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NONGENERIC BIFURCATIONS NEAR HETERODIMENSIONAL CYCLES WITH INCLINATION FLIP IN $\mathbb{R}^4$

Dan Liu

Department of Mathematics, Shanghai Normal University
Shanghai 200234, China

Shigui Ruan

Department of Mathematics, University of Miami
Coral Gables, FL 33124, USA

Deming Zhu

Department of Mathematics, East China Normal University
Shanghai 200062, China

Abstract. Nongeneric bifurcation analysis near rough heterodimensional cycles associated to two saddles in $\mathbb{R}^4$ is presented under inclination flip. By setting up local moving frame systems in some tubular neighborhood of unperturbed heterodimensional cycles, we construct a Poincaré return map under the nongeneric conditions and further obtain the bifurcation equations. Coexistence of a heterodimensional cycle and a unique periodic orbit is proved after perturbations. New features produced by the inclination flip that heterodimensional cycles and homoclinic orbits coexist on the same bifurcation surface are shown. It is also conjectured that homoclinic orbits associated to different equilibria coexist.

1. **Introduction.** Newhouse and Palis [11] were the first to consider heterodimensional cycles in dynamical systems. A heteroclinic cycle is said to be equidimensional if all saddle-type periodic points in the cycle have the same dimension of the stable manifold or unstable manifold. Otherwise, such a cycle is called heterodimensional (Díaz [5]). Since different saddles in $\mathbb{R}^n$ are not necessarily identical with the dimension of their stable manifolds, heterodimensional cycles turn out to be a more general type of heteroclinic cycles than equidimensional heteroclinic cycles in practical problems. In 2005, Lamb et al [9] demonstrated that the reversible vector fields with heterodimensional cycles are dense near Hopf-zero bifurcation, which confirmed indirectly the generality of heterodimensional cases. It is worthy to note that the dimension of the concerned vector field is required to be not less than 3 because 2-dimensional heteroclinic cycles are certainly equidimensional.

For concrete heterodimensional cycles of finite dimension there are few results, especially on the bifurcation, and it is more challenging to study than the non-heterodimensional cases. The heterodimensional cycles involving two saddle-foci were found in Bykov [1] and Deng and Zhu [4]. Rademacher [13] took into account the bifurcations of heterodimensional cycles connecting one hyperbolic equilibrium

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and one hyperbolic periodic orbit. Different from the above results, in an earlier paper (Liu *et al.* [10]) the authors studied the bifurcation of heterodimensional cycles containing two saddles in three dimensional vector fields. Under some generic hypotheses, it was proved that there are a family of homoclinic cycles bifurcated from the heterodimensional cycle through small perturbations of parameters. Nonetheless, that phenomenon has not been observed in the equidimensional cases (see Shui and Zhu [14], Zhang and Zhu [18], Zhu and Xia [20]), which demonstrates that there are very rich dynamical behaviors near heterodimensional cycles although the bifurcation occurs in dimension-3 vector fields. Following this method, generic bifurcation in a very simple 4-dimensional system has also been analyzed in Geng [6].

Bifurcations on inclination flips have been developed in equidimensional cycles (see Shui and Zhu [14], Worfolk [17]). However, there is no research on heterodimensional problems concerning inclination flips in the literature. Motivated by this fact, in this paper we intend to explore possible bifurcations by an inclination flip of heterodimensional cycles in 4-dimensional vector fields and make a comparison with that of equidimensional cycles and 3-dimensional heterodimensional cycles, respectively.

The organization of the rest of this paper is as follows. In the next section, some hypotheses are given for our discussion. Based on these hypotheses, we make qualitative analysis of system (1) using the invariant manifold theory in section 3. Near the heterodimensional cycle of the unperturbed system (2), the Poincaré return map and the successor function are obtained by the establishment of a local moving frame system which was firstly introduced in Zhu [19]. Then bifurcation equations are derived by using the implicit function theorem. Section 4 presents the bifurcation results on different parameter regions and the sufficient conditions for the persistence of heterodimensional cycles, the coexistence of a heterodimensional cycle and a periodic orbit or a homoclinic orbit, and the existence of bifurcation surfaces of homoclinic orbits. A brief discussion ends the paper in section 5.

2. **Hypotheses.**

This paper is concerned with, under an inclination flip, the non-generic bifurcation analysis of the 4-dimensional C^r system

$$\dot{z} = f(z) + g(z, \mu), \tag{1}$$

where $r \geq 6,\ z \in \mathbb{R}^4,\ f(p_i) = 0,\ g(p_i, \mu) = g(z, 0) = 0$ for $i = 1, 2,\ \mu \in \mathbb{R}^l$ $(0 < ||\mu|| \ll 1,\ l \geq 3)$ is a vector of perturbation parameters. Therefore, the unperturbed system of system (1) is

$$\dot{z} = f(z). \tag{2}$$

Besides the above requirements, the following hypotheses (H_1)-(H_5) are also necessary for systems (1) and (2) throughout this paper.

(H_1) $z = p_i$ is a hyperbolic fixed point of (1). W_i^s and W_i^u are the C^r stable and unstable manifolds of $z = p_i$, respectively, for $i = 1, 2$, which satisfy $\dim(W_1^s) = 4 - \dim(W_1^u) = 1$ and $\dim(W_2^s) = \dim(W_2^u) = 2$. In addition, $\mathrm{D}f(p_1)$ has four simple real eigenvalues $-\rho,\ \lambda_1^j$ $(j = 1, 2, 3)$ satisfying $-\rho < 0 < \lambda_1^1 < \lambda_1^2 < \lambda_1^3$ and $Df(p_2)$ has four simple real eigenvalues $-\rho_2^k,\ \lambda_2^k$ $(k = 1, 2)$ satisfying $-\rho_2^2 < -\rho_2^1 < 0 < \lambda_2^1 < \lambda_2^2$, where $Df(z)$ denotes the Jacobian matrix of $f(z)$ (see Fig. 1).

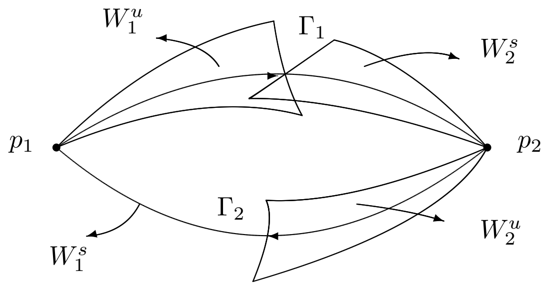

Figure 1. Manifolds of the heterodimensional cycle $\Gamma = \Gamma_1 \cup \Gamma_2$.

(H_2) System (2) has a heteroclinic cycle $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i = \{z = r_i(t) : t \in \mathrm{R}\}$, $r_1(-\infty) = r_2(+\infty) = p_1,\ r_1(+\infty) = r_2(-\infty) = p_2$.

(H_3) Define $e_i^{\pm} = \lim_{t\to\mp\infty} \frac{\dot r_i(t)}{|\dot r_i(t)|}$, then $e_i^+ \in T_{p_i} W_i^u, e_i^- \in T_{p_{i+1}} W_{i+1}^s$ are the unit eigenvectors corresponding to λ_i^1 and $-\rho_i^1$, respectively, where $\rho_1^1 = \rho$, $W_3^s = W_1^s$, $i = 1, 2$.

(H_4) $\dim(T_{r_1(t)} W_1^u \cap T_{r_1(t)} W_2^s) = 1$.

(H_5) $T_{r_2(t)} W_2^u \to \mathrm{span}\{e_2^-, e_1^+\}$ as $t \to +\infty$, where e_1^+ is the unit eigenvector corresponding to λ_1^1 (see Fig. 2).

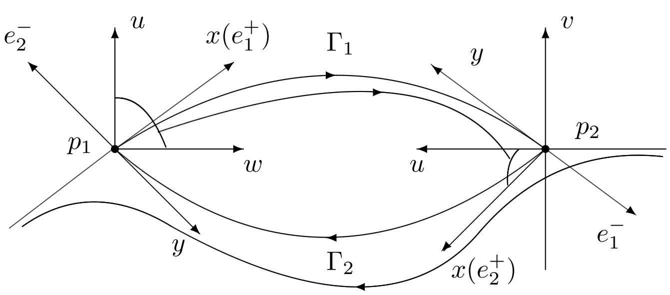

Figure 2. The local coordinate systems of $\Gamma = \Gamma_1 \cup \Gamma_2$
under the inclination flip of W_2^u.

We now introduce a definition of strong inclination property of manifolds.

Definition 2.1. W_i^u is said to have the *strong inclination property* if and only if

$$\lim_{t\to+\infty} T_{\Gamma_i} W_i^u = \mathtt{span}\{\Gamma_i'(+\infty),\ U_{i+1}^{m_i-1}\},$$

where $\Gamma_i'(+\infty)$ stands for the tangent direction of Γ_i at $t = +\infty$, $U_{i+1}^{m_i-1}$ is a subspace of $\mathbb{R}^4$ spanned by $(m_i - 1)$ strongest expanding directions of $T_{p_{i+1}} W_{i+1}^u$, $m_i = \mathtt{dim}(W_i^u)$, $p_3 = p_1$, $W_3^u = W_1^u$, $i = 1, 2$.

Likewise, the strong inclination property of W_i^s can be defined as

$$\lim_{t\to-\infty} T_{\Gamma_{i+1}} W_i^s = \mathtt{span}\{\Gamma_{i+1}'(-\infty),\ U_{i+1}^{n_i-1}\},$$

where $U_{i+1}^{n_i-1}$ is a subspace of $\mathbb{R}^4$ spanned by (n_i-1) strongest contracting directions of $T_{p_{i+1}}W_{i+1}^s$, $\Gamma_3=\Gamma_1$, $n_i=\mathtt{dim}(W_i^s)$, $i=1,2$.

Remark 1. If the invariant manifold of an equilibrium fails to satisfy the strong inclination property, we say it undergoes an *inclination flip*.

Within the above hypotheses, $(H_1)-(H_4)$ are all generic conditions, while (H_5) implies that the inclination flip occurs on W_2^u by Definition 2.1 and Remark 1. (H_1) indicates that Γ is a heterodimensional cycle. Since (H_4) means that Γ_1 is a transversal heteroclinic orbit, we can deduce that the heterodimensional cycle Γ has codimension 3, two comes from the fact that $\text{codim}(T_{r_2(t)}W_1^s\oplus T_{r_2(t)}W_2^u)=2$, and the remaining one is caused by the inclination flip of W_2^u.

3. **Preliminary analysis and bifurcation equations.** By the stable and unstable manifolds theorem and up to two local linear transformations, we see that there are two open neighborhoods $U_i\ni p_i=(0,0,0,0)^T$ $(i=1,2)$ such that p_i $(i=1,2)$ have C^{r-1} local manifolds $W_{i,loc}^s$ and $W_{i,loc}^u$, which are rendered as below:

$$\begin{aligned}
W_{1,loc}^u=\{z=(x,y,u,w)^T\in U_1|&y=y(x,u,w),y(0,0,0)=0,\\
&\frac{\partial y}{\partial(x,u,w)}(0,0,0)=\emptyset\},\\
W_{1,loc}^s=\{z=(x,y,u,w)^T\in U_1|&(x,u,w)=(x,u,w)(y),(x,u,w)(0)=\emptyset,\\
&\frac{\partial(x,u,w)}{\partial y}(0)=\emptyset\},\\
W_{2,loc}^u=\{z=(x,y,u,v)^T\in U_2|&(y,v)=(y,v)(x,u),(y,v)(0,0)=\emptyset,\\
&\frac{\partial(y,v)}{\partial(x,u)}(0,0)=\emptyset\},\\
W_{2,loc}^s=\{z=(x,y,u,v)^T\in U_2|&(x,u)=(x,u)(y,v),(x,u)(0,0)=\emptyset,\\
&\frac{\partial(x,u)}{\partial(y,v)}(0,0)=\emptyset\},
\end{aligned}$$

where the sign T means the transposition of a matrix and $\emptyset$ is a zero matrix.

