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OSCILLATIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider the oscillatory behavior of the second order neutral delay differential equation

$$(a(t)(x(t) + p(t)x(t-\tau))')' + q(t)f(x(t-\sigma)) = 0,$$

where $t \geq t_0$, τ and σ are positive constants, $a, p, q \in C([t_0, \infty), R), f \in C[R, R]$. Some sufficient conditions are established such that the above equation is oscillatory. The obtained oscillation criteria generalize and improve a number of known results about both neutral and delay differential equations.

1. **Introduction.** A *neutral delay differential equation* is a differential equation in which the highest order derivative of the unknown function appears both with and without delay. Second order neutral delay differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems (see Hale [11]).

In this paper, we are interested in the oscillatory behaviour of the following second order neutral delay differential equation

$$\Big(a(t)\big(x(t) + p(t)x(t-\tau)\big)'\Big)' + q(t)f\big(x(t-\sigma)\big) = 0, \tag{1}$$

where $t \geq t_0, \tau$ and σ are positive constants, $a, p, q \in C([t_0, \infty), R), f \in C[R, R]$. Throughout this paper, we assume that

(a) $0 \leq p(t) \leq 1$, $q(t) \geq 0$, $a(t) > 0$;

(b) $\int^{\infty} \frac{1}{a(s)}\, ds = \infty$;

(c) $\frac{f(x)}{x} \geq \gamma > 0$ for $x \neq 0$.

Let $\psi \in C([t_0 - \theta, R)$, where $\theta = \max\{\tau, \sigma\}$, be a given function and let y_0 be a given constant. Using the method of steps, it follows that equation (1) has a unique solution $x \in C([t_0 - \theta, \infty), R)$ in the sense that both $x(t) + p(t)x(t-\tau)$ and $a(t)\big(x(t) + p(t)x(t-\tau)\big)'$ are continuously differentiable for $t \geq t_0$, $x(t)$ satisfies equation (1) and

$$x(s) = \psi(s), \quad \text{for } s \in [t_0 - \theta, t_0],$$
$$[x(t) + p(t)x(t-\tau)]'_{t=t_0} = y_0.$$

For further questions concerning existence and uniqueness of solutions of neutral delay differential equations, see Hale [11].

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A solution of equation (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative. Equation (1) is called *oscillatory* if all solutions of (1) are oscillatory. In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large t.

Special cases of equation (1) are the following delay differential equation

$$\big(a(t)\,x'(t)\big)' + q(t)f\big(x(t-\sigma)\big) = 0 \tag{2}$$

and ordinary differential equation

$$\big(a(t)\,x'(t)\big)' + q(t)f\big(x(t)\big) = 0. \tag{3}$$

If $a(t) = 1, f\big(x(t)\big) = x(t)$, then equations (1)–(3) become to the following linear second order neutral delay differential equation

$$\big(x(t) + p(t)\,x(t-\tau)\big)'' + q(t)\,x(t-\sigma) = 0, \tag{4}$$

delay differential equation

$$x''(t) + q(t)\,x(t-\sigma) = 0, \tag{5}$$

and ordinary differential equation

$$x''(t) + q(t)\,x(t) = 0. \tag{6}$$

Averaging function method is one of the most important techniques in the study of oscillation. By using this technique, many oscillation criteria have been found which involve the behaviour of the integrals of the coefficients. For the linear ordinary differential equation (6), the most simple oscillation criterion is the well-known Fife-Wintner-Leighton theorem which states that if $q(t) \in C([t_0, \infty)$ and

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)\,ds = \infty, \tag{7}$$

then equation (6) is oscillatory (see Leighton [15]). In fact Fife [3] assumed in addition that $q(t) \geq 0$, while Wintner [20] showed a stronger result which required a weaker condition involving the integral average, that is,

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)\,du\,ds = \infty. \tag{8}$$

In a different direction, Hartman [12] proved that the limit in (8) can not be replaced by the upper limit and that the following condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)\,du\,ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)\,du\,ds \leq \infty \tag{9}$$

suffices for the oscillation of equation (6).

