

REACTION-DIFFUSION EQUATIONS WITH INFINITE DELAY

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ABSTRACT. We have developed several results on the existence and asymptotic behavior of mild solutions to reaction-diffusion systems that have infinite delays in the nonlinear reaction terms. We find that the semiflow generated by a cooperative and irreducible reaction-diffusion system with infinite delay is not compact but set-condensing, and not strongly order-preserving but quasi strongly order-preserving. These set-condenseness and quasi strong order-preserving properties allow us to use a modification, recently given by Freedman, Miller and one of the authors of this paper, of the well-known monotone dynamical system theory due to Dancer, Hess, Hirsch, Matano, Smith, Thieme, Poláčik and Takáč to obtain some results about convergence and stability of solutions. Examples of Lotka-Volterra competition-diffusion models with distributed delay are given to illustrate the obtained results.

1. Introduction. A variety of mathematical models for biological processes are most appropriately framed as partial functional differential equations. For example, the reaction-diffusion logistic equation with finite delay

$$(1.1) \quad \begin{aligned} \frac{\partial u(t, x)}{\partial t} &= d \frac{\partial^2 u(t, x)}{\partial x^2} + ru(t, x) \left[1 - \frac{u(t-\tau, x)}{K} \right], & t > 0, x \in (0, 1) \\ \frac{\partial u(t, x)}{\partial x} &= 0, & x = 0, 1, \end{aligned}$$

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where d, r, τ and K are positive constants, has been used to model an one-dimensional herbivorous population and has been studied by many authors, for example, Busenberg and Huang [5], Friesecke [8, 9], Gopalsamy, He and Sun [13], Green and Stech [14], Huang [28], Lin and Khan [33], Luckhaus [35], Memory [44, 45], Morita [47], Yoshida [76] and Yoshida and Kishimoto [77], to name a few.

General partial functional differential equations with finite delay have been extensively studied. We refer to Fitzgibbon [10], Rankin [55] and Travis and Webb [68, 69] for detailed discussions on the existence and asymptotic behavior of solutions; to Kunish and Schappacher [32] for necessary conditions to generate C_0 -semigroups; to Hale [17] for the convergence to solutions of an ordinary functional differential equation; to Fitzgibbon and Parrott [11] and Parrott [50] for the linearized stability; to He [20, 21] for periodic and almost periodic solutions; to Lin, So and Wu [34] for a center manifold theory; to Hale and Ladeira [19] for the differentiability with respect to delays, and to Rey and Mackey [57, 58] for bifurcations, traveling waves and multistability.

Recently Martin and Smith, in their three consecutive papers [36–38], have studied partial functional differential equations in a Banach space. They developed several fundamental results on the existence and asymptotic behavior of solutions to abstract semilinear functional differential equations with finite delay and then apply the results to reaction-diffusion equations which have finite time delays in the nonlinear reaction terms. By employing the monotone dynamical system theory due to Hirsch [25–27], Matano [39–42], Smith [61], and Smith and Thieme [63, 64], they established sufficient conditions for a reaction-diffusion equation with finite delay to generate a (eventually) strongly monotone semiflow on an appropriate space and concluded that almost all orbits converge to the set of equilibria. They also established the existence of an invariant rectangle and obtained certain comparing systems of ordinary functional differential equations relative to the invariant rectangle. As an application, they considered the n -species Lotka-Volterra model of competition with diffusion and finite delay and developed sufficient conditions for the global asymptotic stability of the coexistence state.

It is well known that distributed delay should be used to describe the stochastic element in the delayed response of a biological process. The

following reaction-diffusion equation with infinite delay

$$(1.2) \quad \begin{aligned} \frac{\partial u(t, x)}{\partial t} &= d \frac{\partial^2 u(t, x)}{\partial x^2} + ru(t, x) \left[1 - \frac{1}{K} \int_{-\infty}^t k(t-s)u(s, x)ds \right], \\ & \quad t > 0, x \in (0, 1) \\ \frac{\partial u(t, x)}{\partial x} &= 0, \quad x = 0, 1 \end{aligned}$$

with

$$d > 0, r > 0, K > 0, k(s) > 0 \quad \text{for } s > 0$$

and

$$\int_0^{\infty} k(s)ds = 1$$

also has been used as a modified version of (1.1) and has been studied by Bonilla and Liñán [3], Britton [4], Redlinger [56], Schiaffino [60], Tesi [67] and Yamada [73]. Particular classes of reaction-diffusion systems with infinite delay have been investigated by Gopalsamy [12], Kuang and Smith [30, 31], Pozio [54], Yamada [74], and Yamada and Niikura [75]. However, there are all together very few results on general theory of reaction-diffusion systems with infinite delay.

One of the main purposes of this paper is to develop a general theory of existence, comparison, invariance, monotonicity and set-condenseness for partial functional differential equations with infinite delay and to provide some applications to reaction-diffusion systems with general distributed delays. We shall follow Martin and Smith [36-38] to derive general results for abstract semilinear integral equations in general phase spaces and then draw some conclusions for reaction-diffusion systems with distributed delays. We also establish an invariance principle which generalizes a well-known result due to Haddock and Terjéki [15] from ordinary to partial functional differential equations.

Our approaches and ideas are motivated by those in Martin and Smith [36-38] for reaction-diffusion systems with finite delay and in Hale and Kato [18] for ordinary functional differential equations with infinite delay. However, there are numerous technical difficulties in dealing with partial functional differential equations with infinite delay due to the unboundedness of the delay involved. In particular, the

choice of phase space is a nontrivial job and plays an important role in establishing several inequalities which are essential to build a general theory. Secondly, the corresponding semiflow is no longer compact, but set-condensing in the sense of Nussbaum [49]. Thirdly, an example in Wu [70] indicates that the solution semiflow of a functional differential equation with infinite delay may not be (eventually) strongly monotone even if the usual quasimonotonicity and irreducibility conditions are satisfied. We shall show that the solution semiflow generated by a certain reaction-diffusion system with infinite delay is quasi strongly order-preserving in the sense described in Section 2. This enables us to apply a result due to Freedman, Miller and Wu [7], which represents some modification and generalization of those recent results due to Dancer and Hess [6], to establish some general results on the convergence and stability for a class of reaction-diffusion systems with infinite delay.

As an application of the invariance principle established in this paper, we obtain the convergence of the steady state of the n -species Lotka-Volterra competition-diffusion model with distributed delay. We also find that the two-species Lotka-Volterra competition-diffusion model with infinite delay generates a quasi strongly order-preserving semiflow, hence we can rule out the occurrence of pattern formation in certain situations. It is observed that the asymptotic behavior of solutions of the two-species Lotka-Volterra competition-diffusion model with infinite delay is similar to that of the two-species Lotka-Volterra competition-diffusion model without a delay or with a discrete delay.

The paper is organized as follows. Section 2 contains a short survey of some recent results about set-condensing and quasi strongly order-preserving semiflows. In Section 3, we introduce the definition and examples of a fundamental phase space which will be used throughout this paper. In Section 4, we establish a general existence theorem for the abstract semilinear integral equations. Comparison and monotonicity principles are described, in Section 5, for the abstract semilinear integral equations and, in Section 6, for the reaction-diffusion systems with infinite delay. Finally, in Section 7, we apply our general results to the Lotka-Volterra reaction-diffusion models with distributed delay.

2. Preliminary results on order-preserving and set-condensing semiflows. Consider the Banach space B with a closed ordered cone

B_+ . For $\phi, \psi \in B$, we write $\phi \geq_B \psi$ if $\phi - \psi \in B_+$, $\phi >_B \psi$ if $\phi - \psi \in B_+ \setminus \{0\}$ and $\phi \gg_B \psi$ if $\text{int}(B_+) \neq \emptyset$ and $\phi - \psi \in \text{int}(B_+)$. $\phi \leq_B$ ($<_B, \ll_B$) ψ means $\psi \geq_B$ ($>_B, \gg_B$) ϕ .

Let $R_+ = [0, \infty)$. Assume $\Phi : R_+ \times B \rightarrow B$ is a given semiflow. For each $\phi \in B$, define the positive orbit initiating from ϕ by $\gamma^+(\phi) = \{\Phi_t(\phi); t \geq 0\}$ and the omega limit set of $\gamma^+(\phi)$ by

$$\omega(\phi) = \bigcap_{t \geq 0} \overline{\gamma^+(\Phi_t(\phi))},$$

here and in what follows, $\Phi_t(\phi) = \Phi(t, \phi)$ for $(t, \phi) \in R_+ \times B$. It is well-known that if $\gamma^+(\phi)$ is relatively compact then $\omega(\phi)$ is nonempty, compact, connected and invariant.

The semiflow Φ is said to be *strictly order-preserving* if $(t, \phi), (t, \psi) \in R_+ \times B$ with $\phi >_B \psi$ implies $\Phi_t(\phi) >_B \Phi_t(\psi)$ for $t > 0$. A given point $\phi \in B$ is called a *subequilibrium* of Φ if $\phi \leq_B \Phi_t(\phi)$ for $t \geq 0$, and a *strict subequilibrium* if $\phi <_B \Phi_t(\phi)$ for $t \geq 0$. Similarly, we can define a (strict) *superequilibrium*. An *entire orbit* of Φ is a mapping $u : R \rightarrow B$ such that $u(t+s) = \Phi_t(u(s))$ for $t \geq 0$ and $s \in R$.

Let α be a measure of noncompactness (see, cf. Nussbaum [49]). We say that the semiflow Φ_t is *set-condensing* on a subset E of B for $t > 0$ if, for every bounded subset W of E with $\alpha(W) \neq 0$, we have $\alpha(\Phi_t(W)) < \alpha(W)$ for every $t > 0$.

Theorem 2.1. *Suppose $\Phi : R_+ \times B \rightarrow B$ is strictly order-preserving. Let $\phi_1 <_B \phi_2$ be order-related equilibria of Φ and define*

$$E = [\phi_1, \phi_2]_B = \{\phi \in B; \phi_1 \leq_B \phi \leq_B \phi_2\}.$$

Assume that Φ_t is set-condensing on E for $t > 0$. Then one of the following statements holds:

- (i) *there exists a further fixed point ϕ of Φ such that $\phi_1 <_B \phi <_B \phi_2$;*
- (ii) *there exists an entire orbit of strict subequilibria u connecting ϕ_1 and ϕ_2 such that $u(t) \rightarrow \phi_1$ as $t \rightarrow -\infty$ and $u(t) \rightarrow \phi_2$ as $t \rightarrow \infty$;*
- (iii) *there exists an entire orbit of strict superequilibria u connecting ϕ_1 and ϕ_2 such that $u(t) \rightarrow \phi_2$ as $t \rightarrow -\infty$ and $u(t) \rightarrow \phi_1$ as $t \rightarrow \infty$.*

The above theorem represents a slight improvement of Theorem 8 in Matano [40] and Proposition 1.1 in Dancer and Hess [6] by relaxing the requirement on compactness and the strongly order-preserving property of Φ . We refer to Smith and Thieme [64] for a related result (Proposition 3.7). Theorem 2.1 can be proved by a similar argument to that in Dancer and Hess [6] with some necessary modifications in order to deal with set-condensing semiflows in stead of compact semiflows. We refer to Freedman, Miller and Wu [7] for details.

Recall that a *strongly order-preserving* semiflow Φ is one such that $\phi <_B \psi$ implies $\Phi_t(\phi) \ll_B \Phi_t(\psi)$ for $t > 0$. The following weaker notation of strong order-preserving property of Φ was introduced in Freedman, Miller and Wu [7].

Definition 2.2. A semiflow $\Phi : R_+ \times B \rightarrow B$ is said to be *quasi strongly order-preserving* (QSOP) if it is order-preserving and for every sequence $\{\psi^n\}$ of equilibria and for every compact invariant subset $A \subseteq B$ such that $\lim_{n \rightarrow \infty} \psi^n = \psi <_B A$ and $\psi <_B \psi^n$ for $n = 1, 2, \dots$, there exists an integer n_0 such that $\psi^{n_0} \leq_B A$.

It can be easily shown that a strongly order-preserving semiflow is QSOP, but the inverse is not true. We refer to Freedman, Miller and Wu [7] for some examples and for a sufficient condition guaranteeing an order-preserving semiflow to be QSOP.

For $r > 0$ and $\phi \in B$, define

$$\mathcal{U}_B(\phi, r) = \{\psi \in B; |\psi - \phi|_B < r\}.$$

Suppose that the semiflow Φ has a subequilibrium ϕ_1 and a superequilibrium ϕ_2 with $\phi_1 <_B \phi_2$. Denote $E = [\phi_1, \phi_2]_B$. An equilibrium ϕ in E is said to be *stable with respect to E* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\Phi_t(\psi) \in \mathcal{U}_B(\phi, \varepsilon)$ for all $\psi \in \mathcal{U}_B(\phi, \delta)$ and $t > 0$.

Theorem 2.3. *Suppose that Φ is a QSOP semiflow and that there exists a subequilibrium ϕ_1 and a superequilibrium ϕ_2 such that $\phi_1 <_B \phi_2$ and every equilibrium in $E = [\phi_1, \phi_2]_B$ is stable with respect to E . Assume further that Φ_t is set-condensing on E for $t > 0$. Then every bounded positive orbit converges. That is, for every bounded positive orbit $\gamma^+(\phi)$ in E , $\omega(\phi)$ is a singleton.*

The above result is established by Dancer and Hess [6] for strongly order-preserving semiflows. It turns out (see Freedman, Miller and Wu [7]) that their argument applies to QSOP semiflows as well once Theorem 2.1 is justified. We refer to Alikakos, Hess and Matano [2], Hirsch [27], Paláčik [52, 53], Takáč [65, 66], Smith and Thieme [63, 64] for related results.

Theorem 2.4. *Assume Φ is a QSOP semiflow and ϕ_1, ϕ_2 are strict subequilibrium and strict superequilibrium of Φ , respectively, with $\phi_1 \ll_B \phi_2$. Assume also that Φ_t is set-condensing for $t > 0$ on $E = [\phi_1, \phi_2]_B$. Then there exists a stable equilibrium in E .*

Theorem 2.5. *Suppose Φ is a strictly order-preserving semiflow and $\phi_1 <_B \phi_2$ are order-related equilibria. Furthermore, assume that Φ_t is set-condensing for $t > 0$ on $E = [\phi_1, \phi_2]_B$. If ϕ_1 and ϕ_2 are stable respect to E and ϕ_1 is an isolated equilibrium from above or ϕ_2 is an isolated equilibrium from below, then there exists an unstable equilibrium ϕ such that $\phi_1 <_B \phi <_B \phi_2$.*

In the above result, ϕ_1 (ϕ_2) is isolated from above (below) means that there exists a small neighborhood of ϕ_1 (ϕ_2) in E which contains no other equilibrium ψ such that $\phi_1 <_B \psi$ ($\phi_2 >_B \psi$). The above two results are basically established by Dancer and Hess [6] (they prove these results by assuming that $\Phi_t(E)$ is relatively compact, but their argument also applies to the case where Φ_t is set-condensing in E). We refer to Matano [40–42] and Hirsch [27] for some earlier versions.

3. Phase spaces and examples. Let $R_- = (-\infty, 0]$. Assume that X is a real Banach space with a norm denoted by $|\cdot|_X$ and that \widehat{B} is a linear space of mappings from R_- to X with elements designated by $\hat{\phi}, \hat{\psi}, \dots$. Suppose also there is a seminorm $|\cdot|_{\widehat{B}}$ on \widehat{B} so that the quotient space $B = \widehat{B}/|\cdot|_{\widehat{B}}$ is a Banach space with a norm $|\cdot|_B$ naturally induced by $|\cdot|_{\widehat{B}}$. Throughout this paper, we shall denote by ϕ the corresponding equivalence class of $\hat{\phi} \in \widehat{B}$.

The Banach space B will be employed as the phase space for the study of partial functional differential equations with infinite delay. The first

hypothesis on B is described as follows:

(A1) There exists a constant $L > 0$ such that

$$|\hat{\phi}(0)|_X \leq L|\hat{\phi}|_{\hat{B}} \quad \text{for any } \hat{\phi} \in \hat{B}.$$

Under the above assumption, for every equivalence class ϕ there is a unique $\phi(0) \in X$ defined by $\phi(0) = \hat{\phi}(0)$ for $\hat{\phi} \in \hat{B}$ and $|\phi(0)|_X \leq L|\phi|_B$.

To describe other assumptions on the phase space, we introduce the following notation: suppose $0 \leq t \leq A < \infty$ and the mapping $u : (-\infty, A] \rightarrow X$ is given, then $u_t : R_- \rightarrow B$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in R_-$.

For any given $\hat{\phi} \in \hat{B}$ and $A \geq 0$, we define

$$F_A(\hat{\phi}) = \{\hat{u} : (-\infty, A] \rightarrow X; \hat{u}_0 = \hat{\phi} \text{ and } \hat{u}|_{[0, A]} \text{ is continuous}\}.$$

Moreover, we set

$$F_A = \bigcup_{\hat{\phi} \in \hat{B}} F_A(\hat{\phi}).$$

Other fundamental assumptions on the phase spaces can now be described as follows:

(A2) If $A \geq 0$ and $\hat{u} \in F_A$, then $\hat{u}_t \in \hat{B}$ for all $t \in [0, A]$. Moreover, the mapping $t \in [0, A] \rightarrow u_t \in B$ is continuous.

(A3) There exist a continuous nondecreasing function $K : R_+ \rightarrow R_+$ and a locally bounded function $M : R_+ \rightarrow R_+$ such that

$$|u_t|_B \leq K(t) \sup_{0 \leq s \leq t} |u(s)|_X + M(t)|u_0|_B, \quad 0 \leq t \leq A, \quad u \in F_A.$$

To discuss the monotonicity of semiflows on the phase space B , we also assume that there exists a closed cone X_+ in X with the property that if $u, -u \in X_+$, then $u = 0$. The cone X_+ induces a partial ordering \geq_X on X defined as follows

$$u \geq_X v \quad (\text{or } v \leq_X u) \quad \text{iff } u - v \in X_+.$$

We assume that X with the above ordering is a Banach lattice. For simplicity of notations, we shall employ the following ordering intervals:

$$\begin{aligned} [v, w]_X &= \{u \in X; v \leq_X u \leq_X w\}, \\ [v, \infty)_X &= \{u \in X; v \leq_X u\}, \\ (-\infty, w]_X &= \{u \in X; u \leq_X w\}, \\ (-\infty, \infty)_X &= X, \end{aligned}$$

where $v, w \in X$ with $v \leq_X w$.

Similarly, we assume that there exists a closed cone B_+ in the phase space B with the property that if $\phi, -\phi \in B_+$, then $\phi = 0$. The partial ordering induced by B_+ is denoted by \geq_B , and ordering intervals can be defined similarly.

Finally, we require the following compatibility conditions on the orderings \leq_X and \leq_B :

(A4) $\phi \leq_B \psi$ implies that $\phi(0) \leq_X \psi(0)$.

(A5) If $A \geq 0$ and $\hat{u} \in F_A, \hat{v} \in F_A$ are given such that $u_0 \leq_B v_0$ and $u(t) \leq_X v(t)$ for $t \in [0, A]$, then $u_A \leq_B v_A$.

To present an example of phase space satisfying (A1)–(A5), we suppose $g : (-\infty, 0] \rightarrow [1, \infty)$ is a function satisfying the following conditions:

- (g1) g is continuous, nonincreasing and $g(0) = 1$;
- (g2) $g(s + \mu)/g(s) \rightarrow 1$ uniformly for $s \in (-\infty, 0]$ as $\mu \rightarrow 0^+$;
- (g3) $g(s) \rightarrow \infty$ as $s \rightarrow -\infty$.

For example, if $\gamma > 0$, then the functions $g(s) = e^{-\gamma s}$ and $g(s) = (1 + |s|)^\gamma$ satisfy the above conditions.

Suppose $\Omega \subset R^n$ is bounded, $C(\bar{\Omega}; R^m)$ is the Banach space of continuous functions from $\bar{\Omega}$ to R^m with the supremum norm $|\cdot|_{C(\bar{\Omega}; R^m)}$. Define

$$C_g = \left\{ \phi; \phi : (-\infty, 0] \rightarrow C(\bar{\Omega}; R^m) \text{ is continuous, } \sup_{s \leq 0} \frac{|\phi(s)|_{C(\bar{\Omega}; R^m)}}{g(s)} < +\infty \right\}.$$

Then C_g equipped with norm

$$|\phi|_g = |\phi|_{C_g} := \sup_{s \leq 0} \frac{|\phi(s)|_{C(\bar{\Omega}; R^m)}}{g(s)} \quad \text{for } \phi \in C_g$$

is a Banach space.

