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PERIODIC SOLUTIONS OF AN AGE-STRUCTURED EPIDEMIC MODEL WITH PERIODIC INFECTION RATE

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Abstract. In this paper we consider an age-structured epidemic model of the susceptible-exposed-infectious-recovered (SEIR) type. To characterize the seasonality of some infectious diseases such as measles, it is assumed that the infection rate is time periodic. After establishing the well-posedness of the initial-boundary value problem, we study existence of time periodic solutions of the model by using a fixed point theorem. Some numerical simulations are presented to illustrate the obtained results.

1. **Introduction.** It is well-known that biological, physical, and social conditions of the host population have great impacts on the transmission dynamics of infectious diseases. Age structure is one of such factors that affect the outcome and consequences of the epidemics. In their pioneer series of papers published in the 1920s-1930s, Kermack and McKendrick [24, 25, 26] considered the effect of infection-age and proposed the so-called age-structured epidemic models described by three first-order hyperbolic partial differential equations (see Inaba [20] for a re-derivation and analysis of the age-structured Kermack-McKendrick model). Starting in the 1970s, researchers have noticed that the chronological age of the host population also plays a crucial role in the transmission process of some infectious diseases, such as measles, mumps, and pertussis, in particular when mass vaccination program is targeted at specific age groups. Consequently, many age-structured epidemic models have been proposed and analyzed in the literature, we refer to some early studies by Anderson and May [1], Andreasen [2], Bentil and Murray [5], Busenberg et al. [8, 7], Cha et al. [9], Feng et al. [11], Greenhalgh [12], Hethcote [13], Hoppensteadt

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[15], Iannelli [17], Iannelli et al. [18], Inaba [19], Li et al. [33], Schenzle [36], Tudor [38], the recent monograph by Inaba [23] and the references cited therein.

Periodic outbreak is a common phenomenon observed in the epidemics of many infectious diseases such as chickenpox, influenza, measles, mumps, rubella, etc. (Hethcote and Levin [14]). It is very important to understand such epidemic patterns in order to introduce public health interventions and design control measures for the spread of these infectious diseases. Recent studies have demonstrated that seasonality plays a crucial role in causing periodic outbreaks of some of these infectious diseases and many researchers have developed epidemic models with periodic parameters to investigate the effect of seasonality on the transmission dynamics of some infectious diseases, we refer to Bacaër [3], Bacaër and Guernaoui [4], Earn et al. [10], Huang et al. [16], Wang and Zhao [39], and so on.

It is very natural to study the combined effects of age structure of host population and the periodicity in environmental, seasonal or social changes. In recent years, there has been considerable interest in investigating age-structured epidemic models with periodic parameters, see for examples, Busenberg et al. [7], Kubo and Langlais [28], Kuniya [29], Kuniya and Iannelli [30], Kuniya and Inaba [31], and Langlais and Busenberg [32]. Especially, a threshold value for the existence and uniqueness of a nontrivial endemic periodic solution of an age-structured SIS epidemic model with periodic parameters was obtained in Kuniya and Inaba [31].

The purpose of this paper is to generalize the periodic SEIR epidemic model describing measles (Earn et al. [10], Huang et al. [16]) to an age-structured SEIR model with periodic infection rate. The host population is divided into four epidemiological classes: susceptible, exposed, infectious, and recovered. We assume that individuals who were vaccinated or recovered from infection would obtain immunity and go to the recovered class directly. We will first establish the well-posedness of the initial-boundary value problem for the age-structured SEIR model. Then we will study existence of time periodic solutions of the model by using a fixed point theorem. Finally we will present some numerical simulations to illustrate the obtained results.

The paper is organized as follows: In Section 2, we formulate the age-structured SEIR epidemic model with periodic infection rate. In Section 3, we establish the well-posedness of the time evolution problem. In Section 4, we prove the existence of a nontrivial endemic periodic solution of the system. Finally, numerical illustrations are given in Section 5 and a brief discussion is given in Section 6.

2. **Mathematical model.**

Motivated by the study of periodic solutions of an SEIR epidemic model describing measles (Huang et al. [16]), we consider the following age-structured epidemic model:

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{\partial S}{\partial a} = -S(t,a)\int_0^{a^+} \beta(t;a,a')I(t,a')da' - (\mu(a)+\rho(a))S(t,a), \\ \dfrac{\partial E}{\partial t} + \dfrac{\partial E}{\partial a} = S(t,a)\int_0^{a^+} \beta(t;a,a')I(t,a')da' - (\sigma(a)+\mu(a))E(t,a), \\ \dfrac{\partial I}{\partial t} + \dfrac{\partial I}{\partial a} = \sigma(a)E(t,a) - (\gamma(a)+\mu(a))I(t,a), \\ \dfrac{\partial R}{\partial t} + \dfrac{\partial R}{\partial a} = \rho(a)S(t,a) + \gamma(a)I(t,a) - \mu(a)R(t,a), \end{cases} \tag{2.1}$$

where $S(t,a), E(t,a), I(t,a)$, and $R(t,a)$ represent the densities of susceptible, exposed, infectious, and recovered individuals at time t with age a, respectively. $\mu(a)$

is the mortality rate, $\rho(a)$ is the vaccination rate, $\sigma(a)$ is the reciprocal of the incubation period, $\gamma(a)$ is the reciprocal of the infective period, and $\beta(t;a,a')$ is the rate at which susceptible individuals with age a are infected by infections individuals with age a' and is a nonnegative and periodic function in t with period $T>0$. The associated boundary conditions are

$$S(t,0)=A,\ E(t,0)=0,\ I(t,0)=0,\ R(t,0)=0, \tag{2.2}$$

where A is a positive constant, and initial conditions are given by

$$S(0,a)=S_0(a),\ E(0,a)=E_0(a),\ I(0,a)=I_0(a),\ R(0,a)=R_0(a), \tag{2.3}$$

where $S_0(a),\ E_0(a),\ I_0(a)$, and $R_0(a)$ are nonnegative continuous functions of a. The corresponding ordinary differential equations (ODEs) model is as follows:

$$\begin{cases} \dfrac{dS}{dt}=A(1-\rho)-\beta(t)IS-\mu S, \\ \dfrac{dE}{dt}=\beta(t)IS-(\sigma+\mu)E, \\ \dfrac{dI}{dt}=\sigma E-(\gamma+\mu)I, \\ \dfrac{dR}{dt}=A\rho+\gamma I-\mu R, \end{cases} \tag{2.4}$$

where the parameters are positive constants and have the same interpretations as in (2.1). Huang et al. [16] showed that the existence of a periodic solution for (2.4) when the basic reproduction number $\mathscr{R}_0>1$. In this paper, we will follow the idea in Kuniya [29] and Kuniya and Inaba [31] to establish the existence of periodic solutions in (2.1) under the same condition $\mathscr{R}_0>1$.

