

## GLOBAL DYNAMICS OF A PREDATOR-PREY SYSTEM WITH DENSITY-DEPENDENT MORTALITY AND RATIO-DEPENDENT FUNCTIONAL RESPONSE

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**ABSTRACT.** In this paper, we study the global dynamics of a density-dependent predator-prey system with ratio-dependent functional response. The main features and challenges are that the origin of this model is a degenerate equilibrium of higher order and there are multiple positive equilibria. Firstly, local qualitative behavior of the system around the origin is explicitly described. Then, based on the dynamics around the origin and other equilibria, global qualitative analysis of the model is carried out. Finally, the existence of Bogdanov-Takens bifurcation (cusp case) of codimension two is analyzed. This shows that the system undergoes various bifurcation phenomena, including saddle-node bifurcation, Hopf bifurcation, and homoclinic bifurcation along with different topological sectors near the degenerate origin. Numerical simulations are presented to illustrate the theoretical results.

**1. Introduction.** Since the early twentieth century, modelling predator-prey interactions has attracted great attention from applied mathematicians and theoretical biologists and various types of predator-prey models have been developed. Ratio-dependent predator-prey models were proposed in the late 1980s and have been extensively studied since then. In 1989, Arditi and Ginzburg [3] provided evidence and empirical observations to suggest that ratio-dependent predator-prey models give more reasonable predictions for the complex, heterogeneous systems with slow dynamics, where the final (large scale) outcome of predation is a sharing process, such as large terrestrial carnivores and their prey. Akcakaya et al. [2] also referred to the resource sharing and interference as the mechanistic base of ratio dependence. Gutierrez [10] developed the physiological basis for the ratio-dependent theory and used a metabolic pool model of Nicholson's blowflies for explanation. Akcakaya et al.

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[1] constructed a mathematical model of mammals to predict some specific patterns of population cycles, which showed the reasonability of the ratio-dependent predation hypothesis for some mammals. Based on these studies, in the past two decades the following predator-prey system with ratio-dependent functional response

$$\begin{cases} x'(t) = x(t) \left[ c - bx(t) - \frac{sy(t)}{x(t)+my(t)} \right], \\ y'(t) = y(t) \left[ -d + \frac{fx(t)}{x(t)+my(t)} \right], \end{cases} \quad (1.1)$$

has been analyzed considerably. Here,  $x(t)$  and  $y(t)$  denote the population densities of the prey and predators at time  $t$ , respectively.  $c$  is the intrinsic growth rate of the prey and  $b$  is the mutual interference among prey population so that  $\frac{c}{b}$  is the carrying capacity of the prey in the absence of predators.  $d$  represents the death rate of predators. The predators consume prey according to the ratio-dependent functional response, where  $s, f$  and  $m$  stand for the capturing rate, conversion rate, and half saturation constant, respectively. All parameters of system (1.1) are positive.

Detailed studies on the dynamics of (1.1) have been done in some early studies. For instance, Kuang and Beretta [19] proved that if the positive steady state is locally asymptotically stable, then the system has no nontrivial positive periodic solutions, that is, positive solutions tend either to the origin or to the unique positive equilibrium. Jost et al. [18] investigated the asymptotic behavior of system (1.1) around the origin and demonstrated that  $(0, 0)$  can be either a saddle point or an attractor for certain trajectories, that is,  $(0, 0)$  has its own basin of attraction even when there exists a nontrivial equilibrium. A complete classification of the asymptotic behavior of the solutions for model (1.1) was provided in Hsu et al. [14]. Based on the qualitative behavior of system (1.1) at the origin, global qualitative analysis of the model depending on all parameters was carried out in Xiao and Ruan [30]. Berezovskaya et al. [6] showed the existence of eight qualitatively differential types of system behaviors for various parameter values. Versal unfoldings of system (1.1) were explored and all its possible bifurcations have been discussed in Ruan et al. [23]. Heteroclinic bifurcation in system (1.1) was studied by Ruan et al. [22], Kuang et al. [20], and Tang and Zhang [27].

However, in the natural world, predators can interfere with each other's activities and generate competition for various resources. Hence, it is more natural to incorporate both prey density dependence and predator density dependence into the predator-prey system [5]. In 1976, Bazykin [4] proposed a predator-prey model taking the density-dependent mortality of predators into consideration. Systematically qualitative analysis on the original Bazykin's model can be found in Hainzl [11, 12]. Motivated by these ecological observations, in this paper, we consider the following predator-prey model with both density-dependent mortality and ratio-dependent functional response:

$$\begin{cases} x'(t) = x(t) \left[ c - bx(t) - \frac{sy(t)}{x(t)+my(t)} \right], \\ y'(t) = y(t) \left[ -d - ry(t) + \frac{fx(t)}{x(t)+my(t)} \right]. \end{cases} \quad (1.2)$$

Compared with system (1.1) without density dependent death rate of predator species  $ry(t)^2$ , system (1.2) has richer and more complex dynamics. For system (1.2), local asymptotic stability of the positive steady state cannot guarantee the global asymptotic stability due to the effect of predator density dependence [17].

Inspired by the S-procedure [25] and semi-definite programming [21], by using the sum of squares decomposition (SOS) based method [32] to find polynomial Lyapunov functions, global asymptotic stability of the positive equilibrium of system (1.2) was discussed in [17]. However, the global topological structure of the model is not presented. For this purpose, we have to study the behavior of solutions of system (1.2) in the interior of the first quadrant.

Meanwhile, it is also challenging to investigate the bifurcation phenomena of system (1.2). By exploring the monotonic property of the trace of the Jacobian matrix with respect to the predator density dependence parameter  $r$ , the existence of Hopf bifurcation was analytically verified in [17]. According to the bifurcation theory in [9, 28], we know that under certain nondegeneracy conditions, some equilibrium can be a cusp. If we choose suitable bifurcation parameters, then the system undergoes a Bogdanov-Takens bifurcation. This implies that the system can undergo saddle-node bifurcation, Hopf bifurcation, and homoclinic bifurcation. Bogdanov-Takens bifurcation in different predator-prey models has been studied by some researchers recently. In Xiao and Ruan [29], Bogdanov-Takens bifurcation in predator-prey systems with constant rate harvesting was analyzed. In Ruan and Xiao [24], Bogdanov-Takens bifurcation of cusp type of codimension 2 in a predator-prey system with nonmonotonic functional response was studied. Huang et al. [15] investigated bifurcations in a Leslie type system with generalized Holling type III functional response and proved that the model can exhibit degenerate focus type Bogdanov-Takens bifurcation of codimension 3 for some parametric values. These bifurcation analytical results and corresponding numerical simulations predicted that the populations of the predator-prey systems oscillate and outbreak under different parameter conditions.

In this paper, we will explore the global structure of system (1.2) in  $\mathbb{R}_+^2 := \{(x, y) : x > 0, y > 0\}$ . To this purpose, letting  $t \rightarrow ct$ ,  $x \rightarrow \frac{b}{c}x$ ,  $y \rightarrow \frac{bm}{c}y$ ,  $s = \frac{s}{cm}$ ,  $b = \frac{f}{c}$ ,  $d = \frac{d}{f}$  and  $r = \frac{cr}{bfm}$ , we can simplify system (1.2) as

$$\begin{cases} x'(t) = x(t) \left[ 1 - x(t) - \frac{sy(t)}{x(t)+y(t)} \right], \\ y'(t) = by(t) \left[ -d - ry(t) + \frac{x(t)}{x(t)+y(t)} \right]. \end{cases} \tag{1.3}$$

By geometrical analysis of hyperbolic curves, sufficient and necessary conditions on the existence of positive equilibria of system (1.3) have been given in [17]. By analyzing the bifurcation function, it was proved that the positive equilibrium is a saddle-node under some conditions. This implies the existence of a saddle-node bifurcation. The positive equilibrium can be a cusp under special parametric conditions. Thus, the Bogdanov-Takens bifurcation of system (1.3) can be further analyzed.

We first explore the local qualitative behavior of system (1.3) near the origin in  $\mathbb{R}_+^2$ . The origin is a critical point of higher order and system (1.3) exhibits various dynamic phenomena nearby. There exist parabolic sectors, hyperbolic sectors, elliptic sectors, and some combinations of them in the neighborhood of the origin. These structures have important implications on the global behavior of the model. Based on the dynamics around the origin and equilibria, we study the global dynamics of system (1.3). Lastly, we focus on the existence of bifurcations that occur in a two-dimensional parameter region. Under some conditions, we prove that system (1.3) undergoes Bogdanov-Takens bifurcation, that is, the bifurcation of a cusp of codimension 2 along with different topological sectors near the degenerate origin.

The rest of this paper is organized as follows. In section 2, the qualitative behavior of system (1.3) around the origin is investigated. Section 3 is devoted to the global dynamics of system (1.3). Section 4 explores the Bogdanov-Takens bifurcation. Conclusions can be found in Section 5 .

**2. Local qualitative behavior near the origin.** We analyze the global qualitative behavior and topological structures of system (1.3) in  $\mathbb{R}_+^2$ . However, since the ratio-dependent model (1.3) is not well defined at the origin  $(0, 0)$ , we need to explore its local qualitative behavior around the origin. To this end, similar to the analysis in [30], we redefine system (1.3) as

$$\begin{cases} x'(t) = x(t) \left[ 1 - x(t) - \frac{sy(t)}{x(t)+y(t)} \right], \\ y'(t) = by(t) \left[ -d - ry(t) + \frac{x(t)}{x(t)+y(t)} \right], \\ x'(t) = y'(t) = 0, \text{ when } x(t) = y(t) = 0. \end{cases} \tag{2.1}$$

Moreover, since system (2.1) cannot be linearized at  $(0, 0)$ , local stability of  $(0, 0)$  cannot be studied directly. By making a time scale change  $dt = (x + y)dt$ , we can transform system (2.1) into the following equivalent polynomial system

$$\begin{cases} x'(t) = X_2(x, y) + \Phi(x, y) := x^2 + (1 - s)xy - x^3 - x^2y, \\ y'(t) = Y_2(x, y) + \Psi(x, y) := b(1 - d)xy - bdy^2 - brxy^2 - bry^3, \end{cases} \tag{2.2}$$

where  $X_2$  and  $Y_2$  are homogeneous polynomials in  $x$  and  $y$  of degree 2 and  $\Phi(x, y) := -x^3 - x^2y$ ,  $\Psi(x, y) := -brxy^2 - bry^3$ . The equilibrium  $(0, 0)$  of the polynomial system (2.2) is an isolated critical point of higher order.