Take open neighborhoods V_i such that $p_i\in V_i\subset\bar{V}_i\subset U_i$ for $i=1,2$, we can use successively straightening transformations (including the straightening of the orbit segments $\Gamma_1\cap V_1,\ \Gamma_1\cap V_2,\ \Gamma_2\cap V_2$) such that system (1) has the following C^k normal forms

$$\begin{aligned}
\dot{x}&=[\lambda_1^1(\mu)+o(1)]x+O(y)[O(u)+O(w)]\\
\dot{y}&=[-\rho(\mu)+o(1)]y\\
\dot{u}&=[\lambda_1^3(\mu)+o(1)]u+O(xy)+O(w)[O(x)+O(y)+O(w)]\\
\dot{w}&=[\lambda_1^2(\mu)+o(1)]w+O(xy)+O(u)[O(x)+O(y)+O(u)]
\end{aligned}\tag{3}$$

as $z=(x,y,u,w)^T\in V_1$, and

$$\begin{aligned}
\dot{x}&=[\lambda_2^1(\mu)+o(1)]x+O(u)[O(y)+O(v)]\\
\dot{y}&=[-\rho_2^1(\mu)+o(1)]y+O(v)[O(x)+O(u)]\\
\dot{u}&=[\lambda_2^2(\mu)+o(1)]u+O(x)[O(y)+O(v))]\\
\dot{v}&=[-\rho_2^2(\mu)+o(1)]v+O(y)[O(x)+O(u)]
\end{aligned}\tag{4}$$

as $z = (x, y, u, v)^T \in V_2$, where $k = \min\{r-3, [\lambda_1^2/\lambda_1^1]-1, [\rho_2^2/\rho_2^1]-1, [\lambda_2^2/\lambda_2^1]-1\}$, which is owing to that the curves $\Gamma_1 \cap V_1$, $\Gamma_1 \cap V_2$ and $\Gamma_2 \cap V_2$ are approximately $C^{\lambda_1^2/\lambda_1^1}$, $C^{\rho_2^2/\rho_2^1}$ and $C^{\lambda_2^2/\lambda_2^1}$, respectively.

In order to ensure the above resulting systems are at least C^3, we make another assumption.

(H_6) $\lambda_1^2 \geq 4\lambda_1^1$, $\rho_2^2 \geq 4\rho_2^1$, $\lambda_2^2 \geq 4\lambda_2^1$.

Choose $\delta > 0$ small enough and $T_1, T_2 > 0$ sufficiently large such that $r_i(-T_i) = (\delta, 0, 0, 0)^T \in V_i$, $r_i(T_i) = (0, \delta, 0, 0)^T \in V_{i+1}$, where $V_3 = V_1$. Now let us take into account the linear variational system and its corresponding adjoint system of (1) formed respectively by:

$$\dot{z} = \mathrm{D}f(r_i(t))z \tag{5}$$

and

$$\dot{\phi} = -(\mathrm{D}f(r_i(t)))^T\phi, \tag{6}$$

where $i = 1, 2$. By the above assumptions $(H_1)-(H_6)$, system (5) has exponential dichotomies in $\mathbb{R}^+$ and $\mathbb{R}^-$ (see Palmer [12] and Wiggins [16]) and the following property can be obtained.

Lemma 3.1. *System* (5) *has the fundamental solution matrices*

$$Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t)),\ i = 1, 2,$$

which satisfy, respectively,

$$\begin{array}{lcl} z_1^1(t) & = & \dot{r}_1(t)/|\dot{r}_1(-T_1)| \in T_{r_1(t)}W_1^u \cap T_{r_1(t)}W_2^s, \\ z_1^2(t),\ z_1^3(t) & \in & T_{r_1(t)}W_1^u \cap (T_{r_1(t)}W_2^s)^c, \\ z_1^4(t) & \in & (T_{r_1(t)}W_1^u)^c \cap T_{r_1(t)}W_2^s, \end{array}$$

$$Z_1(-T_1) = \begin{pmatrix} 1 & 0 & 0 & w_1^{14} \\ 0 & 0 & 0 & w_1^{24} \\ 0 & 1 & 0 & w_1^{34} \\ 0 & 0 & 1 & w_1^{44} \end{pmatrix}, \quad Z_1(T_1) = \begin{pmatrix} 0 & w_1^{12} & w_1^{13} & 0 \\ w_1^{21} & w_1^{22} & w_1^{23} & 0 \\ 0 & w_1^{32} & w_1^{33} & 0 \\ 0 & w_1^{42} & w_1^{43} & 1 \end{pmatrix},$$

where $w_1^{21} < 0$, $d_1 = w_1^{12}w_1^{33} - w_1^{13}w_1^{32} \neq 0$, $w_1^{24} \neq 0$, $|\ (w_1^{24})^{-1}w_1^{i4}\ | \ll 1$, $i \neq 2$, $d_1^{-1}w_1^{jk} \ll 1$, $j = 2, 4, k = 2, 3$, *and*

$$\begin{array}{lcl} z_2^1(t) & = & \dot{r}_2(t)/|\dot{r}_2(-T_2)| \in T_{r_2(t)}W_2^u \cap T_{r_2(t)}W_1^s, \\ z_2^2(t) & \in & T_{r_2(t)}W_2^u \cap (T_{r_2(t)}W_1^s)^c, \\ z_2^3(t),\ z_2^4(t) & \in & (T_{r_2(t)}W_2^u)^c, \end{array}$$

$$Z_2(-T_2) = \begin{pmatrix} 1 & 0 & w_2^{13} & w_2^{14} \\ 0 & 0 & w_2^{23} & w_2^{24} \\ 0 & 1 & w_2^{33} & w_2^{34} \\ 0 & 0 & w_2^{43} & w_2^{44} \end{pmatrix}, \quad Z_2(T_2) = \begin{pmatrix} 0 & w_2^{12} & 0 & 0 \\ w_2^{21} & w_2^{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $w_2^{21} < 0, w_2^{12} \neq 0, d_2 = w_2^{23}w_2^{44} - w_2^{24}w_2^{43} \neq 0, |\ (w_2^{12})^{-1}w_2^{22}\ | \ll 1, |\ d_2^{-1}w_2^{jk}\ | \ll 1, j = 1, 3, k = 3, 4$.

Proof. First w_1^{21}, $w_2^{21} < 0$ are evident since $z_i^1(t)$ is taken along the heterodimensional cycle Γ $(i = 1, 2)$. In terms of the transversality hypothesis (H_4), we derive

$$\mathbb{R}^4 = T_{r_1(T_1)}W_1^{uu} + T_{r_1(T_1)}W_2^s = \mathrm{span}\{z_1^2(T_1), z_1^3(T_1)\} + \mathrm{span}\{z_1^1(T_1), z_1^4(T_1)\},$$

where W_1^{uu} is the strong unstable manifold of p_1, which turns out that $d_1 = w_1^{12}w_1^{33} - w_1^{13}w_1^{32} \neq 0$. Then applying the Liouville formula (Hartman [7, Ch.IV, Th.1.2]) to the fundamental solution $Z_1(t)$ leads to $w_1^{24} \neq 0$.

Suppose the initial value of $z_2^2(t)$ is $z_2^2(-T_2) = (0,0,1,0)^T$ and denote $z_2^2(T_2) = (w_2^{12}, w_2^{22}, w_2^{32}, w_2^{42})^T$. Then, based on the fact that, compared with the x component, the u and w components are subject to the stronger expanding rate as $t \to +\infty$, we see that the inclination flip of W_2^u must imply that $w_2^{12} \neq 0$, $w_2^{32} = w_2^{42} = 0$, that is, $T_{r_2(T_2)} W_2^u = \text{span}\{(1,0,0,0)^T, (0,1,0,0)^T\}$. Now the Liouville formula implies that $d_2 = w_2^{23} w_2^{44} - w_2^{24} w_2^{43} \neq 0$.

All the inequalities follow directly from that the associated component is either exponentially expanding or exponentially contracting. □

Let $(z_i^1(t),\ z_i^2(t),\ z_i^3(t),\ z_i^4(t))$ be a local coordinate system along Γ_i. Denote

$$\Phi_i(t) = (\phi_i^1(t),\ \phi_i^2(t),\ \phi_i^3(t),\ \phi_i^4(t)) = (Z_i^{-1}(t))^T. \tag{7}$$

Then it is obvious to find that $\Phi_i(t)$ is a fundamental solution matrix of the adjoint system (6) for $i = 1, 2$.

Now take the transformation of coordinates $z(t) = h_i(t) = r_i(t) + Z_i(t) N_i(t)$ in the neighborhood of Γ_i, where $t \in [-T_i, T_i]$, $N_i(t) = (0, n_i^2(t), n_i^3(t), n_i^4(t))^T$, $i = 1, 2$. Define four cross sections:

$$\begin{aligned} S_1^0 &= \{z = h_1(-T_1) : |x|, |y|, |u|, |w| < 2\delta\}, \\ S_1^1 &= \{z = h_1(T_1) : |x|, |y|, |u|, |v| < 2\delta\}, \\ S_2^0 &= \{z = h_2(-T_2) : |x|, |y|, |u|, |v| < 2\delta\}, \\ S_2^1 &= \{z = h_2(T_2) : |x|, |y|, |u|, |w| < 2\delta\}, \end{aligned}$$

which intersect with the heteroclinic orbit Γ_i at $t = -T_i$ or $t = T_i$, respectively.

In order to obtain the corresponding bifurcation equation, we need to restrict our attention to set up the Poincaré return map of system (1). Consider the mappings $F_i^0 :\ q_{i-1}^1 \in S_{i-1}^1 \mapsto q_i^0 \in S_i^0$, and $F_i^1 :\ q_i^0 \in S_i^0 \mapsto q_i^1 \in S_i^1$, where $S_0^1 = S_2^1$, $q_0^1 = q_2^1$ (see Fig. 3), the construction of the Poincaré return map or successive function consists of three steps.

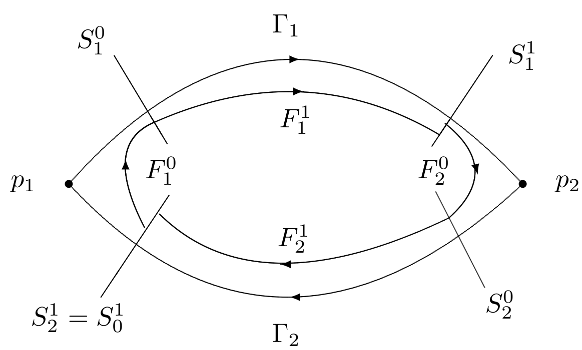

Figure 3. The Poincaré map of system (1).

