SIAM J. APPL. MATH.
Vol. 80, No. 2, pp. 929–950

ON A NETWORK MODEL OF TWO COMPETITORS WITH APPLICATIONS TO THE INVASION AND COMPETITION OF AEDES ALBOPICTUS AND AEDES AEGYPTI MOSQUITOES IN THE UNITED STATES*

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Abstract. Based on the invasion of the *Aedes albopictus* mosquitoes and the competition between *Ae. albopictus* and *Ae. aegypti* mosquitoes in the United States, we consider a two-species competition model in a network, that is, with discrete Laplacian diffusion. In the case of weak-strong competition where the invasive competitor is stronger than the local one, it is shown that solutions converge uniformly to the semipositive equilibrium such that the invasive species survives while the local species becomes extinct, and vice versa. In the case of weak-weak competition, solutions converge uniformly to the positive equilibrium such that both invasive and local species coexist. By using numerical simulations, we apply the two-species competition model in a network to explain the invasion and competition of *Ae. Albopictus* and *Ae. Aegypti* mosquitoes in the United States. It also indicates that discrete Laplacian diffusion induces different spreading speeds in different invasive directions.

Key words. biological invasion, competition, discrete Laplacian operator, global stability, network

AMS subject classifications. 35B35, 35K60, 92B05

DOI. 10.1137/19M1257950

1. Introduction.

The two prominent mosquito species, *Aedes aegypti* and *Ae. albopictus*, are the primary vectors that transmit several arboviral diseases, including chikungunya, dengue fever, yellow fever, and Zika (CDC [5], Kraemer et al. [21]). The world is presently experiencing a series of major outbreaks of these vector-borne diseases, so it is very important and necessary to understand the current distributions and movements of these mosquito vectors for successful surveillance and control programs. As a species with worldwide tropical and subtropical distribution, the *Ae. aegypti* mosquito is an insect intimately involved in the life of human beings. Whereas the *Ae. albopictus* mosquito, a most invasive species native to the tropical and subtropical areas of Southeast Asia, has spread recently to many countries (including the United States) through international travel along with the global transport of goods (Kamal et al. [20], Kraemer et al. [22]).

Ae. aegypti mosquito has been present in the United States since the 17th century (Eisen and Moore [7]). *Ae. albopictus* was first recorded in Harris County, Texas in 1985 (Sprenger and Wuithiranyagool [39]) and in northern counties in Florida

*Received by the editors April 24, 2019; accepted for publication (in revised form) February 5, 2020; published electronically April 16, 2020.
https://doi.org/10.1137/19M1257950
Funding: The work of the first author was partially supported by the NSFC through grant NSFC-11771380. The work of the second author was partially supported by the NSFC through grant NSFC-61877052, the Jiangsu Province 333 Talent, and the Jiangsu Province Qinglan Project. The work of the third author was partially supported by the NSF through grant DMS-1853622 and the CDC Southeastern Regional Center of Excellency in Vector-Borne Diseases through the Gateway Program grant 1U01-CK000510.
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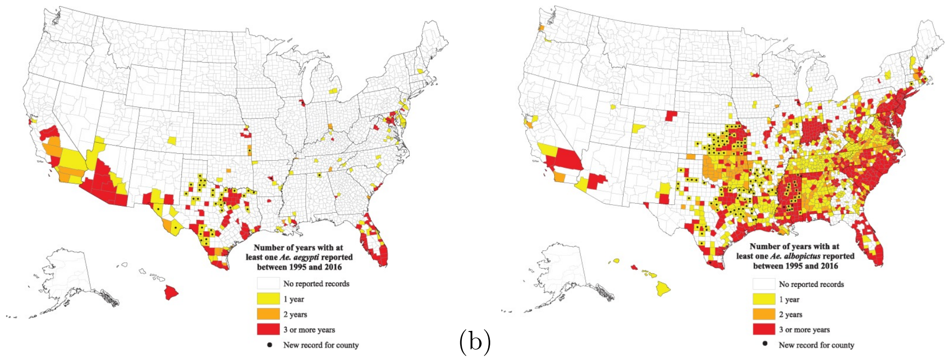

(a) (b)

FIG. 1. *Maps showing the reported occurrence of* Aedes *mosquitoes by county between January* 1995 *and December* 2016 *in the United States.* (a) Ae. aegypti *and* (b) Ae. albopictus *(Hahn et al.* [13]*).*

in 1986 (Peacock et al. [35]). This mosquito subsequently proliferated throughout much of the eastern United States and continues to expand its range (Kraemer et al. [22]). By 2008, *Ae. albopictus* had spread to 36 states and continued to expand its range (Enserink [8]). There exists between *Ae. aegypti* and *Ae. albopictus* an interspecific competition among mosquito larvae on larval, adult, and life-table traits, which affects primarily larva-to-adult survivorship and the larval development time (Noden et al. [30]). The wide spread of *Ae. albopictus* has been detected with a major decline of the local *Ae. aegypti* population in the southern continental United States (Hobbs, Hughes, and Eichold II [16], O'Meara et al. [31]). Since 1995, *Ae. aegypti* has been reported in 220 counties in 28 states and the District of Columbia, and *Ae. albopictus* in 1,368 counties in 40 states and the District of Columbia (Hahn et al. [13]; see Figure 1).

The spreading of invasive species is a central topic in ecology. Many mathematical models described by differential equations have been proposed to describe this phenomenon (Lockwood, Hoopes, and Marchetti [25], Shigesada and Kawasaki [36]). By using Laplacian operators to describe the random population diffusion, reaction-diffusion equations have been used to understand the spreading of species through the investigation of a traveling wave; see, for example, Aronson and Weinberger [2, 3], Fisher [9], Liang and Zhao [23], Skellam [37], Weinberger, Lewis, and Li [42], and the references therein. Besides its effect on traveling wave speed, random diffusion has also been found to play an important role in other ecological processes. For instance, Zhang et al. [43] provided rigorous experimental tests to show that random diffusion could drive the total population to its exceed carrying capacity. Lou and Zhou [27] used mathematical models to show that random diffusion determines the survival or extinction of species.

However, in the above-mentioned studies Laplacian operators were used to describe movements of the species where the direction of diffusion is isotropic, i.e., the probability of moving to any direction is equal. In population dynamics, species can often sense and respond to local environmental cues and resources by moving toward favorable habitats, and these movements usually depend upon a combination of local biotic and abiotic factors such as stream, climate, and food. Hence, the probabilities of moving toward all directions are not equal. In fact there are some field observations showing that the direction of biological invasion is not isotropic. Maidana and Yang [29] studied the propagation of West Nile virus from New York City to California and observed that the virus moved northward 187km, but southward 1100km. After being established in Florida in 1986, the *Ae. albopictus* mosquito was more likely to move northward and westward than southward and eastward because the eastern and southern habitats are the sea.

We plan to consider anisotropic random diffusion and our approach is to consider a few directions depending on different routines, where each routine is a connection of two discrete habitats. In the previous investigations of patch dynamics, Liao and Lou [24] used discrete diffusion to study two discrete habitats that are connected by one routine. Gourley and Ruan [12] divided the whole mosquito community into several patches where each patch is a pool with mosquito eggs living there. In modeling the transmission dynamics of infectious disease, unweighted networks have been used to describe the heterogeneous contact rate (Pastor-Satorras et al. [34]). Grigoryan, Lin, and Yang [10] investigated the dynamical behavior of weighted networks. Inspired by these approaches, we divide the habitats into a finite number of vertices where the adjacent two vertices are connected by an edge. Thus we regard the habitats of mosquitoes as a weighted network.

In this paper, we use discrete Laplacian operators defined on a network to describe the movements of mosquitoes in each vertex which depend on the topological structure of the network. For this purpose, we take into account the dispersed anisotropy and introduce the theoretic graph notions. Recall that a *undirected graph* $G = (V, E)$ contains a set $V = \{1, 2, \dots, n\}$ of vertices and a set E of edges (x, y) connecting vertex x and vertex y. G is called a *finite-dimensional graph* if it has a finite number of edges and vertices. G is called *connected* if for every pair of vertices x and y, there exists a sequence (called a *path*) of vertices $x = x_0, x_1, \dots, x_n = y$ such that x_{j-1} and x_j are connected by an edge (called *adjacent*) for $j = 1, \dots, n$. If vertex y is adjacent to vertex x, we write $y \sim x$. A graph is *weighted* if each adjacent x and y is assigned a weight function ω_{xy}. Here $\omega : V \times V \to [0, \infty)$ satisfies that $\omega_{xy} = \omega_{yx}$ and $\omega_{xy} > 0$ if and only if $x \sim y$.

For a finite subset $\Omega \subset V$, let $\partial\Omega$ denote the boundary of Ω and Ω^0 denote the interior of Ω, which are defined by

$$\partial\Omega := \{x \in \Omega : \ \exists y \in \Omega^c \text{ such that } y \text{ is adjacent to } x\}, \ \Omega^0 := \Omega \setminus \partial\Omega, \tag{1.1}$$

respectively. Throughout this paper, $G = (V, E)$ is assumed to be a connected undirected weighted finite-dimensional graph with no self-loops.