By the analysis results in section II.2 in [31], we know that no orbit of system (2.2) can tend to the critical point  $(0, 0)$  spirally. If the orbits of system (2.2) tend to the origin as a sequence  $\{t_n\}$  of  $t$  tends to  $\infty$  along a direction, then the direction is called a *characteristic direction*. These characteristic directions are given by solutions of the characteristic equation [31]. For this, we introduce the polar coordinate transformation  $x = \gamma \cos \theta$  and  $y = \gamma \sin \theta$ . Then the characteristic function of system (2.2) takes the form

$$\begin{aligned} G(\theta) &:= \cos \theta Y_2(\cos \theta, \sin \theta) - \sin \theta X_2(\cos \theta, \sin \theta) \\ &= \sin \theta \cos \theta [(s - 1 - bd) \sin \theta - (b - 1 - bd) \cos \theta]. \end{aligned}$$

The characteristic equation

$$G(\theta) = 0 \tag{2.3}$$

indicates the following two cases: (i)  $G(\theta) \equiv 0$  or (ii)  $G(\theta)$  has at most three real roots  $\theta_i (i = 1, 2, 3)$  in  $\Theta := \{\theta \mid 0 \leq \theta \leq \frac{\pi}{2}\}$ . If  $G(\theta)$  is not identically zero, then there are at most 3 directions  $\theta_i (i = 1, 2, 3)$  in  $\Theta$  along which at least one orbit of system (2.2) approaches the origin. These orbits of system (2.2) divide the neighborhood of the origin into various open regions, which are called sectors. For system (2.2), there exist three kinds of sectors: hyperbolic sectors, parabolic sectors and elliptic sectors [31]. According to the number of real roots of characteristic equation (2.3) in  $\Theta$ , we consider the following three cases.

**2.1.  $s - 1 - bd = 0$  and  $b - 1 - bd = 0$ .** In this case,  $G(\theta) \equiv 0$ . This is a singular case. By the Briot-Bouquet transformation  $y = ux$ , system (2.2) in  $\mathbb{R}_+^2$  is changed into

$$\begin{cases} x'(t) = x^2 + (1 - s)x^2u - x^3 - x^3u, \\ u'(t) = x^2u + (1 - br)x^2u^2 - brx^2u^3. \end{cases} \tag{2.4}$$

On the  $(x, u)$ -plane, system (2.4) can be written as

$$\frac{dx}{du} = \frac{1 - bdu - x - xu}{u + (1 - br)u^2 - br u^3}. \tag{2.5}$$

By conventional calculation, the solution of system (2.5) can be expressed by

$$x = \begin{cases} 1 + C_1 |br - \frac{1}{u}|^{\frac{br}{1+br}} |1 + \frac{1}{u}|^{\frac{1}{1+br}} + C_2 |\frac{bru-1}{u+1}|^{\frac{1}{1+br}} - bd, & x > 1, \\ 1 - C_1 |br - \frac{1}{u}|^{\frac{br}{1+br}} |1 + \frac{1}{u}|^{\frac{1}{1+br}} + C_2 |\frac{bru-1}{u+1}|^{\frac{1}{1+br}} - bd, & x \leq 1, \end{cases} \tag{2.6}$$

where  $C_i$  ( $i = 1, 2$ ) are arbitrary constants. So a general solution of system (2.2) in  $\mathbb{R}_+^2$  is

$$x = \begin{cases} 1 + C_1 |br - \frac{x}{y}|^{\frac{br}{1+br}} |1 + \frac{x}{y}|^{\frac{1}{1+br}} + C_2 |\frac{bry-x}{y+x}|^{\frac{1}{1+br}} - bd, & x > 1, \\ 1 - C_1 |br - \frac{x}{y}|^{\frac{br}{1+br}} |1 + \frac{x}{y}|^{\frac{1}{1+br}} + C_2 |\frac{bry-x}{y+x}|^{\frac{1}{1+br}} - bd, & x \leq 1. \end{cases}$$

The dynamic behavior can be seen in Figure 1. In the following, we only need to consider the non-singular case.

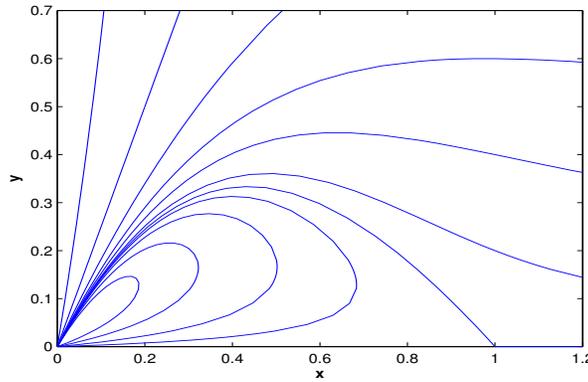


FIGURE 1. Phase diagram of system (2.2) with  $s = b = 2$ ,  $d = 0.5$  and  $r = 0.2$ .

2.2.  $(s - 1 - bd)(b - 1 - bd) = 0$  but one of them is not zero. In this case, characteristic equation (2.3) has two roots in  $\Theta$ :  $\theta_1 = 0$  and  $\theta_2 = \frac{\pi}{2}$ . To verify the existence and number of orbits of system (2.2) which tend to the origin along the direction  $\theta_i$  ( $i = 1, 2$ ) as  $t$  tends to  $\infty$ , we need to introduce the following auxiliary functions [31]

$$\begin{aligned} G'(\theta) &= \cos 2\theta[(s - 1 - bd) \sin \theta + (b - 1 - bd) \cos \theta] \\ &\quad + \sin \theta \cos \theta[(s - 1 - bd) \cos \theta - (b - 1 - bd) \sin \theta], \\ G''(\theta) &= -\frac{5}{2} \sin 2\theta[(s - 1 - bd) \sin \theta + (b - 1 - bd) \cos \theta] \\ &\quad + 2 \cos 2\theta[(s - 1 - bd) \cos \theta - (b - 1 - bd) \sin \theta], \\ H(\theta) &:= \sin \theta Y_2(\cos \theta, \sin \theta) + \cos \theta X_2(\cos \theta, \sin \theta) \\ &= \cos^3 \theta + (1 - s) \cos^2 \theta \sin \theta + b(1 - d) \cos \theta \sin^2 \theta - bd \sin^3 \theta. \end{aligned}$$

Now we classify different situations by virtue of the multiplicity of roots of characteristic equation (2.3).

2.2.1.  $s - 1 - bd = 0$  and  $b - 1 - bd \neq 0$ . In this case,  $\theta_1$  is a simple root of (2.3) and  $\theta_2$  is a multiple root with multiplicity 2. Through the corresponding analysis in [31], we obtain the following result.

**Theorem 2.1.** *Assume that  $s - 1 - bd = 0$  and  $b - 1 - bd \neq 0$ .*

- (I) *There exist  $\epsilon_1 > 0$  and  $\gamma_1 > 0$  such that*
- (i) *if  $b - 1 - bd > 0$ , then all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_1, 0 < \gamma < \gamma_1\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ;*
  - (ii) *if  $b - 1 - bd < 0$ , then there exists a unique orbit of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_1, 0 < \gamma < \gamma_1\}$  which tends to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ;*
- (II) *There exist  $\epsilon_2 > 0$  and  $\gamma_2 > 0$  such that all orbits of system (2.2) in  $\{(\theta, r) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_2, 0 < \gamma < \gamma_2\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ .*

*Proof.* When  $s - 1 - bd = 0$  and  $b - 1 - bd > 0$ , we have  $G'(\theta_1) = b - 1 - bd > 0$  and  $H(\theta_1) = 1 > 0$ . Then,  $G'(\theta_1)H(\theta_1) > 0$ . Thus, by Theorem 3.4 on page 76 of [31], there exist  $\epsilon_1 > 0$  and  $\gamma_1 > 0$  such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_1, 0 < \gamma < \gamma_1\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ .

On the other hand, if  $s - 1 - bd = 0$  and  $b - 1 - bd < 0$ , then  $G'(\theta_1)H(\theta_1) < 0$ . Note that  $\Phi(\gamma, \theta) = -\gamma^3 \cos^3 \theta - \gamma^3 \cos^2 \theta \sin \theta$  and  $\Psi(\gamma, \theta) = -br\gamma^3 \cos \theta \sin^2 \theta - br\gamma^3 \sin^3 \theta$ . Let  $C(\gamma) := \min\{7\gamma, 7br\gamma\}$ , then for  $0 < \theta^1, \theta^2 \ll 1$ , we have

$$\begin{aligned} & \frac{1}{\gamma^2} [\Phi(\gamma, \theta^2) - \Phi(\gamma, \theta^1)] \\ &= \gamma [(\cos \theta^1 - \cos \theta^2)(\cos^2 \theta^1 + \cos \theta^1 \cos \theta^2 + \cos^2 \theta^2) \\ & \quad + (1 - \sin^2 \theta^1) \sin \theta^1 - (1 - \sin^2 \theta^2) \sin \theta^2] \\ &= \gamma [(\sin(\frac{\pi}{2} - \theta^1) - \sin(\frac{\pi}{2} - \theta^2))(\cos^2 \theta^1 + \cos \theta^1 \cos \theta^2 + \cos^2 \theta^2) \\ & \quad + (\sin \theta^1 - \sin \theta^2)(1 + \sin^2 \theta^1 + \sin \theta^1 \sin \theta^2 + \sin^2 \theta^2)] \\ & \leq 7\gamma |\theta^2 - \theta^1| \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\gamma^2} [\Psi(\gamma, \theta^2) - \Psi(\gamma, \theta^1)] \\ &= br\gamma [(\sin \theta^1 - \sin \theta^2)(\sin^2 \theta^1 + \sin \theta^1 \sin \theta^2 + \sin^2 \theta^2) \\ & \quad + (1 - \cos^2 \theta^1) \cos \theta^1 - (1 - \cos^2 \theta^2) \cos \theta^2] \\ &= br\gamma [(\sin \theta^1 - \sin \theta^2)(\sin^2 \theta^1 + \sin \theta^1 \sin \theta^2 + \sin^2 \theta^2) \\ & \quad + (\sin(\frac{\pi}{2} - \theta^1) - \sin(\frac{\pi}{2} - \theta^2))(1 - \cos^2 \theta^1 - \cos \theta^1 \cos \theta^2 - \cos^2 \theta^2)] \\ & \leq 7br\gamma |\theta^2 - \theta^1|. \end{aligned}$$

Furthermore, we can see that

$$\begin{aligned} C(\gamma) &= \min\{7\gamma, 7br\gamma\} \rightarrow 0 \text{ as } \gamma \rightarrow 0, \\ \frac{1}{\gamma^2} \Phi(\gamma, \theta) &= -\gamma \cos^2 \theta (\sin \theta + \cos \theta) \rightarrow 0 \text{ as } \gamma \rightarrow 0, \\ \frac{1}{\gamma^2} \Psi(\gamma, \theta) &= -br\gamma \sin^2 \theta (\sin \theta + \cos \theta) \rightarrow 0 \text{ as } \gamma \rightarrow 0. \end{aligned}$$

Hence, the conclusion (I)(ii) follows from Theorem 3.7 on page 79 of [31].

