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Persistence and Extinction in Two Species Reaction–Diffusion Systems with Delays

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Both uniform persistence and global extinction are established for two species predator–prey and competition reaction–diffusion systems with delays in terms of the principal eigenvalues of the scalar elliptic eigenvalue problems by appealing to the theories of abstract persistence, asymptotically autonomous semiflows, and monotone dynamical systems. © 1999 Academic Press

1. INTRODUCTION

A fundamental problem in population dynamics is to study uniform persistence of the ecosystems, that is, to study the long term survival of interacting species. Abstract persistence theory, started in [BFW, BW], has been well-developed for both continuous and discrete semi-dynamical systems (see, e.g., [FRT, HW, FS, HS1, Th2, YR]) and has been applied to various types of equations including reaction–diffusion equations (see, e.g., [CC1, CCH, FLG, HS2, LG, Zh1, Zh2, ZH]) and functional differential equations (see, e.g., [FR]). For more details and references, we refer to a survey paper [HS2].

In realistic ecosystem models, diffusion and time delay should be taken into account. As argued in [Br] and pointed out by the referee, since individuals in the populations are moving, they may not have been at the same location in space at previous times, and the terms involving delay must be nonlocal in space. We refer to [GB1, GB2] for two-species com-

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petition or predator–prey models with double convolutions in both time and space. As shown in [GB1], by choosing the convolution kernels as delta functions, one can obtain diffusion equations with delay in time only. Although diffusion systems with time delays and nonlocal effects are more realistic (see [CC2] and the references therein), here we only consider the effect of time delays. In fact, recently a great deal of attention has been paid to reaction–diffusion equations with time delays (see, e.g., [FL, FZ, Ha1, Hu, KS, Lu, MS1, MS2, Pa, TW, Wu]). Topics include fundamental theory, comparison, monotonicity, convergence, stability, bifurcations, traveling waves, etc. We refer to a recent monograph [Wu] for the fundamental theory and references.

In this paper, we consider two species predator–prey and competition reaction–diffusion systems with delays. The purpose is first to establish uniform persistence criteria for these two types of systems. Current results on this subject (see [FL, Lu]) were obtained by using a comparison argument. We shall use the abstract persistence theory (see [HW]) and the infinite-dimensional dissipative system theory (see, e.g., [Ha2]). One of the main steps is to analyze the semiflow on the boundary and to show that the boundary equilibria are weakly repelling. Notice that on the boundary the two species system will reduce to a scalar diffusive logistic equation with delay. Most recently (see [FZ, Pa]), very useful results on the global dynamics of diffusive logistic equations with delay were established in terms of the the principal eigenvalues of the scalar elliptic eigenvalue problems. Thus, we can combine the results in [FZ, Pa] and the comparison theorem on reaction–diffusion equations with delays in [MS1, MS2] to establish the abstract uniform persistence for the solution semiflows generated by both predator–prey and competition systems. By a general result on the existence of the stationary coexistence state [Zh1], the abstract uniform persistence enables us to conclude that the delayed reaction–diffusion systems admit at least one positive steady state. However, since the solution semiflow is defined on the positive cone of the Banach space of continuous functions on a compact set with the usual maximum norm, the abstract uniform persistence only implies that there is a $\delta > 0$ such that any solution $u(t, x, \phi)$ with inner initial value ϕ satisfies $\liminf_{t\to\infty} \max_{x\in\bar{\Omega}} u_i(t, x, \phi) \geqslant \delta$, $i = 1, 2$. It is then natural and practical to expect that there is a uniform lower bound for all $x \in \bar{\Omega}$. Notice that it is impossible to have a uniform lower positive constant bound in the case of Dirichlet boundary condition since for all $t \geqslant 0$, the solution is always zero on the boundary of the spatial domain Ω. Therefore we also need to consider the practical uniform persistence for the delayed reaction–diffusion systems (see the statement of our Theorem 3.2 for the precise definition). Thanks to the compactness, invariance, and attractivity of the inner global attractor for the semiflow and the parabolic maximum principle, we are then able to prove that the abstract uniform persistence actually implies the practical persistence. For some discussions related to practical persistence in reaction–diffusion systems, we refer to [CC1, CCH, Co, HS2, Zh1, ZH].

We then would like to know more about the dynamics of these systems. If some or all of the persistence conditions are reversed, we would like to know whether the semi-trivial or the trivial equilibrium is globally attractive, that is, whether one or both species go extinct. For instance, for the predator–prey system, we want to study the global attractivity of the boundary equilibrium $(u_1^*, 0)$. To do this, technically we only need to consider a scalar equation, namely the prey equation. However, even if the limit of u_2 is zero, it still appears in the prey equation. Usually, it is not too much easier to study a scalar equation involving both components than to study the original system. Notice that $u_2 \to 0$ as $t \to 0$. Taking this limit in the prey equation, then, the limiting equation will be a scalar equation in u_1 only. By appealing to the asymptotically autonomous semiflow theory due to Thieme [Th1] and Mischaikow *et al.* [MST], we then can carry the properties for the limiting scalar equation in u_1 onto the prey equation. For the competition system, we shall use Martin and Smith's comparison theorem for delayed reaction–diffusion systems [MS2] and Hirsch's global attractivity theorem for monotone semiflows [Hi1] to show that the "competition exclusion principle" occurs under certain assumptions, that is, only one species wins the competition. For more detailed results on monotone dynamical systems, we refer to [DH, Hi2, Sm] and the references cited therein.

We would like to point out that we could consider distributed delay in the logistic reaction term, since there is no essential difference in our dynamical system approach, we only consider discrete delay for the sake of simplicity (see Remark 2.6).

This paper is organized as follows. In Section 2, we summarize some results in abstract persistence theory [HW], asymptotically autonomous semiflow theory [Th1; MST], and on the scalar diffusive logistic equations with delay [FZ; Pa]), which will be used throughout the paper. In Section 3, we first establish weak repellency of the semi-trivial steady state in general reaction–diffusion systems with delays, then we derive uniform persistence criteria for both predator–prey and competition systems. Global extinction criteria are established in Section 4.

2. PRELIMINARIES

In this section, we recall some results on abstract persistence, asymptotically autonomous semiflows, and scalar diffusive logistic equations with delay, which will be used in the following sections.

2.1. *Uniform Persistence*

Let (X, d) be a complete metric space with metric d. Suppose that $T(t)\colon X \to X$, $t \geqslant 0$, is a C^0-semiflows on X, that is, $T(0) = I$, $T(t+s) = T(t)\, T(s)$ for $t, s \geqslant 0$, and $T(t)x$ is continuous in t and x. $T(t)$ is said to be *point dissipative* in X if there is a bounded nonempty set B in X such that for any $x \in X$, there is a $t_0 = t_0(x, B) > 0$ such that $T(t)x \in B$ for $t \geqslant t_0$.

DEFINITION 2.1. Assume that $X = X_0 \cup \partial X_0$ and $X_0 \cap \partial X_0 = \varnothing$ with X_0 being open in X. The semiflow $T(t)\colon X \to X$ is said to be *uniformly persistent with respect to* $(X_0, \partial X_0)$ if there is an $\eta > 0$ such that for any $x \in X_0$, $\liminf_{t \to \infty} d(T(t)x, \partial X_0) \geqslant \eta$.

Let $\omega(x)$ denote the ω-limit set of $x \in X$ for semiflow $T(t)\colon X \to X$ and let $\tilde{A}_\partial = \bigcup_{x \in \partial X_0} \omega(x)$. The set $\tilde{A}_\partial$ is said to be *acyclic* if there exists an isolated covering $\bigcup_{i=1}^k M_i$ of $\tilde{A}_\partial$ such that no subset of the M_i's forms a cycle. Then we have the following theorem on uniform persistence [HW, Theorem 4.1].

THEOREM 2.2. *Suppose $T(t)\colon X \to X$ satisfies $T(t)\colon X_0 \to X_0$ and $T(t)\colon \partial X_0 \to \partial X_0$ and we have the following*:

- (i) *there is a $t_0 \geqslant 0$ such that $T(t)$ is compact for $t > t_0$;*
- (ii) *$T(t)$ is point dissipative in X;*
- (iii) *$\tilde{A}_\partial$ is isolated and has an acyclic covering $\bigcup_{i=1}^k M_i$.*

Then $T(t)$ is uniformly persistent with respect to $(X_0, \partial X_0)$ if and only if for each M_i, $1 \leqslant i \leqslant k$,

$$W^s(M_i) \cap X_0 = \varnothing, \tag{2.1}$$

where $W^s(M_i) = \{x\colon x \in X, \omega(x) \neq \varnothing, \omega(x) \subset M_i\}$.

