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On first-order hyperbolic partial differential equations with two internal variables modeling population dynamics of two physiological structures

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Abstract
In this paper we develop fundamental theories for a scalar first-order hyperbolic partial differential equation with two internal variables which models single-species population dynamics with two physiological structures such as age–age, age–maturation, age–size, and age–stage. Classical techniques of treating structured models with a single internal variable are generalized to study the double physiologically structured model. First, the semigroup is defined based on the solutions and its infinitesimal generator is determined. Then, the compactness of solution trajectories is established. Finally, spectrum theory is employed to investigate stability of the zero steady state and asynchronous exponential growth of solutions is studied when the zero steady state is unstable.

Keywords Physiological structure · Semigroup theory · Infinitesimal generator · Spectrum theory · Asynchronous exponential growth

Mathematics Subject Classification 35L04 · 92D25 · 47A10

1 Introduction

In populations dynamics, structured models bridge the gap between the individual level and the population level and allow us to study the dynamics of populations from properties of individuals or vice versa (Metz and Diekmann [40]). In order to parametrize the state of individuals as well as to distinguish individuals from one another, we usually take their physiological conditions or physical characteristics such as age, maturation, size, stage, status, location, and movement into consideration and determine their birth, growth and death rates, interactions with each other and with environment, infectivity, and so on. The goal of studying structured population models is to understand how these physiological conditions

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or physical characteristics affect the dynamical properties of these models and thus the outcomes and consequences of the biological and epidemiological processes (Magal and Ruan [34]).

Age structure is the most important characteristic in population dynamics. It is well known for a long time that the age-structure of a population affects the nonlinear dynamics of the species in ecology and the transmission dynamics of infectious diseases in epidemiology. In modeling specific diseases, the age could be chronological age (the age of the population), infection age (the time elapsed since infection), recovery age (the time elapsed since the last infection), class age (the length of time in the present group), etc. The theory of age-structured models has been well developed, and we refer to the classical monographs of Anita [1], Iannelli [23], Inaba [26], and Webb [52] on this subject. Models with other physiological structures, such as size and stage, have also been extensively studied in the literature (Barfield et al. [3], Cushing [7], Ebenman and Persson [17], Manly [36], Metz and Diekmann [40]).

There are some models taking into account the combined effects of two age characteristics. Hoppensteadt [22] proposed a double age-structured epidemic model for a population consisted of susceptible, infectious, quarantined infectious, and immune classes by keeping track of the chronological ages of the individuals as well as their class ages (i.e., the length of time since entering their present state). Existence and uniqueness of solutions to the model were considered. In order to determine the likely success of isolating symptomatic individuals and tracing and quarantining their contacts, Fraser et al. [19] studied a double age-structured model involving individuals who were infected time τ ago by people who themselves were infected time τ' ago. They concluded direct estimation of the proportion of asymptomatic and presymptomatic infections is achievable by contact tracing and should be a priority during an outbreak of a novel infectious agent. Kapitanov [29] considered the coinfection of HIV and tuberculosis in which each disease progresses through several stages. He suggested modeling these stages through a time since-infection tracking transmission rate function and introduced a double age-structured model. By incorporating the chronological ages of the individuals into the Kermack and McKendrick's original infection-age structured endemic model, Inaba [25] developed a double age-structured susceptible–infectious–recovered model and studied some basic properties of the model. Burie et al. [6] investigated an age and infection age structured model for the propagation of fungal diseases in plants and analyzed the asymptotic behavior of the model. Laroche and Perasso [33] studied a generic epidemic model structured by age and, for infected, the time remaining before the end of the incubation where they show detectable clinical signs. Population dynamical models with age and size structures (Sinko and Streifer [44], Webb [55]), age and maturation structures (Dyson et al. [15, 16]), age and stage structures (McNair and Goulden [39], Matucci [38], Delgado et al. [10], Walker [48]), and age and an aggregated variable (Doumic [13]) have also been proposed and studied by many researchers.

Though many epidemic and populations models with two physiological structures have been proposed in the literature (as mentioned above), there are very few theoretical studies on the fundamental properties of such equations. By considering the property of the disease-free steady state of a susceptible–infectious–recovered model with chronological and class ages, Inaba [25] obtained a scalar equation with two age structures and provided some analysis of the model, including global stability of the disease-free steady state, using the strongly continuous semigroup theory. Webb [53] investigated a scalar structured population model with nonlinear boundary conditions in which individuals are distinguished by age and another physical characteristic and proved that populations structured by two internal variables converge to stable distributions.

Motivated by the studies of Inaba [25] and Webb [53], in this paper we aim to develop fundamental results and theories for a scalar equation with two physiological structures and to generalize the techniques in treating single age-structured models (Iannelli [23], Inaba [26], Webb [51, 52]) to study structured models with two internal variables. Consider the following first-order hyperbolic partial differential equation with two internal variables a and a' (which models single-species population dynamics with two physiological structures)

$$\begin{aligned}\frac{\partial u(t,a,a')}{\partial t}+\frac{\partial u(t,a,a')}{\partial a}+\frac{\partial u(t,a,a')}{\partial a'}\\ =-\mu(a,a')u(t,a,a'),\quad t>0,(a,a')\in(0,a^+)\times(0,a^+)\end{aligned}\tag{1.1}$$

under the initial condition

$$u(0,a,a')=\phi(a,a'),\quad a\in(0,a^+),a'\in(0,a^+)\tag{1.2}$$

and boundary conditions

$$u(t,0,a')=\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s,\quad t>0,a'\in(0,a^+),\tag{1.3}$$

$$u(t,a,0)=\int_0^{a^+}\int_0^{a^+}\beta'(a,s,a')u(t,s,a')\mathrm{d}s\mathrm{d}a',\quad t>0,a\in(0,a^+).\tag{1.4}$$

Here $u(t,a,a')$ denotes the density of a population at time t with age a and another characteristic a' (age, size, maturation, stage, etc.), the function ϕ represents the initial distribution of the population with respect to age a and another physiological characteristic a'. $\mu(a,a')$ denotes the mortality rate of the population at age a with characteristic a'. Boundary condition (1.3) accounts for the input at time t of individuals of age 0 with characteristic a' and boundary condition (1.4) describes the input at time t of individuals of age a with characteristic a' at level 0. Here a^+ represents the maximum age or characteristic and could be infinity.

By using semigroup theory, we study the basic properties and dynamics of model (1.1) under the initial condition (1.2) and boundary conditions (1.3)–(1.4), including the solution flow $u(t,a,a')$ and its semigroup $\{S(t)\}_{t\geq 0}$ with infinitesimal generator A. Moreover, we establish the compactness of the solution trajectories, analyze the spectrum of A, investigate stability of the zero equilibrium, and discuss the asynchronous exponential growth of the solutions when the zero equilibrium is unstable. We study the initial-boundary value problem in the two cases: $a^+<\infty$ and $a^+=\infty$. First, we show that for $a^+<\infty$ the semigroup $\{S(t)\}_{t\geq 0}$ with infinitesimal generator A is eventually compact, while for $a^+=\infty$ the semigroup $\{S(t)\}_{t\geq 0}$ is quasi-compact. Next, we study the existence and uniqueness of the principal and simple eigenvalue and spectral bound of A under extra assumptions (see Assumption 2.1(iii)' and (5.7)) for $a^+=\infty$. Finally, we establish the same threshold dynamics for these two cases.

2 The semigroup

In this section we investigate the existence and uniqueness of solutions to the initial-boundary value problem (1.1)–(1.4) via theory of integral functions. By using the method of characteristic lines we obtain the following expression for the density $u(t, a, a')$

$$u(t,a,a') = \begin{cases} \phi(a-t, a'-t)e^{-\int_0^t \mu(s-t+a, s-t+a')\mathrm{d}s}, & t < a < a' < a^+ \quad \text{or} \quad t < a' < a < a^+ \\ u(t-a, 0, a'-a)e^{-\int_0^a \mu(s, s+a'-a)\mathrm{d}s}, & a < t < a' < a^+ \quad \text{or} \quad a < a' < t, a < a' < a^+ \\ u(t-a', a-a', 0)e^{-\int_0^{a'} \mu(s+a-a', s)\mathrm{d}s}, & a' < a < t, a' < a < a^+ \quad \text{or} \quad a' < t < a < a^+. \end{cases} \tag{2.1}$$

Assumption 2.1 Assume that

(i) $\beta, \beta' : [0, a^+) \times [0, a^+) \times [0, a^+) \to [0, \infty)$ are nonnegative L^1 integrable and Lipschitz continuous, and $\mu : [0, a^+) \times [0, a^+) \to [0, \infty)$ is also nonnegative L^1 integrable and Lipschitz continuous.

(ii) The following limits

$$\lim_{h\to 0} \int_0^{a^+} |\beta(a'+h, a, s) - \beta(a', a, s)|\mathrm{d}a' = 0$$

and

$$\lim_{h\to 0} \int_0^{a^+} |\beta'(a+h, s, a') - \beta'(a, s, a')|\mathrm{d}a = 0$$

hold uniformly for $(a, s) \in (0, a^+) \times (0, a^+)$ and $(s, a') \in (0, a^+) \times (0, a^+)$, respectively.

(iii) There exist two nonnegative functions $\epsilon_1(s), \epsilon_2(s)$ such that $\beta(a', a, s) \geq \epsilon_1(s) > 0$ and $\beta'(a, s, a') \geq \epsilon_2(s) > 0$ for all $a', a \in (0, a^+)$ and $a, a' \in (0, a^+)$, respectively.

(iii)' β, β' are separable such that $\beta(a', a, s) = \beta_1(a')\beta_2(a, s)$ and $\beta'(a, s, a') = \beta_1'(a)\beta_2'(s, a')$ with $\beta_1(a'), \beta_1'(a) > 0$ a.e. in $L^1(0, a^+)$, i.e., β_1, β_1' are quasi-interior points in $L^1(0, a^+)$ and $\beta_2(a, s), \beta_2'(s, a')$ satisfy Assumption 2.1(iii), so that there exist two nonnegative functions $\epsilon_1(s)$ and $\epsilon_2(s)$ such that $\beta_2(a, s) \geq \epsilon_1(s) > 0$ and $\beta_2'(s, a') \geq \epsilon_2(s) > 0$ for all $a \in (0, a^+)$ and $a' \in (0, a^+)$, respectively.

(iv) In addition,

$$\sup_{(a,s)\in(0,a^+)\times(0,a^+)} \beta(a', a, s) \leq \overline{\beta}(a'), \quad \text{where} \quad \overline{\beta} \in L^1((0, a^+)),$$

$$\sup_{(s,a')\in(0,a^+)\times(0,a^+)} \beta'(a, s, a') \leq \overline{\beta'}(a), \quad \text{where} \quad \overline{\beta'} \in L^1((0, a^+)).$$

Denote

$$\beta_{\sup} := \int_0^{a^+} \overline{\beta}(a)\mathrm{d}a, \quad \beta'_{\sup} := \int_0^{a^+} \overline{\beta'}(a)\mathrm{d}a$$

and $\beta_{\max} := \max\{\beta_{\sup}, \beta'_{\sup}\}$.

(v) For $\mu(a, a')$, denote

$$\overline{\mu} = \sup_{(a,a')\in(0,a^+)\times(0,a^+)} \mu(a, a'), \quad \underline{\mu} = \inf_{(a,a')\in(0,a^+)\times(0,a^+)} \mu(a, a') > 0.$$

Remark 2.2 In fact, we only use the Lipschitz continuity of β, β' and μ with respect to their own components to show compactness of solution trajectories in Sect. 4, for other sections L^1 integrability is enough. Moreover, (iii) and (iii)' are used for cases $a^+ < \infty$ and $a^+ = \infty$, respectively.

2.1 A *priori* estimate

We use boundary conditions (1.3) and (1.4) and the above solution flow to obtain equations for the fertility rate functions $b_\phi(t, a') := u(t, 0, a')$, $b'_\phi(t, a) := u(t, a, 0)$, where $b_\phi, b'_\phi : (0, \infty) \times (0, a^+) \to \mathbb{R}$ satisfy the following integral equations, respectively, see Fig. 1a for $t < a^+$ (including $a^+ = \infty$ and $t < a^+ < \infty$) and Fig. 1b for $t > a^+$ when $a^+ < \infty$:

$$\begin{aligned} b_\phi(t, a') = & \int_0^{a^+-t}\int_0^{a^+-t} h(a, s)\beta(a', a+t, s+t)\phi(a, s)e^{-\int_0^t \mu(\sigma+a,\sigma+s)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s \\ & + \int_0^t\int_a^{a^+} h(a, s)\beta(a', a, s)b_\phi(t-a, s-a)e^{-\int_0^a \mu(\sigma,\sigma+s-a)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a \\ & + \int_0^t\int_s^{a^+} h(a, s)\beta(a', a, s)b'_\phi(t-s, a-s)e^{-\int_0^s \mu(\sigma+a-s,\sigma)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s \end{aligned} \tag{2.2}$$

and

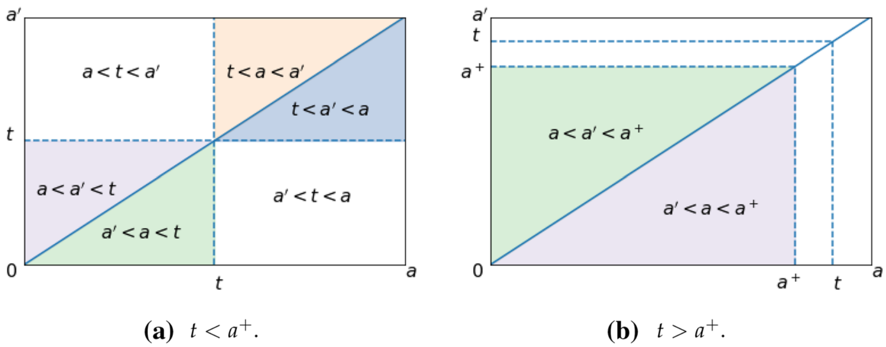


Fig. 1 Integration regions for (2.2)–(2.3)

$$
\begin{aligned}
b'_\phi(t,a) &= \int_0^{a^+-t}\int_0^{a^+-t} h(a',s)\beta'(a,s+t,a'+t)\phi(s,a')e^{-\int_0^t \mu(\sigma+s,\sigma+a')\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s \\
&\quad + \int_0^t \int_{a'}^{a^+} h(a',s)\beta'(a,s,a')b'_\phi(t-a',s-a')e^{-\int_0^{a'} \mu(\sigma+s-a',\sigma)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a' \\
&\quad + \int_0^t \int_s^{a^+} h(a',s)\beta'(a,s,a')b_\phi(t-s,a'-s)e^{-\int_0^s \mu(\sigma,\sigma+a'-s)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s,
\end{aligned} \tag{2.3}
$$

where $h(a,\ s)$ is a cut-off function defined by $h(a,s)=1$ if $a,s\in(0,a^+)$ and $h(a,s)=0$ otherwise.

Now adding up (2.2) and (2.3) and denoting $X(t,a)=b_\phi(t,a)+b'_\phi(t,a)$, we obtain

$$
\begin{aligned}
X(t,a) &\le \int_0^t \int_p^{a^+} [F(a,p,s)+G(a,s,p)]X(t-p,s-p)\mathrm{d}s\mathrm{d}p \\
&\quad + \int_0^{a^+-t}\int_0^{a^+-t} H(a,p,s,t)\phi(p,s)\mathrm{d}p\mathrm{d}s \\
&= \int_0^t \int_0^{a^+-p} [F(a,t-p,s+t-p)+G(a,s+t-p,t-p)]X(p,s)\mathrm{d}s\mathrm{d}p \\
&\quad + \int_0^{a^+-t}\int_0^{a^+-t} H(a,p,s,t)\phi(p,s)\mathrm{d}p\mathrm{d}s,
\end{aligned} \tag{2.4}
$$

where

$$
\begin{aligned}
F(a,p,s) &= \beta(a,p,s)e^{-\int_0^p \mu(\sigma,\sigma+s-p)\mathrm{d}\sigma} + \beta'(a,s,p)e^{-\int_0^p \mu(\sigma+s-p,\sigma)\mathrm{d}\sigma}, \\
G(a,s,p) &= \beta(a,s,p)e^{-\int_0^p \mu(\sigma+s-p,\sigma)\mathrm{d}\sigma} + \beta'(a,p,s)e^{-\int_0^p \mu(\sigma,\sigma+s-p)\mathrm{d}\sigma}, \\
H(a,p,s,t) &= [\beta(a,p+t,s+t)+\beta'(a,p+t,s+t)]e^{-\int_0^t \mu(\sigma+p,\sigma+s)\mathrm{d}\sigma}.
\end{aligned}
$$

Denote $Y(t)=\int_0^{a^+} X(t,a)\mathrm{d}a$ and define $E:=L^1_+((0,a^+)\times(0,a^+))$. Integrating (2.4) on $[0,a^+)$ with respect to a, we obtain

$$
\begin{aligned}
Y(t) &\le \int_0^{a^+}\int_0^t \int_0^{a^+-p} [F(a,t-p,s+t-p)+G(a,s+t-p,t-p)]X(p,s)\mathrm{d}s\mathrm{d}p\mathrm{d}a \\
&\quad + \int_0^{a^+}\int_0^{a^+-t}\int_0^{a^+-t} H(a,p,s,t)\phi(p,s)\mathrm{d}p\mathrm{d}s\mathrm{d}a \\
&\le \int_0^t \int_0^{a^+} X(p,s)\mathrm{d}s \int_0^{a^+} 2[\overline{\beta}(a)+\overline{\beta'}(a)]\mathrm{d}a\mathrm{d}p \\
&\quad + \int_0^{a^+}\int_0^{a^+}\int_0^{a^+} [\overline{\beta}(a)+\overline{\beta'}(a')]\mathrm{d}a\phi(p,s)\mathrm{d}p\mathrm{d}s \\
&\le 4\beta_{\max}\int_0^t Y(p)\mathrm{d}p + 2\beta_{\max}\|\phi\|_E.
\end{aligned} \tag{2.5}
$$

Hence, by Gronwall's inequality, we have the following estimate:

$$Y(t) \leq 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}t}, \tag{2.6}$$

which implies that $\int_0^{a^+} b_\phi(t,a)\mathrm{d}a, \int_0^{a^+} b'_\phi(t,a)\mathrm{d}a \leq 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}t}$.

2.2 Existence and uniqueness of solutions

Now we prove that there exists a unique solution $(b_\phi(t,a), b'_\phi(t,a)) \in C((0,\infty), L^1_+(0,a^+)) \times C((0,\infty), L^1_+(0,a^+))$ to system (2.2)–(2.3). By (2.2) and (2.3), after changing variables, we obtain

$$\begin{aligned} b_\phi(t,a) = &\int_0^t \int_0^{a^+-t+p} f_1(a, t-p, s+t-p) b_\phi(p,s)\mathrm{d}s\mathrm{d}p \\ &+ \int_0^t \int_0^{a^+-t+s} g_1(a, p+t-s, t-s) b'_\phi(s,p)\mathrm{d}p\mathrm{d}s \\ &+ \int_0^{a^+-t} \int_0^{a^+-t} h_1(a,p,s,t)\phi(p,s)\mathrm{d}p\mathrm{d}s \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} b'_\phi(t,a) = &\int_0^t \int_0^{a^+-t+p} f_2(a, t-p+s, t-p) b'_\phi(p,s)\mathrm{d}s\mathrm{d}p \\ &+ \int_0^t \int_0^{a^+-t+s} g_2(a, t-s, p+t-s) b_\phi(s,p)\mathrm{d}p\mathrm{d}s \\ &+ \int_0^{a^+-t} \int_0^{a^+-t} h_2(a,s,p,t)\phi(s,p)\mathrm{d}p\mathrm{d}s, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} f_1(a,p,s) &= h(p,s)\beta(a,p,s)e^{-\int_0^p \mu(\sigma,\sigma+s-p)\mathrm{d}\sigma}, \\ g_1(a,p,s) &= h(p,s)\beta(a,p,s)e^{-\int_0^s \mu(\sigma+p-s,\sigma)\mathrm{d}\sigma}, \\ h_1(a,p,s,t) &= h(p,s)\beta(a,p+t,s+t)e^{-\int_0^t \mu(\sigma+p,\sigma+s)\mathrm{d}\sigma}, \\ f_2(a,s,p) &= h(p,s)\beta'(a,s,p)e^{-\int_0^p \mu(\sigma+s-p,\sigma)\mathrm{d}\sigma}, \\ g_2(a,s,p) &= h(p,s)\beta'(a,s,p)e^{-\int_0^s \mu(\sigma,\sigma+p-s)\mathrm{d}\sigma}, \\ h_2(a,s,p,t) &= h(p,s)\beta'(a,s+t,p+t)e^{-\int_0^t \mu(\sigma+s,\sigma+p)\mathrm{d}\sigma}. \end{aligned}$$

Denote $M = C([0,T], L^1_+(0,a^+))$ and define

$$\mathscr{F} : M \times M \to M \times M$$

by $\mathscr{F}(b_\phi, b'_\phi) = (\mathscr{F}_1(b_\phi, b'_\phi), \mathscr{F}_2(b_\phi, b'_\phi))$, where

$$
\begin{aligned}
\mathscr{F}_1(b_\phi, b'_\phi) = & \int_0^t \int_0^{a^+-t+p} f_1(a, t-p, s+t-p) b_\phi(p, s) \mathrm{d}s \mathrm{d}p \\
& + \int_0^t \int_0^{a^+-t+s} g_1(a, p+t-s, t-s) b'_\phi(s, p) \mathrm{d}p \mathrm{d}s \\
& + \int_0^{a^+-t} \int_0^{a^+-t} h_1(a, p, s, t) \phi(p, s) \mathrm{d}p \mathrm{d}s
\end{aligned} \tag{2.9}
$$

and

$$
\begin{aligned}
\mathscr{F}_2(b_\phi, b'_\phi) = & \int_0^t \int_0^{a^+-t+p} f_2(a, t-p+s, t-p) b'_\phi(p, s) \mathrm{d}s \mathrm{d}p \\
& + \int_0^t \int_0^{a^+-t+s} g_2(a, t-s, p+t-s) b_\phi(s, p) \mathrm{d}p \mathrm{d}s \\
& + \int_0^{a^+-t} \int_0^{a^+-t} h_2(a, s, p, t) \phi(s, p) \mathrm{d}p \mathrm{d}s.
\end{aligned} \tag{2.10}
$$

First it is easy to see that $\mathscr{F}$ is linear and bounded. In fact,

$$
\begin{aligned}
\sup_{t \in [0,T]} \int_0^{a^+} \mathscr{F}_1(b_\phi, b'_\phi) \mathrm{d}a = & \int_0^{a^+} \int_0^t \int_0^{a^+-t+p} f_1(a, t-p, s+t-p) b_\phi(p, s) \mathrm{d}s \mathrm{d}p \mathrm{d}a \\
& + \int_0^{a^+} \int_0^t \int_0^{a^+-t+s} g_1(a, p+t-s, t-s) b'_\phi(s, p) \mathrm{d}p \mathrm{d}s \mathrm{d}a \\
& + \int_0^{a^+} \int_0^{a^+-t} \int_0^{a^+-t} h_1(a, p, s, t) \phi(p, s) \mathrm{d}p \mathrm{d}s \mathrm{d}a \\
\leq & \int_0^{a^+} \overline{\beta}(a) \mathrm{d}a \int_0^t \int_0^{a^+} b_\phi(p, s) \mathrm{d}s \mathrm{d}p \\
& + \int_0^{a^+} \overline{\beta}(a) \mathrm{d}a \int_0^t \int_0^{a^+} b'_\phi(s, p) \mathrm{d}p \mathrm{d}s \\
& + \int_0^{a^+} \overline{\beta}(a) \mathrm{d}a \int_0^{a^+} \int_0^{a^+} \phi(p, s) \mathrm{d}p \mathrm{d}s \\
\leq & \beta_{\sup} \int_0^t \int_0^{a^+} b_\phi(p, s) \mathrm{d}s \mathrm{d}p \\
& + \beta_{\sup} \int_0^t \int_0^{a^+} b'_\phi(s, p) \mathrm{d}p \mathrm{d}s + \beta_{\sup} \|\phi\|_E \\
\leq & \beta_{\sup} T \left[\|b_\phi\|_M + \|b'_\phi\|_M \right] + \beta_{\sup} \|\phi\|_E.
\end{aligned} \tag{2.11}
$$

Similarly,

$$
\begin{aligned}
\sup_{t\in[0,T]}\int_0^{a^+}\mathscr{F}_2(b_\phi,b'_\phi)\mathrm{d}a &\le \beta'_{\sup}\int_0^t\int_0^{a^+}b'_\phi(p,s)\mathrm{d}s\mathrm{d}p \\
&\quad + \beta'_{\sup}\int_0^t\int_0^{a^+}b_\phi(s,p)\mathrm{d}p\mathrm{d}s + \beta'_{\sup}\|\phi\|_E \\
&\le \beta'_{\sup}T\big[\|b'_\phi\|_M + \|b_\phi\|_M\big] + \beta'_{\sup}\|\phi\|_E.
\end{aligned}
\tag{2.12}
$$

Thus, $\mathscr{F}$ maps $M\times M$ into itself for any $T>0$.

