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MONOSTABLE WAVEFRONTS IN COOPERATIVE LOTKA-VOLTERRA SYSTEMS WITH NONLOCAL DELAYS

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Abstract. This paper is concerned with traveling wavefronts in a Lotka-Volterra model with nonlocal delays for two cooperative species. By using comparison principle, some existence and nonexistence results are obtained. If the wave speed is larger than a threshold which can be formulated in terms of basic parameters, we prove the asymptotic stability of traveling wavefronts by the spectral analysis method together with squeezing technique.

1. **Introduction.** In this paper, we are interested in traveling wavefronts of the following cooperative Lotka-Volterra system with nonlocal delays

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1\Delta u_1(x,t) + r_1u_1(x,t)\left[1 - a_1u_1(x,t) + b_1(g_1 * u_2)(x,t)\right], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2\Delta u_2(x,t) + r_2u_2(x,t)\left[1 - a_2u_2(x,t) + b_2(g_2 * u_1)(x,t)\right], \end{cases} \tag{1}$$

hereafter $x \in \mathbb{R}, t > 0, u = (u_1, u_2) \in \mathbb{R}^2$, $u_1(x,t)$ and $u_2(x,t)$ denote the densities of two cooperative species in location $x \in \mathbb{R}$ and at time t, and all parameters are positive. The kernels $(g_1 * u_2)(x,t)$ and $(g_2 * u_1)(x,t)$ are defined by

$$\begin{cases} (g_1 * u_2)(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{t} G_1(x-y,t-s)k_1(t-s)u_2(y,s)dyds, \\ (g_2 * u_1)(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{t} G_2(x-y,t-s)k_2(t-s)u_1(y,s)dyds, \end{cases}$$

respectively, in which the kernel function k_i denotes the impact factor of historical behavior of species u_{3-i} on species u_i, and we assume that it takes the form of the so-called *weak kernel* (see Ruan and Xiao [30]) as follows

$$k_i(s) = \frac{1}{\tau_i}e^{-\frac{1}{\tau_i}s},\ i = 1, 2,\ s \geq 0,$$

herein $\tau_i \geq 0$ is the average delay ([30]). For $i = 1, 2$, G_i is a weighting function describing the distribution at past time of the individuals of the species u_{3-i} who are in position x at time t, and the species u_i diffuses at diffusivity d_i; thus

$$\frac{\partial G_i(x,t)}{\partial t} = d_{3-i}\frac{\partial^2 G_i(x,t)}{\partial x^2},\ G_i(x,0) = \delta(x),\ x \in \mathbb{R},\ t > 0,\ i = 1, 2,$$

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where $\delta(x)$ is the general Dirac function, so that G_i is a fundamental solution of the heat equation with diffusivity d_{3-i}, i.e.,

$$G_i(x,t) = \frac{1}{\sqrt{4d_{3-i}\pi t}} e^{-\frac{x^2}{4d_{3-i}t}},\ i = 1,2.$$

For more details on the choice of kernel functions and the background of nonlocal delay (spatial-temporal delay), we refer to Briton [3] for a single species population model, Gourley and Ruan [8] for a competition model, Ruan and Xiao [30] for an epidemic model, Gourley and Wu [9] and Ruan [28] for surveys on nonlocal delay models.

Let $\theta = t - s$ and $z = x - y$, then

$$(g_j * u_i)(x,t) = \int_0^\infty \int_{-\infty}^\infty \frac{1}{\tau_j} e^{-\frac{1}{\tau_j}\theta} \frac{1}{\sqrt{4d_i\pi\theta}} e^{-\frac{z^2}{4d_i\theta}} u_i(x-z, t-\theta) dz d\theta$$

for $i, j = 1, 2$. With these assumptions, (1) has a trivial equilibrium $E_0 = (0,0)$, two semitrivial spatially homogeneous equilibria $E_1 = (1/a_1, 0)$ and $E_2 = (0, 1/a_2)$, and a positive spatially homogeneous equilibrium defined by

$$E^* = \left(\frac{a_2 + b_1}{a_1a_2 - b_1b_2}, \frac{a_1 + b_2}{a_1a_2 - b_1b_2} \right) := (k_1, k_2)$$

provided that

$$a_1a_2 > b_1b_2. \tag{2}$$

From Li et al. [12], we know that the cooperative Lotka-Volterra system without delay

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1\Delta u_1(x,t) + r_1u_1(x,t)\left[1 - a_1u_1(x,t) + b_1u_2(x,t)\right], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2\Delta u_2(x,t) + r_2u_2(x,t)\left[1 - a_2u_2(x,t) + b_2u_1(x,t)\right], \end{cases} \tag{3}$$

has a traveling wavefront with the speed $c \geq c^* := \max\{2\sqrt{d_1r_1}, 2\sqrt{d_2r_2}\}$ if (2) holds. Here a traveling wavefront of (3) is a solution of the form $u(x,t) = \Phi(x+ct)$ for some $c \in \mathbb{R}$ accounting for the wave speed of propagation and $\Phi \in C^2(\mathbb{R}, \mathbb{R}^2)$ being interpreted as the wave profile, from a stable equilibrium E^* to one of the unstable equilibria $\{E_0, E_1, E_2\}$, which is called a *monostable wavefront*, see [5, 34]. In particular, Li et al. [12] showed that their results on minimal wave speed are coincident with the so-called *linear determinate conjecture* [2, 38]. Notice that (3) is also called a mutualist model, we refer to Huang [11], Mischaikow and Hutson [25], and Volpert et al. [34] for the bistable wavefronts of mutualist models.

Recently, there are many results on the existence and persistence of traveling wavefronts of reaction-diffusion systems with (nonlocal) delay, see, for example, [6, 8, 9, 15, 17, 19, 26, 30, 35, 36, 39]. We should note that the results of Faria et al. [6] and Ou and Wu [26] do not apply to (1) directly because (1) has four equilibria (they required that the system considered has two equilibria). Using the monotone iteration technique, Wang et al. [35] established the existence of traveling wavefronts of reaction-diffusion systems with nonlocal delays under the so-called (exponential) quasimonotone condition, by which the traveling wavefronts of (1) were considered if $c \geq c_0 := \max\left\{2\sqrt{d_1r_1a_1k_1}, 2\sqrt{d_2r_2a_2k_2}\right\}$, see Li and Wang [14]. Similar results were established by Huang and Zou [10] for the discrete delay version of (3). However, the results in [10, 14] did not give precise asymptotic behavior of such traveling wavefronts as traveling wave coordinate $x + ct \to -\infty$, which is necessary in studying the asymptotic stability of monostable wavefronts (see [5, 21, 22, 23, 34]).

Motivated by the linear determine conjecture [12, 38], it is also natural to ask whether the constant $c^*(< c_0)$ is the minimal wave speed of (1). In this paper we will address this problem. In order to establish the existence of traveling wavefronts of (1), we first consider an abstract reaction-diffusion system with nonlocal delays by modifying the techniques in [35] (less restrictive on upper-lower solutions than that of [35]) and using Schauder's fixed point theorem. We then obtain some existence results of traveling wavefronts connecting E^* with E_0 if $c > c^*$. As a byproduct, the precise asymptotic behavior of such traveling wavefronts is obtained. If $c < c^*$, then we confirm that (1) has no traveling wavefronts by the theory of asymptotic spreading. For (3), we also establish the existence of monostable wavefronts connecting E_0 with E^*, which is different from [12, Theorem 4.2] (see Remark 2.12). In particular, our parameters concerning with the existence, nonexistence and precise asymptotic behavior of traveling wavefronts of (1) are dependent only on the real roots of the following characteristic equations

$$d_1\lambda^2 - c\lambda + r_1 = 0 \text{ and } d_2\lambda^2 - c\lambda + r_2 = 0.$$

We should point out that our existence results are invalid for arbitrary τ_1, τ_2 (Theorem 2.8) or d_1, d_2, r_1, r_2 (Remark 2.9 and Theorem 2.10). However, if the wave speed is larger than a threshold formulated in terms of d_1, d_2, r_1 and r_2, we may obtain the existence of traveling wavefronts for *any* positive parameters satisfying (2).

After the existence of monostable wavefronts of (1) is established, it is natural to consider the stability of such monostable wavefronts, which is very important in interpreting some phenomena in physics, biology and other applied subjects [34]. In this respect, there are several methods, e.g., the spectral analysis [34], and energy estimates [22, 23]. Another method is the so-called squeezing technique based on the comparison principle and upper-lower solutions, see [4] for reaction-diffusion equations, [21] for lattice dynamical systems, [31] for delayed reaction-diffusion equations, and [9, 19, 36, 37] for nonlocal reaction-diffusion equations.

Since the time delays in (1) are infinite, it will be very difficult to discuss the stability of traveling wavefronts via spectral analysis. Moreover, when the energy method and squeezing technique are involved, we often need to improve the distance between the traveling wavefronts and the solution of the corresponding Cauchy type problem on the delayed interval as time increases (see, e.g., Ma and Zou [20, Lemma 4.4], Smith and Zhao [31, Lemma 3.1]), this is impossible for (1) due to the unboundedness of time delays. Motivated by the ideas of Gopalsamy [7] and Lin and Li [16], we shall consider the stability of traveling wavefronts of (1) by studying the following auxiliary undelayed system

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1\Delta u_1(x,t) + r_1u_1(x,t)\left[1 - a_1u_1(x,t) + b_1u_3(x,t)\right], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2\Delta u_2(x,t) + r_2u_2(x,t)\left[1 - a_2u_2(x,t) + b_2u_4(x,t)\right], \\ \frac{\partial u_3(x,t)}{\partial t} = d_3\Delta u_3(x,t) + \frac{1}{\tau_1}u_2(x,t) - \frac{1}{\tau_1}u_3(x,t), \\ \frac{\partial u_4(x,t)}{\partial t} = d_4\Delta u_4(x,t) + \frac{1}{\tau_2}u_1(x,t) - \frac{1}{\tau_2}u_4(x,t), \end{cases} \tag{4}$$

where $d_3 = d_2, d_4 = d_1, u_3, u_4 \in \mathbb{R}$, and the other parameters are the same as in (1). Since the monostable and bistable wavefronts have significantly different properties [5, 34], different techniques from those of [16] are required to study the monostable wavefronts of (1). Utilizing the squeezing technique and spectral analysis, we establish two different results on the asymptotic stability of monostable traveling wavefronts of (1) if the wave speed is larger than the threshold that can be formulated in terms of the basic parameters d_1, d_2, r_1 and r_2.

The rest of this paper is organized as follows. In Section 2, the existence and asymptotic behavior of traveling wavefronts of (1) are proved by constructing upper-lower solutions. The corresponding initial value problem of (1) is considered in Section 3. In Section 4, we establish the asymptotic stability of traveling wavefronts of (1). In Section 5, the nonexistence of traveling wavefronts of (1) is investigated by the theory of asymptotic spreading.

2. **Existence of traveling wavefronts.** In this section, we consider the existence and asymptotic behavior of traveling wavefronts of systems (1) and (4). This part is motivated by Li et al. [13], Ma [18] and Wang et al. [35], Wu and Zou [40]. For more results on this topic, we refer to Faria et al. [6], Gourley and Wu [9] and Ou and Wu [26].

2.1. **Preliminaries.** In this paper, we shall use the standard partial ordering in $\mathbb{R}^n$. Moreover, let $\Phi(\xi) = (\phi_1(\xi), \cdots, \phi_n(\xi))$ be a vector-valued function, then $\Phi(\xi)$ is monotone if $\phi_i(\xi)$ is monotone for each $i = 1, 2, \cdots, n$. We also denote $\Phi'(\xi) = (\phi_1'(\xi), \cdots, \phi_n'(\xi))$ and $\Phi''(\xi) = (\phi_1''(\xi), \cdots, \phi_n''(\xi))$ if $\phi_i'(\xi)$ and $\phi_i''(\xi)$ exist for all $i = 1, \cdots, n$.