DEFINITION 1.1. *For a function $u : \Omega^0 \to \mathbb{R}$, the* discrete Laplacian Δ_ω *is defined by*

$$\Delta_\omega u(x) := \sum_{y\sim x,\ y\in\Omega^0} [u(y) - u(x)]\omega_{xy}. \tag{1.2}$$

DEFINITION 1.2. *For a function $D_\omega : \Omega^0 \to [0, \infty)$, the* degree $D_\omega(x)$ *is defined by*

$$D_\omega(x) := \sum_{y\sim x,\ y\in\Omega^0} \omega_{xy}. \tag{1.3}$$

We consider the following two-species competition model in a network:

$$(1.4)\quad \begin{cases} \frac{\partial U_1}{\partial t} - D_1\Delta_\omega U_1 = r_1U_1(1-\frac{U_1}{K_1}) - \tilde{a}_1U_1U_2, & (x,t)\in\Omega^0\times(0,+\infty),\\ \frac{\partial U_2}{\partial t} - D_2\Delta_\omega U_2 = r_2U_2(1-\frac{U_2}{K_2}) - \tilde{a}_2U_1U_2, & (x,t)\in\Omega^0\times(0,+\infty),\\ U_1(x,t)=U_2(x,t)=0, & (x,t)\in\partial\Omega\times[0,+\infty),\\ U_1(x,0)=u_{10}(x)\ge(\not\equiv)0,\ U_2(x,0)=u_{20}(x)\ge(\not\equiv)0, & x\in\Omega^0. \end{cases}$$

The biological meanings of (1.4) are described as follows: $U_1(x,t)$ represents the density of the local species (*Ae. aegypti*) and $U_2(x,t)$ represents the density of the invasive species (*Ae. albopictus*) at space location x and time t, respectively. These two species have a competition relation; D_1 and D_2 are the discrete Laplacian diffusion rates of the two species, respectively; r_1 and r_2 are the intrinsic growth rates of the two species, respectively; and K_1 and K_2 are the carrying capacities of the two species, respectively; and $\tilde{a}_1$ and $\tilde{a}_2$ are the interspecific competition rates. Δ_ω is the discrete Laplacian operator defined in (1.2). Ω^0 and $\partial\Omega$ are interior and boundary of the graph Ω defined in (1.1). Here the habitats of two mosquitoes are discretized to several patches, where each patch is described by a vertex of graph.

In order to minimize the number of parameters involved in the model, we introduce the dimensionless variables. Set

$$(1.5)\qquad u_1 = \frac{1}{K_1}U_1,\ u_2=\frac{1}{K_2}U_2,\ \bar{t}=r_1t.$$

Then omitting the bar of t, system (1.4) is rewritten as follows:

$$(1.6)\quad \begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 = u_1(1-u_1-a_1u_2), & (x,t)\in\Omega^0\times(0,+\infty),\\ \frac{\partial u_2}{\partial t} - d_2\Delta_\omega u_2 = u_2(r-a_2u_1-ru_2), & (x,t)\in\Omega^0\times(0,+\infty),\\ u_1(x,t)=u_2(x,t)=0, & (x,t)\in\partial\Omega\times[0,+\infty),\\ u_1(x,0)=u_{10}(x)\ge(\not\equiv)0,\ u_2(x,0)=u_{20}(x)\ge(\not\equiv)0, & x\in\Omega^0, \end{cases}$$

where $a_1=\frac{K_2\tilde{a}_1}{r_1}$, $r=\frac{r_2}{r_1}$, $a_2=\frac{K_1\tilde{a}_2}{r_1}$, $d_1=\frac{D_1}{r_1}$, $d_2=\frac{D_2}{r_1}$.

Our main purpose in this paper is to study the influence of the discrete Laplacian diffusion on the asymptotic behavior of the competition system (1.6). He and Ni [15] studied system (1.6) with classical Laplacian diffusion in heterogeneous environments. For weak-strong competition ($a_1>1$ and $a_2<r$) and strong-weak competition ($a_1<1$ and $a_2>r$), solutions of system (1.6) with classical Laplacian diffusion converge globally asymptotically to the semipositive equilibria $(0,1)$ and $(1,0)$, respectively. For weak-weak competition ($a_1<1$ and $a_2<r$), solutions of system (1.6) with classical Laplacian diffusion converge globally asymptotically to a unique positive equilibrium. For the corresponding ordinary differential equation (ODE) model with week-weak competition, the global stability was shown in Brown [4]. Some of the results for a reaction-diffusion system were also proved in Zhou and Pao [44] under restricted initial conditions. Goh [11] and Hsu [18] used a Liapunov functional method to study the global dynamics of some ODE models. The extension of a Liapunov functional method to Lotka–Volterra systems with a classical Laplacian operator was discussed in Hastings [14] and Hsu [19]. We will extend the global stability results from the Laplacian diffusion system to the discrete Laplacian diffusion system. Moreover, we will use numerical simulations to illustrate the short time behavior of solutions before they converge to the positive equilibrium. We would like to mention that the discrete Laplacian diffusion in problem (1.6) causes the spreading speed not to be the same in all directions at the initial invasive stage.

The rest of the article is organized as follows. In section 2 we introduce the discrete maximum principle. In section 3, we prove the global existence and uniqueness of solutions to system (1.6). In section 4, we investigate the global stability of solutions to the system according to different competitive strengths. In section 5 we carry out numerical simulations to confirm our analytical findings and illustrate the small time dynamical behavior. We also apply our model to simulate and interpret the invasion of *Ae. albopictus* mosquitoes and the competition between *Ae. aegypti* and *Ae. albopictus* in the United States. A discussion and conclusions are given in section 6.

2. Discrete maximum principle. In this section, we present the well-known maximum principle and strong maximum principle for scalar discrete Laplacian equations.

LEMMA 2.1 (maximum principle). *Suppose that $d > 0$ and K are constants. For any $T > 0$, assume that $u(x,t)$ is continuous with respect to t in $\Omega \times [0,T]$, is differentiable with respect to t in $\Omega \times (0,T]$, and satisfies*

$$\begin{cases} \frac{\partial u}{\partial t} - d\Delta_\omega u + Ku \geq 0, & (x,t) \in \Omega^0 \times (0,T], \\ u(x,t) \geq 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) \geq 0, & x \in \Omega^0. \end{cases} \tag{2.1}$$

Then $u(x,t) \geq 0$ in $\Omega \times [0,T]$.

LEMMA 2.2 (strong maximum principle). *Suppose that $d > 0$ and K are constants. For any $T > 0$, assume that $u(x,t)$ is continuous with respect to t in $\Omega \times [0,T]$, is differentiable with respect to t in $\Omega \times (0,T]$, and satisfies* (2.1). *If $u(x^*,0) > 0$ for some $x^* \in \Omega^0$, then $u(x,t) > 0$ in $\Omega^0 \times (0,T]$.*

Proof. Note that $u(x,t) \geq 0$ in $\Omega \times [0,T]$ by the above maximum principle. By (2.1), we have

$$\left(\frac{\partial u}{\partial t} - d\Delta_\omega u + Ku\right)|_{(x^*,t)} \geq 0. \tag{2.2}$$

Plugging (1.2) and (1.3) into (2.2), we have

$$\begin{aligned} \frac{\partial u(x^*,t)}{\partial t} &\geq \sum_{y\sim x^*,\ y\in\Omega^0} d[u(y,t) - u(x^*,t)]\omega_{x^*y} - Ku(x^*,t) \\ &\geq -\sum_{y\sim x^*,\ y\in\Omega^0} d\omega_{x^*y}u(x^*,t) - Ku(x^*,t) \\ &\geq -(dD_\omega(x^*) + K)u(x^*,t) \quad \text{for } t \in (0,T]. \end{aligned} \tag{2.3}$$

Since $u(x^*,0) > 0$, (2.3) implies that

$$u(x^*,t) \geq u(x^*,0)e^{-(dD_\omega(x^*)+K)t} > 0 \text{ for } t \in (0,T]. \tag{2.4}$$

We prove the lemma by contradiction. We first consider the case where $K > 0$. If $u(x,t) > 0$ in $\Omega^0 \times (0,T]$ cannot hold, there would exist a point $(x_0,t_0) \in \Omega^0 \times (0,T]$ such that $u(x_0,t_0) = \min_{\Omega^0\times(0,T]} u(x,t) = 0$. By (2.1), we have

$$\left(\frac{\partial u}{\partial t} - d\Delta_\omega u + Ku\right)|_{(x_0,t_0)} \geq 0. \tag{2.5}$$

Since u is differentiable with respect to t in $\Omega \times (0,T]$, it follows that $\frac{\partial u}{\partial t}|_{(x_0,t_0)} \leq 0$.