Next, we claim that $\mathscr{F}$ is a contractive operator in $M\times M$ when T is sufficiently small. Indeed, by similar estimates as in (2.11),

$$
\begin{aligned}
&\sup_{t\in[0,T]}\int_0^{a^+}|\mathscr{F}_1(b_\phi,b'_\phi)-\mathscr{F}_1(\tilde b_\phi,\tilde b'_\phi)|\mathrm{d}a \\
&\le \sup_{t\in[0,T]}\Big[\beta_{\sup}\int_0^t\int_0^{a^+}|b_\phi(p,s)-\tilde b_\phi(p,s)|\mathrm{d}s\mathrm{d}p \\
&\quad + \beta_{\sup}\int_0^t\int_0^{a^+}|b'_\phi(s,p)-\tilde b'_\phi(s,p)|\mathrm{d}p\mathrm{d}s\Big] \\
&\le \beta_{\sup}T\|b_\phi-\tilde b_\phi\|_M + \beta_{\sup}T\|b'_\phi-\tilde b'_\phi\|_M.
\end{aligned}
\tag{2.13}
$$

Similarly, we have

$$
\sup_{t\in[0,T]}\int_0^{a^+}|\mathscr{F}_2(b_\phi,b'_\phi)-\mathscr{F}_2(\tilde b_\phi,\tilde b'_\phi)|\mathrm{d}a \le \beta'_{\sup}T\|b_\phi-\tilde b_\phi\|_M + \beta'_{\sup}T\|b'_\phi-\tilde b'_\phi\|_M.
$$

Now let T be sufficiently small, then

$$
\|\mathscr{F}(b_\phi,b'_\phi)-\mathscr{F}(\tilde b_\phi,\tilde b'_\phi)\|_{M\times M} \le \frac{1}{2}\|(b_\phi,b'_\phi)-(\tilde b_\phi,\tilde b'_\phi)\|_{M\times M}.
$$

It follows that $\mathscr{F}$ is contractive, which implies that there exists a unique fixed point to $\mathscr{F}$ in $M\times M$, i.e., there exists a unique solution to system (2.2)–(2.3) for $t\in[0,T]$. But since we have (2.6), it allows us to conclude that the solution $(b_\phi(t,a),b'_\phi(t,a))$ exists globally. In fact, we can extend the solution from T to $2T$ with initial data at T, and by the same argument as above to conclude the existence and uniqueness of the solution on $[T, 2T]$. Continuing this procedure, we obtain the global existence.

2.3 Semigroup generated by the solution flow

Define the family of linear operators $\{S(t)\}_{t\ge0}$ in E by the following formula:

$$
(S(t)\phi)(a,a') = \begin{cases} \phi(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}, & t<a<a'<a^+ \quad \text{or} \quad t<a'<a<a^+ \\ b_\phi(t-a,a'-a)e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}, & a<t<a'<a^+ \quad \text{or} \quad a<a'<t, a<a'<a^+ \\ b'_\phi(t-a',a-a')e^{-\int_0^{a'} \mu(s+a-a',s)\mathrm{d}s}, & a'<a<t, a'<a<a^+ \quad \text{or} \quad a'<t<a<a^+. \end{cases} \tag{2.14}
$$

We have the following theorem.

Theorem 2.3 *Let Assumption 2.1 hold. Then $\{S(t)\}_{t\geq 0}$ defined in (2.14) is a strongly continuous semigroup of bounded linear operators in E. Furthermore, $\{S(t)\}_{t\geq 0}$ is a positive semigroup in E.*

Proof First we can see that the positivity of $\{S(t)\}_{t\geq 0}$ follows immediately from the space M of b_ϕ, b'_ϕ and the space E of ϕ. Next we prove the semigroup property. Motivated by Webb [55], we prove that for $\phi \in E$,

$$
B_{S(t_1)\phi}(t) = B_\phi(t+t_1), \quad B'_{S(t_1)\phi}(t) = B'_\phi(t+t_1), \tag{2.15}
$$

where $B_{S(t_1)\phi}(t) = b_{S(t_1)\phi}(t,a')$, $B'_{S(t_1)\phi}(t) = b'_{S(t_1)\phi}(t,a')$; i.e.,

$$
b_{S(t_1)\phi}(t,a') = b_\phi(t+t_1,a'), \quad b'_{S(t_1)\phi}(t,a) = b'_\phi(t+t_1,a). \tag{2.16}
$$

Observe from (2.2), (2.3) and (2.14) that

$$
\begin{aligned}
b_{S(t_1)\phi}(t,a') = &\int_0^{a^+-t}\int_0^{a^+-t} h(a,s)\beta(a',a+t,s+t)S(t_1)\phi(a,s)e^{-\int_0^t \mu(\sigma+a,\sigma+s)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s \\
&+ \int_0^t\int_a^{a^+} h(a,s)\beta(a',a,s)b_{S(t_1)\phi}(t-a,s-a)e^{-\int_0^a \mu(\sigma,\sigma+s-a)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a \\
&+ \int_0^t\int_s^{a^+} h(a,s)\beta(a',a,s)b'_{S(t_1)\phi}(t-s,a-s)e^{-\int_0^s \mu(\sigma+a-s,\sigma)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s, \\
b'_{S(t_1)\phi}(t,a) = &\int_0^{a^+-t}\int_0^{a^+-t} h(a',s)\beta'(a,s+t,a'+t)S(t_1)\phi(s,a')e^{-\int_0^t \mu(\sigma+s,\sigma+a')\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s \\
&+ \int_0^t\int_{a'}^{a^+} h(a',s)\beta'(a,s,a')b'_{S(t_1)\phi}(t-a',s-a')e^{-\int_0^{a'} \mu(\sigma+s-a',\sigma)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a' \\
&+ \int_0^t\int_s^{a^+} h(a',s)\beta'(a,s,a')b_{S(t_1)\phi}(t-s,a'-s)e^{-\int_0^s \mu(\sigma,\sigma+a'-s)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s.
\end{aligned} \tag{2.17}
$$

We have

$$
\begin{aligned}
b_\phi(t+t_1,a') &= \int_0^{a^+-t-t_1}\int_0^{a^+-t-t_1} h(a,s)\beta(a',a+t+t_1,s+t+t_1)\phi(a,s)e^{-\int_0^{t+t_1}\mu(\sigma+a,\sigma+s)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s\\
&\quad+\int_0^{t+t_1}\int_a^{a^+} h(a,s)\beta(a',a,s)b_\phi(t+t_1-a,s-a)e^{-\int_0^{a}\mu(\sigma,\sigma+s-a)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a\\
&\quad+\int_0^{t+t_1}\int_s^{a^+} h(a,s)\beta(a',a,s)b'_\phi(t+t_1-s,a-s)e^{-\int_0^{s}\mu(\sigma+a-s,\sigma)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s\\
&= \int_{t_1}^{a^+-t}\int_{t_1}^{a^+-t} h(a-t_1,s-t_1)\beta(a',a+t,s+t)\phi(a-t_1,s-t_1)e^{-\int_0^{t+t_1}\mu(\sigma+a-t_1,\sigma+s-t_1)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s\\
&\quad+\int_0^{t_1}\int_a^{a^+-t} h(a+t,s+t)\beta(a',a+t,s+t)b_\phi(t_1-a,s-a)e^{-\int_0^{t+a}\mu(\sigma,\sigma+s-a)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a\\
&\quad+\int_0^{t_1}\int_s^{a^+-t} h(a+t,s+t)\beta(a',a+t,s+t)b'_\phi(t_1-s,a-s)e^{-\int_0^{t+s}\mu(\sigma+a-s,\sigma)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s\\
&\quad+\int_0^{t}\int_a^{a^+} h(a,s)\beta(a',a,s)b_\phi(t+t_1-a,s-a)e^{-\int_0^{a}\mu(\sigma,\sigma+s-a)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a\\
&\quad+\int_0^{t}\int_s^{a^+} h(a,s)\beta(a',a,s)b'_\phi(t+t_1-s,a-s)e^{-\int_0^{s}\mu(\sigma+a-s,\sigma)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s\\
&= \int_0^{a^+-t}\int_0^{a^+-t} h(a,s)\beta(a',a+t,s+t)S(t_1)\phi(a,s)e^{-\int_0^{t}\mu(\sigma+a,\sigma+s)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s\\
&\quad+\int_0^{t}\int_a^{a^+} h(a,s)\beta(a',a,s)b_\phi(t+t_1-a,s-a)e^{-\int_0^{a}\mu(\sigma,\sigma+s-a)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a\\
&\quad+\int_0^{t}\int_s^{a^+} h(a,s)\beta(a',a,s)b'_\phi(t+t_1-s,a-s)e^{-\int_0^{s}\mu(\sigma+a-s,\sigma)\mathrm{d}\sigma}\mathrm{d}a\mathrm{d}s,\\
b'_\phi(t+t_1,a) &= \int_0^{a^+-t-t_1}\int_0^{a^+-t-t_1} h(a',s)\beta'(a,s+t+t_1,a'+t+t_1)\phi(s,a')e^{-\int_0^{t+t_1}\mu(\sigma+s,\sigma+a')\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s\\
&\quad+\int_0^{t+t_1}\int_{a'}^{a^+} h(a',s)\beta'(a,s,a')b'_\phi(t+t_1-a',s-a')e^{-\int_0^{a'}\mu(\sigma+s-a',\sigma)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a'\\
&\quad+\int_0^{t+t_1}\int_s^{a^+} h(a',s)\beta'(a,s,a')b_\phi(t+t_1-s,a'-s)e^{-\int_0^{s}\mu(\sigma,\sigma+a'-s)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s\\
&= \int_{t_1}^{a^+-t}\int_{t_1}^{a^+-t} h(a'+t,s+t)\beta'(a',s+t,a'+t)\phi(s-t_1,a'-t_1)e^{-\int_0^{t+t_1}\mu(\sigma+s-t_1,\sigma+a'-t_1)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s\\
&\quad+\int_0^{t_1}\int_{a'}^{a^+-t} h(a'+t,s+t)\beta'(a,s+t,a'+t)b'_\phi(t_1-a',s-a')e^{-\int_0^{t+a'}\mu(\sigma+s-a',\sigma)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a'\\
&\quad+\int_0^{t_1}\int_s^{a^+-t} h(a'+t,s+t)\beta'(a,s+t,a'+t)b_\phi(t_1-s,a'-s)e^{-\int_0^{t+s}\mu(\sigma,\sigma+a'-s)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s\\
&\quad+\int_0^{t}\int_{a'}^{a^+} h(a',s)\beta'(a,s,a')b'_\phi(t+t_1-a',s-a')e^{-\int_0^{a'}\mu(\sigma+s-a',\sigma)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a'\\
&\quad+\int_0^{t}\int_s^{a^+} h(a',s)\beta'(a,s,a')b_\phi(t+t_1-s,a'-s)e^{-\int_0^{s}\mu(\sigma,\sigma+a'-s)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s\\
&= \int_0^{a^+-t}\int_0^{a^+-t} h(a',s)\beta'(a,s+t,a'+t)S(t_1)\phi(s,a')e^{-\int_0^{t}\mu(\sigma+s,\sigma+a')\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s\\
&\quad+\int_0^{t}\int_{a'}^{a^+} h(a',s)\beta'(a,s,a')b'_\phi(t+t_1-a',s-a')e^{-\int_0^{a'}\mu(\sigma+s-a',\sigma)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}a'\\
&\quad+\int_0^{t}\int_s^{a^+} h(a',s)\beta'(a,s,a')b_\phi(t+t_1-s,a'-s)e^{-\int_0^{s}\mu(\sigma,\sigma+a'-s)\mathrm{d}\sigma}\mathrm{d}a'\mathrm{d}s.
\end{aligned}
\tag{2.18}
$$

By the uniqueness of solutions to (2.2) and (2.3), we then obtain (2.16), which implies that

$$(S(t)S(t_1)\phi)(0,a') = (S(t+t_1)\phi)(0,a'), \quad t,t_1 \geq 0$$

and

$$(S(t)S(t_1)\phi)(a,0) = (S(t+t_1)\phi)(a,0), \quad t,t_1 \geq 0.$$

For $\phi \in E, 0 \leq t < t_1$ and $t+t_1 < a$ or $t+t_1 < a'$,

$$\begin{aligned}(S(t)S(t_1)\phi)(a,a') &= (S(t_1)\phi)(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}\\ &= \phi(a-t-t_1,a'-t-t_1)e^{-\int_0^{t_1} \mu(s-t_1+a-t,s-t_1+a'-t)\mathrm{d}s}\\ &\quad e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}\\ &= \phi(a-(t+t_1),a'-(t+t_1))e^{-\int_0^{t+t_1} \mu(s-(t+t_1)+a,s-(t+t_1)+a')\mathrm{d}s}\\ &= (S(t+t_1)\phi)(a,a').\end{aligned} \tag{2.19}$$

For $t < a < a' < t+t_1$,

$$\begin{aligned}(S(t)S(t_1)\phi)(a,a') &= (S(t_1)\phi)(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}\\ &= b_\phi(t_1-(a-t),a'-a)e^{-\int_0^{a-t} \mu(s,s+a'-a)\mathrm{d}s}e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}\\ &= b_\phi(t+t_1-a,a'-a)e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}\\ &= (S(t+t_1)\phi)(a,a').\end{aligned} \tag{2.20}$$

For $t < a' < a < t+t_1$,

$$\begin{aligned}(S(t)S(t_1)\phi)(a,a') &= (S(t_1)\phi)(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}\\ &= b'_\phi(t_1-(a'-t),a-a')e^{-\int_0^{a'-t} \mu(s+a-a',s)\mathrm{d}s}e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}\\ &= b'_\phi(t+t_1-a',a-a')e^{-\int_0^{a'} \mu(s+a-a',s)\mathrm{d}s}\\ &= (S(t+t_1)\phi)(a,a').\end{aligned} \tag{2.21}$$

For $a < a' < t$, by (2.16)

$$\begin{aligned}(S(t)S(t_1)\phi)(a,a') &= b_{S(t_1)\phi}(t-a,a'-a)e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}\\ &= b_\phi(t+t_1-a,a'-a)e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}\\ &= (S(t+t_1)\phi)(a,a').\end{aligned} \tag{2.22}$$

For $a' < a < t$, by (2.16)

$$
\begin{aligned}
(S(t)S(t_1)\phi)(a,a') &= b'_{S(t_1)\phi}(t-a',a-a')e^{-\int_0^{a'}\mu(s+a-a',s)\mathrm{d}s} \\
&= b'_{\phi}(t+t_1-a',a-a')e^{-\int_0^{a}\mu(s+a-a',s)\mathrm{d}s} \\
&= (S(t+t_1)\phi)(a,a').
\end{aligned}
\tag{2.23}
$$

Thus,

$$
S(t+t_1)\phi = (S(t)S(t_1))\phi, \quad t,t_1 \geq 0, \ \phi \in E.
$$

Now we need to prove the strong continuity property. We have

$$
\begin{aligned}
\|S(t)\phi-\phi\|_E &= \int_0^{a^+}\int_0^{a^+} |(S(t)\phi)(a,a')-\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\
&\leq \int_t^{a^+}\int_t^{a^+} |\phi(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}-\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\
&\quad + \int_0^{t}\int_a^{a^+} |b_\phi(t-a,a'-a)e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}-\phi(a,a')|\mathrm{d}a'\mathrm{d}a \\
&\quad + \int_0^{t}\int_{a'}^{a^+} |b'_\phi(t-a',a-a')e^{-\int_0^{a'} \mu(s+a-a',s)\mathrm{d}s}-\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\
&: = \mathrm{I}+\mathrm{II}+\mathrm{III},
\end{aligned}
\tag{2.24}
$$

where

$$
\begin{aligned}
\mathrm{I} &\leq \int_t^{a^+}\int_t^{a^+} |\phi(a-t,a'-t)-\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\
&\quad + \int_t^{a^+}\int_t^{a^+} |\phi(a-t,a'-t)||e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}-1|\mathrm{d}a\mathrm{d}a' \\
&\to 0 \quad \text{as} \quad t\to 0^+
\end{aligned}
\tag{2.25}
$$

by the absolute continuity and boundedness of $\phi \in E$ and continuity of the exponential function $e^{\underline{\mu}t}$. Next observe that

$$
\begin{aligned}
\mathrm{II} &\leq \int_0^{t}\int_a^{a^+} |b_\phi(t-a,a'-a)-b_\phi(0,a'-a)|+|b_\phi(0,a'-a)-\phi(a,a')|\mathrm{d}a'\mathrm{d}a \\
&\quad + \int_0^{t}\int_a^{a^+} |b_\phi(t-a,a'-a)||e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}-1|\mathrm{d}a'\mathrm{d}a \\
&\to 0 \quad \text{as} \quad t\to 0^+
\end{aligned}
\tag{2.26}
$$

by the boundary condition $b_\phi(0,a') = u(0,0,a') = \phi(0,a')$ and absolute continuity and boundedness of b_ϕ in $C((0,\infty),L^1_+(0,a^+))$ with continuity of the exponential function $e^{\underline{\mu}a}$. Similarly, we can show that $\mathrm{III} \to 0$ as $t \to 0^+$. It follows that

$$
\lim_{t\to 0^+} S(t)\phi = \phi, \quad \forall \phi \in E.
$$

This completes the proof. □

2.4 Solutions of the initial-boundary value problem

In the previous subsections we established the global existence and uniqueness of the solution for (2.1). In this subsection we claim that the solution of (2.14) is indeed a solution of (1.1)–(1.4).

Proposition 2.4 *Let $T > 0$ and let $\phi \in L^1_+((0,a^+)\times(0,a^+))$. If u is a solution of* (2.1) *on* $[0,T]$, *then it is a solution of* (1.1)–(1.4) *on* $[0,T]$.

Proof First, $u(0,\cdot,\cdot)=\phi$ since $u(t,\cdot,\cdot)$ satisfies (2.1) at $t=0$. Next, let $0\le t<T$ and $0<h<T-t$. From (2.1) we have

$$\begin{aligned}
&\int_0^{a^+}\int_0^{a^+}\Big|h^{-1}[u(t+h,a+h,a'+h)-u(t,a,a')]+\mu(a,a')u(t,a,a')\Big|\mathrm{d}a\mathrm{d}a'\\
&=\int_t^{a^+}\int_t^{a^+}|h^{-1}\phi(a-t,a'-t)[e^{-\int_0^{t+h}\mu(s-t+a,s-t+a')\mathrm{d}s}-e^{-\int_0^{t}\mu(s-t+a,s-t+a')\mathrm{d}s}]\\
&\quad+\mu(a,a')\phi(a-t,a'-t)e^{-\int_0^{t}\mu(s-t+a,s-t+a')\mathrm{d}s}|\mathrm{d}a\mathrm{d}a'\\
&\quad+\int_0^{t}\int_a^{a^+}|h^{-1}u(t-a,0,a'-a)[e^{-\int_0^{a+h}\mu(s,s+a'-a)\mathrm{d}s}-e^{-\int_0^{a}\mu(s,s+a'-a)\mathrm{d}s}]\\
&\quad+\mu(a,a')u(t-a,0,a'-a)e^{-\int_0^{a}\mu(s,s+a'-a)\mathrm{d}s}|\mathrm{d}a'\mathrm{d}a\\
&\quad+\int_0^{t}\int_{a'}^{a^+}|h^{-1}u(t-a',a-a',0)[e^{-\int_0^{a'+h}\mu(s+a-a',s)\mathrm{d}s}-e^{-\int_0^{a'}\mu(s+a-a',s)\mathrm{d}s}]\\
&\quad+\mu(a,a')u(t-a',a-a',0)e^{-\int_0^{a'}\mu(s+a-a',s)\mathrm{d}s}|\mathrm{d}a\mathrm{d}a'\\
&\le\int_t^{a^+}\int_t^{a^+}|\phi(a-t,a'-t)||h^{-1}[e^{-\int_t^{t+h}\mu(s-t+a,s-t+a')\mathrm{d}s}-1]+\mu(a,a')||e^{-\int_0^{t}\mu(s-t+a,s-t+a')\mathrm{d}s}|\mathrm{d}a\mathrm{d}a'\\
&\quad+\int_0^{t}\int_a^{a^+}|u(t-a,0,a'-a)||h^{-1}[e^{-\int_a^{a+h}\mu(s,s+a'-a)\mathrm{d}s}-1]+\mu(a,a')||e^{-\int_0^{a}\mu(s,s+a'-a)\mathrm{d}s}|\mathrm{d}a'\mathrm{d}a\\
&\quad+\int_0^{t}\int_{a'}^{a^+}|u(t-a',a-a',0)||h^{-1}[e^{-\int_{a'}^{a'+h}\mu(s+a-a',s)\mathrm{d}s}-1]+\mu(a,a')||e^{-\int_0^{a'}\mu(s+a-a',s)\mathrm{d}s}|\mathrm{d}a\mathrm{d}a'.
\end{aligned}\tag{2.27}$$

Letting $h\to 0$, by continuity of the exponential function $e^{\underline{\mu}h}$, the above expression will approach zero by the uniform boundedness of $b(t,a'), b'(t,a)$ in $C((0,\infty),L^1_+(0,a^+))$ and ϕ in E. Hence, (1.1) holds.