Consider the traveling wavefronts of the following reaction-diffusion system

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + f\left(u(x,t)\right), \tag{5}$$

where $u \in \mathbb{R}^n$, $D = \mathrm{diag}(d_1, \cdots, d_n)$ is a diagonal matrix with $d_i > 0, i = 1, 2, \cdots, n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector-valued function.

Definition 2.1. A *traveling wave solution* of (5) is a solution with special form $u(x,t) = \Phi(x + ct)$ for some $c > 0$ accounting for the wave speed and the twice differentiable vector-valued function $\Phi \in C^2(\mathbb{R}, \mathbb{R}^n)$ being interpreted as the wave profile. Moreover, if $\Phi(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called a *traveling wavefront*.

Remark 2.2. For the bistable model, $c \le 0$ is admissible ([34]). Since our main attention in this paper is on monostable traveling wavefronts, we only consider the case $c > 0$ in Definition 2.1.

Let $\xi = x + ct$, then a traveling wavefront of (5) satisfies

$$D\Phi''(\xi) - c\Phi'(\xi) + f\left(\Phi(\xi)\right) = 0,\ \xi \in \mathbb{R}.$$

Motivated by the meaning of traveling wavefronts in biology, physics and chemical reaction [32, 34], we also require that Φ satisfy the following asymptotic boundary conditions

$$\lim_{\xi\to-\infty} \Phi(\xi) = \Phi_-,\ \lim_{\xi\to\infty} \Phi(\xi) = \Phi_+ \tag{6}$$

with $f(\Phi_-) = f(\Phi_+) = 0$.

A typical nonlocal delay system takes the form

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + f\left(u(x,t), (g * u)(x,t)\right), \tag{7}$$

in which $f = (f_1, \cdots, f_n) : (\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$, $g * u \in \mathbb{R}^n$ is defined by

$$(g * u)(x,t) = \int_0^\infty \int_{-\infty}^\infty g(y,s) u(x-y, t-s) dy ds$$

and we also assume that $g * I = I$ with $I = \mathrm{diag}(1, 1, \cdots, 1)$.

Similar to that of (5), if we denote the traveling wave solution of (7) as

$$u(x, t) = \Phi(x + ct) = (\phi_1(x + ct), \cdots, \phi_n(x + ct)),$$

then $\Phi(\xi)$ must satisfy the following functional differential system

$$D\Phi''(\xi) - c\Phi'(\xi) + f\left(\Phi(\xi), (g * \Phi)(\xi)\right) = 0 \tag{8}$$

and the asymptotic behavior (6) with

$$f\left(\Phi_-, \Phi_-\right) = f\left(\Phi_+, \Phi_+\right) = 0,$$

herein $(g * \Phi)(\xi)$ is defined by

$$(g * \Phi)(\xi) = \int_0^\infty \int_{-\infty}^\infty g(y, s)\Phi(\xi - y - cs)dyds.$$

Without loss of generality, we shall consider the following asymptotic behavior

$$\lim_{t\to-\infty} \Phi(\xi) = 0, \lim_{t\to\infty} \Phi(\xi) = \mathbf{S} \tag{9}$$

where $0 \ll \mathbf{S} \in \mathbb{R}^n$ with $f(0, 0) = f(\mathbf{S}, \mathbf{S}) = 0$.

In order to consider the existence of monotone solutions of (8) and (9), we need the following quasimonotone condition (QM).

(QM) There exists a matrix $\beta = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n), \beta_i > 0, i = 1, 2, \cdots, n$, such that

$$f\left(\Phi(\xi), (g * \Phi)(\xi)\right) - f\left(\Psi(\xi), (g * \Psi)(\xi)\right) + \beta\left(\Phi(\xi) - \Psi(\xi)\right) \geq 0$$

for any $0 \leq \Psi(\xi) \leq \Phi(\xi) \leq \mathbf{S}, \xi \in \mathbb{R}$.

Definition 2.3. Assume that f satisfies (QM). Then a continuous vector-valued function $\Phi(\xi)$ is called an *upper* (*lower*) solution of (8) if $0 \leq \Phi(\xi) \leq \mathbf{S}$ and $\Phi''(\xi), \Phi'(\xi)$ are bounded for $\xi \in \mathbb{R} \setminus \mathbb{T}$ and satisfy the following inequality

$$D\Phi''(\xi) - c\Phi'(\xi) + f\left(\Phi(\xi), (g * \Phi)(\xi)\right) \leq (\geq) 0 \quad \text{for all } \xi \in \mathbb{R} \setminus \mathbb{T}, \tag{10}$$

where $\mathbb{T} = \{T_1, T_2, \cdots, T_m\}$ and $T_1 < T_2 < \cdots < T_m$.

Before proving the existence of solutions to (8) and (9), we introduce a Banach space equipped with the exponential decay norm used by Ma [18] and an integral operator used by Wang et al. [35]. Set

$$\lambda_{i1} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, \ \lambda_{i2} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i} \quad \text{for } i = 1, 2, \cdots, n.$$

For $\Phi(\xi) \in C(\mathbb{R}, \mathbb{R}^n)$ with $0 \leq \Phi(\xi) \leq \mathbf{S}$, we define $P = (P_1, \cdots, P_n)$ by

$$P_i(\Phi)(\xi) = \frac{1}{d_i\left(\lambda_{i2} - \lambda_{i1}\right)}\left[\int_{-\infty}^\xi e^{\lambda_{i1}(\xi - s)} + \int_\xi^\infty e^{\lambda_{i2}(\xi - s)}\right] H_i(\Phi)(s)ds,$$

in which $H_i(\Phi)(\xi)$ are given by

$$H_i(\Phi)(\xi) = \beta_i\phi_i(\xi) + f_i\left(\Phi(\xi), (g * \Phi)(\xi)\right), i = 1, \cdots, n.$$

It is clear that P is well defined and a fixed point of the operator P satisfies (8). We also assume that the kernel function g satisfies the following integrable condition.

(G) There exists a constant $\nu \in (0, \min_{i=1,2,\cdots,n}\{-\lambda_{i1}\})$ such that

$$G(\nu) := \int_{-\infty}^{0}\int_{-\infty}^{\infty} n\|g(y,\theta)I\|e^{\nu|y+c\theta|}dyd\theta < \infty, \tag{11}$$

where $\|\cdot\|$ is the maximal norm of $\mathbb{R}^n \times \mathbb{R}^n$.

Let $|\cdot|$ denote the supremum norm in $\mathbb{R}^n$. Then

$$B_\nu(\mathbb{R},\mathbb{R}^n) = \left\{u : u(\xi) \in C(\mathbb{R},\mathbb{R}^n) \text{ and } \sup_{\xi\in\mathbb{R}} |u(\xi)| e^{-\nu|\xi|} < \infty\right\}$$

is a Banach space with norm $\|\cdot\|_\nu$ defined by

$$\|u\|_\nu = \sup_{\xi\in\mathbb{R}} |u(\xi)| e^{-\nu|\xi|} \text{ for } u(\xi) \in B_\nu(\mathbb{R},\mathbb{R}^n).$$

Moreover, we assume that there exists a constant $L > 0$ such that

$$|f(u_1,v_1) - f(u_2,v_2)| \le L(|u_1-u_2| + |v_1-v_2|) \tag{12}$$

for any $0 \le u_1, u_2, v_1, v_2 \le \mathbf{S}$.

Theorem 2.4. *Assume that (QM), (G) and (12) hold. If (8) has an upper solution $\overline{\Phi}(\xi) = (\overline{\phi}_1, \cdots, \overline{\phi}_n)$ and a lower solution $\underline{\Phi}(\xi) = (\underline{\phi}_1, \cdots, \underline{\phi}_n)$ satisfying*

- *(a)* $\sup_{s\le\xi} \underline{\Phi}(s) \le \inf_{s\ge\xi} \overline{\Phi}(s)$ *for* $\xi \in \mathbb{R}$;
- *(b)* $f(u,u) \ne 0, u \in (0, \inf_{\xi\in\mathbb{R}} \overline{\Phi}(\xi)] \cup [\sup_{\xi\in\mathbb{R}} \underline{\Phi}(\xi), \boldsymbol{S})$;
- *(c)* $\overline{\Phi}'(\xi+) \le \overline{\Phi}'(\xi-)$, $\underline{\Phi}'(\xi+) \ge \underline{\Phi}'(\xi-)$ *for* $\xi \in \mathbb{T}$, *where* $\Phi'(\xi+) = \lim_{t\to\xi+} \Phi'(t)$ *and* $\Phi'(\xi-) = \lim_{t\to\xi-} \Phi'(t)$.

Then (8) and (9) has a monotone solution connecting 0 *with* $\boldsymbol{S}$ *in the sense of*

$$\lim_{\xi\to-\infty} \Phi(\xi) = 0, \ \lim_{\xi\to\infty} \Phi(\xi) = \boldsymbol{S}.$$

Proof. It suffices to prove that the operator P has a fixed point satisfying (9). Now we employ Schauder fixed point theorem to obtain this conclusion.

Define a set Γ by

$$\Gamma = \left\{\Phi = (\phi_1,\cdots,\phi_n) \in C(\mathbb{R},\mathbb{R}^n) \;\middle|\; \begin{array}{l} \text{(i) } \underline{\Phi}(\xi) \le \Phi(\xi) \le \overline{\Phi}(\xi); \\ \text{(ii) } \Phi(\xi) \text{ is nondecreasing for } \xi \in \mathbb{R} \end{array}\right\}.$$

By (a), Γ is nonempty. Moreover, it is evident that Γ is bounded, convex and closed with respect to $\|\cdot\|_\nu$.

We know that $P(\Phi)(\xi)$ is a monotone operator for $0 \le \Phi(\xi) \le \mathbf{S}$, i.e., $P(\Phi_1)(\xi) \le P(\Phi_2)(\xi)$ for $0 \le \Phi_1(\xi) \le \Phi_2(\xi) \le \mathbf{S}$. Furthermore, $P\Gamma \subset \Gamma$. In fact, if $\Phi \in \Gamma$ such that $\Phi(\xi) \le \Phi(\xi + s)$ for any $s \ge 0$ and $\xi \in \mathbb{R}$, then the monotonicity of P implies that

$$P(\Phi)(\xi) \le P(\Phi)(\xi + s), \ s \ge 0, \ \xi \in \mathbb{R}.$$

Thus the condition (ii) of Γ is true. In order to prove the condition (i) of Γ, it suffices to verify that

$$\underline{\Phi}(\xi) \le P(\underline{\Phi})(\xi) \le P(\overline{\Phi})(\xi) \le \overline{\Phi}(\xi), \ \xi \in \mathbb{R} \tag{13}$$

since the monotonicity of P indicates that

$$P(\underline{\Phi})(\xi) \le P(\Phi)(\xi) \le P(\overline{\Phi})(\xi), \ \ \underline{\Phi}(\xi) \le \Phi(\xi) \le \overline{\Phi}(\xi).$$

If $\xi \in \mathbb{R} \setminus \mathbb{T}$, then Definition 2.3 leads to

$$\begin{aligned}
&P_i(\underline{\Phi})(\xi) \\
&= \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\left[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{i2}(\xi-s)}\right] H_i(\underline{\Phi})(s)ds \\
&= \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\sum_{j=1}^{m+1}\int_{T_{j-1}}^{T_j} \min\left\{e^{\lambda_{i1}(\xi-s)}, e^{\lambda_{i2}(\xi-s)}\right\} H_i(\underline{\Phi})(s)ds \\
&\geq \frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\sum_{j=1}^{m+1}\int_{T_{j-1}}^{T_j} \min\left\{e^{\lambda_{i1}(\xi-s)}, e^{\lambda_{i2}(\xi-s)}\right\} \\
&\quad \times \left(\beta_i \underline{\phi}_i(s) + c\underline{\phi}_i'(s) - d_i \underline{\phi}_i''(s)\right) ds \\
&= \underline{\phi}_i(\xi) + \frac{1}{\lambda_{i2}-\lambda_{i1}}\left[\sum_{j=1}^{m} \min\left\{e^{\lambda_{i2}(\xi-T_i)}, e^{\lambda_{i1}(\xi-T_i)}\right\}\left(\underline{\phi}_i'(T_j+) - \underline{\phi}_i'(T_j-)\right)\right] \\
&\geq \underline{\phi}_i(\xi),\ i = 1, 2, \cdots, n,
\end{aligned}$$

where $T_0 = -\infty, T_{m+1} = \infty$. Then the continuity of $\underline{\Phi}(\xi)$ and $P(\underline{\Phi})(\xi)$ implies that $P(\underline{\Phi})(\xi) \geq \underline{\Phi}(\xi)$ for any $\xi \in \mathbb{R}$. Similarly, we can prove that (13) holds for all $\xi \in \mathbb{R}$.