Let $P(t,a):=S(t,a)+E(t,a)+I(t,a)+R(t,a)$ denote the total population at time t with age a. By adding up the four equations in (2.1), we know that $P(t,a)$ satisfies the following equation:

$$\frac{\partial P}{\partial t}+\frac{\partial P}{\partial a}=-\mu(a)P(t,a) \tag{2.5}$$

with boundary condition

$$P(t,0)=A$$

and initial condition

$$P(0,a)=P_0(a):=S_0(a)+E_0(a)+I_0(a)+R_0(a).$$

In this paper we assume that the host population is already in a demographic steady state. Then the dynamics of $P(t,a)$ can be determined independently from the epidemic. Actually,

$$P(t,a)=Ae^{-\int_0^a \mu(a')da'}.$$

If we normalize variables by

$$s(t,a)=\frac{S(t,a)}{P(t,a)},\ e(t,a)=\frac{E(t,a)}{P(t,a)},\ i(t,a)=\frac{I(t,a)}{P(t,a)},\ r(t,a)=\frac{R(t,a)}{P(t,a)},$$

then we obtain the following system for (s,e,i,r):

$$\begin{cases} \dfrac{\partial s}{\partial t}+\dfrac{\partial s}{\partial a}=-\lambda(t,a)s(t,a)-\rho(a)s(t,a), \\ \dfrac{\partial e}{\partial t}+\dfrac{\partial e}{\partial a}=\lambda(t,a)s(t,a)-\sigma(a)e(t,a), \\ \dfrac{\partial i}{\partial t}+\dfrac{\partial i}{\partial a}=\sigma(a)e(t,a)-\gamma(a)i(t,a), \\ \dfrac{\partial r}{\partial t}+\dfrac{\partial r}{\partial a}=\rho(a)s(t,a)+\gamma(a)i(t,a) \end{cases} \tag{2.6}$$

with boundary conditions

$$s(t,0)=1,\ e(t,0)=0,\ i(t,0)=0,\ r(t,0)=0$$

and initial conditions

$$\begin{aligned} s(0,a)=s_0(a):=\frac{S_0(a)}{P_0(a)},\ e(0,a)=e_0(a):=\frac{E_0(a)}{P_0(a)}, \\ i(0,a)=i_0(a):=\frac{I_0(a)}{P_0(a)},\ r(0,a)=r_0(a):=\frac{R_0(a)}{P_0(a)} \end{aligned}$$

where

$$\lambda(t,a):=\int_0^{a^+}\beta(t;a,a')I(t,a')da'=\int_0^{a^+}\beta(t;a,a')P(t,a')i(t,a')da'. \tag{2.7}$$

3. **Well-posedness.** In order to study the well-posedness of the initial-boundary value problem, we first make the following assumptions.

Assumption 3.1. We assume that

(i) $\sigma(\cdot),\rho(\cdot),\gamma(\cdot)\in L_+^\infty(0,a^+)$ and for each $t\in\mathbb{R}_+$, $\beta(t,\cdot,\cdot)\in L_+^\infty([0,a^+]\times[0,a^+])$;

(ii) $\int_0^{a^+}\mu(a)da=+\infty$ and μ is locally integrable.

We now present a semigroup approach to prove the existence and uniqueness of nonnegative solutions of system (2.6). Since $s(t,a)=1-e(t,a)-i(t,a)-r(t,a)$, system (2.6) can be reduced to a three-component system for (e,i,r) as follows:

$$\begin{cases} \dfrac{\partial e}{\partial t}+\dfrac{\partial e}{\partial a}=\lambda(t,a)(1-e(t,a)-i(t,a)-r(t,a))-\sigma(a)e(t,a), \\ \dfrac{\partial i}{\partial t}+\dfrac{\partial i}{\partial a}=\sigma(a)e(t,a)-\gamma(a)i(t,a), \\ \dfrac{\partial r}{\partial t}+\dfrac{\partial r}{\partial a}=\rho(a)(1-e(t,a)-i(t,a)-r(t,a))+\gamma(a)i(t,a) \end{cases} \tag{3.1}$$

with boundary conditions

$$e(t,0)=0,\ i(t,0)=0,\ r(t,0)=0 \tag{3.2}$$

and initial conditions

$$e(0,a)=e_0(a),\ i(0,a)=i_0(a),\ r(0,a)=r_0(a). \tag{3.3}$$

Define the state space of (e,i,r) by

$$M:=L_+^1(0,a^+)\times L_+^1(0,a^+)\times L_+^1(0,a^+).$$

Define the operator A with domain $\mathscr{D}(A)$ and family $\{F(t;\cdot)\}_{t\geq 0}$ acting on E by

$$(A\phi)(a) = \begin{pmatrix} -\dfrac{d}{da} & 0 & 0 \\ 0 & -\dfrac{d}{da} & 0 \\ 0 & 0 & -\dfrac{d}{da} \end{pmatrix} \begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \phi_2(a) \end{pmatrix},$$

$$\mathscr{D}(A) = \left\{ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \in M : \phi_j \in W^{1,1}(0,a^+), \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$F(t,\phi)(a) = \begin{pmatrix} \lambda[t,a|\phi_2](1-\phi_1(a)-\phi_2(a)-\phi_3(a)) - \sigma\phi_1 \\ \sigma\phi_1 - \gamma\phi_2 \\ \rho(1-\phi_1(a)-\phi_2(a)-\phi_3(a)) + \gamma\phi_2 \end{pmatrix},$$

where

$$\lambda[t,a|\phi_2] := \int_0^{a^+} \beta(t;a,a')P(t,a')\phi_2(a')da'$$

and $W^{1,1}(0,a^+)$ is the space of all absolutely continuous functions on $(0,a^+)$.

Let $u = (e,i,r)^T$ be the state vector of the (e,i,r) system. Then the (e,i,r) system can be formulated as a semilinear Cauchy problem on the Banach space E:

$$\frac{du(t)}{dt} = Au(t) + F(t,u(t)), \quad u(0) = u_0. \tag{3.4}$$

We can show that the operator A generates a strongly continuous semigroup e^{tA}. In fact, the semigroup e^{tA} can be expressed as follows:

$$e^{tA}\begin{pmatrix} \phi_1(a) \\ \phi_2(a) \\ \phi_3(a) \end{pmatrix} = \begin{pmatrix} (e^{tA_1}\phi_1)(a) \\ (e^{tA_2}\phi_2)(a) \\ (e^{tA_3}\phi_3)(a) \end{pmatrix},$$

where e^{tA_i} for $i = 1,2,3$ are the semigroups on $L^1(0,a^+)$, which are defined for $u \in L^1(0,a^+)$ as

$$(e^{tA_i}u)(a) = \begin{cases} 0, & t-a>0, \\ u(a-t), & a-t>0, \end{cases} \quad i=1,2,3.$$

In the following, define a region in M by

$$\Omega := \{(e,i,r) \in L^1_+(0,a^+) \times L^1_+(0,a^+) \times L^1_+(0,a^+) : 0 \leq e+i+r \leq 1\}.$$

Then we can conclude that

Lemma 3.2. *The mapping $F(t,\cdot) : \Omega \to M$ is Lipschitz continuous for any fixed $t \in \mathbb{R}_+$ and there exists a number $\alpha \in (0,1)$ such that*

$$(I + \alpha F(t,\cdot))(\Omega) \subset \Omega. \tag{3.5}$$

Proof. The Lipschitz continuity of $F(t,\cdot)$ is clear, let us show (3.5). If we define the vector $v = (v_1,v_2,v_3)^T$ by

$$(I+\alpha F(t,\cdot))(u_1,u_2,u_3)^T = (v_1,v_2,v_3)^T,$$

then we have

$$v_1(a)+v_2(a)+v_3(a) = \alpha(\lambda[t,a|u_2]+\rho)(1-u_1(a)-u_2(a)-u_3(a)) + (u_1(a)+u_2(a)+u_3(a)).$$

Set $\lambda^+ := \sup \lambda$ and $\rho^+ := \sup \rho$ and choose $\alpha : \alpha \le (\lambda^+ + \rho^+)^{-1}$. Then we have $v_1(a) + v_2(a) + v_3(a) \le 1$. Moreover, set $\sigma^+ := \sup \sigma, \gamma^+ := \sup \gamma$ and choose $\alpha : \alpha\sigma^+ \le 1, \alpha\gamma^+ \le 1$. Thus we have $v_1 \ge 0$ and $v_2 \ge 0$. Therefore, if we choose

$$0 < \alpha < \min \left\{ \frac{1}{\lambda^+ + \rho^+}, \frac{1}{\sigma^+}, \frac{1}{\gamma^+} \right\},$$

then (3.5) holds. □

Now following the method of Busenberg et al. [8], we can rewrite the Cauchy problem (3.4) as follows:

$$\frac{du(t)}{dt} = \left(A - \frac{1}{\alpha} \right) u(t) + \frac{1}{\alpha}(I + \alpha F(t,\cdot))u(t), \quad u(0) = u_0, \tag{3.6}$$

where α is chosen such that (3.5) holds. The mild solution of this problem is then given by the variation of constants formula:

$$u(t) = e^{-\frac{1}{\alpha}t} e^{tA} u_0 + \int_0^t e^{-\frac{1}{\alpha}(t-s)} e^{(t-s)A} [u(s) + \alpha F(s, u(s))] ds.$$