Since $g(0) = 1$, C_g satisfies (A1) with $L = 1$. Moreover, as g is nonincreasing, if $u : R \rightarrow C(\bar{\Omega}; R^m)$ so that $u_0 \in C_g$ and $u : [0, \infty) \rightarrow$

$C(\bar{\Omega}; R^m)$ is continuous, then $u_t \in C_g$ for all $t \geq 0$. A similar argument to that in Example 2.1 of Haddock and Terjéki [15] leads to

$$|u_t|_g \leq \sup_{0 \leq s \leq t} |u(s)|_{C(\bar{\Omega}; R^m)} + \sup_{\theta \leq -t} \frac{g(\theta + t)}{g(\theta)} |u_0|_g.$$

Hence C_g satisfies (A3) with $K(t) = 1$ and $M(t) = \sup_{\theta \leq -t} (g(\theta + t)/g(\theta))$ on $[0, \infty)$.

Obviously, $X_+ = C(\bar{\Omega}; R^m_+)$ is a closed cone in $X = C(\bar{\Omega}; R^m)$, where R^m_+ denotes the positive octant of R^m . Hence $C(\bar{\Omega}; R^m_+)$ induces a partial ordering $\geq_{C(\bar{\Omega}; R^m)}$ on $C(\bar{\Omega}; R^m)$ by

$$u \geq_{C(\bar{\Omega}; R^m)} v \quad \text{iff} \quad u - v \in C(\bar{\Omega}; R^m_+).$$

Define a closed order cone C_g^+ in C_g by

$$C_g^+ = \{ \phi \in C_g; \phi(\theta) \geq_{C(\bar{\Omega}; R^m)} 0 \text{ for } \theta \leq 0 \}.$$

Similarly, C_g^+ induces a partial ordering \geq_{C_g} on C_g . Obviously, if $\phi, \psi \in C_g^+$, then $\phi \leq_{C_g} \psi$ implies $\phi(0) \leq_{C(\bar{\Omega}; R^m)} \psi(0)$, i.e., (A4) is satisfied. Moreover, if $u, v : R \rightarrow R^n$ are given continuous functions such that $u_0, v_0 \in C_g, u_0 \leq_{C_g} v_0$ and $u(t) \leq_{C(\bar{\Omega}; R^m)} v(t)$ for $t \geq 0$, we have $u_t \leq_{C_g} v_t$ for $t \geq 0$.

Note that (A2) is not satisfied by C_g . We now define a subspace UC_g of C_g as follows

$$UC_g = \left\{ \phi \in C_g; \frac{\phi}{g} \text{ is uniformly continuous on } R_- \right\}.$$

As a closed subspace of C_g , UC_g is a Banach space and satisfies (A1) and (A3). Employing the same argument as that of Theorem 2.1 of Haddock and Terjéki [15], we can prove that $(UC_g, |\cdot|_g)$ satisfies (A2) as well. Similarly, we can define a closed cone UC_g^+ in UC_g . $UC_g^+ = C_g^+ \cap UC_g$ induces a partial ordering \geq_{UC_g} on UC_g and satisfies (A4) and (A5). Another closed subspace of C_g which satisfies (A1)–(A5) is

$$C_g^0 = \{ \phi \in C_g; \phi(s) \rightarrow 0 \text{ as } s \rightarrow -\infty \}.$$

It should be mentioned that the work of Wu [71] shows that UC_g arises very naturally from reaction-diffusion equations with both discrete and distributed delays in the nonlinear reaction term which satisfies a certain *fading memory* condition. We refer to Atkinson and Haddock [1] for related discussions.

The following axiom on the phase space B will sometime be required:

(A6) Every constant mapping from $(-\infty, 0]$ into X belongs to B .

Denote by \bar{w} the constant mapping from $(-\infty, 0]$ into X with the constant value $w \in X$. We will also require the following conditions.

(A7) For any constant mapping $\bar{v}, \bar{w} \in B$, $\bar{v} \leq_B \bar{w}$ if and only if $v \leq_X w$.

(A8) There exists a constant $Q > 0$ so that for every constant mapping $\bar{w} \in B$, $|\bar{w}|_B \leq Q|w|_X$, $w \in X$.

(A1) and (A8) imply that the induced topology (from B) on the space of constant mappings from $(-\infty, 0]$ into X is equivalent to the topology of X . Also

$$|\overline{\phi(0)}|_B \leq Q|\phi(0)|_X \leq QL|\phi|_B, \quad \phi \in B.$$

It is easy to see that (A6)–(A8) are satisfied by $X = C(\bar{\Omega}; R^m)$ and $B = UC_g$ defined above. Moreover, it is trivial to show that for these two spaces, the following assumption is also satisfied:

(A9) If $\phi, \psi \in B$ and $\hat{\phi}(\theta) \leq_X \hat{\psi}(\theta)$ for $\theta \leq 0$, then $\phi \leq_B \psi$.

4. Existence theorems for semilinear integral equations.

Assume $a \geq 0$ is a given constant. In this section, we shall consider an abstract integral equation in some subset of $[a, \infty) \times B$ satisfying the following properties:

(D1) D is a closed subset of $[a, \infty) \times X$ and $D(t) = \{u \in X; (t, u) \in D\}$ is a nonempty subset of X for each $t \geq a$.

(D2) \mathcal{D} is a closed subset of $[a, \infty) \times B$ such that $\mathcal{D}(t) = \{\phi \in B; (t, \phi) \in \mathcal{D}\}$ is a nonempty subset of B for each $t \geq a$, and for any $\hat{u} \in F_\infty$ and $a \leq \tau \leq t$, if $u_\tau \in \mathcal{D}(\tau)$ and $u(s) \in D(s)$ for $s \in [\tau, t]$, then $u_t \in \mathcal{D}(t)$.

(D3) For each $b > a$, there exist a constant $\hat{K}(a, b) > 0$ and a continuous nondecreasing function $\eta_{a,b} : [0, b - a) \rightarrow R_+$ such that

$\eta_{a,b}(0) = 0$ and if $a \leq t_1 < t_2 \leq b$, $u_1 \in D(t_1)$ and $u_2 \in D(t_2)$ then there is a continuous function $w : [t_1, t_2] \rightarrow X$ such that $w(t_1) = u_1$, $w(t_2) = u_2$, $w(t) \in D(t)$ for $t_1 < t < t_2$ and

$$|w(t) - w(s)|_X \leq \eta_{a,b}(|t - s|) + \widehat{K}(a, b)|t - s| \frac{|u_2 - u_1|_X}{t_2 - t_1}$$

for all $s, t \in [t_1, t_2]$.

Now we consider the following abstract semilinear integral equation

$$(4.1) \quad u(t) = S(t, a)\phi(0) + \int_a^t T(t, r)F(r, u_r)dr, \quad t \geq a$$

$$u_a \in \phi \in \mathcal{D}(a),$$

where F is a continuous mapping from an open neighborhood $\nu(\mathcal{D})$ of \mathcal{D} to X satisfying the following condition:

(F) for any $l \geq 0$ there exist a constant $L_{1,l} > 0$ and a continuous function $L_{2,l} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $L_{2,l}(0) = 0$ such that

$$|F(t, \phi) - F(s, \psi)|_X \leq L_{2,l}(|t - s|) + L_{1,l}|\phi - \psi|_B$$

for $(t, \phi), (s, \psi) \in \mathcal{D}$ with $|\phi|_B \leq l$, $|\psi|_B \leq l$ and $a \leq t, s \leq a + l$.

$T = \{T(t, s); t \geq s \geq a\}$ is a C_0 linear evolution system on X and $S = \{S(t, s); t \geq s \geq a\}$ is an affine evolution system associated with T . That is, the following conditions (T1)–(T3), (S1) and (S2) are satisfied.

(T1) $T(t, t)u = u$ and $T(t, s)T(s, r)u = T(t, r)u$ for all $t \geq s \geq r \geq a$, $u \in X$;

(T2) for each $u \in X$, the mapping $(t, s) \rightarrow T(t, s)u$ is continuous for $t \geq s \geq a$;

(T3) there are numbers $\widehat{M} \geq 1$ and $\omega \in \mathbb{R}$ such that

$$|T(t, s)| = \sup\{|T(t, s)u|_X; |u|_X \leq 1\} \leq \widehat{M}e^{\omega(t-s)}, \quad t, s \geq a;$$

(S1) $t \rightarrow S(t, a)0$ is continuous from $[a, \infty)$ into X , where 0 is the zero of X ;

(S2) $S(t, r)u + T(t, s)v = S(t, s)[S(s, r)u + v]$ for all $u, v \in X$ and $t \geq s \geq r \geq a$.

It can be shown that (see, cf. Martin and Smith [36]) conditions (S1) and (S2) are equivalent to the following

(S3) there is a continuous function $\hat{\mu} : [a, \infty) \rightarrow X$ such that

$$S(t, s)u = T(t, s)[u - \hat{\mu}(s)] + \hat{\mu}(t)$$

for all $u \in X$ and $t \geq s \geq a$.

A function $u : (-\infty, b) \rightarrow X$, $b > a$ is a *solution in \mathcal{D}* of (4.1) if $u : [a, b) \rightarrow X$ is continuous, $(t, u_t) \in \mathcal{D}$ and u satisfies (4.1) for all $t \in [a, b)$. To guarantee the invariance of the set \mathcal{D} , we also need the following *subtangential condition*:

(SC) $\lim_{h \rightarrow 0^+} (1/h)d(S(t+h, t)\phi(0) + \int_t^{t+h} T(t+h, r)F(t, \phi)dr; D(t+h)) = 0$ for all $(t, \phi) \in \mathcal{D}$, where $d(u; D(t)) = \inf\{|u - v|_X; v \in D(t)\}$ for $u \in X$ and $t \geq a$.

We refer to Martin and Smith [36] for detailed discussion on (D1)–(D3), (S1)–(S3), (T1)–(T3) and (SC). Under the above assumptions, we have the following basic result on the existence and uniqueness of a solution of (4.1).

Theorem 4.1. *Suppose that (D1)–(D3), (T1)–(T3), (S1), (S2), (F) and (SC) are satisfied. Then (4.1) has a unique noncontinuable solution u , denoted by $u(t; a, \phi)$, on an interval of the form $[a, b)$, where $a < b \leq \infty$. Moreover, $u(t) \in D(t)$ and $u_t \in \mathcal{D}(t)$ for $a \leq t < b$ and if $b < \infty$, then*

$$\limsup_{t \rightarrow b^-} |u_t|_B = \infty.$$

Proof. The proof is similar to that in Martin and Smith [36] for abstract integral equations with finite delay. However, since our argument involves certain technicalities caused by the unboundedness of the delay, we provide the details of the proof here for the sake of completeness. The basic idea of our proof is to construct a sequence of approximate solutions which converges to a solution. We divide the proof into ten steps.

Step 1. By assumption (D3) there exists a constant $M_1 > 0$ and a continuous nondecreasing function $M_2 : [0, 1] \rightarrow \mathbb{R}_+$ satisfying $M_2(0) = 0$ and if $a \leq t_1 < t_2 \leq a + 1$, $u_1 \in D(t_1)$ and $u_2 \in D(t_2)$, then there exists a continuous function $w : [t_1, t_2] \rightarrow X$ such that $w(t_1) = u_1$, $w(t_2) = u_2$, $w(t) \in D(t)$ for $t_1 < t < t_2$ and

$$(4.2) \quad |w(t) - w(s)|_X \leq M_2(|t - s|) + M_1|t - s| \frac{|u_2 - u_1|_X}{t_2 - t_1}$$

for $s, t \in [t_1, t_2]$.

Step 2. Because of the continuity of $F : \mathcal{D} \rightarrow X$, there exists $\delta_1 > 0$ such that for any $(t, \psi) \in \mathcal{D}$ with $|t - a| < \delta_1$ and $|\psi - \phi|_B < \delta_1$, we have

$$(4.3) \quad |F(t, \psi)|_X \leq N := |F(a, \phi)|_X + 1.$$

On the other hand, since $D(a + 1) \neq \emptyset$, by the result in Step 1, we can find a continuous function $\tilde{w} : [a, a + 1] \rightarrow X$ such that $\tilde{w}(a) = \phi(0)$ and $\tilde{w}(t) \in D(t)$ for $t \in [a, a + 1]$. We now define $\bar{\phi} : (-\infty, a + 1] \rightarrow X$ by

$$\bar{\phi}(t) = \begin{cases} \hat{\phi}(t - a), & t \leq a \\ \tilde{w}(t), & a \leq t \leq a + 1, \end{cases}$$

where $\hat{\phi} \in \hat{B}$ is a representing element of $\phi \in B$. By (A2) and (D2), $\bar{\phi}_t \in \mathcal{D}(t) \subseteq B$ for any $t \in [a, a + 1]$ and $\bar{\phi}_t \rightarrow \phi$ as $t \rightarrow a^+$. Therefore, there exists $\delta_2 > 0$ such that $|\bar{\phi}_t - \phi|_B < \delta_1/2$ for all $t \in [a, a + \delta_2]$. Set $\delta_0 = \min\{1, \delta_1, \delta_2\}$. Then for any $u : (-\infty, a + \delta_0] \rightarrow X$ such that $u_a = \phi$, $u : [a, a + \delta_0] \rightarrow X$ is continuous, $u(s) \in D(s)$ and $|u(s) - \phi(0)|_X < \delta_1/(2[K(\delta_0) + 1] + 1)$ for $s \in [a, a + \delta_0]$, we have for $t \in [a, a + \delta_0]$ that

$$\begin{aligned} |u_t - \phi|_B &\leq |u_t - \bar{\phi}_t|_B + |\bar{\phi}_t - \phi|_B \\ &\leq K(\delta_0) \sup_{a \leq s \leq t} |u(s) - \phi(0)|_X + |\bar{\phi}_t - \phi|_B \\ &< K(\delta_0) \frac{\delta_1}{2[K(\delta_0) + 1]} + \frac{1}{2}\delta_1 \\ &\leq \delta_1. \end{aligned}$$

Hence, from (4.3) it follows that

$$(4.4) \quad |F(t, u_t)|_X \leq N, \quad t \in [a, a + \delta_0].$$

Step 3. Because of the continuity of S and M_2 , we can find constants $\sigma > a$ and $\varepsilon_0 > 0$ such that

$$(4.5) \quad \sigma + \varepsilon_0 < a + \delta_0,$$

$$(4.6) \quad e^{|\omega|} \widehat{M}(e^{|\omega|} \widehat{M}N + \varepsilon_0)(\sigma + \varepsilon_0 - a) < \frac{\delta_1}{8[K(\delta_0) + 1][2M_1 + 1]},$$

$$(4.7) \quad |S(t, a)\phi(0) - \phi(0)|_X < \frac{\delta_1}{8[K(\delta_0) + 1][2M_1 + 1]},$$

$$t \in [a, \sigma + \varepsilon_0],$$

$$(4.8) \quad M_2(\sigma + \varepsilon_0 - a) < \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]},$$

where ω and \widehat{M} were defined in (T3). Now for any given $\varepsilon \in (0, \varepsilon_0]$, we construct an ε -approximate solution w and a corresponding sequence $\{t_j\}$ as follows: set $t_0 = a$ and define $w(a + s) = \hat{\phi}(s)$ for $s \leq 0$, where $\hat{\phi}$ is a fixed representative element of ϕ . Assume that i is a nonnegative integer and w is constructed and continuous on $(-\infty, t_i]$ with $a \leq t_i < \sigma + \varepsilon_0$ and $w(t) \in D(t)$ for $t \in [a, t_i]$. If $t_i \geq \sigma$, then set $t_{i+1} = t_i$. If $t_i < \sigma$, we define Γ_i to be the set of all constants $\gamma \in [0, \varepsilon/2]$ such that

$$(4.9) \quad |S(t, t_i)w(t_i) - w(t_i)|_X \leq \varepsilon, \quad t_i \leq t \leq t_i + \gamma,$$

$$(4.10) \quad d\left(S(t_i + h, t_i)w(t_i) + \int_{t_i}^{t_i+h} T(t_i + h, \theta)F(t_i, w_{t_i})d\theta, D(t_i + h)\right) \leq \frac{\varepsilon}{2}h,$$

where $0 \leq h \leq \gamma$, and

$$(4.11) \quad |w(t) - w(s)|_X \leq \frac{\varepsilon}{4[M_1K(\sigma - a) + 1]}, \quad t, s \in [a, t_i], \quad |t - s| \leq \gamma,$$

$$(4.12) \quad K(\sigma - a)M_2(\gamma) < \varepsilon/2,$$

$$(4.13) \quad |\bar{\phi}_t - \phi|_B < \varepsilon/2, \quad t \in [a, a + \gamma].$$

By assumptions (S3) and (SC), Γ_i is nonempty. We let

$$(4.14) \quad \gamma_i = \frac{3}{4} \sup\{\gamma; \gamma \in \Gamma_i\},$$

and $t_{i+1} = t_i + \gamma_i$. Because of (4.10), we can find an element $w(t_{i+1}) \in D(t_{i+1})$ so that

$$(4.15) \quad \left| S(t_{i+1}, t_i)w(t_i) + \int_{t_i}^{t_{i+1}} T(t_{i+1}, \theta)F(t_i, w_{t_i}) d\theta - w(t_{i+1}) \right|_X \leq \varepsilon(t_{i+1} - t_i).$$

Finally, by Step 1, we can define w on $[t_i, t_{i+1}]$ so that w is continuous, $w(t) \in D(t)$ and (4.2) holds.