2.2. *Asymptotically Autonomous Semiflows*

Let $\Delta = \{(t, s)\colon 0 \leqslant s \leqslant t < \infty\}$. Consider the mapping $\Phi\colon \Delta \times X \to X$. Φ is called a *nonautonomous semiflow* if it is continuous and satisfies $\Phi(s, s, x) = x$, $s \geqslant 0$, and $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x)$, $t \geqslant s \geqslant r \geqslant 0$, $x \in X$.

DEFINITION 2.3. A nonautonomous semiflow Φ on X is called *asymptotically autonomous*—with *limit semiflow* $T(t)$—if $T(t)$ is an autonomous semiflow on X and

$$\Phi(t_j + s_j, s_j, x_j) \to T(t)x, \qquad j \to \infty \tag{2.2}$$

for any three sequences $t_j \to t$, $s_j \to \infty$, $x_j \to x$, $j \to \infty$ with $x, x_j \in X$, $0 \leqslant t$, $t_j < \infty$ and $s_j \geqslant 0$.

The following generalized Markus' theorem is due to Thieme [Th1, Theorem 4.1]. For the chain recurrence and Liapunov functions in asymptotically autonomous semiflows, we refer to a recent paper by Mischaikow *et al.* [MST]. For asymptotically periodic semiflow theory, we refer to [Zh2].

THEOREM 2.4. *Let Φ be an asymptotically autonomous semiflow on X and $T(t)$ its limit semiflow. Let e be a locally asymptotically stable equilibrium of $T(t)$, i.e., $T(t)e = e$ for all $t \geqslant 0$, and $W^s(e)$ its stable set for $T(t)$. Then every pre-compact Φ-orbit whose $\omega - \Phi$-limit set intersects $W^s(e)$ converges to e.*

2.3. *Diffusive Logistic Equations with Delay*

Now we consider a diffusive Logistic equation with time delay:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = d\Delta u + u[a(x) - b(x)\,u(x,t) + c(x)\,u(x,t-\tau)], \\
&\qquad\qquad\qquad t > 0, \quad x \in \Omega, \\
&Bu = 0, \qquad\qquad t > 0, \quad x \in \partial\Omega, \\
&u(x,\theta) = \phi(x,\theta), \qquad -\tau \leqslant \theta \leqslant 0, \quad x \in \Omega,
\end{aligned}
\tag{2.3}
$$

where $d > 0$, Ω is a bounded domain in R^N with a smooth boundary $\partial\Omega$, Δ is the Laplacian operator on Ω, $Bu = u$ (Dirichlet boundary condition) or $Bu = (\partial u/\partial v) + \alpha(x)u$ (Robin or Neumann boundary condition), $\alpha(\cdot) \in C^1(\bar{\Omega}, [0, \infty))$ and $(\partial/\partial v)$ denotes differentiation in the direction of the outward normal to $\partial\Omega$. We assume that $a(x)$ and $b(x)$ are positive functions, and *either* $c(x) \geqslant 0$ for all $x \in \bar{\Omega}$, *or* $c(x) \leqslant 0$ for all $x \in \bar{\Omega}$. Let $\lambda_0 = \lambda_0(a(\cdot))$ be the principal eigenvalue of the eigenvalue problem

$$
\begin{aligned}
d\Delta w(x) + a(x)\,w(x) = \lambda w(x), \qquad & x \in \Omega, \\
Bw(x) = 0, \qquad & w \in \partial\Omega.
\end{aligned}
\tag{2.4}
$$

Let $X^+ = C(\bar{\Omega} \times [-\tau, 0], R^+)$ in the case of Robin or Neumann boundary condition and let

$$
\begin{aligned}
X^+ &= C_0(\bar{\Omega} \times [-\tau, 0], R^+) \\
&= \{\phi \colon \phi \in (\bar{\Omega} \times [-\tau, 0], R^+), \phi(x,\theta) = 0, x \in \partial\Omega, \theta \in [-\tau, 0]\}
\end{aligned}
$$

in the case of Dirichlet boundary condition. Then we have the following threshold result on the global dynamics of (2.3), which is essentially a combination of [FZ, Proposition 3.3] and [Pa, Theorem 5.1].

THEOREM 2.5. *Assume that $b(x) > |c(x)|$ for all $x \in \bar{\Omega}$. For any $\phi \in X^+$, let $u(x, t, \phi)$ be the unique solution of* (2.3).

(a) *If* $\lambda_0(a(\cdot)) \leqslant 0$, *then for any* $\phi \in X^+$, $\lim_{t \to \infty} u(x, t, \phi) = 0$ *uniformly for* $x \in \bar{\Omega}$;

(b) *If* $\lambda_0(a(\cdot)) > 0$, *then* (2.3) *has a unique positive steady state solution* $u^*(x)$ *and for every* $\phi \in X^+$ *with* $\phi(\cdot, 0) \not\equiv 0$, $\lim_{t \to \infty} u(x, t, \phi) = u^*(x)$ *uniformly for* $x \in \bar{\Omega}$.

Proof. In the case where $c(x) > 0$, $x \in \bar{\Omega}$, the conclusion is a direct application of [FZ, Proposition 3.3] for the Robin or Neumann boundary condition and an argument similar to [FZ, Proposition 3.3] applies for the Dirichlet boundary condition with $X^+ = C_0(\bar{\Omega} \times [-\tau, 0], R^+)$. In the case where $c(x) < 0$, $x \in \bar{\Omega}$, the conclusion is essentially the same as [Pa, Theorem 5.1]. Indeed, there $\lambda(0)$ is the smallest eigenvalue of the eigenvalue problem

$$\begin{aligned} d\Delta w(x) + \lambda a(x)\, w(x) &= 0, \qquad x \in \Omega, \\ Bw(x) &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{2.5}$$

It then easily follows that $\lambda_0(a(\cdot))$ and $(1 - \lambda(0))$ have the same sign (see, e.g., [He, Chapter II.15, Theorems 16.1 and 16.3 and Remark 16.5]). This completes the proof. ▮

Remark 2.6. From the proofs of Proposition 3.3 in [FZ] and Theorems 1.2 and 1.3 in [Hu], the threshold global dynamics still holds for (2.3) with $c(x)\, u(x, t-\tau)$ replaced by $c(x) \int_0^\tau u(t-s)(x)\, m(ds)$, where $m(\cdot)$ is a nonnegative measure with $\int_0^\tau m(ds) = 1$.

3. UNIFORM PERSISTENCE

In this section, we first derive a result on weak repellency of the semitrivial steady state in general reaction–diffusion systems with delay. Then we establish uniform persistence criteria for the predator–prey and competition systems, respectively.

3.1. *Weak Repellor for Delayed Reaction–Diffusion Systems*

Let $\tau \geqslant 0$ and m be an integer. Define $C_\tau = C([-\tau, 0], R^m)$. For any $\phi \in C_\tau$, define $\|\phi\| = \max_{\theta \in [-\tau, 0]} |\phi(\theta)|$. Then C_τ is a Banach space. Let ^ denote the inclusion $R^m \to C_\tau$ by $u \to \hat{u}$, $\hat{u}(\theta) = u$, $\theta \in [-\tau, 0]$. Given a function $u(x, t)$: $\bar{\Omega} \times [-\tau, \sigma) \to R^m$ $(\sigma > 0)$, for each $x \in \bar{\Omega}$, define $u_t(x) \in C_\tau$ by $u_t(x)(\theta) = u(x, t+\theta)$, $\theta \in [-\tau, 0]$.

Consider the nonautonomous reaction–diffusion equation with delay

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= d_i\,\Delta u_i + u_i F_i(t, x, u_t(x)), && t>0, \quad x \in \Omega, \quad 1 \leqslant i \leqslant m, \\ B_i u_i &= 0, && t>0, \quad x \in \partial\Omega, \quad i \leqslant i \leqslant m, \end{aligned} \tag{3.1}$$

where for each $1 \leqslant i \leqslant m$, F_i: $R^+ \times \bar{\Omega} \times C_\tau \to R$ is continuous and Lipschitzian on bounded subsets of $R^+ \times C_\tau$ uniformly in $x \in \bar{\Omega}$, $d_i > 0$, and $B_i u_i = (\partial u_i / \partial v) + \alpha_i(x) u_i$, $\alpha_i(\cdot) \in C^1(\bar{\Omega}, R^+)$ or $B_i u_i = u_i$.