First, we intend to find the relationship between the two types of coordinates of $z(t)$ and $N(t)$ in the section S_i^j and furthermore present the expressions of F_i^1 for $i = 1, 2$. Denote:

$$\begin{aligned} q_i^j &= (x_i^j, y_i^j, u_i^j, r_i^j)^T = r_i((-1)^{j+1} T_i) + Z_i((-1)^{j+1} T_i) N_i^j, \\ N_i^j &= (0, n_i^{j,2}, n_i^{j,3}, n_i^{j,4})^T \in S_i^j, \end{aligned}$$

where $i = 1, 2$, $j = 0, 1$ and $r_1^0 = w_1^0$, $r_1^1 = v_1^1$, $r_2^0 = v_2^0$, $r_2^1 = w_2^1$. Then the expressions of $Z_i(-T_i)$ and $Z_i(T_i)$ give us that $x_i^0 = \delta + o(\delta)$, $y_i^1 = \delta + o(\delta)$,

$$\begin{cases} n_1^{0,2} = u_1^0 - w_1^{34}(w_1^{24})^{-1}y_1^0, \\ n_1^{0,3} = w_1^0 - w_1^{44}(w_1^{24})^{-1}y_1^0, \\ n_1^{0,4} = (w_1^{24})^{-1}y_1^0, \end{cases} \tag{8}$$

$$\begin{cases} n_1^{1,2} = d_1^{-1}(w_1^{33}x_1^1 - w_1^{13}u_1^1), \\ n_1^{1,3} = d_1^{-1}(w_1^{12}u_1^1 - w_1^{32}x_1^1), \\ n_1^{1,4} = v_1^1 + d_1^{-1}[(w_1^{32}w_1^{43} - w_1^{33}w_1^{42})x_1^1 + (w_1^{13}w_1^{42} - w_1^{12}w_1^{43})u_1^1], \end{cases} \tag{9}$$

$$\begin{cases} n_2^{0,2} = u_2^0 - d_2^{-1}[(w_2^{33}w_2^{44} - w_2^{34}w_2^{43})y_2^0 + (w_2^{23}w_2^{34} - w_2^{24}w_2^{33})v_2^0], \\ n_2^{0,3} = d_2^{-1}(w_2^{44}y_2^0 - w_2^{24}v_2^0), \\ n_2^{0,4} = d_2^{-1}(w_2^{23}v_2^0 - w_2^{43}y_2^0), \end{cases} \tag{10}$$

and

$$\begin{cases} n_2^{1,2} = (w_2^{12})^{-1}x_0^1, \\ n_2^{1,3} = u_0^1, \\ n_2^{1,4} = w_0^1. \end{cases} \tag{11}$$

Now we suppose that $z = r_i(t) + Z_i(t)N_i(t)$ is a solution of system (1). Putting it into (1), together with the assumption of $\dot r_i(t) = f(r_i(t))$, $\dot Z_i(t) = Df(r_i(t))Z_i(t)$ and $g(r_i(t), 0) = 0$, one gets the following differential equations of $N_i(t)$:

$$\dot N_i(t) = \Phi_i^T(t) g_\mu(r_i(t), 0)\mu + \text{h.o.t.},$$

which implies that

$$N_i(T_i) = N_i(-T_i) + (\int_{-Ti}^{T_i} \Phi_i^T(t) g_\mu(r_i(t), 0)\mathrm{d}t)\mu + \text{h.o.t.} = N_i(-T_i) + M_i^j\mu + \text{h.o.t.},$$

where M_i^j $(i = 1, 2,\ j = 2, 3, 4)$ are called Melnikov vectors. As a result, the Poincaré return maps F_i^1 $(i = 1, 2)$ are solved as

$$n_i^{1,j} = n_i^{0,j} + M_i^j\mu + \text{h.o.t.},\ i = 1, 2,\ j = 2, 3, 4. \tag{12}$$

Remark 2. In fact, M_i^j is independent of the choice of T_i for $i = 1, 2,\ j = 2, 3, 4$, which can be verified similarly as in [18].

Our second task is to seek the Poincaré mappings F_i^0 induced by the flows of systems (3) and (4) in small neighborhoods U_i of p_i $(i = 1, 2)$. Let τ_i be the time it takes from q_{i-1}^1 to q_i^0, $s_1 = e^{-\lambda_1^1(\mu)\tau_1}$ and $s_2 = e^{-\rho_2^1(\mu)\tau_2}$ be the Silnikov times (Deng [3]). Since Γ is an unperturbed heteroclinic cycle, $\tau_i > 0$ will be infinite or a very large constant which corresponds to $0 \le s_i \ll 1$ for the perturbed system (1).

We use the same notations $\beta_1(\mu) = \rho(\mu)/\lambda_1^1(\mu)$ and $\beta_2(\mu) = \rho_2^1(\mu)/\lambda_2^1(\mu)$ as in Liu *et al.* [10] for simplification. From the linearly approximated solutions of (3) and (4) in the neighborhood U_i $(i = 1, 2)$, respectively, the mapping $F_1^0:\ q_0^1 = (x_0^1, y_0^1, u_0^1, w_0^1) \in S_0^1 \mapsto q_1^0 = (x_1^0, y_1^0, u_1^0, w_1^0) \in S_1^0$ can be expressed in the forms:

$$\begin{aligned} &x_0^1 = x(T_2) = s_1\delta + \text{h.o.t.} && y_1^0 = y(T_2 + \tau_1) = s_1^{\beta_1}\delta + \text{h.o.t.} \\ &u_0^1 = u(T_2) = s_1^{\lambda_1^3(\mu)/\lambda_1^1(\mu)}u_1^0 + \text{h.o.t.} && w_0^1 = w(T_2) = s_1^{\lambda_1^2(\mu)/\lambda_1^1(\mu)}w_1^0 + \text{h.o.t.} \end{aligned} \tag{13}$$

and $F_2^0:\ q_1^1 = (x_1^1, y_1^1, u_1^1, v_1^1) \in S_1^1 \mapsto q_2^0 = (x_2^0, y_2^0, u_2^0, v_2^0) \in S_2^0$ is given by

$$\begin{aligned} &x_1^1 = x(T_1) = s_2^{1/\beta_2}\delta + \text{h.o.t.} && y_2^0 = y(T_1 + \tau_2) = s_2\delta + \text{h.o.t.} \\ &u_1^1 = u(T_1) = s_2^{\lambda_2^2(\mu)/\rho_2^1(\mu)}u_2^0 + \text{h.o.t.} && v_2^0 = w(T_1 + \tau_2) = s_2^{\rho_2^2(\mu)/\rho_2^1(\mu)}v_1^1 + \text{h.o.t.} \end{aligned} \tag{14}$$

In the final step, we compose the mappings F_i^1 and F_i^0 by merging (8), (10) and (12) as follows:

$$F_1 = F_1^1 \circ F_1^0 : \begin{cases} n_1^{1,2} = u_1^0 - w_1^{34}(w_1^{24})^{-1}y_1^0 + M_1^2\mu + \text{h.o.t.} \\ n_1^{1,3} = w_1^0 - w_1^{44}(w_1^{24})^{-1}y_1^0 + M_1^3\mu + \text{h.o.t.} \\ n_1^{1,4} = (w_1^{24})^{-1}y_1^0 + M_1^4\mu + \text{h.o.t.} \end{cases} \tag{15}$$

and

$$F_2 = F_2^1 \circ F_2^0 : \begin{cases} n_2^{1,2} = u_2^0 - d_2^{-1}[(w_2^{33}w_2^{44} - w_2^{34}w_2^{43})y_2^0 + (w_2^{23}w_2^{34} \\ \quad -w_2^{24}w_2^{33})v_2^0] + M_2^2\mu + \text{h.o.t.} \\ n_2^{1,3} = d_2^{-1}(w_2^{44}y_2^0 - w_2^{24}v_2^0) + M_2^3\mu + \text{h.o.t.} \\ n_2^{1,4} = d_2^{-1}(w_2^{23}v_2^0 - w_2^{43}y_2^0) + M_2^4\mu + \text{h.o.t.} \end{cases} \tag{16}$$

Let G be the successor function associated with the heterodimensional cycle, then it can be worked out by considering the difference of the transitional variables $n_i^{1,j}$ between (9) and (15), (11) and (16), that is, $G = (G_1, G_2) = (F_1(q_0^1) - q_1^1, F_2(q_1^1) - q_0^1)$. Using the expressions of solutions (13) and (14), the successor function is expressed as

$$\begin{aligned} G_1^2 &= u_1^0 - \delta w_1^{34}(w_1^{24})^{-1}s_1^{\beta_1} + M_1^2\mu - d_1^{-1}(\delta w_1^{33}s_2^{1/\beta_2} - w_1^{13}s_2^{\lambda_2^2/\rho_2^1}u_2^0) + \text{h.o.t.} \\ G_1^3 &= w_1^0 - \delta w_1^{44}(w_1^{24})^{-1}s_1^{\beta_1} + M_1^3\mu - d_1^{-1}(w_1^{12}s_2^{\lambda_2^2/\rho_2^1}u_2^0 - \delta w_1^{32}s_2^{1/\beta_2}) + \text{h.o.t.} \\ G_1^4 &= \delta(w_1^{24})^{-1}s_1^{\beta_1} + M_1^4\mu - v_1^1 - d_1^{-1}[\delta(w_1^{32}w_1^{43} - w_1^{33}w_1^{42})s_2^{1/\beta_2} + (w_1^{13}w_1^{42} \\ &\qquad -w_1^{12}w_1^{43})s_2^{\lambda_2^2/\rho_2^1}u_2^0] + \text{h.o.t.} \\ G_2^2 &= u_2^0 - d_2^{-1}[\delta(w_2^{33}w_2^{44} - w_2^{34}w_2^{43})s_2 + (w_2^{23}w_2^{34} - w_2^{24}w_2^{33})s_2^{\rho_2^2/\rho_2^1}v_1^1] + M_2^2\mu \\ &\quad -\delta(w_2^{12})^{-1}s_1 + \text{h.o.t.} \\ G_2^3 &= d_2^{-1}(\delta w_2^{44}s_2 - w_2^{24}s_2^{\rho_2^2/\rho_2^1}v_1^1) + M_2^3\mu - s_1^{\lambda_1^3/\lambda_1^1}u_1^0 + \text{h.o.t.} \\ G_2^4 &= d_2^{-1}(w_2^{23}s_2^{\rho_2^2/\rho_2^1}v_1^1 - \delta w_2^{43}s_2) + M_2^4\mu - s_1^{\lambda_1^2/\lambda_1^1}w_1^0 + \text{h.o.t.} \end{aligned}$$

where all variables appearing in the exponents depend on the bifurcation parameter μ, which is dropped in denotations here and likewise in the sequel for simplification.

To determine if there are any periodic orbits, homoclinic or heteroclinic cycles bifurcated from the heterodimensional cycle when system (1) is perturbed slightly by μ, it is sufficient to verify if the equation

$$G = (G_1^2, G_1^3, G_1^4, G_2^2, G_2^3, G_2^4) = 0 \tag{17}$$

has any solution $(s_1, s_2, u_1^0, w_1^0, u_2^0, v_1^1)$ satisfying $s_1 \geq 0$ and $s_2 \geq 0$. Concretely, the existence of solution pairs (a) $s_1 > 0,\ s_2 > 0$, (b) $s_1 > 0,\ s_2 = 0$ (or $s_1 = 0,\ s_2 > 0$) and (c) $s_1 = 0,\ s_2 = 0$ of (17) correspond to, respectively, the existence of a periodic orbit, a homoclinic orbit and a heteroclinic cycle of system (1).