Step 4. We claim that the function w constructed as above satisfies the following properties:

$$(4.16) \quad |w(t_i) - \phi(0)|_X < \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]} \quad \text{for } i = 0, 1, 2, \dots;$$

$$(4.17) \quad |w(t) - \phi(0)|_X < \frac{\delta_1}{2[K(\delta_0) + 1]} \quad \text{for } t \in [a, t_i], i = 0, 1, 2, \dots;$$

$$(4.18) \quad |F(t_i, w_{t_i})|_X \leq N, \quad i = 0, 1, 2, \dots$$

These estimates can be proved simultaneously by induction on i . Clearly, (4.16), (4.17) and (4.18) hold for $i = 0$. Assume that $k \geq 0$ is given such that (4.16)-(4.18) hold for all $i = 0, 1, 2, \dots, k$. Then, from (4.15), (T3) and the fact that $t_{i+1} - t_i \leq a + 1 - a = 1$, it follows that

$$(4.19) \quad \begin{aligned} & |S(t_{i+1}, t_i)w(t_i) - w(t_{i+1})|_X \\ & \leq \left| \int_{t_i}^{t_{i+1}} T(t_{i+1}, \theta)F(t_i, w_{t_i}) d\theta \right|_X + \varepsilon(t_{i+1} - t_i) \\ & \leq (e^{|\omega|} \widehat{M}N + \varepsilon_0)(t_{i+1} - t_i) \quad \text{for } i = 0, 1, 2, \dots, k. \end{aligned}$$

Therefore, by (S2), (T1), (T3) and (4.15) we have

$$\begin{aligned} & \sup\{e^{-\omega r}|T(t_{i+1} + r, t_{i+1})[S(t_{i+1}, a)\phi(0) - w(t_{i+1})]|_X; r \geq 0\} \\ & \leq \sup\{e^{-\omega r}|T(t_{i+1} + r, t_{i+1})[S(t_{i+1}, t_i)S(t_i, a)\phi(0) \\ & \qquad \qquad \qquad - S(t_{i+1}, t_i)w(t_i)]|_X; r \geq 0\} \\ & \quad + \sup\{e^{-\omega r}|T(t_{i+1} + r, t_{i+1})[S(t_{i+1}, t_i)w(t_i) - w(t_{i+1})]|_X; r \geq 0\} \\ & \leq \sup\{e^{-\omega r}|T(t_{i+1} + r, t_{i+1})[S(t_i, a)\phi(0) - w(t_i)]|_X; r \geq 0\} \\ & \quad + \widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)(t_{i+1} - t_i) \\ & \leq e^{\omega(t_{i+1}-t_i)} \sup\{e^{-\omega r}|T(t_i + r, t_i)[S(t_i, a)\phi(0) - w(t_i)]|_X; r \geq 0\} \\ & \quad + \widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)(t_{i+1} - t_i), \end{aligned}$$

where $i = 0, 1, 2, \dots, k$. Using the above inequality for $i = 0, 1, 2, \dots, k$, we can easily get

$$\begin{aligned} & \sup\{e^{-\omega r}|T(t_{k+1} + r, t_{k+1})[S(t_{k+1}, a)\phi(0) - w(t_{k+1})]|_X; r \geq 0\} \\ & \leq e^{|\omega|}\widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)(t_{k+1} - a). \end{aligned}$$

Consequently, by (4.6), we obtain

$$\begin{aligned} (4.20) \quad & |S(t_{k+1}, a)\phi(0) - w(t_{k+1})|_X \\ & \leq \sup\{e^{-\omega r}|T(t_{k+1} + r, t_{k+1})[S(t_{k+1}, a)\phi(0) - w(t_{k+1})]|_X; r \geq 0\} \\ & \leq e^{|\omega|}\widehat{M}(e^{|\omega|}\widehat{M}N + \varepsilon_0)(t_{k+1} - a) \\ & < \frac{\delta_1}{8[K(\delta_0) + 1][2M_1 + 1]}. \end{aligned}$$

Hence, by (4.7) we get

$$\begin{aligned} |w(t_{k+1}) - \phi(0)|_X & \leq |w(t_{k+1}) - S(t_{k+1}, a)\phi(0)|_X \\ & \quad + |S(t_{k+1}, a)\phi(0) - \phi(0)|_X \\ & < \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]}. \end{aligned}$$

Moreover, by (4.2) and (4.8), for $t \in [t_k, t_{k+1}]$ we have

$$\begin{aligned}
 |w(t) - \phi(0)|_X &\leq |w(t) - w(t_k)|_X + |w(t_k) - \phi(0)|_X \\
 &\leq M_2(|t - t_k|) + M_1 \frac{|w(t_{k+1}) - w(t_k)|_X}{t_{k+1} - t_k} (t - t_k) \\
 &\quad + |w(t_k) - \phi(0)|_X \\
 &\leq M_2(\sigma + \varepsilon_0 - a) + M_1[|w(t_{k+1}) - \phi(0)|_X \\
 &\quad + |w(t_k) - \phi(0)|_X] + |w(t_k) - \phi(0)|_X \\
 &< \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]} + M_1 \frac{\delta_1}{2[K(\delta_0) + 1][2M_1 + 1]} \\
 &\quad + \frac{\delta_1}{4[K(\delta_0) + 1][2M_1 + 1]} \\
 &< \frac{\delta_1}{2[K(\delta_0) + 1]}.
 \end{aligned}$$

Therefore, by the result of Step 2, $|F(t_{k+1}, w_{t_{k+1}})|_X \leq N$.

Step 5. Let $\rho = \lim_{i \rightarrow \infty} t_i$. Obviously, ρ exists and w is defined on $[a, \rho)$ with $w(t) \in D(t)$ for all $t \in [a, \rho)$. We claim that $z = \lim_{t \rightarrow \rho^-} w(t)$ exists and $z \in D(\rho)$. Indeed, using a similar argument as that for (4.20), we obtain for $j \geq k$,

(4.21)

$$\begin{aligned}
 |S(t_j, t_k)w(t_k) - w(t_j)|_X &\leq e^{|\omega|(t_j - t_k)} \widehat{M}(e^{|\omega|} \widehat{M}N + \varepsilon_0)(t_j - t_k) \\
 &\leq \widehat{N}(t_j - t_k),
 \end{aligned}$$

where $\widehat{N} = e^{|\omega|(\rho - a)} \widehat{M}(e^{|\omega|} \widehat{M}N + \varepsilon_0)$. Let $\bar{\varepsilon} > 0$ be given and choose $k > 0$ so that $2\widehat{N}(\rho - t_k) < \bar{\varepsilon}/2$. By the continuity of S , we can find an integer $n(\bar{\varepsilon}) > k$ such that

(4.22) $|S(t_i, t_k)w(t_k) - S(t_j, t_k)w(t_k)|_X \leq \bar{\varepsilon}/2$ for $i, j \geq n(\bar{\varepsilon})$.

Therefore, for $i, j \geq n(\bar{\varepsilon})$ we have

$$\begin{aligned}
 |w(t_i) - w(t_j)|_X &\leq |w(t_i) - S(t_i, t_k)w(t_k)|_X \\
 &\quad + |S(t_i, t_k)w(t_k) - S(t_j, t_k)w(t_k)|_X \\
 &\quad + |S(t_j, t_k)w(t_k) - w(t_j)|_X \\
 &\leq \widehat{N}(\rho - t_k) + \frac{\bar{\varepsilon}}{2} + \widehat{N}(\rho - t_k) < \bar{\varepsilon}.
 \end{aligned}$$

Hence, $z = \lim_{i \rightarrow \infty} w(t_i)$ exists. Since $(t_i, w(t_i)) \in D$ and D is closed, it follows that $(\rho, z) \in D$. Moreover, by (4.2) we have

$$\begin{aligned} |w(t) - w(t_i)|_X &\leq M_2(|t - t_i|) + M_1|t - t_i| \frac{|w(t_{i+1}) - w(t_i)|_X}{t_{i+1} - t_i} \\ &\leq M_2(|t_{i+1} - t_i|) + M_1|w(t_{i+1}) - w(t_i)|_X, \\ &\quad t_i \leq t \leq t_{i+1}. \end{aligned}$$

This implies that $w(t) \rightarrow z$ as $t \rightarrow \rho^-$.

Step 6. We show that the ε -approximate solution constructed above is defined in $[a, \sigma]$, i.e., there is an integer $n = n(\varepsilon)$ so that $t_n \geq \sigma$. By contradiction, if no such n exists, then $t_i < \sigma$ for all $i \geq 1$, $\lim_{i \rightarrow \infty} t_i = \rho \leq \sigma$ and $\lim_{i \rightarrow \infty} w(t_i) = z$. Let $w(\rho) = z$. Clearly, $w_\rho \in \mathcal{D}(\rho)$. By the subtangential condition (SC), there exists a constant $\delta > 0$ independent of i , such that (4.9), (4.11), (4.12) and (4.13) hold with γ being replaced by δ . Moreover,

$$d\left(S(\rho + \eta, \rho)z + \int_\rho^{\rho+\eta} T(\rho + \eta, \theta)F(\rho, w_\rho)d\theta; D(\rho + \eta)\right) \leq \frac{\varepsilon}{4}\eta$$

for all $\eta \in (0, \delta]$. (A2) implies $\lim_{i \rightarrow \infty} w_{t_i} = w_\rho$. Therefore, we have

$$d\left(S(t_i + \eta, t_i)w(t_i) + \int_{t_i}^{t_i+\eta} T(t_i + \eta, \theta)F(t_i, w_{t_i})d\theta; D(t_i + \eta)\right) \leq \frac{\varepsilon}{2}\eta$$

for sufficiently large i . This implies that $\delta \in \Gamma_i$, hence $\delta \leq 4\gamma_i/3 = 4(t_{i+1} - t_i)/3 \rightarrow 0$ as $i \rightarrow \infty$, a contradiction to the independence of δ on i .

Step 7. We now construct a sequence of approximate solutions. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a decreasing sequence in $[0, \varepsilon_0)$ such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and for each $n \geq 1$ let w^n and $\{t_i^n\}_{i=1}^\infty$ be as constructed above with $\varepsilon = \varepsilon_n$, $t_i = t_i^n$ and $w = w^n$. Therefore, by the result in Step 6, for each $n \geq 1$ there exists an $n_0 = n_0(n)$ such that $t_{n_0}^n \geq \sigma$. For convenience, we define a comparison function v^n for w^n as follows

(4.23)

$$\begin{aligned} v^n(t) &= S(t, a)\phi(0) + \int_a^t T(t, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta, \quad t \in [a, \sigma], \\ v^n(a + \theta) &= \widehat{\phi}(\theta) \quad \text{for } \theta \leq 0, \end{aligned}$$

where $\gamma^n : [a, \sigma] \rightarrow [a, \sigma]$ is given by $\gamma^n(t) = t_i^n$ for $t_i^n \leq t < t_{i+1}^n$. We claim that there exists a constant $P > 0$ independent of n , such that

$$(4.24) \quad |v^n(t) - w^n(t)|_X \leq P \max\{\varepsilon_n, M_2(\varepsilon_n)\}, \quad a \leq t \leq \sigma, n = 1, 2, \dots .$$

Indeed, by (S2) we obtain, for $a \leq s \leq t \leq \sigma$, that

$$(4.25) \quad \begin{aligned} v^n(t) &= S(t, s)S(s, a)\phi(0) \\ &\quad + T(t, s) \int_a^s T(s, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta \\ &\quad + \int_s^t T(t, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta \\ &= S(t, s)v^n(s) + \int_s^t T(t, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta. \end{aligned}$$

Note that for each $n \geq 1$ and $i = 1, 2, \dots$, we have

$$\begin{aligned} &\sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)[v^n(t_{i+1}^n) - w^n(t_{i+1}^n)]|_X; r \geq 0\} \\ &= \sup\{e^{-\omega r} \left| T(t_{i+1}^n + r, t_{i+1}^n)[S(t_{i+1}^n, t_i^n)v^n(t_i^n) \right. \\ &\quad \left. + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta - w^n(t_{i+1}^n)] \right|_X; r \geq 0\} \\ &\leq \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)[S(t_{i+1}^n, t_i^n)v^n(t_i^n) \\ &\quad - S(t_{i+1}^n, t_i^n)w^n(t_i^n)]|_X; r \geq 0\} \\ &\quad + \sup \left\{ e^{-\omega r} \left| T(t_{i+1}^n + r, t_{i+1}^n)[S(t_{i+1}^n, t_i^n)w^n(t_i^n) \right. \right. \\ &\quad \left. \left. + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta - w^n(t_{i+1}^n)] \right|_X; r \geq 0 \right\} \\ &\leq \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_{i+1}^n)T(t_{i+1}^n, t_i^n)[v^n(t_i^n) - w^n(t_i^n)]|_X; r \geq 0\} \\ &\quad + \widehat{M} \left| S(t_{i+1}^n, t_i^n)w^n(t_i^n) + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta \right. \\ &\quad \left. - w^n(t_{i+1}^n) \right|_X \\ &= \sup\{e^{-\omega r}|T(t_{i+1}^n + r, t_i^n)[v^n(t_i^n) - w^n(t_i^n)]|_X; r \geq 0\} \end{aligned}$$

$$\begin{aligned}
 & + \widehat{M} \left| S(t_{i+1}^n, t_i^n) w^n(t_i^n) + \int_{t_i^n}^{t_{i+1}^n} T(t_{i+1}^n, \theta) F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta \right. \\
 & \qquad \qquad \qquad \left. - w^n(t_{i+1}^n) \right|_X.
 \end{aligned}$$

Therefore, by (4.15), we have

$$\begin{aligned}
 & \sup\{e^{-\omega r} |T(t_{i+1}^n + r, t_{i+1}^n)[v^n(t_{i+1}^n) - w^n(t_i^n)]|_X; r \geq 0\} \\
 & \leq e^{|\omega|(t_{i+1}^n - t_i^n)} \sup\{e^{-\omega r} |T(t_i^n + r, t_i^n)[v^n(t_i^n) - w^n(t_i^n)]|_X; r \geq 0\} \\
 & \quad + \widehat{M} \varepsilon_n (t_{i+1}^n - t_i^n), \quad i = 0, 1, 2, \dots
 \end{aligned}$$

Applying the above inequality at $0, 1, 2, \dots, i$ and using $e^{|\omega|(t_{i+1}^n - t_i^n)} \leq e^{|\omega|(t_{i+1}^n - a)}$, we get

$$\begin{aligned}
 & |v^n(t_{i+1}^n) - w^n(t_{i+1}^n)|_X \\
 & \leq \sup\{e^{-\omega r} |T(t_{i+1}^n + r, t_{i+1}^n)[v^n(t_{i+1}^n) - w^n(t_{i+1}^n)]|_X; r \geq 0\} \\
 & \leq e^{|\omega|(t_{i+1}^n - a)} \widehat{M} \varepsilon_n (t_{i+1}^n - a) \\
 & \leq \widehat{Q} \varepsilon_n,
 \end{aligned}$$

where $\widehat{Q} = e^{|\omega|(\sigma - a)} \widehat{M}(\sigma - a)$. If $t_i^n \leq t \leq t_{i+1}^n \leq \sigma$, then by using (4.25) we get

$$\begin{aligned}
 |v^n(t) - w^n(t)|_X & \leq \left| S(t, t_i^n) v^n(t_i^n) \right. \\
 & \quad \left. + \int_{t_i^n}^t T(t, \theta) F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta - S(t, t_i^n) w(t_i^n) \right|_X \\
 & \quad + |S(t, t_i^n) w^n(t_i^n) - w^n(t_i^n)|_X + |w^n(t_i^n) - w^n(t)|_X.
 \end{aligned}$$

By (S3), (T3), (4.2) and (4.9), we have

$$\begin{aligned}
 |v^n(t) - w^n(t)|_X & \leq \widehat{M} e^{|\omega|(\sigma - a)} |v^n(t_i^n) - w^n(t_i^n)|_X \\
 & \quad + \left| \int_{t_i^n}^t T(t, \theta) F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta \right|_X \\
 & \quad + \varepsilon_n + M_2(t_{i+1}^n - t_i^n) + M_1 |w^n(t_{i+1}^n) - w^n(t_i^n)|_X \\
 & \leq \widehat{M} e^{|\omega|(\sigma - a)} \widehat{Q} \varepsilon_n + \widehat{M} e^{|\omega|(\sigma - a)} N \varepsilon_n \\
 & \quad + \varepsilon_n + M_2(\varepsilon_n) + M_1 |w^n(t_{i+1}^n) - w^n(t_i^n)|_X,
 \end{aligned}$$

where we used the fact that $t - t_i^n \leq t_{i+1}^n - t_i^n \leq \varepsilon_n$. By (4.21) and (4.9), we obtain

$$\begin{aligned}
 |w^n(t_{i+1}^n) - w^n(t_i^n)|_X &\leq |w^n(t_{i+1}^n) - S(t_{i+1}^n, t_i^n)w^n(t_i^n)|_X \\
 &\quad + |S(t_{i+1}^n, t_i^n)w^n(t_i^n) - w^n(t_i^n)|_X \\
 (4.26) \qquad &\leq \tilde{N}(t_{i+1}^n - t_i^n) + \varepsilon_n \\
 &\leq (\tilde{N} + 1)\varepsilon_n,
 \end{aligned}$$

where $\tilde{N} = e^{|\omega|(\sigma-a)}(e^{|\omega|}\widehat{M}N + \varepsilon_0)$. Therefore

$$|v^n(t) - w^n(t)|_X \leq [\widehat{M}e^{|\omega|(\sigma-a)}(\widehat{Q} + N) + 1 + M_1(\tilde{N} + 1)]\varepsilon_n + M_2(\varepsilon_n).$$

This proves (4.24).

Step 8. We claim that there exists a constant $Q > 0$ independent of $t \in [a, \sigma]$ and $n \geq 1$, such that

$$|w_t^n - w_{\gamma^n(t)}^n|_B \leq Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} \quad \text{for } t \in [a, \sigma], n = 1, 2, \dots$$

Indeed, for all $n \geq 1$ and $t_i^n \leq t \leq t_{i+1}^n \leq \sigma$, by (A3) we have

$$\begin{aligned}
 |w_t^n - w_{\gamma^n(t)}^n|_B &= |w_t^n - w_{t_i^n}^n|_B \\
 (4.27) \qquad &\leq K(t_i^n - a) \sup_{a \leq s \leq t_i^n} |w^n(t - t_i^n + s) - w^n(s)|_X \\
 &\quad + M(t_i^n - a)|w_{t-t_i^n+a}^n - \phi|_B.
 \end{aligned}$$

On the other hand, by (A3) we get

$$\begin{aligned}
 |w_{t-t_i^n+a}^n - \phi|_B &\leq |w_{t-t_i^n+a}^n - \bar{\phi}_{t-t_i^n+a}|_B + |\bar{\phi}_{t-t_i^n+a} - \phi|_B \\
 &\leq K(t - t_i^n + a) \sup_{a \leq s \leq t-t_i^n+a} |w(s) - \phi(0)|_X \\
 &\quad + |\bar{\phi}_{t-t_i^n+a} - \phi|_B.
 \end{aligned}$$

Therefore, (4.11) and (4.13) imply that

$$(4.28) \quad |w_{t-t_i^n+a}^n - \phi|_B \leq K(\sigma - a) \frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]} + \frac{\varepsilon_n}{2} \leq \frac{3}{4}\varepsilon_n.$$

Moreover, if $t - t_i^n + s \leq t_i^n$, then from (4.11) it follows that

$$(4.29) \quad |w^n(t - t_i^n + s) - w^n(s)|_X \leq \frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]},$$

and if $s \leq t_i^n$, $t - t_i^n + s > t_i^n$, then

$$|w^n(t_i^n) - w^n(s)|_X \leq \frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]}$$

and by (4.2) we have

$$\begin{aligned} & |w^n(t - t_i^n + s) - w^n(s)|_X \\ & \leq |w^n(t - t_i^n + s) - w^n(t_i^n)|_X + |w^n(t_i^n) - w^n(s)|_X \\ & \leq M_2(\varepsilon_n) + M_1|w^n(t_{i+1}^n) - w^n(t_i^n)|_X + \frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]}. \end{aligned}$$

This, together with (4.26), implies that

$$(4.30) \quad |w^n(t - t_i^n + s) - w^n(s)|_X \leq M_2(\varepsilon_n) + M_1(\tilde{N} + 1)\varepsilon_n + \frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]}.$$

Substituting (4.28), (4.29) and (4.30) into (4.27), we obtain

$$\begin{aligned} |w_i^n - w_{\gamma^n(t)}^n|_B & \leq K(\sigma - a) \left\{ M_2(\varepsilon_n) + M_1(\tilde{N} + 1)\varepsilon_n \right. \\ & \quad \left. + \frac{\varepsilon_n}{4[M_1K(\sigma - a) + 1]} \right\} \\ & \quad + \sup_{a \leq s \leq \sigma} M(s - a) \cdot \frac{3}{4}\varepsilon_n. \end{aligned}$$

Step 9. We show that $\{w^n(t)\}_{n=1}^\infty$ is uniformly Cauchy on $[a, \sigma]$. In fact, by (4.17), we have

$$|w^n(t) - \phi(0)|_X \leq \frac{\delta_1}{2[K(\delta_0) + 1]}$$

and hence from the argument in Step 2, we have

$$|w_i^n - \phi|_B \leq \delta_1.$$

This implies that

$$|w_t^n|_B \leq |\phi|_B + \delta_1.$$

Therefore, by the Lipschitz condition of F , there exist a constant $L_1 > 0$ and a continuous function $L_2 : R_+ \rightarrow R_+$ such that $L_2(0) = 0$, and for all $\theta \in [a, \sigma]$, $n, m \geq 1$, we have

$$\begin{aligned} & |F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) - F(\gamma^m(\theta), w_{\gamma^m(\theta)}^m)|_X \\ & \leq L_1 |w_{\gamma^n(\theta)}^n - w_{\gamma^m(\theta)}^m|_B + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|). \end{aligned}$$

By the results in Step 7 and Step 8 as well as assumption (A3), we get

$$\begin{aligned} & |F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) - F(\gamma^m(\theta), w_{\gamma^m(\theta)}^m)|_X \\ & \leq L_1 |w_\theta^n - w_\theta^m|_B + L_1 Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ & \quad + L_1 Q \max\{\varepsilon_m, M_2(\varepsilon_m)\} + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|) \\ & \leq L_1 K(\sigma - a) \sup_{a \leq s \leq \theta} |w^n(s) - w^m(s)|_X \\ & \quad + L_1 Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ & \quad + L_1 Q \max\{\varepsilon_m, M_2(\varepsilon_m)\} + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|) \\ & \leq L_1 K(\sigma - a) \sup_{a \leq s \leq \theta} |v^n(s) - v^m(s)|_X \\ & \quad + L_1 [K(\sigma - a)P + Q] \max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ & \quad + L_1 [K(\sigma - a)P + Q] \max\{\varepsilon_m, M_2(\varepsilon_m)\} \\ & \quad + L_2(|\gamma^n(\theta) - \gamma^m(\theta)|). \end{aligned}$$

By the definition of v^n , it follows that

$$\begin{aligned} & |v^n(t) - v^m(t)|_X \\ & \leq \int_a^t |T(t, \theta)| |F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) - F(\gamma^m(\theta), w_{\gamma^m(\theta)}^m)|_X d\theta \\ & \leq L_1 K(\sigma - a) \widehat{M} e^{|\omega|(\sigma - a)} \int_a^t \sup_{a \leq s \leq \theta} |v^n(s) - v^m(s)|_X d\theta + \eta_{n,m}, \end{aligned}$$

where

$$\begin{aligned} \eta_{n,m} = & L_1 \widehat{M} e^{|\omega|(\sigma - a)} (\sigma - a) [K(\sigma - a)P + Q] (\max\{\varepsilon_n, M_2(\varepsilon_n)\} \\ & \quad + \max\{\varepsilon_m, M_2(\varepsilon_m)\}) \\ & + \widehat{M} e^{|\omega|(\sigma - a)} (\sigma - a) \max_{a \leq r \leq \sigma} L_2(|\gamma^n(r) - \gamma^m(r)|). \end{aligned}$$

Hence, by the well-known Gronwall integral inequality, we have

$$\sup_{a \leq s \leq t} |v^n(s) - v^m(s)|_X \leq \eta_{n,m} \exp(L_1 K(\sigma - a) \widehat{M} e^{|\omega|(\sigma-a)(t-a)}) \rightarrow 0$$

uniformly for $t \in [a, \sigma]$ as $n, m \rightarrow \infty$, since $\eta_{n,m} \rightarrow 0$ as $n, m \rightarrow \infty$. Therefore, $\{v^n(t)\}_{n=1}^\infty$ is uniformly Cauchy on $[a, \sigma]$, and so is $\{w^n(t)\}_{n=1}^\infty$ by (4.24).