Let $C_0(\bar{\Omega}, R^+) = \{\phi\colon \phi \in C(\bar{\Omega}, R),\ \ \phi(x) = 0,\ \ x \in \partial\Omega\}$ and $C_0(\bar{\Omega} \times [-\tau, 0], R) = \{\phi\colon \phi \in C(\bar{\Omega} \times [-\tau, 0], R), \phi(x, \theta) = 0,\ x \in \partial\Omega,\ \theta \in [-\tau, 0]\}$. We will distinguish between Robin or Neumann boundary condition (R.B.C.) and Dirichlet boundary condition (D.B.C.). For each $1 \leqslant i \leqslant m$, let $X_i = C([-\tau, 0], C(\bar{\Omega}, R))$ and $W_i = C(\bar{\Omega} \times [-\tau, 0], R)$ if $B_i u_i = (\partial u_i / \partial v) + \alpha_i(x)\, u_i$; and $X_i = C([-\tau, 0],\ C_0(\bar{\Omega}, R))$ and $W_i = C_0(\bar{\Omega} \times [-\tau, 0], R)$ if $B_i u_i = u_i$. Let $X = \prod_{i=1}^m X_i$. We will identify X with $\prod_{i=1}^m W_i$ when this does not cause confusion. By a standard formulation of (3.1) (see, e.g., [MS1, MS2]), for any $\phi \in X$, there exists a unique solution $\tilde{u}(t, \phi)$ on the maximal interval $[0, \tilde{\sigma}_\phi)$, $\tilde{\phi}_\phi > 0$, satisfying $\tilde{u}_0 = \phi$. Moreover, if $\tau < \tilde{\sigma}_\phi$, then for $t > \tau$, $u(x, t) = u(t, \phi)(x)$ is a classical solution of (3.1). By [MS2, Proposition 1.3], the positive cone X^+ of X is positively invariant for (3.1), i.e., for any $\phi \in X$ with $\phi_i(x, \theta) \geqslant 0$, $1 \leqslant i \leqslant m$, $\tilde{u}_i(t, \phi)(x) \geqslant 0$, $1 \leqslant i \leqslant m$, for all $x \in \Omega$ and $t \in [0, \tilde{\sigma}_\phi)$.

We also consider the autonomous reaction–diffusion system

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= d\Delta u_i + u_i F_i^0(x, u_t), && t>0, \quad x \in \Omega, \quad 1 \leqslant i \leqslant m, \\ B_i u_i &= 0, && t>0, \quad x \in \partial\Omega, \quad 1 \leqslant i \leqslant m, \end{aligned} \tag{3.2}$$

where F_i^0: $\bar{\Omega} \times C_\tau \to R$ is continuous and Lipschitzian on bounded sets of C_τ uniformly in $x \in \bar{\Omega}$, $i \leqslant i \leqslant m$. For any $\phi \in X$, let $u(t, \phi)$ be the unique solution of (3.2) satisfying $u_0 = \phi$ on the maximal interval $[0, \sigma_\phi)$, $\sigma_\phi > 0$.

For each $1 \leqslant i \leqslant m$, let

$$Y_{0i} = \{\phi_i \in X_i\colon \phi_i(x, \theta) \geqslant 0,\ x \in \bar{\Omega},\ \theta \in [-\tau, 0], \text{ and } \phi_i(x, 0) \not\equiv 0\}.$$

Let $Y_0 = \prod_{i=1}^m Y_{0i}$ and $\partial Y_0 = X^+ \backslash Y_0$, where $X^+ = \prod_{i=1}^m X_i^+$ is the positive cone of X. We further assume that for any $\phi \in X^+$, both $\tilde{u}(t, \phi)$ and $u(t, \phi)$ exist globally on $[0, \infty)$. Then we have the following lemma.

LEMMA 3.1. *Assume that there exists some* $1 \leqslant i \leqslant m$ *such that*

(1) $\lim_{t \to \infty} |F_i(t, x, \phi) - F_i^0(x, \phi)| = 0$ *uniformly for* $x \in \bar{\Omega}$ *and* ϕ *in bounded subsets of* C_τ;

(2) $\hat{u}^*(x) = (u_1^*(x), ..., u_{i-1}^*(x), 0, u_{i+1}^*(x), ..., u_n^*(x))$ *is a nonnegative equilibrium of* (3.2) *and* $\lambda_0 = \lambda_0(d_i, F_i^0(x, \hat{u}^*(x))) > 0$, *where* $\lambda_0(d_i, F_i^0(x, \hat{u}^*(x)))$ *is the principal eigenvalue of the eigenvalue problem*

$$\begin{aligned} d_i \Delta w + F_i^0(x, \hat{u}^*(x)) w &= \lambda w(x), \qquad && x \in \Omega, \\ B_i w_i &= 0, && x \in \partial\Omega. \end{aligned} \tag{3.3}$$

Then there exists a $\delta > 0$ *such that for any* $\phi \in Y_0$,

$$\limsup_{t \to \infty} \|\tilde{u}_t(\phi) - \hat{u}^*\| \geqslant \delta.$$

Proof. Since for any $t \geqslant 0$, $x \in \bar{\Omega}$, $\phi \in X$,

$$\begin{aligned} |F_i(t, x, \phi(\cdot, x)) - F_i^0(x, \hat{u}^*(x))| \leqslant{}& |F_i(t, x, \phi(\cdot, x)) - F_i^0(x, \phi(\cdot, x))| \\ &+ |F_i^0(x, \phi(\cdot, x)) - F_i^0(x, \hat{u}^*(x))|, \end{aligned}$$

condition (1) implies that

$$\lim_{t \to \infty, \phi \to \hat{u}^*} (F_i(t, x, \phi(\cdot, x)) - F_i^0(x, \hat{u}^*(x)) = 0$$

uniformly for $x \in \bar{\Omega}$. Choose $0 < \varepsilon_0 < \lambda_0$. Then, there exist $t_0 > 0$ and $\delta_0 > 0$ such that

$$F_i(t, x, \phi(\cdot, x)) \geqslant F_i^0(x, \hat{u}^*(x)) - \varepsilon_0 \tag{3.4}$$

for all $t \geqslant t_0$, $x \in \bar{\Omega}$ and $\|\phi - \hat{u}^*\| < \delta_0$. Suppose that, by contradiction, there exists some $\phi_0 \in Y_0$ such that

$$\limsup_{t \to \infty} \|u_t(\phi_0) - \hat{u}^*\| < \frac{\delta_0}{2}.$$

Then there exists $t_1 \geqslant t_0$ such that

$$\|u_t(\phi_0) - \hat{u}^*\| < \delta_0 \qquad \text{for all} \quad t \geqslant t_1. \tag{3.5}$$

Therefore, by (3.4),

$$F_i(t, x, u_t(\phi_0)(x)) \geqslant F_i^0(x, \hat{u}^*(x)) - \varepsilon_0, \qquad t \geqslant t_1, \quad x \in \bar{\Omega}$$

and hence $u_i(x, t) = u_i(t, \phi)(x)$ satisfies

$$\begin{aligned} &\frac{\partial u_i}{\partial t} \geqslant d_i\, \Delta u_i + u_i[F_i^0(x, \hat{u}^*(x)) - \varepsilon_0], && t \geqslant t_1, \quad x \in \bar{\Omega}, \\ &B_i u_i = 0, && t \geqslant t_1, \quad x \in \partial\Omega. \end{aligned} \tag{3.6}$$

Let $\phi_i(x)) \gg 0$ in $C(\bar{\Omega})$ when $B_i u_i = (\partial u_i/\partial v) + \alpha_i u_i$ or in $C_0^1(\bar{\Omega})$ when $B_i u_i = u_i$, be the principal eigenvalue corresponding to λ_0, that is, ϕ_i satisfies (3.3). Since $u_i(t_1) \gg 0$ in $C(\bar{\Omega})$ or $C_0^1(\bar{\Omega})$, depending on the R.B.C. or D.B.C., respectively, there exists a $k > 0$ such that

$$u_i(x, t_1) \geqslant k\phi_i(x), \qquad x \in \bar{\Omega}. \tag{3.7}$$

It then easily follows that $v_i(x, t) = ke^{(\lambda_0 - \varepsilon_0)(t - t_1)}\phi_i(x)$ satisfies

$$\begin{aligned} &\frac{\partial v_i}{\partial t} = d_i\, \Delta v_i + v_i[F_i^0(x, \hat{u}^*(x)) - \varepsilon_0], && t \geqslant t_1, \quad x \in \Omega, \\ &B_i v_i = 0, && t \geqslant t_1, \quad x \in \partial\Omega. \end{aligned} \tag{3.8}$$