Thus (2.5) implies that

$$\Delta_\omega u(x_0,t_0) \le \frac{1}{d}\left(\frac{\partial u}{\partial t}|_{(x_0,t_0)} + Ku(x_0,t_0)\right) \le 0. \tag{2.6}$$

By (1.2), we also have $\Delta_\omega u|_{(x_0,t_0)} \ge 0$. Thus, we have

$$\Delta_\omega u(x_0,t_0) = 0, \text{ i.e., } \sum_{y\sim x,\ y\in\Omega^0} \omega_{xy} u(y,t_0) = 0.$$

The above equation implies that

$$u(y,t_0) = 0 \text{ for all } y \in \Omega^0 \text{ and } y \sim x_0. \tag{2.7}$$

On the other hand, since Ω^0 is connected, for any $x \in \Omega^0$, there exists a path

$$x_0 \sim x_1 \sim \cdots \sim x_n \equiv x^*.$$

By (2.7), we obtain that $u(x_1,t_0) = 0$. Employing the above argument repeatly, we shall induce $u(x_n,t_0) = 0$ in order. Therefore, we obtain $u(x^*,t_0) = 0$, which contradicts (2.4). In the case of $K \le 0$, by performing a transformation $w = e^{-\gamma t}u$, we also induce a similar contradiction. The proof is completed. ∎

In view of Lemmas 2.1 and 2.2, we obtain the following comparison principle.

Lemma 2.3 (comparison principle). *Suppose that $d > 0$, $\alpha > 0$, and $\beta > 0$ are constants. For any $T > 0$, assume that $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are continuous with respect to t in $\Omega \times [0,T]$, are differentiable with respect to t in $\Omega \times (0,T]$, and satisfy*

$$\begin{cases} \frac{\partial \overline{u}}{\partial t} - d\Delta_\omega \overline{u} \ge \overline{u}(\alpha - \beta\overline{u}), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \underline{u}}{\partial t} - d\Delta_\omega \underline{u} \le \underline{u}(\alpha - \beta\underline{u}), & (x,t) \in \Omega^0 \times (0,T], \\ \overline{u}(x,t) \ge 0 \ge \underline{u}(x,t), & (x,t) \in \partial\Omega \times [0,T], \\ \overline{u}(x,0) \ge \underline{u}(x,0), & x \in \Omega^0. \end{cases} \tag{2.8}$$

Then $\overline{u}(x,t) \ge \underline{u}(x,t)$ in $\Omega \times [0,T]$. Moreover, if $\overline{u}(x^*,0) > \underline{u}(x^*,0)$ for some $x^* \in \Omega^0$, then $\overline{u}(x,t) > \underline{u}(x,t)$ in $\Omega^0 \times (0,T]$.

3. Existence and uniqueness. For the sake of simplicity, throughout this paper we denote $\mathbf{f}(\mathbf{u}) = (f_1(u_1,u_2), f_2(u_1,u_2))$, and here

$$f_1(u_1,u_2) = u_1(1 - u_1 - a_1u_2), \quad f_2(u_1,u_2) = u_2(r - a_2u_1 - ru_2).$$

Our approach to study the existence of solutions is the method of coupled upper and lower solutions, which are defined as follows.

Definition 3.1. *Suppose that $\tilde{u}_i(x,\cdot)$, $\underset{\sim}{u}_i(x,\cdot) \in C[0,T] (i = 1,2)$ are differentiable in $(0,T]$ for each $x \in \Omega^0$, a pair of functions $\tilde{\mathbf{u}} = (\tilde{u}_1,\tilde{u}_2), \underset{\sim}{\mathbf{u}} = (\underset{\sim}{u}_1,\underset{\sim}{u}_2)$ are called* coupled upper and lower solutions *of* (1.6) *if $\tilde{\mathbf{u}} \ge \underset{\sim}{\mathbf{u}} \ge \mathbf{0}$ and*

$$\begin{cases} \frac{\partial \tilde{u}_1}{\partial t} - d_1\Delta_\omega \tilde{u}_1 \ge f_1(\tilde{u}_1, \underset{\sim}{u}_2), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \tilde{u}_2}{\partial t} - d_2\Delta_\omega \tilde{u}_2 \ge f_2(\underset{\sim}{u}_1, \tilde{u}_2), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \underset{\sim}{u}_1}{\partial t} - d_1\Delta_\omega \underset{\sim}{u}_1 \le f_1(\underset{\sim}{u}_1, \tilde{u}_2), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \underset{\sim}{u}_2}{\partial t} - d_2\Delta_\omega \underset{\sim}{u}_2 \le f_2(\tilde{u}_1, \underset{\sim}{u}_2), & (x,t) \in \Omega^0 \times (0,T], \\ \tilde{u}_1(x,t),\ \tilde{u}_2(x,t) \ge 0, & (x,t) \in \partial\Omega \times [0,T], \\ \underset{\sim}{u}_1(x,t),\ \underset{\sim}{u}_2(x,t) \le 0, & (x,t) \in \partial\Omega \times [0,T], \\ \tilde{u}_i(x,0) \ge u_{i0}(x), \underset{\sim}{u}_i(x,0) \le u_{i0}(x) \text{ for } i = 1,2, & x \in \Omega^0. \end{cases} \tag{3.1}$$

For a given pair of coupled upper and lower solutions $\tilde{\mathbf{u}}$ and $\underset{\sim}{\mathbf{u}}$, we set

$$\Lambda_i \equiv \{u_i(x,\cdot) \in C[0,T] : \ \underset{\sim}{u}_i \le u_i \le \tilde{u}_i\}, \quad \Lambda \equiv \{\mathbf{u} : \ \underset{\sim}{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}}\}. \tag{3.2}$$

There exist constants $K_i (i = 1, 2)$ such that

$$K_i u_i + \frac{\partial f_i}{\partial u_i}(\mathbf{u}) \ge 0 \text{ for } \mathbf{u} \in \Lambda. \tag{3.3}$$

In fact, as for system (1.6) it suffices to choose any K_i satisfying

$$K_1 = \max_{\mathbf{u}\in\Lambda} |1 - 2u_1 - a_1 u_2|, \ K_2 = \max_{\mathbf{u}\in\Lambda} |r - a_2 u_1 - 2r u_2|.$$

For each $i = 1, 2$, we define

$$F_1(u_1, u_2) = K_1 u_1 + f_1(u_1, u_2), \ F_2(u_1, u_2) = K_2 u_2 + f_2(u_1, u_2). \tag{3.4}$$

We consider the system

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta_\omega u_1 + K_1 u_1 = F_1(u_1, u_2), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial u_2}{\partial t} - d_2 \Delta_\omega u_2 + K_2 u_2 = F_2(u_1, u_2), & (x,t) \in \Omega^0 \times (0,T], \\ u_1(x,t) = u_2(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u_1(x,0) = u_{10}(x) \ge 0, \ u_2(x,0) = u_{20}(x) \ge 0, & x \in \Omega^0. \end{cases} \tag{3.5}$$

Then system (3.5) is equivalent to system (1.6) in a finite time interval.

By using $\underline{\mathbf{u}}^{(0)} = \underset{\sim}{\mathbf{u}}$ and $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ as the initial iterations we can construct sequences $\{\overline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ from the iterations of scalar equations as follows:

$$\begin{cases} \frac{\partial \overline{u}_1^{(m)}}{\partial t} - d_1 \Delta_\omega \overline{u}_1^{(m)} + K_1 \overline{u}_1^{(m)} = F_1(\overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \overline{u}_2^{(m)}}{\partial t} - d_2 \Delta_\omega \overline{u}_2^{(m)} + K_2 \overline{u}_2^{(m)} = F_2(\underline{u}_1^{(m-1)}, \overline{u}_2^{(m-1)}), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \underline{u}_1^{(m)}}{\partial t} - d_1 \Delta_\omega \underline{u}_1^{(m)} + K_1 \underline{u}_1^{(m)} = F_1(\underline{u}_1^{(m-1)}, \overline{u}_2^{(m-1)}), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \underline{u}_2^{(m)}}{\partial t} - d_2 \Delta_\omega \underline{u}_2^{(m)} + K_2 \underline{u}_2^{(m)} = F_2(\overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}), & (x,t) \in \Omega^0 \times (0,T], \\ \overline{u}_i^{(m)}(x,t) = \underline{u}_i^{(m)}(x,t) = 0 \text{ for } i = 1, 2, & (x,t) \in \partial\Omega \times [0,T], \\ \overline{u}_i^{(m)}(x,0) = \underline{u}_i^{(m)}(x,0) = u_{i0}(x) \text{ for } i = 1, 2, & x \in \Omega^0. \end{cases} \tag{3.6}$$

Since system (3.6) is a scalar discrete Laplacian system on networks, it follows from the local existence theorem (Chung and Choi [6, Lemma 1.8]) that the sequences $\{\overline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ and $\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ exist and are unique for a small T. Since (3.6) is a monotone dynamical system, applying a similar argument as in Smith [38], we have the following monotone property.

LEMMA 3.2. *The sequences* $\{\overline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ *and* $\{\underline{\mathbf{u}}^{(m)}\}_{m=1}^{\infty}$ *governed by* (3.6) *possess the monotone property*

$$\underset{\sim}{\mathbf{u}} \le \underline{\mathbf{u}}^{(\mathbf{m})} \le \underline{\mathbf{u}}^{(\mathbf{m+1})} \le \overline{\mathbf{u}}^{(\mathbf{m+1})} \le \overline{\mathbf{u}}^{(\mathbf{m})} \le \tilde{\mathbf{u}} \textit{ for } m = 1, 2, \ldots \tag{3.7}$$

for $(x,t) \in \Omega^0 \times [0,T]$. *Moreover, for each* $m = 1, 2, \ldots, \overline{\mathbf{u}}^{(\mathbf{m})}$ *and* $\underline{\mathbf{u}}^{(\mathbf{m})}$ *are coupled upper and lower solutions of* (1.6).