Next, let $0\le t<T$ and let $0<h<T-t$ with $0<h<a'$. From (2.1) we have

$$
\begin{aligned}
& h^{-1}\int_0^h\int_0^{a^+}\left|u(t+h,x,a')-\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s\right|\mathrm{d}a'\mathrm{d}x\\
&\quad = h^{-1}\int_0^h\int_0^{a^+}\Big|u(t+h-x,0,a'-x)e^{-\int_0^x\mu(s,s+a'-x)\mathrm{d}s}\\
&\qquad -\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s\Big|\mathrm{d}a'\mathrm{d}x\\
&\quad = h^{-1}\int_0^h\int_0^{a^+}\Big|\int_0^{a^+}\int_0^{a^+}\beta(a'-x,a,s)u(t+h-x,a,s)\mathrm{d}a\mathrm{d}s e^{-\int_0^x\mu(s,s+a'-x)\mathrm{d}s}\\
&\qquad -\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s\Big|\mathrm{d}a'\mathrm{d}x\\
&\quad = h^{-1}\int_0^h\int_0^{a^+}\Bigg|\Bigg[\int_0^{a^+}\int_0^{a^+}\beta(a'-x,a,s)u(t+h-x,a,s)\\
&\qquad -\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s\Bigg]e^{-\int_0^x\mu(s,s+a'-x)\mathrm{d}s}\\
&\qquad +\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s[e^{-\int_0^x\mu(s,s+a'-x)\mathrm{d}s}-1]\Bigg|\mathrm{d}a'\mathrm{d}x\\
&\quad \le \sup_{0\le x\le h}\int_0^{a^+}\int_0^{a^+}\int_0^{a^+}\beta(a'-x,a,s)u(t+h-x,a,s)-\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s\mathrm{d}a'\\
&\qquad +\sup_{0\le x\le h}[e^{-\underline{\mu}x}-1]\int_0^{a^+}\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)u(t,a,s)\mathrm{d}a\mathrm{d}s\mathrm{d}a'\\
&\quad := \mathrm{I}+\mathrm{II},
\end{aligned}
\tag{2.28}
$$

where

$$
\begin{aligned}
\mathrm{I} &\le \sup_{0\le x\le h}\Big|\int_0^{a^+}\int_0^{a^+}\int_0^{a^+}|\beta(a'-x,a,s)-\beta(a',a,s)||u(t+h-x,a,s)|\mathrm{d}a\mathrm{d}s\mathrm{d}a'\\
&\quad +\int_0^{a^+}\int_0^{a^+}\int_0^{a^+}|\beta(a',a,s)||u(t+h-x,a,s)-u(t,a,s)|\mathrm{d}a\mathrm{d}s\mathrm{d}a'\Big|\\
&\le \sup_{0\le x\le h}\Big|\int_0^{a^+}|\beta(a'-x,a,s)-\beta(a',a,s)|\mathrm{d}a'\int_0^{a^+}\int_0^{a^+}u(t+h-x,a,s)\mathrm{d}a\mathrm{d}s\\
&\quad +\int_0^{a^+}\overline{\beta}(a')\mathrm{d}a'\int_0^{a^+}\int_0^{a^+}|u(t+h-x,a,s)-u(t,a,s)|\mathrm{d}a\mathrm{d}s\Big|
\end{aligned}
\tag{2.29}
$$

and

$$
\mathrm{II}\le \sup_{0\le x\le h}[e^{-\underline{\mu}x}-1]\int_0^{a^+}\overline{\beta}(a')\mathrm{d}a'\int_0^{a^+}\int_0^{a^+}u(t,a,s)\mathrm{d}a\mathrm{d}a. \tag{2.30}
$$

Letting $h\to 0$ yields that $\mathrm{I}\to 0$ by the equicontinuity of β with respect to a' in Assumption 2.1(ii), continuity of $u(t,\cdot,\cdot)$ in E with respect to t, and the uniform boundedness of $u(t,\cdot,\cdot)$ in E and $\overline{\beta}$ in Assumption 2.1, while $\mathrm{II}\to 0$ by uniform boundedness of $u(t,\cdot,\cdot)$ in E and $\overline{\beta}$ in Assumption 2.1 with continuity of exponential functions.

Similarly, we can show that

$$\lim_{h\to 0} h^{-1}\int_0^h\int_0^{a^+}\left|u(t+h,a,x)-\int_0^{a^+}\int_0^{a^+}\beta'(a,s,a')u(t,s,a')\mathrm{d}s\mathrm{d}a'\right|dadx=0.$$

This completes the proof. □

3 The infinitesimal generator

The infinitesimal generator of the strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ is defined as the (unbounded) linear operator A in X by (see Hille and Phillips [21])

$$A\phi=\lim_{t\to 0^+}t^{-1}(S(t)\phi-\phi)$$

with $\phi\in D(A)$, where

$$D(A)=\{\phi\in X:\lim_{t\to 0^+}t^{-1}(S(t)\phi-\phi)\text{ exists}\}.$$

One would intuitively imagine that the infinitesimal generator would follow similarly to the single age-structured systems (Webb [52]) as $B\phi=-\frac{\partial\phi}{\partial a}-\frac{\partial\phi}{\partial a'}-\mu\phi$ with domain

$$D(B)=\left\{\phi\in W^{1,1}([0,a^+)^2):\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)\phi(a,s)\mathrm{d}a\mathrm{d}s\right.$$
$$\left.=\phi(0,a'),\int_0^{a^+}\int_0^{a^+}\beta'(a,s,a')\phi(s,a')\mathrm{d}s\mathrm{d}a'=\phi(a,0)\right\}.$$

However, it is unlikely to show the weak differentiability of $\phi\in D(A)$; thus, we cannot conclude $A=B$. The key reason for A and B being different lies in the fact that A takes the directional derivative along $\langle 1,1\rangle$ in the L^1 sense, while B takes partial derivatives with respect to both variables in the L^1 sense. Indeed, the characterization of the infinitesimal generator A is a bit more complicated than the single age-structured case, and here, we adopt the description of A as pointed out in Webb [53, Remark 3.1] and provide a proof.

Proposition 3.1 *If $\phi\in D(A)$, there exists $\chi\in E$ such that for $a\geq 0, a'\geq 0$:*

$$\int_0^a\phi(s,a')\mathrm{d}s+\int_0^{a'}\phi(a,s')\mathrm{d}s'+\int_0^{a'}\int_0^a\chi(s,s')\mathrm{d}s\mathrm{d}s'=\int_0^a\mathcal{F}(\phi)(s)\mathrm{d}s+\int_0^{a'}\mathcal{G}(\phi)(s')\mathrm{d}s',\tag{3.1}$$

where $\mathcal{F}(\phi)(a):=\int_0^{a^+}\int_0^{a^+}\beta'(a,s,a')\phi(s,a')\mathrm{d}s\mathrm{d}a'$ and $\mathcal{G}(\phi)(a'):=\int_0^{a^+}\int_0^{a^+}\beta(a',a,s)\phi(a,s)\mathrm{d}a\mathrm{d}s$.

Proof Let $\phi\in D(A)$, for $t>0$, define $\chi_t\in E,\phi_t\in E$ by

$$\chi_t(a,a') = \begin{cases} 0, & \text{for a.e.} \quad a < t \text{ or } a' < t \\ t^{-1}[\phi(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s} - \phi(a-t,a'-t)], & \text{for a.e. } a > t \text{ and } a' > t \end{cases}$$

and

$$\phi_t(a,a') = \begin{cases} 0, & \text{for a.e.} \quad a < t \text{ or } a' < t \\ t^{-1}[\phi(a-t,a'-t) - \phi(a,a')], & \text{for a.e. } a > t \text{ and } a' > t, \end{cases}$$

respectively. Then, we have

$$\begin{aligned} &\|\chi_t + \mu(a,a')\phi(a,a')\|_E \\ &= \int_t^{a^+}\int_t^{a^+} \left|\left[\phi(a-t,a'-t)\frac{e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s} - 1}{t}\right] + \mu(a,a')\phi(a,a')\right| \mathrm{d}a\mathrm{d}a' \\ &\quad + \int_0^t \int_a^{a^+} |\mu(a,a')\phi(a,a')|\mathrm{d}a'\mathrm{d}a + \int_0^t \int_{a'}^{a^+} |\mu(a,a')\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\ &\leq \int_t^{a^+}\int_t^{a^+} |\mu(a,a')||\phi(a-t,a'-t) - \phi(a,a')|\mathrm{d}a\mathrm{d}a' \\ &\quad + \int_t^{a^+}\int_t^{a^+} |\phi(a-t,a'-t)| \left[\frac{e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s} - 1}{t} + \mu(a,a')\right] \mathrm{d}a\mathrm{d}a' \\ &\quad + \int_0^t \int_a^{a^+} |\mu(a,a')\phi(a,a')|\mathrm{d}a'\mathrm{d}a + \int_0^t \int_{a'}^{a^+} |\mu(a,a')\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\ &\to 0 \quad \text{as} \quad t \to 0^+ \end{aligned} \tag{3.2}$$

by the differentiability of exponential functions and the boundedness and absolute continuity of ϕ in E.

Next, observe that

$$\begin{aligned} &\|\phi_t + \chi_t - A\phi\|_E \\ &\leq \int_0^t \int_a^{a^+} |A\phi(a,a')|\mathrm{d}a'\mathrm{d}a + \int_0^t \int_{a'}^{a^+} |A\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\ &\quad + \int_t^{a^+}\int_t^{a^+} \left|\frac{\phi(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s} - \phi(a,a')}{t} - A\phi(a,a')\right| \mathrm{d}a\mathrm{d}a' \\ &\leq \int_0^t \int_a^{a^+} |A\phi(a,a')|\mathrm{d}a'\mathrm{d}a + \int_0^t \int_{a'}^{a^+} |A\phi(a,a')|\mathrm{d}a\mathrm{d}a' \\ &\quad + \int_t^{a^+}\int_t^{a^+} \left|t^{-1}[S(t)\phi - \phi] - A\phi(a,a')\right| \mathrm{d}a\mathrm{d}a' \\ &\to 0 \quad \text{as} \quad t \to 0^+ \end{aligned} \tag{3.3}$$

by the definition of A. Thus, we have $\lim_{t\to 0^+} \phi_t(a,a') = A\phi(a,a') + \mu(a,a')\phi(a,a')$ *a.e.* in $[0,a^+] \times [0,a^+]$. In addition we get for any $a, a' \geq 0$ that

$$
\begin{aligned}
&\lim_{t\to 0^+} t^{-1}\int\limits_t^{a+t}\int\limits_t^{a'+t}\phi(s,s')\mathrm{d}s'\mathrm{d}s - t^{-1}\int\limits_0^{a}\int\limits_0^{a'}\phi(s,s')\mathrm{d}s'\mathrm{d}s\\
&\quad=\lim_{t\to 0^+} t^{-1}\int\limits_0^{a}\int\limits_0^{a'}\phi(s+t,s'+t)-\phi(s,s')\mathrm{d}s'\mathrm{d}s \qquad (3.4)\\
&\quad=-\int\limits_0^{a}\int\limits_0^{a'}[A\phi(s,s')+\mu(s,s')\phi(s,s')]\mathrm{d}s'\mathrm{d}s.
\end{aligned}
$$

We further observe that for *a.e.* $a\in[0,a^+]$,

$$
\begin{aligned}
&\|t^{-1}[S(t)\phi-\phi]-A\phi\|_E\\
&\geq \int\limits_0^{t}\int\limits_{s'}^{a}\left|t^{-1}[b'_\phi(t-s',s-s')e^{-\int_0^{s'}\mu(r+s-s',r)dr}-\phi(s,s')]-A\phi(s,s')\right|\mathrm{d}s\mathrm{d}s'.
\end{aligned}
$$

By the fact that $\int_0^t\int_a^{a^+}|A\phi(a,a')|\mathrm{d}a'\mathrm{d}a\to 0$ as $t\to 0^+$, we have

$$
t^{-1}\int\limits_0^{t}\int\limits_{s'}^{a}\left|b'_\phi(t-s',s-s')e^{-\int_0^{s'}\mu(r+s-s',r)dr}-\phi(s,s')\right|\mathrm{d}s\mathrm{d}s'\to 0 \text{ as } t\to 0^+. \qquad (3.5)
$$

Then

$$
\begin{aligned}
&\left|t^{-1}\int_0^t\int_0^a\phi(s,s')\mathrm{d}s\mathrm{d}s'-\int_0^a\mathcal{F}(\phi)(s)\mathrm{d}s\right|\\
&\leq t^{-1}\int_0^t\int_0^a\left|\phi(s,s')-\mathcal{F}(\phi)(s)\right|\mathrm{d}s\mathrm{d}s'\\
&\leq t^{-1}\int_0^t\int_0^a\left|\phi(s,s')-b'_\phi(t-s',s-s')e^{-\int_0^{s'}\mu(r+s-s',r)dr}\right|\mathrm{d}s\mathrm{d}s'\\
&\quad+t^{-1}\int_0^t\int_0^a\left|b'_\phi(t-s',s-s')\left(e^{-\int_0^{s'}\mu(r+s-s',r)dr}-1\right)\right|\mathrm{d}s\mathrm{d}s'\\
&\quad+t^{-1}\int_0^t\int_0^a\left|b'_\phi(t-s',s-s')-b'_\phi(t-s',s)\right|\mathrm{d}s\mathrm{d}s'\\
&\quad+t^{-1}\int_0^t\int_0^a\left|b'_\phi(t-s',s)-\mathcal{F}(\phi)(s)\right|\mathrm{d}s\mathrm{d}s'\\
&=:I_1+I_2+I_3+I_4,
\end{aligned}
$$

where I_1 converges to 0 as $t\to 0^+$ because of (3.5). Next note that $b'_\phi\in C([0,\infty),L^1_+(0,a^+))$ and $b'_\phi(0,\cdot)=\mathcal{F}(\phi)(\cdot)$, we have

$$
\lim_{t\to 0^+}\int_0^{a^+}|b'_\phi(t,a)-\mathcal{F}(\phi)(a)|da=0. \qquad (3.6)
$$

Hence, we can find $\delta>0$ such that

$$\sup_{t\in[0,\delta]}\int_0^{a^+} b'_\phi(t,a)da < 2\int_0^{a^+}\mathcal{F}(\phi)(a)da.$$

Therefore, by Assumption 2.1(v),

$$I_2 \le \sup_{s'\in[0,t]}\int_0^{a^+}|b'_\phi(t-s',s-s')|(1-e^{-\underline{\mu}s'})\mathrm{d}s \to 0 \text{ as } t\to 0^+.$$

Clearly, $I_4 \to 0$ as $t \to 0^+$. Further,

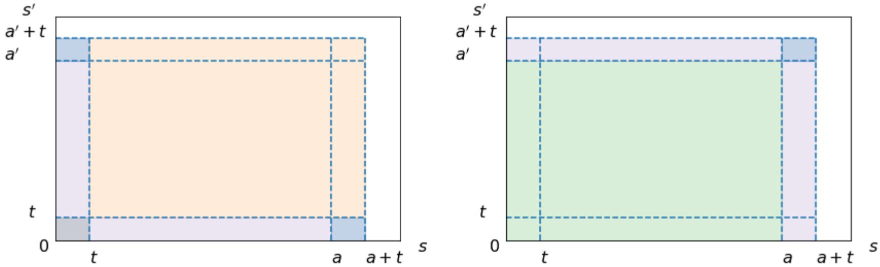
$$\begin{aligned}
I_3 &\le \sup_{s'\in[0,t]}\int_0^{a^+}|b'_\phi(t-s',s-s')-b'_\phi(t-s',s)|\mathrm{d}s\\
&\le \sup_{s'\in[0,t]}\int_0^{a^+}|b'_\phi(t-s',s-s')-\mathcal{F}(\phi)(s-s')|\mathrm{d}s\\
&\quad+\sup_{s'\in[0,t]}\int_0^{a^+}|\mathcal{F}(\phi)(s)-b'_\phi(t-s',s)|\mathrm{d}s\\
&\quad+\sup_{s'\in[0,t]}\int_0^{a^+}|\mathcal{F}(\phi)(s-s')-\mathcal{F}(\phi)(s)|\mathrm{d}s \to 0 \text{ as } t\to 0^+,
\end{aligned}$$

where the first two terms converge to 0 as $t \to 0^+$ because of (3.6) and the last term converges to 0 due to the continuity of the L^1 function $\mathcal{F}(\phi)$ with respect to translation. So we have

$$\lim_{t\to 0^+} t^{-1}\int_0^t\int_0^a \phi(s,s')\mathrm{d}s\mathrm{d}s' = \int_0^a \mathcal{F}(\phi)(s)\mathrm{d}s. \tag{3.7}$$

Similarly

$$\lim_{t\to 0^+} t^{-1}\int_0^t\int_0^{a'} \phi(s,s')\mathrm{d}s'\mathrm{d}s = \int_0^{a'} \mathcal{G}(\phi)(s')\mathrm{d}s'. \tag{3.8}$$



(a) LHS of (3.13). **(b)** RHS of (3.13).

Fig. 2 Illustration of equality (3.13)

Next we observe that $\int_0^a \int_0^{a'} \phi(s,s')\mathrm{d}s'\mathrm{d}s$ is differentiable with respect to the variable a for all $a, a' \geq 0$, thus

$$\lim_{t\to 0^+} t^{-1} \int_a^{a+t} \int_0^{a'} \phi(s,s')\mathrm{d}s'\mathrm{d}s = \int_0^{a'} \phi(a,s')\mathrm{d}s', \tag{3.9}$$

and similarly

$$\lim_{t\to 0^+} t^{-1} \int_0^{a} \int_{a'}^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s = \int_0^{a} \phi(s,a')\mathrm{d}s. \tag{3.10}$$

Further, it is easy for us to have the following estimates:

$$\lim_{t\to 0^+} t^{-1} \int_0^{t} \int_0^{t} \phi(s,s')\mathrm{d}s'\mathrm{d}s = 0, \quad \lim_{t\to 0^+} t^{-1} \int_0^{t} \int_{a'}^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s = 0, \tag{3.11}$$

$$\lim_{t\to 0^+} t^{-1} \int_a^{a+t} \int_0^{t} \phi(s,s')\mathrm{d}s'\mathrm{d}s = 0, \quad \lim_{t\to 0^+} t^{-1} \int_a^{a+t} \int_{a'}^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s = 0. \tag{3.12}$$

Note that the integration of ϕ on $[0, a+t] \times [0, a'+t]$ can be approached in different ways (see Fig. 2) and we have the following equality for any $a, a' \geq 0$:

$$\begin{aligned} & t^{-1}\Big(\int_0^{a} \int_0^{t} \phi(s,s')\mathrm{d}s'\mathrm{d}s + \int_0^{t} \int_0^{a'} \phi(s,s')\mathrm{d}s'\mathrm{d}s - \int_0^{t} \int_0^{t} \phi(s,s')\mathrm{d}s'\mathrm{d}s \\ & + \int_a^{a+t} \int_0^{t} \phi(s,s')\mathrm{d}s'\mathrm{d}s + \int_0^{t} \int_{a'}^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s + \int_t^{a+t} \int_t^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s \Big) \\ & = t^{-1}\Big(\int_0^{a} \int_0^{a'} \phi(s,s')\mathrm{d}s'\mathrm{d}s + \int_a^{a+t} \int_0^{a'} \phi(s,s')\mathrm{d}s'\mathrm{d}s \\ & \quad + \int_0^{a} \int_{a'}^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s + \int_a^{a+t} \int_{a'}^{a'+t} \phi(s,s')\mathrm{d}s'\mathrm{d}s \Big). \end{aligned} \tag{3.13}$$

Let $t \to 0^+$ and apply (3.4)–(3.12), we have (3.1) for *a.e.* $a, a' \geq 0$ with $\chi = A\phi + \mu\phi$. ☐

Remark 3.2 Based on (3.1), one can easily derive a description of operator A:

$$\begin{aligned} (A\phi)(a,a') &= -\frac{\partial}{\partial a}\Big(\phi(a,a') + \frac{\partial}{\partial a'} \int_0^{a} \phi(s,a')\mathrm{d}s\Big) - \mu(a,a')\phi(a,a') \\ &= -\frac{\partial}{\partial a'}\Big(\phi(a,a') + \frac{\partial}{\partial a} \int_0^{a'} \phi(a,s')\mathrm{d}s'\Big) - \mu(a,a')\phi(a,a'). \end{aligned} \tag{3.14}$$

From (3.1) we can conclude that

$$
\begin{aligned}
D(A) \subset \Big\{ & \phi \in E : (a,a') \to \int_0^a \phi(s,a')\mathrm{d}s \text{ is absolutely continuous in } a' \text{ for } a \geq 0,\\
& (a,a') \to \left[\frac{\partial}{\partial a'} \int_0^a \phi(s,a')\mathrm{d}s + \phi(a,a')\right] \text{ is absolutely continuous in } a \text{ for a.e. } a' > 0,\\
& \lim_{a\to 0^+} \left[\frac{\partial}{\partial a'} \int_0^a \phi(s,a')\mathrm{d}s + \phi(a,a')\right] = \mathcal{G}(\phi)(a') \text{ for a.e. } a' > 0,\\
& (a,a') \to \int_0^{a'} \phi(a,s)\mathrm{d}s \text{ is absolutely continuous in } a \text{ for } a' \geq 0,\\
& (a,a') \to \left[\frac{\partial}{\partial a} \int_0^{a'} \phi(a,s)\mathrm{d}s + \phi(a,a')\right] \text{ is absolutely continuous in } a' \text{ for a.e. } a > 0,\\
& \lim_{a'\to 0^+} \left[\frac{\partial}{\partial a} \int_0^{a'} \phi(a,s)\mathrm{d}s + \phi(a,a')\right] = \mathcal{F}(\phi)(a) \text{ for a.e. } a > 0,\\
& \text{and } \frac{\partial}{\partial a}\left[\frac{\partial}{\partial a'} \int_0^a \phi(s,a')\mathrm{d}s + \phi(a,a')\right] \in E \Big\}.
\end{aligned}
\tag{3.15}
$$

Moreover, Webb [53] claimed that the inclusion is in fact an equality. Furthermore, $\phi \in D(A)$ if and only there exists $\chi \in E$ such that for $a \geq 0, a' \geq 0$, (3.1) holds. If in addition ϕ is sufficiently smooth, then $A = B$ and $\phi(a,0) = \mathcal{F}(\phi)(a), a \geq 0, \phi(0,a') = \mathcal{G}(\phi)(a'), a' \geq 0$.