Step 10. Let $u(t) = \lim_{n \rightarrow \infty} w^n(t)$ uniformly for $t \in [a, \sigma]$ and $u(t) = \tilde{\phi}(t - a)$ for $t \leq a$. Clearly $(t, u(t)) \in D$ since $(t, w^n(t)) \in D$ and D is closed. We claim that $u(t)$ is a solution of (4.1). In fact, by the result in Step 8 and assumption (A3), we have

$$\begin{aligned} |w_{\gamma^n(t)}^n - u_t|_B &\leq |w_{\gamma^n(t)}^n - w_t^n|_B + |w_t^n - u_t|_B \\ &\leq Q \max\{\varepsilon_n, M_2(\varepsilon_n)\} + K(\sigma - a) \sup_{a \leq s \leq t} |w^n(s) - u(s)|_X. \end{aligned}$$

This implies $w_{\gamma^n(t)}^n \rightarrow u_t$ in B uniformly for $t \in [a, \sigma]$ as $n \rightarrow \infty$, and hence

$$F(\gamma^n(t), w_{\gamma^n(t)}^n) \rightarrow F(t, u_t)$$

uniformly for $t \in [a, \sigma]$ as $n \rightarrow \infty$. Therefore

$$\begin{aligned} u(t) &= \lim_{n \rightarrow \infty} \left\{ S(t, a)\phi(0) + \int_a^t T(t, \theta)F(\gamma^n(\theta), w_{\gamma^n(\theta)}^n) d\theta \right\} \\ &= S(t, a)\phi(0) + \int_a^t T(t, \theta)F(\theta, u_\theta) d\theta. \end{aligned}$$

This establishes the existence of a solution of (4.1) on $[a, \sigma]$.

The uniqueness and continuation can be proved by using the standard technique. The proof is complete. \square

As a consequence of Theorem 4.1, we obtain the following invariant property:

Corollary 4.2. *Suppose that K_1 and K_2 are nonempty closed, convex subsets of X and B , respectively, such that (D2), (T1)–(T3),*

(S1), (S2) and (F) are satisfied with $D = [a, \infty) \times K_1$ and $\mathcal{D} = [a, \infty) \times K_2$. Moreover, assume that

- (i) $S(t, s)K_1 \subseteq K_1$ for $t \geq s \geq a$;
- (ii) $\lim_{h \rightarrow 0^+} (1/h)d(\phi(0) + hF(t, \phi); K_1) = 0$ for $(t, \phi) \in R \times K_2$.

Then the unique noncontinuable solution $u : (-\infty, b) \rightarrow B$, $b > a$ of (4.1) satisfies that $u(t) \in K_1$ and $u_t \in K_2$ for all $t \in [a, b)$.

Proof. Obviously, (D3) holds since $D(t) = K_1$ is convex. To prove Corollary 4.2. it suffices to verify that

$$\lim_{h \rightarrow 0^+} \frac{1}{h} d \left(S(t+h, t)\phi(0) + \int_t^{t+h} T(t+h, \theta)F(t, \phi)d\theta; K_1 \right) = 0$$

for $(t, \phi) \in [a, \infty) \times K_2$. Indeed, for given $h > 0$ and $(t, \phi) \in [a, \infty) \times K_2$, define $\psi : (-\infty, t+h) \rightarrow X$ by $\psi_t = \phi$ and $\psi(\theta) = S(\theta, t)\phi(0)$ for $\theta \in [t, t+h]$. Since $S(\theta, t)K_1 \subseteq K_1$ for $t \leq \theta \leq t+h$, by assumption (D2) we have $\psi_{t+h} \in K_2$ and ψ_{t+h} is continuous in $h \in R_+$. Therefore, the set $\{(t, \psi_{t+h}); 0 \leq h \leq 1\}$ is a compact set of $[a, \infty) \times K_2$. This, together with the convexity of the function $h \in R_+ \rightarrow d(\phi(0) + hF(t, \phi); K_1) \in R_+$ as well as the continuity of F , shows that assumption (ii) implies that $\lim_{h \rightarrow 0^+} (1/h)d(\psi(t+h) + hF(t, \psi_{t+h}); K_1) = 0$.

On the other hand, by (A3), we have

$$\begin{aligned} |\psi_{t+h} - \psi|_B &\leq |\psi_{t+h} - \bar{\phi}_{t+h}|_B + |\bar{\phi}_{t+h} - \phi|_B \\ &\leq K(h) \sup_{t \leq \theta \leq t+h} |S(\theta, t)\phi(0) - \phi(0)|_X \\ &\quad + |\bar{\phi}_{t+h} - \phi|_B \rightarrow 0 \quad \text{as } h \rightarrow 0^+ \end{aligned}$$

and for any $\varepsilon_1 > 0$ there exists $h_1 > 0$ such that if $0 < h < h_1$, then

$$|T(t+h, \theta)F(t, \phi) - F(t, \phi)|_X < \varepsilon_1, \quad t \leq \theta \leq t+h.$$

Hence, if $0 < h < h_1$, we obtain

$$\begin{aligned} \frac{1}{h} d \left(S(t+h, t)\phi(0) + \int_t^{t+h} T(t+h, \theta)F(t, \phi)d\theta; K_1 \right) \\ \leq \frac{1}{h} d(S(t+h, t)\phi(0) + hF(t, \phi); K_1) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{h} \left| \int_t^{t+h} [T(t+h, \theta)F(t, \phi) - F(t, \phi)] d\theta \right|_X \\
 & \leq \frac{1}{h} d(S(t+h, t)\phi(0) + hF(t, \phi); K_1) + \varepsilon_1 \\
 & \leq \frac{1}{h} d(\psi(t+h) + hF(t, \psi_{t+h}); K_1) \\
 & \quad + |F(t, \psi_{t+h}) - F(t, \phi)|_X + \varepsilon_1 \\
 & \rightarrow 0 \quad \text{as } h \rightarrow 0.
 \end{aligned}$$

This completes the proof. \square

5. Comparison and monotonicity principles for semilinear integral equations. In this section, we show how to apply the general results in Section 4 to obtain some comparison principles.

Let $S^+ = \{S^+(t, s); t \geq s \geq a\}$ and $S^- = \{S^-(t, s); t \geq s \geq a\}$ be given families of mappings from X to X which satisfy the following conditions:

(C1) S^+ and S^- satisfy (S1) and (S2) with S replaced by S^+ and S^- , respectively;

(C2) $S^-(t, s)x \leq S(t, s)x \leq S^+(t, s)x$ for all $t \geq s \geq a$ and $x \in X$.

In addition to (T1), (T2) and (T3), we require that

(T4) $T(t, s)X_+ \subseteq X_+$ for $t \geq s \geq a$.

Suppose that $v^\pm : (-\infty, b] \rightarrow X$, $b > a$ are given such that $v_a^\pm \in B$, $v^\pm : [a, b] \rightarrow X$ are continuous and the following condition is satisfied:

(C3) $v_a^- \leq_B v_a^+$ and $v^-(t) \leq_X v^+(t)$ for $t \in [a, b]$.

Let

$$D(t) = [v^-(t), v^+(t)]_X, \quad \mathcal{D}(t) = [v_t^-, v_t^+]_B \quad \text{for } t \geq a.$$

Define

$$D = \{(t, x); t \geq a, x \in D(t)\} \quad \text{and} \quad \mathcal{D} = \{(t, \phi); t \geq a, \phi \in \mathcal{D}(t)\}.$$

Clearly, (A5) implies that (D2) is satisfied. We assume that there exist continuous mappings $F^\pm : \mathcal{D} \rightarrow X$ such that

$$(C4) \quad v^+(t+h) \geq S^+(t+h, t)v^+(t) + \int_t^{t+h} T(t+h, \theta)F^+(\theta, v_\theta^+)d\theta, \\ a \leq t < t+h < b;$$

$$(C5) \quad v^-(t+h) \leq S^-(t+h, t)v^-(t) + \int_t^{t+h} T(t+h, \theta)F^-(\theta, v_\theta^-)d\theta, \\ a \leq t < t+h < b;$$

$$(C6) \quad \lim_{h \rightarrow 0^+} (1/h)d(v^+(0) - \phi(0) + h[F^+(t, v_t^+) - F(t, \phi)]); X_+ = 0, \\ a \leq t \leq b, (t, \phi) \in \mathcal{D};$$

$$(C7) \quad \lim_{h \rightarrow 0^+} (1/h)d(\phi(0) - v^-(0) + h[F(t, \phi) - F^-(t, v_t^-)]); X_+ = 0, \\ a \leq t \leq b, (t, \phi) \in \mathcal{D}.$$

Now we can state and prove the following general comparison principle:

Theorem 5.1. *Suppose that (T1)–(T4), (S1), (S2), (F), and (C1)–(C7) are satisfied. If $v_a^- \leq_B \phi \leq_B v_a^+$, then the abstract semilinear integral equation (4.1) has a solution u on $[a, \bar{b})$ for some $\bar{b} \in (a, b]$ such that*

$$v^-(t) \leq_X u(t) \leq_X v^+(t) \quad \text{and} \quad v_t^- \leq_B u_t \leq_B v_t^+$$

for $t \in [a, \bar{b})$.

Proof. It has been shown in Martin and Smith [36] that (D3) is satisfied with $D(t) = [v^-(t), v^+(t)]_X$ on $[a, b]$. By using assumptions (C2), (C4), (S2) and (T3), we can show that if $t \in [a, b]$, $\phi \in \mathcal{D}(t)$ and $h > 0$ is sufficiently small, then

$$\begin{aligned} & d\left(v^+(t+h) - S(t+h, t)\phi(0) - \int_t^{t+h} T(t+h, \theta)F(t, \phi) d\theta; X_+\right) \\ & \leq d\left(S^+(t+h, t)v^+(t) - S(t+h, t)\phi(0) \right. \\ & \quad \left. + \int_t^{t+h} T(t+h, \theta)[F^+(\theta, v_\theta^+) - F(t, \phi)] d\theta; X_+\right) \\ & \leq d\left(S(t+h, t)v^+(t) - S(t+h, t)\phi(0) \right. \\ & \quad \left. + \int_t^{t+h} T(t+h, \theta)[F^+(\theta, v_\theta^+) - F(t, \phi)] d\theta; X_+\right) \end{aligned}$$

$$\begin{aligned}
 &= d\left(T(t+h, t)[v^+(t) - \phi(0)] \right. \\
 &\quad \left. + \int_t^{t+h} T(t+h, \theta)[F^+(\theta, v_\theta^+) - F(t, \phi)]d\theta; X_+\right) \\
 &\leq d(T(t+h, t)([v^+(t) - \phi(0)] + h[F^+(t, v_t^+) - F(t, \phi)]); X_+) \\
 &\quad + \left| \int_t^{t+h} T(t+h, \theta)[F^+(\theta, v_\theta^+) - F(t, \phi)] d\theta \right. \\
 &\quad \left. - hT(t+h, t)[F^+(t, v_t^+) - F(t, \phi)] \right|_X \\
 &\leq \widehat{M}e^{\omega h}d(v^+(t) - \phi(0) + h[F^+(t, v_t^+) - F(t, \phi)]); X_+) \\
 &\quad + \left| \int_t^{t+h} T(t+h, \theta)[F^+(\theta, v_\theta^+) - F(t, \phi)] d\theta \right. \\
 &\quad \left. - hT(t+h, t)[F^+(t, v_t^+) - F(t, \phi)] \right|_X.
 \end{aligned}$$

Therefore, by assumption (C6), we get

$$\begin{aligned}
 \lim_{h \rightarrow 0^+} \frac{1}{h} d\left(v^+(t+h) - S(t+h, t)\phi(0) \right. \\
 \left. - \int_t^{t+h} T(t+h, r)F(t, \phi)dr; X_+\right) = 0.
 \end{aligned}$$

That is,

$$\begin{aligned}
 \lim_{h \rightarrow 0^+} \frac{1}{h} d\left(S(t+h, t)\phi(0) \right. \\
 \left. + \int_t^{t+h} T(t+h, \theta)F(t, \phi)d\theta; (-\infty, v^+(t+h)]_X\right) = 0.
 \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
 \lim_{h \rightarrow 0^+} \frac{1}{h} d\left(S(t+h, t)\phi(0) \right. \\
 \left. + \int_t^{t+h} T(t+h, r)F(t, \phi)dr; [v^-(t+h), \infty)_X\right) = 0.
 \end{aligned}$$

These imply that

$$\lim_{h \rightarrow 0^+} \frac{1}{h} d \left(S(t+h, t) \phi(0) + \int_t^{t+h} T(t+h, r) F(t, \phi) dr; [v^-(t+h), v^+(t+h)]_X \right) = 0,$$

which completes the proof, by Theorem 4.1. \square

As an immediate consequence of the general comparison principle, we obtain the following monotonicity principle:

Theorem 5.2. *Suppose that (T1)–(T4), (S1), (S2), (F) and (C3)–(C5) are satisfied with $S^+ = S^- = S$ and $F^+ = F^- = F$. Moreover, suppose that F satisfies the following quasimonotonicity condition*

(QM) $\lim_{h \rightarrow 0^+} (1/h) d(\psi(0) - \phi(0) + h[F(t, \psi) - F(t, \phi)]; X_+) = 0$ for all $(t, \phi), (t, \psi) \in \mathcal{D}$ with $\phi \leq \psi$.

Then for each $\phi \in \mathcal{D}(a)$ with $v_a^- \leq_B \phi \leq_B v_a^+$, equation (4.1) has a unique solution $u(t; a, \phi)$ on $[a, \bar{b}]$, where $a < \bar{b} = \bar{b}(\phi)$. Furthermore, if $v_a^- \leq_B \phi \leq_B \psi \leq_B v_a^+$, then

$$v^-(t) \leq_X u(t; a, \phi) \leq_X u(t; a, \psi) \leq_X v^+(t)$$

and

$$v_t^- \leq_B u_t(a, \phi) \leq_B u_t(a, \psi) \leq_B v_t^+$$

for all $t \in [a, \hat{b}]$, where $\hat{b} = \min\{\bar{b}(\phi), \bar{b}(\psi)\}$.

Throughout the remainder of this section, in order to establish some strict inequalities leading to the so-called quasi strongly order-preserving property, we assume that the Banach lattice X is a product space $X = \prod_{i=1}^m X_i$, where m is a positive integer, X_i is a Banach space with a cone $X_i^+ \subseteq X_i$ such that $X_+ = \prod_{i=1}^m X_i^+$. We shall use \geq_{X_i} and \geq_X to denote the partial order on X_i and X induced by X_i^+ and X_+ , respectively. Therefore, if $x = (x_i)^m, y = (y_i)^m \in X$, then $x \geq_X y$ if and only if $x_i \geq_{X_i} y_i$ for $i \in \{1, 2, \dots, m\}$.

We first assume that (T1)–(T3), (S1) and (S2) are satisfied by T and S which are defined as follows:

(CT1) $T(t, s)x = (T_i(t, s)x_i)_1^m$, where $T_i(t, s) : X_i \rightarrow X_i$ for all $t \geq s \geq a$ and $x = (x_i)_1^m \in X$;

(CT2) $S(t, s)x = (S_i(t, s)x_i)_1^m$, where $S_i(t, s) : X_i \rightarrow X_i$ for all $t \geq s \geq a$ and $x = (x_i)_1^m \in X$.

Moreover, we assume that (C1)–(C7) and (F) are satisfied with $v^\pm = (v_i^\pm)_1^m$, $F = (F_i)_1^m$, $F^\pm = (F_i^\pm)_1^m$, where v_i^\pm maps $(-\infty, b)$ into X_i , F_i^\pm maps \mathcal{D} into X_i , and F_i maps a neighborhood of \mathcal{D} into X_i .

Under these assumptions, system (4.1) can be reformulated as

$$(5.1) \quad \begin{aligned} u_i(t) &= S_i(t, a)\phi_i(0) + \int_a^t T_i(t, r)F_i(r, u_r)dr, & t \geq a \\ u_i(a + \theta) &= \hat{\phi}_i(\theta), & \theta \leq 0, \end{aligned}$$

where $\hat{\phi}_i = (\hat{\phi}_i)_1^m$ and $u = (u_i)_1^m$ is the solution of (4.1). From Theorem 5.1, if $v_a^- \leq_B \phi \leq_B v_a^+$, then $v_i^-(t) \leq_{X_i} u_i(t) \leq_{X_i} v_i^+(t)$ and $v_i^- \leq_B u_i \leq_B v_i^+$ for $t \in [a, \bar{b})$ and $i = 1, 2, \dots, m$.

The following one side Lipschitz condition will be useful in establishing strict inequalities:

(L1) for each $l > 0$ there exists $L_l > 0$ such that

$$F_i(t, \phi) - F_i^-(t, v_i^-) \geq -L_l[\phi_i(0) - v_i^-(t)]$$

for all $i \in \{1, 2, \dots, m\}$ and $(t, \phi) \in [a, a + l] \times B$ with $v_i^- \leq_B \phi \leq_B v_i^+$ and $|\phi|_B \leq l$.

Under this assumption, it can be shown (by using the same argument as that for Lemma 3.1 in Martin and Smith [36]) that, if $u = (u_i)_1^m$ is the solution of (4.1), $b \in (a, \bar{b})$ with $b - a \leq l$ and $|u_t|_B \leq l$ for all $t \in [a, b]$, then

$$(5.2) \quad u_i(t) - v_i^-(t) \geq_{X_i} e^{-L_l(t-t_0)}T_i(t, t_0)[u_i(t_0) - v_i^-(t_0)]$$

for all $a \leq t_0 \leq t \leq b$ and $i = 1, 2, \dots, m$.

For each $i \in \{1, 2, \dots, m\}$, let X_i^* be the dual space of X_i and

$$P_i^* = \{\Phi_i \in X_i^*; \Phi_i(x_i) \geq 0 \text{ for all } x_i \in X_i^+\}.$$

Following Martin and Smith [36], to achieve generality we introduce a nonempty index set Λ_i and assume that for each $\rho \in \Lambda_i$ there is a given $\Phi_i^\rho \in P_i^*$. Let

$$\Lambda_i^* = \{\Phi_i^\rho; \rho \in \Lambda_i\}.$$

The inequality (5.2) motivates the following abstract “maximum principle”:

(MP) if $i \in \{1, 2, \dots, m\}$ and $x_i \in X_i^+$, then $\Phi_i^\sigma(x_i) > 0$ for some $\sigma \in \Lambda_i$ implies that $\Phi_i^\rho(T_i(t, t_0)x_i) > 0$ for all $t > t_0 \geq a$ and all $\rho \in \Lambda_i$.