Besides, when  $b - 1 - bd > 0$ , we obtain  $G'(\theta_2) = 0$ ,  $G''(\theta_2) = 2(b - 1 - bd) > 0$  and  $H(\theta_2) = -bd < 0$ . Then  $G''(\theta_2)H(\theta_2) \neq 0$ .

Let

$$D := \left(\frac{H(\theta_2)}{2}\right)^2 (G''(\theta_2))^{-1} = \frac{b^2 d^2}{8(b - 1 - bd)} > 0,$$

$$A(\gamma) := \gamma^2 \left(\ln \frac{1}{\gamma}\right)^{-2} = \left(\frac{\gamma}{\ln \gamma}\right)^2 > 0,$$

and

$$\begin{aligned} \eta(\gamma, \theta) &:= \cos \theta \Psi(\gamma, \theta) - \sin \theta \Phi(\gamma, \theta) \\ &= \gamma^3 \cos \theta \sin \theta [(1 - br) \cos \theta \sin \theta + \cos^2 \theta - br \sin^2 \theta] \\ &< 0 \text{ as } \theta \rightarrow \frac{\pi}{2}. \end{aligned}$$

We choose  $C_1 = \frac{1}{2}D$ , then  $\eta(\gamma, \theta) < C_1 A(\gamma)$  for  $0 < C_1 < D$ . Thus, due to Theorem 3.8 on page 85 of [31], if  $b - 1 - bd > 0$ , then there exist  $\epsilon_2 > 0$  and  $\gamma_2 > 0$ , such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_2, 0 < \gamma < \gamma_2\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ .

When  $b - 1 - bd < 0$ , the conclusion (II) can also be obtained in a similar manner. □

2.2.2.  $s - 1 - bd \neq 0$  and  $b - 1 - bd = 0$ . In this case,  $\theta_1$  is a multiple root with multiplicity 2 and  $\theta_2$  is a simple root of (2.3). By using a similar analysis as the proof of Theorem 2.1, we obtain the following theorem.

**Theorem 2.2.** *Suppose that  $s - 1 - bd \neq 0$  and  $b - 1 - bd = 0$ .*

- (I) *There exist  $\epsilon_3 > 0$  and  $\gamma_3 > 0$  such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_3, 0 < \gamma < \gamma_3\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ;*
- (II) *There exist  $\epsilon_4 > 0$  and  $\gamma_4 > 0$  such that*
  - (i) *if  $s - 1 - bd > 0$ , then all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_4, 0 < \gamma < \gamma_4\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ ;*
  - (ii) *if  $s - 1 - bd < 0$ , then there exists a unique orbit of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_4, 0 < \gamma < \gamma_4\}$  which tends to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ .*

2.3.  $(s - 1 - bd)(b - 1 - bd) \neq 0$ . In this case, since we only need to analyze equation (2.3) in  $\Theta$ , we discuss the following two subcases:

- (A) If  $(b - 1 - bd)(s - 1 - bd) > 0$ , then (2.3) has two simple roots:  $\theta_1 = 0$  and  $\theta_2 = \frac{\pi}{2}$ ;
- (B) If  $(b - 1 - bd)(s - 1 - bd) < 0$ , then (2.3) has three simple roots:  $\theta_1, \theta_2$  and  $\theta_3 = \arctan \frac{1+bd-b}{s-1-bd}$ .

Firstly, we analyze the orbits along  $\theta_1$  and  $\theta_2$  in cases (A) and (B). According to Theorems 3.4 and 3.7 in [31], the following theorem can be obtained.

**Theorem 2.3.** *Suppose that  $(b - 1 - bd)(s - 1 - bd) \neq 0$ .*

- (I) *There exist  $\epsilon_5 > 0$  and  $\gamma_5 > 0$  such that*
  - (i) *if  $b - 1 - bd > 0$ , then all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_5, 0 < \gamma < \gamma_5\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ;*
  - (ii) *if  $b - 1 - bd < 0$ , then there exists a unique orbit of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_5, 0 < \gamma < \gamma_5\}$  which tends to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ;*
- (II) *There exist  $\epsilon_6 > 0$  and  $\gamma_6 > 0$  such that*

- (i) if  $s - 1 - bd > 0$ , then all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_6, 0 < \gamma < \gamma_6\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ ;
- (ii) if  $s - 1 - bd < 0$ , then there exists a unique orbit of system (2.2) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_6, 0 < \gamma < \gamma_6\}$  which tends to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ .

In the following, we consider  $\theta_3$  in case (B).

**Theorem 2.4.** *Suppose that  $b - 1 - bd > 0$  and  $s - 1 - bd < 0$ .*

- (I) *If  $s < 1 + sd$ , then there exist  $\epsilon_7 > 0$  and  $\gamma_7 > 0$  such that there exists a unique orbit of system (2.2) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_3| < \epsilon_7, 0 < \gamma < \gamma_7\}$  that tends to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow -\infty$ ;*
- (II) *If  $s > 1 + sd$ , then there exist  $\epsilon_8 > 0$  and  $\gamma_8 > 0$  such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_3| < \epsilon_8, 0 < \gamma < \gamma_8\}$  tend to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow +\infty$ .*

*Proof.* By introducing the Briot-Bouquet transformation  $x = x, y = ux$  and  $dt = xdt$ , system (2.2) is transformed into

$$\begin{cases} x'(t) = x + (1 - s)xu - x^2 - x^2u, \\ u'(t) = (b - 1 - bd)xu + (s - 1 - bd)u^2 + [1 + (1 - br)u - bru^2]x^2u. \end{cases} \tag{2.7}$$

Note that the inverse transformation condenses the  $u$ -axis into one point. Thus, we should investigate the equilibria of system (2.7) in the  $u$ -axis. For system (2.7), there exist two equilibria,  $(0, 0)$  and  $(0, -\frac{b-1-bd}{s-1-bd})$ , on the  $u$ -axis.  $(0, 0)$  is an unstable node and we need to analyze the other one.

Letting  $x_1 = x$  and  $x_2 = u - \frac{b-1-bd}{s-1-bd}$ , we translate system (2.7) into

$$\begin{cases} x_1'(t) = A_1x_1 + (1 - s)x_1x_2 - \frac{s-b}{s-1-bd}x_1^2 - x_1^2x_2, \\ x_2'(t) = A_2x_1 + A_3x_2 + A_4x_1x_2 + A_5x_2^2 + A_6x_1x_2^2 - brx_1x_2^3, \end{cases} \tag{2.8}$$

where  $A_1 = \frac{b(s-1-sd)}{s-1-bd}$ ,  $A_2 = \frac{(s-b)(1+bd-b)}{(s-1-bd)^2} \left(1 + br\frac{b-1-bd}{s-1-bd}\right)$ ,  $A_3 = 1 + bd - b$ ,  $A_4 = 1 + 2(1 - br)\frac{1+bd-b}{s-1-bd} - 3br\left(\frac{1+bd-b}{s-1-bd}\right)^2$ ,  $A_5 = s - 1 - bd$  and  $A_6 = 1 - br + 3br\frac{b-1-bd}{s-1-bd}$ .

We can see that if  $b - 1 - bd > 0$  and  $s - 1 - bd < 0$ , then  $A_3 < 0$  holds. Moreover, if  $s - 1 - sd < 0$ , then  $A_1 > 0$  also holds. This implies that  $(0, 0)$  is a saddle point of system (2.8). Thus, the equilibrium  $(0, -\frac{b-1-bd}{s-1-bd})$  of system (2.7) is a saddle point. Hence, there exists a unique separatrix of this equilibrium in the interior of the first quadrant of system (2.7), which tends to  $(0, -\frac{b-1-bd}{s-1-bd})$  as  $t \rightarrow -\infty$ . Through the inverse Briot-Bouquet transformation, we can see that there exist  $\epsilon_7 > 0$  and  $\gamma_7 > 0$  such that system (2.2) has one unique orbit in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_3| < \epsilon_7, 0 < \gamma < \gamma_7\}$  that tends to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow -\infty$ .

However, when  $s - 1 - sd > 0$ , there holds  $A_1 < 0$ .  $(0, 0)$  is a stable node of system (2.8). Hence, the equilibrium  $(0, -\frac{b-1-bd}{s-1-bd})$  of system (2.7) is a stable node. By the inverse Briot-Bouquet transformation, we obtain that there exist  $\epsilon_8 > 0$  and  $\gamma_8 > 0$  such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_3| < \epsilon_8, 0 < \gamma < \gamma_8\}$  tend to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow +\infty$ . This completes the proof of the theorem.  $\square$

By a similar argument as the proof in Theorem 2.4, we have

**Theorem 2.5.** *Assume that  $b - 1 - bd < 0$  and  $s - 1 - bd > 0$ .*

- (I) If  $s < 1 + sd$ , then there exist  $\epsilon_9 > 0$  and  $\gamma_9 > 0$  such that there exists a unique orbit of system (2.2) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_9| < \epsilon_9, 0 < \gamma < \gamma_9\}$  that tends to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow +\infty$ ;
- (II) If  $s > 1 + sd$ , then there exist  $\epsilon_{10} > 0$  and  $\gamma_{10} > 0$  such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_{10}| < \epsilon_{10}, 0 < \gamma < \gamma_{10}\}$  tend to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow -\infty$ .