4 Compactness of solution trajectories

First we introduce the α-measure of non-compactness of a bounded linear operator in the Banach space X from Nussbaum [41] or Webb [52]. If T is a bounded linear operator in the Banach space X, then the *Kuratowski measure of non-compactness* of T, denoted by $\alpha[T]$, is the infimum of $\epsilon > 0$ such that $\alpha[T(M)] \leq \epsilon\alpha[M]$ for all bounded sets M in X, where $\alpha[M]$ is the measure of non-compactness of M. The following result is proved in Webb [52, Proposition 4.9]:

Proposition 4.1 *Let T_1 and T_2 be bounded linear operator in Banach space X. The following hold:*

(i) $\alpha(T_1) \leq |T_1|$;
(ii) $\alpha[T_1T_2] \leq \alpha[T_1]\alpha[T_2]$;
(iii) $\alpha[T_1 + T_2] \leq \alpha[T_1] + \alpha[T_2]$;
(iv) $\alpha[T_1] = 0$ *if and only if T_1 is compact.*

To establish the compactness of solution trajectories, we need the following proposition which was proved by Webb [50].

Proposition 4.2 (Webb [50]) *Let $\{S(t)\}_{t\geq 0}$ be a dynamical system in the Banach space X satisfying*

(i) $S(t) = S_1(t) + S_2(t)$ *for each $t \geq 0$, where $S_1(t), S_2(t)$ are mappings from X to X;*
(ii) $\|S_1(t)x\| \leq c(t,r)$ *for all $t \geq 0$ and all $x \in X$ such that $\|x\| \leq r$, where $c : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that for all $r > 0$, $\lim_{t\to\infty} c(t,r) = 0$;*

(iii) $S_2(t)$ *is compact (that is, maps bounded sets into precompact sets) for* t *sufficiently large.*

If the trajectory $\gamma(x_0)$ *of* $x_0 \in X$ *is bounded, then it is also precompact.*

We now apply this proposition to show that bounded trajectories of $\{S(t)\}_{t\ge 0}$ are precompact. Define

$$S_1(t)\phi(a,a') = \begin{cases} \phi(a-t,a'-t)e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s}, & t<a<a'<a^+ \quad \text{or} \quad t<a'<a<a^+ \\ 0, & \text{otherwise} \end{cases}$$
$$S_2(t)\phi(a,a') = \begin{cases} b_\phi(t-a,a'-a)e^{-\int_0^a \mu(s,s+a'-a)\mathrm{d}s}, & a<t<a'<a^+ \quad \text{or} \quad a<a'<t, a<a'<a^+ \\ 0, & \text{otherwise} \end{cases}$$
$$S_3(t)\phi(a,a') = \begin{cases} b'_\phi(t-a',a-a')e^{-\int_0^{a'} \mu(s+a-a',s)\mathrm{d}s}, & a'<a<t, a'<a<a^+ \quad \text{or} \quad a'<t<a<a^+ \\ 0, & \text{otherwise.} \end{cases}$$

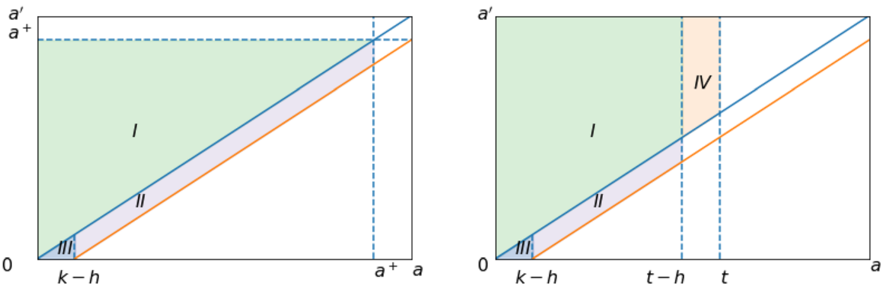
Proposition 4.3 *If* $\underline{\mu} > 0$, *then* $S_1(t)$ *satisfies the hypothesis (ii) of Proposition 4.2 and* $S_2(t), S_3(t)$ *satisfy (iii) of Proposition 4.2.*

Proof We assume $a^+ < \infty$ throughout this proof, the conclusion also holds for $a^+ = \infty$ and the corresponding proof is presented in Appendix, Proposition A.1.

Obviously $S_i(t), t \ge 0, i = 1,2,3$, are mappings from E to E. We have

$$\begin{aligned} \|S_1(t)\phi\|_E &\le \left| \int_t^{a^+} \int_t^{a^+} e^{-\int_0^t \mu(s-t+a,s-t+a')\mathrm{d}s} \phi(a-t,a'-t) \right| \mathrm{d}a\mathrm{d}a' \\ &\le \int_t^{a^+} \int_t^{a^+} e^{-\underline{\mu}t} |\phi(a-t,a'-t)| \mathrm{d}a\mathrm{d}a' \\ &\le e^{-\underline{\mu}t}\|\phi\|_E. \end{aligned} \tag{4.1}$$

Now we show that $S_2(t)$ is compact for $t > 2a^+$, it is equivalent to show that for a bounded set K of E,



(a) $a^+ < \infty$. **(b)** $a^+ = \infty$.

Fig. 3 Integration regions for (4.4) and (A.3)

$$\lim_{\substack{h \to 0 \\ k \to 0}} \int_0^{a^+} \int_0^{a^+} \left| S_2(t)\phi(a+h, a'+k) - S_2(t)\phi(a, a') \right| \mathrm{d}a\mathrm{d}a' = 0, \tag{4.2}$$

$$\lim_{\substack{h \to a^+ \\ k \to a^+}} \int_h^{a^+} \int_k^{a^+} \left| S_2(t)\phi(a, a') \right| \mathrm{d}a\mathrm{d}a' = 0 \tag{4.3}$$

uniformly for $\phi \in K$ (which can be found in Dunford and Schwartz [14, Theorem 21, p. 301]). Without loss of generality, assume that $k > h$ and $h, k \to 0^+$, we have

$$\begin{aligned}
&\int_0^{a^+} \int_0^{a^+} \left| S_2(t)\phi(a+h, a'+k) - S_2(t)\phi(a, a') \right| \mathrm{d}a'\mathrm{d}a \\
&\quad \le \underbrace{\int_0^{a^+} \int_a^{a^+} \left| S_2(t)\phi(a+h, a'+k) - S_2(t)\phi(a, a') \right| \mathrm{d}a'\mathrm{d}a}_{\text{region I}} \\
&\quad\quad + \underbrace{\int_{k-h}^{a^+} \int_{a+h-k}^{a} \left| S_2(t)\phi(a+h, a'+k) \right| \mathrm{d}a'\mathrm{d}a}_{\text{region II}} \\
&\quad\quad + \underbrace{\int_0^{k-h} \int_0^{a} \left| S_2(t)\phi(a+h, a'+k) \right| \mathrm{d}a'\mathrm{d}a,}_{\text{region III}}
\end{aligned} \tag{4.4}$$

as illustrated in Fig. 3a: $S_2(t)\phi(a, a')$ is non-trivial for points (a, a') in regions I, and $S_2(t)\phi(a+h, a'+k)$ is non-trivial for points (a, a') in regions I, II, and III. First recall from (2.7) and (2.8) that when $a^+ < \infty$ and $t > a^+$,

$$\begin{aligned}
b_\phi(t, a) = &\int_{t-a^+}^{t} \int_0^{a^+-t+p} f_1(a, t-p, s+t-p) b_\phi(p, s)\mathrm{d}s\mathrm{d}p \\
&+ \int_{t-a^+}^{t} \int_0^{a^+-t+s} g_1(a, p+t-s, t-s) b'_\phi(s, p)\mathrm{d}p\mathrm{d}s
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
b'_\phi(t, a) = &\int_{t-a^+}^{t} \int_0^{a^+-t+p} f_2(a, t-p+s, t-p) b'_\phi(p, s)\mathrm{d}s\mathrm{d}p \\
&+ \int_{t-a^+}^{t} \int_0^{a^+-t+s} g_2(a, t-s, p+t-s) b_\phi(s, p)\mathrm{d}p\mathrm{d}s.
\end{aligned} \tag{4.6}$$

We then show

$$
\begin{aligned}
&\int_0^{a^+}\int_a^{a^+}\big|S_2(t)\phi(a+h,a'+k)-S_2(t)\phi(a,a')\big|\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_0^{a^+}\int_a^{a^+}\big|b_\phi(t-a-h,a'+k-a-h)-b_\phi(t-a,a'-a)\big|e^{-\int_0^{a+h}\mu(s,s+a'+k-a-h)\mathrm{d}s}\mathrm{d}a'\mathrm{d}a\\
&\qquad+\int_0^{a^+}\int_a^{a^+}\big|b_\phi(t-a,a'-a)[e^{-\int_0^{a+h}\mu(s,s+a'+k-a-h)\mathrm{d}s}-e^{-\int_0^{a}\mu(s,s+a'-a)\mathrm{d}s}]\big|\mathrm{d}a'\mathrm{d}a\\
&\quad:=\mathrm{I}+\mathrm{II},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{II}&\le \int_0^{a^+}\int_a^{a^+} b_\phi(t-a,a'-a)\big|e^{-\int_0^{a+h}\mu(s,s+a'-a)\mathrm{d}s}[1-e^{\int_0^{a+h}\mu(s,s+a'-a)-\mu(s,s+a'+k-a-h)\mathrm{d}s}]\\
&\quad+e^{-\int_0^{a}\mu(s,s+a'-a)\mathrm{d}s}[1-e^{-\int_a^{a+h}\mu(s,s+a'-a)\mathrm{d}s}]\big|\mathrm{d}a'\mathrm{d}a\\
&\le \int_0^{a^+}\int_a^{a^+} b_\phi(t-a,a'-a)\big(\max\{1-e^{-K_\mu(k-h)a^+},e^{K_\mu(k-h)a^+}-1\}+(1-e^{-\bar{\mu}h})\big)\mathrm{d}a'\mathrm{d}a\\
&\le \big(\max\{1-e^{-K_\mu(k-h)a^+},e^{K_\mu(k-h)a^+}-1\}+(1-e^{-\bar{\mu}h})\big)\int_0^{a^+}\int_0^{a^+} b_\phi(t-a,s)\mathrm{d}s\mathrm{d}a\\
&\le 2\beta_{\max}\|\phi\|_E\big(\max\{1-e^{-K_\mu(k-h)a^+},e^{K_\mu(k-h)a^+}-1\}+(1-e^{-\bar{\mu}h})\big)\int_0^{a^+} e^{4\beta_{\max}(t-a)}\mathrm{d}a
\end{aligned}
$$

based on our prior estimate in Sect. 2.1 and with K_μ being the Lipschitz constant for μ. Thus, $\mathrm{II}\to 0$ uniformly for $\phi\in K$ as $h,k\to 0^+$.

Next, we need to show that

$$
\lim_{\substack{h\to 0\\ k\to 0}}\int_0^{a^+}\int_a^{a^+}|b_\phi(t-a-h,a'+k-a-h)-b_\phi(t-a,a'-a)|\mathrm{d}a'\mathrm{d}a=0. \tag{4.7}
$$

By (4.5) and (4.6), we have (note that we consider $t>2a^+$)

$$
\begin{aligned}
&\int_0^{a^+}\int_a^{a^+}|b_\phi(t-a-h,a'+k-a-h)-b_\phi(t-a,a'-a)|\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_0^{a^+}\int_a^{a^+}\Big|\int_{t-a-h-a^+}^{t-a-h}\int_0^{a^+-t+a+h+p} f_1(a'+k-a-h,t-a-h-p,s+t-a-h-p)b_\phi(p,s)\mathrm{d}s\mathrm{d}p\\
&\qquad-\int_{t-a-a^+}^{t-a}\int_0^{a^+-t+a+p} f_1(a'-a,t-a-p,s+t-a-p)b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big|\mathrm{d}a'\mathrm{d}a\\
&\qquad+\int_0^{a^+}\int_a^{a^+}\Big|\int_{t-a-h-a^+}^{t-a-h}\int_0^{a^+-t+a+h+s} g_1(a'+k-a-h,p+t-a-h-s,t-a-h-s)b'_\phi(s,p)\mathrm{d}p\mathrm{d}s\\
&\qquad-\int_{t-a-a^+}^{t-a}\int_0^{a^+-t+a+s} g_1(a'-a,p+t-a-s,t-a-s)b'_\phi(s,p)\mathrm{d}p\mathrm{d}s\Big|\mathrm{d}a'\mathrm{d}a\\
&\quad:=\mathrm{J}_1+\mathrm{J}_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{J}_1 \leq & \int_0^{a^+}\int_a^{a^+}\Big(\int_{t-a-a^+}^{t-a-h}\int_0^{a^+-t+a+p} |f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p)\\
& -f_1(a'-a, t-a-p, s+t-a-p)| b_\phi(p,s)\mathrm{d}s\mathrm{d}p\\
& +\int_{t-a-a^+}^{t-a-h}\int_{a^+-t+a+p}^{a^+-t+a+h+p} f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p) b_\phi(p,s)\mathrm{d}s\mathrm{d}p\\
& +\int_{t-a-h}^{t-a}\int_0^{a^+-t+a+p} f_1(a'-a, t-a-p, s+t-a-p) b_\phi(p,s)\mathrm{d}s\mathrm{d}p\\
& +\int_{t-a-a^+-h}^{t-a-a^+}\int_0^{a^+-t+a+p+h} f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p) b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
:= & \mathrm{J}_1^1+\mathrm{J}_1^2+\mathrm{J}_1^3+\mathrm{J}_1^4,
\end{aligned}
$$

in which

$$
\begin{aligned}
& |f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p) - f_1(a'-a, t-a-p, s+t-a-p)|\\
& \quad \leq |\beta(a'+k-a-h, t-a-h-p, s+t-a-h-p)(1-e^{-\int_{s+t-a-p-h}^{s+t-a-p}\mu(\sigma,\sigma+s)\mathrm{d}\sigma})|\\
& \qquad + |\beta(a'+k-a-h, t-a-h-p, s+t-a-h-p) - \beta(a'-a, t-a-p, s+t-a-p)|\\
& \quad \leq \bar{\beta}(a'+k-a-h)(1-e^{\bar{\mu}h}) + 3\max\{k,h\}K_\beta
\end{aligned}
$$

by Assumption 2.1(i) on β being Lipschitz continuous and K_β as the Lipschitz constant. Thus,

$$
\begin{aligned}
J_1^1 \leq & \int_0^{a^+}\int_a^{a^+}\big[\bar{\beta}(a'+k-a-h)(1-e^{\bar{\mu}h}) + 3\max\{k,h\}K_\beta\big]\Big(\int_{t-a-a^+}^{t-a-h}\int_0^{a^+-t+a+p} b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
\leq & \int_0^{a^+}\int_a^{a^+}\big[\bar{\beta}(a'+k-a-h)(1-e^{\bar{\mu}h}) + 3\max\{k,h\}K_\beta\big]\Big(\int_{t-a-a^+}^{t-a-h} 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}p}\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
\leq & \big[a^+(1-e^{\bar{\mu}h})\beta_{\sup} + 3\max\{k,h\}K_\beta(a^+)^2\big]\Big(\int_0^t 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}p}\mathrm{d}p\Big)\\
\to & 0 \text{ uniformly for } \phi\in K \text{ as } h,k\to 0^+
\end{aligned}
$$

based on the assumption of $a^+ < \infty$. From (4.5) we first estimate that

$$
\begin{aligned}
\int_x^{x+h} b_\phi(t,a)da \leq & \int_x^{x+h}\Big(\int_0^t\int_0^{a^+-t+p}\bar{\beta}(a)b_\phi(p,s)\mathrm{d}s\mathrm{d}p + \int_0^t\int_0^{a^+-t+s}\bar{\beta}(a)b'_\phi(s,p)\mathrm{d}pds\Big)\mathrm{d}a\\
\leq & \Big(\int_x^{x+h}\bar{\beta}(a)da\Big)\Big(4\|\phi\|_E\beta_{\max}\int_0^t e^{4\beta_{\max}p}\mathrm{d}p\Big)\\
\to & 0, \text{ as } h\to 0^+ \text{ uniformly for } \phi\in K,\ x\geq 0.
\end{aligned}
\tag{4.8}
$$

Then, we have

$$
\begin{aligned}
J_1^2 \leq & \int_0^{a^+}\int_a^{a^+}\bar{\beta}(a'+k-a-h)\Big(\int_{t-a-a^+}^{t-a-h}\int_{a^+-t+a+p}^{a^+-t+a+p+h} b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
\leq & \Big(\int_0^{a^+}\bar{\beta}(a')\mathrm{d}a'\Big)\Big(\int_0^{a^+}\int_{t-a-a^+}^{t-a-h}\int_{a^+-t+a+p}^{a^+-t+a+p+h} b_\phi(p,s)\mathrm{d}s\mathrm{d}p\mathrm{d}a\Big)\\
\to & 0, \text{ as } h,k\to 0^+ \text{ uniformly for } \phi\in K
\end{aligned}
$$

and

$$
\begin{aligned}
J_1^3 &\le \int_0^{a^+}\int_a^{a^+} \bar\beta(a'-a)\Big(\int_{t-a-h}^{t-a}\int_0^{a^+-t+a+p} b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
&\le \Big(\int_0^{a^+}\bar\beta(a')\mathrm{d}a'\Big)\Big(\int_0^{a^+}\int_{t-a-h}^{t-a} 2\beta_{\max}\|\phi\|_E e^{\beta_{\max}p}\mathrm{d}p\mathrm{d}a\Big)\\
&\to 0,\ \text{as } h,k\to 0^+ \text{ uniformly for } \phi\in K.
\end{aligned}
$$

Similarly, one can show that $J_1^4 \to 0$. Therefore, we have $J_1 \to 0$ as $h, k \to 0^+$ uniformly for $\phi \in K$. The fact that $J_2 \to 0$ uniformly for $\phi \in K$ as $h, k \to 0^+$ can be proved by a similar argument.

Therefore, we know that the first term in (4.4) goes to 0 uniformly for $\phi \in K$. Secondly, based on estimate in (4.8) we have

$$
\begin{aligned}
&\int_{k-h}^{a^+}\int_{a+h-k}^{a} \big|S_2(t)\phi(a+h,a'+k)\big|\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_0^{a^+}\int_{a+h-k}^{a} b_\phi(t-a-h,a'+k-a-h)\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_0^{a^+}\Big|\int_{a+h-k}^{a}\bar\beta(a')\mathrm{d}a'\Big|\Big(4\|\phi\|_E\beta_{\max}\int_0^t e^{4\beta_{\max}p}\mathrm{d}p\Big)da\\
&\quad\le \sup_{0<a<t}\Big|\int_{a-h+k}^{a}\bar\beta(a')\mathrm{d}a'\Big|\cdot a^+\cdot\Big(4\|\phi\|_E\beta_{\max}\int_0^t e^{4\beta_{\max}p}\mathrm{d}p\Big)\\
&\quad\to 0 \text{ as } h,k\to 0^+ \text{ uniformly for } \phi\in K.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_0^{k-h}\int_0^{a} \big|S_2(t)\phi(a+h,a'+k)\big|\mathrm{d}a'\mathrm{d}a\\
&\le \int_0^{k-h}\int_0^{a} b_\phi(t-a-h,a'+k-a-h)\mathrm{d}a'\mathrm{d}a\\
&\le \int_0^{k-h}\int_0^{a^+} b_\phi(t-a-h,s)\mathrm{d}s\mathrm{d}a \le 2\beta_{\max}\|\phi\|_E\int_0^{k-h} e^{4\beta_{\max}(t-a-h)}\mathrm{d}a\\
&\to 0 \text{ as } h,k\to 0^+ \text{ uniformly for } \phi\in K,
\end{aligned}
$$

we thus have (4.2). For (4.3),

$$
\begin{aligned}
\int_h^{a^+}\int_k^{a^+}\big|S_2(t)\phi(a,a')\big|\mathrm{d}a\mathrm{d}a' &\le \int_h^{a^+}\int_0^{a^+} b_\phi(t-a,s)\mathrm{d}s\mathrm{d}a\\
&\le 2\|\phi\|_E\beta_{\max}\int_h^{a^+} e^{4\beta_{\max}(t-a)}\mathrm{d}a \to 0
\end{aligned}
$$

as $h, k \to a^+$ uniformly for $\phi \in K$. We can show that $\{S_3(t)\}_{t\ge 0}$ is compact for sufficiently large t in the same way. □

Remark 4.4 Similarly, in the case of $a^+ = \infty$ and with Assumption 2.1(iii)', one can show that $S_1(t), S_2(t)$, and $S_3(t)$ satisfy the hypothesis of Proposition 4.2, see Appendix. In particular, when $a^+ < \infty$, $\{S_2(t)\}_{t\ge 0}$ and $\{S_3(t)\}_{t\ge 0}$ are eventually compact, while in the case of

$a^+ = \infty$ we can actually show that $\{S_2(t)\}_{t\geq 0}$ and $\{S_3(t)\}_{t\geq 0}$ are compact for all $t > 0$. Figure 3b illustrates the estimation similar to (4.4) for all $t > 0$.