Kamenev [13] established a new integral criterion for the oscillation of the linear ordinary differential equation (6), which has (8), the result of Wintner, as a particular case. More precisely, Kamenev showed that if for some positive integer $n > 2$,

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds = \infty, \tag{10}$$

then equation (6) is oscillatory.

Kamenev's criterion has been extended in various directions by many authors, of particular interest, we refer to Yan [22] and Philos [16]. Yan [22] proved a new oscillation criterion for equation (6), which is the following: Suppose there exist a continuous function ϕ on $[t_0, \infty)$ and an integer $n > 2$, if Kamenev's condition (10) does not hold, *i.e.* if

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds < \infty, \tag{11}$$

but the following conditions

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{u}^{t} (t-s)^{n-1} q(s)\, ds > \phi(u), \quad \forall u \geq t_0 \tag{12}$$

and

$$\int_{t_0}^{\infty} \phi_+(u)\, du = \infty, \quad \phi_+(u) = \max_{u \geq t_0}\{\phi(u), 0\} \tag{13}$$

hold, then equation (6) is oscillatory. Recently, Ruan [17] extended Yan's results to the nth-order delay differential equations.

Instead of using the auxiliary function $(t-s)^{n-1}$ as Kamenev and Yan did, Philos [16] used a general class of function $H(t, s)$ to establish oscillation criteria for the linear ordinary differential equation (6). Due to the generality of the function $H(t, s)$, Philos' results include many well-known criteria as particular cases. Some of Philos' criteria have been generalized to the second order linear matrix differential systems by Erbe, Kong and Ruan [1].

For the oscillation of the nonlinear ordinary differential equation (3), we refer to the recent paper of Wong [21] and the references cited therein.

On the other hand, it is quite natural to expect that a delay differential equation is still oscillatory under the assumptions for the corresponding ordinary differential equation. In fact, Waltman [19] extended Fife's criterion to the linear delay differential equation (5), that is, conditions $q(t) \geq 0$ and (7) imply that all solutions of equation (5) oscillate. But Travis [18] showed that Fife-Wintner-Leighton's criterion does not hold for the delay differential equation (5), *i.e.* the condition (7) alone (without the assumption that $q(t) \geq 0$) is not enough to ensure the oscillation of equation (5). Hence, the oscillation analysis of the delay differential equations is more complicated than that of ordinary differential equations.

Recently, there has been some interest in the oscillation of solutions of the second order neutral delay differential equations. The results of Waltman and Travis have been extended to neutral equations by Grammatikopoulos, Ladas and Meimaridou [8]. They proved that if

$$0 \le p(t) \le 1, \quad q(t) \ge 0 \tag{14}$$

and

$$\int_{t_0}^{\infty} q(s)[1 - p(s-\sigma)]\,ds = \infty, \tag{15}$$

then every solution of the linear neutral delay differential equation (4) is oscillatory. Grace and Lalli [4] showed that if conditions (a), (b), (c) hold and there exists a function $\rho \in C^1\big([t_0,\infty),(0,\infty)\big)$ such that

$$\int^{\infty}\left[\gamma\rho(s)q(s)\big(1-p(s-\sigma)\big) - \frac{\big(\rho'(s)\big)^2 a(s-\sigma)}{4\rho(s)}\right] ds = \infty, \tag{16}$$

then every solution of the nonlinear neutral delay differential equation (1) is oscillatory. For more results about oscillations and asymptotic properties of second order neutral delay differential equations, we refer to Grammatikopoulos, Ladas and Meimaridou [8][9], Graef, Grammatikopoulos and Spikes [6][7], Grace and Lalli [4] [5], Erbe and Zhang [2], and the recent book by Györi and Ladas [10] and the references cited therein.

In this article, by using Riccati technique and averaging functions method and following the results of Yan and Philos, we establish some general oscillation criteria for second order neutral delay differential equations. Our oscillation criteria have a general class of functions $H(t,s)$ as the parameter function. By choosing various specific functions $H(t,s)$, we are able to derive several useful corollaries. The corollaries generalize Kamenev's criterion to the neutral equations and improve the results of Grace and Lalli [4] and Grammatikopoulos, Ladas and Meimaridou [8]. The obtained oscillation criteria are new even for the delay differential equations.