Obviously, we can see from the Jacobian matrix that the equation $(G_1^2, G_1^3, G_1^4, G_2^2) = 0$ has a unique solution $(u_1^0, w_1^0, v_1^1, u_2^0)$ in a sufficiently small neighborhood of $(s_1, s_2, \mu) = (0, 0, 0)$ due to the implicit function theorem, where

$$\begin{aligned} u_1^0 &= \delta w_1^{34}(w_1^{24})^{-1}s_1^{\beta_1} + \delta w_1^{33}d_1^{-1}s_2^{1/\beta_2} - M_1^2\mu + \text{h.o.t.} \\ w_1^0 &= \delta w_1^{44}(w_1^{24})^{-1}s_1^{\beta_1} - \delta w_1^{32}d_1^{-1}s_2^{1/\beta_2} - M_1^3\mu + \text{h.o.t.} \\ v_1^1 &= \delta(w_1^{24})^{-1}s_1^{\beta_1} + \delta(w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{1/\beta_2} + M_1^4\mu + \text{h.o.t.} \\ u_2^0 &= \delta(w_2^{12})^{-1}s_1 + \delta(w_2^{33}w_2^{44} - w_2^{34}w_2^{43})d_2^{-1}s_2 - M_2^2\mu + \text{h.o.t.} \end{aligned} \tag{18}$$

Substituting (18) into the equations $G_2^3 = 0$ and $G_2^4 = 0$, the next lemma is followed.

Lemma 3.2. *The bifurcation equations corresponding to the heterodimensional cycle* Γ *consist of the following two equations:*

$$\begin{aligned} w_2^{44}s_2 &= w_1^{34}(w_1^{24})^{-1}d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} + w_1^{33}d_1^{-1}d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}} s_2^{\frac{1}{\beta_2}} - \delta^{-1}d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}} M_1^2\mu - \delta^{-1}d_2 M_2^3\mu \\ &\quad + w_2^{24}s_2^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + (w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{\frac{1}{\beta_2}} + \delta^{-1}M_1^4\mu] + h.o.t. \\ w_2^{43}s_2 &= -w_1^{44}(w_1^{24})^{-1}d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + w_1^{32}d_1^{-1}d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1}} s_2^{\frac{1}{\beta_2}} + \delta^{-1}d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3\mu + \delta^{-1}d_2 M_2^4\mu \\ &\quad + w_2^{23}s_2^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + (w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{\frac{1}{\beta_2}} + \delta^{-1}M_1^4\mu] + h.o.t., \end{aligned} \tag{19}$$

where $0 \leq s_1,\ s_2 \ll 1,\ |\mu| \ll 1.$

Remark 3. The two equations of (19) have a symmetrical structure which distinguish from previous bifurcation equations of unsymmetrical cycles and can exhibit richer bifurcation behaviors.

4. **Bifurcation results.** In this section we first study the nonnegative solutions (s_1, s_2) of system (19) to determine the bifurcation surfaces for the persistence of the primary heterodimensional cycle or occurrence of new bifurcated singular orbits after perturbation. Then we give bifurcation results on the coexistence of a heterodimensional cycle and a periodic or homoclinic orbit. The next theorem provides conditions to ensure that the heterodimensional cycle persists even if undergoing some slight perturbation of bifurcation parameters.

Theorem 4.1. *Suppose that hypotheses* $(H_1) - (H_6)$ *hold and* M_2^3 *and* M_2^4 *are two linearly independent vectors. Then there exists an* $(l-2)$*-dimensional surface*

$$L_{12} = \{\mu :\ M_2^3\mu + h.o.t. = M_2^4\mu + h.o.t. = 0\}$$

with a normal plane $\Sigma_{12} = span\{M_2^3,\ M_2^4\}$ *at* $\mu = 0$ *such that system (1) has only one heteroclinic cycle* $\Gamma(\mu)$ *in the small tube neighborhood of* Γ *when* $\mu \in L_{12}$ *and* $0 < |\mu| \ll 1$.

The proof is a direct application of the implicit function theorem after setting $s_1 = s_2 = 0$ in equation (19), so it is omitted here.

In the following, we will develop our discussion under the condition $1/\beta_2 > \beta_1 > 1$ to look for other possible singular orbits near the heterodimensional cycle on the bifurcation surface L_{12}. The bifurcation analysis for other cases, such as $1/\beta_2 > 1 > \beta_1$, $1 > 1/\beta_2 > \beta_1$ as well as $\beta_1\beta_2 > 1$, can be investigated in a similar way.

From the expression of $Z_2(-T_2)$ in Lemma 3.1, we have $d_2 \neq 0$, which shows that $(w_2^{43})^2 + (w_2^{44})^2 \neq 0$. In other words, there are three possible situations: $w_2^{43}w_2^{44} \neq 0$, $w_2^{43} = 0, w_2^{44} \neq 0$ or $w_2^{43} \neq 0, w_2^{44} = 0$. First of all, we discuss the bifurcation analysis when neither w_2^{43} nor w_2^{44} is vanished.

Theorem 4.2. *Suppose that hypotheses* $(H_1) - (H_6)$ *are valid,* Rank(M_2^3, M_2^4)=2, $1/\beta_2 > \beta_1 > 1$ *and* $w_2^{43}w_2^{44} \neq 0$. *Then for* $\mu \in L_{12}$ *and* $0 < |\mu| \ll 1$, *the following results hold.*

(1) If $w_1^{44} \neq 0$ *and* $\rho + \lambda_1^2 < \lambda_1^3$, *then when* μ *lies in the region* $\{\mu : w_1^{24}w_1^{44}M_1^3\mu > 0,\ \omega(s_1^*, \mu) > 0\}$, *there exists one unique periodic orbit. When* $\mu \in \{\mu : w_1^{24}w_1^{44}M_1^3\mu > 0,\ \omega(s_1^*, \mu) = 0\}$, *there exists one unique homoclinic orbit associated to* p_2, *where* $\omega(s_1^*, \mu)$ *is defined as in the following (23). When*

$\mu \in \{\mu : w_1^{24}w_1^{44}M_1^3\mu > 0,\ \omega(s_1^*,\mu) < 0\}$ or $\{\mu : w_1^{24}w_1^{44}M_1^3\mu < 0\}$, there are no periodic or homoclinic orbits near Γ.

(2) If $w_1^{44} \neq 0$ and $\rho+\lambda_1^2 > \lambda_1^3$, then when μ lies in the region $\{\mu :\ d_2w_2^{44}M_1^2\mu < 0,\ |M_1^3\mu| \ll |M_1^2\mu|^{\frac{\alpha_1}{\alpha_1-1}},\ w_1^{24}w_1^{44}w_2^{43}w_2^{44}M_1^2\mu > 0\}$ or $\{\mu :\ d_2w_2^{44}M_1^2\mu < 0,\ |M_1^3\mu| \gg |M_1^2\mu|^{\frac{\alpha_1}{\alpha_1-1}},\ w_1^{24}w_1^{44}M_1^3\mu > 0\}$, there exists one unique periodic orbit, where $\alpha_1 = \frac{\lambda_1^1\beta_1}{\lambda_1^3-\lambda_1^2} > 1$. When $d_2w_2^{44}M_1^2\mu > 0$, system (1) has no periodic or homoclinic orbits near Γ.

(3) If $w_1^{44} = 0$ and $w_1^{34} \neq 0$, then when μ lies in the region $\{\mu :\ d_2w_2^{43}M_1^3\mu > 0,\ |M_1^3\mu| \ll |M_1^2\mu|^{\frac{\alpha_2}{\alpha_2-1}},\ w_1^{24}w_1^{34}M_1^2\mu > 0\}$ or $\{\mu :\ d_2w_2^{43}M_1^3\mu > 0,\ |M_1^3\mu| \ll |M_1^2\mu|^{\frac{\alpha_2}{\alpha_2-1}},\ w_1^{24}w_1^{34}w_2^{43}w_2^{44}M_1^3\mu > 0\}$, there exists one unique periodic orbit, where $\alpha_2 = 1+\frac{\lambda_1^1\beta_1}{\lambda_1^3-\lambda_1^2} > 1$. When $d_2w_2^{43}M_1^3\mu < 0$, system (1) has no periodic or homoclinic orbits near Γ.

(4) If $w_1^{44} = w_1^{34} = 0$, then when μ satisfies $d_2w_2^{43}M_1^3\mu > 0$, $|M_1^3\mu| \ll |M_1^2\mu|$ and $w_2^{43}w_2^{44}M_1^2\mu M_1^3\mu < 0$, there exists one unique periodic orbit. Otherwise, system (1) has neither periodic orbits nor homoclinic orbits near Γ.

Proof. When $1/\beta_2 > 1$ and $w_2^{43}w_2^{44} \neq 0$, bifurcation equations (19) are reduced to

$$\begin{aligned} w_2^{44}s_2 &= w_1^{34}(w_1^{24})^{-1}d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} - \delta^{-1}d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}}M_1^2\mu + \text{h.o.t.}\\ w_2^{43}s_2 &= -w_1^{44}(w_1^{24})^{-1}d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + \delta^{-1}d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + \text{h.o.t.}\end{aligned} \tag{20}$$

for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$. Eliminating s_2 in (20), it follows that:

$$\begin{aligned} &w_1^{44}(w_1^{24}w_2^{43})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + w_1^{34}(w_1^{24}w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1}\\ &= \delta^{-1}[(w_2^{43})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + (w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}}M_1^2\mu] + \text{h.o.t.}\end{aligned} \tag{21}$$

(1) When $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1}+\beta_1 < \frac{\lambda_1^3}{\lambda_1^1}$ (i.e. $\lambda_1^2+\rho < \lambda_1^3$) hold, (21) reads as

$$w_1^{44}(w_1^{24})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} = \delta^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + \text{h.o.t.} \tag{22}$$

As $w_1^{24}w_1^{44}M_1^3\mu < 0$, (22) has no nonnegative solutions which means that system (1) has neither periodic orbits nor homoclinic orbits near Γ. Whereas as $w_1^{24}w_1^{44}M_1^3\mu > 0$, it is easy to see that (22) has only one positive solution $s_1^* = [\delta^{-1}w_1^{24}(w_1^{44})^{-1}M_1^3\mu]^{\frac{1}{\beta_1}} + \text{h.o.t.} \ll 1$. Substituting s_1^* into the first equation of (20), we can find that

$$s_2^* = \delta^{-1}(s_1^*)^{\frac{\lambda_1^3}{\lambda_1^1}}(w_2^{44})^{-1}d_2[w_1^{34}(w_1^{44})^{-1}M_1^3\mu - M_1^2\mu] + \text{h.o.t.} = \omega(s_1^*,\mu), \tag{23}$$

which implies the conclusion of (1).

(2) In case $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1}+\beta_1 > \frac{\lambda_1^3}{\lambda_1^1}$ (i.e. $\lambda_1^2+\rho > \lambda_1^3$), (21) can be simplified to

$$w_1^{44}(w_1^{24}w_2^{43})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} = \delta^{-1}[(w_2^{43})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + (w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}}M_1^2\mu] + \text{h.o.t.} \tag{24}$$

Since we are interested in the coexistence of periodic orbits or homoclinic orbits with the heterodimensional cycle, the case $s_1 = 0$ is not considered any more. Suppose

$s_1 > 0$ is small enough, by eliminating the term $s_1^{\lambda_1^2/\lambda_1^1}$ from both sides of (24), we obtain

$$w_1^{44}(w_1^{24}w_2^{43})^{-1}s_1^{\beta_1} = \delta^{-1}[(w_2^{43})^{-1}M_1^3\mu + (w_2^{44})^{-1}s_1^{\frac{\lambda_1^3-\lambda_1^2}{\lambda_1^1}}M_1^2\mu] + \text{h.o.t.}$$

Set $t_1 = s_1^{(\lambda_1^3-\lambda_1^2)/\lambda_1^1}$ and $\alpha_1 = \frac{\lambda_1^1\beta_1}{\lambda_1^3-\lambda_1^2}$. Then the above equation yields

$$w_1^{44}(w_1^{24}w_2^{43})^{-1}t_1^{\alpha_1} = \delta^{-1}(w_2^{43})^{-1}M_1^3\mu + [\delta^{-1}(w_2^{44})^{-1}M_1^2\mu]t_1 + \text{h.o.t.} \tag{25}$$

Hypothesis (H_1) and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 > \frac{\lambda_1^3}{\lambda_1^1}$ guarantee that α_1 is a positive constant greater than 1 and $0 < t_1 \ll 1$.