Now we have shown that if $a^+ < \infty$, for sufficiently large $t > a^+$,

$$\alpha[S(t)] \leq \alpha[S_1(t)] + \alpha[S_2(t)] + \alpha[S_3(t)] = 0 + 0 + 0 = 0,$$

which implies that the semigroup $\{S(t)\}_{t\geq 0}$ is eventually compact, hence the essential growth bound

$$\omega_1(A) := \lim_{t\to\infty} t^{-1} \log(\alpha[S(t)]) = -\infty, \tag{4.9}$$

while if $a^+ = \infty$,

$$\alpha[S(t)] \leq \alpha[S_1(t)] + \alpha[S_2(t)] + \alpha[S_3(t)] \leq e^{-\underline{\mu}t} + 0 + 0 = e^{-\underline{\mu}t}, \ t \geq 0,$$

which implies the estimate of the essential growth bound

$$\omega_1(A) := \lim_{t\to\infty} t^{-1} \log(\alpha[S(t)]) \leq -\underline{\mu}.$$

Moreover, the essential spectral radius of A satisfies that

$$r_e(S(t)) = \exp[\omega_1(A)t] \leq e^{-\underline{\mu}t} < 1, \quad t \geq 0.$$

It follows that $\{S(t)\}_{t\geq 0}$ is quasi-compact. The following theorem from Engel and Nagel [18, Theorem 2.5, Chapter VI] will be used to show the stability of an equilibrium for a C_0 -semigroup.

Theorem 4.5 (Engel and Nagel [18]) *Let $\{S(t)\}_{t\geq 0}$ be a positive strongly continuous semigroup with generator A on a Banach lattice $L^P(\Omega, \mu)$, $1 \leq p < \infty$. Then $s(A) = \omega_0$, where*

$$s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$$

is the spectral bound of A and

$$\omega_0 := \lim_{t\to\infty} t^{-1} \log(\|S(t)\|)$$

is the growth bound of A.

5 Spectrum analysis

In order to study the spectral theory, we introduce some definitions and results in positive operator theory on ordered Banach spaces from Inaba [24]. For more complete exposition, we refer to Daners and Koch-Medina [8], Heijmans [20], Marek [37], and Sawashima [43].

Let E be a real or complex Banach space and E^* be its dual (the space of all linear functionals on E). Write the value of $f \in E^*$ at $\psi \in E$ as $\langle f, \psi \rangle$. A non-empty closed subset E_+ is called a *cone* if the following hold: (1) $E_+ + E_+ \subset E_+$, (2) $\lambda E_+ \subset E_+$ for $\lambda \geq 0$, (3) $E_+ \cap (-E_+) = \{0\}$. Let us define the *order* in E such that $x \leq y$ if and only if $y - x \in E_+$ and $x < y$ if and only if $y - x \in E_+ \setminus \{0\}$. The cone E_+ is called *total* if the set $\{\psi - \phi : \psi, \phi \in E_+\}$ is dense in E. The *dual cone* E_+^* is the subset of E^* consisting of all positive linear functionals on E; that is, $f \in E_+^*$ if and only if $\langle f, \psi \rangle \ge 0$ for all $\psi \in E_+$. $\psi \in E_+$ is called a *quasi-interior point* if $\langle f, \psi \rangle > 0$ for all $f \in E_+^* \setminus \{0\}$. $f \in E_+^*$ is said to be *strictly positive* if $\langle f, \psi \rangle > 0$ for all $\psi \in E_+ \setminus \{0\}$. The cone E_+ is called *generating* if $E = E_+ - E_+$ and is called *normal* if $E^* = E_+^* - E_+^*$.

An ordered Banach space $(E, \le)$ is called a *Banach lattice* if (1) any two elements $x, y \in E$ have a supremum $x \vee y = \sup\{x, y\}$ and an infimum $x \wedge y = \inf\{x, y\}$ in E; and (2) $|x| \le |y|$ implies $\|x\| \le \|y\|$ for $x, y \in E$, where the modulus of x is defined by $|x| = x \vee (-x)$.

Let $B(E)$ be the set of bounded linear operators from E to E. $T \in B(E)$ is said to be *positive* if $T(E_+) \subset E_+$. $T \in B(E)$ is said to be *strongly positive* if $\langle f, T\psi \rangle > 0$ for every pair $\psi \in E_+ \setminus \{0\}, f \in E_+^* \setminus \{0\}$. For $T, S \in B(E)$, we say $T \ge S$ if $(T - S)(E_+) \subset E_+$. A positive operator $T \in B(E)$ is called *non-supporting* if for every pair $\psi \in E_+ \setminus \{0\}, f \in E_+^* \setminus \{0\}$, there exists a positive integer $p = p(\psi, f)$ such that $\langle f, T^n \psi \rangle > 0$ for all $n \ge p$. The *spectral radius* and *spectral bound* of $T \in B(E)$ are denoted as $r(T)$ and $s(T)$, respectively. $\sigma(T)$ denotes the *spectrum* of T and $\sigma_P(T)$ denotes the *point spectrum* of T.

From results in Sawashima [43], Marek [37], and Inaba [24], we state the following proposition.

Proposition 5.1 *Let E be a Banach lattice and let $T \in B(E)$ be compact and non-supporting. Then the following statements hold:*

(i) *$r(T) \in \sigma_P(T) \setminus \{0\}$ and $r(T)$ is a simple pole of the resolvent, that is $r(T)$ is an algebraically simple eigenvalue of T;*
(ii) *The eigenspace of T corresponding to $r(T)$ is one-dimensional, and the corresponding eigenvector $\psi \in E_+$ is a quasi-interior point. The relation $T\phi = \mu\phi$ with $\phi \in E_+$ implies that $\phi = c\psi$ for some constant c;*
(iii) *The eigenspace of T^* corresponding to $r(T)$ is also a one-dimensional subspace of E^* spanned by a strictly positive functional $f \in E_+^*$;*
(iv) *Let $S, T \in B(E)$ be compact and non-supporting. Then $S \le T, S \ne T$ and $r(T) \ne 0$ imply $r(S) < r(T)$.*

5.1 Point spectrum and stability analysis

In this subsection we study the spectrum of A. Note that we will not solve the characteristic or resolvent equation of A directly, since A with its domain $D(A)$ seems very complicated as shown in Sect. 3. But thanks to the solution flow $\{S(t)\}_{t \ge 0}$, we can still characterize the eigenfunctions or resolvent solutions of A, see the following theorems from Webb [53]. Moreover, we only consider the case $a^+ < \infty$ in this section but the main results presented here remain true in the case $a^+ = \infty$ (see Remark 5.8).

Theorem 5.2 (Webb [53]) *Let $\mathcal{F}$ and $\mathcal{G}$ be defined in Proposition 3.1. Then $\phi \in D(A)$ and $A\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$ if and only if ϕ (in the complexification of E) satisfies*

$$\phi(a, a') = \begin{cases} e^{-\lambda a'} \Pi(a, a', a') \mathcal{F}(\phi)(a - a'), & \text{a.e. } a > a' \\ e^{-\lambda a} \Pi(a, a', a) \mathcal{G}(\phi)(a' - a), & \text{a.e. } a' > a, \end{cases} \tag{5.1}$$

where

$$\Pi(a,a',t) = \exp\left[-\int_0^t \mu(a-\tau,a'-\tau)d\tau\right].$$

Theorem 5.3 (Webb [53]) *Let $\lambda \in \rho(A)$ (the resolvent set of A) such that* $\mathrm{Re}\lambda > \beta_{\sup} + \beta'_{\sup} - \underline{\mu}$, *let $\psi \in E$ and satisfy $\phi = (\lambda I - A)^{-1}\psi$. Then*

$$\phi(a,a') = \begin{cases} e^{-\lambda a'}\Pi(a,a',a')\mathcal{F}(\phi)(a-a') + \int_0^{a'} e^{-\lambda\sigma}\Pi(a,a',\sigma)\psi(a-\sigma,a'-\sigma)\mathrm{d}\sigma, & \text{a.e. } a > a' \\ e^{-\lambda a}\Pi(a,a',a)\mathcal{G}(\phi)(a'-a) + \int_0^{a} e^{-\lambda\sigma}\Pi(a,a',\sigma)\psi(a-\sigma,a'-\sigma)\mathrm{d}\sigma, & \text{a.e. } a' > a, \end{cases} \tag{5.2}$$

where $\beta_{\sup} + \beta'_{\sup}$ is from the norm of linear boundary conditions.

Plugging (5.1) into the integral conditions $\mathcal{F}(\phi)$ and $\mathcal{G}(\phi)$ defined in Proposition 3.1, we obtain

$$\begin{aligned} \mathcal{G}(\phi)(a') &= \int_0^{a^+}\int_0^{a^+} \beta(a',a,s)\phi(a,s)\mathrm{d}a\mathrm{d}s \\ &= \int_0^{a^+}\int_0^{a} \beta(a',a,s)\Pi(a,s,s)e^{-\lambda s}\mathcal{F}(\phi)(a-s)\mathrm{d}s\mathrm{d}a \\ &\quad + \int_0^{a^+}\int_0^{s} \beta(a',a,s)\Pi(a,s,a)e^{-\lambda a}\mathcal{G}(\phi)(s-a)\mathrm{d}a\mathrm{d}s, \\ \mathcal{F}(\phi)(a) &= \int_0^{a^+}\int_0^{a^+} \beta'(a,s,a')\phi(s,a')\mathrm{d}s\mathrm{d}a' \\ &= \int_0^{a^+}\int_0^{a'} \beta'(a,s,a')\Pi(s,a',s)e^{-\lambda s}\mathcal{G}(\phi)(a'-s)\mathrm{d}s\mathrm{d}a' \\ &\quad + \int_0^{a^+}\int_0^{s} \beta'(a,s,a')\Pi(s,a',a')e^{-\lambda a'}\mathcal{F}(\phi)(s-a')\mathrm{d}a'\mathrm{d}s. \end{aligned} \tag{5.3}$$

Denote

$$\begin{aligned} &\alpha(t) = \mathcal{G}(\phi)(t), \quad \eta(t) = \mathcal{F}(\phi)(t), \\ &f_1(a',a,s) = \beta(a',a,s)\Pi(a,s,s), \quad f_2(a',a,s) = \beta(a',a,s)\Pi(a,s,a), \\ &f_3(a,s,a') = \beta'(a,s,a')\Pi(s,a',s), \quad f_4(a,s,a') = \beta'(a,s,a')\Pi(s,a',a'). \end{aligned}$$

So

$$\begin{aligned} \alpha(t) &= \int_0^{a^+}\int_0^{a} f_1(t,a,s)\eta(a-s)e^{-\lambda s}\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^{s} f_2(t,a,s)\alpha(s-a)e^{-\lambda a}\mathrm{d}a\mathrm{d}s, \\ \eta(t) &= \int_0^{a^+}\int_0^{a'} f_3(t,s,a')\alpha(a'-s)e^{-\lambda s}\mathrm{d}s\mathrm{d}a' + \int_0^{a^+}\int_0^{s} f_4(t,s,a')\eta(s-a')e^{-\lambda a'}\mathrm{d}a'\mathrm{d}s. \end{aligned} \tag{5.4}$$

If we can solve non-trivial α and η from the above equations, we would find the non-trivial solution of the characteristic equation. In the following context, the function space that appeared is itself or its complexification based on the values of λ in $\mathbb{R}$ or $\mathbb{C}$.

Define $F_\lambda : L^1(0,a^+) \times L^1(0,a^+) \to L^1(0,a^+) \times L^1(0,a^+)$, $\lambda \in \mathbb{C}$, by

$$F_\lambda(\alpha,\eta) = (F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta)),$$

where

$$F_{1\lambda}(\alpha,\eta)=\int_0^{a^+}\int_0^a f_1(t,a,s)\eta(a-s)e^{-\lambda s}\mathrm{d}s\mathrm{d}a+\int_0^{a^+}\int_0^s f_2(t,a,s)\alpha(s-a)e^{-\lambda a}\mathrm{d}a\mathrm{d}s$$

and

$$F_{2\lambda}(\alpha,\eta)=\int_0^{a^+}\int_0^{a'} f_3(t,s,a')\alpha(a'-s)e^{-\lambda s}\mathrm{d}s\mathrm{d}a'+\int_0^{a^+}\int_0^s f_4(t,s,a')\eta(s-a')e^{-\lambda a'}\mathrm{d}a'\mathrm{d}s.$$

Now the problem becomes finding a non-trivial fixed point of F_λ in $L^1(0,a^+)\times L^1(0,a^+)$. Furthermore, it is easy to check that F_λ maps $L^1(0,a^+)\times L^1(0,a^+)$ into itself since

$$\begin{aligned}\|F_{1\lambda}(\alpha,\eta)\|_{L^1(0,a^+)}&=\int_0^{a^+}\int_0^{a^+}\int_0^a|f_1(t,a,s)\eta(a-s)e^{-\lambda s}|\mathrm{d}s\mathrm{d}a\mathrm{d}t\\&\quad+\int_0^{a^+}\int_0^{a^+}\int_0^s|f_2(t,a,s)\alpha(s-a)e^{-\lambda s}|\mathrm{d}a\mathrm{d}s\mathrm{d}t\\&\le\int_0^{a^+}\overline{\beta}(t)\mathrm{d}t\int_0^{a^+}|\eta(a-s)|da\int_0^a e^{-(\mathrm{Re}\lambda+\underline{\mu})s}\mathrm{d}s\\&\quad+\int_0^{a^+}\overline{\beta}(t)\mathrm{d}t\int_0^{a^+}|\alpha(s-a)|\mathrm{d}s\int_0^s e^{-(\mathrm{Re}\lambda+\underline{\mu})a}\mathrm{d}a\\&\le\frac{\beta_{\sup}}{\mathrm{Re}\lambda+\underline{\mu}}[1-e^{-(\mathrm{Re}\lambda+\underline{\mu})a^+}]\|(\alpha,\eta)\|.\end{aligned}\tag{5.5}$$

Similarly, for $F_{2\lambda}$, we also have the following estimate:

$$\|F_{2\lambda}(\alpha,\eta)\|_{L^1(0,a^+)}\le\frac{\beta'_{\sup}}{\mathrm{Re}\lambda+\underline{\mu}}[1-e^{-(\mathrm{Re}\lambda+\underline{\mu})a^+}]\|(\alpha,\eta)\|.\tag{5.6}$$

In the following we give some properties of F_λ.

Lemma 5.4 *Let Assumption 2.1 hold. Then the operator F_λ is compact for all $\lambda\in\mathbb{C}$ and non-supporting for all $\lambda\in\mathbb{R}$.*

Proof For the compactness of F_λ, it is equivalent to show that for a bounded set K of $L^1(0,a^+)\times L^1(0,a^+)$,

$$\lim_{h\to0}\int_0^{a^+}|F_{i\lambda}(\alpha,\eta)(t+h)-F_{i\lambda}(\alpha,\eta)(t)|\mathrm{d}t=0\quad\text{uniformly for}\quad(\alpha,\eta)\in L^1(0,a^+)\times L^1(0,a^+),$$

where $i=1,2$. Now let us consider $F_{1\lambda}$, that is,

$$
\begin{aligned}
&\Big|\int_0^{a^+}\int_0^{a^+}\int_0^a f_1(t+h,a,s)e^{-\lambda s}\eta(a-s)\mathrm{d}s\mathrm{d}a\mathrm{d}t\\
&\quad+\int_0^{a^+}\int_0^{a^+}\int_0^s f_2(t+h,a,s)e^{-\lambda a}\alpha(s-a)\mathrm{d}a\mathrm{d}s\mathrm{d}t\\
&\quad-\int_0^{a^+}\int_0^{a^+}\int_0^a f_1(t,a,s)e^{-\lambda s}\eta(a-s)\mathrm{d}s\mathrm{d}a\mathrm{d}t\\
&\quad-\int_0^{a^+}\int_0^{a^+}\int_0^s f_2(t,a,s)e^{-\lambda a}\alpha(s-a)\mathrm{d}a\mathrm{d}s\mathrm{d}t\Big|\\
&\quad\le\int_0^{a^+}\int_0^{a^+}\int_0^a |f_1(t+h,a,s)-f_1(t,a,s)||e^{-\lambda s}||\eta(a-s)|\mathrm{d}s\mathrm{d}a\mathrm{d}t\\
&\qquad+\int_0^{a^+}\int_0^{a^+}\int_0^s |f_2(t+h,a,s)-f_2(t,a,s)||e^{-\lambda a}||\alpha(s-a)|\mathrm{d}a\mathrm{d}s\mathrm{d}t\\
&\quad\to 0\quad\text{as}\quad h\to 0
\end{aligned}
\tag{5.7}
$$

by Assumption 2.1(ii) on β, β'. Similarly, we can show the convergence for $F_{2\lambda}$, which implies that F_λ is a compact operator for all $\lambda \in \mathbb{C}$.

Next, for $\lambda \in \mathbb{R}$, define a positive functional $\mathscr{F}_\lambda = (\mathscr{F}_{1\lambda}, \mathscr{F}_{2\lambda})$ by

$$
\begin{aligned}
\langle \mathscr{F}_{1\lambda}, (\alpha,\eta)\rangle &:= \int_0^{a^+}\int_0^a \epsilon_1(s)\Pi(a,s,s)e^{-\lambda s}\eta(a-s)\mathrm{d}s\mathrm{d}a\\
&\quad+\int_0^{a^+}\int_0^s \epsilon_1(s)\Pi(a,s,a)e^{-\lambda a}\alpha(s-a)\mathrm{d}a\mathrm{d}s,\\
\langle \mathscr{F}_{2\lambda}, (\alpha,\eta)\rangle &:= \int_0^{a^+}\int_0^{a'} \epsilon_2(s)\Pi(s,a',s)e^{-\lambda s}\alpha(a'-s)\mathrm{d}s\mathrm{d}a'\\
&\quad+\int_0^{a^+}\int_0^s \epsilon_2(s)\Pi(s,a',a')e^{-\lambda a}\eta(s-a')\mathrm{d}a'\mathrm{d}s.
\end{aligned}
\tag{5.8}
$$

From Assumption 2.1(iii), $\mathscr{F}_\lambda$ is a strictly positive functional and we have

$$
\begin{aligned}
&F_\lambda(\alpha,\eta) = (F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta)) \ge (\langle \mathscr{F}_{1\lambda}, (\alpha,\eta)\rangle e_1, \langle \mathscr{F}_{2\lambda}, (\alpha,\eta)\rangle e_2),\\
&\lim_{\lambda\to-\infty}(\langle \mathscr{F}_{1\lambda}, (e_1,e_2)\rangle, \langle \mathscr{F}_{2\lambda}, (e_1,e_2)\rangle) = (+\infty,+\infty),
\end{aligned}
\tag{5.9}
$$

where $(e_1, e_2) \equiv 1$ is a quasi-interior point in $L^1(0,a^+)\times L^1(0,a^+)$ when $a^+ < \infty$. Moreover, we have

$$
\begin{aligned}
F_\lambda^2(\alpha,\eta) &= F_\lambda(F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta))\\
&= (F_{1\lambda}(F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta)), F_{2\lambda}(F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta))),
\end{aligned}
$$

where

$$
\begin{aligned}
F_{i\lambda}(F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta)) &\ge \langle \mathscr{F}_{i\lambda}, (F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta))\rangle e_i\\
&\ge \langle \mathscr{F}_{i\lambda}, \big(\langle \mathscr{F}_{1\lambda}, (\alpha,\eta)\rangle e_1, \langle \mathscr{F}_{2\lambda}, (\alpha,\eta)\rangle e_2\big)\rangle e_i\\
&\ge \min\{\langle \mathscr{F}_{1\lambda}, (\alpha,\eta)\rangle, \langle \mathscr{F}_{2\lambda}, (\alpha,\eta)\rangle\}\langle \mathscr{F}_{i\lambda}, (e_1,e_2)\rangle e_i\\
&:= \min\langle \mathscr{F}_{\lambda}, (\alpha,\eta)\rangle\langle \mathscr{F}_{i\lambda}, (e_1,e_2)\rangle e_i,\quad i=1,2.
\end{aligned}
$$

It follows that

$$\begin{aligned} F_\lambda^2(\alpha,\eta) &\geq \min\langle \mathscr{F}_\lambda, (\alpha,\eta)\rangle(\langle \mathscr{F}_{1\lambda}, (e_1,e_2)\rangle e_1, \langle \mathscr{F}_{2\lambda}, (e_1,e_2)\rangle e_2) \\ &\geq \min\langle \mathscr{F}_\lambda, (\alpha,\eta)\rangle \min\langle \mathscr{F}_\lambda, (e_1,e_2)\rangle (e_1,e_2). \end{aligned}$$

By induction for any integer n we have

$$F_\lambda^{n+1}(\alpha,\eta) \geq \min\langle \mathscr{F}_\lambda, (\alpha,\eta)\rangle \left[\min\langle \mathscr{F}_\lambda, (e_1,e_2)\rangle\right]^n (e_1,e_2).$$

Then we obtain $\langle \mathscr{F}, F_\lambda^n(\alpha,\eta)\rangle > 0, n \geq 1$, for every pair $(\alpha,\eta) \in L_+^1(0,a^+) \times L_+^1(0,a^+) \setminus \{(0,0)\}$, $\mathscr{F} \in (L_+^1(0,a^+))^* \times (L_+^1(0,a^+))^* \setminus \{(0,0)\}$; that is, we know that F_λ is a non-supporting operator. In summary, F_λ is a compact and non-supporting operator. □

Remark 5.5 Note that in the above proof of non-supporting of F_λ, we chose a constant function $e \equiv 1$ as the lower bound of F_λ. But if $a^+ = \infty$, $e \equiv 1$ is no longer in $L^1(0,\infty) \times L^1(0,\infty)$. Fortunately, we can still prove it under Assumption 2.1(iii)'.

Still define the same positive functional $\mathscr{F}_\lambda = (\mathscr{F}_{1\lambda}, \mathscr{F}_{2\lambda})$ by (5.8). From Assumption 2.1(iii)', $\mathscr{F}_\lambda$ is a strictly positive functional and we have

$$\begin{aligned} &F_\lambda(\alpha,\eta) = (F_{1\lambda}(\alpha,\eta), F_{2\lambda}(\alpha,\eta)) \geq (\langle \mathscr{F}_{1\lambda}, (\alpha,\eta)\rangle \beta_1, \langle \mathscr{F}_{2\lambda}, (\alpha,\eta)\rangle \beta_1'), \\ &\lim_{\lambda\to-\infty} (\langle \mathscr{F}_{1\lambda}, (\beta_1,\beta_1')\rangle, \langle \mathscr{F}_{2\lambda}, (\beta_1,\beta_1')\rangle) = (+\infty,+\infty), \end{aligned} \tag{5.10}$$

where (β_1,β_1') is obviously a quasi-interior point in $L^1(0,a^+) \times L^1(0,a^+)$. The estimates are the same as above by just changing (e_1,e_2) into (β_1,β_1'). Hence, F_λ is still a non-supporting operator when $a^+ = \infty$.

