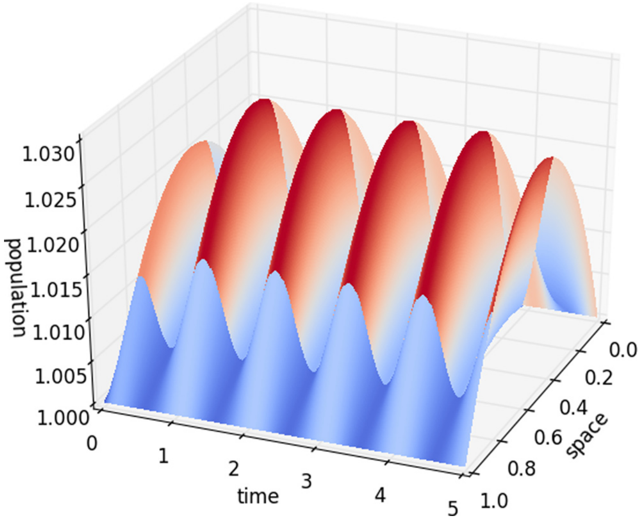

Fig. 5. A T-periodic solution of the diffusive logistic equation (4.7) starting at $u(0, a) = 1$ and with boundary condition $u(t, 0) = u(t, 1) = 1$, where $r(t) = 0.15 + 0.1\cos(2\pi t)$, $T = 1$ and $K(t) = 15 + \sin(2\pi t)$.

Then

$$N = \frac{1}{\|I - U_A(1)\|} < \frac{1}{1 - \frac{6}{\pi^3}} \approx 1.24.$$

For $\|\varphi\| \le \rho$ and $t \in [0, 1]$

$$\left\| r(t)(\varphi + 1)(1 - \frac{1+\varphi}{K(t)}) \right\| \le 0.25(\rho + 1)(1 + \frac{1+\rho}{14}).$$

So $r_+ = 0.25$.

Then from $(\frac{T}{\|I - U_A(T)\|} + T)r_+(\rho + 1)(1 + \frac{1+\rho}{k_-}) \le \rho$, we get $2.24 \times 0.25(\rho + 1)(1 + \frac{1+\rho}{14}) \le \rho$, i.e. $(\rho + 1)(\rho + 15) \le 25\rho$, which is also equivalent to $\rho^2 - 9\rho + 15 \le 0$, where we get $\frac{9-\sqrt{21}}{2} \le \rho \le \frac{9+\sqrt{21}}{2}$, such ρ exists.

Now all the assumptions in Proposition 4.7 are satisfied, we conclude that (4.8) has a mild 1-periodic solution, i.e., (4.7) has a mild 1-periodic solution. The graph in Fig. 5 shows the mild 1-periodic solution to the first equation and second boundary condition in (4.7) with initial value $u \equiv 1$, which confirms our result.

Remark 4.8. The results and techniques developed in this paper can be used to study the existence of periodic solutions in other structured population models in time-periodic environments, such as age-structured periodic models in Aniţa et al. [1], phenotype-structured periodic models in Lorenzi et al. [25], as well as periodic reaction-diffusion competition models in Zhao and Ruan [45,46] and Du et al. [11].

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