2. **Main results.**

The following theorem provides sufficient conditions for the oscillation of the nonlinear neutral delay differential equation (1).

THEOREM 1. *Suppose conditions (a), (b) and (c) hold. Let $H(t,s), h(t,s): D = \{(t,s) : t \ge s \ge t_0\} \longrightarrow R$ be continuous with $H(t,t) = 0$, $\forall t \ge t_0$, $H(t,s) > 0$, $\forall t > s \ge t_0$, $H'_t(t,s) \ge 0$, $H'_s(t,s)$ nonpositive and continuous, and such that*

$$-H'_s(t,s) = h(t,s)[H(t,s)]^{\frac{1}{2}}, \quad \forall (t,s) \in D.$$

Assume further that

(d) $\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \{H(t,s)\gamma q(s)[1-p(s-\sigma)] - \frac{1}{4}h^2(t,s)a(s-\sigma)\}\,ds = \infty.$

Then the nonlinear neutral equation (1) is oscillatory.

PROOF. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of equation (1), set

$$z(t) = x(t) + p(t)x(t-\tau). \tag{17}$$

Then, in view of condition (a), we see that $z(t) > 0$ for $t \geq t_1 \geq t_0$ and

$$\big(a(t)\, z'(t)\big)' \leq 0 \quad \text{for } t \geq t_1. \tag{18}$$

Therefore, $a(t)\, z'(t)$ is a decreasing function of t. We claim that

$$z'(t) \geq 0 \quad \text{for } t \geq t_1. \tag{19}$$

Otherwise, $z'(t) < 0$, which, together with (18), implies that

$$z(t) \leq z(t_1) + a(t_1)\, z'(t_1) \int_{t_1}^{t} \frac{1}{a(s)}\, ds.$$

Hence by condition (b), we have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that $z(t) \geq 0$. Now observe that from (1) we have

$$\big(a(t)\, z'(t)\big)' + q(t) f\big(x(t-\sigma)\big) = 0. \tag{20}$$

Using (c) and (17) in (20), we get

$$\big(a(t)\, z'(t)\big)' + \gamma q(t)[z(t-\sigma) - p(t-\sigma)\, x(t-\tau-\sigma)] \leq 0,$$

which, in view of the fact that $z(t) \geq x(t)$ and $z(t)$ is monotone increasing, yields

$$\big(a(t)\, z'(t)\big)' + \gamma q(t)[1 - p(t-\sigma)]z(t-\sigma) \leq 0.$$

Define

$$w(t) = \frac{a(t)\, z'(t)}{z(t-\sigma)}, \tag{21}$$

then

$$w'(t) \leq -\gamma q(t)[1 - p(t-\sigma)] - \frac{a(t)\, z'(t)\, z'(t-\sigma)}{z^2(t-\sigma)}.$$

Using the fact that $a(t)\, z'(t)$ is decreasing, we get

$$a(t)\, z'(t) \leq a(t-\sigma)\, z'(t-\sigma), \quad t \geq t_1.$$

Thus

$$w'(t) \leq -\gamma q(t)[1 - p(t-\sigma)] - \frac{w^2(t)}{a(t-\sigma)},$$

i.e.

$$\gamma q(t)[1 - p(t-\sigma)] \leq -w'(t) - \frac{w^2(t)}{a(t-\sigma)}, \quad t \geq t_1. \tag{22}$$

For every t, u with $t > u \geq t_1$, we obtain

$$\begin{aligned}
&\int_u^t H(t,s)\gamma q(s)[1-p(s-\sigma)]\,ds \\
&\qquad \leq -\int_u^t H(t,s)\,w'(s)\,ds - \int_u^t \frac{H(t,s)}{a(s-\sigma)} w^2(s)\,ds \\
&\qquad = H(t,u)\,w(u) - \int_u^t [-H_s'(t,s)]w(s)\,ds - \int_u^t \frac{H(t,s)}{a(s-\sigma)} w^2(s)\,ds \\
&\qquad = H(t,u)\,w(u) - \int_u^t h(t,s)[H(t,s)]^{\frac{1}{2}} w(s)\,ds - \int_u^t \frac{H(t,s)}{a(s-\sigma)} w^2(s)\,ds \\
&\qquad = H(t,u)\,w(u) + \frac{1}{4}\int_u^t h^2(t,s)\,a(s-\sigma)\,ds \\
&\qquad\quad - \int_u^t \left\{ \left[\frac{H(t,s)}{a(s-\sigma)}\right]^{\frac{1}{2}} w(s) + \frac{1}{2}h(t,s)[a(s-\sigma)]^{\frac{1}{2}} \right\}^2 ds.
\end{aligned}$$