From inequality (5.2) it easily follows that assumptions (L1) and (MP) imply that

(5.3) if $\Phi_k^\sigma(u_k(t_0)) > \Phi_k^\sigma(v_k^-(t_0))$ for some $t_0 \in [a, b)$ and $\sigma \in \Lambda_k$, then $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $t \in (t_0, b]$ and $\rho \in \Lambda_k$.

In order to obtain strict inequalities for other components, we need the following “irreducibility” condition:

(I1) there exists a constant $\tau_1 \geq 0$ such that if t_1 and t_2 are given constants such that $a \leq t_1 < t_1 + \tau_1 < t_2$, Σ is a proper nonempty subset of $\{1, 2, \dots, m\}$ and $w = (w_j)_1^m : (-\infty, t_2] \rightarrow B$ is given such that $w : [t_1, t_2] \rightarrow B$ is continuous, $w_{t_1} \in B$ and

(a) $v_{t_1}^- \leq_B w_{t_1} \leq_B v_{t_1}^+$ and $v^-(t) \leq_X w(t) \leq_X v^+(t)$ for all $t \in [t_1, t_2]$;

(b) $\Phi_j^\rho(w_j(t)) = \Phi_j^\rho(v_j^-(t))$ for all $j \in \Sigma^c$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

(c) $\Phi_j^\rho(w_j(t)) > \Phi_j^\rho(v_j^-(t))$ for all $j \in \Sigma$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

then there exist a $k \in \Sigma^c$ and a $\sigma \in \Lambda_k$ such that

$$\sup\{\Phi_k^\sigma(F_k(t, w_t)) - \Phi_k^\sigma(F_k^-(t, v_t^-)); t_1 \leq t \leq s\} > 0$$

for all $t_1 + \tau < s \leq t_2$.

We are now in the position to state our first result on strict inequalities relative to $v^-(t)$. Similar results hold for $v^+(t)$.

Theorem 5.3. *Suppose that, in addition to all conditions of Theorem 5.1, (MP), (L1) and (I1) are satisfied. Consider the solution u of (4.1) defined on $[a, \bar{b}]$. If there exists a $t_1 \geq a$ such that $t_1 + (m-1)\tau_1 < \bar{b}$ and $\Phi_j^\sigma(u_j(t_1)) > \Phi_j^\sigma(v_j^-(t_1))$ for some $j \in \{1, 2, \dots, m\}$ and some $\sigma \in \Lambda_j$, then there exists a $t_m \in [t_1, t_1 + (m-1)\tau_1]$ such that*

$$\Phi_i^\rho(u_i(t)) > \Phi_i^\rho(v_i^-(t))$$

for all $t \in (t_m, \bar{b})$, $i \in \{1, 2, \dots, m\}$ and $\rho \in \Lambda_i$.

Proof. Let $\Sigma = \{j\}$. If $m = 1$, then by (5.3) we are done. If $m > 1$, then Σ is a proper and nonempty subset of $\{1, 2, \dots, m\}$. We now fix $\varepsilon \in (0, \bar{b} - t_1 - \tau_1)$ and claim that there exists $k \in \Sigma^c$, $\sigma \in \Lambda_k$ and some $t_2 \in [t_1, t_1 + \tau_1 + \varepsilon]$ such that $\Phi_k^\sigma(u_k(t_2)) > \Phi_k^\sigma(v_k^-(t_2))$. By the way of contradiction, if the claim is false, then $\Phi_k^\rho(u_k(t)) = \Phi_k^\rho(v_k^-(t))$ for all $k \in \Sigma^c$, $\rho \in \Lambda_k$ and $t \in [t_1, t_1 + \tau_1 + \varepsilon]$. By (I1), we can find $k \in \Sigma^c$, $\sigma \in \Lambda_k$ and a sequence $\{\varepsilon_j\}_1^\infty$ in $(0, \varepsilon)$ such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ and

$$\begin{aligned} &\Phi_k^\sigma(F_k(t_1 + \tau_1 + \varepsilon_j, u_{t_1 + \tau_1 + \varepsilon_j})) \\ &\quad - \Phi_k^\sigma(F_k^-(t_1 + \tau_1 + \varepsilon_j, v_{t_1 + \tau_1 + \varepsilon_j}^-)) > 0. \end{aligned}$$

By continuity, one can find $\bar{t}_j \in [t_1 + \tau_1, t_1 + \tau_1 + \varepsilon_j]$ such that

$$\Phi_k^\sigma(T_k(t_1 + \tau_1 + \varepsilon_j, \theta)[F_k(\theta, u_\theta) - F_k^-(\theta, u_\theta)]) > 0$$

for $\bar{t}_j \leq \theta \leq t_1 + \tau_1 + \varepsilon_j$. Therefore

$$\begin{aligned} &\Phi_k^\sigma(u_k(t_1 + \tau_1 + \varepsilon_j) - v_k^-(t_1 + \tau_1 + \varepsilon_j)) \\ &\geq \Phi_k^\sigma(S(t_1 + \tau_1 + \varepsilon_j, \bar{t}_j)[u_k(\bar{t}_j) - v_k^-(\bar{t}_j)]) \\ &\quad + \int_{\bar{t}_j}^{\bar{t}_j + \tau_1 + \varepsilon_j} \Phi_k^\sigma(T(t_1 + \tau_1 + \varepsilon_j, t\theta)[F_k(\theta, u_\theta) - F_k^-(\theta, u_\theta)]) d\theta \\ &> 0, \end{aligned}$$

a contradiction to the assumption that $\Phi_k^\sigma(u_k(t)) = \Phi_k^\sigma(v_k^-(t))$ for all $t \in [t_1, t_1 + \tau_1 + \varepsilon]$.

Therefore, there exist $k \in \Sigma^c$, $\sigma \in \Lambda_k$ and some $t_2 \in [t_1, t_1 + \tau_1 + \varepsilon]$ such that $\Phi_k^\sigma(u_k(t_2)) > \Phi_k^\sigma(v_k^-(t_2))$. By (5.3), we have $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $t > t_2$ and thus for $t > t_1 + \tau_1 + \varepsilon$. Because of the arbitrary choice of ε , we have $\Phi_k^\rho(u_k(t)) > \Phi_k^\rho(v_k^-(t))$ for all $\rho \in \Lambda_k$ and $t > t_2$, where t_2 is some number in $[t_1, t_1 + \tau_1]$.

If $m = 2$, then we are done. If $m > 2$, by repeating the above argument m times, we will arrive at the conclusion. \square

In order to obtain strict inequalities between compatible solutions of (4.1), we assume

(L2) for each $l > 0$, there is an $L_l > 0$ such that

$$F_i(t, \psi) - F_i(t, \phi) \geq -L_l[\psi_i(0) - \phi_i(0)]$$

for all $i = 1, 2, \dots, m$ and $(t, \phi), (t, \psi) \in [a, a + l] \times B$ with $v_i^- \leq_B \phi \leq_B \psi \leq_B v_i^+$ and $|\phi|_B, |\psi|_B \leq l$;

(I2) there exists a constant $\tau_1 > 0$ such that if t_1 and t_2 are given constants with $a \leq t_1 < t_1 + \tau_1 < t_2$, Σ is a nonempty and proper subset of $\{1, 2, \dots, m\}$ and $w^\pm = (w_j^\pm)_1^m : (-\infty, t_2] \rightarrow X$ are given such that $w^\pm : [t_1, t_2] \rightarrow X$ are continuous, $w_{t_1}^\pm \in B$ and

(a) $v_{t_1}^- \leq_B w_{t_1}^- \leq_B w_{t_1}^+ \leq_B v_{t_1}^+$ and $v^-(t) \leq_X w^-(t) \leq_X w^+(t) \leq_X v^+(t)$ for all $t \in [t_1, t_2]$;

(b) $\Phi_j^\rho(w_j^+(t)) = \Phi_j^\rho(w_j^-(t))$ for all $j \in \Sigma^c$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

(c) $\Phi_j^\rho(w_j^+(t)) > \Phi_j^\rho(w_j^-(t))$ for all $j \in \Sigma$, $\rho \in \Lambda_j$ and $t \in [t_1, t_2]$;

then there exist a $k \in \Sigma^c$ and a $\sigma \in \Lambda_k$ such that

$$\sup\{\Phi_k^\sigma(F_k(t, w_t^+)) - \Phi_k^\sigma(F_k(t, w_t^-)); t_1 \leq t \leq s\} > 0$$

for all $t_1 + \tau_1 < s \leq t_2$.

Theorem 5.4. *Suppose that, in addition to the assumptions for Theorem 5.2, (L2) and (I2) are satisfied. Then for $\psi^\pm \in B$ with $v_a^- \leq_B \psi^- <_B \psi^+ \leq_B v_a^+$, if u^\pm denotes the solutions of (4.1) on $[a, \bar{b}(\psi^\pm))$ with $\phi = \psi^\pm$, respectively, then*

$$\begin{aligned} v^-(t) &\leq_X u^-(t) \leq_X u^+(t) \leq_X v^+(t), \\ v_i^- &\leq_B u_i^- \leq_B u_i^+ \leq_B v_i^+, \quad a \leq t \leq \hat{b}, \end{aligned}$$

where $\hat{b} = \min\{\bar{b}(\psi^-), \bar{b}(\psi^+)\}$. Also, if there is a $t_1 \geq a$ such that $t_1 + (m - 1)\tau_1 < \hat{b}$ and $\Phi_j^\sigma(u_j^+(t_1)) > \Phi_j^\sigma(u_j^-(t_1))$ for some $j \in \{1, 2, \dots, m\}$ and some $\sigma \in \Lambda_j$, then there is a $t_m \in [t_1, t_1 + (m - 1)\tau_1]$ such that

$$\Phi_i^\rho(u_i^+(t)) > \Phi_i^\rho(u_i^-(t))$$

for all $t \in (t_m, \hat{b})$, $i \in \{1, 2, \dots, m\}$ and $\rho \in \Lambda_i$.

Proof. This is an immediate consequence of Theorems 5.1 and 5.3.

□

6. Applications to reaction-diffusion equations with distributed delay. Suppose Ω is a bounded region in R^n with $\partial\Omega$ smooth, $C(\bar{\Omega}; R^m)$ is the Banach space of continuous functions from $\bar{\Omega}$ to R^m with the supremum norm, Δ is the Laplacian operator on Ω and $\partial/\partial n$ is the outward normal derivative on $\partial\Omega$. Let $X = C(\bar{\Omega}; R^m)$ and B be a phase space satisfying (A1)–(A5) with $X_+ = C(\bar{\Omega}; R_+^m)$ as specified in Section 3. Consider the following nonlinear reaction-diffusion system with infinite delay

$$\begin{aligned}
 & \frac{\partial}{\partial t} u^i(t, x) = F_i(t, u_t)(x), \quad t > a, x \in \Omega, i \in \Sigma_0 \\
 & u_a^i(\cdot, x) = \hat{\phi}^i(\cdot, x), \quad x \in \bar{\Omega}, i \in \Sigma_0 \\
 (6.1) \quad & \frac{\partial}{\partial t} u^i(t, x) = d_i \Delta u^i(t, x) + F_i(t, u_t)(x), \quad t > a, x \in \Omega, i \in \Sigma_0^c \\
 & \alpha_i(x) u^i(t, x) + \frac{\partial}{\partial n} u^i(t, x) = \beta_i(t, x), \quad t > a, x \in \partial\Omega, i \in \Sigma_0^c \\
 & u_a^i(\cdot, x) = \hat{\phi}^i(\cdot, x), \quad x \in \bar{\Omega}, i \in \Sigma_0^c,
 \end{aligned}$$

where

(H1) Σ_0 is a given subset of $\{1, 2, \dots, m\}$ and $d_i > 0$, $\alpha_i \in C^1(\bar{\Omega}; R_+)$ and $\beta_i \in C^2(R_+ \times \bar{\Omega}; R)$ for $i \in \Sigma_0^c$;

(H2) $a \geq 0$ is a constant, $\phi = (\phi_i)_1^m \in B$ is a given initial function and $\hat{\phi}$ is a representing element of ϕ ;

(H3) $F = (F_i)_1^m : R_+ \times B \rightarrow C(\bar{\Omega}; R^m)$ is continuous and for each $l > 0$, there exists $L_{1,l} > 0$ and $L_{2,l} : R_+ \rightarrow R_+$ with $L_{2,l}(0) = 0$ such that

$$|F(t, \phi) - F(s, \psi)|_{C(\bar{\Omega}; R^m)} \leq L_{1,l} |\phi - \psi|_B + L_{2,l} (|t - s|)$$

for all $0 \leq t, s \leq l$ and $\phi, \psi \in B$ with $|\phi|_B, |\psi|_B \leq l$.

For $i \in \Sigma_0^c$, let $\mathcal{A}^i : \text{dom}(\mathcal{A}^i) \subseteq C(\bar{\Omega}; R) \rightarrow C(\bar{\Omega}; R)$ be the linear operator defined by

$$\begin{aligned}
 \text{dom}(\mathcal{A}^i) &= \left(u^i \in C^2(\Omega; R) \cap C^1(\bar{\Omega}; R); \alpha_i u^i + \frac{\partial}{\partial n} u^i = 0 \text{ on } \partial\Omega \right), \\
 \mathcal{A}^i u^i &= d_i \Delta u^i, \quad u^i \in \text{dom}(\mathcal{A}^i).
 \end{aligned}$$

Then the closure of \mathcal{A}^i generates an analytic, nonexpansive and positive semigroup $\{T_i(t)\}_{t \geq 0}$ on $C(\bar{\Omega}; R)$ (see, cf. Mora [46] and Rothe [59]). Assume that $\gamma^i : [0, \infty) \times \bar{\Omega} \rightarrow R$ is a smooth mapping satisfying

$$\alpha_i(x)\gamma^i(t, x) + \frac{\partial}{\partial n}\gamma^i(t, x) = \beta_i(t, x) \quad \text{on } (0, \infty) \times \partial\Omega.$$

Define

$$S_i(t, s)w_0^i = T_i(t-s)[w_0^i - \hat{\mu}_i(s)] + \hat{\mu}_i(t), \quad t \geq s \geq 0, w_0^i \in C(\bar{\Omega}; R),$$

where

$$\hat{\mu}_i(t) = \gamma^i(t) + \int_0^t T_i(t-s) \left[d_i \Delta \gamma^i(s) - \frac{\partial}{\partial t} \gamma^i(s) \right] ds.$$

For $i \in \Sigma_0$ and $v_0^i, w_0^i \in C(\bar{\Omega}; R)$, define

$$\begin{aligned} T_i(t)v_0^i &= v_0^i, & t \geq 0, \\ S_i(t, s)w_0^i &= w_0^i, & t \geq s \geq 0. \end{aligned}$$

Let

$$(6.2) \quad v^i(t; \cdot, v_0^i) = T_i(t)v_0^i, \quad v_0^i \in C(\bar{\Omega}; R), t \geq 0,$$

and

$$(6.3) \quad w^i(t, s, \cdot, w_0^i) = S_i(t, s)w_0^i, \quad w_0^i \in C(\bar{\Omega}; R), t > s \geq 0.$$

Then $v^i(t; x, v_0^i)$, $i \in \Sigma_0^c$, is a solution of the following linear system

$$(6.4) \quad \begin{aligned} \frac{\partial}{\partial t} v^i(t, x) &= d_i \Delta v^i(t, x), & t > 0, x \in \Omega, i \in \Sigma_0^c \\ \alpha_i(x)v^i(t, x) + \frac{\partial}{\partial n} v^i(t, x) &= 0, & t > 0, x \in \partial\Omega, i \in \Sigma_0^c \\ v^i(0, x) &= v_0^i(x), & x \in \bar{\Omega}, i \in \Sigma_0^c, \end{aligned}$$

and $w^i(t, s, x, w_0^i)$, $i \in \Sigma_0^c$, is a solution of the following nonhomogeneous system

$$(6.5) \quad \begin{aligned} \frac{\partial}{\partial t} w^i(t, x) &= d_i \Delta w^i(t, x), & t > s \geq 0, x \in \Omega, i \in \Sigma_0^c \\ \alpha_i(x)w^i(t, x) + \frac{\partial}{\partial n} w^i(t, x) &= \beta_i(t, x), & t > s \geq 0, x \in \partial\Omega, i \in \Sigma_0^c \\ w^i(s, x) &= w^i(x), & s \geq 0, x \in \bar{\Omega}, i \in \Sigma_0^c. \end{aligned}$$

It is known that (T1)–(T4) and (S1)–(S2) are satisfied (see, cf. Martin and Smith [36, 37]). We will call a solution of the abstract integral equation

$$(6.6) \quad u(t) = S(t, a)\phi(0) + \int_a^t T(t - \theta)F(\theta, u_\theta)d\theta, \quad t \geq a$$

$$u_a \in \phi, \quad (a, \phi) \in R_+ \times B$$

a *mild solution* of (6.1). Hence by Theorem 4.1, for given closed subsets $D \subset [a, \infty) \times X$ and $\mathcal{D} \subset [a, \infty) \times B$ satisfying (D1)–(D3), we have the following result about the existence and uniqueness of a solution of the general nonlinear reaction-diffusion system (6.1).

Theorem 6.1 (Existence). *Suppose that the operators $T = \{T(t)\}_{t \geq 0}$ and $S = \{S(t, s)\}_{t \geq s \geq 0}$ defined above satisfy (SC). Then system (6.1) has a unique noncontinuable mild solution u , denoted by $u(t; a, \phi)$, on an interval of the form $[a, b)$, where $a < b \leq \infty$. Moreover, $u(t) \in D(t)$ and $u_t \in \mathcal{D}(t)$ for $t \in [a, b)$ and if $b < \infty$, then*

$$\limsup_{t \rightarrow b^-} |u_t|_B = \infty.$$

We now show the existence of solutions of the nonlinear equation (6.1) relative to given upper and lower solutions. Suppose that $F^\pm = (F_i^\pm)_1^m : [a, \infty) \times B \rightarrow C(\bar{\Omega}; R^m)$ are continuous functions, $v^\pm = (v_i^\pm)_1^m : (-\infty, c) \rightarrow C(\bar{\Omega}; R^m)$ are given mappings, $a < c \leq \infty$, such that

(UL1) $v_a^\pm \in B$ with $v_a^- \leq_B v_a^+$ and $v^\pm(t) \in X$ with $v^-(t) \leq_{C(\bar{\Omega}; R^m)} v^+(t)$ for $a \leq t < c$;

(UL2) if $i \in \Sigma_0^c$, $(\partial/\partial t)v_i^\pm, (\partial^2/\partial x^2)v_i^\pm : (a, c) \rightarrow C(\Omega; R)$ are continuous;

(UL3) if $i \in \Sigma_0$, $(\partial/\partial t)v_i^\pm : [a, c) \rightarrow C(\bar{\Omega}; R)$ is continuous;

(UL4) there exist $\beta_i^\pm \in C^2(R_+ \times \bar{\Omega}; R)$ for $i \in \Sigma_0^c$ such that

$$\frac{\partial}{\partial t} v_i^+(t, x) \geq d_i \Delta v_i^+(t, x) + F_i^+(t, v_i^+)(x),$$

$$a < t < c, \quad x \in \Omega, \quad i \in \Sigma_0^c$$

$$\begin{aligned} \frac{\partial}{\partial t} v_i^+(t, x) &\geq F_i^+(t, v_i^+)(x), \\ a \leq t < c, x \in \bar{\Omega}, i \in \Sigma_0 \\ \alpha_i(x)v_i^+(t, x) + \frac{\partial}{\partial n} v_i^+(t, x) &= \beta_i^+(t, x) \geq \beta_i(t, x), \\ a < t < c, x \in \partial\Omega, i \in \Sigma_0^c \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} v_i^-(t, x) &\leq d_i \Delta v_i^-(t, x) + F_i^-(t, v_i^-)(x), \\ a < t < c, x \in \Omega, i \in \Sigma_0^c \\ \frac{\partial}{\partial t} v_i^-(t, x) &\leq F_i^-(t, v_i^-)(x), \\ a \leq t < c, x \in \bar{\Omega}, i \in \Sigma_0 \\ \alpha_i(x)v_i^-(t, x) + \frac{\partial}{\partial n} v_i^-(t, x) &= \beta_i^-(t, x) \leq \beta_i(t, x), \\ a < t < c, x \in \partial\Omega, i \in \Sigma_0^c. \end{aligned}$$

We have the following result about the existence of a solution for equation (6.1) relative to an upper solution (v^+) and a lower solution (v^-).