The mild solution defines an evolutionary system $\{U(t,s)\}_{t \ge s \ge 0}$ by $U(t,0)u_0 = u(t)$. Define an iterative sequence $\{u^n(t)\}$ by

$$u^0(t) = u_0, \quad u^{n+1}(t) = e^{-\frac{1}{\alpha}t} e^{tA} u_0 + \int_0^t e^{-\frac{1}{\alpha}(t-s)} e^{(t-s)A} [u^n(s) + \alpha F(s, u^n(s))] ds.$$

If $u^n \in \Omega$, it follows that $e^{tA}u_0, e^{(t-s)A}[u^n(s) + \alpha F(s, u^n(s))] \in \Omega$. Hence, $u^{n+1} \in \Omega$ because it is the convex sum of two elements of the convex set Ω. It follows from the Lipschitz continuity that the iterative sequence $\{u^n(t)\}$ converges uniformly to the mild solution $U(t,0)u_0 \in \Omega$. Thus, we have the following existence and uniqueness result.

Proposition 3.3. *The Cauchy problem (3.4) has a unique mild solution $U(t,0)u_0$, where $\{U(t,s)\}_{t \ge s \ge 0}$ defines an evolutionary system with the following property:*

$$U(s,s) = I, \quad U(t,\sigma)U(\sigma,s) = U(t,s), \quad U(t,s)(\Omega) \subset \Omega,$$

If the initial data are in the domain $\mathscr{D}(A)$, then the mild solution becomes a classical solution.

4. Existence of an endemic periodic solution.

In this section we investigate the existence of an endemic T-periodic solution of system (2.6). Let X_T be the space of all locally integrable T-periodic $L^1(0, a^+)$-valued functions with norm

$$\|\varphi\|_{X_T} := \int_0^T \|\varphi(t)\|_{L^1} dt = \int_0^T \int_0^{a^+} |\varphi(t,a)| da dt$$

and $X_{T,+}$ be its positive cone. Let C_T be the state subspace defined by

$$C_T := \{\varphi \in X_{T,+} : 0 \le \varphi(t,a) \le 1\}.$$

Since $r(t,a) = 1 - s(t,a) - e(t,a) - i(t,a)$ and the equations of (s,e,i) are independent of r, system (2.6) can be reduced to a three-component system in

(s, e, i) as follows

$$\begin{cases} \dfrac{\partial s}{\partial t} + \dfrac{\partial s}{\partial a} = -\lambda(t,a)s(t,a) - \rho(a)s(t,a), \\ \dfrac{\partial e}{\partial t} + \dfrac{\partial e}{\partial a} = \lambda(t,a)s(t,a) - \sigma(a)e(t,a), \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial a} = \sigma(a)e(t,a) - \gamma(a)i(t,a) \end{cases} \tag{4.1}$$

with boundary conditions

$$s(t,0) = 1, e(t,0) = 0, i(t,0) = 0 \tag{4.2}$$

and initial conditions

$$s(0,a) = s_0(a), e(0,a) = e_0(a), i(0,a) = i_0(a). \tag{4.3}$$

Let $s^*, e^*, i^*, \lambda^* \in X_{T,+} \setminus \{0\}$ be a nontrivial periodic solution satisfying system (4.1). Then from the integration along the characteristic line $t - a$ =constant, we have

$$\begin{aligned} s^*(t,a) =& e^{-\int_0^a (\lambda^*(t-a+\tau,\tau)+\rho(\tau))d\tau}, \\ e^*(t,a) =& \int_0^a \lambda^*(t-a+\tau,\tau)s^*(t-a+\tau,\tau)e^{-\int_\tau^a \sigma(\eta)d\eta}d\tau \\ =& \int_0^a \lambda^*(t-\tau,a-\tau)e^{-\int_\tau^a (\lambda^*(t-\eta,a-\eta)+\rho(a-\eta))d\eta}e^{-\int_0^\tau \sigma(a-\eta)d\eta}d\tau \end{aligned}$$

and

$$\begin{aligned} i^*(t,a) =& \int_0^a \sigma(\xi)e^*(t-a+\xi,\xi)e^{-\int_\xi^a \gamma(\delta)d\delta}d\xi \\ =& \int_0^a \sigma(a-\xi)e^*(t-\xi,a-\xi)e^{-\int_0^\xi \gamma(a-\eta)d\eta}d\xi \\ =& \int_0^a \sigma(a-\xi)\int_\xi^a \lambda^*(t-\tau,a-\tau) \\ & \times e^{-\int_\tau^a (\lambda^*(t-\eta,a-\eta)+\rho(a-\eta))d\eta}e^{-\int_\xi^\tau \sigma(a-\eta)d\eta}d\tau e^{-\int_0^\xi \gamma(a-\eta)d\eta}d\xi. \end{aligned}$$

From the above equation of $i^*(t,a)$ and the equation of λ, we obtain an integral equation of λ^* as follows:

$$\begin{aligned} \lambda^*(t,a) =& \int_0^{a^+} \beta(t;a,a')P^*(t,a')\int_0^{a'} \sigma(a'-\xi)\int_\xi^{a'} \lambda^*(t-\tau,a'-\tau) \\ & \times e^{-\int_\tau^{a'} (\lambda^*(t-\eta,a'-\eta)+\rho(a'-\eta))d\eta}e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta}d\tau e^{-\int_0^\xi \gamma(a'-\eta)d\eta}d\xi da', \end{aligned} \tag{4.4}$$

where $P^*(t,a) = Ae^{-\int_0^a \mu(\tau)d\tau}$ was assumed in Section 2. Thus, in order to find a nontrivial periodic solution of system (4.1), we only have to show the existence of a nontrivial periodic solution λ^* of (4.4). To this end, we define a nonlinear operator

$$\begin{aligned} F(\varphi)(t,a) :=& \int_0^{a^+} \beta(t;a,a')P^*(t,a')\int_0^{a'} \sigma(a'-\xi)\int_\xi^{a'} \varphi(t-\tau,a'-\tau) \\ & \times e^{-\int_\tau^{a'} (\varphi(t-\eta,a'-\eta)+\rho(a'-\eta))d\eta}e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta}d\tau \\ & \times e^{-\int_0^\xi \gamma(a'-\eta)d\eta}d\xi da' \end{aligned} \tag{4.5}$$

on $X_{T,+}$ and investigate the existence of its nontrivial fixed point in $X_{T,+} \setminus \{0\}$. It is easy to see that F has the strong Fréchet derivative

$$\mathscr{F}(\varphi)(t,a) := \int_0^{a^+} \beta(t;a,a')P^*(t,a') \int_0^{a'} \sigma(a'-\xi) \int_\xi^{a'} \varphi(t-\tau,a'-\tau) \times e^{-\int_\tau^{a'} \rho(a'-\eta)d\eta} e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta} d\tau e^{-\int_0^\xi \gamma(a'-\eta)d\eta} d\xi da' \tag{4.6}$$

at $0 \in X_{T,+}$. Now we make the following technical assumption, which is needed to ensure the compactness of the fixed point operator and its derivative.