In view of Lemma 3.2, the pointwise limits

$$\lim_{m\to\infty} \overline{\mathbf{u}}^{(\mathbf{m})} = \overline{\mathbf{u}}, \quad \lim_{m\to\infty} \underline{\mathbf{u}}^{(\mathbf{m})} = \underline{\mathbf{u}} \tag{3.8}$$

exist for $(x,t) \in \Omega^0 \times [0,T]$. In the following theorem we show that $(\overline{u}_1, \underline{u}_2)$ and $(\underline{u}_1, \overline{u}_2)$ are the solutions of system (1.6).

THEOREM 3.3. *Let $\tilde{\mathbf{u}}$ and $\utilde{\mathbf{u}}$ be a pair of coupled upper and lower solutions of system* (1.6) *that are bounded on $\Omega^0 \times [0,T]$. Let $(\overline{u}_1, \underline{u}_2)$ and $(\underline{u}_1, \overline{u}_2)$ be given by* (3.8). *Then the following hold:*

(i) $(\overline{u}_1, \underline{u}_2)$ *and* $(\underline{u}_1, \overline{u}_2)$ *are the solutions of system* (1.6). *Moreover for all $m \geq 1$*

$$\utilde{\mathbf{u}} \leq \underline{\mathbf{u}}^{(\mathbf{m})} \leq \underline{\mathbf{u}}^{(\mathbf{m+1})} \leq \underline{\mathbf{u}} \leq \overline{\mathbf{u}} \leq \overline{\mathbf{u}}^{(\mathbf{m+1})} \leq \overline{\mathbf{u}}^{(\mathbf{m})} \leq \tilde{\mathbf{u}} \text{ in } \Omega^0 \times [0,T]. \tag{3.9}$$

(ii) *If $\underline{\mathbf{u}} = \overline{\mathbf{u}} (\equiv \mathbf{u}^*)$, then $\mathbf{u}^*$ is a solution of system* (1.6).

Proof. (i) By (3.6), we know that $\overline{u}_1^{(m)}$ and $\underline{u}_2^{(m)}$ are solutions of the following two scalar equations:

$$\begin{cases} \frac{\partial \overline{u}_1^{(m)}}{\partial t} - d_1\Delta_\omega \overline{u}_1^{(m)} + K_1 \overline{u}_1^{(m)} = F_1(\overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial \underline{u}_2^{(m)}}{\partial t} - d_2\Delta_\omega \underline{u}_2^{(m)} + K_2 \underline{u}_2^{(m)} = F_2(\overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}), & (x,t) \in \Omega^0 \times (0,T], \\ \overline{u}_1^{(m)}(x,t) = \underline{u}_2^{(m)}(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ \overline{u}_1^{(m)}(x,0) = u_{10}(x),\ \underline{u}_2^{(m)}(x,0) = u_{20}(x), & x \in \Omega^0. \end{cases} \tag{3.10}$$

By the local existence theorem [6, Lemma 1.8], for $(x,t) \in \Omega^0 \times (0,T]$ we have

$$\begin{aligned} \overline{u}_1^{(m)}(x,t) &= u_{10}(x) + \int_0^t (d_1\Delta_\omega \overline{u}_1^{(m)} - K_1\overline{u}_1^{(m)} + F_1(\overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}))ds, \\ \underline{u}_2^{(m)}(x,t) &= u_{20}(x) + \int_0^t (d_2\Delta_\omega \underline{u}_2^{(m)} - K_2\underline{u}_2^{(m)} + F_2(\overline{u}_1^{(m-1)}, \underline{u}_2^{(m-1)}))ds. \end{aligned} \tag{3.11}$$

Since $\utilde{u}_1 \leq \overline{u}_1^{(m)} \leq \tilde{u}_1$ and $\utilde{u}_2 \leq \underline{u}_2^{(m)} \leq \tilde{u}_2$ for $(x,t) \in \Omega^0 \times [0,T]$, the dominated convergence theorem implies that for $t \in [0,T]$ the limits $\overline{u}_1(x,t)$ and $\underline{u}_2(x,t)$ in (3.8) satisfy the relation

$$\begin{aligned} \overline{u}_1(x,t) &= u_{10}(x) + \int_0^t (d_1\Delta_\omega \overline{u}_1 - K_1\overline{u}_1 + F_1(\overline{u}_1, \underline{u}_2))ds, \\ \underline{u}_2(x,t) &= u_{20}(x) + \int_0^t (d_2\Delta_\omega \underline{u}_2 - K_2\underline{u}_2 + F_2(\overline{u}_1, \underline{u}_2))ds. \end{aligned} \tag{3.12}$$

Thus, $(\overline{u}_1, \underline{u}_2)$ is a solution of system (1.6). A similar argument shows that $(\underline{u}_1, \overline{u}_2)$ is also a solution of system (1.6). Equation (3.9) can be immediately deduced from (3.7) and (3.8).

(ii) Since $\underline{\mathbf{u}} = \overline{\mathbf{u}} (\equiv \mathbf{u}^*)$, for $(x,t) \in \Omega^0 \times (0,T]$, (3.12) becomes

$$\begin{aligned} u_1^*(x,t) &= u_{10}(x) + \int_0^t (d_1\Delta_\omega u_1^* - K_1 u_1^* + F_1(u_1^*, u_2^*))ds, \\ u_2^*(x,t) &= u_{20}(x) + \int_0^t (d_2\Delta_\omega u_2^* - K_2 u_2^* + F_2(u_1^*, u_2^*))ds. \end{aligned} \tag{3.13}$$

Hence, $\mathbf{u}^*$ is a solution of system (1.6). This completes the proof of the theorem. ∎

We extend the local solution obtained in Theorem 3.3 to the maximal time. To do so, we need the following priori estimate.

LEMMA 3.4. *Let* (u_1, u_2) *be a solution to system* (1.6) *defined for* $t \in [0, T]$ *for some* $T \in (0, \infty)$. *Then there exist constants* M_1 *and* M_2 *independent of* T *such that*

$$\begin{aligned} &0 \le u_1(x,t) \le M_1 \text{ for } (x,t) \in \Omega^0 \times [0,T], \\ &0 \le u_2(x,t) \le M_2 \text{ for } (x,t) \in \Omega^0 \times [0,T]. \end{aligned} \tag{3.14}$$

Proof. As the initial conditions $u_{i0}(x) \ge 0$ for $i = 1, 2$, we use the comparison principle to get

$$u_i(x,t) \ge 0 \text{ for } (x,t) \in \Omega^0 \times [0,T]. \tag{3.15}$$

Consequently, since (u_1, u_2) satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta_\omega u_1 = u_1(1 - u_1 - a_1 u_2) \le u_1(1 - u_1), & (x,t) \in \Omega^0 \times (0,T], \\ \frac{\partial u_2}{\partial t} - d_2 \Delta_\omega u_2 = u_2(r - a_2 u_1 - r u_2) \le u_2(r - r u_2), & (x,t) \in \Omega^0 \times (0,T], \\ u_1(x,t) = u_2(x,t) = 0, & (x,t) \in \Omega \times [0,T], \\ u_1(x,0) = u_{10}(x) \ge 0,\ u_2(x,0) = u_{20}(x) \ge 0, & x \in \Omega^0, \end{cases}$$

by choosing

$$M_1 = \max\left\{\max_{x \in \Omega^0} u_{10}(x),\ 1\right\} \text{ and } M_2 = \max\left\{\max_{x \in \Omega^0} u_{20}(x),\ 1\right\}, \tag{3.16}$$

we know that (M_1, M_2) and $(0, 0)$ are a pair of upper and lower solutions of system (1.6). Applying Theorem 3.3 immediately induces (3.14). The proof is complete. ∎

Owing to the priori estimate of Lemma 3.4, we present the following global existence theorem.

THEOREM 3.5. *System* (1.6) *possesses a unique solution for all* $t \in [0, \infty)$.

4. Stability of solutions. The main purpose of this section is to show global asymptotic stability of solutions for system (1.6). According to the strength of competitive interaction, we will discuss three types of competition relation: weak-strong competition, strong-weak competition, and weak-weak competition.

4.1. Weak-strong competition and strong-weak competition. In this subsection, we first examine the case that u_1 is an inferior competitor and u_2 is a superior competitor, namely,

$$a_1 > 1 \text{ and } a_2 < r. \tag{4.1}$$

In order to show global stability of solutions for system (1.6), we give the following lemma.