Theorem 6.2. *Suppose that conditions (UL1)–(UL4) are satisfied and*

(FC) *for any $(t, \phi) \in [a, \infty) \times B$ with $v_t^- \leq_B \phi \leq_B v_t^+$, we have*

(a) *if $\phi_k(0)(x) = v_k^+(t)(x)$ for some $k \in \{1, 2, \dots, m\}$ and $x \in \bar{\Omega}$, then $F_k(t, \phi)(x) \leq F_k^+(t, v_t^+)(x)$;*

(b) *if $\phi_k(0)(x) = v_k^-(t)(x)$ for some $k \in \{1, 2, \dots, m\}$ and $x \in \bar{\Omega}$, then $F_k(t, \phi)(x) \geq F_k^-(t, v_t^-)(x)$.*

Then the nonlinear system (6.1) has a unique noncontinuable mild solution u on $[a, b)$, where $b \geq c$, and this solution satisfies

$$v^-(t) \leq_{C(\bar{\Omega}; R^m)} u(t) \leq_{C(\bar{\Omega}; R^m)} v^+(t) \quad \text{and} \quad v_t^- \leq_B u_t \leq_B v_t^+$$

for all $t \in [a, c)$.

Proof. As $\{T(t)\}_{t \geq 0}$ is a positive semigroup on $C(\bar{\Omega}; R^m)$, (T4) is satisfied. Moreover, (UL4) implies (C4) and (C5), and (FC) implies (C6) and (C7). Therefore, the theorem follows from Theorem 5.1 immediately. \square

One side inequality in Theorem 6.2 holds when the corresponding condition (a) or (b) of (FC) is satisfied. In particular, we have the following nonnegative property of solutions of equation (6.1).

Theorem 6.3 (Nonnegativeness). *Suppose that*

(i) $\beta_i \geq 0$ on $\partial\Omega \times R_+$ for all $i \in \Sigma_0^c$;

(ii) F is quasipositive in the sense that if $k \in \{1, 2, \dots, m\}$ and $(t, \phi) \in R_+ \times B_+$, then $\phi_k(0)(x) = 0$ at some $x \in \bar{\Omega}$ implies that $F(t, \phi)(x) \geq 0$.

Then for each $(t, \phi) \in [a, \infty) \times B_+$, system (6.1) has a unique non-continuable mild solution $u : (-\infty, b) \rightarrow C(\bar{\Omega}; R_+^m)$, $b > a$, such that $u(t) \in C(\bar{\Omega}; R_+^m)$ and $u_t \in B_+$ for all $t \in [a, b)$.

To state a criterion for invariant rectangles, we will require that the phase space B satisfies (A6) in Section 3. Recall that we denote by \bar{w} the constant mapping from $(-\infty, 0]$ into X with the constant value $w \in X$.

Theorem 6.4 (Invariant Rectangle Criterion). *Suppose that $M = (M_i)_i^m$, $N = (N_i)_i^m \in R^m$ with $-\infty \leq M_i < N_i \leq \infty$ for $i \in \{1, 2, \dots, m\}$ and $\bar{M} \leq_B \bar{N}$ are given so that $\alpha_i(x)M_i \leq \beta_i(t, x) \leq \alpha_i(x)N_i$ for $i \in \Sigma_0^c$, $x \in \bar{\Omega}$ and that for any $(t, \phi) \in [a, \infty) \times B$ with $\bar{M} \leq_B \phi \leq_B \bar{N}$, we have*

(a) if $\phi_k(0)(x) = M_k$ for some $k \in \{1, 2, \dots, m\}$, $x \in \bar{\Omega}$, then $F_k(t, \phi)(x) \geq 0$;

(b) if $\phi_k(0)(x) = N_k$ for some $k \in \{1, 2, \dots, m\}$, $x \in \bar{\Omega}$, then $F_k(t, \phi)(x) \leq 0$.

Then for each $(t, \phi) \in [a, \infty) \times B$ such that $\bar{M} \leq_B \phi \leq_B \bar{N}$, system (6.1) has a unique noncontinuable mild solution $u : (-\infty, b) \rightarrow C(\bar{\Omega}; R_+^m)$, $b > a$, such that $\bar{M} \leq_B u_t \leq_B \bar{N}$ for $t \in [a, b)$.

Let us assume that all conditions of Theorem 6.4 are satisfied. We want to construct comparing systems for (6.1) relative to the rectangle $[\overline{M}, \overline{N}]_B$. For $\phi \in [\overline{M}, \overline{N}]_B$, $x \in \overline{\Omega}$ and $i \in \{1, 2, \dots, m\}$, define

$$h_i(t, \phi)(x) = \inf\{F_i(t, \psi)(x); \phi \leq_B \psi \leq_B \overline{N}, \phi_i(0)(x) = \psi_i(0)(x)\},$$

$$H_i(t, \phi)(x) = \sup\{F_i(t, \psi)(x); \overline{M} \leq_B \psi \leq_B \phi, \phi_i(0)(x) = \psi_i(0)(x)\}.$$

We assume that $h, H : R_+ \times [\overline{M}, \overline{N}]_B \rightarrow C(\overline{\Omega}; R^m)$ are continuous, can be extended to a neighborhood of $R_+ \times [\overline{M}, \overline{N}]_B$, and satisfy (H3) with F replaced by h and H , respectively and $R_+ \times B$ replaced by $R_+ \times [\overline{M}, \overline{N}]_B$. Consider the following comparing systems

(6.7)

$$\frac{\partial}{\partial t} v^i(t, x) = h_i(t, v_t)(x), \quad t > a, x \in \overline{\Omega}, i \in \Sigma_0$$

$$\frac{\partial}{\partial t} v^i(t, x) = d_i \Delta v^i(t, x) + h_i(t, v_t)(x), \quad t > a, x \in \Omega, i \in \Sigma_0^c$$

$$\alpha_i(x) v^i(t, x) + \frac{\partial}{\partial n} v^i(t, x) = \beta_i(t, x), \quad t > a, x \in \partial\Omega, i \in \Sigma_0^c$$

$$v_a^i(s, x) = \phi_i^-(s, x), \quad i \in \{1, 2, \dots, m\}, -\infty < s \leq 0, x \in \overline{\Omega}$$

and

(6.8)

$$\frac{\partial}{\partial t} w^i(t, x) = H_i(t, w_t)(x), \quad t > a, x \in \overline{\Omega}, i \in \Sigma_0$$

$$\frac{\partial}{\partial t} w^i(t, x) = d_i \Delta w^i(t, x) + H_i(t, w_t)(x), \quad t > a, x \in \Omega, i \in \Sigma_0^c$$

$$\alpha_i(x) w^i(t, x) + \frac{\partial}{\partial n} w^i(t, x) = \beta_i(t, x), \quad t > a, x \in \partial\Omega, i \in \Sigma_0^c$$

$$w_a^i(s, x) = \phi_i^+(s, x), \quad i \in \{1, 2, \dots, m\}, -\infty < s \leq 0, x \in \overline{\Omega}.$$

By Theorem 6.2, we have the following comparison theorem

Theorem 6.5 (Comparison Principle). *Assume that all conditions of Theorem 6.4 are satisfied. If $\overline{M} \leq_B \phi^- \leq_B \phi \leq_B \phi^+ \leq_B \overline{N}$, then $\overline{M} \leq_B v_t \leq_B u_t(a, \phi) \leq_B w_t \leq_B \overline{N}$ for all $t \in [a, b]$ provided u, v and w are defined on $[a, b]$.*

We now discuss the monotonicity of solutions of system (6.1). By Theorem 6.2, we have

Theorem 6.6 (Monotonicity Principle). *Suppose that all conditions of Theorem 6.2 hold and $F : R_+ \times B \rightarrow C(\bar{\Omega}; R^m)$ satisfies the following quasi-monotonicity property*

(QMP) *if $k \in \{1, 2, \dots, m\}$ and $(t, \phi), (t, \psi) \in R_+ \times B$ are given such that $\phi \leq_B \psi$ and $\phi_k(0)(x) = \psi_k(0)(x)$ at some $x \in \bar{\Omega}$, then $F_k(t, \phi)(x) \leq F_k(t, \psi)(x)$.*

Then for any $(a, \phi), (a, \psi) \in R_+ \times B$ with $\phi \leq_B \psi$, we have

$$u(t; a, \phi) \leq_{C(\bar{\Omega}; R^m)} u(t; a, \psi) \quad \text{and} \quad u_i(a, \phi) \leq_B u_i(a, \psi)$$

for all $t \in [a, \min\{b(\phi), b(\psi)\}]$.

In order to obtain strict inequalities of solutions of (6.1), we suppose F satisfies the following one-side Lipschitz condition, ignition condition and irreducibility condition:

(LR) for each $l > 0$, there is $L_l > 0$ such that

$$F_i(t, \psi) - F_i(t, \phi) \geq_{C(\bar{\Omega}; R^m)} -L_l[\psi_i(0) - \phi_i(0)]$$

for all $i = 1, 2, \dots, m$ and $(t, \phi), (t, \psi) \in [a, a+l] \times B$ with $v_i^- \leq_B \phi \leq_B \psi \leq_B v_i^+$ and $|\phi|_B, |\psi|_B \leq l$;

(IG) there exists $\tau_0 > 0$ such that for any continuous functions $u, v : [a, a + \tau_0] \rightarrow C(\bar{\Omega}; R^m)$ with $u_a <_B v_a$ and $u(t) = v(t)$ for $t \in [a, a + \tau_0]$, there exists $k \in \{1, 2, \dots, m\}$ and $x \in \bar{\Omega}$ such that

$$\sup\{F_k(t, v_t)(x) - F_k(t, u_t)(x); a \leq t \leq a + \tau_0\} > 0;$$

(IR) there exists a constant $\tau_1 > 0$ such that if Σ is a proper, nonempty subset of $\{1, 2, \dots, m\}$, $\tau > a + \tau_1$ and $u, v : (-\infty, \tau] \rightarrow C(\bar{\Omega}; R^m)$ are given so that

- (a) $u_j(t)(x) < v_j(t)(x)$ for all $j \in \Sigma, x \in \bar{\Omega}$ and $t \in [\tau - \tau_1, \tau]$;
- (b) $u_j(t)(x) = v_j(t)(x)$ for all $j \in \Sigma^c, x \in \bar{\Omega}$ and $t \in [\tau - \tau_1, \tau]$;
- (c) $v_i^- \leq_B u_i \leq_B v_i \leq_B v_i^+$ for $t \in [a, \tau - \tau_1]$;

then there exists a $k \in \Sigma^c$ and $x \in \bar{\Omega}$ such that

$$\sup\{F_k(t, v_t)(x) - F_k(t, u_t)(x); \tau - \tau_1 \leq t \leq \tau\} > 0.$$

Applying Theorem 5.4 and the classical maximal principle to system (6.1) with $\Lambda_i = \bar{\Omega}$ and $\Phi_i^x(u_i) = u_i(x)$ for $x \in \bar{\Omega}$, $i \in \{1, 2, \dots, m\}$ and $u_i \in C(\bar{\Omega}; R)$, we get the following result.

Theorem 6.7 (Strict Inequality Principle). *Suppose that, in addition to the conditions of Theorem 6.6, (LR), (IG) and (IR) are satisfied. Assume that $\phi, \psi \in B$ are given so that $v_a^- \leq_B \phi <_B \psi \leq_B v_a^+$, and $u(t; a, \phi)$ and $u(t; a, \psi)$ are defined on $[a, b)$ with $b > a + \tau_0 + (m - 1)\tau_1$. Then*

$$u^i(t; a, \phi)(x) < u^i(t; a, \psi)(x)$$

for all $i \in \{1, 2, \dots, m\}$, $x \in \bar{\Omega}$ and $t \in (a + \tau_0 + (m - 1)\tau_1, b)$.

In the following, we suppose that $F(t, u_i)(x) \equiv F(u_i)(x)$, i.e., we consider the following autonomous reaction-diffusion system with infinite delay

(6.9)

$$\frac{\partial}{\partial t} u^i(t, x) = F_i(u_t)(x), \quad t > a, x \in \Omega, i \in \Sigma_0$$

$$u_a^i(\cdot, x) = \hat{\phi}^i(\cdot, x), \quad x \in \bar{\Omega}, i \in \Sigma_0$$

$$\frac{\partial}{\partial t} u^i(t, x) = d_i \Delta u^i(t, x) + F_i(u_t)(x),$$

$$t > a, x \in \Omega, i \in \Sigma_0^c$$

$$\alpha_i(x) u^i(t, x) + \frac{\partial}{\partial n} u^i(t, x) = \beta_i(t, x),$$

$$t > a, x \in \partial\Omega, i \in \Sigma_0^c$$

$$u_a^i(\cdot, x) = \hat{\phi}^i(\cdot, x), \quad x \in \bar{\Omega}, i \in \Sigma_0^c,$$

Clearly, all the above theorems hold for system (6.9) with necessary modifications. Let $u(t; a, \phi)$ be a solution of the autonomous system (6.9). In the following, we always assume that all solutions of system (6.9) can be extended to infinity. Let $\Phi : [0, \infty) \times B \rightarrow B$ be defined by $\Phi(t, \phi) = u_t(\phi) = u_t(0, \phi)$ for all $(t, \phi) \in [0, \infty) \times B$. Then Φ is a semiflow generated by (6.9). The above strict inequality principle (Theorem 6.7) implies the following quasi strongly order-preserving property for the semiflow Φ .

Theorem 6.8 (Quasi Strongly Order-Preserving Principle). *Assume that all the conditions of Theorem 6.7 hold for autonomous system (6.9) and assume axioms (A7) and (A9) are satisfied by the phase space B . Then the solution semiflow Φ is quasi strongly order-preserving.*

Proof. For any $\phi, \psi \in B$ with $\phi <_B \psi$, by Theorems 6.6 and 6.7, we have

$$u(t, \phi) \leq_{C(\bar{\Omega}; R^m)} u(t, \psi), \quad u_t(\phi) \leq_B u_t(\psi) \quad \text{for } t \geq 0$$

and

$$u(t, \phi) \ll_{C(\bar{\Omega}; R^m)} u(t, \psi) \quad \text{for all } t \geq \tau_0 + (m - 1)\tau_1,$$

provided both solutions are defined.

Let E_0 denote the space of all constant mappings from $(-\infty, 0]$ into $C(\bar{\Omega}; R^m)$. Clearly, E_0 can be identified with $C(\bar{\Omega}; R^m)$, and all equilibria of Φ belong to E_0 , the norm of E_0 is weaker than the induced topology from B , and by axiom (A7), for any $u, v \in E_0$, we have $u(t) \leq_{C(\bar{\Omega}; R^m)} v(t)$ if and only if $u_t \leq_B v_t$.

Suppose $v \in E_0$ and A is a given compact set invariant with respect to the semiflow and such that $v <_B A$.

Claim I. $v = v(t) \ll_{C(\bar{\Omega}; R^m)} \psi(t)$ for every $\psi \in A$ and $t \leq 0$.

If the claim is false, then there exist $t^* \leq 0, k \in \{1, 2, \dots, m\}, x \in \bar{\Omega}$ and $\psi \in A$ such that

$$v_k(x) = v_k(t^*)(x) = \psi_k(t^*)(x).$$

Let $\tau = 1 + \tau_0 + (m - 1)\tau_1 - t^*$. Since A is invariant, there exists $\psi^* \in A$ such that $\psi = u_\tau(\psi^*)$, so

$$u(\tau + t^*, \psi^*) = u_\tau(\psi^*)(t^*) = \psi(t^*).$$

On the other hand, since $v <_B \psi^*$, we have

$$\begin{aligned} v_k(y) &= u_k(\tau + t^*; v)(y) < u_k(\tau + t^*; \psi^*)(y) \\ &= \psi_k(t^*)(y) \quad \text{for all } y \in \bar{\Omega}, \end{aligned}$$

which contradicts the choice of k, t^* and $x \in \bar{\Omega}$ such that $v_k(x) = \psi_k(t^*)(x)$.

Claim II. *There exists $\delta > 0$ such that*

$$v(t) + \delta \hat{e} \leq_{C(\bar{\Omega}; R^m)} \psi(t)$$

for every $t \leq 0$ and $\psi \in A$, where $\hat{e} \in C(\bar{\Omega}; R^m)$ has the constant value $\hat{e} = (1, 1, \dots, 1)^T$.

In fact, if Claim II is not true, there must exist sequences $\{t_k\} \subset R_-$, $\{x_k\} \subseteq \bar{\Omega}$, $\{\psi^k\} \subseteq A$ and an integer $l \in \{1, 2, \dots, m\}$ such that

$$v_l(x_k) + \frac{1}{k} \geq \psi_l^k(t_k)(x_k)$$

for infinitely many k . Since $\psi_{t_k}^k \in A$ and A is compact, there exists a subsequence, also denoted by $\{\psi_{t_k}^k\}$, such that $\psi_{t_k}^k \rightarrow \psi \in A$. So $\psi^k(t_k) \rightarrow \psi(0)$ in $C(\bar{\Omega}; R^m)$. We can also assume, without loss of generality, that $x_k \rightarrow x \in \bar{\Omega}$. Hence, by taking $k \rightarrow \infty$ in the above inequality, we get $v_l(x) \geq \psi_l(0)(x)$, a contradiction to Claim I.

Let $v_0 = v + \delta \hat{e}/2$. Then $v_0 \in E_0$ and $v \ll_{C(\bar{\Omega}; R^m)} v_0$, and $v_0 \leq_B A$ by using (A9).

Now we can prove the quasi strongly order-preserving property of Φ . Suppose that $\{\psi^n\}$ is a sequence of equilibria of Φ such that

$$\lim_{n \rightarrow \infty} \psi^n = \psi <_B A \quad \text{and} \quad \psi <_B \psi^n \quad \text{for } n = 1, 2, \dots$$

By Claim II, there exists $\psi^0 \in E_0$ such that

$$\psi \ll_{C(\bar{\Omega}; R^m)} \psi^0, \quad \text{and} \quad \psi^0 \leq_B A.$$

On the other hand, since $\psi^n \rightarrow \psi$ in E_0 , there exists an integer n_0 such that $\psi^{n_0} \ll_{C(\bar{\Omega}; R^m)} \psi^0$. Hence, by (A7), we get

$$\psi <_B \psi^{n_0} \leq_B \psi^0 \leq_B A.$$

This completes the proof. \square

We now consider the set-condensing property of the semiflow $\Phi_t : B \rightarrow B$, $t > 0$.

Theorem 6.9 (Set-Condensing Principle). *Assume that the following fading memory condition is satisfied*

(FM) $M(t) < 1$ for all $t > 0$.

Assume further that E is a given subset of B such that

- (i) F maps bounded subsets of E into bounded subsets of X ;
- (ii) For all $t > 0$, Φ_t is defined and maps bounded subsets of E into bounded subsets of X .

Then Φ_t is set-condensing on E for $t > 0$ with respect to the Kuratowski measure of noncompactness.

Proof. Clearly, $\Phi_t(\phi) = P_1(t)\phi + P_2(t)\phi$ for $(t, \phi) \in R_+ \times E$, where

$$\begin{aligned} (P_1(t)\phi)(\theta) &= \begin{cases} S(t + \theta, 0)\phi(0), & -t \leq \theta \leq 0 \\ \phi(t + \theta), & \theta < -t, \end{cases} \\ (P_2(t)\phi)(\theta) &= \begin{cases} \int_0^{t+\theta} T(t + \theta - s)F(\Phi_s(\phi)) ds, & -t \leq \theta \leq 0 \\ 0, & \theta < -t. \end{cases} \end{aligned}$$

By using the same argument as that in Travis and Webb [69], we can show that $P_2(t) : E \rightarrow B$ is compact for $t > 0$. Moreover, using (A3) we can prove that if (FM) holds, then $P_1(t) : E \rightarrow B$ is a set-condensing mapping with respect to the Kuratowski measure of noncompactness for each $t > 0$. Therefore, the conclusion follows. \square

Note that if $B = UC_g$ with

$$(6.10) \quad \sup_{\theta \leq -t} \frac{g(t + \theta)}{g(\theta)} < 1, \quad t > 0,$$

then (FM) is satisfied.