Lastly, we consider a special case for  $s - 1 - sd = 0$ . In this case,  $A_1 = 0$  and system (2.8) reduces to

$$\begin{cases} x'_1(t) = (1 - s)x_1x_2 - \frac{1}{d}x_1^2 - x_1^2x_2, \\ x'_2(t) = A_2x_1 + A_3x_2 + A_4x_1x_2 + A_5x_2^2 + A_6x_1x_2^2 - brx_1x_2^3. \end{cases} \tag{2.9}$$

Let  $x_3 = x_1$  and  $x_4 = A_2x_1 + A_3x_2$ , then system (2.9) is transformed into

$$\begin{cases} x'_3(t) = \left[ (s - 1)\frac{A_2}{A_3} - \frac{1}{d} \right] x_3^2 + (1 - s)\frac{1}{A_3}x_3x_4 + \frac{A_2}{A_3}x_3^3 - \frac{1}{A_3}x_3^2x_4, \\ x'_4(t) = A_3x_4 + \left[ -(1 - s)\frac{A_2^2}{A_3} - \frac{A_2}{d} - A_2A_4 + \frac{A_2^2A_5}{A_3} \right] x_3^2 \\ \quad + \left[ (1 - s)\frac{A_2}{A_3} + A_4 - \frac{2A_2A_5}{A_3} \right] x_3x_4 + \frac{A_5}{A_3}x_4^2 \\ \quad + \left( \frac{A_2^2}{A_3} + \frac{A_2^2A_6}{A_3} \right) x_3^3 + \left( -\frac{A_2}{A_3} - \frac{2A_2A_6}{A_3} \right) x_3^2x_4 + \frac{A_6}{A_3}x_3x_4^2 \\ \quad + \frac{brA_2^3}{A_3^2}x_3^4 - \frac{3brA_2^2}{A_3^2}x_3^3x_4 + \frac{3brA_2}{A_3^2}x_3^2x_4^2 - \frac{br}{A_3^2}x_3x_4^3. \end{cases} \tag{2.10}$$

To simplify the system, we consider the special case  $A_2 = 0$ , that is, the condition (H6)  $s - 1 - bd = br(1 + bd - b)$ .

Under conditions  $s - 1 - sd = 0$  and (H6), system (2.10) becomes

$$\begin{cases} x'_3(t) = -\frac{1}{d}x_3^2 + (1 - s)\frac{1}{A_3}x_3x_4 - \frac{1}{A_3}x_3^2x_4, \\ x'_4(t) = A_3x_4 + A_4x_3x_4 + \frac{A_5}{A_3}x_4^2 + \frac{A_6}{A_3}x_3x_4^2 - \frac{br}{A_3^2}x_3x_4^3. \end{cases} \tag{2.11}$$

We can see that, in the first equation of system (2.11), the degree of the lowest power term is 2 and the corresponding coefficient is  $-\frac{1}{d} < 0$ . Thus, due to Theorem 7.1 in page 131 of [31],  $(0, 0)$  is a saddle-node (see Figure 2). Hence, we obtain the following theorem.

**Theorem 2.6.** *Suppose that conditions  $(s - 1 - bd)(b - 1 - bd) \neq 0$ ,  $s - 1 - sd = 0$  and (H6) hold. Then there exist  $\epsilon_{11} > 0$  and  $\gamma_{11} > 0$  such that all orbits of system (2.2) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_3| < \epsilon_{11}, 0 < \gamma < \gamma_{11}\}$  tend to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow +\infty$  (see Figure 3).*

**3. Global dynamics.** In this section, by summarizing the results on the local qualitative behavior around the critical point  $(0, 0)$  in Sections 3 and the results on the positive and boundary equilibria in [17], we discuss the global dynamics of system (1.3) depending on various parameters.

Note that  $d < 1$  is the necessary condition for the existence of positive equilibria. When  $d < 1$ , the boundary equilibrium point  $(1, 0)$  is a saddle point; and when  $d > 1$ , it is a stable node. The sufficient and necessary conditions for the existence of positive equilibria of system (1.3) were also given.

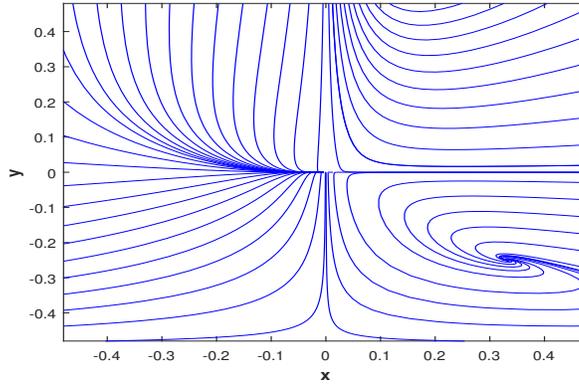


FIGURE 2. Saddle-node  $(0,0)$  of system (2.11) with  $s = 2, d = 0.5, b = 1, r = 1$ .

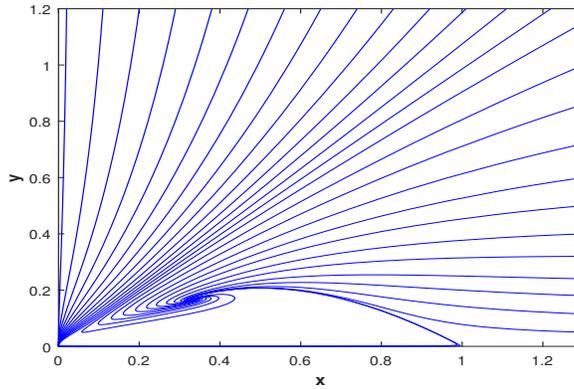


FIGURE 3. Phase diagram of system (2.2) with  $s = 2, d = 0.5, b = 1, r = 1$ .

**Lemma 3.1** ([17]). *Let  $\wedge, \vee$  and the bar  $-$  represent the intersection, union and complement of the corresponding conditions. Then for system (1.3), there exists a unique positive equilibrium if and only if one of the following conditions is satisfied:*

- (H1)  $s < \frac{1}{1-d}$ ;
- (H2)  $s = \frac{1}{1-d}$  and  $s(2-d-r) < 2$ ;
- (H3)  $s = \frac{1}{1-d} + \frac{(d-r)^2}{4r(1-d)}$  and  $s(2-d-r) < 2$ .

*There exist two positive equilibria if and only if*

- (H4)  $\frac{1}{1-d} < s < \frac{1}{1-d} + \frac{(d-r)^2}{4r(1-d)}$  and  $s(2-d-r) < 2$ .

*There exists no positive equilibrium if and only if  $\overline{(H1) \vee (H2) \vee (H3) \vee (H4)}$  holds.*

Under the condition  $(H1) \vee (H2) \vee (H3)$ , denote  $E^* = (x^*, y^*)$  as the unique positive equilibrium. Under the condition (H4), denote  $E_1 = (x_1, y_1)$  and  $E_2 = (x_2, y_2)$  as the two positive equilibria. Under condition  $(H1) \vee (H2) \vee (H3)$ , let

$$a_1^* = \frac{(b-s)x^*y^*}{(x^*+y^*)^2} + x^* + by^* \text{ and } a_2^* = \left[ r + \frac{x^* - rsy^*}{(x^*+y^*)^2} \right] bx^*y^*,$$

and under condition (H4), we just need to replace  $(x^*, y^*)$  with  $(x_i, y_i)$  to obtain  $a_1^i$  and  $a_2^i$ ,  $i = 1, 2$ . Then, the local stability of those positive equilibria is shown as follows.

**Lemma 3.2** ([17]). *Under the condition  $(H1) \vee (H2)$ , the positive equilibrium  $E^*$  is locally asymptotically stable if  $a_1^* > 0$ . Under condition (H3), if the following condition*

$$(H5) \quad 4(1 + sr)(2s - 1 - sd) \neq (s - b)(r + 2 - d)^2 + 4b(1 - d)(1 + sr)^2$$

*also holds, then the unique positive equilibrium  $E^*$  is a saddle-node. Under the condition (H4), the positive equilibrium  $E_1$  is a saddle point; if  $a_1^2 > 0$  also holds, then the positive equilibrium  $E_2$  is locally asymptotically stable.*

**3.1. Global dynamics of system (1.3) with no positive equilibrium.** Firstly, we consider the global dynamic behavior of system (1.3) when it has no positive equilibria. In this case, the critical point  $(0, 0)$  and the boundary equilibrium  $(1, 0)$  may be attractors and even a global attractor.

**Theorem 3.3.** *Suppose that the following conditions hold*

$$\begin{aligned} & b - 1 - bd \geq 0 \text{ and } s - 1 - bd \geq 0, \\ & \text{or } b - 1 - bd < 0 \text{ and } s - 1 - bd > 0, \\ & \text{or } b - 1 - bd > 0 \text{ and } s - 1 - bd < 0, \end{aligned}$$

*then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of an elliptic sector and a parabolic sector. Moreover, if  $(H1) \vee (H2) \vee (H3) \vee (H4)$  and  $d < 1$  also hold, then system (1.3) has no interior equilibrium and  $(0, 0)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 4).*

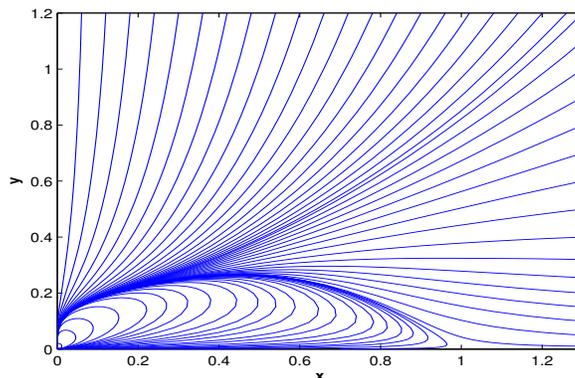


FIGURE 4. Phase diagram of system (1.3) with  $s = r = 2, d = 0.1, b = 1$ .

*Proof.* From Theorem 2.1 (I)(i),(II), Theorem 2.2 (I),(II)(i), and Theorem 2.3 (I)(i), (II)(i), if  $b - 1 - bd \geq 0$  and  $s - 1 - bd \geq 0$ , then there exist  $\bar{\epsilon}_1 > 0$  and  $\bar{\gamma}_1 > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \bar{\epsilon}_1, 0 < \gamma < \bar{\gamma}_1\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\bar{\epsilon}_2$  and  $\bar{\gamma}_2 > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \bar{\epsilon}_2, 0 < \gamma < \bar{\gamma}_2\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, no other orbits tend to  $(0, 0)$ . Thus, the topological structure

of the origin in  $\mathbb{R}_+^2$  consists of an elliptic sector and a parabolic sector. Under the conditions  $b - 1 - bd < 0, s - 1 - bd > 0$  or  $b - 1 - bd > 0, s - 1 - bd < 0$ , the conclusions can be obtained similarly.