Assume that $\Phi, \Psi \in \Gamma$. Then (G) and (12) indicate that

$$|H_i(\Phi)(\xi) - H_i(\Psi)(\xi)| e^{-\nu|\xi|} \leq [L + \beta_i + L|G(\nu)|] \|\Phi - \Psi\|_\nu .$$

We further have the following estimate

$$\begin{aligned}
&[d_i(\lambda_{i2}-\lambda_{i1})] |P_i(\Phi)(\xi) - P_i(\Psi)(\xi)| e^{-\nu|\xi|} \\
&\leq e^{-\nu|\xi|}\left[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi-s)} + \int_{\xi}^{\infty} e^{\lambda_{i2}(\xi-s)}\right] |H_i(\Phi)(s) - H_i(\Psi)(s)| ds \\
&\leq [L + \beta_i + L|G(\nu)|] \|\Phi - \Psi\|_\nu \left[\int_{-\infty}^{\xi} e^{(\lambda_{i1}+\nu)(\xi-s)} + \int_{\xi}^{\infty} e^{(\lambda_{i2}-\nu)(\xi-s)}\right] ds \\
&= [L + \beta_i + L|G(\nu)|] \left[\frac{1}{\lambda_{i2}-\nu} - \frac{1}{\lambda_{i1}+\nu}\right] \|\Phi - \Psi\|_\nu
\end{aligned}$$

for $i = 1, \cdots, n$, which implies that $P : \Gamma \to \Gamma$ is continuous in the decay norm. Furthermore, note that $P(\Phi)(\xi) \to 0$ is uniformly convergent as $\xi \to \pm\infty$ in the decay norm and $P(\Phi)(\xi)$ is equicontinuous and totally bounded, so $P : \Gamma \to \Gamma$ is compact, hence, is completely continuous in the decay norm.

Thus, P has a fixed point Φ^* in Γ by Schauder fixed point theorem, which makes (8) true. Furthermore, the condition (b) and the monotonicity of Φ^* imply that the fixed point Φ^* also satisfies (9). The proof is complete. □

Remark 2.5. In Theorem 2.4, we do not require the monotonicity of upper and lower solutions, which is weaker than the conditions in Wang et al. [35] and makes the construction of upper and lower solutions easier.

2.2. **Traveling wavefronts of (1).** From the previous section we know that a traveling wavefront of (1) must satisfy

$$\begin{cases} d_1\phi_1''(\xi) - c\phi_1'(\xi) + r_1\phi_1(\xi)\left[1 - a_1\phi_1(\xi) + b_1(g_1 * \phi_2)(\xi)\right] = 0, \\ d_2\phi_2''(\xi) - c\phi_2'(\xi) + r_2\phi_2(\xi)\left[1 - a_2\phi_2(\xi) + b_2(g_2 * \phi_1)(\xi)\right] = 0, \end{cases} \tag{14}$$

and we are interested in the following asymptotic boundary conditions

$$\lim_{\xi\to-\infty}(\phi_1(\xi),\phi_2(\xi))=(0,0)\,,\ \lim_{\xi\to\infty}(\phi_1(\xi),\phi_2(\xi))=(k_1,k_2)\,, \tag{15}$$

where $(g_1*\phi_2)(\xi)$ and $(g_2*\phi_1)(\xi)$ are defined by

$$\begin{cases}(g_1*\phi_2)(\xi)=\int_0^{+\infty}\int_{-\infty}^{\infty}\frac{1}{\tau_1}e^{-\frac{\theta}{\tau_1}}\frac{1}{\sqrt{4\pi d_2\theta}}e^{-\frac{s^2}{4d_2\theta}}\phi_2(\xi-c\theta-s)\,dsd\theta,\\ (g_2*\phi_1)(\xi)=\int_0^{+\infty}\int_{-\infty}^{\infty}\frac{1}{\tau_2}e^{-\frac{\theta}{\tau_2}}\frac{1}{\sqrt{4\pi d_1\theta}}e^{-\frac{s^2}{4d_1\theta}}\phi_1(\xi-c\theta-s)\,dsd\theta.\end{cases} \tag{16}$$

For any given d_1,d_2,c,τ_1,τ_2, we always can choose $\nu>0$ such that

$$\tau_{3-i}\nu c+d_i\nu^2\tau_{3-i}<1, i=1,2,$$

which implies that g_1 and g_2 satisfy (G). Moreover, it is easy to see that f satisfies (QM). Thus, in order to prove the existence of traveling wavefronts, it suffices to construct proper upper and lower solutions of (14).

For $c>c^*$, let λ_i be the smaller positive root of $d_i\lambda^2-c\lambda+r_i=0$. Now, we define the following continuous functions

$$\overline{\phi}_1(\xi)=\min\left\{k_1\left[e^{\lambda_1\xi}+qe^{\eta\lambda_1\xi}\right],k_1\right\},\quad \underline{\phi}_1(\xi)=\max\left\{k_1\left[e^{\lambda_1\xi}-qe^{\eta\lambda_1\xi}\right],0\right\},$$
$$\overline{\phi}_2(\xi)=\min\left\{k_2\left[e^{\lambda_2\xi}+qe^{\eta\lambda_2\xi}\right],k_2\right\},\quad \underline{\phi}_2(\xi)=\max\left\{k_2\left[e^{\lambda_2\xi}-qe^{\eta\lambda_2\xi}\right],0\right\},$$

where $1<\eta<2$ such that $\eta\lambda_1<\lambda_1+\lambda_2,\eta\lambda_2<\lambda_1+\lambda_2$ and

$$\Delta_i(\eta\lambda_i,c):=d_i(\eta\lambda_i)^2-c\eta\lambda_i+r_i<0,\ i=1,2,$$

and $q\geq\max\{q_1,q_2,1\}$ with

$$q_1=\max\left\{\frac{-4r_2}{\Delta_2(\eta\lambda_2,c)},\frac{-4r_1}{\Delta_1(\eta\lambda_1,c)},\left(\frac{-4r_2}{\Delta_2(\eta\lambda_2,c)}\right)^{\frac{\eta\lambda_2}{\lambda_1}},\left(\frac{-4r_1}{\Delta_1(\eta\lambda_1,c)}\right)^{\frac{\eta\lambda_1}{\lambda_2}}\right\},$$
$$q_2=\max\left\{-\frac{r_2a_2k_2}{\Delta_2(\eta\lambda_2,c)},-\frac{r_1a_1k_1}{\Delta_1(\eta\lambda_1,c)}\right\}.$$

Without loss of generality, we assume that

$$\overline{\phi}_1(\xi)=\begin{cases}k_1[e^{\lambda_1\xi}+qe^{\eta\lambda_1\xi}],\xi<\xi_1,\\ k_1,\ \xi\geq\xi_1,\end{cases}\qquad \overline{\phi}_2(\xi)=\begin{cases}k_2[e^{\lambda_2\xi}+qe^{\eta\lambda_2\xi}],\xi<\xi_2,\\ k_2,\ \xi\geq\xi_2,\end{cases}$$
$$\underline{\phi}_1(\xi)=\begin{cases}k_1[e^{\lambda_1\xi}-qe^{\eta\lambda_1\xi}],\xi<\xi_3,\\ 0,\xi\geq\xi_3,\end{cases}\qquad \underline{\phi}_2(\xi)=\begin{cases}k_2[e^{\lambda_2\xi}-qe^{\eta\lambda_2\xi}],\xi<\xi_4,\\ 0,\xi\geq\xi_4.\end{cases}$$

Lemma 2.6. *For any given $c>c^*$, assume that τ_1 (τ_2) is large enough if $\lambda_1(c)>\lambda_1(c)$ ($\lambda_1(c)<\lambda_1(c)$). Then $(\overline{\phi}_1,\overline{\phi}_2)$ is an upper solution of (14).*

Proof. It suffices to verify the definition of the upper solution. We first give estimates of $(g_1*\overline{\phi}_2)(\xi)$ and $(g_2*\overline{\phi}_1)(\xi)$. It is clear that

$$\begin{aligned}&(g_2*\overline{\phi}_1)(\xi)\leq\left(g_2*\left(e^{\lambda_1\cdot}+qe^{\eta\lambda_1\cdot}\right)\right)(\xi)\\ &=k_1\int_0^{+\infty}\int_{-\infty}^{\infty}\frac{1}{\tau_2}e^{-\frac{\theta}{\tau_2}}\frac{1}{\sqrt{4\pi d_1\theta}}e^{-\frac{s^2}{4d_1\theta}}e^{\lambda_1(\xi-c\theta-s)}dsd\theta\\ &\quad+qk_1\int_0^{+\infty}\int_{-\infty}^{\infty}\frac{1}{\tau_2}e^{-\frac{\theta}{\tau_2}}\frac{1}{\sqrt{4\pi d_1\theta}}e^{-\frac{s^2}{4d_1\theta}}e^{\eta\lambda_1(\xi-c\theta-s)}dsd\theta\\ &=I_1+I_2,\end{aligned}$$

in which the definitions of I_1, I_2 are clear. Then it follows that

$$\begin{aligned}
I_1 &= k_1 \int_0^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2}} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{s^2}{4d_1\theta}} e^{\lambda_1(\xi - c\theta - s)} ds d\theta \\
&= k_1 e^{\lambda_1 \xi} \int_0^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2} - \lambda_1 c\theta} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{s^2}{4d_1\theta} - \lambda_1 s} ds d\theta \\
&= k_1 e^{\lambda_1 \xi} \int_0^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2} - \lambda_1 c\theta + d_1 \lambda_1^2 \theta} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{(s+2d_1\lambda_1\theta)^2}{4d_1\theta}} ds d\theta \\
&= k_1 e^{\lambda_1 \xi} \int_0^{+\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2} - r_1 \theta} d\theta \\
&= \frac{k_1 e^{\lambda_1 \xi}}{1 + r_1 \tau_2}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= qk_1 \int_0^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2}} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{s^2}{4d_1\theta}} e^{\eta\lambda_1(\xi - c\theta - s)} ds d\theta \\
&= qk_1 e^{\eta\lambda_1 \xi} \int_0^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2} - c\eta\lambda_1 \theta} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{s^2}{4d_1\theta} - \eta\lambda_1 s} ds d\theta \\
&= qk_1 e^{\eta\lambda_1 \xi} \int_0^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2} - c\eta\lambda_1 \theta + d_1 \eta^2 \lambda_1^2 \theta} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{(s+2d_1\eta\lambda_1\theta)^2}{4d_1\theta}} ds d\theta \\
&= qk_1 e^{\eta\lambda_1 \xi} \int_0^{+\infty} \frac{1}{\tau_2} e^{-\frac{\theta}{\tau_2} - c\eta\lambda_1\theta + d_1\eta^2\lambda_1^2\theta} d\theta \\
&= \frac{qk_1 e^{\eta\lambda_1 \xi}}{1 + \tau_2 \left(c\eta\lambda_1 - d_1 \eta^2 \lambda_1^2\right)}.
\end{aligned}$$

We now use the above estimate to confirm the inequality on $\overline{\phi}_2(\xi)$.