We are now in the position to discuss some applications of Theorem 2.1, 2.3–2.5 to the reaction-diffusion system (6.9).

Theorem 6.10. *Assume that F satisfies (QMP) and $\phi <_B \psi$ are order-related equilibria of Φ such that there is no equilibrium in $[\phi, \psi]_B$ except ϕ and ψ , and all conditions of Theorem 6.9 are satisfied with $E = [\phi, \psi]_B$. Then there exists a monotone orbit connecting ϕ and ψ .*

Theorem 6.11. *Suppose that all conditions of Theorem 6.8 are satisfied, and there exist a subequilibrium ϕ and a superequilibrium ψ of Φ with $\phi <_B \psi$, such that all conditions of Theorem 6.9 are satisfied with $E = [\phi, \psi]_B$. Then*

(a) *If all equilibria of Φ in E are stable with respect to E , then every bounded solution in E converges.*

(b) *If ϕ and ψ are strict subequilibrium and superequilibrium of Φ , respectively, then there exists a stable equilibrium in E .*

(c) *If ϕ and ψ are equilibria stable with respect to E and ϕ is isolated from above or ψ is isolated from below, then there exists an unstable equilibrium in E .*

We conclude this section with an invariance principle for system (6.9) of Liapunov-Razumikhin type. We will state our results for $B = UC_g$ with g satisfying (g1)-(g3) and (6.10). We leave extensions of this invariance principle in general phase spaces to a further paper.

Let $V : C(\bar{\Omega}; R^m) \rightarrow R$ be a given continuous function. The derivative of V with respect to a solution u of (6.9) is defined by

$$V'_{(6.9)}(u(t)) = \limsup_{h \rightarrow 0^+} \frac{1}{-h} \{V(u(t-h)) - V(u(t))\}, \quad t > 0.$$

Define

$$D(W) = \{\phi \in UC_g; V(\phi(\cdot)) : (-\infty, 0] \rightarrow R \text{ is bounded}\},$$

$$W(\phi) = \sup_{\theta \leq 0} \frac{V(\phi(\theta))}{g(\theta)}, \quad \phi \in D(W).$$

For a constant C , define

$$G_C = \{\phi \in D(W); W(u_t(\phi)) \equiv W(\phi) = C, t \geq 0\}$$

$$= \left\{ \phi \in D(W); \sup_{\theta \leq 0} \frac{V(u(\phi)(t+\theta))}{g(\theta)} \equiv \sup_{\theta \leq 0} \frac{V(\phi(\theta))}{g(\theta)} = C, t \geq 0 \right\},$$

M_C = the largest subset of G_C that is invariant with respect to (6.9).

Theorem 6.12 (Invariance Principle). *Suppose that there exists a continuous function $V : C(\bar{\Omega}; R^m) \rightarrow R$ such that for all $\phi \in D(W)$*

and $t > 0$ with $V(u(t, \phi)) = W(u_t(\phi))$, $V'_{(6,9)}(u(t, \phi)) \leq 0$. Assume further that F maps bounded subsets of $D(W)$ into bounded subsets of X , and that Φ_t is defined for $t > 0$ and maps bounded subsets of $D(W)$ into bounded subsets of X . Then for any given $\phi \in D(W)$ such that $u_t(\phi)$ and $V(u_t(\phi))$ are bounded and $Cl\{u_t(\phi); t \geq 0\} \subseteq D(W)$, there exists $C \in R$ such that

$$u_t(\phi) \rightarrow M_C \quad \text{as } t \rightarrow \infty.$$

Proof. By Theorem 6.10, the semiflow Φ_t is set-condensing on $D(W)$ for $t > 0$. So, for any $\phi \in D(W)$ such that $u_t(\phi)$ is bounded and $Cl\{u_t(\phi); t \geq 0\} \subseteq D(W)$, $\omega(\phi)$ is nonempty, compact, connected and invariant. We claim that the map $t \rightarrow W(u_t)$ is nonincreasing on $[0, \infty)$, where $u_t = u_t(\phi)$.

In fact, if it is not nonincreasing on $[0, \infty)$, then for some $t_0 > 0$,

$$\liminf_{h \rightarrow 0^+} \frac{1}{-h} \{W(u_{t_0-h}) - W(u_{t_0})\} > 0,$$

hence there exist $h_n \rightarrow 0^+$ as $n \rightarrow \infty$ and $\alpha > 0$ such that

$$\frac{W(u_{t_0-h_n}) - W(u_{t_0})}{-h_n} \geq \alpha, \quad n \geq 1,$$

which implies that $W(u_{t_0}) > W(u_{t_0-h_n})$. We shall show that $W(u_{t_0}) = V(u(t_0))$.

Suppose that $W(u_{t_0}) > V(u(t_0))$. Then the boundedness of $V(u(\cdot)) : (-\infty, 0] \rightarrow R$ and $\lim_{s \rightarrow -\infty} g(s) = \infty$ guarantee that there exists $\theta_0 < 0$ such that

$$W(u_{t_0}) = \frac{V(u(t_0 + \theta_0))}{g(\theta_0)}.$$

Choose sufficiently large n so that $\theta_0 + h_n < 0$. Hence

$$\begin{aligned} W(u_{t_0-h_n}) &\geq \frac{V(u(t_0 + \theta_0))}{g(\theta_0 + h_n)} \\ &= \frac{V(u(t_0 + \theta_0))}{g(\theta_0)} \cdot \frac{g(\theta_0)}{g(\theta_0 + h_n)} \end{aligned}$$

$$\begin{aligned} &\geq \frac{V(u(t_0 + \theta_0))}{g(\theta_0)} \\ &\geq W(u_{t_0}), \end{aligned}$$

a contradiction to $W(u_{t_0}) > W(u_{t_0-h_n})$.

By assumption, we have

$$\limsup_{h \rightarrow 0^+} \frac{V(u(t_0 - h)) - V(u(t_0))}{-h} \leq 0.$$

On the other hand, we have

$$\begin{aligned} V(u(t_0 - h_n)) - V(u(t_0)) &= V(u(t_0 - h_n)) - W(u_{t_0}) \\ &\leq W(u_{t_0-h_n}) - W(u_{t_0}), \end{aligned}$$

which implies that

$$\frac{V(u(t_0 - h_n)) - V(u(t_0))}{-h_n} \geq \frac{W(u_{t_0-h_n}) - W(u_{t_0})}{-h_n} \geq \alpha,$$

a contradiction. This proves that $W(u_t)$ is nonincreasing on $[0, \infty)$.

As $V(u_t(\phi))$ is bounded for all $t \geq 0$, $\inf_{t \geq 0} W(u_t) \geq \inf_{t \geq 0} V(u_t) > -\infty$. Therefore,

$$\lim_{t \rightarrow \infty} W(u_t) = C \quad \text{exists.}$$

Let $\psi \in \omega(\phi)$. Then there is a sequence $\{t_n\}$ such that $t_n \rightarrow \infty$ and $u_{t_n}(\phi) \rightarrow \psi$ as $n \rightarrow \infty$. This leads to

$$W(\psi) = \lim_{n \rightarrow \infty} W(u_{t_n}) = C.$$

Thus

$$W(u_t(\psi)) = W(\psi) = C \quad \text{for all } t \geq 0,$$

which implies that $\psi \in G_C$ and we have $\omega(\phi) \subseteq G_C$. It follows that $u_t(\phi) \rightarrow M_C$ as $t \rightarrow \infty$ and the proof is complete. \square

The above result generalizes the invariance principle of Haddock and Terjéki [15] for functional differential equations with infinite delay to reaction-diffusion systems with distributed delay.

7. Lotka-Volterra models with diffusion and distributed delay. Consider the Lotka-Volterra competition-diffusion model with distributed delay

$$(7.1) \quad \frac{\partial}{\partial t} u^i = d_i \Delta u^i + u^i \left[r_i - a_{ii} u^i - \sum_{j=1}^m b_{ij} \int^t -\infty k_{ij}(t-s) u^j(s, x) ds \right],$$

$$t > 0, x \in \Omega$$

$$\frac{\partial}{\partial n} u^i = 0, \quad t > 0, x \in \partial\Omega$$

$$u^i(s, x) = \phi_i(s, x), \quad s \in (-\infty, 0], x \in \Omega, i \in \{1, 2, \dots, m\},$$

where Ω is a bounded region in R^n with smooth boundary $\partial\Omega$, the parameters satisfy

$$d_i > 0, \quad r_i > 0, \quad a_{ii} > 0, \quad b_{ij} \geq 0,$$

and each $k_{ij} : [0, \infty) \rightarrow R_+$ is a nonincreasing continuous function and can be normalized so that

$$\int_0^\infty k_{ij}(s) ds = 1.$$

Let $\mathcal{A} : \text{dom}(\mathcal{A}) \subseteq C(\bar{\Omega}; R^m) \rightarrow C(\bar{\Omega}; R^m)$ be the linear operator defined by

$$\text{dom}(\mathcal{A}) = \left\{ u \in C^2(\Omega; R^m) \cap C^1(\bar{\Omega}; R^m), \frac{\partial}{\partial n} u^i = 0 \text{ on } \partial\Omega, 1 \leq i \leq m \right\},$$

$$\mathcal{A}u = (d_i \Delta u^i)_1^m, \quad u \in \text{dom}(\mathcal{A}).$$

It is well known that (see, cf. Rothe [59]) the closure of the operator \mathcal{A} generates an analytic semigroup $T = \{T(t)\}_{t \geq 0}$ in the space $C(\bar{\Omega}; R^m)$. In [1], it has shown that there exists $g : (-\infty, 0] \rightarrow [1, \infty)$ satisfying conditions (g1)–(g3) in Section 3 and such that

$$\int_{-\infty}^0 k_{ij}(-s) g(s) ds < \infty, \quad i, j \in \{1, 2, \dots, m\}.$$

Clearly, the mapping $F = (F_i)_1^m : UC_g \rightarrow C(\bar{\Omega}; R^m)$ defined by

$$(7.2) \quad F_i(\phi) = \phi_i(0) \left[r_i - a_{ii}\phi_i(0) - \sum_{j=1}^m b_{ij} \int_{-\infty}^0 k_{ij}(-s)\phi_j(s)ds \right]$$

satisfies (H3) in Section 6.

We now consider the abstract integral equation

$$(7.3) \quad u(t) = T(t-s)u(s) + \int_s^t T(t-\theta)F(u_\theta)d\theta, \quad t \geq s \geq 0.$$

Theorem 6.1 implies that for any $\phi \in UC_g$, there exists a unique solution $u(t, \phi)$ of (7.3) subject to the initial condition

$$(7.4) \quad u_0 = \phi.$$

For any $R = (R_i)_1^m \in \text{int}R_+^m$, we set

$$K_R = \prod_{i=1}^m [0, R_i] \subseteq R_+^m,$$

$$UC_{g,R} = \{\phi \in UC_g; \phi(\theta) \in C(\bar{\Omega}; K_R) \text{ for } \theta \leq 0\}.$$

Lemma 7.1. *If $R_i \geq r_i/a_{ii}$ for $i \in \{1, 2, \dots, m\}$, then $UC_{g,R}$ is positively invariant for system (7.1). That is, if $\phi \in UC_{g,R}$, then $u_t(\phi) \in UC_{g,R}$ for all $t \geq 0$.*

Proof. Assume that $\phi \in UC_{g,R}$. If $\phi_i(0)(x) = 0$ for some $x \in \bar{\Omega}$, then clearly $F_i(\phi)(x) = 0$. If $\phi_i(0)(x) = R_i$, then $F_i(\phi)(x) \leq \phi_i(0)(x)[r_i - a_{ii}\phi_i(0)(x)] \leq 0$. Consequently, the conclusion follows from the invariant rectangular criterion (Theorem 6.4). \square

For given $\phi \in UC_{g,R}$ with $R_i \geq r_i/a_{ii}$ for $i \in \{1, 2, \dots, m\}$, define

$$\begin{aligned}
 F_i^-(\phi)(x) &= \inf \left\{ \psi_i(0)(x) \left[r_i - a_{ii}\psi_i(0)(x) - \sum_{j=1}^m b_{ij} \int_{-\infty}^0 k_{ij}(-s)\psi_j(s)(x)ds \right]; \right. \\
 &\quad \left. \phi \leq UC_g \psi \leq UC_g \bar{R}, \psi_i(0)(x) = \phi_i(0)(x) \right\} \\
 &= \phi_i(0)(x) \left[r_i - a_{ii}\phi_i(0)(x) - \sum_{j=1}^m b_{ij}R_j \right]
 \end{aligned}$$

and

$$\begin{aligned}
 F_i^+(\phi)(x) &= \sup \left\{ \psi_i(0)(x) \left[r_i - a_{ii}\psi_i(0)(x) - \sum_{j=1}^m b_{ij} \int_{-\infty}^0 k_{ij}(-s)\psi_j(s)(x) ds \right]; \right. \\
 &\quad \left. 0 \leq UC_g \psi \leq UC_g \phi, \psi_i(0)(x) = \phi_i(0)(x) \right\} \\
 &= \phi_i(0)(x)[r_i - a_{ii}\phi_i(0)(x)].
 \end{aligned}$$

Let $v^i(t, \phi)$ and $w^i(t, \phi)$ be the solutions of the following systems

$$\begin{aligned}
 (7.5) \quad \frac{\partial}{\partial t} v^i &= d_i \Delta v^i - v^i \left[\left(-r_i + \sum_{j=1}^m b_{ij}R_j \right) + a_{ii}v^i \right], \quad x \in \Omega \\
 \frac{\partial}{\partial n} v^i &= 0, \quad x \in \partial\Omega, \quad 1 \leq i \leq m \\
 v^i(0, \phi)(x) &= \phi_i(0, x), \quad x \in \Omega
 \end{aligned}$$

and

$$\begin{aligned}
 (7.6) \quad \frac{dw^i}{dt} &= w^i[r_i - a_{ii}w^i], \quad 1 \leq i \leq n \\
 w^i(0, \phi) &= \sup_{x \in \Omega} \phi_i(0, x),
 \end{aligned}$$

respectively. We have the following result.

Proposition 7.2. *If $u(t, \phi)$ is a solution of system (7.3) such that $\phi \in UC_{g,R}$ with $R_i \geq r_i/a_{ii}$ for $i \in \{1, 2, \dots, m\}$, then*

$$(7.7) \quad v^i(t, \phi)(x) \leq u^i(t, \phi)(x) \leq w^i(t, \phi)$$

for all $i \in \{1, 2, \dots, m\}$, $x \in \bar{\Omega}$ and $t > 0$. In particular, if $\phi_i(0)(\bar{x}) > 0$ for some $\bar{x} \in \Omega$, then

$$(7.8) \quad u^i(t, \phi)(x) > 0$$

for all $i \in \{1, 2, \dots, m\}$, $x \in \Omega$ and $t > 0$. Moreover, $u(\cdot, \phi) : R_+ \rightarrow C(\bar{\Omega}; R^m)$ is bounded.

Proof. The inequality (7.7) is an immediate consequence of the comparison principle (Theorem 6.5). If $\phi_i(0)(x) > 0$ for some $x \in \Omega$, then the classical maximum principle implies that $v^i(t, \phi)(x) > 0$ for all $(t, x) \in (0, \infty) \times \Omega$, thus $u^i(t, \phi)(x) > 0$ for all $(t, x) \in (0, \infty) \times \Omega$.

Clearly, $\lim_{t \rightarrow \infty} w^i(t, \phi) \leq r_i/a_{ii} < \infty$ implies that

$$\limsup_{t \rightarrow \infty} u^i(t, \phi)(x) \leq r_i/a_{ii}, \quad 1 \leq i \leq m.$$

This implies the boundedness of the solutions $u(t, \phi)$ of (7.3). \square

Theorem 7.3. *Suppose that system (7.1) has a unique, spatially homogeneous steady state solution $u^* = (u^{*i})_1^n \in \text{int } R_+^n$. Furthermore, assume that*

$$(7.9) \quad a_{ii} > \sum_{j=1}^m b_{ij} \int_{-\infty}^0 k_{ij}(-s)g(s)ds, \quad 1 \leq i \leq m.$$

Then for any $\phi \in UC_{g,R}$ so that $R_i \geq r_i/a_{ii}$ and $\phi_i(0) >_{C(\bar{\Omega}; R)} 0$, $i = 1, 2, \dots, m$, we have $u_t(\phi) \rightarrow u^*$ as $t \rightarrow \infty$.

Proof. Throughout the proof, we assume, without loss of generality, that $u^{*i} \neq u^{*j}$ for $i \neq j$. For otherwise we can choose $\delta_i > 0$ such that $\delta_i u^{*i} \neq \delta_j u^{*j}$, $i \neq j$, and make the change of variables $\bar{u}^i = \delta_i u^i$ in system (7.1).

For $\xi \in R^m$, define

$$V_1(\xi) = \max\{|\xi^i - u^{*i}| : 1 \leq i \leq m\}, \quad I(\xi) = \{i : V_1(\xi) = |\xi^i - u^{*i}|\}.$$

Also, for $u \in C(\bar{\Omega}, R^m)$, define

$$V(u) = \max\{V_1(u(x)) : x \in \bar{\Omega}\}, \quad J(u) = \{x : V(u) = V_1(u(x))\}.$$

It is easy to check that if

$$D_-V(u)(v) = \limsup_{h \rightarrow 0^+} \frac{V(u - hv) - V(u)}{-h}$$

for all $u, v \in C(\bar{\Omega}, R^m)$, then for $u(x) \not\equiv u^*$ on $\bar{\Omega}$, we have

(7.10)

$$D_-V(u)(v) = \max\{\text{sign}[u^i(x) - u^{*i}]v^i(x) : x \in J(u), i \in I(u(x))\},$$

where

$$\text{sign}(r) = \begin{cases} 1 & \text{if } r > 0 \\ -1 & \text{if } r < 0. \end{cases}$$

Note that $T(t)u^* \equiv u^*$ for $t \geq 0$. Applying the maximum principle, we can show that

$$V(T(h)u) \leq V(u) \quad \text{for all } h \geq 0 \quad \text{and} \quad u \in C(\bar{\Omega}; R^m).$$

Assume $\phi \in UC_{g,R}$ and denote $u(t) = u(t, \phi)$. Then

$$u(t) - hF(u_t) = T(h)u(t-h) + o(h) \quad \text{for } t \geq 0,$$

where $h^{-1}|o(h)| \rightarrow 0$ as $h \rightarrow 0^+$. Hence,

$$V(u(t) - hF(u_t)) \leq V(T(h)u(t-h)) + o(h) \leq V(u(t-h)) + o(h).$$

Consequently,

$$(7.11) \quad V'_{(7.3)}(u(t)) \leq D_-V(u(t))(F(u_t)).$$

Now for $\phi \in UC_{g,R}$, define

$$(7.12) \quad W(\phi) = \sup_{\theta \leq 0} \frac{V(\phi(\theta))}{g(\theta)}.$$

We want to show that if $W(u_t) = V(u(t))$, then

$$(7.13) \quad V'_{(7.3)}(u(t)) \leq 0.$$

Pick up $i \in \{1, 2, \dots, m\}$ and $x_0 \in \bar{\Omega}$ so that $V(u(t)) = |u^i(t)(x_0) - u^{*i}|$. Assume at this moment that $V(u(t)) \neq 0$. We consider

$$\begin{aligned} & \text{sign} [u^i(t)(x_0) - u^{*i}] F_i(u_t)(x_0) \\ &= \text{sign} [u^i(t)(x_0) - u^{*i}] u^i(t)(x_0) \\ & \quad \times \left\{ r_i - a_{ii} u^i(t)(x_0) - \sum_{j=1}^m b_{ij} \int_{-\infty}^t k_{ij}(t-s) u^j(s)(x_0) ds \right\} \\ &= \text{sign} [u^i(t)(x_0) - u^{*i}] u^i(t)(x_0) \\ & \quad \times \left\{ -a_{ii} [u^i(t)(x_0) - u^{*i}] - \sum_{j=1}^m b_{ij} \int_{-\infty}^t k_{ij}(t-s) [u^j(s)(x_0) - u^{*j}] ds \right\} \\ &= u^i(t)(x_0) |u^i(t)(x_0) - u^{*i}| \\ & \quad \times \left\{ -a_{ii} - \frac{\sum_{j=1}^m b_{ij} \int_{-\infty}^t k_{ij}(t-s) [u^j(s)(x_0) - u^{*j}] ds}{u^i(t)(x_0) - u^{*i}} \right\}. \end{aligned}$$

As $W(u_t) = V(u(t))$, we have

$$|u^j(s)(x_0) - u^{*j}| \leq |u^i(t)(x_0) - u^{*i}| g(s-t)$$

for all $j \in \{1, 2, \dots, m\}$ and $s \leq t$. Therefore,

$$\begin{aligned} -\frac{b_{ij} \int_{-\infty}^t k_{ij}(t-s) [u^j(s)(x_0) - u^{*j}] ds}{u^i(t)(x_0) - u^{*i}} &\leq b_{ij} \int_{-\infty}^t k_{ij}(t-s) g(s-t) ds \\ &= b_{ij} \int_{\infty}^0 k_{ij}(-s) g(s) ds. \end{aligned}$$

Consequently,

$$\begin{aligned} & \text{sign} [u^i(t)(x_0) - u^{*i}] F_i(u_t)(x_0) \\ & \leq u^i(t)(x_0) |u^i(t)(x_0) - u^{*i}| \left\{ -a_{ii} + \sum_{j=1}^m b_{ij} \int_{-\infty}^0 k_{ij}(-s) g(s) ds \right\}. \end{aligned}$$

This, together with (7.9)–(7.11), implies (7.13).