When $|M_1^3\mu| \ll |M_1^2\mu|^{\frac{\alpha_1}{\alpha_1-1}}$ and $w_1^{24}w_1^{44}w_2^{43}w_2^{44}M_1^2\mu > 0$ are valid, we can conclude that system (25) has a unique sufficiently small positive solution

$$t_1 = [\delta^{-1}w_1^{24}w_2^{43}(w_1^{44}w_2^{44})^{-1}M_1^2\mu]^{\frac{1}{\alpha_1-1}} + \text{h.o.t.}$$

This follows the fact that $|M_1^2\mu|t_1,\ t_1^{\alpha_1} = O(|M_1^2\mu|^{\frac{\alpha_1}{\alpha_1-1}}) \gg |M_1^3\mu|$.

When $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{\alpha_1}{\alpha_1-1}}$ and $w_1^{24}w_1^{44}M_1^3\mu > 0$, equation (25) has a unique sufficiently small positive solution

$$t_1 = [\delta^{-1}w_1^{24}(w_1^{44})^{-1}M_1^3\mu]^{\frac{1}{\alpha_1}} + \text{h.o.t.}$$

which is because $|M_1^2\mu|t_1 = O(|M_1^2\mu||M_1^3\mu|^{\frac{1}{\alpha_1}}) \ll |M_1^3\mu|$.

The existence of a positive solution $t_1 \ll 1$ indicates that system (24) has a unique sufficiently small positive solution s_1. Putting this solution s_1 into the second equation of (20), we have

$$s_2 = -\delta^{-1}d_2(w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}}M_1^2\mu + \text{h.o.t.}$$

which implies the conclusion of (2).

(3) If $w_1^{44} = 0$, the second equation of (20) becomes

$$w_2^{43}s_2 = \delta^{-1}d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + \text{h.o.t.} \tag{26}$$

which has no positive solution s_2 if $d_2w_2^{43}M_1^3\mu < 0$ and has at most one nonnegative solution s_2 if $d_2w_2^{43}M_1^3\mu > 0$.

When $d_2w_2^{43}M_1^3\mu > 0$, we need to obtain a positive solution s_1 from (21). If $w_1^{44} = 0$ and $w_1^{34} \neq 0$, system (21) can be transformed into

$$w_1^{34}(w_1^{24}w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}+\beta_1} = \delta^{-1}[(w_2^{43})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + (w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}}M_1^2\mu] + \text{h.o.t.} \tag{27}$$

Let $\alpha_2 = 1 + \frac{\lambda_1^1\beta_1}{\lambda_1^3-\lambda_1^2} > 1$, then (27) can be changed in a similar way as (24) by eliminating the common factor $s_1^{\lambda_1^2/\lambda_1^1}$. The new equation is

$$w_1^{34}(w_1^{24}w_2^{44})^{-1}t_1^{\alpha_2} = \delta^{-1}(w_2^{43})^{-1}M_1^3\mu + [\delta^{-1}(w_2^{44})^{-1}M_1^2\mu]t_1 + \text{h.o.t.} \tag{28}$$

Using the same techniques as in equation (25) for (28), we obtain the conclusion of (3).

(4) If $w_1^{44} = w_1^{34} = 0$, system (21) is transformed into

$$(w_2^{43})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + (w_2^{44})^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1}}M_1^2\mu + \text{h.o.t.} = 0. \tag{29}$$

After eliminating the common factor $s_1^{\lambda_1^2/\lambda_1^1}$ in the left side of (29), we have

$$s_1^{\frac{\lambda_1^3-\lambda_1^2}{\lambda_1^1}} M_1^2\mu = -w_2^{44}(w_2^{43})^{-1}M_1^3\mu + \text{h.o.t.} \tag{30}$$

Obviously, (30) has a unique sufficiently small positive solution when $|M_1^3\mu| \ll |M_1^2\mu|$ and $w_2^{44}w_2^{43}M_1^2\mu M_1^3\mu < 0$, but has no solutions under the condition $|M_1^3\mu| \gg |M_1^2\mu|$ or $|M_1^3\mu| \ll |M_1^2\mu|$ and $w_2^{44}w_2^{43}M_1^2\mu M_1^3\mu > 0$. This ends the proof of conclusion (4). □

Remark 4. Theorem 4.2 (1) shows that a heterodimensional cycle can coexist with a homoclinic orbit in the heteroclinic bifurcation surface L_{12} (see Fig. 4), which indicates that there is a great difference between inclination-flip bifurcation and the bifurcation without inclination flip in heterodimensional cycles (Liu *et al.* [10]).

Now look back at the bifurcation equations (19), then we can observe evidently that, between the two equations, there is some kind of symmetry with regard to the variables s_1 and s_2. Therefore, the bifurcation discussion for the case $(w_2^{43})^2 + (w_2^{44})^2 \neq 0$ can be simplified without loss of generality in the situation $w_2^{43} = 0$ and $w_2^{44} \neq 0$. The opposite case can be dealt with by analogous techniques.

Theorem 4.3. *Suppose that hypotheses $(H_1) - (H_6)$ are valid,* Rank*(M_2^3, M_2^4)=2, $1/\beta_2 > \beta_1 > 1$, $w_2^{44} \neq 0$ and $w_2^{43} = w_1^{34} = 0$. Then for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, the following results hold.*

(1) If $w_2^{44}d_2M_1^2\mu > 0$, then system (1) has no periodic or homoclinic orbits near Γ.

(2) If $w_2^{44}d_2M_1^2\mu < 0$, $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, then when μ satisfies $w_1^{24}w_1^{44}M_1^3\mu > 0$, there exists one unique periodic orbit (see Fig. 4). System (1) has neither periodic orbits nor homoclinic orbits when $w_1^{24}w_1^{44}M_1^3\mu < 0$.

(3) If $w_2^{44}d_2M_1^2\mu < 0$, $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 > \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, then when μ lies in the region $\{\mu: \ |M_1^3\mu| \ll (|M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}|M_1^4\mu|)^{\frac{\alpha_3}{\alpha_3-1}},\ w_1^{24}w_1^{44}w_2^{23}d_2M_1^4\mu > 0\}$ or $\{\mu: |M_1^3\mu| \gg (|M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}|M_1^4\mu|)^{\frac{\alpha_3}{\alpha_3-1}},\ w_1^{24}w_1^{44}M_1^3\mu > 0\}$, there exists one unique periodic orbit, where $\alpha_3 = \frac{\lambda_1^1\rho_2^1\beta_1}{\lambda_1^3\rho_2^2-\lambda_1^2\rho_2^1} > 1$ (see Fig. 4).

(4) If $w_2^{44}d_2M_1^2\mu < 0$, $w_1^{44} = 0$ and $w_1^{32} \neq 0$, then when μ satisfies $|M_1^3\mu| \ll |M_1^2\mu|^m$, there exists at most one periodic orbit and when $|M_1^3\mu| \gg |M_1^2\mu|^m$ system (1) has neither periodic nor homoclinic orbits, where m=min$\{1/\beta_2, \rho_2^2/\rho_2^1\}$.

(5) If $w_2^{44}d_2M_1^2\mu < 0$, $w_1^{44} = w_1^{32} = 0$, then when μ satisfies $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$, system (1) has no periodic orbits or homoclinic orbits near Γ. When μ lies in the region $\{\mu: \ |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}} \gg |M_1^3\mu| \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}|M_1^4\mu|^{\frac{\alpha_4}{\alpha_4-1}}$ and $w_2^{23}w_1^{24}M_1^3\mu < 0\}$ or $\{\mu: \ |M_1^3\mu| \ll |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}|M_1^4\mu|^{\frac{\alpha_4}{\alpha_4-1}}$ and $w_1^{24}M_1^4\mu < 0\}$, there exists one unique periodic orbit bifurcated from Γ, where $\alpha_4 = (\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1 - \frac{\lambda_1^2}{\lambda_1^1})/(\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} - \frac{\lambda_1^2}{\lambda_1^1}) > 1$.

Proof. For any $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, when $1/\beta_2 > \beta_1 > 1$, $w_2^{43} = w_1^{34} = 0$ but $w_2^{44} \neq 0$, bifurcation equations (19) are changed into

$$\begin{aligned} s_2 &= -\delta^{-1}(w_2^{44})^{-1} d_2 s_1^{\frac{\lambda_1^3}{\lambda_1^1}} M_1^2\mu + \text{h.o.t.} \\ 0 &= -w_1^{44}(w_1^{24})^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + w_1^{32} d_1^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} s_2^{\frac{1}{\beta_2}} + \delta^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3\mu \\ &\quad + w_2^{23} d_2^{-1} s_2^{\frac{\rho_2^2}{\rho_2^1}} [(w_1^{24})^{-1} s_1^{\beta_1} + \delta^{-1} M_1^4 \mu] + \text{h.o.t.} \end{aligned} \tag{31}$$

where the term $s_2^{\frac{\rho_2^2}{\rho_2^1}+\frac{1}{\beta_2}} = o(s_2^{\frac{\rho_2^2}{\rho_2^1}} s_1^{\beta_1})$ in the right side of the second equation has been merged in the higher order term by the first equation $s_2 = o(s_1)$ and $1/\beta_2 > \beta_1$.

(1) The first equation of (31) obviously shows that there is no nonnegative solutions if $w_2^{44} d_2 M_1^2 \mu > 0$. Thus there is no periodic orbits or homoclinic orbits near the heterodimensional cycle Γ.

In the following, we set $\nu_1 = -\delta^{-1}(w_2^{44})^{-1} d_2 M_1^2 \mu \ll 1$, which is positive when $w_2^{44} d_2 M_1^2 \mu < 0$. Then $s_2 = \nu_1 s_1^{\lambda_1^3/\lambda_1^1} + \text{h.o.t.} = o(s_1^{\lambda_1^3/\lambda_1^1}) \geq 0$, where $s_2 = 0$ if and only if $s_1 = 0$. Putting this simple expression of s_2 into the second equation of (31), we have

$$\begin{aligned} &-w_1^{44}(w_1^{24})^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + w_1^{32} d_1^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\frac{\lambda_1^3}{\lambda_1^1\beta_2}} \nu_1^{\frac{1}{\beta_2}} + \delta^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3\mu \\ &+ w_2^{23} d_2^{-1} s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}} \nu_1^{\frac{\rho_2^2}{\rho_2^1}} [(w_1^{24})^{-1} s_1^{\beta_1} + \delta^{-1} M_1^4\mu] + \text{h.o.t.} = 0. \end{aligned} \tag{32}$$

(2) If $w_2^{44} d_2 M_1^2 \mu < 0$, $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, then system (32) is simplified into

$$w_1^{44}(w_1^{24})^{-1} s_1^{\beta_1} = \delta^{-1} M_1^3 \mu + \text{h.o.t.} \tag{33}$$

(33) has a unique positive solution when $w_1^{24} w_1^{44} M_1^3 \mu > 0$ which corresponds to a unique pair of positive solutions (s_1, s_2) of (31). Thus the conclusion is obvious.