LEMMA 4.1. *Suppose that for each* $x \in \Omega^0$, $w(x, \cdot) \in C([0, \infty))$ *is differentiable in* $(0, \infty)$. *Assume that*

$$d > 0,\ \alpha > 0,\ \beta > 0 \tag{4.2}$$

are constants. If w satisfies

$$
\begin{cases} \frac{\partial w}{\partial t} - d\Delta_\omega w \geq \ (\leq) \ w(\alpha - \beta w), & (x,t) \in \Omega^0 \times (0,\infty), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ w(x,0) = w_0(x) \geq \ 0, \ \textit{and } w_0(x) \not\equiv 0, & x \in \Omega^0, \end{cases} \tag{4.3}
$$

then for any given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$
w(x,t) > \frac{\alpha}{\beta} - \varepsilon \ \left(w(x,t) < \frac{\alpha}{\beta} + \varepsilon \right) \textit{ for } (x,t) \in \Omega^0 \times [t_\varepsilon, \infty). \tag{4.4}
$$

Moreover,

$$
\liminf_{t\to\infty} w(x,t) > \frac{\alpha}{\beta} - \varepsilon \ \left(\limsup_{t\to\infty} w(x,t \right) < \frac{\alpha}{\beta} + \varepsilon) \textit{ uniformly in } x \in \Omega^0. \tag{4.5}
$$

Proof. We first show that solutions of the scalar equation

$$
\begin{cases} \frac{\partial z}{\partial t} - d\Delta_\omega z = z(\alpha - \beta z), & (x,t) \in \Omega^0 \times (0,\infty), \\ z(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ z(x,0) = w_0(x) \not\equiv 0, & x \in \Omega^0, \end{cases} \tag{4.6}
$$

converge to $\frac{\alpha}{\beta}$ uniformly in $x \in \Omega^0$.

Since $w_0(x) \not\equiv 0$ for $x \in \Omega^0$, the strong maximum principle (Lemma 2.2) implies that $z(x,t) > 0$ for $(x,t) \in \Omega^0 \times (0,\infty)$. For any small $t_1 > 0$, we set $\delta = \min_{x\in\Omega^0} z(x,t_1)$, then $\delta > 0$. Consider $\underline{z}(x,t)$ satisfying the following equation:

$$
\begin{cases} \frac{d\underline{z}}{dt} = \underline{z}(\alpha - \beta \underline{z}), & x \in \Omega^0, \ t \in (t_1,\infty), \\ \underline{z}(x,t) = 0, & x \in \partial\Omega, \ t \in [t_1,\infty), \\ \underline{z}(x,t_1) = \delta, & x \in \Omega^0. \end{cases} \tag{4.7}
$$

Since Ω^0 is finite, we have

$$
\lim_{t\to\infty} \underline{z}(x,t) = \frac{\alpha}{\beta} \text{ uniformly in } x \in \Omega^0. \tag{4.8}
$$

Moreover, owing to Definition 1.1, we have $\Delta_\omega \underline{z}(t,x) = \sum_{y\sim x} \omega_{xy}(\underline{z}(t,y) - \underline{z}(t,x)) \equiv 0$. Hence $\underline{z}$ is a lower solution of system (4.6) with $t \in [t_1,\infty)$. The comparison principle implies that $z(x,t) \geq \underline{z}(x,t)$ for $(x,t) \in \Omega^0 \times [t_1,\infty)$. Combining with (4.8), we obtain

$$
\liminf_{t\to\infty} z(x,t) \geq \frac{\alpha}{\beta} \text{ uniformly in } x \in \Omega^0. \tag{4.9}
$$

On the other hand, consider $\overline{z}(x,t)$ satisfying the following equation:

$$
\begin{cases} \frac{d\overline{z}}{dt} = \overline{z}(\alpha - \beta \overline{z}), & x \in \Omega^0, \ t \in (0,\infty), \\ \overline{z}(x,t) = 0, & x \in \partial\Omega, \ t \in [0,\infty), \\ \overline{z}(x,t_1) = \max_{x\in S} w_0(x), & x \in \Omega^0. \end{cases} \tag{4.10}
$$

Since Ω^0 is finite, we have

$$
\lim_{t\to\infty} \overline{z}(x,t) = \frac{\alpha}{\beta} \text{ uniformly in } x \in \Omega^0. \tag{4.11}
$$

Moreover, since $\overline{z}$ is an upper solution of system (4.6) with $t \in [0, \infty)$, we have $z(x,t) \leq \overline{z}(x,t)$ for $(x,t) \in \Omega^0 \times [0,\infty)$. Combining with (4.10), we obtain

$$\limsup_{t\to\infty} z(x,t) \geq \frac{\alpha}{\beta} \text{ uniformly in } x \in \Omega^0. \tag{4.12}$$

Combining (4.9) and (4.12), we deduce that

$$\lim_{t\to\infty} z(x,t) = \frac{\alpha}{\beta} \text{ uniformly in } x \in \Omega^0. \tag{4.13}$$

Next, since w satisfies (4.3), the comparison principle (Lemma 2.3) implies (4.4), which immediately shows that (4.5) holds. This completes the proof. ∎

Applying a similar argument, we obtain the following lemma.

LEMMA 4.2. *Suppose that for each* $x \in \Omega^0$*,* $w(x,\cdot) \in C([0,\infty))$ *is differentiable in* $(0,\infty)$*. Assuming that*

$$d > 0,\ \ \alpha < 0,\ \ \beta > 0 \tag{4.14}$$

are constants. If w *satisfies*

$$\begin{cases} \frac{\partial w}{\partial t} - d\Delta_\omega w \leq \ w(\alpha - \beta w), & (x,t) \in \Omega^0 \times (0,\infty), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ w(x,0) = w_0(x) \geq \ 0, & x \in \Omega^0, \end{cases} \tag{4.15}$$

then

$$\liminf_{t\to\infty} w(x,t) \leq 0 \textit{ uniformly in } x \in \Omega^0. \tag{4.16}$$

THEOREM 4.3 (weak-strong competition). *Assuming that* (4.1) *holds, then the solution* (u_1,u_2) *to system* (1.6) *satisfies*

$$\lim_{t\to\infty} (u_1,u_2) = (0,1) \textit{ uniformly in } x \in \Omega^0. \tag{4.17}$$

Proof. By Lemma 3.4, $0 \leq u_2(x,t) \leq M_2$ for $(x,t) \in \Omega_0 \times [0,\infty)$. Then we find that u_1 satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 \leq u_1(1-u_1), & (x,t) \in \Omega^0 \times (0,\infty), \\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u_1(x,0) = u_{10}(x) \not\equiv 0, & x \in \Omega^0. \end{cases}$$

Applying Lemma 4.1, for any $0 < \varepsilon_1 << 1$, we have

$$\limsup_{t\to\infty} u_1(x,t) < 1 + \varepsilon_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, there exists $t_1 > 0$ such that

$$u_1(x,t) < 1 + \varepsilon_1 \text{ for } t \geq t_1,\ x \in \Omega^0. \tag{4.18}$$

By (4.1), we can choose $\varepsilon_1 = \frac{r-a_2}{2a_2} > 0$. Plugging (4.18) into system (1.6), we see that u_2 satisfies

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_2\Delta_\omega u_2 \geq u_2(\frac{r-a_2}{2} - ru_2), & (x,t) \in \Omega^0 \times (t_1,\infty), \\ u_2(x,t) = 0, & (x,t) \in \partial\Omega \times [t_1,\infty), \\ u_2(x,t)|_{t=t_1} = u_2(x,t_1), & x \in \Omega^0. \end{cases}$$

Using Lemma 4.1, for any $0 < \varepsilon_2 << 1$, we have

$$\liminf_{t\to\infty} u_2(x,t) > \frac{r-a_2}{2r} - \varepsilon_2 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_2, we have

$$\liminf_{t\to\infty} u_2(x,t) \geq \frac{r-a_2}{2r} := \underline{v}_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, for any $0 < \varepsilon_2 << 1$, there exists $t_2 > t_1$ such that

$$u_2(x,t) > \underline{v}_1 - \varepsilon_2 \text{ for } t \geq t_2,\ x \in \Omega^0. \tag{4.19}$$

(i) Plugging (4.19) into system (1.6), we see that u_1 satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 \leq u_1(1 - u_1 - a_1\underline{v}_1 + a_1\varepsilon_2), & (x,t) \in \Omega^0 \times (t_2,\infty), \\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times [t_2,\infty), \\ u_1(x,t)|_{t=t_2} = u_1(x,t_2), & x \in \Omega^0. \end{cases}$$

Employing Lemma 4.1, for any $0 < \varepsilon_3 << 1$, we have

$$\limsup_{t\to\infty} u_1(x,t) < 1 - a_1\underline{v}_1 + a_1\varepsilon_2 + \varepsilon_3 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_2 and ε_3, it immediately follows that

$$\limsup_{t\to\infty} u_1(x,t) \leq 1 - a_1\underline{v}_1 := \overline{v}_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, for any $0 < \varepsilon_3 << 1$, there exists $t_3 > t_2$ such that

$$u_1(x,t) < \overline{v}_1 + \varepsilon_3 \text{ for } t \geq t_3,\ x \in \Omega^0. \tag{4.20}$$

(ii) Plugging (4.20) into system (1.6), we see that u_2 satisfies

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_2\Delta_\omega u_2 \geq u_2(r - a_2\overline{v}_1 - a_2\varepsilon_3 - ru_2), & (x,t) \in \Omega^0 \times (t_3,\infty), \\ u_2(x,t) = 0, & (x,t) \in \partial\Omega \times [t_3,\infty), \\ u_2(x,t)|_{t=t_3} = u_2(x,t_3), & x \in \Omega^0. \end{cases}$$