Since (3.7) implies $u_i(x, t_1) \geqslant v_i(x, t_1)$ for $x \in \bar{\Omega}$, by (3.6), (3.8) and the standard comparison theorem,

$$u_i(x, t) \geqslant v_i(x, t) = ke^{(\lambda_0 - \varepsilon_0)(t - t_1)}\phi_i(x), \qquad t \geqslant t_1, \quad x \in \bar{\Omega}, \tag{3.9}$$

which contradicts (3.5) when we let $t \to \infty$. Therefore the lemma holds for $\delta = \delta_0/2$. This completes the proof. ∎

3.2. *The Predator–Prey System*

Consider the predator–prey reaction–diffusion system with delays:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t) + \bar{a}_{11}(x)\, u_1(x, t - \tau_{11}) \\ &\quad - a_{12}(x)\, u_2(x, t - \tau_{12})], \qquad t > 0, \quad x \in \Omega, \\ \frac{\partial u_2}{\partial t} &= d_2\, \Delta u_2 + u_2[b_2(x) + a_{21}(x)\, u_1(x, t - \tau_{21}) - a_{22}(x)\, u_2(x, t) \\ &\quad + \bar{a}_{22}(x)\, u_2(x, t - \tau_{22})], \qquad t > 0, \quad x \in \Omega, \\ B_1 u_1 &= B_2 u_2 = 0, \qquad t > 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.10}$$

where for every $1 \leqslant i, j \leqslant 2$, d_i is a positive constant, $\tau_{ij} \geqslant 0$, $b_i(x)$ and $a_{ij}(x)$ are positive continuous functions on $\bar{\Omega}$, $\bar{a}_{ii}(\cdot) \in C(\bar{\Omega}, R)$, and $B_i u_i = (\partial u_i/\partial v) + \alpha_i(x)\, u_i$, $\alpha_i \in C(\bar{\Omega}, R^+)$, or $B_i u_i = u_i$. We will further impose the following condition on $\bar{a}_{ii}(\cdot)$, $1 \leqslant i \leqslant 2$.

(H) For each $1 \leqslant i \leqslant 2$, $a_{ii}(x) > |\bar{a}_{ii}(x)|$, $x \in \bar{\Omega}$, and *either* $\bar{a}_{ii}(x) \geqslant 0$, $x \in \bar{\Omega}$, *or* $\bar{a}_{ii}(x) \leqslant 0$, $x \in \bar{\Omega}$.

Notice that we assume that the delays are nonnegative. This includes the special case when the delay involved in the predator term in the prey equation is zero. Moreover, the delay involved in the prey term in the predator equation could be positive. For example, if the prey is regarded as the nutrient in the chemostat, the species (predator) may have a time delay in digesting the nutrient (see, e.g., [SW]). Biologically, for the single population growth in the absence of the other population, $\bar{a}_{ii}(x) \geqslant 0$, $x \in \bar{\Omega}$ implies that the positive feedback is $b_i(x) + \bar{a}_{ii}(x)\, u_i(x, t - \tau_{ii})$ while the negative feedback (i.e., the self-limitation effect) is $a_{ii}(x)\, u_i(x, t)$; and $\bar{a}_{ii}(x) \leqslant 0$, $x \in \bar{\Omega}$ implies that the negative feedback is $a_{ii}(x)\, u_i(x, t) - \bar{a}_{ii}(x)\, u_i(x, t - \tau_{ii})$. Moreover, $a_{ii}(x) > |\bar{a}_{ii}(x)|$, $x \in \bar{\Omega}$ implies that the instantaneous self-limitation effect dominates the corresponding delayed effect.

In what follows, we let $\lambda_0(d_i, m(\cdot))$ denote the principal eigenvalue of the eigenvalue problem

$$\begin{aligned} d_i\, \Delta w + m(x) w &= \lambda w, && x \in \Omega, \\ B_i w &= 0, && x \in \partial\Omega \end{aligned} \tag{3.11}$$

and $u_1^*(x)$ be the unique positive steady state of the diffusive Logistic equation

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = d_1\, \Delta u_1 + u_1[b_1(x) + (\bar{a}_{11}(x)) - a_{11}(x)\, u_1], && t > 0, \quad x \in \Omega, \\ &B_1 u_1 = 0, && t > 0, \quad x \in \partial\Omega. \end{aligned} \tag{3.12}$$

We then have the following result on the uniform persistence of u_1 and u_2.

THEOREM 3.2. *Let* (H) *hold. Assume that*

(A1) $\lambda_0(d_2, b_2(\cdot)) \leqslant 0$, $\lambda_0(d_1, b_1(\cdot)) > 0$ *and* $\lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)\, u_1^*(\cdot)) > 0$.

Then system (3.10) *admits at least one positive steady state and is uniformly persistent. More precisely, there exists a* $\beta_0 > 0$ *such that for any* $\phi \in Y_0$ (*with* $\tau = \max_{1 \leqslant i, j \leqslant 2}\{\tau_{ij}\}$ *and* $m = 2$), *there exists a* $t_0 = t_0(\phi)$ *such that* $u(t, \phi)(x) = (u_1(t, \phi)(x), u_2(t, \phi)(x))$ *satisfies*

$$u_i(t, \phi)(x) \geqslant \beta_0 e_i(x) \qquad \text{for all} \quad t \geqslant t_0, \qquad x \in \bar{\Omega}, \quad i = 1, 2,$$

where

$$e_i(x) = \begin{cases} e(x), & \text{if} \quad B_i u_i = u_i \\ 1, & \text{if} \quad B_i u_i = \dfrac{\partial u_i}{\partial \nu} + \alpha_i(x)\, u_i, \end{cases}$$

in which $e \in C^2(\bar{\Omega})$ *is given such that for* $x \in \Omega$, $e(x) > 0$ *and for* $x \in \partial\Omega$, $e(x) = 0$ *and* $(\partial e/\partial v) < -\gamma < 0$.

Proof. Let $\tau = \max_{1 \leqslant i,\, j \leqslant 2}\{\tau_{ij}\}$ and let X^+, Y_0 and ∂Y_0 be defined as before with $m = 2$. For any $\phi \in X^+$, let $u(t, \phi)(x)$ be the unique solution of (3.10), satisfying $u_0(\phi) = \phi$, on the maximal interval of existence $[0, \sigma_\phi)$. Then for any $\phi \in X^+$, as claimed before, $u(t, \phi)(x) = (u_1(t, \phi)(x), u_2(t, \phi)(x))$ satisfies $u_i(t, \phi)(x) \geqslant 0$, $x \in \bar{\Omega}$, $t \in [0, \sigma_\phi)$, $i = 1, 2$. Then in the case where $\bar{a}_{11}(x) \geqslant 0$, $x \in \bar{\Omega}$, $u_1(t, \phi)(x)$ satisfies

$$\frac{\partial u_1}{\partial t} \leqslant d_1\, \Delta u_1 + u_1[b_1(x) - a_{11} u_1(x, t) + \bar{a}_{11}(x)\, u_1(x, t - \tau_{11})]$$

for $x \in \Omega$ and $t \in [0, \sigma_\phi)$. By a comparison theorem for the quasimonotone reaction–diffusion system with delays [MS2, Theorem 2.2],

$$u_1(t, \phi)(x) \leqslant \bar{u}_1(x, t), \qquad x \in \bar{\Omega}, \quad t \in [0, \sigma_\phi),$$

where $\bar{u}_1(x, t)$ is the unique solution of diffusive logistic equation with delay

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t) + \bar{a}_{11}(x)\, u_1(x, t - \tau_{11})], \\ &\qquad\qquad t > 0, \quad x \in \Omega, \\ &B_1 u_1 = 0, \qquad t > 0, \quad x \in \partial\Omega \end{aligned}$$

satisfying $\bar{u}_1(x, \theta) = \phi(x, \theta)$, $x \in \bar{\Omega}$, $\theta \in [-\tau, 0]$. In the case where $\bar{a}_{11}(x) \leqslant 0$, $x \in \bar{\Omega}$, $u_1(t, \phi)(x)$ satisfies

$$\frac{\partial u_1}{\partial t} \leqslant d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t)], \qquad x \in \Omega, \quad t \in [0, \sigma_\phi).$$