Assumption 4.1. *(i) For the infection rate $\beta(t,a,a')$, it satisfies that*

$$\lim_{h\to 0} \int_0^T \int_0^{a^+} |\beta(t+h,a+h,a') - \beta(t,a,a')| da dt = 0 \text{ uniformly for } a' \in [0,a^+]$$

and $\beta(t,a,a') = 0$ for $a, a' \notin [0,a^+] \times [0,a^+]$;

(ii) For the reciprocal of the incubation period σ, it is integrable in $L^1(0,a^+)$ and $\sigma(a) = 0$ for $a \notin [0,a^+]$;

(iii) There exists a positive constant $\epsilon_0 > 0$ such that $\beta(t;a,a') \geq \epsilon_0$ for almost all $t \in \mathbb{R}_+$ and $a, a' \in [0,a^+]$;

(iv) The maximum attainable age is greater than or equal to the period of coefficients; that is, $a^+ \geq T$.

Lemma 4.2. *Under the Assumption 4.1, if $r(\mathscr{F}) > 0$, where $r(\mathscr{F})$ is the spectral radius of the operator $\mathscr{F}$, then it is a positive eigenvalue of $\mathscr{F}$ associated with a positive eigenvector $v_0 \in X_{T,+} \setminus \{0\}$.*

Proof. First we show that $\mathscr{F}$ is regarded as a compact operator in $L^1([0,T]\times[0,a^+])$. Observe that $\mathscr{F}$ is a linear map from $X_{T,+}$ into itself preserving the cone invariant, and we have

$$\begin{aligned}
(\mathscr{F}\varphi)(t,a) &= \int_0^{a^+} \int_0^{a'} \int_\xi^{a'} \beta(t;a,a')P^*(t,a')\sigma(a'-\xi)\varphi(t-\tau,a'-\tau) \\
&\quad \times e^{-\int_\tau^{a'} \rho(a'-\eta)d\eta} e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta} e^{-\int_0^\xi \gamma(a'-\eta)d\eta} d\tau d\xi da' \\
&= \int_0^{a^+} \int_\xi^{a^+} \int_\tau^{a^+} \beta(t;a,a')P^*(t,a')\sigma(a'-\xi)\varphi(t-\tau,a'-\tau) \\
&\quad \times e^{-\int_\tau^{a'} \rho(a'-\eta)d\eta} e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta} e^{-\int_0^\xi \gamma(a'-\eta)d\eta} da' d\tau d\xi \\
&= \int_0^{a^+} \int_\xi^{a^+} \int_0^{a^+-\tau} \beta(t;a,\tau+\chi)P^*(t,\tau+\chi)\sigma(\tau+\chi-\xi)\varphi(t-\tau,\chi) \\
&\quad \times e^{-\int_\tau^{\tau+\chi} \rho(\tau+\chi-\eta)d\eta} e^{-\int_\xi^\tau \sigma(\tau+\chi-\eta)d\eta} e^{-\int_0^\xi \gamma(\tau+\chi-\eta)d\eta} d\chi d\tau d\xi \\
&= \int_0^{a^+} \int_{t-a^+}^{t-\xi} \int_0^{a^+-t+s} \Big[\beta(t;a,t-s+\chi)P^*(t,t-s+\chi) \\
&\quad \times \sigma(t-s+\chi-\xi)\varphi(s,\chi)e^{-\int_{t-s}^{t-s+\chi} \rho(t-s+\chi-\eta)d\eta} \\
&\quad \times e^{-\int_\xi^{t-s} \sigma(t-s+\chi-\eta)d\eta} e^{-\int_0^\xi \gamma(t-s+\chi-\eta)d\eta}\Big] d\chi ds d\xi \\
&= \int_{t-a^+}^t \int_0^{a^+-t+s} \int_0^{t-s} \Big[\beta(t;a,t-s+\chi)P^*(t,t-s+\chi)
\end{aligned}$$

$$\times \sigma(t-s+\chi-\xi)\varphi(s,\chi)e^{-\int_{t-s}^{t-s+\chi}\rho(t-s+\chi-\eta)d\eta}$$
$$\times e^{-\int_{\xi}^{t-s}\sigma(t-s+\chi-\eta)d\eta}e^{-\int_0^{\xi}\gamma(t-s+\chi-\eta)d\eta}\Big]d\xi d\chi ds, \tag{4.7}$$

where in the last step we changed the order of triple integral. Next, define

$$G(t-s,\chi):=\int_0^{t-s}\sigma(t-s+\chi-\xi)e^{-\int_{\xi}^{t-s}\sigma(t-s+\chi-\eta)d\eta}e^{-\int_0^{\xi}\gamma(t-s+\chi-\eta)d\eta}d\xi.$$

Then we can rewrite (4.7) as

$$(\mathscr{F}\varphi)(t,a)=\int_{t-a^+}^{t}\int_0^{a^+-t+s}\beta(t;a,t-s+\chi)P^*(t,t-s+\chi)G(t-s,\chi)$$
$$\times e^{-\int_{t-s}^{t-s+\chi}\rho(t-s+\chi-\eta)d\eta}\varphi(s,\chi)d\chi ds. \tag{4.8}$$

If we extend the domain of parameters as $\beta(t;a,a')=0$ for $(a,a')\notin[0,a^+]\times[0,a^+]$ and $\sigma(a)=0$ for $a\notin[0,a^+]$, we can rewrite (4.8) as

$$(\mathscr{F}\varphi)(t,a)=\int_{-\infty}^{t}\int_0^{a^+}\beta(t;a,t-s+\chi)P^*(t,t-s+\chi)G(t-s,\chi)$$
$$\times e^{-\int_{t-s}^{t-s+\chi}\rho(t-s+\chi-\eta)d\eta}\varphi(s,\chi)d\chi ds. \tag{4.9}$$

Next, we can use the periodization technique as in [31, Lemma 4.2], [23, 9.2.2] or [3, Section 2] to write (4.9) as

$$(\mathscr{F}\varphi)(t,a)=\int_0^T\int_0^{a^+}\hat{\Psi}(t,a,s,\chi)\varphi(s,\chi)d\chi ds,$$

where

$$\hat{\Psi}(t,a,s,\chi):=\begin{cases}\sum_{n=0}^{\infty}\Psi(t,a,t-s+nT,\chi) & \text{for} \quad t>s,\\ \sum_{n=1}^{\infty}\Psi(t,a,t-s+nT,\chi) & \text{for} \quad t<s\end{cases} \tag{4.10}$$

and

$$\Psi(t,a,z,\chi):=\beta(t;a,z+\chi)P^*(t,z+\chi)G(z,\chi)e^{-\int_z^{z+\chi}\rho(z+\chi-\eta)d\eta}.$$

Hence, we can regard $\mathscr{F}$ as an operator in $L^1([0,T]\times[0,a^+])$. From Assumption 4.1 and the well-known compactness criterion in L^1, for example see [6, Theorem 4.26], we see that $\mathscr{F}$ is compact in $L^1([0,T]\times[0,a^+])$. Since $\mathscr{F}$ is positive, linear and compact, it follows from the Krein-Rutman theorem that if $r(\mathscr{F})>0$, then it is a positive eigenvalue of $\mathscr{F}$ associated with a positive eigenvector $\tilde{v}_0\in L^1_+([0,T]\times[0,a^+])\setminus\{0\}$. That is,

$$(\mathscr{F}\tilde{v}_0)(t,a)=r(\mathscr{F})\tilde{v}_0(t,a).$$

Hence, we can see that there exists a periodic eigenvector v_0 in $X_{T,+}\setminus\{0\}$ of $\mathscr{F}$, which is associated with the eigenvalue $r(\mathscr{F})$ and is the periodization of $\tilde{v}_0$. □