Hence, for $t > u \geq t_1 \geq t_0$ we have

$$\begin{aligned}
&\int_u^t \left\{ H(t,s)\gamma q(s)[1-p(s-\sigma)] - \frac{1}{4}h^2(t,s)\,a(s-\sigma) \right\} ds \\
&\qquad \leq H(t,s)\,w(u) - \int_u^t \left\{ \left[\frac{H(t,s)}{a(s-\sigma)}\right]^{\frac{1}{2}} w(s) + \frac{1}{2}h(t,s)[a(s-\sigma)]^{\frac{1}{2}} \right\}^2 ds. \qquad (23)
\end{aligned}$$

Since $H_s'(t,s) \leq 0$, $t_1 \geq t_0$ implies $H(t,t_1) \leq H(t,t_0)$. So for every $t \geq t_1$,

$$\int_{t_1}^t \left\{ H(t,s)\gamma q(s)[1-p(s-\sigma)] - \frac{1}{4}h^2(t,s)\,a(s-\sigma) \right\} ds \leq H(t,t_1)\,w(t_1) \leq H(t,t_0)\,w(t_1).$$

Therefore

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^t \left\{ H(t,s)\gamma q(s)[1-p(s-\sigma)] - \frac{1}{4}h^2(t,s)\,a(s-\sigma) \right\} ds \\
&\qquad \leq w(t_1) + \int_{t_0}^{t_1} \frac{H(t,s)}{H(t,t_0)}\gamma q(s)[1-p(s-\sigma)]\,ds \\
&\qquad \leq w(t_1) + \int_{t_0}^{t_1} \gamma q(s)[1-p(s-\sigma)]\,ds
\end{aligned}$$

for all $t \geq t_1$. This gives that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \left\{ H(t,s)\gamma q(s)[1-p(s-\sigma)] - \frac{1}{4}h^2(t,s)\,a(s-\sigma) \right\} ds \\
&\qquad \leq w(t_1) + \int_{t_0}^{t_1} \gamma q(s)[1-p(s-\sigma)]\,ds,
\end{aligned}$$

which contradicts condition (d). This completes the proof of Theorem 1.

REMARK 1. Theorem 1 extends Theorem 1 of Philos [16] to the second order nonlinear neutral delay differential equation (1).

We used a general class of functions $H(t, s)$ as the parameter function in Theorem 1. By choosing various specific functions $H(t, s)$, we can derive several very useful oscillation criteria.

First, let us consider the function H defined by

$$H(t, s) = (t - s)^{n-1}, \quad t \geq s \geq t_0,$$

where n is an integer with $n > 2$. Then H is continuous on $D = \{(t, s) : t \geq s \geq t_0\}$, and $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ for $t > s \geq t_0$, $H'_s(t, s)$ is continuous and nonpositive. Let

$$h(t, s) = (n - 1)(t - s)^{\frac{n-3}{2}}, \quad t \geq s \geq t_0.$$

Then we have

$$\begin{aligned} -H'_s(t, s) &= (n - 1)(t - s)^{n-2} \\ &= (n - 1)(t - s)^{\frac{n-3}{2}}(t - s)^{\frac{n-1}{2}} \\ &= h(t, s)[H(t, s)]^{\frac{1}{2}}, \quad \forall (t, s) \in D. \end{aligned}$$

Hence we have the following corollary.

COROLLARY 1. *Assume conditions (a), (b) and (c) hold. Let n be an integer with $n > 2$ such that*

$$\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \{(t - s)^{n-1} \gamma q(s)[1 - p(s - \sigma)] - \frac{(n - 1)^2}{4}(t - s)^{n-3} a(s - \sigma)\}\, ds = \infty.$$

Then the nonlinear neutral equation (1) is oscillatory.