(3) If $w_2^{44} d_2 M_1^2 \mu < 0$, $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 > \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, then system (32) is reduced to

$$w_1^{44}(w_1^{24})^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} = \delta^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + \delta^{-1} w_2^{23} d_2^{-1} s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}} \nu_1^{\frac{\rho_2^2}{\rho_2^1}} M_1^4 \mu + \text{h.o.t.} \tag{34}$$

Setting $t_2 = s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} - \frac{\lambda_1^2}{\lambda_1^1}}$ and $\alpha_3 = \frac{\lambda_1^1\rho_2^1\beta_1}{\lambda_1^3\rho_2^2 - \lambda_1^2\rho_2^1} > 1$ and taking similar techniques to (25), we obtain the conclusion (3).

(4) If $w_2^{44} d_2 M_1^2 \mu < 0$, $w_1^{44} = 0$ but $w_1^{32} \neq 0$, then system (32) becomes

$$\begin{aligned} &w_1^{32} d_1^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\frac{\lambda_1^3}{\lambda_1^1\beta_2}} \nu_1^{\frac{1}{\beta_2}} + \delta^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3\mu + w_2^{23} d_2^{-1} s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}} \nu_1^{\frac{\rho_2^2}{\rho_2^1}} [(w_1^{24})^{-1} s_1^{\beta_1} + \delta^{-1} M_1^4\mu] \\ &+ \text{h.o.t.} = 0. \end{aligned} \tag{35}$$

For system (35), if $\frac{\lambda_1^2}{\lambda_1^1} + \frac{\lambda_1^3}{\lambda_1^1\beta_2} < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, which implies that $\frac{1}{\beta_2} < \frac{\rho_2^2}{\rho_2^1}$ and therefore $s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}} \nu_1^{\frac{\rho_2^2}{\rho_2^1}} = o(s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\frac{\lambda_1^3}{\lambda_1^1\beta_2}} \nu_1^{\frac{1}{\beta_2}})$, then we can rewrite it as

$$w_1^{32} d_1^{-1} s_1^{\frac{\lambda_1^3}{\lambda_1^1\beta_2}} \nu_1^{\frac{1}{\beta_2}} = -\delta^{-1} M_1^3 \mu + \text{h.o.t.} = 0. \tag{36}$$

From (36) we can see directly that this equation has only one positive solution $s_1 \ll 1$ if and only if $|M_1^3\mu| \ll |M_1^2\mu|^{\frac{1}{\beta_2}}$ and $w_1^{32}d_1M_1^3\mu < 0$. When $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{1}{\beta_2}}$ or $w_1^{32}d_1M_1^3\mu > 0$ there is no sufficiently small positive solutions.

If $\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} < \frac{\lambda_1^2}{\lambda_1^1} + \frac{\lambda_1^3}{\lambda_1^1\beta_2} < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1$ and $\frac{1}{\beta_2} < \frac{\rho_2^2}{\rho_2^1}$, then (35) turns out to be

$$w_1^{32}d_1^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\frac{\lambda_1^3}{\lambda_1^1\beta_2}}\nu_1^{\frac{1}{\beta_2}} + \delta^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + \delta^{-1}w_2^{23}d_2^{-1}s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu + \text{h.o.t.} = 0,$$

which can be simplified to

$$w_1^{32}d_1^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1\beta_2}}\nu_1^{\frac{1}{\beta_2}} + \delta^{-1}M_1^3\mu + \delta^{-1}w_2^{23}d_2^{-1}s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}-\frac{\lambda_1^2}{\lambda_1^1}}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu + \text{h.o.t.} = 0. \tag{37}$$

When $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{1}{\beta_2}} \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$, (37) has no solutions. To assure the existence of small enough positive solutions, $|M_1^3\mu| \ll |M_1^2\mu|^{\frac{1}{\beta_2}}$ must be valid at any rate.

If $\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} < \frac{\lambda_1^2}{\lambda_1^1} + \frac{\lambda_1^3}{\lambda_1^1\beta_2} < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1$ and $\frac{1}{\beta_2} > \frac{\rho_2^2}{\rho_2^1}$, then (35) is equivalent to

$$\begin{aligned} &w_1^{32}d_1^{-1}s_1^{\frac{\lambda_1^3}{\lambda_1^1\beta_2}}\nu_1^{\frac{1}{\beta_2}} + \delta^{-1}M_1^3\mu + w_2^{23}d_2^{-1}s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}-\frac{\lambda_1^2}{\lambda_1^1}}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + \delta^{-1}M_1^4\mu] \\ &+\text{h.o.t.} = 0. \end{aligned} \tag{38}$$

When $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}} \gg |M_1^2\mu|^{\frac{1}{\beta_2}}$, (38) has no solutions. Whereas in case that $|M_1^3\mu| \ll |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$, there is at most one sufficiently small positive solution for (38).

If $\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1 < \frac{\lambda_1^2}{\lambda_1^1} + \frac{\lambda_1^3}{\lambda_1^1\beta_2}$ and $\frac{1}{\beta_2} < \frac{\rho_2^2}{\rho_2^1}$, then the simplified equation here is the same as (38). The only difference lies in $\nu_1^{\rho_2^2/\rho_2^1} \ll \nu_1^{\frac{1}{\beta_2}}$ as ν approaches zero, so in the case that $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{1}{\beta_2}} \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$, (37) has no solutions. In the other case, there is at most one positive solution for system (35).

If $\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1 < \frac{\lambda_1^2}{\lambda_1^1} + \frac{\lambda_1^3}{\lambda_1^1\beta_2}$ and $\frac{1}{\beta_2} > \frac{\rho_2^2}{\rho_2^1}$, we can rewrite system (35) into

$$\delta^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + w_2^{23}d_2^{-1}s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} = 0,$$

which is followed by

$$\delta^{-1}M_1^3\mu + w_2^{23}d_2^{-1}s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}-\frac{\lambda_1^2}{\lambda_1^1}}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} = 0. \tag{39}$$

Obviously system (39) has no sufficiently small positive solutions in case that $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}} \gg |M_1^2\mu|^{\frac{1}{\beta_2}}$ and has at most one such solution in case that $|M_1^3\mu| \ll |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$.

Let m=min$\{1/\beta_2,\ \rho_2^2/\rho_2^1\}$. Summarizing the above discussion, we finish the proof of the conclusion (4).

(5) If $w_2^{44}d_2M_1^2\mu < 0$ and $w_1^{44} = w_1^{32} = 0$, we obtain the following equation from system (32)

$$\delta^{-1}M_1^3\mu + w_2^{23}d_2^{-1}s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}-\frac{\lambda_1^2}{\lambda_1^1}}\nu_1^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} = 0. \tag{40}$$

As a consequence, in case of $|M_1^3\mu| \gg |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$, no solutions exist for system (40). If $|M_1^3\mu| \ll |M_1^2\mu|^{\frac{\rho_2^2}{\rho_2^1}}$, we set $t_2 = s_1^{\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} - \frac{\lambda_1^2}{\lambda_1^1}}$ and $\alpha_4 = (\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} + \beta_1 - \frac{\lambda_1^2}{\lambda_1^1})/(\frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1} - \frac{\lambda_1^2}{\lambda_1^1})$, then system (40) becomes

$$w_2^{23}(w_1^{24})^{-1}d_2^{-1}t_2^{\alpha_4} = -\delta^{-1}\nu_1^{-\frac{\rho_2^2}{\rho_2^1}}M_1^3\mu - \delta^{-1}w_2^{23}d_2^{-1}t_2M_1^4\mu + \text{h.o.t.} \tag{41}$$

Applying analogous techniques used for (25) to the above equation, one can complete the proof. □

Theorem 4.4. *Suppose that hypotheses $(H_1) - (H_6)$ are valid,* Rank*(M_2^3, M_2^4)=2, $1/\beta_2 > \beta_1 > 1$, $w_2^{43} = 0$, $w_2^{44}w_1^{34} \neq 0$. Then for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, the following results hold.*

(1) *If $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1}+\beta_1 < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, then when μ lies in the region $\{\mu : w_1^{24}w_1^{44}M_1^3\mu > 0,\ \omega(\hat{s}_1, \mu) > 0\}$, there exists one unique periodic orbit. When $\mu \in \{\mu : w_1^{24}w_1^{44}M_1^3\mu > 0,\ \omega(\hat{s}_1, \mu) = 0\}$, there exists one unique homoclinic orbit associated to p_2, where $\omega(\hat{s}_1, \mu)$ is defined same as in (23). When $\mu \in \{\mu : w_1^{24}w_1^{44}M_1^3\mu > 0,\ \omega(\hat{s}_1, \mu) < 0\}$ or $\{\mu : w_1^{24}w_1^{44}M_1^3\mu < 0\}$, system (1) has neither periodic nor homoclinic orbits near Γ (see Fig. 4).*

(2) *If $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 > \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, then there exists at most one periodic orbit or homoclinic orbit associated to p_2.*

(3) *If $w_1^{44} = 0$, then there exists at most one periodic orbit.*

Proof. (1) In case $w_1^{44} \neq 0$, for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, bifurcation equations (19) can be changed into

$$\begin{aligned} w_2^{44}s_2 &= s_1^{\frac{\lambda_1^3}{\lambda_1^1}}[w_1^{34}(w_1^{24})^{-1}d_2s_1^{\beta_1} - \delta^{-1}d_2M_1^2\mu] + \text{h.o.t.} \\ 0 &= -w_1^{44}(w_1^{24})^{-1}d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + w_1^{32}d_1^{-1}d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}}s_2^{\frac{1}{\beta_2}} + \delta^{-1}d_2s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu \\ &\quad + w_2^{23}s_2^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{24})^{-1}s_1^{\beta_1} + (w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{\frac{1}{\beta_2}} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} \end{aligned} \tag{42}$$

The first equation of (42) shows $s_2 = o(s_1^{\lambda_1^3/\lambda_1^1})$. Under the condition that $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 < \frac{\lambda_1^3\rho_2^2}{\lambda_1^1\rho_2^1}$, the second equation of (42) is simplified into

$$w_1^{44}(w_1^{24})^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} = \delta^{-1}s_1^{\frac{\lambda_1^2}{\lambda_1^1}}M_1^3\mu + \text{h.o.t.}$$

which meets the same situation as (22) and has a sufficiently small positive solution $\hat{s}_1 = [\delta^{-1}w_1^{24}(w_1^{44})^{-1}M_1^3\mu]^{\frac{1}{\beta_1}} + \text{h.o.t.}$ only as $w_1^{24}w_1^{44}M_1^3\mu > 0$, which has the same leading term as s_1^*. Since (42) and (20) have the same first equation, the conclusion is followed by Theorem 3.2 (1).