Due to Lemma 3.1, system (1.3) has no positive equilibria under condition  $(H1) \vee (H2) \vee (H3) \vee (H4)$  and the boundary equilibrium is a saddle point under condition  $d < 1$ . Hence,  $(0, 0)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$ .  $\square$

**Theorem 3.4.** *If  $d > 1$  and  $s - 1 - bd \geq 0$ , then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a parabolic sector and a hyperbolic sector. Moreover, system (1.3) has no interior equilibrium, and  $(0, 0)$  and  $(1, 0)$  are attractors of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 5).*

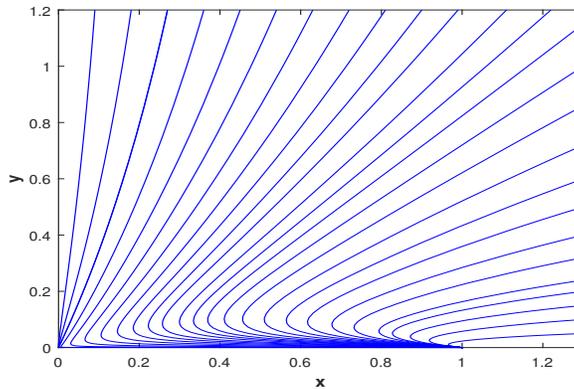


FIGURE 5. Phase diagram of system (1.3) with  $s = 3, d = 1.5, b = 1, r = 1$ .

*Proof.* Note that  $d > 1$  implies  $b - 1 - bd < 0$  and  $s - 1 - sd < 0$ . Thus, from Theorem 2.3 (I)(ii) and (II)(i), if  $b - 1 - bd < 0$  and  $s - 1 - bd > 0$ , then there exist  $\bar{\epsilon}_3 > 0$  and  $\bar{\gamma}_3 > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \bar{\epsilon}_3, 0 < \gamma < \bar{\gamma}_3\}$  which tends to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\bar{\epsilon}_4 > 0$  and  $\bar{\gamma}_4 > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \bar{\epsilon}_4, 0 < \gamma < \bar{\gamma}_4\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, from Theorem 2.5 (I), if  $s < 1 + sd$  also holds, then there exist  $\bar{\epsilon}_5 > 0$  and  $\bar{\gamma}_5 > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq |\theta - \theta_3| < \bar{\epsilon}_5, 0 < \gamma < \bar{\gamma}_5\}$  that tends to  $(0, 0)$  along  $\theta_3$  as  $t \rightarrow +\infty$ . Furthermore, no other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a parabolic sector and a hyperbolic sector. Under the conditions  $d > 1$  and  $s - 1 - bd = 0$ , the similar conclusion can also be obtained.

Under condition  $d > 1, (H1) \vee (H2) \vee (H3) \vee (H4)$  holds. This implies that system (1.3) has no positive equilibria and the boundary equilibrium is a stable node. Thus, the global conclusion of the theorem holds.  $\square$

**Theorem 3.5.** *If  $d > 1$  and  $s - 1 - bd < 0$ , then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a hyperbolic sector. Moreover, system (1.3) has no interior equilibrium, and the boundary equilibrium  $(1, 0)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 6).*

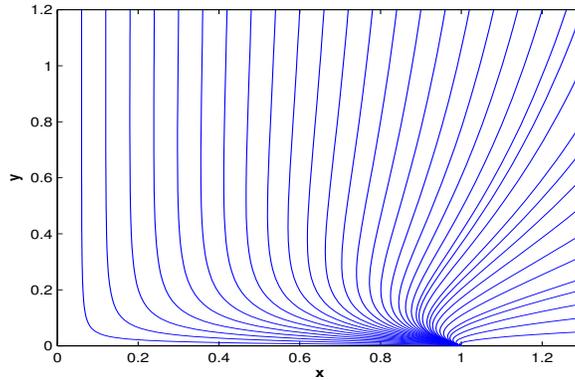


FIGURE 6. Phase diagram of system (1.3) with  $b = s = 2, d = 1.5, r = 2$ .

*Proof.*  $d > 1$  implies that  $b - 1 - bd < 0$  holds. Thus, from Theorem 2.3 (I)(ii) and (II)(ii), if  $b - 1 - bd < 0$  and  $s - 1 - bd < 0$ , then there exist  $\bar{\epsilon}_6 > 0$  and  $\bar{\gamma}_6 > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \bar{\epsilon}_6, 0 < \gamma < \bar{\gamma}_6\}$  tending to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\bar{\epsilon}_7 > 0$  and  $\bar{\gamma}_7 > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \bar{\epsilon}_7, 0 < \gamma < \bar{\gamma}_7\}$  tending to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, no other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a hyperbolic sector

Due to Lemma 3.1, under condition  $d > 1$ , system (1.3) has no positive equilibria and the boundary equilibrium is a stable node. Thus,  $(1, 0)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$ .  $\square$

**3.2. Global dynamics of system (1.3) with a unique positive equilibrium.**

Now we consider the global dynamics for the case (H1). In this case, the critical point  $(0, 0)$  or the positive equilibrium  $(x^*, y^*)$  may be an attractor and even a global attractor. Note that when  $s - 1 - sd < 0$  and  $s - 1 - bd \geq 0$ , we have  $s > b$  and thus  $s - 1 - sd > b - 1 - bd > 0$ . This contradicts with  $s - 1 - sd < 0$ , and thus we need not to consider the case  $b - 1 - bd > 0, s - 1 - bd > 0$  under condition (H1).

**Theorem 3.6.** *Suppose that  $b - 1 - bd < 0$  and  $s - 1 - bd \geq 0$ , then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a parabolic sector and a hyperbolic sector. Moreover, if (H1) and  $a_1^* > 0$  also hold, then both  $(0, 0)$  and  $(x^*, y^*)$  are attractors of system (1.3) (see Figure 7).*

*Proof.* From Theorem 2.3 (I)(ii) and (II)(i), if  $b - 1 - bd < 0$  and  $s - 1 - bd > 0$ , then there exist  $\bar{\epsilon}_8 > 0$  and  $\bar{\gamma}_8 > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \bar{\epsilon}_8, 0 < \gamma < \bar{\gamma}_8\}$  that tends to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\bar{\epsilon}_9 > 0$  and  $\bar{\gamma}_9 > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \bar{\epsilon}_9, 0 < \gamma < \bar{\gamma}_9\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, no other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a parabolic sector and a hyperbolic sector. Under the conditions  $b - 1 - bd < 0$  and  $s - 1 - bd = 0$ , the similar conclusion can also be obtained.

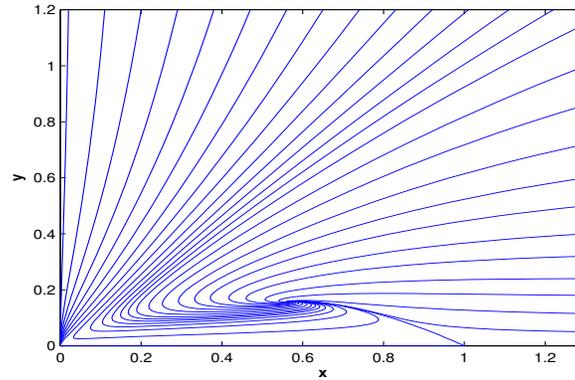


FIGURE 7. Phase diagram of system (1.3) with  $s = 2, d = 0.625, r = 1, b = 0.8$ .

Under condition (H1) and  $a_1^* > 0$ , from Lemmas 3.1 and 3.2, system (1.3) has a unique local stable positive equilibrium and the boundary equilibrium is a saddle point. Thus, the global conclusion of the theorem holds.  $\square$

**Remark 1.** Under condition (H1), only for the case that  $b - 1 - bd < 0$  and  $s - 1 - bd \geq 0$ , there may exist Hopf bifurcation. Theorem 6 in [17] gave specific conditions under which system (1.3) exhibits Hopf bifurcation.

**Theorem 3.7.** Suppose that  $b - 1 - bd \geq 0$  and  $s - 1 - bd < 0$ , then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a parabolic sector and a hyperbolic sector. Moreover, if (H1) also holds, then the unique positive equilibrium  $(x^*, y^*)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 8).

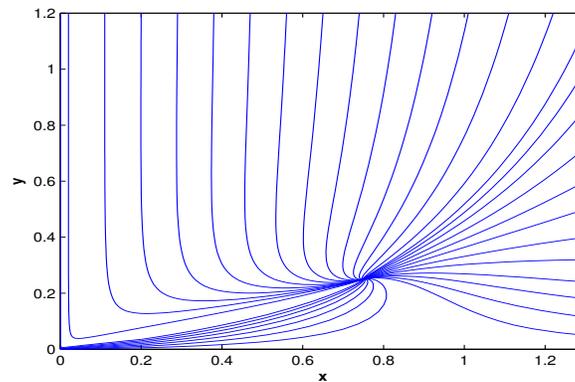


FIGURE 8. Phase diagram of system (1.3) with  $s = 1, d = 0.5, b = 2.5, r = 1$ .

*Proof.* From Theorem 2.3 (I)(i) and (II)(ii), if  $b - 1 - bd > 0$  and  $s - 1 - bd < 0$ , then there exist  $\epsilon_{10} > 0$  and  $\gamma_{10} > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_{10}, 0 < \gamma < \gamma_{10}\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\epsilon_{11} > 0$  and  $\gamma_{11} > 0$  such that there exists a unique orbit of system (1.3) in

$\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_{11}, 0 < \gamma < \gamma_{11}\}$  which tends to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . No other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a parabolic sector and a hyperbolic sector. Under the conditions  $b - 1 - bd = 0$  and  $s - 1 - bd < 0$ , the similar conclusion can also be obtained.

Since  $b - 1 - bd \geq 0$  and  $s - 1 - bd < 0$  imply that  $s < b$  and thus  $a_1^* > 0$ . Then  $(x^*, y^*)$  is local asymptotically stable (and thus system (1.3) does not exhibit Hopf bifurcation). Assume that  $l(t) = (x(t), y(t))$  is an arbitrary nontrivial periodic orbit of system (1.3) with period  $T > 0$ . Then under condition  $s < b$ ,  $\oint_0^T \text{tr} J(x(t), y(t)) dt = \oint_0^T [-x(t) - bry(t) + \frac{(s-b)x(t)y(t)}{(x(t)+y(t))^2}] dt < 0$ . By the divergency criterion [13], the closed orbit  $l(t)$  is stable, which contradicts with the local asymptotic stability of  $(x^*, y^*)$ . Thus, system (1.3) has no closed orbits.