REMARK 2. Corollary 1 gives an extension of Kamenev's criterion [13] to the second order nonlinear neutral equation (1).

REMARK 3. If in some cases that $h(t, s)$ is not well-defined for $t = s$, then it needs to be redefined as in the following corollary.

Consider the function

$$H(t, s) = \rho(t - s), \quad t \geq s \geq 0,$$

where $\rho \in C^1([0, \infty), R^+)$, $\rho(0) = 0$ and $\rho'(u) \geq 0$ for $u \geq 0$. Let

$$h(t, s) = \begin{cases} -\frac{\rho'(t-s)}{[\rho(t-s)]^{\frac{1}{2}}} & \text{if } t > s \\ 0 & \text{if } t = s\,. \end{cases}$$

Then we have

$$-H'_s(t, s) = -\rho'(t - s) = -\frac{\rho'(t - s)}{[\rho(t - s)]^{\frac{1}{2}}}[\rho(t - s)]^{\frac{1}{2}} = h(t, s)[H(t, s)]^{\frac{1}{2}}.$$

By Theorem 1, we obtain the following corollary.

COROLLARY 2. *Assume that conditions (a), (b), (c) hold and*

$$\limsup_{t\to\infty} \frac{1}{\rho(t)} \int_0^t \left\{ \gamma\rho(t-s)q(s)[1-p(s-\sigma)] - \frac{\left(\rho'(t-s)\right)^2 a(s-\sigma)}{4\rho(t-s)} \right\} ds = \infty.$$

Then the nonlinear neutral equation (1) is oscillatory.

REMARK 4. Corollary 2 improves Theorem 1 of Grace and Lalli [4].

Now, let us consider the function

$$H(t,s) = [\ln \frac{t}{s}]^{n-1}, \quad t \geq s \geq t_0, \ n > 2.$$

Clearly, $H(t,s)$ is continuous for $t \geq s \geq t_0$, $H(t,t) = 0$, $H(t,s) > 0$, $t > s \geq t_0$ and

$$H_t'(t,s) = \frac{n-1}{t}\left[\ln \frac{t}{s}\right]^{n-2} \geq 0, \quad H_s'(t,s) = -\frac{n-1}{s}\left[\ln \frac{t}{s}\right]^{n-2} \leq 0, \quad \forall t \geq s \geq t_0.$$

Let

$$h(t,s) = -\frac{n-1}{s}\left[\ln \frac{t}{s}\right]^{\frac{n-3}{2}}, \quad t \geq s \geq t_0.$$

Then

$$\begin{aligned} -H_s'(t,s) &= -\frac{n-1}{s}\left[\ln \frac{t}{s}\right]^{\frac{n-3}{2}}\left[\ln \frac{t}{s}\right]^{\frac{n-1}{2}} \\ &= h(t,s)[H(t,s)]^{\frac{1}{2}}. \end{aligned}$$

Hence, by Theorem 1 we get the following corollary.

COROLLARY 3. *Assume conditions (a), (b), (c) hold and*

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{[\ln t]^{n-1}} \int_{t_0}^t \Bigg\{ &\left[\ln \frac{t}{s}\right]^{n-1} \gamma\rho(s)\, q(s)[1-p(s-\sigma)] \\ &- \frac{(n-1)^2}{4s^2}\left[\ln \frac{t}{s}\right]^{n-3} a(s-\sigma) \Bigg\} ds = \infty. \end{aligned}$$

Then the nonlinear neutral equation (1) is oscillatory.

If $f\big(x(t)\big) = x(t)$, $a(t) = 1$, from Theorem 1 and Corollaries 1–3, we can get several oscillation criteria for the linear neutral equation (4). For example, from Corollary 1, we have the following result.

COROLLARY 4. *Assume that $0 \leq p(t) \leq 1$ and $q(t) \geq 0$. Let n be an integer with $n > 2$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t \left\{ (t-s)^{n-1} q(s)[1-p(s-\sigma)] - \frac{(n-1)^2}{4}(t-s)^{n-3} \right\} ds = \infty.$$

Then the linear neutral equation (4) is oscillatory.

REMARK 5. Corollary 4 generalizes the Kamenev's oscillation criterion to the linear neutral equation and improves Theorem 1 of Grammatikopoulos, Ladas and Meimaridou [8].