Then by the standard comparison theorem,

$$u_1(t, \phi)(x) \leqslant v_1(x, t), \qquad x \in \bar{\Omega}, \quad t \in [0, \sigma_\phi),$$

where $v_1(x, t)$ is the unique solution of diffusive logistic equation

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t)], \qquad t > 0, \quad x \in \Omega, \\ &B_1 u_1 = 0, \qquad t > 0, \quad x \in \partial\Omega \end{aligned} \tag{3.13}$$

satisfying $v_1(x, 0) = \phi(x, 0)$, $x \in \bar{\Omega}$. By Theorem 2.5, it then follows that there exists a $M_1 > 0$ such that

$$0 \leqslant u_1(t, \phi)(x) \leqslant M_1, \qquad x \in \bar{\Omega}, \quad t \in [0, \sigma_\phi).$$

Therefore, $u_2(t, \phi)(x)$ satisfies

$$\frac{\partial u_2}{\partial t} \leqslant d_2\, \Delta u_2 + u_2[b_2(x) + a_{21}(x)\, M_1 - a_{22}(x)\, u_2(x, t) + \bar{a}_{22}(x)\, u_2(x, t - \tau_{22})]$$

for $x \in \Omega$ and $t \in [0, \sigma_\phi)$. By a similar argument, it follows that there is an $M_2 > 0$ such that

$$0 \leqslant u_2(t, \phi) \leqslant M_2, \qquad x \in \bar{\Omega}, \quad t \in [0, \sigma_\phi).$$

Therefore, for any $\phi \in X^+$, $\sigma_\phi = +\infty$. Define the semiflow $S(t)$: $X^+ \to X^+$ by $S(t)\phi = u_t(\phi)$, $t \geqslant 0$. Clearly, $X^+ = Y_0 \cup \partial Y_0$, $S(t)$: $Y_0 \to Y_0$ and $S(t)$: $\partial Y_0 \to \partial Y_0$. From our previous argument, it easily follows that $S(t)$: $X^+ \to X^+$ is point dissipative. By [MS2, Proposition 1.2], $S(t)$: $X^+ \to X^+$ is compact for each $t > \tau$.

We first prove the abstract uniform persistence of the semiflow $S(t)$ with respect to $(Y_0, \partial Y_0)$. By the first two conditions in (A1) and Theorem 2.5, it follows that $\tilde{A}_\partial = \bigcup_{\phi \in \partial Y_0} \omega(\phi) = \{(\hat{0}, \hat{0}),\ (\hat{u}_1^*(x), \hat{0})\}$. Let $M_1 = (\hat{0}, \hat{0})$, $M_2 = (\hat{u}_1^*(x), \hat{0})$. Then $\tilde{A}_\partial = M_1 \cup M_2$, and M_1 and M_2 are disjoint, compact and isolated invariant sets for $S_\partial(t) = S(t)|_{\partial Y_0}$. Since $\lambda_0(d_1, b_1(\cdot)) > 0$ and $\lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)\, u_1^*(\cdot)) > 0$, Lemma 3.1 (with $F_i(t, x, \phi) \equiv F_i^0(x, \phi)$, $i = 1, 2$) implies that each M_i $(i = 1, 2)$ is isolated for $S(t)$ in Y_0 and hence, isolated for $S(t)$ in X^+ since M_i is isolated for $S_\partial(t)$ in ∂Y_0 and $S(t)$: $Y_0 \to Y_0$ and $S_\partial(t)$: $\partial Y_0 \to \partial Y_0$. Again by Theorem 2.5, M_1 and M_2 are acyclic in ∂Y_0. Thus, $M_1 \cup M_2$ is an isolated and acyclic covering of $\tilde{A}_\partial$ in ∂Y_0. Since Lemma 3.1 also implies $W^s(M_i) \cap Y_0 = \emptyset$, $i = 1, 2$, by Theorem 2.2, $S(t)$ is uniformly persistent with respect to $(Y_0, \partial Y_0)$. Therefore, by [HW, Theorem 3.2], there is a global attractor A_0 in Y_0 relatively to strongly bounded sets in Y_0. In particular, let $T(t)$ be the semiflow generated by the reaction–diffusion system (3.10) with all $\tau_{ij} = 0$, $1 \leqslant i, j \leqslant 2$. Thus, $T(t)$: $X^+ \to X^+$ is point dissipative, compact for each $t > 0$ and uniformly persistent with respect to $(Y_0, \partial Y_0)$. Then by [Zh1, Theorem 2.4], $T(t)$ has an equilibrium $u^*(x)$ in Y_0, i.e., $T(t)\, u^*(\cdot) = u^*(\cdot)$ for all $t \geqslant 0$. Clearly, $u^*(x)$ is also a steady state of system (3.10).

It remains to prove practical uniform persistence for system (3.10). Let $e(\cdot) = (e_1(\cdot), e_2(\cdot))$ and $Z_i = C(\bar{\Omega}, R)$ if $B_i u_i = (\partial u_i/\partial \nu) + \alpha_i u_i$ and $Z_i = C_0^1(\bar{\Omega}, R) = C^1(\bar{\Omega}, R) \cap C_0(\bar{\Omega}, R)$ if $B_i u_i = u_i$, $1 \leqslant i \leqslant 2$, and let $Z = Z_1 \times Z_2$ and $Z^+ = Z_1^+ \times Z_2^+$. Clearly, $\hat{e} \in int(C^+([-\tau, 0], Z))$. By the abstract integral formulation of (3.10) and the smooth property of the analytic semigroup generated by the Laplacian operator, it then follows that $S(2\tau + 1)$: $X^+ \to C([-\tau, 0], Z)$ is continuous, and hence, $A_0 = S(2\tau + 1)\, A_0$ is also compact in $C([-\tau, 0], Z)$. Moreover, by the global attractivity of A_0 in Y_0, for any $\phi \in Y_0$, $S(t + 2\tau + 1)\phi = S(2\tau + 1)(S(t)\phi) \to S(2\tau + 1)(A_0) = A_0$ in $C([-\tau, 0], Z)$ as $t \to \infty$, that is, A_0 attracts points in Y_0 with respect to the topology of $C([-\tau, 0], Z)$. It then suffices to prove that there exists an $\alpha > 0$ such that for any $\phi = (\phi_1, \phi_2) \in A_0$, $\phi \geqslant \alpha \hat{e}$ in $C([-\tau, 0), Z)$, that is, $\phi_i(x, \theta) \geqslant \alpha e_i(x)$, $x \in \bar{\Omega}$ and $\theta \in [-\tau, 0]$, $i = 1, 2$. We first claim that $S(2\tau + 1)\, Y_0 \subset int(C^+([-\tau, 0], Z))$. Indeed, for every $\psi \in Y_0$, let $u(x, t, \psi) = (u_1(x, t, \psi), u_2(x, t, \psi))$. Then, by the form of (3.10), for each $i = 1, 2$, $u_i(x, t, \psi)$ can be regarded as a solution of a parabolic equation of the form

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = d_i\, \Delta u_i + c_i(x, t) u_i, && t > 0, \quad x \in \Omega, \\ &B_i u_i = 0, && t > 0, \quad x \in \partial\Omega \end{aligned}$$

with $c_i(x, t)$ being smooth for $t > \tau$. Then, by the standard maximum principle for parabolic equations, for all $t > \tau$ we have $u_i(\cdot, t, \psi) \gg 0$ in Z_i, that is, $u_i(\cdot, t, \psi) \in int(Z_i)$, $i = 1, 2$. In particular, for all $\theta \in [-\tau, 0]$, $u_i(\cdot, 2\tau + 1 + \theta, \psi) \gg 0$ in Z_i, $i = 1, 2$, and hence $S(2\tau + 1)\psi \in int(C^+([-\tau, 0], Z))$. Thus, we have $A_0 = S(2\tau + 1)\, A_0 \subset S(2\tau + 1)\, Y_0 \subset int(C^+([-\tau, 0], Z))$. It then follows that for any $\phi \in A_0$, there exists a $\beta(\phi) > 0$ such that $\phi \gg \beta(\phi)\hat{e}$ in $C([-\tau, 0], Z)$. Therefore, the compactness of A_0 with respect to the topology of $C([-\tau, 0], Z)$ implies that there exists a $\beta = \beta(A_0) > 0$ such that for any $\phi \in A_0$, we have $\phi \gg \beta\hat{e}$ in $C([-\tau, 0], Z)$. This completes the proof. ∎

3.3. *The Competition System*

We then consider the following two species competition-diffusion system with delays:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t) + \bar{a}_{11}(x)\, u_1(x, t - \tau_{11}) \\ &\quad - a_{12}(x)\, u_2(x, t - \tau_{12})], \qquad t > 0, \quad x \in \Omega, \\ \frac{\partial u_2}{\partial t} &= d_2\, \Delta u_2 + u_2[b_2(x) - a_{21}(x)\, u_1(x, t - \tau_{21}) - a_{22}(x) u_2(x, t) \\ &\quad + \bar{a}_{22}(x)\, u_2(x, t - \tau_{22})], \qquad t > 0, \quad x \in \Omega, \\ B_1 u_1 &= B_2 u_2 = 0, \qquad t > 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.14}$$

here all parameters and B_1 and B_2 are as in (3.10). Let $u_i^*(x)$, $i = 1, 2$, be the unique positive steady state of the diffusive logistic equation

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = d_i\, \Delta u_i + u_i[b_i(x) - (a_{ii}(x) - \bar{a}_{ii}(x))\, u_i(x, t)], \qquad t > 0, \quad x \in \Omega, \\ &B_i u_i = 0, \qquad t > 0, \quad x \in \partial\Omega, \quad i = 1, 2. \end{aligned} \tag{3.15}$$

We have the following result for competition model (3.14).