Next, we use the idea in Kuniya [29] and Kuniya and Inaba [31] to prove our main result. First we introduce a fixed point theorem in Inaba [19], based on the Krasnoselskii fixed point theorem (Krasnoselskii [27]). Let E be a real Banach space and E^* be its dual space. Let E_+ be the positive *cone* of E and E^*_+ be the *dual cone* of E; that is, the subset of E^* consisting of all positive linear functionals on E. For $\phi\in E$ and $f\in E^*$, we write the value of f at ϕ as $\langle f,\phi\rangle$. Then we say that $f\in E^*_+\setminus\{0\}$ is *strictly positive* if $\langle f,\phi\rangle>0$ holds for every $\phi\in E_+\setminus\{0\}$. Moreover, we say that $\phi \in E_+$ is a *quasi-interior point* if $\langle f, \phi \rangle > 0$ holds for all $f \in E_+^*$. Denote the set of bounded linear operators from E to E by $B(E)$. Then we say that a positive operator $T \in B(E)$ is *nonsupporting* if for every pair $\phi \in E_+ \setminus \{0\}$ and $f \in E_+^* \setminus \{0\}$, there exists a positive integer $p = p(\phi, f)$, such that $\langle f, T^n \phi \rangle > 0$ for all $n \geq p$. From the results in Marek [34] and Sawashima [35], we obtain the following proposition.

Proposition 4.3 (Marek [34] and Sawashima [35])**.** *Let E be a real Banach space and $T \in B(E)$ be compact and nonsupporting. Then*

(i) $r(T) \in P_\sigma(T) \setminus \{0\}$ and $r(T)$ is a simple pole of the resolvent, where $P_\sigma(\cdot)$ denotes the point spectrum of an operator;
(ii) The eigenspace corresponding to $r(T)$ is one-dimensional and the corresponding eigenvector $\varphi \in E_+$ is a quasi-interior point. The relation $T\phi = \mu\phi$ with $\phi \in E_+$ implies that $\phi = c\varphi$ for a constant c.

The fixed point theorem in Inaba [19] based on the Krasnoselskii fixed point theorem (Krasnoselskii [27]) can be stated as follows:

Theorem 4.4 (Inaba [19])**.** *Let E be a real Banach space with positive cone E_+ and F be a positive nonlinear operator on E. Suppose that*

(i) $F(0) = 0$ and the strong Fréchet derivative $\mathscr{F} := F'[0]$ exists at 0;
(ii) $\mathscr{F}$ has a positive eigenvector $v_0 \in X_+ \setminus \{0\}$ corresponding to an eigenvalue $\lambda_0 > 1$ and no eigenvector corresponding to eigenvalue 1;
(iii) F is compact and $F(E_+)$ is bounded.

Then F has a nontrivial fixed point in $E_+ \setminus \{0\}$.

We now use Proposition 4.3 and Theorem 4.4 to prove our main result.

Theorem 4.5. *Let F and $\mathscr{F}$ be defined as (4.5) and (4.6), respectively. If $r(\mathscr{F}) > 1$, then F has a nontrivial fixed point in $X_{T,+} \setminus \{0\}$.*

Proof. From (i) of Assumption 3.1 and (iii) of Assumption 4.1, we have an inequality

$$\mathscr{F}(\varphi)(t,a) \geq \epsilon_0 e^{-(\rho^+ + \sigma^+ + \gamma^+)a^+} \int_0^{a^+} \int_0^{a'} \int_\xi^{a'} P^*(t,a')\sigma(a'-\xi)\varphi(t-\tau, a'-\tau) d\tau d\xi da'.$$

Then we have

$$\begin{aligned}
\mathscr{F}^2(\varphi)(t,a) \geq & \epsilon_0^2 e^{-2(\rho^+ + \sigma^+ + \gamma^+)a^+} \int_0^{a^+} \int_0^{a'} \int_\xi^{a'} P^*(t,a')\sigma(a'-\xi) \\
& \times \int_0^{a^+} \int_0^{\eta} \int_\omega^{\eta} P^*(t-\tau,\eta)\sigma(\eta-\omega) \\
& \times \varphi(t-\tau-\delta, \eta-\delta) d\delta d\omega d\eta d\tau d\xi da' \\
\geq & \epsilon_0^2 A^2 e^{-2(\rho^+ + \sigma^+ + \gamma^+)a^+} \int_0^{a^+} \int_0^{a'} \int_\xi^{a'} e^{-\int_0^{a'} \mu(s)ds} \sigma(a'-\xi) \\
& \times \int_0^{a^+} \int_0^{\eta} \int_\omega^{\eta} e^{-\int_0^{\eta} \mu(s)ds} \sigma(\eta-\omega) \\
& \times \varphi(t-\tau-\delta, \eta-\delta) d\delta d\omega d\eta d\tau d\xi da'.
\end{aligned} \tag{4.11}$$

Now let us define a linear functional $H \in X_T^*$ on X_T by

$$\langle H, \varphi \rangle := \epsilon_0^2 A^2 e^{-2(\rho^+ + \sigma^+ + \gamma^+)a^+} \int_0^{a^+} \int_0^{a'} \int_\xi^{a'} e^{-\int_0^{a'} \mu(s)ds} \sigma(a'-\xi)$$

$$\times \int_0^{a^+} \int_0^{\eta} \int_{\omega}^{\eta} e^{-\int_0^{\eta} \mu(s)ds} \sigma(\eta - \omega)$$
$$\times \varphi(t - \tau - \delta, \eta - \delta) d\delta d\omega d\eta d\tau d\xi da', \quad \varphi \in X_T.$$

Then from (ii) and (iv) of Assumption 4.1, we see that $H \in X^*_{T,+}$ is a strictly positive linear functional. Thus, since $\mathscr{F}^2\varphi \geq \langle H, \varphi \rangle \mathbf{e}$, $\mathbf{e} = 1 \in X_{T,+}$ follows from (4.11), we have $\langle f, \mathscr{F}^2\varphi \rangle \geq \langle H, \varphi \rangle \langle f, \mathbf{e} \rangle > 0$ for every pair $\varphi \in X_{T,+} \setminus \{0\}$ and $f \in X^*_{T,+} \setminus \{0\}$, which implies that $\mathscr{F}\varphi \in X_{T,+} \setminus \{0\}$, $\forall \varphi \in X_{T,+} \setminus \{0\}$. It follows that $\mathscr{F}^3\varphi = \mathscr{F}^2(\mathscr{F}\varphi) \geq \langle H, \mathscr{F}\varphi \rangle \mathbf{e}$ and hence $\langle f, \mathscr{F}^3\varphi \rangle \geq \langle H, \mathscr{F}\varphi \rangle \langle f, \mathbf{e} \rangle > 0$ for every pair $\varphi \in X_{T,+} \setminus \{0\}$ and $f \in X^*_{T,+} \setminus \{0\}$. Similarly, we can prove that for all $n \geq 2$ that $\langle f, \mathscr{F}^n\varphi \rangle > 0$ holds for every pair $\varphi \in X_{T,+} \setminus \{0\}$ and $f \in X^*_{T,+} \setminus \{0\}$. Consequently, $\mathscr{F}$ is nonsupporting.

Next we can show the compactness of F similarly by the case of $\mathscr{F}$ in Lemma 4.2. And the boundedness of $F(X_{T,+})$ follows from the inequality

$$\begin{aligned}
F(\varphi)(t,a) \leq & \int_0^{a^+} \beta(t; a, a') P^*(t, a') \int_0^{a'} \sigma(a' - \xi) \\
& \times \int_{\xi}^{a'} \varphi(t - \tau, a' - \tau) e^{-\int_{\tau}^{a'} \varphi(t-\eta, a'-\eta) d\eta} d\tau d\xi da' \\
\leq & \beta^+ \int_0^{a^+} P^*(t, a') \int_0^{a^+} \sigma(a' - \xi) d\xi \left(1 - e^{-\int_{\xi}^{a'} \varphi(t-\eta, a'-\eta) d\eta} \right) da' \\
\leq & \beta^+ \int_0^{a^+} P^*(t, a') \int_0^{a^+} \sigma(a' - \xi) d\xi da' < \infty, \quad \forall \varphi \in X_{T,+},
\end{aligned}$$

where $\beta^+ = \sup_{t \in [0,T]} \operatorname{ess\,sup}_{(a,a') \in [0,a^+]^2} \beta(t; a, a') < \infty$.