Under condition (H1), from Lemma 3.1, system (1.3) has a unique positive equilibrium and the boundary equilibrium is a saddle point. Hence,  $(x^*, y^*)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$ . □

**Theorem 3.8.** *Suppose that  $b - 1 - bd < 0$  and  $s - 1 - bd < 0$ , then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a hyperbolic sector. Moreover, if (H1) and  $s \leq b$  also hold, then the unique positive equilibrium  $(x^*, y^*)$  is a global attractor of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 9).*

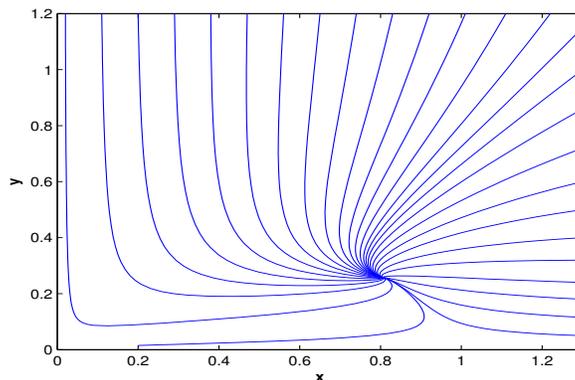


FIGURE 9. Phase diagram of system (1.3) with  $s = 0.8, d = 0.5, r = 1, b = 1$ .

*Proof.* From Theorem 2.3 (I)(ii) and (II)(ii), if  $b - 1 - bd < 0$  and  $s - 1 - bd < 0$ , then there exist  $\epsilon_{12} > 0$  and  $\gamma_{12} > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_{12}, 0 < \gamma < \gamma_{12}\}$  which tends to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\epsilon_{13} > 0$  and  $\gamma_{13} > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_{13}, 0 < \gamma < \gamma_{13}\}$  which tends to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, no other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a hyperbolic sector.

We can see that when  $s \leq b$ , we have  $a_1^* > 0$ . Thus, if condition (H1) and  $s \leq b$  hold, system (1.3) has a unique positive equilibrium, the boundary equilibrium is a saddle point and there exist no closed orbits. Hence, the conclusion of the theorem holds. □

**Remark 2.** When  $s > b$  and  $s-1-bd < 0$ , we have  $a_1^*|_{r=0} = (s-b)d^2+bd+1-s > 0$ . Moreover, under conditions (H1) and  $s > b$ ,  $\frac{\partial a_1^*}{\partial r} > 0$  [17]. Thus, we can conclude that when  $s-1-bd < 0$ ,  $a_1^* > 0$ . Hence, for the case  $b-1-bd < 0$  and  $s-1-bd < 0$  under conditions (H1),  $(x^*, y^*)$  is local asymptotically stable and thus system (1.3) does not undergo Hopf bifurcation.

Now we consider the global dynamics for the case (H2)  $\wedge$  (H6). In this case, the critical point  $(0, 0)$  and the positive equilibrium  $(x^*, y^*)$  may be attractors. Note that when  $s-1-sd = 0$  and  $s-1-bd > 0$ , we have  $s > b$  and thus  $s-1-sd > b-1-bd > 0$ . This contradicts with  $s-1-sd = 0$ , and thus we need not to consider the case  $b-1-bd > 0$ ,  $s-1-bd > 0$ . When  $s-1-bd = 0$  (or  $b-1-bd = 0$ ), we have  $s = b$  and thus  $b-1-bd = 0$  (or  $s-1-bd = 0$ ). This is the first item of this section and we can obtain the solution directly. Thus, we do not consider the case  $s-1-bd = 0$  (or  $b-1-bd = 0$ ). When  $b-1-bd > 0$  and  $s-1-bd < 0$ , we have  $s < b$ . From condition (H2)  $\wedge$  (H6), we have  $b = \frac{d}{r(1-d)}$ . Combining  $s = \frac{1}{1-d}$  and  $d < r$ , we can obtain that  $s > b$ . This is a contradiction. Thus, we do not consider the case  $b-1-bd > 0$ ,  $s-1-bd < 0$ . When  $s-1-sd = 0$  and  $b-1-bd < 0$ , we have  $s > b$  and thus  $s-1-bd > s-1-sd = 0$ . This contradicts with  $s-1-bd < 0$ , and thus we need not to consider the case  $b-1-bd < 0$ ,  $s-1-bd < 0$ . Hence, under condition (H2)  $\wedge$  (H6), we only need to consider the case  $b-1-bd < 0$ ,  $s-1-bd > 0$ .

**Theorem 3.9.** *Suppose that  $b-1-bd < 0$ ,  $s-1-bd > 0$  and (H6) hold, then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a hyperbolic sector and a parabolic sector. Moreover, if (H2) and  $a_1^* > 0$  also hold, then both  $(0, 0)$  and  $(x^*, y^*)$  are attractors of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 3).*

*Proof.* From Theorem 2.3 (I)(ii), (II)(i) and Theorem 2.6, if  $s-1-bd > 0$ ,  $b-1-bd < 0$  and (H6) hold, then there exist  $\epsilon_{14} > 0$  and  $\gamma_{14} > 0$  such that there exists a unique orbit of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_{14}, 0 < \gamma < \gamma_{14}\}$  that tends to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\epsilon_{15} > 0$  and  $\gamma_{15} > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_{15}, 0 < \gamma < \gamma_{15}\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . No other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of a hyperbolic sector and a parabolic sector.

Under condition (H2) and  $a_1^* > 0$ , due to Lemmas 3.1 and 3.2, system (1.3) has a unique local stable positive equilibrium and the boundary equilibrium is a saddle point. Hence,  $(0, 0)$  and  $(x^*, y^*)$  are attractors of system (1.3) in  $\mathbb{R}_+^2$ .  $\square$

**Remark 3.** Under the condition (H2)  $\wedge$  (H6), only for the case that  $b-1-bd < 0$  and  $s-1-bd > 0$ , system (1.3) may have Hopf bifurcation. Corresponding existence conditions can be found in Theorem 7 of [17].

In the following, we consider the case (H3)  $\wedge$  (H5). Note that when  $b-1-bd \geq 0$  and  $s-1-bd \leq 0$ , we have  $s \leq b$  and thus  $a_1^* > 0$ . This contradicts with the condition (H3), and thus we need not to consider the case  $b-1-bd \geq 0$ ,  $s-1-bd \leq 0$ . When  $b-1-bd < 0$  and  $s-1-sd > 0$ , we have  $s > b$  and thus  $s-1-bd > s-1-sd > 0$ . This contradicts with  $s-1-bd \leq 0$ , and thus we need not to consider the case  $b-1-bd < 0$ ,  $s-1-bd \leq 0$ . Hence, under condition (H3)  $\wedge$  (H5), we only need to consider the case  $s-1-bd > 0$ .

**Theorem 3.10.** *Suppose  $s-1-bd > 0$ , then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of an elliptic sector and a parabolic sector. Moreover, if (H3)  $\wedge$  (H5)*

also holds, then  $(x^*, y^*)$  is a saddle-node and  $(0, 0)$  is an attractor of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 10).

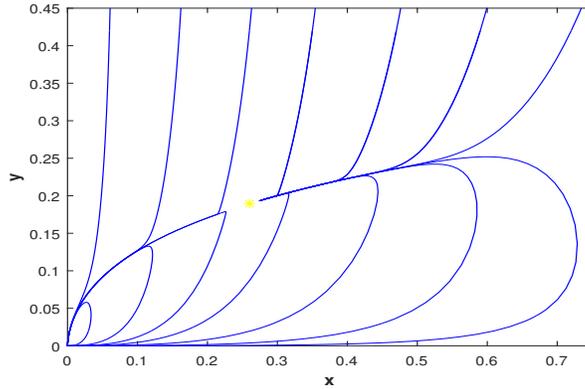


FIGURE 10. Phase diagram of system (1.3) with  $s = 1.75625, d = 0.2, r = 2, b = 3$ .

*Proof.* From Theorem 2.2 (I),(II)(i) and Theorem 2.3 (I)(i),(II)(i), if  $b - 1 - bd \geq 0$  and  $s - 1 - bd > 0$ , then there exist  $\epsilon_{16}^- > 0$  and  $\gamma_{16}^- > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \epsilon_{16}^-, 0 < \gamma < \gamma_{16}^-\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\epsilon_{17}^- > 0$  and  $\gamma_{17}^- > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \epsilon_{17}^-, 0 < \gamma < \gamma_{17}^-\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, no other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of an elliptic sector and a parabolic sector. Under the conditions  $b - 1 - bd < 0$  and  $s - 1 - bd > 0$ , a similar conclusion can also be obtained.

Under condition  $(H3) \wedge (H5)$ , from Lemmas 3.1 and 3.2, system (1.3) has  $(x^*, y^*)$  as the unique saddle-node and the boundary equilibrium as a saddle point. Hence,  $(0, 0)$  is an attractor of system (1.3) in  $\mathbb{R}_+^2$ .  $\square$

**Remark 4.** Under condition  $(H3) \wedge \overline{(H5)}$ ,  $(x^*, y^*)$  is a cusp point and system (1.3) may undergo a Bogdanov-Takens bifurcation, which we will analyze in Section 4.

**3.3. Global dynamics of system (1.3) with two positive equilibria.** For the case  $(H4)$ , the positive equilibrium  $(x_1, y_1)$  is a saddle point, the critical point  $(0, 0)$  and another positive equilibria  $(x_2, y_2)$  may be attractors. In this case, we need not to consider the case  $b - 1 - bd \leq 0, s - 1 - bd \leq 0$ . The reason is similar to the case  $(H3) \wedge (H5)$ .

**Theorem 3.11.** *Suppose that  $(H4)$  and the following conditions hold*

$$\begin{aligned} & b - 1 - bd \geq 0, \quad s - 1 - bd \geq 0, \\ & \text{or } b - 1 - bd < 0, \quad s - 1 - bd > 0, \\ & \text{or } b - 1 - bd > 0, \quad s - 1 - bd < 0, \end{aligned}$$

*then the topological structure of the origin in  $\mathbb{R}_+^2$  consists of an elliptic sector and a parabolic sector.  $(x_1, y_1)$  is a saddle point. Moreover, if  $a_1^2 > 0$  also holds, then both  $(0, 0)$  and  $(x_2, y_2)$  are attractors of system (1.3) in  $\mathbb{R}_+^2$  (see Figure 11).*

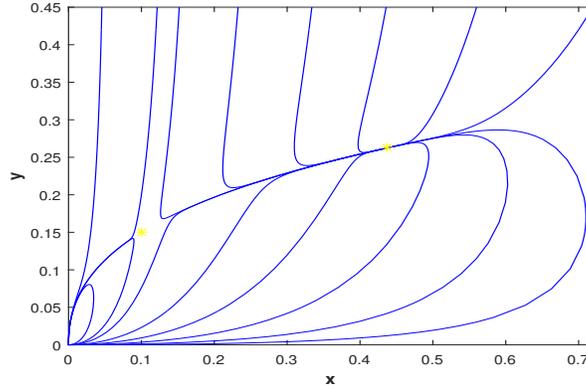


FIGURE 11. Phase diagram of system (1.3) with  $s = 1.5, d = 0.1, r = 2, b = 2$ .