If $\xi > \xi_2$, then $\overline{\phi}_2(\xi) = k_2$ and $\left(g_2 * \overline{\phi}_1\right)(\xi) \le k_1$, it is easy to see that

$$\begin{aligned}
& d_2 \overline{\phi}_2''(\xi) - c\overline{\phi}_2'(\xi) + r_2 \overline{\phi}_2(\xi) \left(1 - a_2 \overline{\phi}_2(\xi) + b_2 \left(g_2 * \overline{\phi}_1\right)(\xi)\right) \\
& \quad = r_1 k_1 \left(1 - a_1 k_1 + b_1 \left(g * \overline{\phi}_1\right)(\xi)\right) \le 0.
\end{aligned}$$

If $\xi < \xi_2 < -\frac{\ln q}{\eta\lambda_2} < 0$, then $\overline{\phi}_2(\xi) = k_2 \left[e^{\lambda_2 \xi} + q e^{\eta\lambda_2 \xi}\right]$, and we shall show that

$$\begin{aligned}
& d_2 \overline{\phi}_2''(\xi) - c\overline{\phi}_2'(\xi) + r_2 \overline{\phi}_2(\xi) \left(1 - a_2 \overline{\phi}_2(\xi) + b_2 \left(g_2 * \overline{\phi}_1\right)(\xi)\right) \\
& \quad \le \Delta_2(\eta\lambda_2, c) q k_2 e^{\eta\lambda_2 \xi} + k_2 \left(e^{\lambda_2 \xi} + q e^{\eta\lambda_2 \xi}\right) \\
& \qquad \times \left[-a_2 k_2 \left(e^{\lambda_2 \xi} + q e^{\eta\lambda_2 \xi}\right) + b_2 \left(\frac{k_1 e^{\lambda_1 \xi}}{1 + r_1 \tau_2} + \frac{qk_1 e^{\eta\lambda_1 \xi}}{1 + \tau_2 \left(c\eta\lambda_1 - d_1 \eta^2 \lambda_1^2\right)}\right)\right] \\
& \quad \le 0.
\end{aligned} \tag{17}$$

If $\lambda_2 \le \lambda_1$, then $a_2 k_2 > b_2 k_1$ implies that

$$-a_2 k_2 \left(e^{\lambda_2 \xi} + q e^{\eta\lambda_2 \xi}\right) + b_2 \left(\frac{k_1 e^{\lambda_1 \xi}}{1 + r_1 \tau_2} + \frac{qk_1 e^{\eta\lambda_1 \xi}}{1 + \tau_2 \left(c\eta\lambda_1 - d_1 \eta^2 \lambda_1^2\right)}\right) < 0$$

and (17) is clear by the definition of q_1. If $\lambda_2 > \lambda_1$ holds and τ_2 is large enough (e.g., $r_1 \tau_2 \ge q b_1 k_2$) such that

$$-a_2 \overline{\phi}_2(\xi) + b_2 \left(\frac{k_1 e^{\lambda_1 \xi}}{1 + r_1 \tau_2} + \frac{qk_1 e^{\eta\lambda_1 \xi}}{1 + \tau_2 \left(c\eta\lambda_1 - d_1 \eta^2 \lambda_1^2\right)}\right) \le 2 e^{\lambda_1 \xi}, \;\; \xi = \xi_2.$$

Then

$$\begin{aligned}
&\Delta_2(\eta\lambda_2, c)qk_2e^{\eta\lambda_2\xi} + k_2\left(e^{\lambda_2\xi} + qe^{\eta\lambda_2\xi}\right) \\
&\quad\times\left[-a_2k_2\left(e^{\lambda_2\xi} + qe^{\eta\lambda_2\xi}\right) + b_2\left(\frac{k_1e^{\lambda_1\xi}}{1+r_1\tau_2} + \frac{qk_1e^{\eta\lambda_1\xi}}{1+\tau_2\left(c\eta\lambda_1 - d_1\eta^2\lambda_1^2\right)}\right)\right] \\
&\quad\le \Delta_2(\eta\lambda_2, c)qk_2e^{\eta\lambda_2\xi} + 2r_2k_2e^{\lambda_1\xi}\left(e^{\lambda_2\xi} + qe^{\eta\lambda_2\xi}\right).
\end{aligned}$$

The definition of q_1 implies that

$$-q\Delta_2(\eta\lambda_2, c) \ge 4r_2, \quad -\Delta_2(\eta\lambda_2, c) \ge 4r_2e^{\lambda_1\xi_2},$$

which further indicates that (17) holds and completes the verification on $\overline{\phi}_2(\xi)$.

In a similar way, we can confirm that $(\overline{\phi}_1(\xi), \overline{\phi}_2(\xi))$ is an upper solution of (8). The proof is complete. □

Lemma 2.7. *$(\underline{\phi}_1, \underline{\phi}_2)$ is a lower solution of (14).*

Proof. We first verify the inequality on $\underline{\phi}_2(\xi)$. If $\xi \ge \xi_4$, then the result is clear. Otherwise, $\xi < \xi_4$ implies $\left(g_2 * \underline{\phi}_1\right)(\xi) \ge 0$ and

$$\begin{aligned}
&d_2\underline{\phi}_2''(\xi) - c\underline{\phi}_2'(\xi) + r_2\underline{\phi}_2(\xi)\left[1 - a_2\underline{\phi}_2(\xi) + b_2\left(g_2 * \underline{\phi}_1\right)(\xi)\right] \\
&\quad\ge d_2k_2\left(e^{\lambda_2\xi} - qe^{\eta\lambda_2\xi}\right)'' - ck_2\left(e^{\lambda_2\xi} - qe^{\eta\lambda_2\xi}\right)'' \\
&\qquad + r_2k_2\left(e^{\lambda_2\xi} - qe^{\eta\lambda_2\xi}\right)\left[1 - a_2k_2\left(e^{\lambda_2\xi} - qe^{\eta\lambda_2\xi}\right)\right] \\
&\quad\ge -qk_2\Delta_2(\eta\lambda_2, c)e^{\eta\lambda_2\xi} - r_2a_2k_2^2e^{2\lambda_2\xi} \\
&\quad\ge 0, \ \xi < \xi_4 \le 0,
\end{aligned}$$

in which the last inequality can be seen from the definition of q_2.

In a similar way, we can prove that $(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi))$ is a lower solution of (14). The proof is complete. □

By what we have done, we may obtain the following result.

Theorem 2.8. *Assume that $c > c^*$ holds and Lemma 2.6 is true. Then (14) has a monotone solution $(\phi_1(\xi), \phi_2(\xi))$ satisfying (15) and*

$$\lim_{\xi\to-\infty}\left(e^{-\lambda_1\xi}\phi_1(\xi), e^{-\lambda_2\xi}\phi_2(\xi)\right) = (k_1, k_2), \tag{18}$$

$$\lim_{\xi\to-\infty}\left(e^{-\lambda_1\xi}\phi_1'(\xi), e^{-\lambda_2\xi}\phi_2'(\xi)\right) = (k_1\lambda_1, k_2\lambda_2). \tag{19}$$

In Theorem 2.8, we have some requirements on τ_i. If the wave speed is large, the requirements are not necessary. For the purpose, we further define constants $\lambda_3 = \frac{c+\sqrt{c^2-4d_1r_1}}{2d_1}$, $\lambda_4 = \frac{c+\sqrt{c^2-4d_2r_2}}{2d_2}$ and

$$\Lambda = (\lambda_1, \min\{\lambda_3, \lambda_1+\lambda_2\}) \bigcap (\lambda_2, \min\{\lambda_4, \lambda_1+\lambda_2\}).$$

Remark 2.9. For any given $d_i, r_i, i = 1, 2$, there exists $\overline{c} \ge c^*$ such that Λ is nonempty for all $c > \overline{c}$. In addition, $\overline{c} = c^*$ holds for some cases, such as $d_1 = d_2$.

Theorem 2.10. *Assume that $c > c^*$ holds such that Λ is nonempty. Then the results of Theorem 2.8 hold.*

Proof. Let $\lambda \in \Lambda$. Define continuous functions as follows

$$\overline{\chi}_1(\xi) = \min\left\{k_1\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right], k_1\right\}, \quad \underline{\chi}_1(\xi) = \max\left\{k_1\left[e^{\lambda_1\xi} - pe^{\eta\lambda_1\xi}\right], 0\right\},$$
$$\overline{\chi}_2(\xi) = \min\left\{k_2\left[e^{\lambda_2\xi} + pe^{\lambda\xi}\right], k_2\right\}, \quad \underline{\chi}_2(\xi) = \max\left\{k_2\left[e^{\lambda_2\xi} - pe^{\eta\lambda_2\xi}\right], 0\right\}.$$

We shall prove that $(\overline{\chi}_1(\xi), \overline{\chi}_2(\xi)), (\underline{\chi}_1(\xi), \underline{\chi}_2(\xi))$ are a pair of upper and lower solutions of (8) if $p > 1$ is large enough. We now verify the inequality on $\overline{\chi}_1(\xi)$. If $\overline{\chi}_1(\xi) = k_1$, then the result is clear. If $\overline{\chi}_1(\xi) = k_1\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right]$, then $(g_1 * \overline{\chi}_2)(\xi) \leq k_2\left[\frac{e^{\lambda_2\xi}}{1+r_2\tau_1} + \frac{pe^{\lambda\xi}}{1+\tau_1(c\lambda - d_2\lambda^2)}\right]$ such that

$$\begin{aligned}
&d_1\overline{\chi}_1''(\xi) - c\overline{\chi}_1'(\xi) + r_1\overline{\chi}_1(\xi)\left[1 - a_1\overline{\chi}_1(\xi) + b_1(g_1 * \overline{\chi}_2)(\xi)\right] \\
&\leq pk_1\Delta_1(\eta\lambda_1, c)e^{\lambda\xi} + k_1\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right]\left\{-a_1k_1\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right]\right. \\
&\quad \left. + b_1k_2\left[\frac{e^{\lambda_2\xi}}{1 + r_2\tau_1} + \frac{pe^{\lambda\xi}}{1 + \tau_1(c\lambda - d_2\lambda^2)}\right]\right\} \\
&\leq pk_1\Delta_1(\eta\lambda_1, c)e^{\lambda\xi} + k_1\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right]\left\{-a_1k_1e^{\lambda_1\xi} + \frac{b_1k_2e^{\lambda_2\xi}}{1 + r_2\tau_1}\right\} \\
&\leq pk_1\Delta_1(\eta\lambda_1, c)e^{\lambda\xi} + \frac{k_1b_1k_2e^{\lambda_2\xi}\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right]}{1 + r_2\tau_1}.
\end{aligned}$$

Note that $\lambda < \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda\}$, then

$$p\Delta_1(\eta\lambda_1, c)e^{\lambda\xi} + \frac{b_1k_2e^{\lambda_2\xi}\left[e^{\lambda_1\xi} + pe^{\lambda\xi}\right]}{1 + r_2\tau_1} \leq 0$$

is true if

$$p > \max\left\{\frac{2b_1k_2}{-\Delta_1(\eta\lambda_1, c)(1 + r_2\tau_1)}, \left(\frac{2b_1k_2}{-\Delta_1(\eta\lambda_1, c)(1 + r_2\tau_1)}\right)^{\frac{\lambda}{\lambda_2}}, 1\right\}.$$

The other part can be confirmed by a similar way, we also refer to Lin et al. [17] for some estimates. Using Theorem 2.4, we complete the proof. ☐

Note that Theorem 2.10 is independent of the size of τ_1, τ_2, letting $\tau_1 = \tau_2 = 0$ in Theorem 2.10, we immediately get the following result.