(2) In case $w_1^{44} \neq 0$ and $\frac{\lambda_1^2}{\lambda_1^1} + \beta_1 > \frac{\lambda_1^3 \rho_2^2}{\lambda_1^1 \rho_2^1}$, (19) becomes

$$
\begin{aligned}
s_2 &= (w_2^{44})^{-1} s_1^{\frac{\lambda_1^3}{\lambda_1^1}} [w_1^{34}(w_1^{24})^{-1} d_2 s_1^{\beta_1} - \delta^{-1} d_2 M_1^2 \mu] + \text{h.o.t.} \\
0 &= -w_1^{44}(w_1^{24})^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}+\beta_1} + w_1^{32} d_1^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} s_2^{\frac{1}{\beta_2}} + \delta^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu \\
&\quad + \delta^{-1} w_2^{23} d_2^{-1} s_2^{\frac{\rho_2^2}{\rho_2^1}} M_1^4 \mu + \text{h.o.t.}
\end{aligned} \tag{43}
$$

for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$. From the above first equation, we see immediately that if $s_1 = 0$, then s_2 must be zero. If $s_2 = 0$, then the second equation of (43) shows that $s_1 = 0$ or $0 < s_1 = [\delta^{-1} w_1^{24}(w_1^{44})^{-1} M_1^3 \mu]^{\frac{1}{\beta_1}} + \text{h.o.t.} \ll 1$ only as $w_1^{24} w_1^{44} d_2 M_1^3 \mu > 0$. If s_2 in the first equation of (43) is positive, then substituting its expression into the second equation of (43), we find that there is at most one sufficiently small positive solution and no zero solution with regard to s_1. Thus, the conclusion of (2) follows.

(3) In case $w_1^{44} = 0$, for $\mu \in L_{12}$ and $0 < |\mu| \ll 1$, bifurcation equations (19) are transformed into

$$
\begin{aligned}
s_2 &= (w_2^{44})^{-1} s_1^{\frac{\lambda_1^3}{\lambda_1^1}} [w_1^{34}(w_1^{24})^{-1} d_2 s_1^{\beta_1} - \delta^{-1} d_2 M_1^2 \mu] + \text{h.o.t.} \\
0 &= w_1^{32} d_1^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} s_2^{\frac{1}{\beta_2}} + \delta^{-1} s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + w_2^{23} d_2^{-1} s_2^{\frac{\rho_2^2}{\rho_2^1}} [(w_1^{24})^{-1} s_1^{\beta_1} + \delta^{-1} M_1^4 \mu] + \text{h.o.t.}
\end{aligned} \tag{44}
$$

which imply that $s_1 = 0$ holds if and only if $s_2 = 0$. Therefore, (44) has at most a positive solution pair (s_1, s_2) small enough. The proof is complete. □

Remark 5. From Theorems 4.2, 4.3 and 4.4, Figure 4 presents the possible orbits bifurcated from the heterodimensional cycle on the bifurcation surface L_{12}. When $w_2^{43} w_2^{44} \neq 0$ or $w_2^{43} = 0$ and $w_2^{44} w_1^{34} \neq 0$, the coexisting orbits on L_{12} correspond to (a) and (b). When $w_2^{43} = w_1^{34} = 0$ and $w_2^{44} \neq 0$, the coexisting case corresponds to (b).

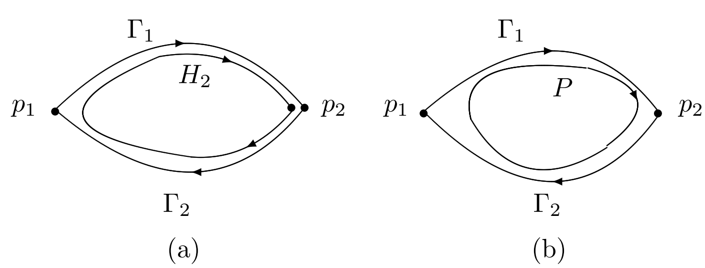

Figure 4. Coexisting diagram of bifurcated orbits as $\mu \in L_{12}$, H_2–homoclinic orbit connecting to p_2, P–periodic orbit.

Now our consideration turns to the existence of homoclinic orbits bifurcated from the heterodimensional cycle as μ is not confined on the heterodimensional bifurcation surface L_{12}.

Theorem 4.5. *Assume that hypotheses* $(H_1) - (H_6)$ *hold,* $\text{Rank}(M_2^3, M_2^4) \geq 1$ *and* $1/\beta_2 > \beta_1 > 1$*. Then for* $0 < |\mu| \ll 1$*, we have the following conclusions.*

(1) When $w_2^{43}w_2^{44} \neq 0$, system (1) has a unique $(l-1)$-dim homoclinic bifurcation surface

$$L_1^1 = \{\mu: \ (w_2^{43}M_2^3 + w_2^{44}M_2^4)\mu + h.o.t. = 0, \ w_2^{43}d_2M_2^4\mu > 0\}$$

such that there is a unique homoclinic orbit Γ_1^1 to p_1 as $\mu \in L_1^1$, which is tangent to L_{12} at $\mu = 0$.

(2) When $w_2^{43} = 0$ and $w_2^{44} \neq 0$, system (1) has a unique $(l-1)$-dim homoclinic bifurcation surface

$$\begin{aligned} L_1^2 = \{\mu: \ & M_2^4\mu + w_2^{23}d_2^{-1}[-\delta^{-1}(w_2^{44})^{-1}d_2M_2^3\mu]^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu \\ & +\delta w_2^{23}(w_1^{33}w_1^{42} - w_1^{32}w_1^{43})(d_1d_2)^{-1}[-\delta^{-1}(w_2^{44})^{-1}d_2M_2^3\mu]^{\frac{\rho_2^2}{\rho_2^1}+\frac{1}{\beta_2}} \\ & +h.o.t. = 0, \ w_2^{44}d_2M_2^3\mu < 0\} \end{aligned}$$

such that there is a unique homoclinic orbit Γ_1^2 to p_1 as $\mu \in L_1^2$, which is tangent to L_{12} at $\mu = 0$.

(3) When $w_2^{43} \neq 0$ and $w_2^{44} = 0$, system (1) has a unique $(l-1)$-dim homoclinic bifurcation surface

$$\begin{aligned} L_1^3 = \{\mu: \ & M_2^3\mu - w_2^{24}d_2^{-1}[\delta^{-1}(w_2^{43})^{-1}d_2M_2^4\mu]^{\frac{\rho_2^2}{\rho_2^1}}M_1^4\mu \\ & -\delta w_2^{24}(w_1^{33}w_1^{42} - w_1^{32}w_1^{43})(d_1d_2)^{-1}[\delta^{-1}(w_2^{43})^{-1}d_2M_2^4\mu]^{\frac{\rho_2^2}{\rho_2^1}+\frac{1}{\beta_2}} \\ & +h.o.t. = 0, \ w_2^{43}d_2M_2^4\mu > 0\} \end{aligned}$$

such that there is a unique homoclinic orbit Γ_1^3 to p_1 as $\mu \in L_1^3$, which is tangent to L_{12} at $\mu = 0$.

Proof. The existence of a homoclinic orbit which is associated to the left saddle p_1 corresponds to the existence of a solution pair $s_1 = 0, \ 0 < s_2 \ll 1$ of the bifurcation equations (19). Fix $s_1 = 0$ in system (19), we have

$$\begin{aligned} w_2^{44}s_2 &= -\delta^{-1}d_2M_2^3\mu + w_2^{24}s_2^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{\frac{1}{\beta_2}} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} \\ w_2^{43}s_2 &= \delta^{-1}d_2M_2^4\mu + w_2^{23}s_2^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{\frac{1}{\beta_2}} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} \end{aligned} \tag{45}$$

(1) If $w_2^{43}w_2^{44} \neq 0$, then (45) is simplified into

$$\begin{aligned} w_2^{44}s_2 &= -\delta^{-1}d_2M_2^3\mu + \text{h.o.t.} \\ w_2^{43}s_2 &= \delta^{-1}d_2M_2^4\mu + \text{h.o.t.} \end{aligned} \tag{46}$$

Thus we have $s_2 = \delta^{-1}(w_2^{43})^{-1}d_2M_2^4\mu + \text{h.o.t.}$ from the second equation of (46). This solution is positive if and only if $w_2^{43}d_2M_2^4\mu > 0$, which guarantees the existence of the homoclinic bifurcation surface L_1^1. At $\mu = 0$, the normal vector of the surface L_1^1 is $w_2^{43}M_2^3 + w_2^{44}M_2^4$, which is contained in Σ_{12}, thus it is inescapably tangent to L_{12}.

(2) If $w_2^{43} = 0, \ w_2^{44} \neq 0$, (45) has the simplified form

$$\begin{aligned} s_2 &= -\delta^{-1}(w_2^{44})^{-1}d_2M_2^3\mu + \text{h.o.t.} \\ 0 &= \delta^{-1}d_2M_2^4\mu + w_2^{23}s_2^{\frac{\rho_2^2}{\rho_2^1}}[(w_1^{33}w_1^{42} - w_1^{32}w_1^{43})d_1^{-1}s_2^{\frac{1}{\beta_2}} + \delta^{-1}M_1^4\mu] + \text{h.o.t.} \end{aligned} \tag{47}$$

Obviously, s_2 is meaningful only in case $w_2^{44}d_2M_2^3\mu < 0$. Putting $s_2 > 0$ into the second equation of (47), we obtain the bifurcation surface L_1^2, which has a normal vector M_2^4 at $\mu = 0$ so this surface is tangent to L_{12} at $\mu = 0$ as well.

(3) If $w_2^{43} \neq 0$ and $w_2^{44} = 0$, we can rewrite (45) as

$$\begin{aligned} 0 &= -\delta^{-1} d_2 M_2^3 \mu + w_2^{24} s_2^{\frac{\rho_2^2}{\rho_2^1}} [(w_1^{33} w_1^{42} - w_1^{32} w_1^{43}) d_1^{-1} s_2^{\frac{1}{\beta_2}} + \delta^{-1} M_1^4 \mu] + \text{h.o.t.} \\ s_2 &= \delta^{-1} (w_2^{43})^{-1} d_2 M_2^4 \mu + \text{h.o.t.} \end{aligned} \tag{48}$$

which is completely symmetrical with system (47), so the homoclinic bifurcation surface L_1^3 can be deduced in an analogous way. □

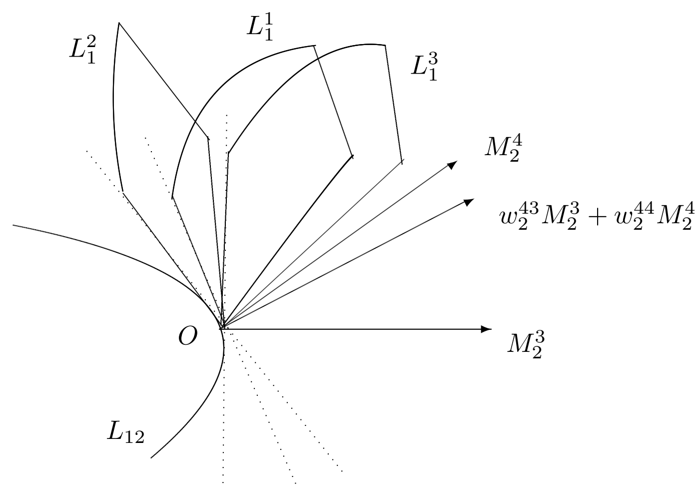

Figure 5. Bifurcation surfaces of orbits homoclinic to p_1 as $w_2^{43} \geq 0,\ w_2^{44} \geq 0,\ d_2 > 0$.

Remark 6. L_1^j $(j = 1, 2, 3)$ correspond to different homoclinic bifurcation surfaces associated to p_1 of system (1) under different conditions. Figure 5 shows the position sketch of bifurcation surfaces L_{12} and L_1^j $(j = 1, 2, 3)$ in the same coordinate system of the μ space as $w_2^{43} > 0,\ w_2^{44} > 0,\ d_2 > 0$.