EXAMPLE. Consider the following neutral delay differential equation

$$\left(\frac{1}{t}\left(x(t)+\frac{1}{t}x(t-2\pi)\right)'\right)'+\frac{2}{t}x(t-\pi)=0,\quad t\geq 2\pi. \tag{24}$$

Choose $n=3$. Then we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}\left\{(t-s)^{n-1}q(s)[1-p(s-\sigma)]-\frac{(n-1)^2}{4}(t-s)^{n-3}\right\}ds\\
&\qquad=\limsup_{t\to\infty}\frac{1}{t^2}\int_{2\pi}^{t}\left[(t-s)^2\frac{2}{s}\left(1-\frac{1}{s-\pi}\right)-\frac{1}{s-\pi}\right]ds\\
&\qquad=\limsup_{t\to\infty}2\left[\ln t+\frac{1}{\pi}\ln\frac{t}{t-\pi}\right]+\text{constant}\\
&\qquad=\infty.
\end{aligned}$$

Hence, by Corollary 1, the neutral delay equation (24) is oscillatory.

THEOREM 2. *Let $H(t,s)$ and $h(t,s)$ be as in Theorem 1, and conditions (a), (b), (c) hold. Assume that there exists a continuous function ϕ on $[t_0,\infty)$, such that*

(e) $\limsup_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\{H(t,s)\gamma q(s)[1-p(s-\sigma)]-\frac{1}{4}h^2(t,s)\,a(s-\sigma)]\}\,ds\geq\phi(u)$, $\forall u\geq t_0$;

(f) $\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\frac{H(t,u)\phi_+^2(u)}{a(u-\sigma)}\,du=\infty$, $\phi_+(u)=\max_{u\geq t_0}\{\phi(u),0\}$.

Then the nonlinear neutral equation (1) is oscillatory.

PROOF. Suppose that $x(t)$ is an eventually positive solution of equation (1). Set $z(t)=x(t)+p(t)x(t-\tau)$, as in the proof of Theorem 1, (23) holds for all t,u with $t>u\geq t_1\geq t_0$. Hence, for $t>u\geq t_1$, we have

$$\begin{aligned}
&\frac{1}{H(t,u)}\int_u^t\left\{H(t,s)\gamma q(s)[1-p(s-\sigma)]-\frac{1}{4}h^2(t,s)a(s-\sigma)\right\}ds\\
&\qquad\leq w(u)-\frac{1}{H(t,u)}\int_u^t\left\{\left[\frac{H(t,s)}{a(s-\sigma)}\right]^{\frac{1}{2}}w(s)+\frac{1}{2}h(t,s)[a(s-\sigma)]^{\frac{1}{2}}\right\}^2ds
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\left\{H(t,s)\gamma q(s)[1-p(s-\sigma)]-\frac{1}{4}h^2(t,s)a(s-\sigma)\right\}ds\\
&\qquad\leq w(u)-\liminf_{t\to\infty}\frac{1}{H(t,u)}\int_u^t\left\{\left[\frac{H(t,s)}{a(s-\sigma)}\right]^{\frac{1}{2}}w(s)+\frac{1}{2}h(t,s)[a(s-\sigma)]^{\frac{1}{2}}\right\}^2ds
\end{aligned}$$

for all $t > u \geq t_1$. By condition (e), we obtain for $t > u \geq t_1$ that

$$w(u) \geq \phi(u) + \liminf_{t\to\infty} \frac{1}{H(t,u)} \int_u^t \left\{ \left[\frac{H(t,s)}{a(s-\sigma)} \right]^{\frac{1}{2}} w(s) + \frac{1}{2} h(t,s)[a(s-\sigma)]^{\frac{1}{2}} \right\}^2 ds,$$

which means that

$$w(u) \geq \phi(u), \quad \forall u \geq t_1 \tag{25}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,u)} \int_u^t \left\{ \left[\frac{H(t,s)}{a(s-\sigma)} \right]^{\frac{1}{2}} w(s) + \frac{1}{2} h(t,s)[a(s-\sigma)]^{\frac{1}{2}} \right\}^2 ds < \infty, \quad u \geq t_1.$$