Now we study the resolvent set of A. Plugging (5.2) into the integral conditions $\mathcal{F}(\phi)$ and $\mathcal{G}(\phi)$ defined in Proposition 3.1, we obtain that

$$\begin{aligned} \alpha(t) =& \int_0^{a^+}\int_0^a f_1(t,a,s)\eta(a-s)e^{-\lambda s}\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^s f_2(t,a,s)\alpha(s-a)e^{-\lambda a}\mathrm{d}a\mathrm{d}s \\ &+ \int_0^{a^+}\int_0^a K_1(t,a,s)\psi(a,s)\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^s K_2(t,a,s)\psi(a,s)\mathrm{d}a\mathrm{d}s, \\ \eta(t) =& \int_0^{a^+}\int_0^{a'} f_3(t,s,a')\alpha(a'-s)e^{-\lambda s}\mathrm{d}s\mathrm{d}a' + \int_0^{a^+}\int_0^s f_4(t,s,a')\eta(s-a')e^{-\lambda a'}\mathrm{d}a'\mathrm{d}s \\ &+ \int_0^{a^+}\int_0^{a'} K_3(t,s,a')\psi(s,a')\mathrm{d}s\mathrm{d}a' + \int_0^{a^+}\int_0^s K_4(t,s,a')\psi(s,a')\mathrm{d}a'\mathrm{d}s, \end{aligned} \tag{5.11}$$

where

$$
\begin{aligned}
K_1(t,a,s)\psi(a,s) &= \beta(t,a,s)\int_0^s e^{-\lambda\sigma}\Pi(a,s,\sigma)\psi(a-\sigma,s-\sigma)\mathrm{d}\sigma,\\
K_2(t,a,s)\psi(a,s) &= \beta(t,a,s)\int_0^a e^{-\lambda\sigma}\Pi(a,s,\sigma)\psi(a-\sigma,s-\sigma)\mathrm{d}\sigma,\\
K_3(t,s,a')\psi(s,a') &= \beta'(t,s,a')\int_0^s e^{-\lambda\sigma}\Pi(s,a',\sigma)\psi(s-\sigma,a'-\sigma)\mathrm{d}\sigma,\\
K_4(t,s,a')\psi(s,a') &= \beta'(t,s,a')\int_0^s e^{-\lambda\sigma}\Pi(s,a',\sigma)\psi(s-\sigma,a'-\sigma)\mathrm{d}\sigma.
\end{aligned}
\tag{5.12}
$$

One can rewrite (5.11) as the following functional equations.

$$
\begin{pmatrix}\alpha\\ \eta\end{pmatrix} = F_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix} + \begin{pmatrix}G_\lambda^1\psi\\ G_\lambda^2\psi\end{pmatrix},
\tag{5.13}
$$

where

$$
\begin{aligned}
G_\lambda^1\psi &= \int_0^{a^+}\int_0^a K_1(t,a,s)\psi(a,s)\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^s K_2(t,a,s)\psi(a,s)\mathrm{d}a\mathrm{d}s,\\
G_\lambda^2\psi &= \int_0^{a^+}\int_0^a K_3(t,s,a)\psi(s,a)\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^s K_4(t,s,a)\psi(s,a)\mathrm{d}a\mathrm{d}s.
\end{aligned}
\tag{5.14}
$$

Next we analyze the spectra of F_λ and A together with their relations via the continuity of $r(F_\lambda)$ with respect to λ and the sign of $r(F_0)-1$.

Proposition 5.6 *Let Assumption 2.1 hold. For $a^+<\infty$, we have the following statements*

- (i) $\Gamma := \{\lambda\in\mathbb{C} : 1\in\sigma(F_\lambda)\} = \{\lambda\in\mathbb{C} : 1\in\sigma_P(F_\lambda)\}$, *where $\sigma(A)$ and $\sigma_P(A)$ denote the spectrum and point spectrum of the operator A, respectively;*
- (ii) *There exists a unique real number $\lambda_0\in\Gamma$ such that $r(F_{\lambda_0})=1$ and $\lambda_0>0$ if $r(F_0)>1$; $\lambda_0=0$ if $r(F_0)=1$; and $\lambda_0<0$ if $r(F_0)<1$;*
- (iii) $\lambda_0>\sup\{\mathrm{Re}\lambda : \lambda\in\Gamma\setminus\{\lambda_0\}\}$;
- (iv) $\{\lambda\in\mathbb{C} : \lambda\in\rho(A)\} = \{\lambda\in\mathbb{C} : 1\in\rho(F_\lambda)\}$, *where $\rho(A)$ denote the resolvent set of A;*
- (v) *λ_0 is the dominant eigenvalue of A, i.e., λ_0 is greater than all real parts of the eigenvalues of A. Moreover, it is an algebraically simple eigenvalue of A;*
- (vi) $\lambda_0 = s(A) := \sup\{\mathrm{Re}\lambda : \lambda\in\sigma(A)\}$.

Proof (i) Since F_λ is compact, $\sigma(F_\lambda)\setminus\{0\} = \sigma_P(F_\lambda)\setminus\{0\}$, hence conclusion (i) follows.

(ii) Next, F_λ, $\lambda\in\mathbb{R}$ is strictly decreasing in the operator sense, which implies that the spectral radius $r(F_\lambda)$, $\lambda\in\mathbb{R}$, is strictly decreasing by Lemma 5.4 and Proposition 5.1 (see also Inaba [24, Proposition 3.3]). On the one hand, for $\lambda\in\mathbb{R}$, let f_λ be a positive eigenfunctional corresponding to the eigenvalue $r(F_\lambda)$ of positive operator F_λ. Then, we have

$$
\langle f_\lambda, F_\lambda(e_1,e_2)\rangle = r(F_\lambda)\langle f_\lambda,(e_1,e_2)\rangle \ge \min\langle\mathscr{F}_\lambda,(e_1,e_2)\rangle\langle f_\lambda,(e_1,e_2)\rangle.
$$

Since f_λ is strictly positive, we obtain $r(F_\lambda)\ge\min\langle\mathscr{F}_\lambda,(e_1,e_2)\rangle$. It follows from (5.9) that $\lim_{\lambda\to-\infty} r(F_\lambda)=+\infty$. On the other hand, it is easy to see that $\lim_{\lambda\to\infty} r(F_\lambda)=0$. Moreover, by the fact that the spectral radius of a compact operator is continuous with respect to the parameter from Nussbaum [42] or Degla [9], we conclude the result (ii).

(iii) Next, we can use the idea in Inaba [24, Proposition 3.3] to show result (iii). For any $\lambda \in \Gamma$, there exists an eigenfunction ϕ_λ such that $F_\lambda \phi_\lambda = \phi_\lambda$, i.e.,

$$\begin{pmatrix} F_{1\lambda}(\phi_{1\lambda}, \phi_{2\lambda}) \\ F_{2\lambda}(\phi_{1\lambda}, \phi_{2\lambda}) \end{pmatrix} = \begin{pmatrix} \phi_{1\lambda} \\ \phi_{2\lambda} \end{pmatrix}.$$

Then, we have $|\phi_\lambda| = |F_\lambda \phi_\lambda| \leq F_{\mathrm{Re}\lambda}|\phi_\lambda|$ where $|\phi_\lambda| := \begin{pmatrix} |\phi_{1\lambda}| \\ |\phi_{2\lambda}| \end{pmatrix}$. Let $f_{\mathrm{Re}\lambda}$ be the positive eigenfunctional corresponding to the eigenvalue $r(F_{\mathrm{Re}\lambda})$ of $F_{\mathrm{Re}\lambda}$, we obtain that

$$\langle f_{\mathrm{Re}\lambda}, F_{\mathrm{Re}\lambda}|\phi_\lambda|\rangle = r(F_{\mathrm{Re}\lambda})\langle f_{\mathrm{Re}\lambda}, |\phi_\lambda|\rangle \geq \langle f_{\mathrm{Re}\lambda}, |\phi_\lambda|\rangle.$$

Hence, we have $r(F_{\mathrm{Re}\lambda}) \geq 1$ and $\mathrm{Re}\lambda \leq \lambda_0$ since $r(F_\lambda)$ is strictly decreasing with respect to $\lambda \in \mathbb{R}$ and $r(F_{\lambda_0}) = 1$. If $\mathrm{Re}\lambda = \lambda_0$, then $F_{\lambda_0}|\phi_\lambda| = |\phi_\lambda|$. In fact, if $F_{\lambda_0}|\phi_\lambda| > |\phi_\lambda|$, taking duality paring with the eigenfunctional f_{λ_0} corresponding to the eigenvalue $r(F_{\lambda_0}) = 1$ on both sides yields $\langle f_{\lambda_0}, F_{\lambda_0}|\phi_\lambda|\rangle = \langle f_{\lambda_0}, |\phi_\lambda|\rangle > \langle f_{\lambda_0}, |\phi_\lambda|\rangle$, which is a contradiction. Then we can write that $|\phi_\lambda| = c\phi_{\lambda_0}$, where ϕ_{λ_0} is the eigenfunction corresponding to the eigenvalue $r(F_{\lambda_0}) = 1$. Hence, without loss of generality we can assume that $c = 1$ and write $\phi_\lambda = \begin{pmatrix} \phi_{1\lambda}(t) \\ \phi_{2\lambda}(t) \end{pmatrix} = \begin{pmatrix} \phi_{10}(t)e^{i\gamma(t)} \\ \phi_{20}(t)e^{i\zeta(t)} \end{pmatrix}$ for some real function $\gamma(t)$ and $\zeta(t)$, where $\phi_{\lambda_0} = \begin{pmatrix} \phi_{10}(t) \\ \phi_{20}(t) \end{pmatrix}$. If we substitute this relation into

$$F_{\lambda_0}\phi_{\lambda_0} = \phi_{\lambda_0} = |\phi_\lambda| = |F_\lambda \phi_\lambda|,$$

then we have

$$\begin{aligned}
&\int_0^{a^+}\int_0^a f_1(t,a,s)\phi_{20}(a-s)e^{-\lambda_0 s}\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^s f_2(t,a,s)\phi_{10}(s-a)e^{-\lambda_0 a}\mathrm{d}a\mathrm{d}s \\
&\quad = \Big|\int_0^{a^+}\int_0^a f_1(t,a,s)\phi_{20}(a-s)e^{i\zeta(a-s)}e^{-(\lambda_0+i\mathrm{Im}\lambda)s}\mathrm{d}s\mathrm{d}a \\
&\qquad + \int_0^{a^+}\int_0^s f_2(t,a,s)\phi_{10}(s-a)e^{i\gamma(s-a)}e^{-(\lambda_0+i\mathrm{Im}\lambda)a}\mathrm{d}a\mathrm{d}s\Big|
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{a^+}\int_0^{a'} f_3(t,s,a')\phi_{10}(a'-s)e^{-\lambda_0 s}\mathrm{d}s\mathrm{d}a' + \int_0^{a^+}\int_0^s f_4(t,s,a')\phi_{20}(s-a')e^{-\lambda_0 a'}\mathrm{d}a'\mathrm{d}s \\
&\quad = \Big|\int_0^{a^+}\int_0^{a'} f_3(t,s,a')\phi_{10}(a'-s)e^{i\gamma(a'-s)}e^{-(\lambda_0+i\mathrm{Im}\lambda)s}\mathrm{d}s\mathrm{d}a' \\
&\qquad + \int_0^{a^+}\int_0^s f_4(t,s,a')\phi_{20}(s-a')e^{i\zeta(s-a')}e^{-(\lambda_0+i\mathrm{Im}\lambda)a'}\mathrm{d}a'\mathrm{d}s\Big|
\end{aligned}$$

which follows after changing variables that

$$\int_0^{a^+}\int_0^a f_1(t,a,s)\phi_{20}(a-s)e^{-\lambda_0 s}+f_2(t,s,a)\phi_{10}(a-s)e^{-\lambda_0 s}\mathrm{d}s\mathrm{d}a$$
$$=\Big|\int_0^{a^+}\int_0^a f_1(t,a,s)\phi_{20}(a-s)e^{i\zeta(a-s)}e^{-(\lambda_0+i\mathrm{Im}\lambda)s}+f_2(t,s,a)\phi_{10}(a-s)e^{i\gamma(a-s)}e^{-(\lambda_0+i\mathrm{Im}\lambda)s}\mathrm{d}s\mathrm{d}a\Big|$$

and

$$\int_0^{a^+}\int_0^{a'} f_3(t,s,a')\phi_{10}(a'-s)e^{-\lambda_0 s}+f_4(t,a',s)\phi_{20}(a'-s)e^{-\lambda_0 s}\mathrm{d}s\mathrm{d}a'$$
$$=\Big|\int_0^{a^+}\int_0^{a'} f_3(t,s,a')\phi_{10}(a'-s)e^{i\gamma(a'-s)}e^{-(\lambda_0+i\mathrm{Im}\lambda)s}+f_4(t,a',s)\phi_{20}(a'-s)e^{i\zeta(a'-s)}e^{-(\lambda_0+i\mathrm{Im}\lambda)s}\mathrm{d}s\mathrm{d}a'\Big|.$$

From Heijmans [20, Lemma 6.12], we have that $\zeta(a-s)-\mathrm{Im}\lambda s=\gamma(a-s)-\mathrm{Im}\lambda s=\theta_1$ and $\zeta(a'-s)-\mathrm{Im}\lambda s=\gamma(a'-s)-\mathrm{Im}\lambda s=\theta_2$ for some constants θ_1 and θ_2. From $F_\lambda\phi_\lambda=\phi_\lambda$, we have

$$\begin{pmatrix} e^{i\theta_1}F_{1\lambda_0}(\phi_{10},\phi_{20})\\ e^{i\theta_2}F_{2\lambda_0}(\phi_{10},\phi_{20})\end{pmatrix}=\begin{pmatrix} e^{i\gamma}\phi_{10}\\ e^{i\zeta}\phi_{20}\end{pmatrix},$$

which implies that $\theta_1=\gamma$ and $\theta_2=\zeta$, hence $\mathrm{Im}\lambda=0$. Then, there is no element $\lambda\in\Gamma$ such that $\mathrm{Re}\lambda=\lambda_0$ and $\lambda\neq\lambda_0$; thus, result (iii) is desired.

(iv) For result (iv), when $1\in\rho(F_\lambda)$, $(I-F_\lambda)^{-1}$ exists and is well defined, then from (5.13), one can obtain that

$$\begin{pmatrix}\alpha\\ \eta\end{pmatrix}=(I-F_\lambda)^{-1}\begin{pmatrix}G_\lambda^1\psi\\ G_\lambda^2\psi\end{pmatrix}. \tag{5.15}$$

Now plugging (5.15) into (5.2), we will obtain the expression of $\varphi=(\lambda I-A)^{-1}\psi$, which is well defined. It follows that $\lambda\in\rho(A)$. Conversely, if $\lambda\in\rho(A)$, the resolvent solution (5.2) exists and is well defined, then the system of integral equations (5.11) on (α,η) has a solution. It follows that (5.13) has a solution, which implies that $1\in\rho(F_\lambda)$.

(v) First claim that $\lambda\in\sigma_P(A)$ with geometric multiplicity m if and only if $1\in\sigma_P(F_\lambda)$ with geometric multiplicity m for all $m\in\mathbb{N}$. In fact, if $\lambda\in\sigma_P(A)$ corresponding to linearly independent eigenfunctions $\phi_1,\dots,\phi_m$, then $\phi_1,\dots,\phi_m$ satisfy (5.1) which implies that (5.3) holds and equivalently (5.4) holds. It follows that $F_\lambda(\mathcal{G}(\phi_i),\mathcal{F}(\phi_i))=(\mathcal{G}(\phi_i),\mathcal{F}(\phi_i))$ for all $i=1,\dots,m$. Hence, $(\mathcal{G}(\phi_i),\mathcal{F}(\phi_i)), i=1,\dots,m$, are necessarily linearly independent eigenfunctions of F_λ corresponding to eigenvalue 1 and so $1\in\sigma_P(F_\lambda)$ with geometric multiplicity $n\geq m$. Conversely, if $(\alpha_i,\eta_i), i=1,\dots,n$, are eigenfunctions of F_λ corresponding to eigenvalue 1, i.e., $F_\lambda(\alpha_i,\eta_i)=(\alpha_i,\eta_i), i=1,\dots,n$, and set

$$\phi_i(a,a')=\begin{cases} e^{-\lambda a'}\Pi(a,a',a')\eta_i(a-a'), & a'<a,\\ e^{-\lambda a}\Pi(a,a',a)\alpha_i(a'-a), & a<a',\end{cases}\quad i=1,\dots,n. \tag{5.16}$$

Then it is easy to verify $\mathcal{F}(\phi_i)=\eta_i, \mathcal{G}(\phi_i)=\alpha_i, i=1,\dots,n$. It follows that $\lambda\phi_i=A\phi_i, i=1,\dots,n$, by Theorem 5.2. Moreover, (5.16) ensures that $\phi_1,\dots,\phi_n$ are linearly independent. Hence, $\lambda\in\sigma_P(A)$ with geometric multiplicity $m\geq n$. Thus, $n=m$. It follows from the claim that

$$\Gamma=\{\lambda\in\mathbb{C}:\lambda\in\sigma_P(A)\}.$$

Now from (iii), we conclude that λ_0 is dominant. Next, we need to prove that λ_0 is simple. Plugging (5.15) into (5.2), we obtain

$$(\lambda I - A)^{-1}\psi = \begin{cases} e^{-\lambda a'}\Pi(a,a',a')(I-F_\lambda)_1^{-1}(G_\lambda^1\psi, G_\lambda^2\psi)(a-a') \\ \qquad + \int_0^{a'} e^{-\lambda\sigma}\Pi(a,a',\sigma)\psi(a-\sigma,a'-\sigma)\mathrm{d}\sigma \ \text{ a.e. } a > a' \\ e^{-\lambda a}\Pi(a,a',a)(I-F_\lambda)_2^{-1}(G_\lambda^1\psi, G_\lambda^2\psi)(a'-a) \\ \qquad + \int_0^{a} e^{-\lambda\sigma}\Pi(a,a',\sigma)\psi(a-\sigma,a'-\sigma)\mathrm{d}\sigma \ \text{ a.e. } a' > a, \end{cases}$$

where $(I-F_\lambda)^{-1} = ((I-F_\lambda)_1^{-1}, (I-F_\lambda)_2^{-1})$. From the above formula, we see that $(\lambda I - A)^{-1}$ does not hold for all λ such that $r(F_\lambda) = 1$. Thus, 1 is a pole of $(I-F_\lambda)^{-1}$ of order m if and only if λ is a pole of $(\lambda I - A)^{-1}$ of order m. However, by Proposition 5.1(i), we know that 1 is a simple pole of $(I-F_\lambda)^{-1}$ which implies that λ_0 is a simple pole of $(\lambda I - A)^{-1}$. Thus, it follows from Webb [52, Proposition 4.11] that λ_0 is an algebraically simple eigenvalue of A.

(vi) Finally, we show result (vi). Let $\hat{\lambda}_0 := s(A)$ denote the spectral bound of A. Then $\hat{\lambda}_0 \geq \lambda_0$ and so $\hat{\lambda}_0 > \omega_1(A) = -\infty$. Thus, $\sigma_0(A) = \{\hat{\lambda}_0\}$ by Webb [54, Proposition 2.5] which states that the peripheral spectrum σ_0 of the generator of a strongly continuous positive semigroup in a Banach lattice consists exactly of the generator's spectral bound provided the latter is strictly greater than the essential growth bound. Then $\hat{\lambda}_0 \in \sigma(A)$ and thus by (i) and (iv), $1 \in \sigma_p(F_{\hat{\lambda}_0})$, which implies that $1 \leq r(F_{\hat{\lambda}_0})$. However, due to $\hat{\lambda}_0 \geq \lambda_0$ we have $r(F_{\hat{\lambda}_0}) \leq r(F_{\lambda_0}) = 1$, Hence, $\hat{\lambda}_0 = \lambda_0$. Thus, (vi) is desired. □

To address the case when $a^+ = \infty$, we make the following assumption.

Assumption 5.7 $r(F_\gamma) > 1$ for some $\gamma \in \mathbb{R}$ with $\gamma > -\underline{\mu}$.

Remark 5.8 For $a^+ = \infty$, if in addition we have Assumption 5.7, all statements in Proposition 5.6 still hold.

It guarantees the existence of λ_0 such that $r(F_{\lambda_0}) = 1$ since for now the domain of F_λ is changing into $\mathrm{Re}\lambda > -\underline{\mu}$ instead of $\mathbb{C}$ to make F_λ be well defined. Further, for Proposition 5.6(vi) one can see that $\hat{\lambda}_0 \geq \lambda_0 > \gamma > -\underline{\mu} \geq \omega_1(A)$ when $a^+ = \infty$. Note Assumption 5.7 is only used here to show the existence of λ_0 under the case $a^+ = \infty$. All other results in this Sect. 5 are still valid without this assumption. Moreover, Assumption 5.7 can be verified explicitly given the separable mixing Assumption 2.1(iii') plus $\beta_2 \equiv \beta_2'$. In fact, it is easy to compute $F_0(\beta_1, \beta_1') = r(F_0)(\beta_1, \beta_1')$ under the condition and then $r(F_0) > 1$ will implies Assumption 5.7, where

$$r(F_0) = \int_0^{a^+}\int_0^a \beta_2(a,s)\Pi(a,s,s)\beta_1'(a-s)\mathrm{d}s\mathrm{d}a + \int_0^{a^+}\int_0^s \beta_2(a,s)\Pi(a,s,a)\beta_1(s-a)\mathrm{d}a\mathrm{d}s.$$

Remark 5.9 In fact, we can prove (vi) by using a different method. Observe that for any $\lambda \in \mathbb{R}$, when $\lambda > \lambda_0$ and so $r(F_\lambda) < r(F_{\lambda_0}) = 1$, $(I-F_\lambda)^{-1}$ exists and is positive. Moreover, $1 \in \rho(F_\lambda) \Rightarrow \lambda \in \rho(A)$. Therefore, λ_0 is larger than any other real spectral values in $\sigma(A)$. It follows that $\lambda_0 = s_{\mathbb{R}}(A) := \sup\{\lambda \in \mathbb{R} : \lambda \in \sigma(A)\}$. Next we claim A is a *resolvent positive operator*. In fact, it is easy to see that the resolvent set of A contains an infinite ray (λ_0, ∞) and $(\lambda I - A)^{-1}$ is a positive operator for $\lambda > \lambda_0$ by (5.2) and the positivity of $(I - F_\lambda)^{-1}$. But since $L^1(0, a^+) \times L^1(0, a^+)$ is a Banach lattice with normal and generating cone $K := L^1_+(0, a^+) \times L^1_+(0, a^+)$ and $s(A) \geq \lambda_0 > -\infty$ due to $\lambda_0 \in \sigma(A)$, we can conclude from Thieme [47, Theorem 3.5] that $s(A) = s_{\mathbb{R}}(A) = \lambda_0$.