Finally, by Proposition 4.3 we see that $r(\mathscr{F})$ is a simple eigenvalue of $\mathscr{F}$ and the corresponding eigenvector is positive. Moreover, since $r(\mathscr{F}) > 1$, it follows from (ii) of Proposition 4.3 that $\mathscr{F}$ has no eigenvector corresponding to eigenvalue 1. Thus (ii) of Theorem 4.4 holds. Since (i) is obvious and (iii) follows from the above argument, we obtain that F has a nontrivial fixed point in $X_{T,+} \setminus \{0\}$. $\square$

Now the existence of a nontrivial fixed point of operator F implies the existence of a nontrivial periodic solution of the original system (2.1). Moreover, we are interested in using the sign of $\mathscr{R}_0 - 1$ to check the existence of a nontrivial periodic solution of the original system (2.1).

Suppose that the system is in the diseases-free state $(S^*(t,a), 0, 0, R^*(t,a))$, where $S^*(t,a) + R^*(t,a) = P^*(t,a)$. From the first equation of (2.1) we can compute $S^*(t,a) = Ae^{-\int_0^a (\mu(\sigma)+\rho(\sigma))d\sigma}$. Further, from the second and third equations of (2.1), we obtain a linearized system

$$\begin{cases} \dfrac{\partial E}{\partial t} + \dfrac{\partial E}{\partial a} = \lambda(t,a) S^*(t,a) - (\sigma(a) + \mu(a)) E(t,a), \\ \dfrac{\partial I}{\partial t} + \dfrac{\partial I}{\partial a} = \sigma(a) E(t,a) - (\gamma(a) + \mu(a)) I(t,a). \end{cases} \tag{4.12}$$

Integrating the first equation of (4.12) along the characteristic line $t - a =$constant yields

$$E(t,a) = \int_0^a \lambda(t - a + \tau, \tau) S^*(t - a + \tau, \tau) e^{-\int_{\tau}^{a} (\sigma(\rho)+\mu(\rho)) d\rho} d\tau$$

$$
\begin{aligned}
&= A \int_0^a \lambda(t-\tau, a-\tau) e^{-\int_\tau^a (\mu(a-\eta)+\rho(a-\eta))d\eta} e^{-\int_0^\tau (\sigma(a-\eta)+\mu(a-\eta))d\eta} d\tau \\
&= A \int_0^a \lambda(t-\tau, a-\tau) e^{-\int_0^a \mu(a-\eta)d\eta} e^{-\int_\tau^a \rho(a-\eta)d\eta} e^{-\int_0^\tau \sigma(a-\eta)d\eta} d\tau. \quad (4.13)
\end{aligned}
$$

Substituting it into the second equation of (4.12), we obtain via integrating along the characteristic line $t-a$ =constant that

$$
\begin{aligned}
I(t,a) &= \int_0^a \sigma(\xi) E(t-a+\xi, \xi) e^{-\int_\xi^a (\gamma(\delta)+\mu(\delta))d\delta} d\xi \\
&= \int_0^a \sigma(a-\xi) E(t-\xi, a-\xi) e^{-\int_0^\xi (\gamma(a-\eta)+\mu(a-\eta))d\eta} d\xi \\
&= A \int_0^a \sigma(a-\xi) \int_\xi^a \lambda(t-\tau, a-\tau) e^{-\int_\xi^a \mu(a-\eta)d\eta} e^{-\int_\tau^a \rho(a-\eta)d\eta} \\
&\quad \times e^{-\int_\xi^\tau \sigma(a-\eta)d\eta} d\tau e^{-\int_0^\xi (\gamma(a-\eta)+\mu(a-\eta))d\eta} d\xi \\
&= A \int_0^a \sigma(a-\xi) \int_\xi^a \lambda(t-\tau, a-\tau) e^{-\int_0^a \mu(a-\eta)d\eta} e^{-\int_\tau^a \rho(a-\eta)d\eta} \\
&\quad \times e^{-\int_\xi^\tau \sigma(a-\eta)d\eta} d\tau e^{-\int_0^\xi \gamma(a-\eta)d\eta} d\xi. \quad (4.14)
\end{aligned}
$$

Insert it into the equation of λ defined in (2.7), we obtain by changing the integral order that

$$
\begin{aligned}
\lambda(t,a) &= A \int_0^{a^+} \beta(t; a, a') \int_0^{a'} \sigma(a'-\xi) \int_\xi^{a'} \lambda(t-\tau, a'-\tau) \\
&\quad \times e^{-\int_0^{a'} \mu(a'-\eta)d\eta} e^{-\int_\tau^{a'} \rho(a'-\eta)d\eta} e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta} d\tau e^{-\int_0^\xi \gamma(a'-\eta)d\eta} d\xi da' \\
&= A \int_0^{a^+} \int_\tau^{a^+} \int_0^\tau \beta(t; a, a') \sigma(a'-\xi) \lambda(t-\tau, a'-\tau) e^{-\int_0^{a'} \mu(a'-\eta)d\eta} \\
&\quad \times e^{-\int_\tau^{a'} \rho(a'-\eta)d\eta} e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta} e^{-\int_0^\xi \gamma(a'-\eta)d\eta} d\xi da' d\tau. \quad (4.15)
\end{aligned}
$$

Define a linear operator $B(t,\tau)$ from $L^1(0,a^+)$ into itself by

$$
\begin{aligned}
(B(t,\tau)\psi)(a) :=& A \int_\tau^{a^+} \int_0^\tau \beta(t; a, a') \sigma(a'-\xi) e^{-\int_0^{a'} \mu(a'-\eta)d\eta} e^{-\int_\tau^{a'} \rho(a'-\eta)d\eta} \\
& \times e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta} e^{-\int_0^\xi \gamma(a'-\eta)d\eta} \psi(a'-\tau) d\xi da'.
\end{aligned}
$$

Then (4.15) can be written as an abstract homogeneous renewal equation:

$$
\lambda(t,a) := \int_0^{a^+} (B(t,\tau)\lambda(t-\tau))(a) d\tau.
$$

From the periodic renewal theorem [37, 21], we see that $\lambda(t,a)$ is asymptotically proportional to an exponential solution $e^{r_0 t} w(t)$ growing with a Malthusian parameter r_0. The Malthusian parameter r_0 is a real root of the characteristic equation $r(\hat{B}(z)) = 1$, where $\hat{B}(z), z \in \mathbb{C}$, is a linear operator on X_T defined by

$$
(\hat{B}(z)\psi)(t) := \int_0^{a^+} e^{-zt} B(t,\tau) \psi(t-\tau) d\tau,
$$

and $w \in X_T$ is a positive eigenfunction of $\hat{B}(r_0)$ associated with the positive eigenvalue unity. Therefore, the sign relation $\mathrm{sign}(r_0) = \mathrm{sign}(r(\hat{B}(0)) - 1)$ holds. Then

$\mathscr{R}_0$ in periodic environment is defined as the spectral radius of the next generation operator $K : X_T \to X_T$ given by

$$K(\varphi)(t,a) := (\hat{B}(0)\varphi)(t) = \int_0^{a^+} B(t,\tau)\varphi(t-\tau)d\tau, \quad \varphi \in X_T.$$

We see that K is a positive operator from X_T into itself:

$$(K\varphi)(t,a) = A\int_0^{a^+}\int_\tau^{a^+}\int_0^\tau \beta(t;a,a')\sigma(a'-\xi)e^{-\int_0^{a'}\mu(a'-\eta)d\eta}e^{-\int_\tau^{a'}\rho(a'-\eta)d\eta} \times e^{-\int_\xi^\tau \sigma(a'-\eta)d\eta}e^{-\int_0^\xi \gamma(a'-\eta)d\eta}\varphi(t-\tau,a'-\tau)d\xi da' d\tau. \tag{4.16}$$

As in [4, 22], we obtain the asymptotic per generation growth factor of the infected population

$$\mathscr{R}_0 = r(K). \tag{4.17}$$

Moreover, assuming that the Malthusian parameter r_0 for population growth equals zero, it follows from [31, Proposition 7.1] that $r(\mathscr{F}) = r(K) = \mathscr{R}_0$. In summary, we have the following main theorem in this paper.