Theorem 4.6. *Suppose that hypotheses* $(H_1) - (H_6)$ *hold,* $\mathrm{Rank}(M_2^3, M_2^4) \geq 1$ *and* $1/\beta_2 > \beta_1 > 1$. *Then for* $0 < |\mu| \ll 1$, *we have the following conclusions.*

(1) If $w_1^{34} \neq 0$, *then in the region* $|M_2^3 \mu| \ll |M_1^2 \mu|^{\frac{\tilde{\alpha}}{\tilde{\alpha}-1}}$, *there exists one unique bifurcation surface*

$$L_2^1 = \{\mu : \ M_2^4 \mu = \delta w_1^{44} (w_1^{24})^{-1} \tilde{s}_1^{\frac{\lambda_1^2}{\lambda_1^1} + \beta_1} - \tilde{s}_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + h.o.t.,\ w_1^{24} w_1^{34} M_1^2 \mu > 0\},$$

where $\tilde{\alpha} = (\frac{\lambda_1^3}{\lambda_1^1} + \beta_1)/(\frac{\lambda_1^2}{\lambda_1^1}) > 1$ *and* $\tilde{s}_1 = [\delta^{-1} w_1^{24} (w_1^{34})^{-1} M_1^2 \mu]^{\frac{1}{\beta_1}}$, *such that for* $\mu \in L_2^1$ *and* $0 < |\mu| \ll 1$, *system (1) has a unique homoclinic orbit* Γ_2^1 *to* p_2 *in a neighborhood of the heterodimensional cycle* Γ.

(2) If $w_1^{34} \neq 0$, *then in the region* $|M_2^3 \mu| \gg |M_1^2 \mu|^{\frac{\tilde{\alpha}}{\tilde{\alpha}-1}}$, *there exists one unique bifurcation surface*

$$L_2^2 = \{\mu : \ M_2^4 \mu = \delta w_1^{44} (w_1^{24})^{-1} \hat{s}_1^{\frac{\lambda_1^2}{\lambda_1^1} + \beta_1} - \hat{s}_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + h.o.t.,\ w_1^{24} w_1^{34} M_2^3 \mu > 0\},$$

where $\tilde{\alpha}$ is same as in (1) and $\hat{s}_1 = [\delta^{-1} w_1^{24} (w_1^{34})^{-1} M_2^3 \mu]^{\frac{\lambda_1^1}{\lambda_1^3 + \lambda_1^1 \beta_1}}$, such that for $\mu \in L_2^2$ and $0 < |\mu| \ll 1$, system (1) has a unique homoclinic orbit Γ_2^2 to p_2 in a neighborhood of the heterodimensional cycle Γ.

(3) *If $w_1^{34} = 0$, then in the region $|M_2^3 \mu| \ll |M_1^2 \mu|$, there exists one unique bifurcation surface*

$$L_2^3 = \{\mu : \ M_2^4 \mu = \delta w_1^{44} (w_1^{24})^{-1} \bar{s}_1^{\frac{\lambda_1^2}{\lambda_1^1} + \beta_1} - \bar{s}_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + h.o.t., \ M_1^2 \mu M_2^3 \mu < 0\},$$

where $\bar{s}_1 = (-\frac{M_2^3 \mu}{M_1^2 \mu})^{\frac{\lambda_1^1}{\lambda_1^3}}$, such that for $\mu \in L_2^3$ and $0 < |\mu| \ll 1$, system (1) has a unique homoclinic orbit Γ_2^3 to p_2 in a neighborhood of the heterodimensional cycle Γ.

(4) *If $w_1^{34} = 0$, then in the region $|M_2^3 \mu| \gg |M_1^2 \mu|$, there exist no homoclinic bifurcation surfaces such that system (1) has a homoclinic orbit to p_2 when μ lies on that surfaces.*

Proof. In order to bifurcate a cycle homoclinic to the right saddle p_2 from the heterodimensional cycle Γ, bifurcation equations (19) need to have the solution pair $0 < s_1 \ll 1, \ s_2 = 0$. Substituting $s_2 = 0$ into system (19), we get

$$\begin{aligned} 0 &= w_1^{34} (w_1^{24})^{-1} s_1^{\frac{\lambda_1^3}{\lambda_1^1} + \beta_1} - \delta^{-1} s_1^{\frac{\lambda_1^3}{\lambda_1^1}} M_1^2 \mu - \delta^{-1} M_2^3 \mu + \text{h.o.t.} \\ 0 &= -w_1^{44} (w_1^{24})^{-1} d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1} + \beta_1} + \delta^{-1} d_2 s_1^{\frac{\lambda_1^2}{\lambda_1^1}} M_1^3 \mu + \delta^{-1} d_2 M_2^4 \mu + \text{h.o.t.} \end{aligned} \tag{49}$$

When $w_1^{34} \neq 0$, we will work out s_1 from the above equations. Let $\tilde{t} = s_1^{\lambda_1^3/\lambda_1^1}$, then the first equation of (49) is turned to be

$$w_1^{34} (w_1^{24})^{-1} \tilde{t}^{\tilde{\alpha}} = \delta^{-1} M_2^3 \mu + \delta^{-1} \tilde{t} M_1^2 \mu + \text{h.o.t.} \tag{50}$$

where $\tilde{\alpha}$ is defined as in (1). Proceeding along the same techniques as (25) to equation (50), we obtain that in case $|M_2^3 \mu| \ll |M_1^2 \mu|^{\frac{\tilde{\alpha}}{\tilde{\alpha}-1}}$ and $w_1^{24} w_1^{34} M_1^2 \mu > 0$, (50) has a unique sufficiently small positive solution $\tilde{t} = [\delta^{-1} w_1^{24} (w_1^{34})^{-1} M_1^2 \mu]^{\frac{1}{\tilde{\alpha}-1}} + \text{h.o.t.}$ In case $|M_2^3 \mu| \gg |M_1^2 \mu|^{\frac{\tilde{\alpha}}{\tilde{\alpha}-1}}$ and $w_1^{24} w_1^{34} M_2^3 \mu > 0$, (50) has a unique sufficiently small positive solution $\tilde{t} = [\delta^{-1} w_1^{24} (w_1^{34})^{-1} M_2^3 \mu]^{\frac{1}{\tilde{\alpha}}} + \text{h.o.t.}$ Uniting $\tilde{t}$ and the second equation of (49), the conclusions of (1) and (2) are proved.

When $w_1^{34} = 0$, the first equation of (49) becomes

$$s_1^{\frac{\lambda_1^3}{\lambda_1^1}} M_1^2 \mu = -M_2^3 \mu + \text{h.o.t.} \tag{51}$$

It is clear to observe that equation (51) has no sufficiently small solution in case $|M_2^3 \mu| \gg |M_1^2 \mu|$. Therefore, there are no homoclinic bifurcation surfaces. In case $|M_2^3 \mu| \ll |M_1^2 \mu|$ and $M_1^2 \mu M_2^3 \mu < 0$, (51) has a unique positive solution $s_1 = (-\frac{M_2^3 \mu}{M_1^2 \mu})^{\frac{\lambda_1^1}{\lambda_1^3}} \ll 1$. Putting this solution into the second equation of (49), we get the expression of L_2^3, which completes the proof of Theorem 3.6. □

Remark 7. Figure 6 depicts the position sketch of homoclinic bifurcation surfaces associated to p_2 in the μ-space when $w_1^{24} > 0, \ w_1^{34} > 0, \ M_1^2 \mu > 0$. Comparing Figure 5 with Figure 6, we observe that there possibly exist some codimension-2 subsurfaces in a small neighborhood of $\mu = 0$ which are the intersection of L_1^j

with L_2^k $(j = 1, 3, k = 1, 2)$ and correspond to the coexistence of homoclinic orbits associated to p_1 and p_2, respectively (see Fig. 7).

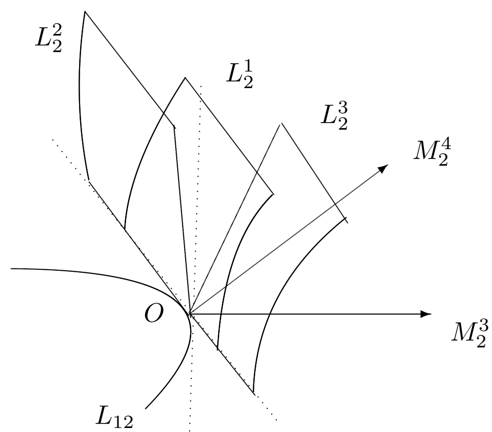

Figure 6. Bifurcation surfaces of orbits homoclinic to p_2 as $w_1^{24} \geq 0,\ w_1^{34} \geq 0,\ M_1^2\mu > 0$.

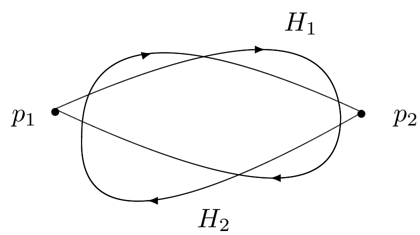

Figure 7. Diagram of coexistence of homoclinic orbits as $\mu \in L_1^j \cap L_2^k$ for $j = 1, 3, k = 1, 2$.

5. **Discussion.** The paper is concerned with codimension 3 bifurcations of heterodimensional cycles with inclination flip in the case of $1/\beta_2 > \beta_1 > 1$. By constructing the local moving frame system in a neighborhood of the heterodimenaional cycle, we constructed the successor functions and the Poincaré return map by using the fundamental solution matrix of the linearly variational system with respect to the primary cycle. Bifurcation equations were also given.

In the forementioned bifurcation analysis, we attempted to prove the possible bifurcations from the heterodimensional cycles associated to two hyperbolic critical points in the four dimensional space. It is interesting to find the persistence of heterodimensional cycles, the coexistence of heterodimensional cycles and bifurcated periodic orbits, and the remarkable coexistence of heterodimensional cycles and bifurcated homoclinic orbits for system (1) when an unstable manifold undergoes inclination flip as moving along the nontransversal heteroclinic orbit. Furthermore, we conjectured the coexistence of orbits homoclinic to p_1 and p_2 at the end of the third section. However, because the bifurcation equations (19) deeply depend on too many coefficients, it is very challenging to conduct the bifurcation analysis. A few cases where the coexistence of heterodimensional cycles and periodic or homoclinic orbits could not be uniquely determined are shown in Theorem 4.3 (4) and

Theorem 4.4 (2) and (3). The inclination flip of W_2^u is essentially responsible for the complicated bifurcations.

It is worthy to mention that our techniques in the present paper can be extended to investigate more general situations. Jin and Zhu [8] proved bifurcations occurring in rough equidimensional cycles with three saddles. As a matter of fact, we may analyze bifurcations of multi-saddle heterodimensional cycles in any finite dimensional vector fields by the above method.

In Sun [15], the problem of persistence of generic heteroclinic orbits connecting nonhyperbolic equilibria was investigated. Bifurcation of homoclinic orbits with a nonhyperbolic equilibrium, such as for a saddle-node, has been dealt with (see Chow and Lin [2]). Motivated by this note, we can study the bifurcation of heterodimensional cycles associated to nonhyperbolic equilibria by constructing local moving frame systems. However, the appearance of center manifolds of nonhyperbolic equilibria will increase tremendous difficulty for studying the global bifurcation problem, since the perturbed system will exhibit local bifurcation at an equilibrium of the unperturbed system besides the orbit bifurcation. Finally, it will be interesting to apply the results obtained in this paper to study some biological and epidemiological models. We leave these for future research.

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E-mail address: liudan_ecnu@163.com
E-mail address: ruan@math.miami.edu
E-mail address: dmzhu@math.ecnu.edu.cn