Using Lemma 4.1, for any $0 < \varepsilon_4 << 1$, we have

$$\liminf_{t\to\infty} u_2(x,t) > 1 - \frac{a_2\overline{v}_1}{r} - \frac{a_2\varepsilon_3}{r} - \varepsilon_4 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_3 and ε_4, we have

$$\liminf_{t\to\infty} u_2(x,t) \geq 1 - \frac{a_2\overline{v}_1}{r} := \underline{v}_2 \text{ uniformly in } x \in \Omega^0. \tag{4.21}$$

Consequently, for any $0 < \varepsilon_4 << 1$, there exists $t_4 > t_3$ such that

$$u_2(x,t) > \underline{v}_2 - \varepsilon_4 \text{ for } t \geq t_4,\ x \in \Omega^0. \tag{4.22}$$

(iii) Plugging (4.22) into system (1.6), we see that u_1 satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 \leq u_1(1 - u_1 - a_1\underline{v}_2 + a_1\varepsilon_4), & (x,t) \in \Omega^0 \times (t_4, \infty), \\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times [t_4, \infty), \\ u_1(x,t)|_{t=t_4} = u_1(x, t_4), & x \in \Omega^0. \end{cases}$$

Employing Lemma 4.1, for any $0 < \varepsilon_5 << 1$, we have

$$\limsup_{t\to\infty} u_1(x,t) < 1 - a_1\underline{v}_4 + a_1\varepsilon_4 + \varepsilon_5 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_4 and ε_5, it immediately follows that

$$\limsup_{t\to\infty} u_1(x,t) \leq 1 - a_1\underline{v}_2 := \overline{v}_2 \text{ uniformly in } x \in \Omega^0.$$

Consequently, for any $0 < \varepsilon_5 << 1$, there exists $t_5 > t_4$ such that

$$u_1(x,t) < \overline{v}_2 + \varepsilon_5 \text{ for } t \geq t_5,\ x \in \Omega^0. \tag{4.23}$$

(iv) Plugging (4.23) into system (1.6), we see that u_2 satisfies

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_2\Delta_\omega u_2 \geq u_2(r - a_2\overline{v}_2 - a_2\varepsilon_5 - ru_2), & (x,t) \in \Omega^0 \times (t_5, \infty), \\ u_2(x,t) = 0, & (x,t) \in \partial\Omega \times [t_5, \infty), \\ u_2(x,t)|_{t=t_5} = u_2(x, t_5), & x \in \Omega^0. \end{cases}$$

Using Lemma 4.1, for any $0 < \varepsilon_6 << 1$, we have

$$\liminf_{t\to\infty} u_2(x,t) > 1 - \frac{a_2\overline{v}_2}{r} - \frac{a_2\varepsilon_5}{r} - \varepsilon_6 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_5 and ε_6, we have

$$\liminf_{t\to\infty} u_2(x,t) \geq 1 - \frac{a_2\overline{v}_2}{r} := \underline{v}_3 \text{ uniformly in } x \in \Omega^0. \tag{4.24}$$

We obtain that (4.21) and (4.24) have the same iterative relation. Therefore, as long as the sequence $\{\underline{v}_n\}$ is monotone increasing and the sequence $\{\overline{v}_n\}$ is monotone decreasing, the condition (4.2) is naturally satisfied. We can apply Lemma 4.1 again. Repeating the above procedure such as (i), (ii), (iii), and (iv), we obtain two sequences $\{\underline{v}_n\}$ and $\{\overline{v}_n\}$, which satisfy

$$\underline{v}_1 = \frac{r - a_2}{2r},\ \overline{v}_n = 1 - a_1\underline{v}_n,\ \text{and } \underline{v}_{n+1} = 1 - \frac{a_2}{r}\overline{v}_n \text{ for } n = 1, 2 \ldots. \tag{4.25}$$

We now claim that $\{\underline{v}_n\}$ is monotone increasing and $\{\overline{v}_n\}$ is monotone decreasing under the conditions $\underline{v}_n \geq 0$ and $\overline{v}_n \geq 0$. We prove it by using an induction argument. For the case $n = 1$, since $a_2 < r$, it is easy to see that

$$\begin{aligned} &\underline{v}_2 - \underline{v}_1 = 1 - \frac{a_2}{r} - \left(1 - \frac{a_1 a_2}{r}\right)\overline{v}_1 > 1 - \frac{a_2}{r} - \overline{v}_1 = \frac{1 - a_2}{2r} > 0, \\ &\overline{v}_2 - \overline{v}_1 = -a_1(\underline{v}_2 - \underline{v}_1) < 0. \end{aligned}$$

Suppose that $\underline{v}_n - \underline{v}_{n-1} > 0$ and $\overline{v}_n - \overline{v}_{n-1} < 0$. By (4.25), we have

$$\begin{aligned} \underline{v}_{n+1} - \underline{v}_n &= -\frac{a_2}{r}(\overline{v}_n - \overline{v}_{n-1}) > 0, \\ \overline{v}_{n+1} - \overline{v}_n &= -a_1(\underline{v}_{n+1} - \underline{v}_n) < 0. \end{aligned}$$

Thus, the induction principle implies the claim.

Since the sequence $\{\underline{v}_n\}$ is monotone increasing and the sequence $\{\overline{v}_n\}$ is monotone decreasing, the limits $\lim_{n\to\infty} \underline{v}_n$ and $\lim_{n\to\infty} \overline{v}_n$ exist, denoted by $\underline{v}$ and $\overline{v}$, respectively. We now show the claim that $\underline{v} > \frac{1}{a_1}$ by contradiction. Assume that $\underline{v} \le \frac{1}{a_1}$ on the contrary. By letting $n \to \infty$, (4.25) implies that

$$\begin{cases} \underline{v} = 1 - \frac{a_2}{r}\overline{v}, \\ \overline{v} = 1 - a_1\underline{v}. \end{cases}$$

Solving the above equations, we have

$$(\overline{v}, \underline{v}) = \Big(\frac{r(1-a_1)}{r - a_1 a_2}, \frac{r - a_2}{r - a_1 a_2}\Big). \tag{4.26}$$

Since $a_1 > 1$ and $a_2 < r$, (4.26) implies that $\overline{v} < 0$ or $\underline{v} < 0$, which is a contradiction. We have shown the claim.

Since $\underline{v} > \frac{1}{a_1}$, for any given ε, we have

$$\liminf_{t\to\infty} u_2(x,t) > \underline{v} - \varepsilon \text{ uniformly in } x \in \Omega^0. \tag{4.27}$$

Consequently, there exists fixed t_n such that

$$u_2(x,t) > \underline{v} - \varepsilon \text{ for } t \ge t_n,\ x \in \Omega^0. \tag{4.28}$$

Plugging (4.28) into system (1.6), we see that u_1 satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 \le u_1(1 - u_1 - a_1\underline{v} + a_1\varepsilon), & (x,t) \in \Omega^0 \times (t_n, \infty), \\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times [t_n, \infty), \\ u_1(x,t)|_{t=t_n} = u_1(x, t_n), & x \in \Omega^0. \end{cases}$$

Since $\underline{v} > \frac{1}{a_1}$, we can choose sufficiently small ε such that $1 - a_1\underline{v} + a_1\varepsilon < 0$. Employing Lemma 4.2, we have

$$\limsup_{t\to\infty} u_1(x,t) \le 0,\ \text{uniformly in } x \in \Omega^0. \tag{4.29}$$

Similarly, we obtain

$$\liminf_{t\to\infty} u_2(x,t) \ge 1 \text{ uniformly in } x \in \Omega^0. \tag{4.30}$$

On the other hand, applying Lemmas 3.4 and 4.1, we have

$$\begin{aligned} \limsup_{t\to\infty} u_1(x,t) &\ge 0 \text{ uniformly in } x \in \Omega^0, \\ \liminf_{t\to\infty} u_2(x,t) &\le 1 \text{ uniformly in } x \in \Omega^0. \end{aligned} \tag{4.31}$$

Combining (4.29), (4.30), and (4.31), we conclude that (4.17) holds. We complete the proof. ∎

We next examine the case that u_1 is a superior competitor and u_2 is an inferior competitor, namely,

$$a_1 < 1 \text{ and } a_2 > r. \tag{4.32}$$

Since the proof of Theorem 4.3 is valid for the case of strong-weak competition, we obtain the following theorem.