Theorem 2.11. *Assume that $c > c^*$ holds such that Λ is nonempty. Then (3) has a traveling wavefront connecting E_0 with E^*.*

Remark 2.12. Theorem 2.11 is different from that in Li et al. [12, Theorem 4.1] since we knew the equilibria involved. Unfortunately, such a result cannot be proved for arbitrary d_1, d_2, r_1, r_2, and we shall further consider the problem in our future papers.

2.3. **Traveling wavefronts of the auxiliary system.** As we have mentioned in Section 1, (1) can be rewritten as (4) formally. Let $\phi_3(\xi) = (g_1 * \phi_2)(\xi)$ and $\phi_4(\xi) = (g_2 * \phi_1)(\xi)$. Then

$$\begin{cases}
c\phi_1'(\xi) = d_1\phi_1''(\xi) + r_1\phi_1(\xi)\left[1 - a_1\phi_1(\xi) + b_1\phi_3(\xi)\right], \\
c\phi_2'(\xi) = d_2\phi_2''(\xi) + r_2\phi_2(\xi)\left[1 - a_2\phi_2(\xi) + b_2\phi_4(\xi)\right], \\
c\phi_3'(\xi) = d_3\phi_3''(\xi) + \frac{1}{\tau_1}\phi_2(\xi) - \frac{1}{\tau_1}\phi_3(\xi), \\
c\phi_4'(\xi) = d_4\phi_4''(\xi) + \frac{1}{\tau_2}\phi_1(\xi) - \frac{1}{\tau_2}\phi_4(\xi),
\end{cases} \tag{20}$$

with the asymptotical boundary conditions

$$\begin{cases} \lim_{\xi\to-\infty}(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi),\phi_4(\xi)) = (0,0,0,0)\,, \\ \lim_{\xi\to\infty}(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi),\phi_4(\xi)) = (k_1,k_2,k_3,k_4)\,, \end{cases} \tag{21}$$

where $k_3 = k_2, k_4 = k_1$.

Theorem 2.13. *Assume that the conditions in Theorem 2.8 hold. Then (4) has a traveling wavefront $\Phi = (\phi_1,\phi_2,\phi_3,\phi_4)$ satisfying (20) and (21). Furthermore,*

$$\begin{aligned} &\lim_{\xi\to-\infty}\left(e^{-\lambda_1\xi}\phi_1(\xi), e^{-\lambda_2\xi}\phi_2(\xi), e^{-\lambda_2\xi}\phi_3(\xi), e^{-\lambda_1\xi}\phi_4(\xi)\right) \\ &\quad = \left(k_1, k_2, \frac{k_2}{1+r_2\tau_1}, \frac{k_1}{1+r_1\tau_2}\right), \\ &\lim_{\xi\to-\infty}\left(e^{-\lambda_1\xi}\phi_1'(\xi), e^{-\lambda_2\xi}\phi_2'(\xi), e^{-\lambda_2\xi}\phi_3'(\xi), e^{-\lambda_1\xi}\phi_4'(\xi)\right) \\ &\quad = \left(k_1\lambda_1, k_2\lambda_2, \frac{k_2\lambda_2}{1+r_2\tau_1}, \frac{k_1\lambda_1}{1+r_1\tau_2}\right). \end{aligned}$$

Proof. We further define the continuous functions as follows

$$\begin{aligned} \overline{\phi}_3(\xi) &= \min\left\{k_3\left[\frac{e^{\lambda_2\xi}}{1+r_2\tau_1} + \frac{qe^{\eta\lambda_2\xi}}{1+\tau_1\left(c\eta\lambda_2 - d_2\eta^2\lambda_2^2\right)}\right], k_3\right\}, \\ \overline{\phi}_4(\xi) &= \min\left\{k_4\left[\frac{e^{\lambda_1\xi}}{1+r_1\tau_2} + \frac{qe^{\eta\lambda_1\xi}}{1+\tau_2\left(c\eta\lambda_1 - d_1\eta^2\lambda_1^2\right)}\right], k_4\right\}, \\ \underline{\phi}_3(\xi) &= \max\left\{k_3\left[\frac{e^{\lambda_2\xi}}{1+r_2\tau_1} - \frac{qe^{\eta\lambda_2\xi}}{1+\tau_1\left(c\eta\lambda_2 - d_2\eta^2\lambda_2^2\right)}\right], 0\right\}, \\ \underline{\phi}_4(\xi) &= \max\left\{k_4\left[\frac{e^{\lambda_1\xi}}{1+r_1\tau_2} - \frac{qe^{\eta\lambda_1\xi}}{1+\tau_2\left(c\eta\lambda_1 - d_1\eta^2\lambda_1^2\right)}\right], 0\right\}. \end{aligned}$$

Then $(\overline{\phi}_1,\overline{\phi}_2,\overline{\phi}_3,\overline{\phi}_4)$ and $(\underline{\phi}_1,\underline{\phi}_2,\underline{\phi}_3,\underline{\phi}_4)$ are upper and lower solutions of (20), respectively. By Lemma 2.6, we only consider the inequalities of $\overline{\phi}_3,\overline{\phi}_4,\underline{\phi}_3,\underline{\phi}_4$.

If $\overline{\phi}_3(\xi) = k_3$, then the result is clear. Otherwise, we see that

$$\begin{aligned} &d_2\overline{\phi}_3''(\xi) - c\overline{\phi}_3'(\xi) - \frac{1}{\tau_1}\overline{\phi}_3(\xi) + \frac{1}{\tau_1}\overline{\phi}_2(\xi) \\ &\le -\frac{1+\tau_1 r_2}{\tau_1}\overline{\phi}_3(\xi) + \frac{1}{\tau_1}\overline{\phi}_2(\xi) + \frac{\Delta_2(\eta\lambda_2,c)\,qk_2e^{\eta\lambda_2\xi}}{1+\tau_1\left(c\eta\lambda_2 - d_2\eta^2\lambda_2^2\right)} \\ &\le k_2\left[\frac{\Delta_2(\eta\lambda_2,c)\,qe^{\eta\lambda_2\xi}}{1+\tau_1\left(c\eta\lambda_2 - d_2\eta^2\lambda_2^2\right)} + \frac{1}{\tau_1}qe^{\eta\lambda_2\xi} - \frac{1+\tau_1 r_2}{\tau_1}\frac{qe^{\eta\lambda_2\xi}}{1+\tau_1\left(c\eta\lambda_2 - d_2\eta^2\lambda_2^2\right)}\right] \\ &= 0, \end{aligned}$$

which confirm the inequality for $\overline{\phi}_3$. In a similar way, we can finish the verification.

Clearly, Theorem 2.4 implies what we wanted. The proof is complete. □

Similar to Theorem 2.11, we have the following result.

Theorem 2.14. *Assume that $c > c^*$ holds such that Λ is nonempty. Then the results of Theorem 2.13 hold.*

3. **Initial value problem.** In order to study the stability of traveling wavefronts of (1), we need to consider the corresponding initial value problem of (1)

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t)\left[1 - a_1 u_1(x,t) + b_1 (g_1 * u_2)(x,t)\right], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + r_2 u_2(x,t)\left[1 - a_2 u_2(x,t) + b_2 (g_2 * u_1)(x,t)\right], \\ u_1(x,s) = \psi_1(x,s),\ u_2(x,s) = \psi_2(x,s), \end{cases} \tag{22}$$

in which $x \in \mathbb{R}, s \leq 0, t > 0$ and $(\psi_1(x,s), \psi_2(x,s)) \in C(\mathbb{R} \times (-\infty, 0], \mathbb{R}^2)$ and $E_0 \leq (\psi_1(x,s), \psi_2(x,s)) \leq E^*$ for all $x \in \mathbb{R}, t \leq 0$.

We first introduce some notations. For $n = 1, 2$ or 4, denote $X = C(\mathbb{R}, \mathbb{R}^n)$ as

$$X = \{u :\ u(x) \text{ is a bounded and uniformly continuous function from } \mathbb{R} \text{ to } \mathbb{R}^n\}.$$

Then X is a Banach space under the general supremum norm $\|\cdot\|$. Define

$$X^+ = \{u \in X : u(x) \geq 0, x \in \mathbb{R}\}$$

and

$$C_{[0,K]} = \{u(x) : u(x) \in C(\mathbb{R}, \mathbb{R}^4) \text{ and } 0 \leq u(x) \leq K \text{ for all } x \in \mathbb{R}\}$$

with $K = (k_1, k_2, k_3, k_4)$. Fix constants

$$\beta_1 = 2r_1 a_1 k_1,\ \beta_2 = 2r_2 a_2 k_2,\ \beta_3 = \frac{1}{\tau_2},\ \beta_4 = \frac{1}{\tau_1},$$

and $T(t) = (T_1(t), \cdots, T_n(t)) : X \to X$ as

$$T_i(t)\psi_i(x) = \frac{e^{-\beta_i t}}{\sqrt{4\pi d_i t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4 d_i t}} \psi_i(x-y)dy,$$

where $\Psi(x) = (\psi_1(x), \cdots, \psi_n(x)) \in X$ and $t > 0$. It is clear that $T(t)$ is a real analytic semigroup on X (see Pazy [27], Smith and Zhao [31] and Smoller [32]).

Using semigroup theory, upper and lower solutions technique and the theory of abstract functional differential equations (see Martin and Smith [24], Ruan and Wu [29]), we have the following result.

Theorem 3.1. *For any $x \in \mathbb{R}$ and $t > 0$, (22) has a mild solution which is continuous in $x \in \mathbb{R},\ t > 0$ and is given by*

$$\begin{cases} u_1(x,t) = T_1(t)\psi_1(x,0) + \int_0^t T_1(t-s)F_1(u_1,u_2)(x,s)ds, \\ u_2(x,t) = T_2(t)\psi_2(x,0) + \int_0^t T_2(t-s)F_2(u_1,u_2)(x,s)ds, \end{cases}$$

in which F_1 and F_2 are defined by

$$F_1(u_1,u_2)(x,t) = \beta_1 u_1(x,t) + r_1 u_1(x,t)\left[1 - a_1 u_1(x,t) + b_1(g_1 * u_2)(x,t)\right],$$
$$F_2(u_1,u_2)(x,t) = \beta_2 u_2(x,t) + r_2 u_2(x,t)\left[1 - a_2 u_2(x,t) + b_2(g_2 * u_1)(x,t)\right].$$

Moreover, assume that $(v_1(x,t), v_2(x,t))$ and $(u_1(x,t), u_2(x,t))$ are mild solutions of (22) with initial values $(\psi_1(x,s), \psi_2(x,s))$ and $(\varphi_1(x,s), \varphi_2(x,s))$, respectively. Then

$$E_0 \leq (v_1(x,t), v_2(x,t)) \leq (u_1(x,t), u_2(x,t)) \leq E^*,\ x \in \mathbb{R},\ t > 0$$

provided that

$$E_0 \leq (\psi_1(x,s), \psi_2(x,s)) \leq (\varphi_1(x,s), \varphi_2(x,s)) \leq E^*,\ x \in \mathbb{R},\ s \leq 0.$$

In particular, the mild solution also satisfies the following property.

Theorem 3.2. *Assume that $u_1(x,t), u_2(x,t)$ are defined by (22). Let*

$$u_3(x,t) = (g_1 * u_2)(x,t), u_4(x,t) = (g_2 * u_1)(x,t) \text{ for all } x \in \mathbb{R}, t \geq 0.$$

Then $(u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))$ satisfies

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t) \left[1 - a_1 u_1(x,t) + b_1 u_3(x,t)\right], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + r_2 u_2(x,t) \left[1 - a_2 u_2(x,t) + b_2 u_4(x,t)\right], \\ \frac{\partial u_3(x,t)}{\partial t} = d_3 \Delta u_3(x,t) + \frac{1}{\tau_1} u_2(x,t) - \frac{1}{\tau_1} u_3(x,t), \\ \frac{\partial u_4(x,t)}{\partial t} = d_4 \Delta u_4(x,t) + \frac{1}{\tau_2} u_1(x,t) - \frac{1}{\tau_2} u_4(x,t), \\ (u_1(x,0), u_2(x,0), u_3(x,0), u_4(x,0)) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)), \end{cases} \tag{23}$$

in which $\psi_3(x) = (g_1 * \psi_2)(x,0), \psi_4(x) = (g_2 * \psi_1)(x,0)$.