Remark 5.10 As we know, non-supporting is a generalization of strong positivity in the Banach space with a positive cone which may have empty interior. In fact, we can give an assumption on β and β' such that F_λ is strongly positive in the sense of dual space (see the definition in Daners and Koch-Medina [8]), for example, Assumption 2.1(iii)'. Now F_λ itself is strongly positive thus irreducible in $L^1(0, a^+) \times L^1(0, a^+)$ which is a Banach lattice, then by [8, Theorem 12.3], one can still conclude that $r(F_\lambda)$ is an algebraically simple eigenvalue of F_λ with a positive eigenfunction and a simple pole of the resolvent of F_λ. Moreover, $\lambda \to r(F_\lambda)$ is continuous by the compactness of F_λ and strictly decreasing by showing that $\lambda \to r(F_\lambda)$ is log-convex (Thieme [47]) or super-convex (Kato [30]), for details see [2, Lemma 1]. Hence, we can still obtain the same results in Proposition 5.6.

Taking a closer look at the operator F_0, we have its first output element illustrated as:

$$
\begin{aligned}
F_{10}(\alpha, \eta)(b) = & \underbrace{\int_0^{a^+} \int_0^{a} \beta(b, a, s)\Pi(a, s, s)\eta(a - s)\mathrm{d}s\mathrm{d}a}_{\substack{\text{next generation population density with structure } (0, b) \\ \text{produced by the first generation with structure } (\cdot, 0) \\ \text{and density function } \eta(\cdot)}} \\
& + \underbrace{\int_0^{a^+} \int_0^{s} \beta(b, a, s)\Pi(a, s, a)\alpha(s - a)\mathrm{d}a\mathrm{d}s}_{\substack{\text{next generation population density with structure } (0, b) \\ \text{produced by the first generation with structure } (0, \cdot) \\ \text{and density function } \alpha(\cdot)}},
\end{aligned}
$$

where $\eta(a - s)$ is the first generation population density with structure $(a - s, 0)$, $\Pi(a, s, s)$ is the survival probability for individuals born with structure $(a - s, 0)$ to reach structure (a, s), and $\beta(b, a, s)$ is the reproduction rate for mothers with structure (a, s) to give birth to daughters with structure $(0, b)$. And the interpretation of the second integral follows similarly.

Biologically speaking, F_0 is the next generation operator: given any population density functions $(\alpha(\cdot), \eta(\cdot))$ on both boundaries (first generation densities), $F_0(\alpha(\cdot), \eta(\cdot))$ represents the offspring density functions (second generation) on both boundaries generated by the first generation during their entire life periods. Thus, the spectral radius of F_0 can be interpreted as the basic reproductive number of the population, where a detailed mathematical interpretation can be adopted directly from the widely known discussion on $\mathscr{R}_0$ for single-structured infectious disease models (Diekmann et al. [12]) or scalar age-structured population dynamical models (Kot [32]). Therefore, we have the following theorem by Theorem 4.5 on the basic reproduction number $\mathscr{R}_0$.

Theorem 5.11 *Define* $\mathscr{R}_0 := r(F_0)$. *If* $\mathscr{R}_0 < 1$, *then the zero equilibrium is globally exponentially stable. Otherwise, if* $\mathscr{R}_0 > 1$, *then the zero equilibrium is unstable.*

Proof If $\mathscr{R}_0 = r(F_0) < 1$, by Proposition 5.6(ii) there exists a unique real $\lambda_0 < 0$ such that $r(F_{\lambda_0}) = 1$. Hence, by Proposition 5.1 we have $F_{\lambda_0}\varphi_{\lambda_0} = r(F_{\lambda_0})\varphi_{\lambda_0} = \varphi_{\lambda_0}$ which φ_{λ_0} is a positive function in $L^1(0,a^+)\times L^1(0,a^+)$; that is, we find a non-trivial solution φ_{λ_0} of the characteristic equations for some $\lambda < 0$ when $r(F_0) < 1$. Moreover, there is no non-trivial solution for all $\lambda > 0$ since 0 is the only fixed point of F_λ by Banach fixed point theorem. And from Proposition 5.6(iv) we know that $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} = \lambda_0 < 0$. It follows from Theorem 4.5 that the zero equilibrium is globally exponentially stable.

If $\mathscr{R}_0 = r(F_0) > 1$, by Proposition 5.6(v) there exists a real $\lambda_0 > 0$ such that $r(F_{\lambda_0}) = 1$. Similarly, by Proposition 5.1 we have $F_{\lambda_0}\varphi_{\lambda_0} = r(F_{\lambda_0})\varphi_{\lambda_0} = \varphi_{\lambda_0}$, in which φ_{λ_0} is a positive function in $L^1(0,a^+)\times L^1(0,a^+)$; that is, we find a non-trivial solution φ_{λ_0} of the characteristic equation for some $\lambda > 0$ when $r(F_0) > 1$. Moreover, by Proposition 5.6(vi), $s(A) = \lambda_0 > 0$. Motivated by Thieme [46, Corollary 4.3], let Σ denote the set of spectral values with positive real parts. As these are normal eigenvalues, Σ is finite and bounded away from the imaginary axis and can so be separated from the rest of the spectrum by a rectifiable simple closed curve. According to Kato [31, Chapter III, Theorem 6.17], there exists a decomposition of E into invariant subspaces E_1 and E_2 such that the restriction of A to E_1 has the spectrum Σ and its restriction to E_2 has a spectral bound $\le 0 < \inf\{\mathrm{Re}\lambda : \lambda \in \Sigma\}$ and thus a non-positive type. Now the instability of the zero equilibrium follows from Desch and Schappacher [11, Theorem 2.2]. □

Furthermore, we have the following estimates when $a^+ = \infty$

$$\begin{aligned}
\|F_{1\lambda}(\alpha,\eta)\|_{L^1(0,+\infty)} &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{a} |f_1(t,a,s)\eta(a-s)e^{-\lambda s}|\mathrm{d}s\mathrm{d}a\mathrm{d}t\\
&\quad + \int_0^{+\infty}\int_0^{+\infty}\int_0^{s} |f_2(t,a,s)\alpha(s-a)e^{-\lambda s}|\mathrm{d}a\mathrm{d}s\mathrm{d}t\\
&\le \int_0^{+\infty}\overline{\beta}(t)\mathrm{d}t\int_0^{+\infty}|\eta(a-s)|da\int_0^{a} e^{-(\mathrm{Re}\lambda+\underline{\mu})s}\mathrm{d}s\\
&\quad + \int_0^{+\infty}\overline{\beta}(t)\mathrm{d}t\int_0^{+\infty}|\alpha(s-a)|\mathrm{d}s\int_0^{s} e^{-(\mathrm{Re}\lambda+\underline{\mu})a}\mathrm{d}a\\
&\le \frac{\beta_{\sup}}{\mathrm{Re}\lambda+\underline{\mu}}\|(\alpha,\eta)\|.
\end{aligned} \tag{5.17}$$

Similarly, for $F_{2\lambda}$, we also have the following estimate:

$$\|F_{2\lambda}(\alpha,\eta)\|_{L^1(0,+\infty)} \le \frac{\beta'_{\sup}}{\mathrm{Re}\lambda+\underline{\mu}}\|(\alpha,\eta)\|. \tag{5.18}$$

Remark 5.12 In general $r(F_0)$ is not easy to compute, but we have the following estimates combining (5.5), (5.6), (5.17) and (5.18) when $a^+ < \infty$:

$$\|F_\lambda\| \le \frac{\beta_{\max}}{\mathrm{Re}\lambda+\underline{\mu}}[1-e^{-(\mathrm{Re}\lambda+\underline{\mu})a^+}], \tag{5.19}$$

and when $a^+ = \infty$,

$$\|F_\lambda\| \le \frac{\beta_{\max}}{\mathrm{Re}\lambda + \underline{\mu}}. \tag{5.20}$$

Then by the Gelfand's formula $r(F) = \lim_{k\to\infty} \|F^k\|^{\frac{1}{k}}$ for bounded linear operators and linearity of F_λ, we have

$$r(F_0) \le \begin{cases} \frac{\beta_{\max}(1-e^{-\underline{\mu}a^+})}{\underline{\mu}}, & \text{when} \quad a^+ < \infty, \\ \frac{\beta_{\max}}{\underline{\mu}}, & \text{when} \quad a^+ = \infty. \end{cases}$$

Thus, we have the following corollary.

Corollary 5.13

(i) *When* $a^+ < \infty$, *if* $\frac{\beta_{\max}(1-e^{-\underline{\mu}a^+})}{\underline{\mu}} < 1$, *then the zero equilibrium is globally exponentially stable;*
(ii) *When* $a^+ = \infty$, *if* $\frac{\beta_{\max}}{\underline{\mu}} < 1$, *then the zero equilibrium is globally exponentially stable.*

5.2 Asynchronous exponential growth

In this subsection, we study the asynchronous exponential growth of $\{S(t)\}_{t\ge0}$ when $r(F_0) > 1$. We study the two cases when $a^+ < \infty$ and when $a^+ = \infty$ together and give an extra Assumption 5.7 in the latter case. First let us recall the definition.

Definition 5.14 Let $\{S(t)\}_{t\ge0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. We say that $\{S(t)\}_{t\ge0}$ has *asynchronous exponential growth* with intrinsic growth constant $\lambda_0 \in \mathbb{R}$ if there exists a nonzero finite rank operator $P_0 \in X$ such that

$$\lim_{t\to\infty} e^{-\lambda_0 t} S(t) = P_0.$$

We introduce a theorem in Magal and Ruan [35, Theorem 4.6.2] which was proved by Webb [54].

Theorem 5.15 (Webb [54]) Let $\{S(t)\}_{t\ge0}$ *be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. Then* $\{S(t)\}_{t\ge0}$ *has asynchronous exponential growth with intrinsic growth constant* $\lambda_0 \in \mathbb{R}$ *if and only if*

(i) $\omega_1(A) < \lambda_0$;
(ii) $\lambda_0 = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$;
(iii) λ_0 *is a simple pole of* $(\lambda I - A)^{-1}$,

where $\omega_1(A)$ *denotes the essential growth bound of A which is defined by* (4.9).

Now we use this theorem to show that our semigroup $\{S(t)\}_{t\ge0}$ defined in (2.14) has asynchronous exponential growth.

Theorem 5.16 *If $\mathscr{R}_0 > 1$, then there exists a unique positive and real number $\lambda_1 > -\underline{\mu}$ satisfying $r(F_{\lambda_1}) = 1$ such that $\{S(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant λ_1. Moreover,*

$$\lim_{t\to\infty} e^{-\lambda_1 t} S(t)\phi = P_{\lambda_1}\phi \quad \text{for all} \quad \phi \in E,$$

where $P_{\lambda_1} : E \to E$ is defined by

$$P_{\lambda_1}\phi = (2\pi i)^{-1} \int_\Gamma (\lambda I - A)^{-1}\phi d\lambda$$

and Γ is a positively oriented closed curve in $\mathbb{C}$ enclosing λ_1, but no other point of $\sigma(A)$.

Proof First it is easy to see that if $r(F_0) > 1$, then there exists a unique real number $\lambda_1 > 0$ such that $r(F_{\lambda_1}) = 1$ and $\lambda_1 = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ by Proposition 5.6, which shows that condition (ii) in Theorem 5.15 holds. Next, in Sect. 4 we have shown that $\omega_1(A) \leq -\underline{\mu}$ for $a^+ = \infty$ and $\omega_1(A) = -\infty$ when $a^+ < \infty$. Thus, condition (i) in Theorem 5.15 is satisfied. Moreover, $\sup\{\mathrm{Re}\lambda : \lambda \in \sigma_e(A)\} \leq \omega_1(A)$, which implies $\lambda_1 \in \sigma_P(A) \setminus \sigma_e(A)$, thus is a pole of $(\lambda I - A)^{-1}$ by Webb [52, Proposition 4.11], where $\sigma_e(A)$ represents the essential spectrum of A. Also λ_1 is a simple eigenvalue of A and a simple pole of $(\lambda I - A)^{-1}$, see the proof in Proposition 5.6(v). Thus, condition (iii) in Theorem 5.15 is also satisfied. Hence, our result is desired. □

What we do next is to derive a formula for the projection $P_{\lambda_1} : E \to \ker(A - \lambda_1 I)$, inspired by Walker [49]. Observe that there is a quasi-interior element $\Phi_0 = (\alpha, \eta) \in L^1_+(0, a^+) \times L^1_+(0, a^+)$ such that $\ker(1 - F_{\lambda_1})$ =span$\{\Phi_0\}$. Denote

$$\Pi_\lambda(a, a')(\alpha, \eta) = \begin{cases} e^{-\lambda a'}\Pi(a, a', a')\eta(a - a'), & a > a' \\ e^{-\lambda a}\Pi(a, a', a)\alpha(a' - a), & a < a \end{cases},$$

then $\ker(A - \lambda_1 I)$ =span$\{\Pi_{\lambda_1}(a, a')\Phi_0\}$. Let $\phi \in E$ be fixed and let $c(\phi) \in \mathbb{R}$ be such that $P_{\lambda_1}\phi = c(\phi)\Pi_{\lambda_1}(a, a')\Phi_0$. Recall that λ_1 is a simple pole of the resolvent $(\lambda I - A)^{-1}$. Denote

$$H_\lambda\phi := \begin{cases} \int_0^{a'} e^{-\lambda\sigma}\Pi(a, a', \sigma)\phi(a - \sigma, a' - \sigma)\mathrm{d}\sigma, & a' < a, \\ \int_0^{a} e^{-\lambda\sigma}\Pi(a, a', \sigma)\phi(a - \sigma, a' - \sigma)\mathrm{d}\sigma, & a < a'. \end{cases}$$

Then $H_\lambda\phi$ is holomorphic in λ, it follows from (5.2), (5.15) and Residue theorem that

$$P_{\lambda_1}\phi = \lim_{\lambda\to\lambda_1} (\lambda - \lambda_1)\Pi_\lambda(a, a')(1 - F_\lambda)^{-1} G_\lambda\phi,$$

where

$$G_\lambda\phi = \left(G^1_\lambda\phi, G^2_\lambda\phi\right),$$

in which $G^i_\lambda, i = 1, 2$, is defined in (5.14). Let $w' \in (L^1_+(0, a^+) \times L^1_+(0, a'))'$ be a positive eigenfunctional of the dual operator F'_{λ_1} of F_{λ_1} corresponding to the eigenvalue $r(F_{\lambda_1}) = 1$. Then for $f' \in E'$ defined by

$$\langle f', \psi\rangle := \langle w', (\mathcal{G}\psi, \mathcal{F}\psi)\rangle, \quad \psi \in E,$$

we have due to $F'_{\lambda_1} w' = w'$ that

$$\begin{aligned} c(\phi)\langle w', \Phi_0\rangle &= \langle f', P_{\lambda_1}\phi\rangle = \lim_{\lambda\to\lambda_1}\langle f', (\lambda-\lambda_1)\Pi_\lambda(a,a')(1-F_\lambda)^{-1}G_\lambda\phi\rangle \\ &= \lim_{\lambda\to\lambda_1}\langle w', (\lambda-\lambda_1)(1-(1-F_\lambda))(1-F_\lambda)^{-1}G_\lambda\phi\rangle \\ &= \lim_{\lambda\to\lambda_1}\langle w', (\lambda-\lambda_1)(1-F_\lambda)^{-1}G_\lambda\phi\rangle. \end{aligned}$$

Writing

$$G_\lambda\phi = d(G_\lambda\phi)\Phi_0 \oplus (1-F_{\lambda_1})g(G_\lambda\phi) \tag{5.21}$$

according to the decomposition $L^1_+(0,a^+)\times L^1_+(0,a^+) = \mathbb{R}\cdot\Phi_0 \oplus \mathrm{rg}(1-F_{\lambda_1})$, it follows that

$$\lim_{\lambda\to\lambda_1}\langle w', (\lambda-\lambda_1)(1-F_\lambda)^{-1}G_\lambda\phi\rangle = d(G_{\lambda_1}\phi)\lim_{\lambda\to\lambda_1}\langle w', (\lambda-\lambda_1)(1-F_\lambda)^{-1}\Phi_0\rangle$$

due to the continuity of F_λ in λ. But from (5.21)

$$\langle w', G_{\lambda_1}\phi\rangle = d(G_{\lambda_1}\phi)\langle w', \Phi_0\rangle$$

since $F'_{\lambda_1}w' = w'$, whence $d(G_{\lambda_1}\phi) = \xi\langle w', G_{\lambda_1}\phi\rangle$ with $\xi^{-1} = \langle w', \Phi_0\rangle$. Similarly, decomposing

$$Z_\lambda := (\lambda-\lambda_1)(1-F_\lambda)^{-1}\Phi_0,$$

we find

$$\lim_{\lambda\to\lambda_1}\langle w', Z_\lambda\rangle = \left(\lim_{\lambda\to\lambda_1} d(Z_\lambda)\right)\langle w', \Phi_0\rangle.$$

With these observations, we derive that

$$c(\phi)\langle w', \Phi_0\rangle = C_0\langle w', G_{\lambda_1}\phi\rangle\langle w', \Phi_0\rangle$$

for some constant C_0. Consequently,

$$P_{\lambda_1}\phi = C_0\langle w', G_{\lambda_1}\phi\rangle\Pi_{\lambda_1}(a,a')\Phi_0.$$

Since P_{λ_1} is a projection, i.e., $P^2_{\lambda_1} = P_{\lambda_1}$, the constant C_0 is easily computed and we obtain the following result.

Proposition 5.17 *Under the assumptions of Theorem 5.16, the projection P_{λ_1} is given by*

$$P_{\lambda_1}\phi = \frac{\langle w', G_{\lambda_1}\phi\rangle}{\langle w', G_{\lambda_1}\Pi_{\lambda_1}(a,a')\Phi_0\rangle}\Pi_{\lambda_1}(a,a')\Phi_0 \tag{5.22}$$

for $\phi\in E$, where $G_{\lambda_1}\phi = (G^1_{\lambda_1}\phi, G^2_{\lambda_1}\phi)$ and $w' \in (L^1_+(0,a^+)\times L^1_+(0,a'))'$ is a positive eigenfunctional of the dual operator F'_{λ_1} of F_{λ_1} corresponding to the eigenvalue $r(F_{\lambda_1}) = 1$.

Remark 5.18 In fact, the expression of $P_{\lambda_1}\phi$ does not look explicitly like the one for single age-structured models, since the resolvent of A cannot be expressed explicitly for our double age-structure model.

5.3 Asymptotic behavior for $\mathscr{R}_0 = 1$

In the previous subsections, we have shown that if $\mathscr{R}_0 < 1$, then $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable and if $\mathscr{R}_0 > 1$, then $\{S(t)\}_{t\geq 0}$ has asynchronous exponential growth. It is natural to ask what happens if $\mathscr{R}_0 = 1$.

Remark 5.19 When $\{S(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant λ_1, the intrinsic growth constant λ_1 can be any real number, it means that

(i) If $\lambda_1 < 0$, then $S(t)$ decays exponentially; thus, the zero equilibrium is exponentially stable, as Theorem 5.11 states;
(ii) If $\lambda_1 > 0$, as $\{S(t)\}_{t\geq 0}$ has asynchronous exponential growth, it describes the divergent rate at which the system blows up when the zero equilibrium is unstable for a linear system;
(iii) If $\lambda_1 = 0$, then $S(t)\phi$ converges to $P\phi$ as $t \to \infty$. This is the case when $\mathscr{R}_0 = 1$ that we are concerned in this subsection.

First, note that when $\mathscr{R}_0 = 1$, i.e., $r(F_0) = 1$, by the previous argument we know that the spectral bound $s(A) = 0$. Now let us recall a theorem from Engel and Nagel [18, Chapter VI, Theorem 3.5].

Theorem 5.20 (Engel and Nagel [18]) *Let $\{S(t)\}_{t\geq 0}$ be a quasi-compact, irreducible, positive strongly continuous semigroup with generator A and assume that $s(A) = 0$. Then 0 is a dominant eigenvalue of A and a first-order pole of $(\lambda I - A)^{-1}$. Moreover, there exist strictly positive elements $0 \ll h \in X, 0 \ll \varphi \in X'$ and constants $M \geq 1, \epsilon > 0$ such that*

$$\|S(t)f - \langle f, \varphi\rangle \cdot h\| \leq Me^{-\epsilon t}\|f\| \quad \text{for all} \quad t \geq 0, f \in X,$$

where $\langle \cdot, \cdot \rangle$ is the dual product in Banach space.

In Sect. 4, we have shown that $\{S(t)\}_{t\geq 0}$ is quasi-compact when $a^+ = \infty$; in Sect. 2, we have shown that $\{S(t)\}_{t\geq 0}$ is positive. Next, we claim that $\{S(t)\}_{t\geq 0}$ is also irreducible. Recall that a positive semigroup with generator A on the Banach lattice X is *irreducible* if for some $\lambda > s(A)$ and all $0 < f \in X$, the resolvent satisfies $(\lambda I - A)^{-1}f \gg 0$, where $\gg$ represents strictly positivity.

By strict positivity in E, we need to require that $\varphi = (\lambda I - A)^{-1}\psi > 0$ for almost all $(a, a') \in (0, a^+) \times (0, a^+)$ for every $0 < \psi \in E$. Look at the resolvent solution (5.2), we only need that α and η are positive almost everywhere for $(a, a') \in (0, a^+) \times (0, a^+)$. Since α and η are determined by (5.11), motivated by Engel and Nagel [18, Theorem 4.4], only for β and β', there exists no $a_0 \geq 0$ such that

$$\beta\big|_{[a_0,a^+)\times[a_0,a^+)\times[a_0,a^+)} = 0, \quad \beta'\big|_{[a_0,a^+)\times[a_0,a^+)\times[a_0,a^+)} = 0 \quad \text{almost everywhere.} \tag{5.23}$$

In fact, Assumption 2.1(iii)' satisfies the above conditions, and then α and η will be positive almost everywhere. Thus, $(\lambda I - A)^{-1}\psi \gg 0$, which implies that $\{S(t)\}_{t\geq 0}$ is irreducible. By Theorem 5.20, we have the following theorem.