Theorem 4.4 (strong-weak competition). *Assuming that* (4.32) *holds, then the solution* (u_1, u_2) *to system* (1.6) *satisfies*

$$\lim_{t\to\infty}(u_1, u_2) = (1, 0) \textit{ uniformly in } x \in \Omega^0. \tag{4.33}$$

4.2. Weak-weak competition. We now examine the case that both u_1 and u_2 are inferior competitors, namely,

$$a_1 < 1 \text{ and } a_2 < r. \tag{4.34}$$

Theorem 4.5 (weak-weak competition). *Assuming that* (4.34) *holds, then the solution* (u_1, u_2) *to system* (1.6) *satisfies*

$$\lim_{t\to\infty}(u_1, u_2) = \Big(\frac{r(1-a_1)}{r-a_1a_2}, \frac{r-a_2}{r-a_1a_2}\Big) \textit{ uniformly in } x \in \Omega^0. \tag{4.35}$$

Proof. By Lemma 3.4, we have $0 \le u_1(x,t) \le M_1$ for $(x,t) \in \Omega^0 \times [0,\infty)$. Then we find that u_2 satisfies

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_2\Delta_\omega u_2 \le u_2(r - ru_2), & (x,t) \in \Omega^0 \times (0,\infty),\\ u_2(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty),\\ u_2(x,0) = u_{20}(x) \not\equiv 0, & x \in \Omega^0. \end{cases}$$

Applying Lemma 4.1, for any $0 < \varepsilon_1 << 1$, we have

$$\limsup_{t\to\infty} u_2(x,t) < 1 + \varepsilon_1 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_1, we have

$$\limsup_{t\to\infty} u_2(x,t) \le 1 := \overline{v}_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, there exists $t_1 > 0$ such that

$$u_2(x,t) < \overline{v}_1 + \varepsilon_1 \text{ for } t \ge t_1,\ x \in \Omega^0. \tag{4.36}$$

(i) Plugging (4.36) into system (1.6), we see that u_1 satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 \ge u_1(1 - u_1 - a_1\overline{v}_1 - a_1\varepsilon_1), & (x,t) \in \Omega^0 \times (t_1,\infty),\\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times [t_1,\infty),\\ u_1(x,t)|_{t=t_1} = u_1(x,t_1), & x \in \Omega^0. \end{cases}$$

Using Lemma 4.1, for any $0 < \varepsilon_2 << 1$, we have

$$\liminf_{t\to\infty} u_1(x,t) > 1 - a_1\overline{v}_1 - a_1\varepsilon_1 - \varepsilon_2 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_1 and ε_2, we have

$$\liminf_{t\to\infty} u_1(x,t) \geq 1 - a_1\overline{v}_1 := \underline{u}_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, for any $0 < \varepsilon_2 << 1$, there exists $t_2 > t_1$ such that

$$u_1(x,t) > \underline{u}_1 - \varepsilon_2 \text{ for } t \geq t_2,\ x \in \Omega^0. \tag{4.37}$$

After a similar argument of (ii), (iii), and (iv) in Theorem 4.3, we can obtain the sequences $\{\overline{v}_n\}$ and $\{\underline{u}_n\}$, which satisfy

$$\overline{v}_1 = 1,\ \underline{u}_n = 1 - a_1\overline{v}_n,\ \text{and } \overline{v}_{n+1} = 1 - \frac{a_2}{r}\underline{u}_n \text{ for } n = 1, 2 \ldots. \tag{4.38}$$

Similar to the proof of Theorem 4.3, we can show that $\{\underline{u}_n\}$ is monotone increasing and $\{\overline{v}_n\}$ is monotone decreasing. Thus the limits $\lim_{n\to\infty}\underline{u}_n$ and $\lim_{n\to\infty}\overline{v}_n$ exist and are denoted by $\underline{u}$ and $\overline{v}$, respectively. Moreover, (4.38) implies that

$$\begin{cases} \overline{v} = 1 - \frac{a_2}{r}\underline{u}, \\ \underline{u} = 1 - a_1\overline{v}. \end{cases}$$

Solving the above equations, we have $\underline{u} = \frac{r(1-a_1)}{r-a_1a_2}$ and $\overline{v} = \frac{r-a_2}{r-a_1a_2}$. Therefore, the above argument shows that

$$\begin{aligned} &\liminf_{t\to\infty} u_1(x,t) \geq \underline{u} = \frac{r(1-a_1)}{r-a_1a_2} \text{ uniformly in } x \in \Omega^0, \\ &\limsup_{t\to\infty} u_2(x,t) \leq \overline{v} = \frac{r-a_2}{r-a_1a_2} \text{ uniformly in } x \in \Omega^0. \end{aligned} \tag{4.39}$$

It remains to show

$$\begin{aligned} &\limsup_{t\to\infty} u_1(x,t) \leq \frac{r(1-a_1)}{r-a_1a_2} \text{ uniformly in } x \in \Omega^0, \\ &\liminf_{t\to\infty} u_2(x,t) \geq \frac{r-a_2}{r-a_1a_2} \text{ uniformly in } x \in \Omega^0. \end{aligned} \tag{4.40}$$

We can employ a similar argument as in (4.39). By Lemma 3.4, we have $0 \leq u_2(x,t) \leq M_2$ for $(x,t) \in \Omega^0 \times [0,\infty)$. Then we find that u_1 satisfies

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1\Delta_\omega u_1 \leq u_1(1-u_1), & (x,t) \in \Omega^0 \times (0,\infty), \\ u_1(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u_1(x,0) = u_{10}(x) \not\equiv 0, & x \in \Omega^0. \end{cases}$$

Applying Lemma 4.1, for any $0 < \varepsilon_1 << 1$, we have

$$\limsup_{t\to\infty} u_1(x,t) < 1 + \varepsilon_1 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_1, we have

$$\limsup_{t\to\infty} u_1(x,t) \leq 1 := \overline{u}_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, there exists $t_1 > 0$ such that

$$u_1(x,t) < \overline{u}_1 + \varepsilon_1 \text{ for } t \geq t_1,\ x \in \Omega^0. \tag{4.41}$$

(v) Plugging (4.41) into system (1.6), we see that u_w satisfies

$$\begin{cases} \frac{\partial u_2}{\partial t} - d_2\Delta_\omega u_2 \geq u_2(r - a_2\overline{u}_1 - ru_2 - a_2\varepsilon_1), & (x,t) \in \Omega^0 \times (t_1, \infty), \\ u_2(x,t) = 0, & (x,t) \in \partial\Omega \times [t_1, \infty), \\ u_2(x,t)|_{t=t_1} = u_2(x,t_1), & x \in \Omega^0. \end{cases}$$

Using Lemma 4.1, for any $0 < \varepsilon_2 << 1$, we have

$$\liminf_{t\to\infty} u_2(x,t) > 1 - \frac{a_1}{r}\overline{u}_1 - \frac{a_1}{r}\varepsilon_1 - \varepsilon_2 \text{ uniformly in } x \in \Omega^0.$$

By the arbitrariness of ε_1 and ε_2, we have

$$\liminf_{t\to\infty} u_2(x,t) \geq 1 - \frac{a_1}{r}\overline{u}_1 := \underline{v}_1 \text{ uniformly in } x \in \Omega^0.$$

Consequently, for any $0 < \varepsilon_2 << 1$, there exists $t_2 > t_1$ such that

$$u_2(x,t) > \underline{v}_1 - \varepsilon_2, \text{ for } t \geq t_2, \ x \in \Omega^0. \tag{4.42}$$

Using a similar argument as in (ii), (iii), and (iv) in Theorem 4.3 again, we can obtain the sequences $\{\overline{u}_n\}$ and $\{\underline{v}_n\}$, which satisfy

$$\overline{u}_1 = 1, \ \underline{v}_n = 1 - \frac{a_2}{r}\overline{u}_n, \ \overline{u}_{n+1} = 1 - a_1\underline{v}_n \text{ for } n = 1, 2 \dots. \tag{4.43}$$

In view of the fact that $\{\overline{u}_n\}$ is monotone decreasing and $\{\underline{v}_n\}$ is monotone increasing, the limits $\lim_{n\to\infty} \overline{u}_n$ and $\lim_{n\to\infty} \underline{v}_n$ exist, and

$$\lim_{n\to\infty} \overline{u}_n = \frac{r(1-a_1)}{r - a_1a_2}, \quad \lim_{n\to\infty} \underline{v}_n = \frac{r - a_2}{r - a_1a_2}.$$

Thus, we conclude that (4.40) holds, which completes the proof. ∎

5. Numerical simulations: Applications to the invasion and competition of *Ae. albopictus* and *Ae. aegypti* in the U.S. In order to deal with the numerical solutions of system (1.6), we assume that Ω^0 has n vertices. The adjacent matrix of the network Ω^0 is denoted by G. Set $u_1(x,t) = U_i(t)$ and $u_2(x,t) = V_i(t)$. Then System (1.6) is rewritten as the following ordinary differential equations:

$$\begin{cases} \frac{dU_i}{dt} = U_i(1 - U_i - a_1V_i) + d_1\sum_{j=1}^n L_{ij}U_j & \text{for } i = 1, 2 \dots, n, \\ \frac{dV_i}{dt} = V_i(r - a_2U_i - rV_i) + d_2\sum_{j=1}^n L_{ij}V_j & \text{for } i = 1, 2 \dots, n, \\ U_i(0) = u_{10}(x)|_{x\in\Omega^0}, \ V_i(0) = u_{20}(x)|_{x\in\Omega^0} & \text{for } i = 1, 2 \dots, n, \end{cases} \tag{5.1}$$

where L is called the *Laplacian matrix*, which is defined by

$$L_{ij} = \begin{cases} G_{ij}\omega_{ij}, & j \neq i, \\ -\Sigma_{j=1}^n G_{ij}\omega_{ij}, & j = i. \end{cases}$$

Here ω_{ij} means the distance from the ith state to the jth state.