Theorem 4.6. *Let $\mathscr{R}_0$ be defined in (4.17).*

(i) If $\mathscr{R}_0 > 1$, then system (2.1) has a nontrivial periodic solution $(S^*, E^*, I^*, R^*) \in (X_{T,+} \setminus \{0\})^4$.

(ii) If $\mathscr{R}_0 < 1$, then system (2.1) has no nontrivial periodic solution in $(X_{T,+} \setminus \{0\})^4$.

Proof. First (i) follows immediately from Theorem 4.5. Now let us prove (ii). On the contrary, if system (2.1) has a nontrivial periodic solution $(S^*, E^*, I^*, R^*) \in (X_{T,+}\setminus\{0\})^4$, then operator F has a nontrivial fixed point $\varphi^* = F(\varphi^*)$ in $X_{T,+}\setminus\{0\}$. Since $\varphi^* = F(\varphi^*) \le \mathscr{F}(\varphi^*)$, we have $r(\mathscr{F}) = \mathscr{R}_0 \ge 1$, which is a contradiction. ☐

5. Numerical examples.

In this section, we provide some numerical examples to illustrate our results obtained in the previous sections. For simplicity, we consider the case where $\beta(t;a,a') = \beta(t) = c_1(1 + c_2\cos(2\pi t))$ is only time-periodic and independent of the age variable a, $\rho(a) = \rho$, $\gamma(a) = \gamma$, and $\sigma(a) = \sigma$ are constant. We also choose $\mu(a) = 10^{-4} \times (a-30)^2$. We consider two scenarios in the following.

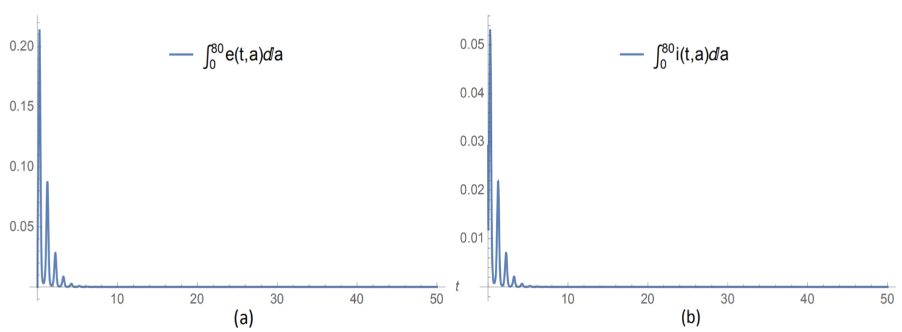

FIGURE 1. Behavior of the model when $\mathscr{R}_0 < 1$: (a) Total exposed population $\int_0^{80} e(t,a)da$ versus time t; (b) Total infected population $\int_0^{80} i(t,a)da$ versus time t.

5.1. **No vaccination.** When the vaccination rate $\rho = 0$, from our previous results we expect that system (2.6) has a unique periodic solution with period 1 when $\mathscr{R}_0 > 1$. In fact, system (2.6) has no nontrivial periodic solution when $\mathscr{R}_0 < 1$ in Figure 1. Figure 2 demonstrates that solutions $\int_0^{80} s(t,a)da$, $\int_0^{80} e(t,a)da$, and $\int_0^{80} i(t,a)da$ converge to periodic solutions with period 1.

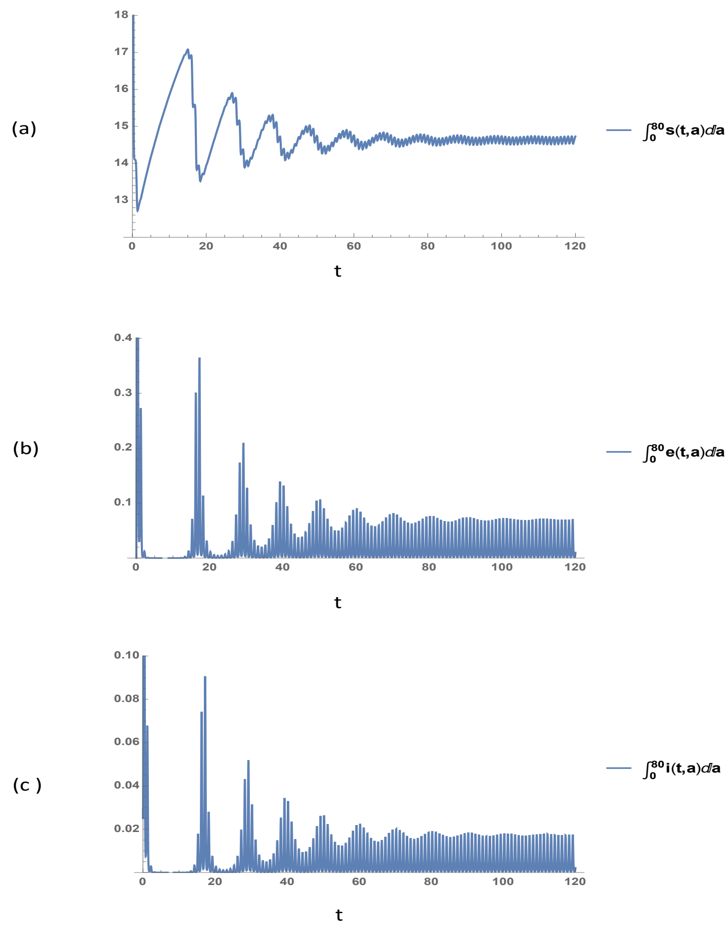

FIGURE 2. Behavior of solutions when $\mathscr{R}_0 > 1$ and the vaccination rate is zero: (a) Total susceptible population $\int_0^{80} s(t,a)da$ versus time t; (b) Total exposed population $\int_0^{80} e(t,a)da$ versus time t; (c) Total infected population $\int_0^{80} i(t,a)da$ versus time t.