The proof of Theorem 3.2 is similar to that of Lin and Li [16], so we omit it here. Moreover, since the solution of (23) is unique if $(\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \in X$, then Theorems 3.1 and 3.2 imply the following important fact.

Remark 3.3. The mild solution of (22) is also a classical solution that can be formulated by (23) if $\psi_3(x) = (g_1 * \psi_2)(x,0), \psi_4(x) = (g_2 * \psi_1)(x,0)$. Thus, we can investigate some properties of (1) by those of (4), which is our main idea in the rest of this paper.

Lemma 3.4. *Assume that*

$$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t)),$$
$$v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))$$

are solutions of (23) with the initial values

$$\Psi_1(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \text{ and } \Psi_2(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)),$$

respectively. Then $\Psi_1(x), \Psi_2(x) \in X^+$ with $\Psi_1(x) \leq \Psi_2(x), x \in \mathbb{R}$ implies that

$$0 \leq u(x,t) \leq v(x,t) \text{ for all } x \in \mathbb{R}, t > 0.$$

Lemma 3.4 implies a weaker version of the comparison principle formulated by Theorem 3.1, we give it as follows (we also refer to Ruan and Wu [29]).

Corollary 3.5. *Assume that $(v_1(x,t), v_2(x,t))$ and $(u_1(x,t), u_2(x,t))$ are mild solutions of (22) with initial value $(\psi_1(x,s), \psi_2(x,s))$ and $(\varphi_1(x,s), \varphi_2(x,s))$, respectively. Then*

$$0 \leq (v_1(x,t), v_2(x,t)) \leq (u_1(x,t), u_2(x,t)) \leq E^*, x \in \mathbb{R}, t > 0$$

provided that for all $x \in \mathbb{R}$

$$0 \leq (\psi_1(x,0), \psi_2(x,0)) \leq (\varphi_1(x,0), \varphi_2(x,0)) \leq E^*,$$
$$0 \leq \left((g_2 * \psi_1)(x,0), (g_1 * \psi_2)(x,0)\right) \leq \left((g_2 * \varphi_1)(x,0), (g_1 * \varphi_2)(x,0)\right) \leq E^*.$$

Lemma 3.6. *Assume that u and v are defined by Lemma 3.4. Then*

$$u_i(x,t) - v_i(x,t) \geq \frac{e^{-\beta_i(t-t_0)}}{\sqrt{4\pi d_i(t-t_0)}} e^{-\frac{(J+1)^2}{4d_i(t-t_0)}} \int_z^{z+1} \left[u_i(y,t_0) - v_i(y,t_0)\right] dy \geq 0$$

for $i = 1,2,3,4$, and any $J \geq 0$, $x, z \in \mathbb{R}$ with $|x - z| \leq J$, and $t > t_0 \geq 0$.

The proof is similar to that of Smith and Zhao [31, Theorem 2.2], so we omit it here. Note that the norm of $T(t)$ is less than 1, so the following lemma is clear by the Gronwall's inequality (we also refer to Wang et al. [37, Lemma 3.6]).

Lemma 3.7. *Assume that u, v are defined by Lemma 3.4 if $\Psi_2(x) \leq K$. Then there exists a constant $\mu > 0$ such that*

$$u_i(x,t) - v_i(x,t) \leq \min\left\{ e^{\mu t} \sum_{j=1}^{4} \|\psi_j(\cdot) - \varphi_j(\cdot)\|, k_i \right\}$$

for any $x \in \mathbb{R}, t \geq 0$, where $\|\cdot\|$ denotes the supremum norm in $C(\mathbb{R},\mathbb{R})$.

Definition 3.8. Assume that a continuous vector-valued function

$$\overline{u}(x,t) = (\overline{u}_1(x,t), \overline{u}_2(x,t), \overline{u}_3(x,t), \overline{u}_4(x,t)) \in X^+ \text{ for } t \in (0,T)$$

satisfies the inequalities

$$\begin{cases} \frac{\partial \overline{u}_1(x,t)}{\partial t} \geq d_1 \Delta \overline{u}_1(x,t) + r_1 \overline{u}_1(x,t)\left[1 - a_1 \overline{u}_1(x,t) + b_1 \overline{u}_3(x,t)\right], \\ \frac{\partial \overline{u}_2(x,t)}{\partial t} \geq d_2 \Delta \overline{u}_2(x,t) + r_2 \overline{u}_2(x,t)\left[1 - a_2 \overline{u}_2(x,t) + b_2 \overline{u}_4(x,t)\right], \\ \frac{\partial \overline{u}_3(x,t)}{\partial t} \geq d_3 \Delta \overline{u}_3(x,t) + \frac{1}{\tau_1} \overline{u}_2(x,t) - \frac{1}{\tau_1} \overline{u}_3(x,t), \\ \frac{\partial \overline{u}_4(x,t)}{\partial t} \geq d_4 \Delta \overline{u}_4(x,t) + \frac{1}{\tau_2} \overline{u}_1(x,t) - \frac{1}{\tau_2} \overline{u}_4(x,t), \\ \left(\overline{u}_1(x,0), \overline{u}_2(x,0), \overline{u}_3(x,0), \overline{u}_4(x,0)\right) \geq (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \end{cases} \tag{24}$$

for $x \in \mathbb{R}, t \in (0,T)$. Then $\overline{u}(x,t)$ is called an *upper* solution of (23) on $x \in \mathbb{R}$, $t \in (0,T)$. By reversing all the inequalities in (24), we can define a *lower* solution.

Lemma 3.9. *Assume that $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are upper and lower solutions of (23). Then $\underline{u}(x,0) \leq \overline{u}(x,0)$ implies that $\underline{u}(x,t) \leq \overline{u}(x,t)$ for all $x \in \mathbb{R}$ and $t \in (0,T)$. Furthermore, (23) has a unique classical solution $u(x,t)$ satisfying $\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t)$.*

Lemma 3.10. *Assume that $\overline{v}(x,t)$ and $\overline{w}(x,t)$ are two upper solutions of (23) and $\underline{u}(x,t)$ is a lower solution of (23). Suppose that $\underline{u}(x,0) \leq \min\{\overline{v}(x,0), \overline{w}(x,0)\}$ is also true. Then Lemma 3.9 holds if we replace $\overline{u}(x,t)$ by $\min\{\overline{v}(x,t), \overline{w}(x,t)\}$.*

Remark 3.11. By Lemma 3.10, $\min\{\overline{v}(x,t), \overline{w}(x,t)\}$ is also called an upper solution of (23). Moreover, Lemmas 3.9 and 3.10 are clear and we refer to Martin and Smith [24] and Smoller [32].

4. **Asymptotical stability of traveling wavefronts.** In this section, we always assume that Λ is nonempty such that Theorems 2.10 and 2.14 hold, and we shall prove the asymptotic stability of traveling wavefronts of (1) by proving the corresponding results for (4). Two results on the stability of traveling wavefronts of (1) will be given, one is global and another is local in suitable sense.

4.1. **Globally asymptotic stability.** In this part, we shall employ the squeezing technique to prove the stability of traveling wavefronts of (23). Our main result in this section is listed as follows.

Theorem 4.1. *Assume that the initial value $\Psi(x)$ of (23) satisfies*

(i) $\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \in C_{[0,K]}$;

(ii) $\underline{\phi}_i(x) \leq \psi_i(x) \leq \overline{\phi}_i(x)$ *for* $x \in \mathbb{R}, i = 1,2,3,4$, *herein* $\underline{\phi}_i(x)$ *and* $\overline{\phi}_i(x)$ *are defined by Theorem 2.10;*

(iii) $\liminf_{x\to\infty} \psi_i(x) > 0$ *for* $i = 1,2,3,4$.

Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be formulated by Theorem 2.14. Then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{u_i(x,t)}{\phi_i(x+ct)} - 1 \right| = 0 \text{ for } i = 1,2,3,4.$$

Before proving Theorem 4.1, we establish some estimates formulated by the following lemmas, through which the conditions in Theorem 4.1 will be imposed.

Lemma 4.2. *$\phi_i(\xi)$ is strictly monotone such that $\phi_i'(\xi) > 0, \xi \in \mathbb{R}, i = 1, 2, 3, 4$.*

Lemma 4.2 is clear by Lemma 3.6, and the proof is omit.

Lemma 4.3. *Let $\delta_1 = \delta_2 = 1, \delta_3 = \delta_4 \in \left(1, \frac{1}{2}\left[1 + \min\left\{\frac{a_2k_2}{b_2k_1}, \frac{a_1k_1}{b_1k_2}\right\}\right]\right)$ be given. Then, for any $\xi^+ \in \mathbb{R}$ and $\varepsilon \in (0, \overline{\varepsilon}]$ with given $\overline{\varepsilon} \in (0, 1)$,*

$$\overline{u}_i(x,t) = \min\left\{\left(1 + \varepsilon\delta_i e^{-\gamma t}\right)\phi_i\left(x + ct + \xi^+ - \varepsilon\sigma e^{-\gamma t}\right), k_i\right\}$$

is an upper solution of (23) provided that $\sigma > 0$ is large enough and $\gamma > 0$ is small enough.

Proof. It suffices to verify the definition of an upper solution. If $\overline{u}_i(x,t) = k_i$ for some $i = 1, 2, 3, 4$, then the result is clear. Otherwise

$$\frac{\partial \overline{u}_i(x,t)}{\partial t} = -\varepsilon\delta_i\gamma e^{-\gamma t}\phi_i(\varsigma) + c\left(1 + \varepsilon\delta_i e^{-\gamma t}\right)\phi_i'(\varsigma) + \varepsilon\sigma\gamma e^{-\gamma t}\left(1 + \varepsilon\delta_i e^{-\gamma t}\right)\phi_i'(\varsigma),$$
$$\Delta\overline{u}_i(x,t) = \left(1 + \varepsilon\delta_i e^{-\gamma t}\right)\phi_i''(\varsigma),$$

where $\varsigma = x + ct + \xi^+ - \varepsilon\sigma e^{-\gamma t}$. If $i = 1$, then

$$\begin{aligned} &r_1\overline{u}_1(x,t)\left[1 - a_1\overline{u}_1(x,t) + b_1\overline{u}_3(x,t)\right] \\ &= \left(1 + \varepsilon\delta_1 e^{-\gamma t}\right)\phi_1(\varsigma)\left[1 - a_1\phi_1(\varsigma) + b_1\phi_3(\varsigma)\right] \\ &\quad + \left(1 + \varepsilon\delta_1 e^{-\gamma t}\right)e^{-\gamma t}\phi_1(\varsigma)\left[-a_1\varepsilon\delta_1\phi_1(\varsigma) + b_1\varepsilon\delta_3\phi_3(\varsigma)\right]. \end{aligned}$$

Using the definition of traveling wavefronts, we need to verify that

$$\begin{aligned} &-\varepsilon\delta_1\gamma\phi_1(\varsigma) + \varepsilon\sigma\gamma\left(1 + \delta_1 e^{-\gamma t}\right)\phi_1'(\varsigma) \\ &\quad \geq \left(1 + \varepsilon\delta_1 e^{-\gamma t}\right)\phi_1(\varsigma)\left[-a_1\varepsilon\delta_1\phi_1(\varsigma) + b_1\varepsilon\delta_3\phi_3(\varsigma)\right]. \end{aligned} \tag{25}$$

Let $M > 0$ be a sufficiently large constant. We now confirm (25) by three steps.