Theorem 5.21 *If $\mathscr{R}_0 = 1$ and (5.23) holds, then there exist a strictly positive linear function $h \in E$, a linear form $\zeta \in \mathrm{fix}(S(t)')$, $t \geq 0$, and constants $M \geq 1, \epsilon > 0$ such that*

$$\|S(t)\phi - \langle \phi, \zeta \rangle \cdot h\| \le Me^{-\epsilon t}\|\phi\| \quad \text{for all} \quad t \ge 0, \phi \in E,$$

where

$$\text{fix}(S(t))_{t\ge 0} := \{x \in E : S(t)x = x \quad \text{for all} \quad t \ge 0\},$$

which coincides with $\ker A$ *and* $\text{fix}(S(t)')_{t\ge 0} = \ker(A')$.

6 Discussion

In this paper we considered a linear first-order hyperbolic partial differential equation that models the single-species population dynamics with two physiological structures. By using semigroup theory, we studied the basic properties and dynamics of the model, including the solution flow $u(t, a, a')$ and its semigroup $\{S(t)\}_{t\ge 0}$ with infinitesimal generator A. Moreover, we established the compactness of solution trajectories, analyzed the spectrum of A, and investigated stability of the zero equilibrium with asynchronous exponential growth.

We would like to point out the differences between single physiologically structured models and double physiologically structured models. For a double physiologically structured model, first, the state space becomes $L^1_+((0, a^+) \times (0, a^+))$ instead of $L^1_+(0, a^+)$. It follows that the Volterra integral equations generated by the boundary conditions become a system of integral equations in the function space $L^1_+(0, a^+)$ instead of $\mathbb{R}$, and the characteristic equation becomes an operator equation instead of a scalar equation. Accordingly, the principal eigenvalue is changed into a point spectrum of an operator. Second, the integral region for an eigenfunction becomes a plane in $\mathbb{R}^2$ instead of a line in $\mathbb{R}$ and correspondingly the characteristic plane in $\mathbb{R}^3$ for a solution flow instead of the characteristic line in $\mathbb{R}^2$. More importantly, the infinitesimal generator A of the semigroup $\{S(t)\}_{t\ge 0}$ for a double physiologically structured model is much more complicated than that for a single physiologically structured model. Thanks to the solution flow (see Theorems 5.2 and 5.3 which give equivalent characterizations of eigenvalues and eigenfunctions of A), we can still study the spectrum of A without solving the characteristic equation $A\phi = \lambda\phi, \phi \in D(A)$.

The novelty and difficulty of the analysis lie in the non-trivial conditions for both boundaries (1.3)–(1.4). Such a setup not only brings extra complications in the proof of existence of solutions and trajectory compactness, but also requires alternative tools in the spectrum analysis. Therefore, it is very natural for one to ask for the motivation in terms of real-world applications with both boundaries being non-trivial (as most of the existing models with double physiological structures assume one trivial boundary condition). Our techniques and ideas can be applied to study multi-dimensional structured models with two physiological structures, such as epidemic models with chronological age and infection age (Hoppensteadt [22], Inaba [25], Burie et al. [6], Laroche and Perasso [33]), population dynamical models with age and size structures (Webb [55]), age and maturation structures (Dyson et al. [15, 16]), and age and stage structures (McNair and Goulden [39], Matucci [38]), or structured cell population models with continuous cell age and another continuous cell status such as cyclin content (Bekkal Brikci et al. [4]), maturity level (Bernard et al. [5]), plasmid copies (Stadler [45]), and telomere length (Kapitanov [28]). For instance, the model developed in Kapitanov [28] describes cell population structured with continuous cell age and discrete telomere length, which can be easily derived into models with both structures being continuous and with both boundary conditions being non-trivial (since newly generated cells could have telomere with any length and there indeed exist aging cells with 0-length telomere). We leave these for future consideration.

Note that our formulation of the problem allows more mortality processes than (1.2) and more general birth processes than (1.3) and (1.4). For example, one can study the following double physiologically structured model with G, F, H under appropriate conditions

$$\begin{cases} Du(t,a,a') = G(t,u(t,\ ))(a,a'), & \text{for} \quad t>0, (a,a') \in (0,a^+)\times(0,a^+) \\ u(t,0,a') = F(t,u(t,\ ),a'), & \text{for} \quad t>0, a' \in (0,a^+) \\ u(t,a,0) = H(t,u(t,\ ),a), & \text{for} \quad t>0, a \in (0,a^+) \\ u(0,a,a') = \phi(a,a'), & \text{for} \quad (a,a') \in (0,a^+)\times(0,a^+), \end{cases} \tag{6.1}$$

where $Du(t,a,a') := \frac{\partial u}{\partial t}(t,a,a') + \frac{\partial u}{\partial a}(t,a,a') + \frac{\partial u}{\partial a'}(t,a,a')$. In an upcoming paper [27], we will study the following nonlinear double physiologically structured model via a different approach: integrated semigroups and non-densely defined operators (where we can study the characteristic and resolvent equations directly),

$$\begin{cases} Du(t,a,a') = -\mu(a,a',P(t))u(t,a,a'), & \text{for} \quad t>0, (a,a') \in (0,a^+)\times(0,a^+) \\ u(t,0,a') = \int_0^{a^+}\int_0^{a^+} \beta(a',a,s,P(t))u(t,a,s)\mathrm{d}a\mathrm{d}s, & \text{for} \quad t>0, a' \in (0,a^+) \\ u(t,a,0) = \int_0^{a^+}\int_0^{a^+} \beta'(a,s,a',P(t))u(t,s,a')\mathrm{d}s\mathrm{d}a', & \text{for} \quad t>0, a \in (0,a^+) \\ u(0,a,a') = \phi(a,a'), & \text{for} \quad (a,a') \in (0,a^+)\times(0,a^+) \\ P(t) = \int_0^{a^+}\int_0^{a^+} u(t,a,a')\mathrm{d}a\mathrm{d}a', & \text{for} \quad t>0. \end{cases} \tag{6.2}$$

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Appendix

In this Appendix, we prove some statements that were used in Sect. 4.

Proposition A.1 *If $a^+ = \infty$, $\underline{\mu} > 0$, and Assumption 2.1 holds, then $S_1(t)$ satisfies the hypothesis (ii) and $S_2(t), S_3(t)$ satisfy the hypothesis (iii) of Proposition 4.2.*

Proof We only need to show that $S_2(t)$ is compact for $t > 0$, which is equivalent to show that for a bounded set K of E,

$$\lim_{h\to 0, k\to 0} \int_0^\infty \int_0^\infty \left|S_2(t)\phi(a+h,a'+k) - S_2(t)\phi(a,a')\right| \mathrm{d}a\mathrm{d}a' = 0 \tag{A.1}$$

$$\lim_{h\to\infty, k\to\infty} \int_h^\infty \int_k^\infty \left|S_2(t)\phi(a,a')\right| \mathrm{d}a\mathrm{d}a' = 0 \tag{A.2}$$

uniformly for $\phi \in K$ (which can be found in [14, Theorem 21, p. 301]). Without loss of generality, assume $k > h$ and $h, k \to 0^+$, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left|S_2(t)\phi(a+h, a'+k) - S_2(t)\phi(a,a')\right| \mathrm{d}a' \mathrm{d}a \\
&\quad \le \underbrace{\int_0^{t-h} \int_a^\infty \left|S_2(t)\phi(a+h, a'+k) - S_2(t)\phi(a,a')\right| \mathrm{d}a' \mathrm{d}a}_{\text{region I}} \\
&\qquad + \underbrace{\int_{k-h}^{t-h} \int_{a+h-k}^{a} \left|S_2(t)\phi(a+h, a'+k)\right| \mathrm{d}a' \mathrm{d}a}_{\text{region II}} \\
&\qquad + \underbrace{\int_0^{k-h} \int_0^{a} \left|S_2(t)\phi(a+h, a'+k)\right| \mathrm{d}a' \mathrm{d}a}_{\text{region III}} + \underbrace{\int_{t-h}^{t} \int_a^\infty \left|S_2(t)\phi(a,a')\right| \mathrm{d}a' \mathrm{d}a}_{\text{region IV}},
\end{aligned} \tag{A.3}$$

as illustrated in Fig. 3b: $S_2(t)\phi(a,a')$ is non-trivial for points (a,a') in regions I and IV, and $S_2(t)\phi(a+h, a'+k)$ is non-trivial for points (a,a') in regions I, II, and III.

We first show

$$\begin{aligned}
&\int_0^{t-h} \int_a^\infty \left|S_2(t)\phi(a+h, a'+k) - S_2(t)\phi(a,a')\right| \mathrm{d}a' \mathrm{d}a \\
&\le \int_0^{t} \int_a^\infty \left|b_\phi(t-a-h, a'+k-a-h) - b_\phi(t-a, a'-a)\right| e^{-\int_0^{a+h} \mu(s,s+a'+k-a-h)\mathrm{d}s} \mathrm{d}a' \mathrm{d}a \\
&\quad + \int_0^{t-h} \int_a^\infty \left|b_\phi(t-a, a'-a)[e^{-\int_0^{a+h} \mu(s,s+a'+k-a-h)\mathrm{d}s} - e^{-\int_0^{a} \mu(s,s+a'-a)\mathrm{d}s}]\right| \mathrm{d}a' \mathrm{d}a \\
&:= \mathrm{I} + \mathrm{II},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{II} &\le \int_0^{t-h} \int_a^\infty b_\phi(t-a, a'-a) \Big| e^{-\int_0^{a+h} \mu(s,s+a'-a)\mathrm{d}s} [1 - e^{\int_0^{a+h} \mu(s,s+a'-a) - \mu(s,s+a'+k-a-h)\mathrm{d}s}] \\
&\quad + e^{-\int_0^{a} \mu(s,s+a'-a)\mathrm{d}s} [1 - e^{-\int_a^{a+h} \mu(s,s+a'-a)\mathrm{d}s}] \Big| \mathrm{d}a' \mathrm{d}a \\
&\le \int_0^{t-h} \int_a^\infty b_\phi(t-a, a'-a) \big(\max\{1 - e^{-K_\mu (k-h)t}, e^{K_\mu (k-h)t} - 1\} + (1 - e^{-\bar{\mu} h}) \big) \mathrm{d}a' \mathrm{d}a \\
&\le \big(\max\{1 - e^{-K_\mu (k-h)t}, e^{K_\mu (k-h)t} - 1\} + (1 - e^{-\bar{\mu} h}) \big) \int_0^t \int_0^\infty b_\phi(t-a, s) \mathrm{d}s \mathrm{d}a \\
&\le 2\beta_{\max} \|\phi\|_E \big(\max\{1 - e^{-K_\mu (k-h)t}, e^{K_\mu (k-h)t} - 1\} + (1 - e^{-\bar{\mu} h}) \big) \int_0^t e^{4\beta_{\max}(t-a)} \mathrm{d}a
\end{aligned}$$

based on our prior estimate in Sect. 2.1 and with K_μ being the Lipschitz constant for μ, thus $\mathrm{II} \to 0$ uniformly for $\phi \in K$ as $h, k \to 0^+$. Next, we need to show that

$$\lim_{h\to 0, k\to 0}\int_0^t\int_a^\infty |b_\phi(t-a-h, a'+k-a-h)-b_\phi(t-a, a'-a)|\mathrm{d}a'\mathrm{d}a = 0. \quad \text{(A.4)}$$

Now by an alternative version of (2.7) and (2.8), we have

$$\begin{aligned}
&\int_0^t\int_a^\infty |b_\phi(t-a-h, a'+k-a-h)-b_\phi(t-a, a'-a)|\mathrm{d}a'\mathrm{d}a \\
&\le \int_0^t\int_a^\infty \Big|\int_0^{t-a-h}\int_0^\infty f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p)b_\phi(p,s)\mathrm{d}s\mathrm{d}p \\
&\quad -\int_0^{t-a}\int_0^\infty f_1(a'-a, t-a-p, s+t-a-p)b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big|\mathrm{d}a'\mathrm{d}a \\
&\quad +\int_0^t\int_a^\infty \Big|\int_0^{t-a-h}\int_0^\infty g_1(a'+k-a-h, p+t-a-h-s, t-a-h-s)b'_\phi(s,p)\mathrm{d}p\mathrm{d}s \\
&\quad -\int_0^{t-a}\int_0^\infty g_1(a'-a, p+t-a-s, t-a-s)b'_\phi(s,p)\mathrm{d}p\mathrm{d}s\Big|\mathrm{d}a'\mathrm{d}a \\
&\quad +\int_0^t\int_a^\infty \Big|\int_0^\infty\int_0^\infty h_1(a'+k-a-h, p, s, t-a-h)\phi(p,s)\mathrm{d}p\mathrm{d}s \\
&\quad -\int_0^\infty\int_0^\infty h_1(a'-a, p, s, t-a)\phi(p,s)\mathrm{d}p\mathrm{d}s\Big|\mathrm{d}a'\mathrm{d}a \\
&:= \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{J}_1 \le &\int_0^t\int_a^\infty \Big(\int_0^{t-a-h}\int_0^\infty |f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p) \\
&-f_1(a'-a, t-a-p, s+t-a-p)|b_\phi(p,s)\mathrm{d}s\mathrm{d}p \\
&+\int_{t-a}^{t-a-h}\int_0^\infty f_1(a'-a, t-a-p, s+t-a-p)b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a \\
&:= \mathrm{J}_1^1+\mathrm{J}_1^2,
\end{aligned}$$

in which

$$\begin{aligned}
&|f_1(a'+k-a-h, t-a-h-p, s+t-a-h-p)-f_1(a'-a, t-a-p, s+t-a-p)| \\
&\le \beta_1(a'+k-a-h)\Big|\beta_2(t-a-h-p, s+t-a-h-p)e^{-\int_0^{t-a-p-h}\mu(\sigma,\sigma+s)\mathrm{d}\sigma} \\
&\quad -\beta_2(t-a-p, s+t-a-p)e^{-\int_0^{t-a-p}\mu(\sigma,\sigma+s)\mathrm{d}\sigma}\Big| \\
&\quad +|\beta_1(a'+k-a-h)-\beta_1(a'-a)|\beta_2(t-a-p, s+t-a-p) \\
&\le \beta_1(a'+k-a-h)\big|\beta_2(t-a-h-p, s+t-a-h-p)-\beta_2(t-a-p, s+t-a-p)\big| \\
&\quad +\beta_1(a'+k-a-h)\beta_2(t-a-p, s+t-a-p)\Big|1-e^{-\int_{t-a-h-p}^{t-a-p}\mu(\sigma,\sigma+s)\mathrm{d}\sigma}\Big| \\
&\quad +\big|\beta_1(a'+k-a-h)-\beta_1(a'-a)\big|\beta_2(t-a-p, s+t-a-p) \\
&\le \beta_1(a'+k-a-h)K_\beta 2h+\bar{\beta}(a'+k-a-h)(1-e^{\bar{\mu}h}) \\
&\quad +\big|\beta_1(a'+k-a-h)-\beta_1(a'-a)\big|\beta_2(t-a-p, s+t-a-p) \\
&\le \beta_1(a'+k-a-h)K_\beta 2h+\bar{\beta}(a'+k-a-h)(1-e^{\bar{\mu}h}) \\
&\quad +\big|\beta_1(a'+k-a-h)-\beta_1(a'-a)\big|\beta_2^{\sup}
\end{aligned}$$

by Assumption 2.1(i) on β being Lipschitz continuous and K_β as the Lipschitz constant, where

$$\beta_2^{\sup} := \sup_{(a,s)\in(0,\infty)\times(0,\infty)} \beta_2(a,s) < \infty$$

because Assumption 2.1(iv) holds. Thus,

$$\begin{aligned}
J_1^1 \le& \int_0^t \int_a^\infty \big(\beta_1(a'+k-a-h)K_\beta 2h + \bar\beta(a'+k-a-h)(1-e^{\bar\mu h})\big)\Big(\int_0^{t-a-h}\int_0^\infty b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
&+ \int_0^t \int_a^\infty |\beta_1(a'+k-a-h)-\beta_1(a'-a)|\beta_2^{\sup}\Big(\int_0^{t-a-h}\int_0^\infty b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
\le& \int_0^t \int_a^\infty \big(\beta_1(a'+k-a-h)K_\beta 2h + \bar\beta(a'+k-a-h)(1-e^{\bar\mu h})\big)\Big(\int_0^{t-a-h} 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}p}\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
&+ \int_0^t \int_a^\infty |\beta_1(a'+k-a-h)-\beta_1(a'-a)|\beta_2^{\sup}\Big(\int_0^{t-a-h} 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}p}\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
\to& \,0 \ \text{ uniformly for } \phi\in K \text{ as } h,k\to 0^+
\end{aligned}$$

and

$$\begin{aligned}
J_1^2 &\le \int_0^t \int_a^\infty \bar\beta(a'-a)\Big(\int_{t-a}^{t-a-h}\int_0^\infty b_\phi(p,s)\mathrm{d}s\mathrm{d}p\Big)\mathrm{d}a'\mathrm{d}a\\
&\le \Big(\int_0^\infty \bar\beta(a')\mathrm{d}a'\Big)\Big(\int_0^t\int_{t-a-h}^{t-a} 2\beta_{\max}\|\phi\|_E e^{4\beta_{\max}p}\mathrm{d}p\mathrm{d}a\Big)\\
&\to 0 \ \text{ as } h,k\to 0^+ \text{ uniformly for } \phi\in K,
\end{aligned}$$

Therefore, we have $J_1 \to 0$ as $h,k\to 0^+$ uniformly for $\phi\in K$. And the fact that $J_2\to 0$ uniformly for $\phi\in K$ as $h,k\to 0^+$ can be proved by using a similar argument. To show $J_3\to 0$, we first

$$\begin{aligned}
&\big|h_1(a'+k-a-h,p,s,t-a-h)-h_1(a'-a,p,s,t-a)\big|\\
&\quad\le \big|\beta(a'+k-a-h,p+t-a-h,s+t-a-h)-\beta(a'-a,p+t-a,s+t-a)\big|e^{-\int_0^{t-a-h}\mu(\sigma+p,\sigma+s)\mathrm{d}\sigma}\\
&\qquad+\beta(a'-a,p+t-a,s+t-a)e^{-\int_0^{t-a-h}\mu(\sigma+p,\sigma+s)\mathrm{d}\sigma}\big|1-e^{-\int_{t-a-h}^{t-a}\mu(\sigma+p,\sigma+s)\mathrm{d}\sigma}\big|\\
&\quad\le \big|\beta(a'+k-a-h,p+t-a-h,s+t-a-h)-\beta(a'-a,p+t-a-h,s+t-a-h)\big|\\
&\qquad+\beta_1(a'-a)\big|\beta_2(p+t-a-h,s+t-a-h)-\beta_2(p+t-a,s+t-a)\big|\\
&\qquad+\beta(a'-a,p+t-a,s+t-a)(1-e^{-\underline{\mu}h})\\
&\quad\le \big|\beta(a'+k-a-h,p+t-a-h,s+t-a-h)-\beta(a'-a,p+t-a-h,s+t-a-h)\big|\\
&\qquad+\beta_1(a'-a)K_\beta 2h+\bar\beta(a'-a)(1-e^{-\underline{\mu}h}).
\end{aligned}$$

Then by applying Assumption 2.1(ii), we have $J_3\to 0$ uniformly for $\phi\in K$.

Therefore, we have the first term in (A.3) goes to 0 uniformly for $\phi\in K$. Secondly, based on estimate (4.8) in the main text we have

$$
\begin{aligned}
&\int_{k-h}^{t-h}\int_{a+h-k}^{a}\left|S_2(t)\phi(a+h,a'+k)\right|\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_{k-h}^{t-h}\int_{a+h-k}^{a} b_\phi(t-a-h,a'+k-a-h)\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_{k-h}^{t-h}\left|\int_{a+h-k}^{a}\bar{\beta}(a')\mathrm{d}a'\right|\left(4\|\phi\|_E\beta_{\max}\int_0^t e^{4\beta_{\max}p}\mathrm{d}pp+\|\phi\|_E\right)\mathrm{d}a\\
&\quad\le \sup_{0<a<t}\left|\int_{a+h-k}^{a}\bar{\beta}(a')\mathrm{d}a'\right|\cdot t\cdot\left(4\|\phi\|_E\beta_{\max}\int_0^t e^{4\beta_{\max}p}\mathrm{d}p+\|\phi\|_E\right)\\
&\quad\to 0 \ \text{ as } h,k\to 0^+ \text{ uniformly for } \phi\in K.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\int_0^{k-h}\int_0^{a}\left|S_2(t)\phi(a+h,a'+k)\right|\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_0^{k-h}\int_0^{a} b_\phi(t-a-h,a'+k-a-h)\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_0^{k-h}\int_0^{\infty} b_\phi(t-a-h,s)\mathrm{d}s\mathrm{d}a\le 2\beta_{\max}\|\phi\|_E\int_0^{k-h}e^{4\beta_{\max}(t-a-h)}\mathrm{d}a\\
&\quad\to 0 \ \text{ as } h,k\to 0^+ \text{ uniformly for } \phi\in K.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\int_{t-h}^{t}\int_a^{\infty}\left|S_2(t)\phi(a,a')\right|\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_{t-h}^{t}\int_a^{\infty} b_\phi(t-a,a'-a)\mathrm{d}a'\mathrm{d}a\\
&\quad\le \int_{t-h}^{t}\int_0^{\infty} b_\phi(t-a,s)\mathrm{d}s\mathrm{d}a\le 2\beta_{\max}\|\phi\|_E\int_{t-h}^{t}e^{4\beta_{\max}(t-a)}\mathrm{d}a\\
&\quad\to 0 \text{ as } h\to 0^+ \text{ uniformly for } \phi\in K.
\end{aligned}
$$

We thus proved (A.1). For (A.2), we have

$$
\begin{aligned}
&\int_h^{\infty}\int_k^{\infty}\left|S_2(t)\phi(a,a')\right|\mathrm{d}a\mathrm{d}a'\\
&\quad\le \int_h^{\infty}\int_0^{\infty} b_\phi(t-a,s)\mathrm{d}s\mathrm{d}a\\
&\quad\le 2\|\phi\|_E\beta_{\max}\int_h^{\infty}e^{4\beta_{\max}(t-a)}\mathrm{d}a\to 0
\end{aligned}
$$

as $h,k\to 0$ uniformly for $\phi\in K$. We can show that $S_3(t)$ is compact for sufficiently large t in the same way. □

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