Hence

$$\begin{aligned} \infty > &\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \left\{ \left[\frac{H(t,s)}{a(s-\sigma)} \right]^{\frac{1}{2}} w(s) + \frac{1}{2} h(t,s)[a(s-\sigma)]^{\frac{1}{2}} \right\}^2 ds \\ \geq &\liminf_{t\to\infty} \left\{ \frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)}{a(s-\sigma)} w^2(s)\,ds + \frac{1}{H(t,t_1)} \int_{t_1}^t h(t,s)[H(t,s)]^{\frac{1}{2}} w(s)\,ds \right\}. \end{aligned} \tag{26}$$

Define

$$u(t) = \frac{1}{H(t,t_1)} \int_{t_1}^t h(t,s)[H(t,s)]^{\frac{1}{2}} w(s)\,ds, \quad t > t_1.$$

$$v(t) = \frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)}{a(s-\sigma)} w^2(s)\,ds, \quad t > t_1.$$

Then (26) implies that

$$\liminf_{t\to\infty} [u(t) + v(t)] < \infty. \tag{27}$$

Consider a sequence $\{t_n\}_1^\infty$ in the interval (t_1, ∞) with $\lim_{n\to\infty} t_n = \infty$ and such that

$$\lim_{n\to\infty} [u(t_n) + v(t_n)] = \liminf_{t\to\infty} [u(t) + v(t)].$$

In view of (27), for all sufficiently large n, there exists a constant K so that

$$u(t_n) + v(t_n) < K, \quad (n = 1, 2, \ldots). \tag{28}$$

Since $u(t) > 0$, $v(t) > 0$, and $v(t)$ is monotone, (28) implies that

$$\lim_{t\to\infty} v(t) = C < \infty.$$

Using (25) we get that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)}{a(s-\sigma)} \phi_+^2(s)\,ds &\leq \lim_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)}{a(s-\sigma)} w^2(s)\,ds \\ &= C < \infty, \end{aligned}$$

which contradicts condition (f). This completes the proof.

REMARK 6. The above oscillation criterion generalizes the results of Yan [22] and Philos [16] to the nonlinear neutral equation (1).

Similar to Corollaries 1 and 3, we have the following corollaries.

COROLLARY 5. *Suppose that conditions (a), (b) and (c) hold. Let n be an integer with $n > 2$, and ϕ be a continuous function on* $[t_0, \infty)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_u^t \left\{ (t-s)^{n-1} q(s)[1-p(s-\sigma)] \right. \\ \left. - \frac{(n-1)^2}{4} (t-s)^{n-3} a(s-\sigma) \right\} ds \geq \phi(u), \quad \forall u \geq t_0$$

and

$$\lim_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^t \frac{(t-u)^{n-1} \phi_+^2(u)}{a(u-\sigma)} du = \infty.$$

Then the nonlinear neutral equation (1) is oscillatory.

COROLLARY 6. *Suppose that conditions (a), (b) and (c) hold. Let n be an integer with $n > 2$, and ϕ be a continuous function on* $[t_0, \infty)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{[\ln t]^{n-1}} \int_{t_0}^t \left\{ \left[\ln \frac{t}{s}\right]^{n-1} \gamma q(s)[1-p(s-\sigma)] \right. \\ \left. - \frac{(n-1)^2}{4s^2} \left[\ln \frac{t}{s}\right]^{n-3} a(s-\sigma) \right\} ds \geq \phi(u), \quad \forall u \geq t_0$$

and

$$\lim_{t\to\infty} \frac{1}{[\ln t]^{n-1}} \int_{t_0}^t \left[\ln \frac{t}{u}\right]^{n-1} \frac{(t-u)^{n-1} \phi_+^2(u)}{a(u-\sigma)} du = \infty.$$

Then the nonlinear neutral equation (1) is oscillatory.

REMARK 7. Similarly, from Theorem 2 and Corollaries 5 and 6, we can derive several oscillation criteria for the linear neutral delay differential equation (4).

REMARK 8. The above results also hold for the delay differential equations (2) and (5). For example, if $p(t) = 0$, Corollary 4 improves the results of Waltman [19] and Travis [18].

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