(i) If $\varsigma \leq -M$, then the asymptotic behavior of traveling wavefronts implies (25) if $\sigma > 0$ is large enough.

(ii) If $\varsigma \geq M$, then $-a_1\delta_1\phi_1(\varsigma) + b_1\delta_3\phi_3(\varsigma) \to -a_1\delta_1k_1 + b_1\delta_3k_3$ as $\varsigma \to \infty$, so (25) holds provided that $a_1\delta_1k_1 - b_1\delta_3k_3 > 0$ and $\gamma > 0$ is small enough.

(iii) If $|\varsigma| \leq M$, then the fact that ϕ_1 is strictly monotone and large $\sigma > 0$ is large implies (25).

Similarly, we can prove that $\overline{u}_2$ is an upper solution since $a_2\delta_2k_2 - b_1\delta_4k_4 > 0$.

For $i = 3$, we need to prove that

$$-\varepsilon\delta_3\gamma\phi_3(\varsigma) + \varepsilon\sigma\gamma\left(1 + \varepsilon\delta_3 e^{-\gamma t}\right)\phi_3'(\varsigma) \geq \frac{\varepsilon}{\tau_1}\left[-\delta_3\phi_3(\varsigma) + \delta_2\phi_2(\varsigma)\right],$$

which is clear if $\delta_3 > \delta_2$ holds, $\sigma > 0$ is large and $\gamma > 0$ is small.

In a similar way, $\overline{u}_4$ is the upper solution if $\delta_4 > \delta_1$. The proof is complete. □

Lemma 4.4. *Assume that the constants $\varepsilon, \delta_i, \sigma, \gamma$ are given by Lemma 4.3 such that $\overline{\varepsilon}\delta_3 = \overline{\varepsilon}\delta_4 < 1$. Then for any $\xi^- \in \mathbb{R}$,*

$$\underline{u}_i(x,t) = \left(1 - \varepsilon\delta_i e^{-\gamma t}\right)\phi_i\left(x + ct + \xi_- + \varepsilon\sigma e^{-\gamma t}\right),\ i = 1, 2, 3, 4,$$

is a lower solution of (23).

The proof of Lemma 4.4 is similar to that of Lemma 4.3, so we omit it here. From the proof of Lemmas 4.3-4.4, we also obtain the following important fact.

Remark 4.5. We can fix σ and γ which only depend on $\overline{\varepsilon}$.

Lemma 4.6. *For any $\varepsilon > 0$, there exists $\xi_1 = \xi_1(\varepsilon)$ such that*

$$\sup_{t\geq 0} u_i(\xi - ct - 2\varepsilon, t) \leq \phi_i(\xi) \leq \inf_{t\geq 0} u_i(\xi - ct + 2\varepsilon, t)$$

holds for any $\xi \leq \xi_1$ and $i = 1, 2, 3, 4$.

The proof of the lemma depends on the upper and lower solutions given in Theorem 2.10 since (ii) holds, and we omit it here.

Lemma 4.7. *There exist positive constants $\varepsilon \in (0,1), \delta_i, \gamma, \sigma$ and z_0 such that*

$$\left(1 - \varepsilon\delta_i e^{-\gamma t}\right)\phi_i(\xi - z_0 + \varepsilon\sigma e^{-\gamma t}) \leq u_i(x,t) \leq \left(1 + \varepsilon\delta_i e^{-\gamma t}\right)\phi_i(\xi + z_0 - \varepsilon\sigma e^{-\gamma t})$$

holds for all $x \in \mathbb{R}$ and $t \geq 1$. Then for all $t \geq 1$, we obtain

$$1 - \varepsilon\delta_i e^{-\gamma t} \leq \inf_{\mathbb{R}} \frac{u_i(\cdot - ct, t)}{\phi_i(\cdot + z_0)} \leq \sup_{\mathbb{R}} \frac{u_i(\cdot - ct, t)}{\phi_i(\cdot - z_0)} \leq 1 + \varepsilon\delta_i e^{-\gamma t}.$$

Proof. From Lemmas 3.6, 4.3-4.6, we know that there exist constants $\varepsilon \in (0,1), \gamma > 0, \sigma > 0$ and $z_0 \geq 0$ such that

$$\left(1 - \varepsilon\delta_i e^{-\gamma}\right)\phi_i(\xi - z_0 + \varepsilon\sigma e^{-\gamma}) \leq u_i(\xi - c, 1) \leq \left(1 + \varepsilon\delta_i e^{-\gamma}\right)\phi_i(\xi + z_0 - \varepsilon\sigma e^{-\gamma})$$

for all $\xi \in \mathbb{R}$. Moreover, these constants can satisfy the conditions in Lemmas 4.3-4.4 if $z_0 > 0$ is large enough. Then Lemma 3.9 implies the conclusion. The proof is complete. ☐

Lemma 4.8. *For any $\varepsilon \in (0,1)$, there exists $M_0 > 0$ such that*

$$(1 - \varepsilon\delta_i)\phi_i(\xi + 3\varepsilon\sigma) \leq \phi_i(\xi) \leq (1 + \varepsilon\delta_i)\phi_i(\xi - 3\varepsilon\sigma), \xi \geq M_0.$$

Proof. Let us consider the function $(1 + s\delta_i)\phi_i(\xi - 3s\sigma)$, it is clear that

$$\frac{d}{ds}\{(1 + s\delta_i)\phi_i(\xi - 3s\sigma)\} = \delta_i\phi_i(\xi - 3s\sigma) - 3\sigma(1 + s\delta_i)\phi_i'(\xi - 3s\sigma).$$

By asymptotic behavior of traveling wavefronts, there exists $M_0 > 0$ such that

$$\delta_i\phi_i(\xi - 3s\sigma) - 3\sigma(1 + s\delta_i)\phi_i'(\xi - 3s\sigma) \geq 0$$

for all $\xi \geq M_0$ and $i = 1, 2, 3, 4$. Since $(1 + s\delta_i)\phi_i(\xi - 3s\sigma)|_{s=0} = \phi_i(\xi)$, then the result is clear. The proof is complete. ☐

Lemma 4.9. *Let z and M be any given positive constants and*

$$u^+(x,t) = \left(u_1^+(x,t), u_2^+(x,t), u_3^+(x,t), u_4^+(x,t)\right),$$
$$u^-(x,t) = \left(u_1^-(x,t), u_2^-(x,t), u_3^-(x,t), u_4^-(x,t)\right)$$

be the solutions of (4) with initial values

$$u_i^+(x,0) = \phi_i(x+z)\chi(x+M) + \phi_i(x+2z)\left[1 - \chi(x+M)\right],$$
$$u_i^-(x,0) = \phi_i(x-z)\chi(x+M) + \phi_i(x-2z)\left[1 - \chi(x+M)\right],$$

respectively, where $x \in \mathbb{R}$, $\chi(y) = \min\{\max\{0, -y\}, 1\}$ for all $y \in \mathbb{R}$. Then there exists a constant $\epsilon \in (0, \min\{1/2, z/(3\sigma)\})$ such that for any $x \in [-M, \infty)$,

$$u_i^+(x - c, 1) \leq (1 + \epsilon\delta_i)\phi_i(x + 2z - 3\epsilon\sigma),$$
$$u_i^-(x - c, 1) \geq (1 - \epsilon\delta_i)\phi_i(x - 2z + 3\epsilon\sigma).$$

Proof. From the definition of $\chi(y)$, we know that $u_i^+(x,0) \le \phi_i(x+2z)$ on $\mathbb{R}$ and $u_i^+(x,0) < \phi_i(x+2z)$ on a nonempty subset of $\mathbb{R}$, so $u_i^+(x-c,1) < \phi_i(x+2z)$ on $\mathbb{R}$ by the positivity of $T(t)$ and comparison principle. Since u_i^+ and ϕ_i are continuous functions, they are also uniformly continuous on any bounded interval. Furthermore, there exists a constant $\epsilon \in (0, \min\{1/2, z/(3\sigma)\})$ such that

$$u_i^+(x-c,1) \le (1+\epsilon\delta_i)\phi_i(x+2z-3\epsilon\sigma)$$

for $x \in [-M, M_0-2z]$, where $M_0 > 0$ is defined as in Lemma 4.8. We further have

$$u_i^+(x-c,1) < \phi_i(x+2z) \le (1+\epsilon\delta_i)\phi_i(x+2z-3\epsilon\sigma)$$

on $[M_0-2z, +\infty)$ by Lemma 4.8.

Similarly, we can prove that our result holds for u_i^-. The proof is complete. $\square$

We are now in the position to prove Theorem 4.1.

Proof of Theorem 4.1. Define

$$z^+ := \inf\left\{z \,\middle|\, z \in A^+\right\}, \quad A^+ = \left\{ z \ge 0 \,\middle|\, \sup_{i=1,2,3,4} \limsup_{t\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u_i(\xi-ct,t)}{\phi_i(\xi+2z)} \le 1 \right\},$$

$$z^- := \inf\left\{z \,\middle|\, z \in A^-\right\}, \quad A^- = \left\{ z \ge 0 \,\middle|\, \inf_{i=1,2,3,4} \liminf_{t\to\infty} \inf_{\xi\in\mathbb{R}} \frac{u_i(\xi-ct,t)}{\phi_i(\xi-2z)} \ge 1 \right\}.$$

By what we have done in Lemma 4.7, $\frac{1}{2}z_0 \in A^\pm$, which implies that $A^\pm$ are well defined. Thus $[\frac{1}{2}z_0, +\infty) \subset A^\pm$, $z^\pm \in [0, \frac{1}{2}z_0]$. If $z^\pm = 0$, then we finish our proof. Now we assume that $z^+ > 0$. Fix $z = z^+$, $M = -\xi_1(z^+/2) + z^+$ and ϵ defined in Lemma 4.9. Since $z^+ \in A^+$, there exists $T \ge 0$ such that

$$\sup_{\xi\in\mathbb{R}} \frac{u_i(\xi-cT,T)}{\phi_i(\xi+2z^+)} \le 1 + \frac{\bar{\epsilon}}{k_i},$$

where $4\bar{\epsilon} = \min_{i=1,2,3,4}\{\phi_i(-M-3\epsilon\sigma)\} \times \epsilon e^{-\mu}$ and $\mu > 0$ is defined by Lemma 3.7. From Lemma 4.9, we obtain

$$u_i(\xi-cT,T) \le \phi_i(\xi+2z^+) + \bar{\epsilon} = u_i^+(\xi,0) + \bar{\epsilon} \text{ for } \xi \in [-M,+\infty).$$

Furthermore, Lemma 4.6 follows that

$$u_i(\xi-cT,T) \le \phi_i(\xi+z^+) \le u_i^+(\xi,0) \text{ if } \xi \in (-\infty,-M].$$

Therefore, by virtue of Lemma 3.7, we have

$$u_i(\xi-cT,T+1) < u_i^+(\xi,1) + 4\bar{\epsilon}e^{\mu} \le u_i^+(\xi,1) + \epsilon\phi_i(-M-3\epsilon\sigma)$$

where $\xi \in \mathbb{R}$, and Lemma 4.7 indicates that

$$\begin{aligned} & u_i(\xi-c(T+1),T+1) \\ & \quad \le u_i^+(\xi-c,1) + \epsilon\phi_i(-M-3\epsilon\sigma) \\ & \quad \le (1+\epsilon\delta_i)\phi_i(\xi+2z^+-3\epsilon\sigma) + \epsilon\delta_i\phi_i(-M-3\epsilon\sigma) \\ & \quad \le (1+2\epsilon\delta_i)\phi_i(\xi+2z^+-3\epsilon\sigma) \end{aligned}$$

if $\xi \in [-M,+\infty)$. On the other hand, since $3\epsilon\sigma \le z^+$, then

$$u_i(\xi-c(T+1),T+1) \le \phi_i(\xi+z^+) \le \phi_i(\xi+2z^+-3\epsilon\sigma)$$

for all $\xi \in (-\infty,-M]$, which further implies that

$$u_i(\xi-c(T+1),T+1) \le \min\{(1+2\epsilon\delta_i)\phi_i(\xi+2z^+-3\epsilon\sigma), k_i\}, \; \xi \in \mathbb{R}.$$

By comparison principle (also see Remark 4.5), it follows that

$$\begin{aligned} & u_i(\xi - c\,(T+1+t)\,, T+1+t) \\ & \quad \le \min\{(1+2\epsilon\delta_i e^{-\gamma t})\phi_i(\xi + 2z^+ - \epsilon\sigma - 2\epsilon\sigma e^{-\gamma t}), k_i\} \end{aligned}$$

if $\xi \in \mathbb{R}$ and $t \ge 0$, which asserts that

$$\sup_{i=1,2,3,4} \limsup_{t\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u_i\,(\xi - ct, t)}{\phi_i\,(\xi + 2z^+ - \epsilon\sigma)} \le 1.$$

Furthermore, the above inequalities imply $z^+ - \epsilon\sigma/2 \in A^+$, which is a contradiction and also means that $z^+ = 0$.