Monitored data suggest that the two species *Ae. aegypti* and *Ae. albopictus* reach relative steady states of coexistence in urban areas (Ayllón et al. [1]). The most recent survey indicates that both *Ae. aegypti* and *Ae. albopictus* occur throughout the entire state of Florida (Parker, Ramirez, and Connelly [33]). We aim to simulate

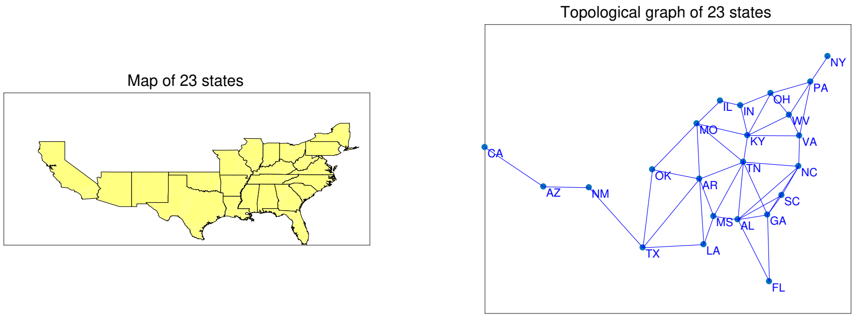

Fig. 2. *The map and corresponding topological graph of* 23 *states.*

the invasion from Florida to the nearby 22 states (Figure 2). Moreover, according to the adjacent geographical relation, we plot the topological graph Ω^0. The adjacent matrix of the network Ω^0 is defined as G, which is a matrix with 23 rows and 23 columns. For system (1.6), we consider the values for the dimensional parameters in the following unit system ($Space = [x] =$ km, $Time = [t] =$ one day):

$$\begin{aligned} &D_1 = 0.0125, D_2 = 0.025, r_1 = 0.02, r_2 = 0.02, \\ &K_1 = 25, K_2 = 25, \tilde{a}_1 = 2.5 \times 10^{-5}, \tilde{a}_2 = 4 \times 10^{-5}. \end{aligned} \tag{5.2}$$

It was reported that (Verdonschot and Besse-Lototskaya [41]) the average flight distances of *Ae. aegypti* and *Ae. albopictus* are 333m (standard deviation 384m) and 676m (standard deviation 458m), respectively. So it is biologically reasonable to choose $D_1 = 0.0125$km and $D_2 = 0.025$km, which are in agreement with the short dispersal experiments and field studies (Oteroa, Schweigmann, and Solari [32]). The other parameters are also chosen in a significant scope such as in Takahashi et al. [40].

By (1.5), we obtain the values for nondimensional parameters

$$d_1 = 0.625, d_2 = 1.25, a_1 = 0.05, a_2 = 0.08, r = 1. \tag{5.3}$$

It follows from (5.3) that *Ae. aegypti* and *Ae. albopictus* are weak-weak competitors. Theorem 4.5 implies that in the case of weak-weak competition the two mosquito species coexist and their population densities converge to the positive equilibrium values (Parker, Ramirez, and Connelly [33]). Moreover, besides the long time stability, our numerical simulations present the intermediate states during the invasion.

Figure 3 shows the invasion process of *Ae. albopictus* while competing with *Ae. aegypti*. Since we assumed that the whole state of Florida had *Ae. albopictus* (the initial data), it took 1500 days for *Ae. albopictus* to spread over the other 22 states, which was less than the real invasion time. Figure 2 shows that the degree of northward states is larger than the degree of westward states. For example, the degree of Georgia is 5 while the degree of Mississippi is 4. From Figure 3(b)–(c), we see that the direction of invasion northward is faster than westward. From a biological point of view, we see that even if in the same habitat the *Ae. albopictus* mosquito is more likely to invade cluster points than remote points. Our discrete Laplacian model (1.6) demonstrates that the degree of the graph drives *Ae. albopictus* moving faster northward than westward. Moreover, in Figure 3(b)–(c), the habitats of *Ae. albopictus* are larger than that of *Ae. aegypti*, which shows that *Ae. albopictus* is more successful than *Ae. aegypti* in invading new territories. This also agrees with field

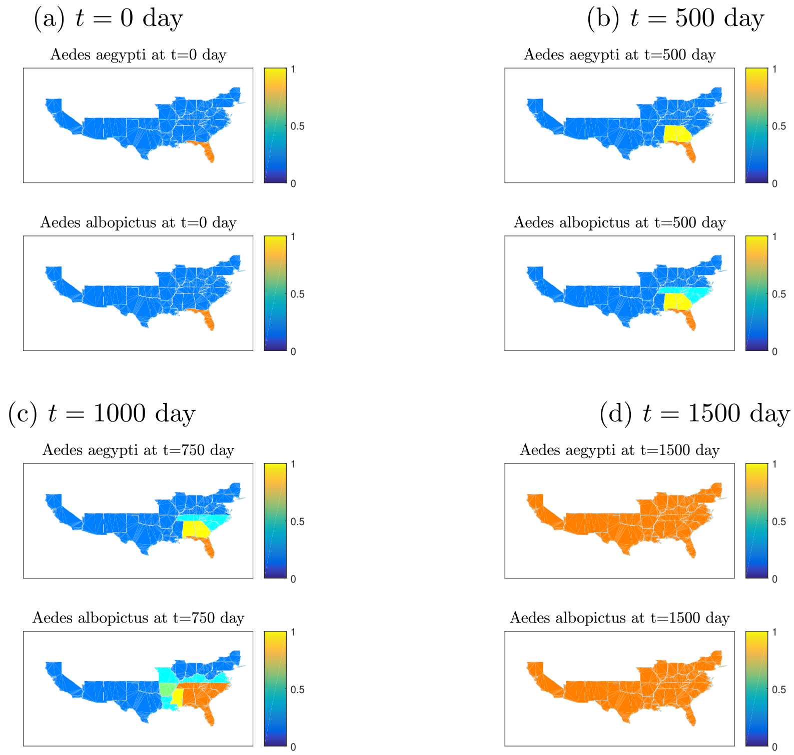

FIG. 3. *Invasion processes of* Aedes *mosquitoes over* 23 *states in* 1500 *days. The initial surviving habit is assumed to be only Florida.*

observations which indicate that *Ae. albopictus* is superior to *Ae. aegypti* in spreading into neglected and densely urbanized areas if close to vegetated areas, being capable of dispersing great distances inside forests near human dwellings, and moving easily between sylvatic and urban environments (Ayllón et al. [1]).

6. Discussions. Invasions by insect vectors of human diseases such as mosquitoes have profound effects on global public health (Lounibos [28]). *Ae. aegypti* and *Ae. albopictus* mosquitoes are two prominent transmitters of dengue fever virus, chikungunya virus, yellow fever virus, Zika virus, etc. Understanding the dispersal and invasive behavior of *Aedes* mosquitoes is essential in implementing vector control strategies and preventing and controlling mosquito-borne diseases. The *Ae. aegypti* mosquito is an invasive domestic species with tropical and subtropical worldwide distribution and *Ae. albopictus* is a most recent invasive species that has spread recently to many countries. After arriving in the U.S. in the middle 1980s (Sprenger and Wuithiranyagool [39], Peacock et al. [35]), *Ae. albopictus* mosquitoes have been competing with *Ae. aegypti* mosquitoes, coexisting with *Ae. aegypti* where *Ae. aegypti* is present, and spreading beyond the boundaries of *Ae. aegypti*'s habitats (Hahn et al. [13]). Our competition model with network (1.6) can be applied to describe the invasion of *Ae. albopictus* and the competition between *Ae. aegypti* and *Ae. albopictus*.

Though the process of biological invasion is not well-understood, researchers have been trying to explain it by using different population dynamical models. Mathematical models included reaction-diffusion equations (Aronson and Weinberger [2, 3], Fisher [9], Liang and Zhao [23], Skellam [37], Weinberger, Lewis, and Li [42]) and heterogeneous environment models (Lou [26], He and Ni [15], and Zhang et al. [43]) have been used to study the spreading of species via the investigation of traveling waves. In these studies Laplacian operators were used to describe the population dispersal in which every individual is assumed to obey the principle of Gaussian random walk, i.e., the probability of moving in any direction is equal. Our competition model in a network is different since the movement of mosquitoes in each vertex depends on the topological structure of the network. We studied the short time and long time dynamical behaviors of the invasive *Ae. albopictus* competing with *Ae. aegypti* by using a discrete Laplacian diffusion operator. To the best of our knowledge, there is no population dynamical model that takes into account the network structure to describe the biological invasion in the literature.

In the case of weak-strong competition or strong-weak competition, our results (Theorems 4.3 and 4.4) indicate that one species will win the competition. The more interesting and practical case is weak-weak competition. By Theorem 4.5, we see that the invasive *Ae. albopictus* and the local *Ae. aegypti* coexist and their densities converge to their positive equilibrium values in long time. The asymptotic behaviors are parallel to that of the reaction-diffusion model. However, our numerical simulations (Figure 3) demonstrated that the spreading speed is not the same in every direction in short time. Consequently, *Ae. albopictus* not only coexists with *Ae. aegypti* in habitats where *Ae. aegypti* presents but also expands to new territories. This dynamical feature of our model seems to agree with the empirical evidence observed in Honório et al. [17], Maidana and Yang [29], and Ayllón et al. [1].

It is known that temperature, humidity, and rainfall impact the survival of adult *Aedes* mosquitoes and the availability of oviposition sites. It will be interesting to study the effect of climate change on the dynamics of network competition models.

Acknowledgments. We would like to thank the two anonymous reviewers and the Associate Editor for their helpful comments and suggestions.

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