By the invariance principle (Theorem 6.12), we know that there exists $C \in R$ such that

$$u_t \rightarrow M_C \subseteq G_C \quad \text{as } t \rightarrow \infty.$$

We will show that $C \equiv 0$.

For the sake of contradiction, we suppose that $C > 0$. Since $\omega(\phi)$ of $u_t(\phi)$ is nonempty, compact and invariant, if $\psi \in \omega(\phi)$ and $v(t) = v(t, \psi)$ is the solution of (7.3), then

$$(7.14) \quad W(v_t) = C > 0 \quad \text{for all } t \geq 0.$$

For each $t > 0$, there exists $x^t \in \bar{\Omega}$ and $i \in \{1, 2, \dots, m\}$ so that

$$V(v(t)) = |v^i(t, \psi)(x^t) - u^{*i}|.$$

Claim I. *For any $t > 0$, if $V(v(t)) = C = W(v_t)$ then $v^i(t)(x^t) = 0$.*

In fact, if $V(v(t)) = W(v_t)$ and $v^i(t)(x^t) > 0$ hold simultaneously at some $t > 0$, then the above argument for (7.13) implies that

$$\limsup_{h \rightarrow 0^+} \frac{V(v(t-h)) - V(v(t))}{-h} < 0.$$

Consequently, for some sufficiently small $h > 0$ with $t-h > 0$, we have

$$W(v_{t-h}) \geq V(v(t-h)) > V(v(t)) = C,$$

a contradiction to $W(v_s) \equiv C$ for all $s \geq 0$. This justifies the claim.

Claim II. *It is impossible for $V(v(t)) < C$ for all $t > 0$.*

By the way of contradiction, we assume that $V(v(t)) < C$ for all $t > 0$. Fix $s_1 < 0$ so that $g(s_1) > 1$. Hence, for a fixed $t > -s_1$, we have

$$\bar{C} := \max\{V(v(s)); t + s_1 \leq s \leq t\} < C$$

and

$$W(v_t) \leq \max\{\bar{C}, C/g(s_1)\} < C,$$

a contradiction to $W(v_s) \equiv C$ for all $s \geq 0$.

Putting Claim I and Claim II together, we obtain $t_0 > 0$, $i_0 \in \{1, 2, \dots, m\}$ and $x^{t_0} \in \bar{\Omega}$ so that $V(v(t_0)) = W(v_{t_0}) = C = |v^{i_0}(t_0)(x^{t_0}) - u^{i_0*}|$ and $v^{i_0}(t_0)(x^{t_0}) = 0$. Hence, $C = u^{i_0*}$.

Claim III. $|v^j(t)(x) - u^{*j}| < C = u^{*i_0}$ for all $t > 0$, $x \in \bar{\Omega}$ and $j \neq i_0$.

For otherwise, there exists $t^* > 0$, $x^* \in \bar{\Omega}$ and $j \neq i_0$ so that $V(v(t^*)) = |v^j(t^*)(x^*) - u^{*j}| = C = W(v_{t^*})$. By the result in Claim I, $v^j(t^*)(x^*) = 0$ and thus, $u^{*j} = C = u^{*i_0}$, a contradiction to our assumption $u^{*i} \neq u^{*j}$ for $i \neq j$.

Since $g(s) \rightarrow +\infty$ as $s \rightarrow -\infty$, we can choose $\sigma > 0$ so that $R_i/g(-\sigma) < C/2$. For this chosen $\sigma > t_0$, $v_\sigma \in \omega(\phi)$ and hence there exists a sequence $\{t_k\}$ so that $t_k \rightarrow \infty$ and $u_{t_k}(\phi) \rightarrow v_\sigma$ in UC_g as $k \rightarrow \infty$. Consequently, we can use Claim III to find $K_1 > 0$ so that

$$\max\{|u^j(t_k + \theta)(x) - u^{*j}|; \theta \in [-\sigma, 0], x \in \bar{\Omega}, j \neq i_0\} < u^{*i_0} = C$$

for $k \geq K_1$.

On the other hand, by Proposition 7.2, we derive from $v^{i_0}(t_0)(x^{t_0}) = 0$ that $v^{i_0}(\sigma + s)(x) = 0$ for all $s < 0$ (and hence for $s \leq 0$ by continuity) and $x \in \bar{\Omega}$. Since $u_{t_k}(\phi) \rightarrow v_\sigma$ as $k \rightarrow \infty$, we can find $K_2 > 0$ so that $0 < u^{i_0}(t_k + \theta)(x) \leq u^{*i_0}$ for all $x \in \bar{\Omega}$, $\theta \in [-\sigma, 0]$ and $k \geq K_2$. Hence, for $k \geq K_2$, we have

$$\max\{|u^{i_0}(t_k + \theta)(x) - u^{*i_0}|; \theta \in [-\sigma, 0], x \in \bar{\Omega}\} < u^{*i_0} = C.$$

Therefore,

$$\max\{|u^j(t_k + \theta)(x) - u^{*j}|; \theta \in [-\sigma, 0], x \in \bar{\Omega}, j = 1, 2, \dots, m\} < C$$

and

$$W(u_{t_k}) \leq \max \left\{ \max\{|u^j(t_k + \theta)(x) - u^{*j}|; \theta \in [-\sigma, 0], x \in \bar{\Omega}, j = 1, 2, \dots, m\}, \frac{2R_i}{g(-\sigma)} \right\} < C.$$

Therefore, $W(u_t) \leq W(u_{t_{K_2}}) < C$ for all $t \geq t_{K_2}$ and $\lim_{t \rightarrow \infty} W(u_t) < C$, a contradiction to $\lim_{t \rightarrow \infty} W(u_t) = C$.

Hence $C \equiv 0$ and then $G_C = \{u^*\}$. This completes the proof. \square

In general, system (7.1) does not satisfy the quasimonotone hypothesis even allowing the nonstandard partial ordering. However, if $m = 2$, then system (7.1) generates a monotone semiflow in the sense described below. For $u, v \in R^2$, define $u \leq_Q v$ if $u_1 \leq v_1$ and $u_2 \geq v_2$. The subscript Q indicates that the order is generated by the second quadrant of R^2 . This generates a new partial ordering in UC_g in the obvious pointwise sense, that is, $\phi = (\phi_1, \phi_2) \leq_Q (\psi_1, \psi_2) = \psi$ in UC_g if and only if $\phi_1(s) \leq \psi_1(s)$ and $\phi_2(s) \geq \psi_2(s)$ for all $s \in (-\infty, 0]$.

Now we consider the following two-species Lotka-Volterra diffusion-competition model with infinite delay

$$\begin{aligned}
 (7.15) \quad & \frac{\partial}{\partial t} u_1 = d_1 \Delta u_1 + u_1 \left[r_1 - a_1 u_1 - b_1 \int_{-\infty}^t k_1(t-s) u_2(s) ds \right], \quad x \in \Omega \\
 & \frac{\partial}{\partial t} u_2 = d_2 \Delta u_2 + u_2 \left[r_2 - b_2 \int_{-\infty}^t k_2(t-s) u_1(s) ds - a_2 u_2 \right], \quad x \in \Omega \\
 & \frac{\partial}{\partial n} u_1 = \frac{\partial}{\partial n} u_2 = 0, \quad x \in \partial \Omega \\
 & u_1(s, x) = \phi_1(s, x) \geq 0, \quad u_2(s, x) = \phi_2(s, x) \geq 0, \\
 & \quad s \in (-\infty, 0], \quad x \in \Omega.
 \end{aligned}$$

Proposition 7.4. *If $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ are elements of UC_g^+ and $\phi \leq_Q \psi$, then $u(t, x; \phi) \leq_Q u(t, x; \psi)$ for all $t \geq 0, x \in \Omega$.*

Proof. Setting $v_1 = u_1$ and $v_2 = -u_2$ in (7.15), we obtain

$$\begin{aligned}
 (7.16) \quad & \frac{\partial}{\partial t} v_1 = d_1 \Delta v_1 + v_1 \left[r_1 - a_1 v_1 + b_1 \int_{-\infty}^t k_1(t-s) v_2(s, x) ds \right] \\
 & \frac{\partial}{\partial t} v_2 = d_2 \Delta v_2 + v_2 \left[r_2 - b_2 \int_{-\infty}^t k_2(t-s) v_1(s, x) ds + a_2 v_2 \right].
 \end{aligned}$$

It can be easily verified that (QMP) in Section 6 holds for the standard partial ordering, hence the conclusion follows by the monotonicity principle (Theorem 6.6). \square

System (7.15) has two boundary equilibria $(r_1/a_1, 0)$, $(0, r_2/a_2)$ and an interior equilibrium

$$(u_1^*, u_2^*) = \left(\frac{r_1 a_2 - r_2 b_1}{a_1 a_2 - b_2 b_1}, \frac{r_2 a_1 - r_1 b_2}{a_1 a_2 - b_2 b_1} \right) \quad \text{if } a_1 a_2 - b_2 b_1 \neq 0.$$

Proposition 7.5. *Suppose*

$$(7.17) \quad \max\{a_1/b_2, b_1/a_2\} < r_1/r_2.$$

If $\phi = (\phi_1, \phi_2) \in UC_{g,R}$ with $R_1 = R_2 \geq \max\{\frac{r_1}{a_1}, \frac{r_2}{a_2}\}$ and $\phi_1(0, x) > 0$ for some $x \in \Omega$, then

$$(u_1(t, x), u_2(t, x)) \rightarrow (r_1/a_1, 0)$$

uniformly in $x \in \Omega$ as $t \rightarrow \infty$.

Proof. By (7.17) and the fact that $\int_0^\infty k_i(s)ds = 1$, $i = 1, 2$, for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$\begin{aligned} b_1 \int_{T_\varepsilon}^\infty k_1(s)ds &< R_1^{-1}\varepsilon, & \frac{a_2}{r_2} &> \frac{b_1 \int_0^{T_\varepsilon} k_1(s)ds}{r_1 - \varepsilon}, \\ \frac{a_1}{r_1 - \varepsilon} &< \frac{b_2 \int_0^{T_\varepsilon} k_1(s)ds}{r_2}. \end{aligned}$$

By Lemma 7.1 and Proposition 7.2, $0 < u_1(t, x) \leq R_1$ and $u_2(t, x) \leq R_2$ for all $t > 0$ and $x \in \bar{\Omega}$. Note that

$$\begin{aligned} &u_1 \left[r_1 - a_1 u_1 - b_1 \int_{-\infty}^t k_1(t-s)u_2(s) ds \right] \\ &= u_1 \left[r_1 - a_1 u_1 - b_1 \int_{-\infty}^{t-T_\varepsilon} k_1(t-s)u_2(s) ds \right. \\ &\quad \left. - b_1 \int_{t-T_\varepsilon}^t k_1(t-s)u_2(s) ds \right] \\ &\geq u_1(r_1 - \varepsilon) \left[1 - \frac{a_1}{r_1 - \varepsilon} u_1 - \frac{b_1}{r_1 - \varepsilon} \int_{t-T_\varepsilon}^t k_1(t-s)u_2(s) ds \right] \end{aligned}$$

and similarly

$$\begin{aligned} & u_2 \left[r_2 - a_2 u_2 - b_2 \int_{-\infty}^t k_2(t-s)u_1(s) ds \right] \\ &= u_2 \left[r_2 - a_2 u_2 - b_2 \int_{-\infty}^{t-T_\epsilon} k_2(t-s)u_1(s) ds \right. \\ &\quad \left. - b_2 \int_{t-T_\epsilon}^t k_2(t-s)u_1(s) ds \right] \\ &\leq u_2 \left[r_2 - a_2 u_2 - b_2 \int_{t-T_\epsilon}^t k_2(t-s)u_1(s) ds \right] \\ &= u_2 r_2 \left[1 - \frac{a_2}{r_2} u_2 - \frac{b_2}{r_2} \int_{t-T_\epsilon}^t k_2(t-s)u_1(s) ds \right]. \end{aligned}$$

Therefore, by Theorem 6.2 we get

$$u_1(t, x) \geq u_1^*(t, x), \quad u_2(t, x) \leq u_2^*(t, x)$$

for all $t \geq 0$ and $x \in \bar{\Omega}$, where (u_1^*, u_2^*) solves the following reaction-diffusion equation with delay

(7.18)

$$\begin{aligned} \partial_t u_1^* = d_1 \Delta u_1^* + (r_1 - \epsilon) u_1^* & \left[1 - \frac{a_1}{r_1 - \epsilon} u_1^* \right. \\ & \left. - \frac{b_1}{r_1 - \epsilon} \int_{t-T_\epsilon}^t k_1(t-s)u_2^*(s) ds \right], \quad x \in \bar{\Omega} \end{aligned}$$

$$\partial_t u_2^* = d_2 \Delta u_2^* + r_2 u_2^* \left[1 - \frac{a_2}{r_2} u_2^* - \frac{b_2}{r_2} \int_{t-T_\epsilon}^t k_2(t-s)u_1^*(s) ds \right], \quad x \in \bar{\Omega}$$

$$\partial_n u_1^* = \partial_n u_2^* = 0, \quad x \in \partial\Omega$$

$$u_1^*(s) = \phi_1(s), \quad u_2^*(s) = \phi_2(s), \quad s \in [-T_\epsilon, 0].$$

Now, Proposition 5.6 in [37] can be applied to (7.18) (as delay T_ϵ is finite). It follows that

$$\lim_{t \rightarrow \infty} u_1^*(t, x) = \frac{r_1 - \epsilon}{a_1}, \quad \lim_{t \rightarrow \infty} u_2^*(t, x) = 0$$

uniformly for $x \in \bar{\Omega}$. Since $u_2(t, x) \leq u_2^*(t, x)$, we have that $\lim_{t \rightarrow \infty} u_2(t, x) = 0$ uniformly for $x \in \bar{\Omega}$. On the other hand, by Proposition 7.2 we get $\limsup_{t \rightarrow \infty} u_1(t, x) = r_1/a_1$, and by $u_1(t, x) \geq u_1^*(t, x)$

we have $\liminf_{t \rightarrow \infty} u_1(t, x) = (r_1 - \varepsilon)/a_1$. Consequently, due to the arbitrary choice of ε , we have $\lim_{t \rightarrow \infty} u_1(t, x) = r_1/a_1$ uniformly for $x \in \bar{\Omega}$. This completes the proof. \square

Similarly we can prove the following result.

Proposition 7.6. *Suppose*

$$(7.19) \quad \min\{a_1/b_2, b_1/a_2\} > r_1/r_2.$$

If $\phi = (\phi_1, \phi_2) \in UC_{g,R}$ with $R_1 = R_2 \geq \max\{r_1/a_1, r_2/a_2\}$ and $\phi_2(0, x) > 0$ for some $x \in \Omega$, then

$$(u_1(t, x), u_2(t, x)) \rightarrow (0, r_2/a_2)$$

uniformly in $x \in \Omega$ as $t \rightarrow \infty$.

Propositions 7.5 and 7.6 indicate that if the intraspecies competition is weaker than the interspecies competition, then one of the two competitors will go to extinction, that is, competition exclusion principle applies to system (7.15). For the positive interior equilibrium, we can employ a similar argument to establish the following result.

Proposition 7.7. *Suppose*

$$(7.20) \quad b_1/a_2 < r_1/r_2 < a_1/b_2.$$

If $\phi = (\phi_1, \phi_2) \in UC_{g,R}$ with $R_1 = R_2 \geq \max\{r_1/a_1, r_2/a_2\}$ and $\phi_1(0, x) > 0$, $\phi_2(0, x) > 0$ for some $x \in \Omega$, then

$$(u_1(t, x), u_2(t, x)) \rightarrow (u_1^*, u_2^*)$$

uniformly in $x \in \Omega$ as $t \rightarrow \infty$.

The above propositions demonstrate that if one of the conditions (7.17), (7.19) and (7.20) holds, then the asymptotic behavior of (7.16) is very similar to that of the ordinary differential equation model

$$(7.21) \quad \begin{aligned} \frac{du_1}{dt} &= u_1[r_1 - a_1 u_1 - b_1 u_2] \\ \frac{du_2}{dt} &= u_2[r_2 - b_2 u_1 - a_2 u_2] \end{aligned}$$

and the reaction-diffusion model

$$\begin{aligned}
 (7.22) \quad & \frac{\partial}{\partial t} u_1 = d_1 \Delta u_1 + u_1[r_1 - a_1 u_1 - b_1 u_2], & x \in \Omega \\
 & \frac{\partial}{\partial t} u_2 = d_2 \Delta u_2 + u_2[r_2 - b_2 u_1 - a_2 u_2], & x \in \Omega \\
 & \frac{\partial}{\partial n} u_1 = \frac{\partial}{\partial n} u_2 = 0, & x \in \partial\Omega.
 \end{aligned}$$

In particular, there are no stable spatially inhomogeneous solutions.

In the remaining case that

$$(7.23) \quad a_1/b_2 < r_1/r_2 < b_1/a_2,$$

Matano and Mimura [43] showed that there may exist a stable spatially inhomogeneous solution for the reaction-diffusion model (7.22) provided the domain Ω is suitably nonconvex, for example, dumbbell shaped with a narrow handle. Matano and Mimura only considered the reaction-diffusion model where there are no delays, but we believe that their conclusion is still true for (7.16), the reaction-diffusion model with infinite delay. This should be proved by following Matano and Mimura’s procedure and by applying the techniques and results in this paper.

However, if the domain Ω is convex and condition (7.23) holds, then there cannot exist stable spatially inhomogeneous solutions (for general reaction-diffusion systems, this was proved by Kishimoto and Weinberger [29]). In fact, we have the following results when (7.23) is satisfied.

Proposition 7.8. *Suppose that (7.23) holds.*

(i) *If ϕ satisfies that $u_1^* < \phi_1(t, x) < R_1$, $0 < \phi_2(t, x) < u_2^*$ for $(t, x) \in (-\infty, 0] \times \bar{\Omega}$ with $R_1 \geq r_1/a_1$, then*

$$(u_1(t, x), u_2(t, x)) \rightarrow (r_1/a_1, 0)$$

uniformly in $x \in \Omega$ as $t \rightarrow \infty$.

(ii) *If ϕ satisfies that $0 < \phi_1(t, x) < u_1^*$, $u_2^* < \phi_2(t, x) < R_2$ for $(t, x) \in (-\infty, 0] \times \bar{\Omega}$ with $R_2 \geq r_2/a_2$, then*

$$(u_1(t, x), u_2(t, x)) \rightarrow (0, r_2/a_2)$$

uniformly in $x \in \Omega$ as $t \rightarrow \infty$.

(iii) (u_1^*, u_2^*) is unstable.

Proof. (i) and (ii) can be proved following the previous procedure. To see (iii), we know that

$$(0, r_2/a_2) <_Q (u_1^*, u_2^*) <_Q (r_1/a_1, 0),$$

where $(0, r_2/a_2)$ and $(r_1/a_1, 0)$ are (locally) stable with respect to $E = [(0, r_2/a_2), (r_1/a_1, 0)]_Q$, and $(0, r_2/a_2)$ is isolated from above, Theorem 6.11 implies that there is an unstable equilibrium in E , while (u_1^*, u_2^*) is the only steady state in E , hence it is unstable. \square

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