Similarly, we can prove that $z^- = 0$. The proof is complete. □

Theorem 4.10. *Assume that the initial values $(\psi_1(x,s), \psi_2(x,s))$ satisfy*

(i) $E_0 \le (\psi_1(x,s), \psi_2(x,s)) \le E^*$;

(ii) $\underline{\phi}_i(x) \le \psi_i(x,0) \le \overline{\phi}_i(x)$ *for* $i = 1,2,3,4$ *where* $\psi_3(x,0) = (g_1 * \psi_2)\,(x,0)$, $\psi_4(x,0) = (g_2 * \psi_1)\,(x,0)$;

(iii) $\liminf_{x\to\infty} \psi_i(x,0) > 0, i = 1,2.$

Let $\Phi = (\phi_1, \phi_2)$ be given by Theorem 2.10. Then

$$\sup_{i=1,2} \lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{u_i(x,t)}{\phi_i(x+ct)} - 1 \right| = 0.$$

As in the proof of Theorem 2.10 or as in 2.14, we can choose $p > 1$ large enough and obtain following conclusion.

Theorem 4.11. *Assume that the initial value $\Psi(x)$ of (23) satisfies*

(i) $\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \in C_{[0,K]}$;

(ii) $\lim_{x\to-\infty} \psi_1(x)e^{-\lambda_1 x} = k_1$, $\lim_{x\to-\infty} \psi_2(x)e^{-\lambda_2 x} = k_2$, $\lim_{x\to-\infty} \psi_3(x)e^{-\lambda_2 x} = \frac{k_2}{1+r_2\tau_1}$, $\lim_{x\to-\infty} \psi_4(x)e^{-\lambda_1 x} = \frac{k_1}{1+r_1\tau_2}$.

(iii) $\liminf_{x\to\infty} \psi_i(x) > 0.$

Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be formulated by Theorem 2.14. Then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{u_i(x,t)}{\phi_i(x+ct)} - 1 \right| = 0 \text{ for } i = 1,2,3,4.$$

The proof of Theorem 4.11 is similar to that of Theorem 4.1, so we omit it here. Moreover, Theorem 4.11 implies a natural stability result of traveling wavefronts of (1). Since this is clear from Theorem 3.2, we omit it here.

Remark 4.12. Note that Theorem 2.14 is independent of the size of τ_1 and τ_2, thus Theorem 4.11 also indicates the stability of traveling wavefronts of (3).

4.2. **Locally exponential stability.** In this subsection, we shall give a stability result on traveling wavefronts of (1), which is different from that of Section 4.1.

Let $\sigma > 0$ and define a subset of uniformly continuous functions as follows

$$C_\sigma = \left\{ u(x) : u(x) \in C\,(\mathbb{R}, \mathbb{R}^n)\,,\ \lim_{x\to\pm\infty} \left| u(x)\left(1 + e^{-\sigma x}\right) \right| = 0 \right\},$$

which is a Banach space with norm $\|\cdot\|_\sigma$ given by

$$\|u\|_\sigma = \sup_{x\in\mathbb{R}} \left| u(x)\left(1 + e^{-\sigma x}\right) \right| \text{ for } u(x) \in C_\sigma.$$

Theorem 4.13. *Assume that $c > c^*$ such that $\Sigma = (\lambda_1, \lambda_3) \cap (\lambda_2, \lambda_4)$ is nonempty. Also suppose that $\Psi(x) \in C_{[0,K]}$, $U(x) = \Phi(x) - \Psi(x) \in C_\nu$ for some $\nu \in \Sigma$ and $n = 4$. Then there exists a constant $\kappa > 0$ such that for any $U(x)$ with $\|U(x)\|_\nu < \kappa$, the unique solution $u(x,t)$ of (23) with initial value $\Psi(x)$ satisfies*

$$\|u(x,t) - \Phi(x+ct)\|_\nu \le Me^{-bt},$$

where $M > 0, b > 0$ are constants independent of $\Psi(x)$ and $t > 0$, and $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is formulated by Theorem 2.14.

Proof. Let f be the reaction term of (23). Then $f'(\Phi)$ is given by

$$\begin{bmatrix} r_1(1 - 2a_1\phi_1 + b_1\phi_3) & 0 & r_1 b_1 \phi_1 & 0 \\ 0 & r_2(1 - 2a_2\phi_2 + b_2\phi_4) & 0 & r_2 b_2 \phi_2 \\ 0 & \frac{1}{\tau_1} & -\frac{1}{\tau_1} & 0 \\ \frac{1}{\tau_2} & 0 & 0 & -\frac{1}{\tau_2} \end{bmatrix}.$$

It is easy to see that $f'(\Phi)$ is irreducible in the functional sense due to the strict monotonicity of traveling wavefronts (see Lemma 4.2). For the matrix

$$f'(K) = \begin{bmatrix} -r_1 a_1 k_1 & 0 & r_1 b_1 k_1 & 0 \\ 0 & -r_2 a_2 k_2 & 0 & r_2 b_2 k_2 \\ 0 & \frac{1}{\tau_1} & -\frac{1}{\tau_1} & 0 \\ \frac{1}{\tau_2} & 0 & 0 & -\frac{1}{\tau_2} \end{bmatrix},$$

all of its eigenvalues have negative real parts. Moreover, consider the matrix

$$f'(0) = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & \frac{1}{\tau_1} & -\frac{1}{\tau_1} & 0 \\ \frac{1}{\tau_2} & 0 & 0 & -\frac{1}{\tau_2} \end{bmatrix},$$

although the eigenvalues of $f'(0)$ have positive real part, all eigenvalues of $D\nu^2 - c\nu + f'(0)$ have negative real part if $\nu \in \Sigma$. By Volpert et al. [34, Theorem 5.4.1], we see that the conclusion is true. The proof is complete. $\square$

Remark 4.14. Λ is nonempty if and only if Σ is nonempty.

Theorem 4.15. *Assume that $c > c^*$ such that Theorem 2.10 holds, Σ is defined by Theorem 4.13 and the initial values $(\psi_1(x,s), \psi_2(x,s))$ of (22) satisfy*

$$(\psi_1(x,0) - \phi_1(x), \psi_2(x,0) - \phi_2(x)) \in C_\nu,$$
$$((g_1 * \psi_2)(x,0) - (g_1 * \phi_2)(x), (g_2 * \psi_1)(x,0) - (g_2 * \phi_1)(x)) \in C_\nu,$$

where $\nu \in \Sigma$ and $n = 2$. Then there exists $\varepsilon > 0$ small enough such that

$$\begin{aligned} &\|(g_1 * \psi_2)(x,0) - \phi_3(x), (g_2 * \psi_1)(x,0) - \phi_4(x)\|_\nu \\ &\quad + \|\psi_1(x,0) - \phi_1(x), \psi_2(x,0) - \phi_2(x)\|_\nu < \varepsilon \end{aligned}$$

implies that the unique bounded solution $u(x,t)$ of (1) satisfies

$$\|u(x,t) - \Phi(x+ct)\|_\nu \le Me^{-bt}$$

where $M > 0, b > 0$ are constants independent of $(\psi_1(x,s), \psi_2(x,s))$.

Remark 4.16. The results in subsections 4.1 and 4.2 indicate the asymptotic stability of traveling wavefronts in the weighted functional space due to the asymptotic behavior of traveling wavefronts. And it is clear that the squeezing technique (see Theorem 4.11) is a better method in the choice of weighted functional spaces while the spectral analysis (see Theorem 4.13) is a better one in the estimate of convergence rates.

5. **Nonexistence of traveling wavefronts.** In this section, we prove that (1) has no positive traveling wave solutions connecting E_0 with E^* if $c < c^*$. For the purpose, we consider

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = d_1 \Delta w(x,t) + r_1 w(x,t)\left[1 - a_1 w(x,t)\right], \\ w(x,0) = w(x), \end{cases} \tag{26}$$

where all constants are positive, $w \in \mathbb{R}, x \in \mathbb{R},\ t > 0$, and $w(x) \in C(\mathbb{R},\mathbb{R})$. Then the following result on asymptotic spreading is true (see [1, 15, 33, 38]).

Lemma 5.1. *Assume that $w(x)$ admits a nonempty compact support and $w(x,t)$ is defined by (26). Then*

(i) $\lim_{t\to\infty} \sup_{|x|>ct} w(x,t) = 0$ *holds for any* $c > 2\sqrt{d_1 r_1}$;

(ii) $\lim_{t\to\infty} \inf_{|x|<ct} w(x,t) = \frac{1}{a_1}$ *holds for any* $c \in \left(0, 2\sqrt{d_1 r_1}\right)$.

Theorem 5.2. *Assume that $d_1 r_1 \geq d_2 r_2$ holds and the wave speed $c < 2\sqrt{d_1 r_1}$. Then (1) has no traveling wavefronts connecting E_0 with E^*.*

Proof. We prove this by contradiction. If the stated statement is false, then there exists some $c_1 \in (0, c^*)$ such that (14)-(15) has a monotone solution $(\phi_1(x+c_1t), \phi_2(x+c_1t))$. Then it is easy to choose a constant $0 < \delta < \min\{\frac{1}{a_1}, \frac{1}{a_2}\}$ such that

$$(\phi_1(x), \phi_2(x)) \geq (\delta, \delta)$$

for all $x \in [-2, 2]$. Moreover, $\phi_1(x + c_1t)$ also satisfies

$$\frac{\partial \phi_1(x + c_1t)}{\partial t} \geq d_1 \Delta \phi_1(x + c_1t) + r_1 \phi_1(x + c_1t)\left[1 - a_1 \phi_1(x + c_1t)\right]$$

since $(g_1 * \phi_2)(x,t) > 0$ for all $x \in \mathbb{R}, t > 0$. Let $w(x,t)$ be defined by (26) if $w(x) > 0$ such that

(a) $w(x) = 0$ if $|x| \geq 2$;

(b) $0 < w(x) \leq \delta$ if $|x| < 2$.

Then Lemma 3.9 (which remains true if $b_1 = b_2 = 0$) implies that $\phi_1(x + c_1t) \geq w(x,t)$ for all $x \in \mathbb{R}, t \geq 0$.

Let $c = \frac{c_1 + c^*}{2}$. Then Lemma 5.1 and the asymptotic behavior (15) indicate a contradiction as $x + ct \to -\infty$. The proof is complete. □

Similarly, we can prove the following result.

Theorem 5.3. *Assume that $c < c^*$ holds. Then (1) has no positive traveling wave solutions connecting E_0 with E^*.*

Remark 5.4. For some parameters, e.g., $(d_1, r_1) \geq (\leq)(d_2, r_2)$, we have proved that c^* is the minimal wave speed such that (1) has (no) traveling wave solutions connecting E_0 with E^* if $c > (<) c^*$.

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