*Proof.* From Theorem 2.1 (I)(i), (II), Theorem 2.2 (I)(i), (II) and Theorem 2.3 (I)(i), (II)(i), if  $b - 1 - bd \geq 0$  and  $s - 1 - bd \geq 0$ , then there exist  $\bar{\epsilon}_{18} > 0$  and  $\bar{\gamma}_{18} > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \theta < \bar{\epsilon}_{18}, 0 < \gamma < \bar{\gamma}_{18}\}$  tend to  $(0, 0)$  along  $\theta_1$  as  $t \rightarrow -\infty$ ; and there exist  $\bar{\epsilon}_{19} > 0$  and  $\bar{\gamma}_{19} > 0$  such that all orbits of system (1.3) in  $\{(\theta, \gamma) : 0 \leq \frac{\pi}{2} - \theta < \bar{\epsilon}_{19}, 0 < \gamma < \bar{\gamma}_{19}\}$  tend to  $(0, 0)$  along  $\theta_2$  as  $t \rightarrow +\infty$ . Moreover, no other orbits tend to  $(0, 0)$ . Thus, the topological structure of the origin in  $\mathbb{R}_+^2$  consists of an elliptic sector and a parabolic sector. Under the conditions  $b - 1 - bd < 0, s - 1 - bd > 0$  or  $b - 1 - bd > 0, s - 1 - bd < 0$ , the similar conclusion can also be obtained.

Under condition (H4) and  $a_1^2 > 0$ , from Lemmas 3.1 and 3.2, system (1.3) has  $(x_1, y_1)$  and the boundary equilibrium as saddle points, and  $(x_2, y_2)$  as a stable positive equilibrium. Thus the conclusion of the theorem holds.  $\square$

**Remark 5.** Under condition (H4), for the case that  $b - 1 - bd \geq 0$  and  $s - 1 - bd \geq 0$  or the case that  $b - 1 - bd < 0$  and  $s - 1 - bd > 0$ , there may exist Hopf bifurcation around  $(x_2, y_2)$  in  $\mathbb{R}_+^2$ . The conclusion can be seen in Theorem 8 of [17].

Combining Theorem 3.3-Theorem 3.11 and corresponding existence results on Hopf bifurcation, the global dynamics of system (1.3) can be summarized in Table 1.

**4. Bogdanov-Takens bifurcation.** Note that system (1.3) can have a saddle-node or a cusp under condition (H3). Especially, the positive equilibrium  $(x^*, y^*)$  is a cusp under the condition  $(H3) \wedge \overline{(H5)}$  [17]. This implies that there may exist Bogdanov-Takens bifurcation.

In order to determine the Bogdanov-Takens bifurcation of system (1.3), we need to fix some values of parameters since the conditions given in [17] are too complicated to analyze. Thus, in this section, we fix  $b$  as a constant and investigate the Bogdanov-Takens bifurcation of system (1.3) on the parameters  $s, r$  and  $d$ .

Let

$$q_{20} = a_{10}a_{20} + a_{01}\beta_{20} - \frac{a_{10}}{a_{01}}(a_{10}a_{11} + a_{01}\beta_{11}) + \frac{a_{10}^2}{a_{01}^2}(a_{10}a_{02} + a_{01}\beta_{02})$$

and

$$q_{11} = 2a_{20} + \beta_{11} - \frac{a_{10}}{a_{01}}(a_{11} + 2\beta_{02}),$$

TABLE 1. The global dynamics of system (1.3).

Condition 1		Condition 2	Global Results	Hopf bifurcation
(H0)	d < 1	(K1)	Theorem 3.3	Does not exist
		(K2)	Theorem 3.3	Does not exist
		(K3)	Theorem 3.3	Does not exist
		(K4)	∅	Does not exist
	d > 1	(K1)	∅	Does not exist
		(K2)	Theorem 3.4	Does not exist
		(K3)	∅	Does not exist
		(K4)	Theorem 3.5	Does not exist
(H1)	(K1)	∅	Does not exist	
	(K2)	Theorem 3.6	Remark 1	
	(K3)	Theorem 3.7	Does not exist	
	(K4)	Theorem 3.8	Does not exist	
(H2) ∧ (H6)	(K1)	∅	Does not exist	
	(K2)	Theorem 3.9	Remark 3	
	(K3)	∅	Does not exist	
	(K4)	∅	Does not exist	
(H3) ∧ (H5)	(K1)	Theorem 3.10	Does not exist	
	(K2)	Theorem 3.10	Does not exist	
	(K3)	∅	Does not exist	
	(K4)	∅	Does not exist	
(H4)	(K1)	Theorem 3.11	Remark 5	
	(K2)	Theorem 3.11	Remark 5	
	(K3)	Theorem 3.11	Does not exist	
	(K4)	∅	Does not exist	

Here (H0) := (H1) ∨ (H2) ∨ (H3) ∨ (H4),

(K1) := {b - 1 - bd ≥ 0, s - 1 - bd ≥ 0}, (K2) := {b - 1 - bd < 0, s - 1 - bd ≥ 0},  
 (K3) := {b - 1 - bd ≥ 0, s - 1 - bd < 0} and (K4) := {b - 1 - bd < 0, s - 1 - bd < 0}.

where  $a_{10} = \frac{sx^*y^*}{(x^*+y^*)^2} - x^*$ ,  $a_{01} = -\frac{sx^{*2}}{(x^*+y^*)^2}$ ,  $a_{20} = \frac{sy^{*2}}{(x^*+y^*)^3} - 1$ ,  $a_{11} = -\frac{2sx^*y^*}{(x^*+y^*)^3}$ ,  
 $a_{02} = \frac{sx^{*2}}{(x^*+y^*)^3}$ ,  $\beta_{10} = \frac{by^{*2}}{(x^*+y^*)^2}$ ,  $\beta_{01} = -bry^* - \frac{bx^*y^*}{(x^*+y^*)^2}$ ,  $\beta_{20} = -\frac{by^{*2}}{(x^*+y^*)^3}$ ,  $\beta_{11} = \frac{2bx^*y^*}{(x^*+y^*)^3}$  and  $\beta_{02} = -br - \frac{bx^{*2}}{(x^*+y^*)^3}$ .

**Lemma 4.1** ([17]). *For system (1.3) satisfying condition (H3) ∧ (H5), if  $q_{20} \neq 0$  and  $q_{11} \neq 0$ , then  $(x^*, y^*) = (\frac{1}{2(1+sr)}(sd + sr + 2 - 2s), \frac{1}{2r(1+sr)}(2sr - d - r - 2sdr))$  is a cusp of codimension 2.*

We aim to show that system (1.3) undergoes Bogdanov-Takens bifurcation on the parameter space  $BT := \{(s, r, d) \mid (H3) \wedge (H5), q_{20} \neq 0 \text{ and } q_{11} \neq 0\}$ . Here, we choose  $d$  and  $r$  as bifurcation parameters. We need to find the universal unfolding of  $(x^*, y^*)$  on  $BT$ . For this point, we consider the following system

$$\begin{cases} x'(t) = x(t) \left[ 1 - x(t) - \frac{sy(t)}{x(t)+y(t)} \right], \\ y'(t) = by(t) \left[ -(d + \mu_1) - (r + \mu_2)y(t) + \frac{x(t)}{x(t)+y(t)} \right]. \end{cases} \tag{4.1}$$

Let  $x = x + x^*$  and  $y = y + y^*$ . By using Taylor expansions, system (4.1) becomes

$$\begin{cases} x'(t) = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + A_1^3(x, y), \\ y'(t) = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + B_1^3(x, y), \end{cases} \quad (4.2)$$

where  $b_{00} = -b(\mu_1 + \mu_2 y^*)y^*$ ,  $b_{10} = \frac{by^{*2}}{(x^* + y^*)^2}$ ,  $b_{01} = -b\mu_1 - b(r + 2\mu_2)y^* - \frac{bx^*y^*}{(x^* + y^*)^2}$ ,  $b_{20} = -\frac{by^{*2}}{(x^* + y^*)^3}$ ,  $b_{11} = \frac{2bx^*y^*}{(x^* + y^*)^3}$ ,  $b_{02} = -b(r + \mu_2) - \frac{bx^{*2}}{(x^* + y^*)^3}$ , and  $A_1^3(x, y)$  and  $B_1^3(x, y)$  are  $o(|x, y|^3)$ .

Consider changes of variables as follows

$$X = x \text{ and } Y = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + A_1^3(x, y),$$

that is,

$$x = X \text{ and } y = k_{10}X + k_{01}Y + k_{20}X^2 + k_{11}XY + k_{02}Y^2 + o(|x, y|^3),$$

where  $k_{10} = \frac{(s-1)-2(s-1)x^*-x^{*2}}{(s-1+x^*)^2}$ ,  $k_{01} = \frac{-s}{(s-1+x^*)^2}$ ,  $k_{20} = \frac{-s(s-1)}{(s-1+x^*)^3}$ ,  $k_{11} = \frac{2s}{(s-1+x^*)^3}$  and  $k_{02} = \frac{s}{x^*(s-1+x^*)^3}$ . Then system (4.2) can be written as

$$\begin{cases} X'(t) = Y, \\ Y'(t) = L_0 + L_{10}X + L_{01}Y + L_{20}X^2 + L_{11}XY + L_{02}Y^2 + L^3(X, Y), \end{cases} \quad (4.3)$$

where

$$L_0 = a_{01}b_{00},$$

$$L_{10} = a_{01}b_{10} + a_{11}b_{00} + k_{10}(2a_{02}b_{00} + a_{01}b_{01}),$$

$$L_{01} = a_{10} + k_{01}(2a_{02}b_{00} + a_{01}b_{01}),$$

$$\begin{aligned} L_{20} &= a_{01}b_{20} + k_{20}(2a_{02}b_{00} + a_{01}b_{01}) + a_{01}b_{11}k_{10} + k_{10}(2a_{02}b_{10} + a_{11}b_{01}) \\ &\quad + k_{10}^2(2a_{02}b_{01} + a_{01}b_{02}), \end{aligned}$$

$$\begin{aligned} L_{11} &= 2a_{02} + k_{11}(2a_{02}b_{00} + a_{01}b_{01}) + a_{11}k_{10} + a_{01}b_{11}k_{01} \\ &\quad + 2k_{10}k_{01}(2a_{02}b_{01} + a_{01}b_{02}) + k_{01}(2a_{02}b_{10} + a_{11}b_{01}), \end{aligned}$$

$$L_{02} = k_{02}(2a_{02}b_{00} + a_{01}b_{01}) + k_{01}^2(2a_{02}b_{01} + a_{01}b_{02}) + a_{11}k_{01}.$$

Following the derivation procedure of the normal form in [29], let  $x = X, y = (1 - L_{02}X)Y$  and  $s = \frac{t}{1 - L_{02}X}$ . Then system (4.3) is strongly topologically equivalent to

$$\begin{cases} x'(s) = y, \\ y'(s) = \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 x^2 + \lambda_4 xy, \end{cases} \quad (4.4)$$

where  $\lambda_0 = L_0$ ,  $\lambda_1 = L_{10} - 2L_0L_{02}$ ,  $\lambda_2 = L_{01}$ ,  $\lambda_3 = L_{20} - 2L_{10}L_{02} + L_0L_{02}^2$  and  $\lambda_4 = L_{11} - L_{01}L_{02}$ .