THEOREM 3.3. *Let* (H) *hold. Assume that*

(B1) $\lambda_0(d_1, b_1(\cdot)) > 0$, $\lambda_0(d_2, b_2(\cdot) - a_{21}(\cdot)\, u_1^*(\cdot)) > 0$, $\lambda_0(d_2, b_2(\cdot)) > 0$, *and* $\lambda_0(d_1, b_1(\cdot) - a_{12}(\cdot)\, u_2^*(\cdot)) > 0$.

Then system (3.14) *admits at least one positive steady state and is uniformly persistent in the sense of Theorem* 3.2.

Proof. We use the same abstract formulation and notations as in Theorem 3.2. Let $S(t)$: $X^+ \to X^+$ be the semiflow generated by (3.14). By comparison theorems, it easily follows that $S(t)$: $X^+ \to X^+$ is point dissipative. By condition (B1), Theorem 2.5 implies that $\tilde{A}_\partial = \bigcup_{\phi \in \partial Y_0} \omega(\phi) = \{(\hat{0}, \hat{0}), (\hat{u}_1^*(x), \hat{0}), (\hat{0}, \hat{u}_2^*(x))\}$. Let $M_1 = (\hat{0}, \hat{0})$, $M_2 = (\hat{u}_1^*(x), \hat{0})$ and $M_3 = (\hat{0}, \hat{u}_2^*(x))$. Then Theorem 2.5 and Lemma 3.1 (with $F = F^0$) imply that $\bigcup_{i=1}^3 M_i$ is an isolated and acyclic covering. Now the conclusion follows from Theorem 2.2 and a similar argument in the proof of Theorem 3.2. ∎

4. GLOBAL EXTINCTION

In this section, we discuss the global extinction of systems (3.10) and (3.14), that is, we study the global attractivity of the boundary equilibria.

4.1. *The Predator–Prey System*

We first have the following result on the predator–prey system (3.10).

THEOREM 4.1. *Let* (H) *hold with* $a_{11}(x) \geqslant 0$, $x \in \Omega$. *Assume that*

(A2) $\lambda_0(d_2, b_2(\cdot)) \leqslant 0$, $\lambda_0(d_1, b_1(\cdot)) > 0$ *and* $\lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)\, u_1^*(\cdot)) < 0$, *where* $u_1^*(x)$ *is the unique positive steady state of equation* (3.12).

Then for any $\phi = (\phi_1, \phi_2) \in X^+$ *with* $\phi_1(x, 0) \not\equiv 0$, *the unique solution* $u(t, \phi)$ *of system* (3.10) *satisfies*

$$\lim_{t \to \infty} u(t, \phi)(x) = (u_1^*(x), 0)$$

uniformly for $x \in \bar{\Omega}$.

Proof. By the proof of Theorem 3.2, it follows that the semiflow $S(t)$: $X^+ \to X^+$, $t \geqslant 0$, generated by (3.10), is point dissipative. For any $\phi \in X^+$, $u(t, \phi)(x) = (u_1(x, t),\ u_2(x, t))$ satisfies $u_i(x, t) \geqslant 0$, $t \geqslant 0$, $x \in \Omega$, $i = 1, 2$. Since

$$\lim_{\varepsilon \to 0} \lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)(u_1^*(\cdot) + \varepsilon)) = \lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)\, u_1^*(\cdot)) < 0,$$

we can choose some $\varepsilon_0 > 0$ such that $\lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)(u_1^*(\cdot) + \varepsilon_0)) < 0$. Then $u_1(x, t)$ satisfies

$$\frac{\partial u_1}{\partial t} \leqslant d_1 \, \Delta u_1 + u_1[b_1(x) - a_{11}(x) \, u_1(x, t) + \bar{a}_{11}(x) \, u_1(x, t - \tau_{11})]$$

for $t > 0$ and $x \in \Omega$. Since $\bar{a}_{11}(x) \geqslant 0$, $x \in \Omega$, by the comparison theorem for quasimonotone reaction–diffusion systems with delays [MS2, Theorem 2.2], we have

$$u_1(x, t) \leqslant \bar{u}_1(x, t), \qquad t \geqslant 0, \quad x \in \bar{\Omega}, \tag{4.1}$$

where $\bar{u}_1(x, t)$ is the unique solution of the logistic diffusion equation with delay

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= d_1 \, \Delta u_1 + u_1[b_1(x) - a_{11}(x) \, u_1(x, t) + \bar{a}_{11}(x) \, u_1(x, t - \tau_{11})], \\ & \qquad t > 0, \quad x \in \Omega, \\ B_1 u_1 &= 0, \qquad t > 0, \quad x \in \partial\Omega \end{aligned}$$

satisfying $\bar{u}_1(x, \theta) = \phi_1(x, \theta)$, $x \in \Omega$, $\theta \in [-\tau_{11}, 0]$. Since $\lambda_0(d_1, b_1(\cdot)) > 0$, Theorem 2.5 implies that there exists $t_0 > 0$ such that

$$u_1(x, t) \leqslant u_1^*(x) + \varepsilon_0 \qquad \text{for} \quad x \in \Omega \quad \text{and} \quad t \geqslant t_0.$$

Then $U_2(x, t) = u_2(x, t + t_0)$, $t \geqslant 0$, satisfies

$$\begin{aligned} \frac{\partial U_2}{\partial t} &\leqslant d_2 \Delta U_2 + U_2(x, t)[b_2(x) + a_{21}(x)(u_1^*(x) + \varepsilon_0) - a_{22}(x) \, U_2(x, t) \\ & \quad + \bar{a}_{22}(x) \, U_2(x, t - \tau_{22})], \qquad t > 0, \quad x \in \Omega. \end{aligned} \tag{4.2}$$

We first consider the case where $\bar{a}_{22}(x) \geqslant 0$, $x \in \bar{\Omega}$. Using the comparison theorem [MS2, Theorem 2.2] one more time, we have

$$U_2(x, t) \leqslant \bar{U}_2(x, t), \qquad t \geqslant 0, \quad x \in \Omega,$$

where $U_2(x, t)$ is the unique solution of the following diffusive logistic equation with delay

$$\begin{aligned} \frac{\partial U_2}{\partial t} &= d_2 \, \Delta U_2 + U_2(x, t)[b_2(x) + a_{21}(x)(u_1^*(x) + \varepsilon_0) - a_{22}(x) \, U_2(x, t) \\ & \quad + \bar{a}_{22}(x) \, U_2(x, t - \tau_{22})], \qquad t > 0, \quad x \in \Omega, \\ B_2 U_2 &= 0, \qquad t > 0, \quad x \in \partial\Omega \end{aligned} \tag{4.3}$$

satisfying $\bar{U}_2(x, \theta) = U_2(x, \theta)$, $x \in \bar{\Omega}$, $\theta \in [-\tau_{22}, 0]$. By the choice of ε_0, $\lambda_0(d_2, b_2(\cdot) + a_{21}(\cdot)(u_1^*(\cdot) + \varepsilon_0)) < 0$. Hence, Theorem 2.5 implies that $\lim_{t\to\infty} \bar{U}_2(x, t) = 0$ uniformly in $x \in \bar{\Omega}$. Therefore, $\lim_{t\to\infty} U_2(x, t) = 0$ and hence $\lim_{t\to\infty} u_2(x, t) = 0$ uniformly in $x \in \bar{\Omega}$.