5.2. **Effect of vaccination.** In Figure 3, we explore the effect of vaccination on the behavior of solutions. When there is no vaccination, Figure 3(a) indicates that there is a periodic solution with period 1. In particular, Figure 4 shows that the infected population and exposed population of this periodic solution versus age and time (in three periods), respectively, and Figure 5 presents the age distribution of the

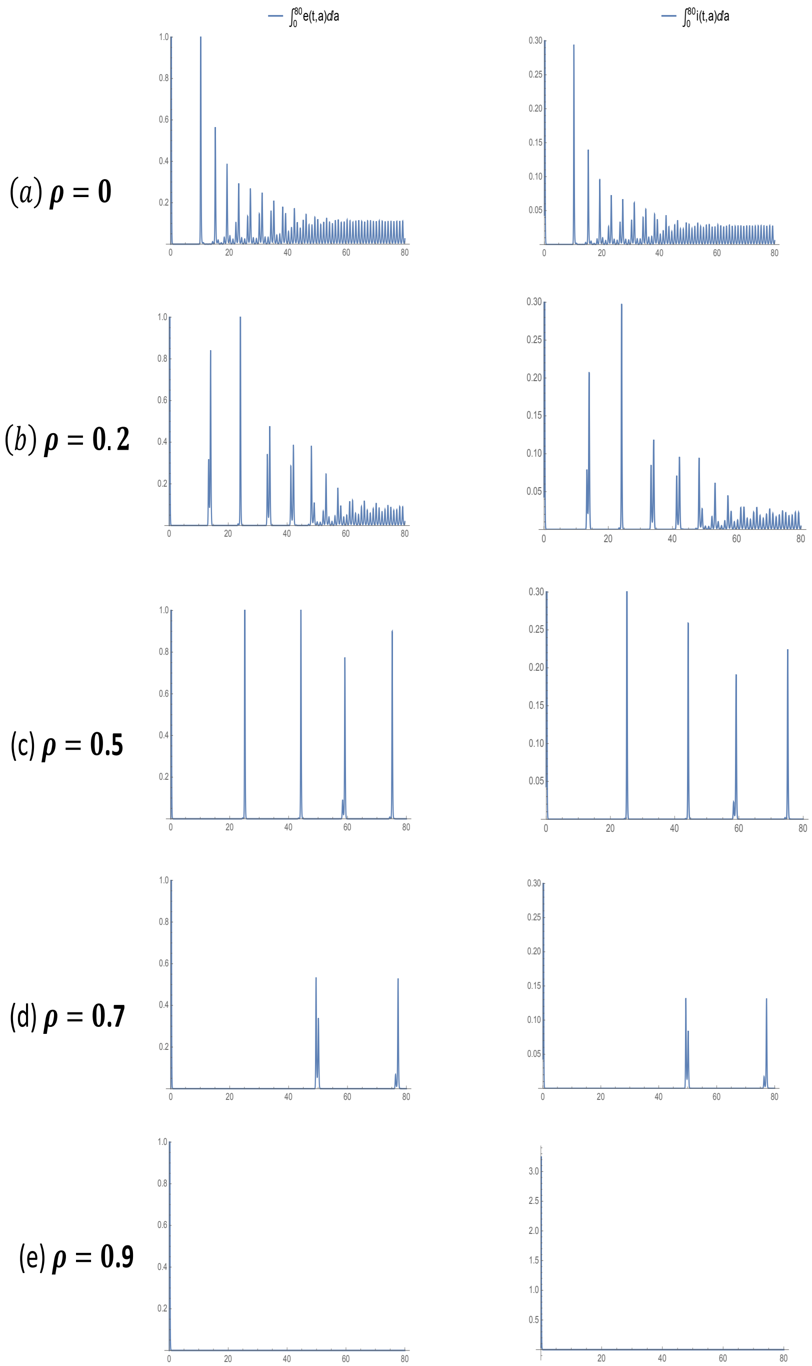

FIGURE 3. Effect of vaccination on the behavior of the solutions (the total exposed population $\int_0^{80} e(t,a)da$ and the total infected population $\int_0^{80} i(t,a)da$ versus time t) and different vaccination rate ρ: (a) $\rho = 0$; (b) $\rho = 0.5$; (c) $\rho = 0.7$; (d) $\rho = 0.9$. All other parameters are fixed.

infected population at peak time ($t = 60.3$), where most infected individuals come from young ages. By increasing the vaccination rate (Figure 3(b)-(e)), which is an essential control strategy for many infectious diseases, it is possible to successfully control the infectious disease.

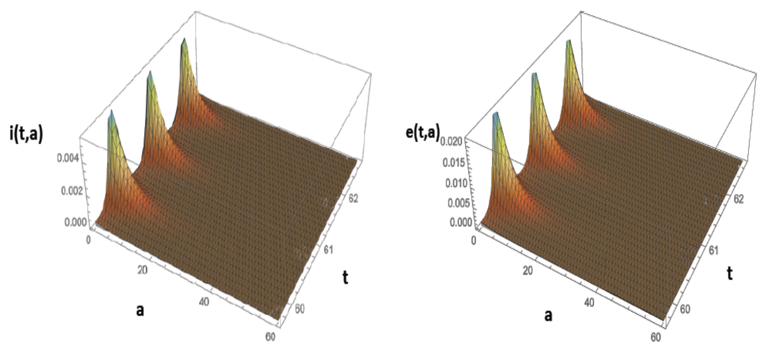

FIGURE 4. Plots of the infected population $i(t, a)$ and exposed population $e(t, a)$ versus age a and time t (in three periods).

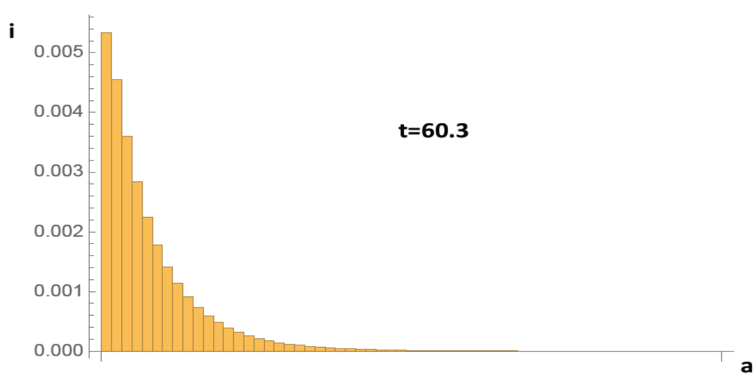

FIGURE 5. Age distribution of the infected population at the peak of a periodic solution ($t = 60.3$).

6. **Discussion.** Since one of the main issues in controlling some infectious diseases, such as measles, mumps, and pertussis, is to find the optimal age to vaccinate children in order to have the maximum impact on the incidence of disease-related morbidity and mortality for a given rate of vaccination coverage, age-structured epidemic models have been extensively used to study the transmission dynamics and control of infectious diseases (see Anderson and May [1], Andreasen [2], Bentil and Murray [5], Busenberg et al. [8, 7], Cha et al. [9], Feng et al. [11], Greenhalgh [12], Hethcote [13], Hoppensteadt [15], Iannelli [17], Iannelli et al. [18], Inaba [19, 23], Li et al. [33], Schenzle [36], and Tudor [38]). The outbreaks of some infectious diseases exhibit seasonal patterns, recently there has been considerable interest in investigating age-structured epidemic models with periodic parameters (Busenberg

et al. [7], Kubo and Langlais [28], Kuniya [29], Kuniya and Inaba [31], Kuniya and Iannelli [30], and Langlais and Busenberg [32]). For an age-structured SIS epidemic model with periodic parameters, Kuniya and Inaba [31] obtained a threshold value for the existence and uniqueness of a nontrivial endemic periodic solution.

Based on a periodic SEIR epidemic model describing measles (Earn et al. [10], Huang et al. [16]), in this paper we considered an age-structured SEIR model with periodic infection rate. We first established the well-posedness of the initial-boundary value problem for the age-structured SEIR model. Then we studied existence of time periodic solutions of the model by using a fixed point theorem and showed that there is also a threshold value for the existence and uniqueness of a nontrivial endemic periodic solution. Finally we provided some numerical simulations to illustrate the obtained results. Note that in this paper we only assumed that the infection rate β is time periodic. In fact the results still remain valid if all parameters are time periodic, that is $\mu(a) = \mu(a,t), \rho(a) = \rho(a,t), \sigma(a) = \sigma(a,t)$ and $\gamma(a) = \gamma(a,t)$, see [31, 29].

It will be interesting to consider the stability of periodic solutions and optimal age vaccinations in age-structured epidemic model with periodic parameters. Also, it will be interesting to find age distribution data on some infectious diseases, such as measles, mumps, and pertussis, and use model (2.1) to calibrate the data.

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