In the case where $\bar{a}_{22}(x) \leqslant 0$, $x \in \bar{\Omega}$, by (4.2), $U_2(x, t)$ satisfies

$$\frac{\partial U_2}{\partial t} \leqslant d_2\, \Delta U_2 + U_2(x, t)[b_2(x) + a_{21}(x)(u_1^*(x) + \varepsilon_0) - a_{22}(x)\, U_2(x, t)]$$

for $t > 0$ and $x \in \Omega$. By the standard parabolic comparison theorem, we have

$$U_2(x, t) \leqslant V_2(x, t), \qquad t \geqslant 0, \quad x \in \bar{\Omega},$$

where $V_2(x, t)$ is the unique solution of the diffusive logistic equation

$$\begin{aligned} &\frac{\partial V_2}{\partial t} = d_2\, \Delta V_2 + V_2[b_2(x) + a_{21}(x)(u_1^*(x) + \varepsilon_0) - a_{22}(x)\, V_2], \\ &\qquad\qquad t > 0, \quad x \in \Omega, \\ &B_2 V_2 = 0, \qquad t > 0, \quad x \in \partial\Omega \end{aligned} \tag{4.4}$$

satisfying $V_2(x, 0) = U_2(x, 0)$. Since $\lambda_0(d_2, b_2(\cdot) + a_{21}(x)(u_1^*(x) + \varepsilon_0)) < 0$, Theorem 2.5 with $c(x) \equiv 0$ implies that $\lim_{t\to\infty} V_2(x, t) = 0$ uniformly for $x \in \bar{\Omega}$. Therefore, $\lim_{t\to\infty} U_2(x, t) = 0$ and hence $\lim_{t\to\infty} u_2(x, t) = 0$ uniformly for $x \in \bar{\Omega}$.

For any given $\phi^0 = (\phi_1^0, \phi_2^0) \in X^+$ with $\phi_1^0(x, 0) \not\equiv 0$, let $u(t, \phi^0)(x) = (u_1(x, t), u_2(x, t))$, $t \geqslant 0$. Then we can regard $u_2(x, t)$ as a fixed function on $R^+ \times \bar{\Omega}$. Therefore, $u_1(x, t)$ satisfies the following nonautonomous diffusive equation with delays:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = d_1\, \Delta u_1 + u_1(x, t)[b_1(x) - a_{11}(x)\, u_1(x, t) + \bar{a}_{11}(x)\, u_1(x, t - \tau_{11}) \\ &\qquad - a_{12}(x)\, u_2(x, t - \tau_{12})], \qquad t > 0, \quad x \in \Omega, \\ &B_1 u_1 = 0, \qquad\qquad t > 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.5}$$

Since we have proved that $\lim_{t\to\infty} u_2(x, t) = 0$ uniformly for $x \in \bar{\Omega}$, (4.5) is asymptotic to the following autonomous diffusive equation with delay

$$\begin{aligned} &\frac{\partial u_1}{\partial t} = d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1 + \bar{a}_{11}(x)\, u_1(x, t - \tau_{11})], \\ &\qquad\qquad t > 0, \quad x \in \Omega, \\ &B_1 u_1 = 0, \qquad t > 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.6}$$

For any $\phi_1 \in X_1^+$ and $s \geqslant 0$, let $\tilde{u}_1(t, s, \phi_1)(x)$, $t \geqslant s$, be the unique solution of (4.5) satisfying $\tilde{u}_1(t, s, \phi_1)(x) = \phi_1(x, t-s)$, $t \in [s-\tau_{11}, s]$, $x \in \Omega$. Define Φ: $\Delta \times X_1^+ \to X_1^+$ by $\Phi(t, s, \phi_1) = \tilde{u}_{1t}(\cdot, s, \phi_1)$. Then Φ is a nonautonomous semiflow. Let $u_1(t, \phi_1)(x)$, $t \geqslant 0$, be the unique solution of (4.6) satisfying $u_1(t, \phi_1)(x) = \phi_1(x, t)$, $t \in [-\tau_{11}, 0]$, $x \in \bar{\Omega}$. Define the autonomous semiflow $T(t)$: $X_1^+ \to X_1^+$, $t \geqslant 0$, by $T(t)\,\phi_1 = u_{1t}(\phi_1)$. It then easily follows that $\Phi(t, s, \phi_1)$ is asymptotically autonomous with limit semiflow $T(t)$ (see, e.g., [MST] for ordinary differential equations with bounded delay). Let $\tilde{\omega}(\phi_1)$ be the omega Φ-limit set, i.e.,

$$\tilde{\omega}(\phi_1) = \{\psi_1 \in X_1^+;\ \text{there exists } t_j \to \infty \text{ such that } \lim_{t \to \infty} \Phi(t_j, 0, \phi_1) = \psi_1\}.$$

Since $\phi_1 \in Y_1^0 = \{\phi_1 \in X_1^+ : \phi_1(x, 0) \not\equiv 0\}$ and $\lambda_0(d_1, b_1(\cdot)) > 0$, Theorem 2.5 implies that for the semiflow $T(t)$: $X_1^+ \to X_1^+$, $t \geqslant 0$, $W^s(\hat{u}_1^*(\cdot)) = Y_1^0$; and Lemma 3.1 with $m = 1$ (by taking $u^*(x) \equiv 0$) implies that $\tilde{\omega}(\phi_1^0) \cap Y_1^0 \neq \emptyset$. Then by Theorem 2.4, $\lim_{t \to \infty} \Phi(t, 0, \phi_1^0) = \hat{u}_1^*(\cdot)$ and hence $\lim_{t \to \infty} u_1(x, t) = u^*(x)$ uniformly for $x \in \bar{\Omega}$. Therefore, for any $\phi^0 = (\phi_1^0, \phi_2^0) \in X^+$ with $\phi_1(x, 0) \not\equiv 0$, $\lim_{t \to \infty} u(t, \phi)(x) = (u_1^*(x), 0)$ uniformly for $x \in \bar{\Omega}$. This completes the proof. ∎

THEOREM 4.2. *Let* (H) *hold. Assume that*

(A3) $\lambda_0(d_1, b_1(\cdot)) \leqslant 0$ *and* $\lambda_0(d_2, b_2(\cdot)) \leqslant 0$.

Then for any $\phi \in X^+$, *the unique solution* $u(t, \phi)$ *of system* (3.10) *satisfies*

$$\lim_{t \to \infty} u(t, \phi)(x) = (0, 0)$$

uniformly for $x \in \bar{\Omega}$.

Proof. For any $\phi \in X^+$, let $u(t, \phi)(x) = (u_1(x, t), u_2(x, t))$, $t \geqslant 0$. Then $u_1(x, t)$ satisfies

$$\frac{\partial u_1}{\partial t} \leqslant d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t) + \bar{a}_{11}(x)\, u_1(x, t-\tau_{11})],$$
$$t > 0, \quad x \in \Omega.$$

In the case where $\bar{a}_{11}(x) \geqslant 0$, $x \in \bar{\Omega}$, by the comparison theorem [MS2, Theorem 2.2] and Theorem 2.5, it follows that $\lim_{t \to \infty} u_1(x, t) = 0$ uniformly for $x \in \bar{\Omega}$. In the case where $\bar{a}_{11}(x) \leqslant 0$, $x \in \Omega$, then $u_1(x, t)$ also satisfies

$$\frac{\partial u_1}{\partial t} \leqslant d_1\, \Delta u_1 + u_1[b_1(x) - a_{11}(x)\, u_1(x, t)], \qquad t > 0, \quad x \in \Omega.$$

By standard parabolic comparison theorem and Theorem 2.5 (with $c(x) \equiv 0$), it follows that $\lim_{t \to \infty} u_1(x, t) = 0$ uniformly for $x \in \Omega$. We regard $u_2(x, t)$

as a solution of the following nonautonomous diffusive equation with delay:

$$\begin{aligned} \frac{\partial u_2}{\partial t} &= d_2\, \Delta u_2 + u_2[b_2(x) + a_{21}(x)\, u_1(x, t-\tau_{21}) - a_{22}(x)\, u_2(x, t) \\ &\quad + \bar{a}_{22}(x)\, u_2(x, t-\tau_{22})], \qquad t>0, \quad x \in \Omega, \\ B_2 u_2 &= 0, \qquad t>0, \quad x \in \partial\Omega. \end{aligned} \tag{4.7}$$

Since $\lim_{t\to\infty} u_1(x, t) = 0$ uniformly for $x \in \bar{\Omega}$, (4.7) is asymptotic to the following autonomous diffusive equation with delay

$$\begin{aligned} \frac{\partial u_2}{\partial t} &= d_2\, \Delta u_2 + u_2[b_2(x) - a_{22}(x)\, u_2(x, t) + \bar{a}_{22}(x)\, u_2(x, t-\tau_{22})], \\ &\qquad t>0, \quad x \in \Omega, \\ B_2 u_2 &= 0, \qquad t>0, \quad x \in \partial\Omega. \end{aligned}$$

Since $\lambda_0(d_2, b_2(\cdot)) \leqslant 0$, by Theorem 2.5 and the asymptotically autonomous semiflow theory, it follows that $\lim_{t\to\infty} u_2(x, t) = 0$ uniformly for $x \in \Omega$. Therefore, $\lim_{t\to\infty} u(t, \phi) = (0, 0)$ uniformly for $x \in \bar{\Omega}$. This completes the proof. ∎

4.2. *The Competition System*

Now we turn to the competition model (3.14). We first seek conditions on extinction of one of the competitors, that is, we would like to see when the competition exclusion principle occurs.

THEOREM 4.3. *Let* (H) *hold with* $\bar{a}_{ii}(x) \geqslant 0$, $x \in \bar{\Omega}$, $i = 1, 2$. *Assume that*

(B2) $\lambda_0(d_1, b_1(\cdot)) > 0$, $\lambda_0(d_2, b_2(\cdot)) > 0$, $\lambda_0(d_1, b_1(\cdot) - a_{12}(\cdot)\, u_2^*(\cdot)) > 0$ *and* (3.14) *admits no positive steady state.*

Then for any $\phi = (\phi_1, \phi_2) \in X^+$ *with* $\phi_1(x, 0) \not\equiv 0$, *the unique solution* $u(t, \phi)(x)$ *of* (3.14) *satisfies*

$$\lim_{t\to\infty} u(t, \phi)(x) = (u_1^*(x), 0)$$

uniformly for $x \in \bar{\Omega}$, *where* $u_i^*(x)$ $(i = 1, 2)$ *are as in Theorem* 3.3.

Proof. Let $S(t)\colon X^+ \to X^+$, $t \geqslant 0$, be the semiflow generated by (3.14). It then follows that $S(t)$ is point dissipative and is compact for every $t > \tau$. Let $W_0 = \{(\phi_1, \phi_2) \in X^+\colon\ \phi_1(x, 0) \not\equiv 0\}$ and $\partial W_0 = X^+ \setminus W_0 = \{(\phi_1, \phi_2) \in X^+;\ \phi_1(x, 0) \equiv 0\}$. Then $X^+ = W_0 \cup \partial W_0$, $S(t)\colon W_0 \to W_0$ and $S(t)\colon \partial W_0 \to \partial W_0$, $t \geqslant 0$. By Theorem 2.5, $\bigcup_{\phi \in \partial W_0} \omega(\phi) = \{(\hat{0}, \hat{0}), (\hat{0}, \hat{u}_2^*(x))\}$. Denote

$M_1 = (\hat{0}, \hat{0})$ and $M_2 = (\hat{0}, \hat{u}_2^*(x))$. As argued in Section 3, Theorem 2.5 and Lemma 3.1 (with $F \equiv F^0$) imply that $M_1 \cup M_2$ is an isolated and acyclic covering of $\bigcup_{\phi \in \partial W_0} \omega(\phi)$. It thus follows from Theorem 2.2 that $S(t)\colon X^+ \to X^+$ is uniformly persistent with respect to $(W_0, \partial W_0)$. Therefore, by [HW, Theorem 3.2], there exists a global attractor A_0 in W_0 relative to strongly bounded sets in W_0. Clearly, $(\hat{u}_1^*(x), \hat{0}) \in A_0$.

Let $P = X_1^+ \times (-X_2^+)$. By a change of variables $u_1 = v_1$ and $u_2 = -v_2$, [MS2, Proposition 1.4] and the assumption that $\bar{a}_{ii}(x) \geqslant 0$ $(i = 1, 2)$ for $x \in \bar{\Omega}$, the comparison theorem [MS2, Theorem 2.2] implies that $S(t)\colon X^+ \to X^+$ is a monotone semiflow with respect to the order generated by P, i.e., for any $\phi, \psi \in X^+$ with $\phi \leqslant \psi$ in (X, P), $S(t)\phi \leqslant S(t)\psi$ in (X, P), $t \geqslant 0$. By [Hi1, Theorem 3.1] and condition (B2) on the nonexistence of positive steady state of (3.14), the global attractor A_0 contains only one equilibrium $(\hat{u}_1^*(x), \hat{0})$. Thus, by [Hi1, Theorem 3.3], $(\hat{u}_1^*(x), \hat{0})$ attracts any point $\phi \in W_0$. Consequently, $\lim_{t \to \infty} u(t, \phi)(x) = (u_1^*(x), 0)$ uniformly for $x \in \bar{\Omega}$. This completes the proof. ∎

Using a similar argument as in Theorem 4.1, we can also prove the following result.

THEOREM 4.4. *Let* (H) *hold. Assume that*

(B3) $\lambda_0(d_1, b_1(\cdot)) > 0$ *and* $\lambda_0(d_2, b_2(\cdot)) \leqslant 0$.

Then for any $\phi = (\phi_1, \phi_2) \in X^+$ *with* $\phi_1(x, 0) \not\equiv 0$, *the unique solution* $u(t, \phi)(x)$ *of* (3.14) *satisfies*

$$\lim_{t \to \infty} u(t, \phi)(x) = (u_1^*(x), 0)$$

uniformly for $x \in \bar{\Omega}$.

Remark 4.5. There are of course symmetric results on the global attractivity of $(0, u_2^*(x))$. Moreover, by Theorem 3.3, it follows that condition (B2) implies that $\lambda_0(d_2, b_2(\cdot) - a_{21}(\cdot)\, u_1^*(\cdot)) \leqslant 0$.

Similarly argued as in Theorem 4.2, we can prove the following result on extinction of both species.

THEOREM 4.6. *Let* (H) *hold. Assume that*

(B4) $\lambda_0(d_1, b_1(\cdot)) \leqslant 0$ *and* $\lambda_0(d_2, b_2(\cdot)) \leqslant 0$.

Then for any $\phi = (\phi_1, \phi_2) \in X^+$, *the unique solution* $u(t, \phi)(x)$ *of* (3.14) *satisfies*

$$\lim_{t \to \infty} u(t, \phi)(x) = (0, 0)$$

uniformly for $x \in \bar{\Omega}$.

Remark 4.7. From the proofs of the theorems in Sections 3 and 4, it follows that if we replace $a_{21}(x)\, u_1(x, t-\tau_{21})$ and $a_{12}(x)\, u_2(x, t-\tau_{12})$ in systems (3.10) and (3.14) by $g_1(x, u_{1t})$ and $g_2(x, u_{2t})$, respectively, where $g_i(x, \phi_i) \in C(\bar{\Omega} \times C_\tau, R)$ with $g_i(x, \hat{0}) \equiv 0$ and $g_i(x, \phi_i)$ is increasing with respect to $\phi_i \in C_\tau$, $i=1, 2$, and replace $a_{21}(x)\, u_1^*(x)$ and $a_{12}(x)\, u_2^*(x)$ in (A1)–(A2) and (B1)–(B2) by $g_1(x, \hat{u}_1^*(x))$ and $g_2(x, \hat{u}_2^*(x))$, respectively, then all theorems in Sections 3 and 4 are still valid.

Remark 4.8. Finally, we would like to remark that the above ideas and techniques can be used to study multiple species reaction–diffusion systems with delay. Also, taking the nonlocal effect into consideration as pointed out by the referee, the interaction term $u_2(t, x)\, a_{21} u_1(x, t-\tau_{21})$ in the predator–prey model can be replaced by

$$u_2(t, x) \int_{-\tau_{21}}^{0} \frac{\int_\Omega G(s, x, y)\, u_1(t-s, y)\, dy}{\int_\Omega G(s, x, y)\, dy}\, m(ds),$$

where G is the Green's function of $\partial_t - d_2 \Delta$ with appropriate boundary condition (see, e.g., [GM]), m is a nonnegative measure, and $G(s, x, y)/\int_\Omega G(s, x, y)\, dy$ is the density describing the probability that a predator which is at location x at time t has been at location y at time $t-s$. This leads to a very interesting nonlocal reaction–diffusion system with delays which we leave for future consideration. For related nonlocal reaction–diffusion models without time delay, we refer to a recent paper [CC2].

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