Further, letting  $X = (x + \frac{\lambda_1}{2\lambda_3})\frac{\lambda_4^2}{\lambda_3}$ ,  $Y = \frac{\lambda_4^3}{\lambda_3}y$  and  $t = \frac{\lambda_4^3}{\lambda_3}s$ , we can rescale system (4.4) as

$$\begin{cases} X'(t) = Y, \\ Y'(t) = v_1 + v_2Y + X^2 + XY, \end{cases} \quad (4.5)$$

where  $v_1 = \frac{\lambda_0 \lambda_4^4}{\lambda_3^3} - \frac{\lambda_1^2 \lambda_4^4}{4 \lambda_3^4}$  and  $v_2 = \frac{\lambda_2 \lambda_4}{\lambda_3} - \frac{\lambda_1 \lambda_3^2}{2 \lambda_3^2}$ . Due to the results in [7, 8, 26], if  $\frac{\partial(v_1, v_2)}{\partial(\lambda_1, \lambda_2)} \neq 0$ , then system (4.1) undergoes Bogdanov-Takens bifurcation when  $\mu_1$  and  $\mu_2$  vary in a small neighborhood of the origin.

As an example, now we consider the following system

$$\begin{cases} x'(t) = x(t) \left[ 1 - x(t) - \frac{281}{160} \frac{y(t)}{x(t)+y(t)} \right], \\ y'(t) = \frac{605}{2248} y(t) \left[ -\left(\frac{1}{5} + \mu_1\right) - (2 + \mu_2)y(t) + \frac{x(t)}{x(t)+y(t)} \right]. \end{cases} \tag{4.6}$$

Note that when  $\mu_1 = \mu_2 = 0$ , there exists a cusp in system (4.6) [17]. By the above analysis, we know that system (4.6) undergoes the Bogdanov-Takens bifurcation when the parameters  $\mu_1$  and  $\mu_2$  vary in a small neighborhood of the origin. The bifurcation curves in the small neighborhood of the origin in the  $(\mu_1, \mu_2)$ -plane are sketched in Figure 12. The local representations of these bifurcation curves in the small neighborhood of the origin are given as follows.

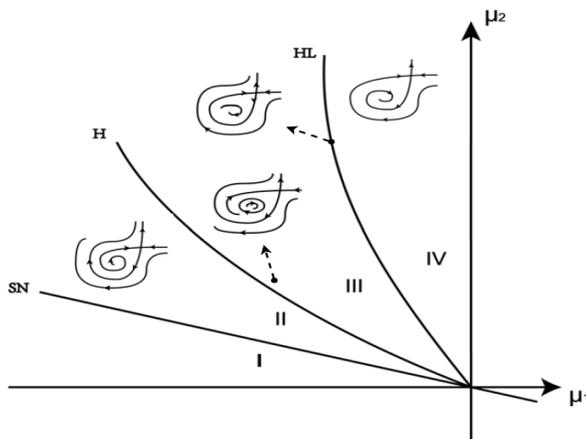


FIGURE 12. Bifurcation sets and the corresponding phase portraits of system (4.6).

1. The saddle-node bifurcation curve  $SN = \{(\mu_1, \mu_2) : v_1 = 0\}$ , that is,

$$SN = \{(\mu_1, \mu_2) : 211.6125\mu_1 + 40.095\mu_2 + 1212.609431\mu_1^2 + 74.85284\mu_2^2 + 615.85864\mu_1\mu_2 = 0\}.$$

When the parameter values lie on the saddle-node bifurcation curve  $SN$ , system (4.6) has one unique equilibrium which is a saddle-node. If the parameters  $\mu_1$  and  $\mu_2$  satisfy  $v_1(\mu_1, \mu_2) > 0$  (that is, in region I), then system (4.6) has no equilibrium (see Figure 13-I). When parameters  $\mu_1$  and  $\mu_2$  pass through the curve  $SN$  into region II, then two equilibria appear (see Figure 13-II).

2. The Hopf bifurcation curve  $H = \{(\mu_1, \mu_2) : v_1 + v_2^2 = 0, v_1 < 0\}$ , that is,

$$H = \{(\mu_1, \mu_2) : 182.32895\mu_1 + 34.54654\mu_2 + 8.43406\mu_1^2 - 3.01678\mu_2^2 + 0.34917\mu_1\mu_2 = 0, v_1 < 0\}.$$

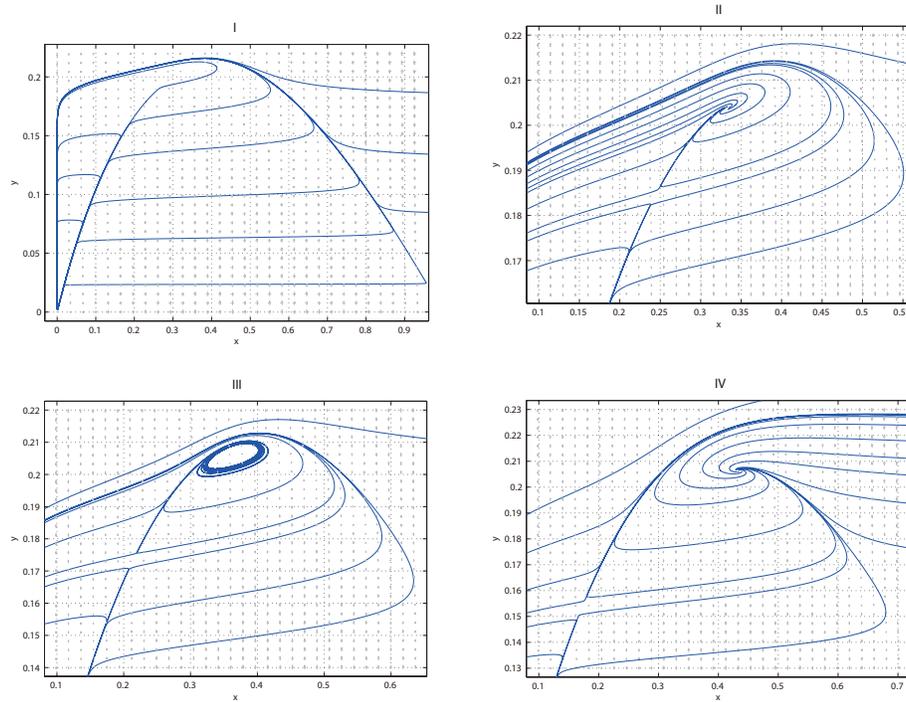


FIGURE 13. (I): When  $u_1 = -0.1$  and  $u_2 = 0.515$  lie in the region I, there exists no positive equilibrium; (II): When  $u_1 = -0.1$  and  $u_2 = 0.55$  lie in the region II, there exist a saddle point and an unstable focus; (III): When  $u_1 = -0.1$  and  $u_2 = 0.605$  lie in the region III, there exist a saddle point, a stable focus and an unstable limit cycle; (IV): When  $u_1 = -0.1$  and  $u_2 = 0.8$  lie in the region IV, there exist a saddle point and a stable focus.

When the parameter values lie on the Hopf bifurcation curve  $H$ , there exist an unstable focus and no periodic orbit. When the parameters cross the Hopf bifurcation curve  $H$  into region III, an unstable limit cycle appears and system (4.6) has a saddle point and a stable focus (see Figure 13-III).

3. The homoclinic bifurcation curve  $HL = \{(\mu_1, \mu_2) : 25v_1 + 49v_2^2 = 0, v_1 < 0\}$ , that is,

$$HL = \{(\mu_1, \mu_2) : 13.22578\mu_1 + 2.50594\mu_2 + 3.204707\mu_1^2 + 4.004325\mu_2^2 + 29.82948\mu_1\mu_2 = 0, v_1 < 0\}.$$

When the parameter values pass region III and lie on the homoclinic bifurcation curve  $HL$ , an unstable homoclinic loop is generated and system (4.6) has a stable focus. However, system (4.6) has a hyperbolic saddle point and a stable focus when parameters lie in region IV (see Figure 13-IV).

When  $\mu_1 = \mu_2 = 0$ , system (4.6) has a cusp of codimension 2 as the unique positive equilibrium (see Fig.5 in [17]). As  $\mu_1$  and  $\mu_2$  vary, specific numerical simulations of system (4.6) are depicted in the following Figure 13, which is corresponding to Figure 12.

**5. Conclusions.** In this paper, we investigated the global dynamics of a predator-prey system with density-dependent mortality and ratio-dependent functional response and showed that the model has complicated qualitative behavior in the interior of the first quadrant. Firstly, the qualitative behavior of the system at the origin in  $\mathbb{R}_+^2$  was studied. It was shown that the origin is indeed a critical point of higher order. There can exist numerous kinds of topological structures in the neighborhood of the origin, including parabolic sectors, elliptic sectors, hyperbolic sectors and some combinations of them. These structures have critical implications on the global behavior of system (1.3).

Afterwards, combining the dynamic behavior around the boundary equilibrium, positive equilibria and the origin, global qualitative analysis of the model depending on various parameters was carried out. A summary of these results was given in Table 1. Numerical simulations were presented to illustrate the conclusions.

Lastly, we explored Bogdanov-Takens bifurcation of system (1.3). Bifurcation sets of system (1.3) were given. An example was given to show the existence of Bogdanov-Takens bifurcation. In the framework in Figure 12, we demonstrated that system (1.3) can exhibit various bifurcations, including saddle-node bifurcation, Hopf bifurcation, and homoclinic bifurcation. The corresponding bifurcation diagrams and phase portraits were sketched in Figure 12. Figure 13 was presented to support the specific numerical simulations.

The results in [17] showed that system (1.3) can have a cusp of codimension 3 under some conditions. Thus, it is challenging to investigate the existence of Bogdanov-Takens bifurcation of codimension 3 [16] in our future work. Moreover, it is also interesting for us to deal with the versal unfoldings [23, 28] of the equivalent polynomial system (2.2) (and thus system (1.3)), and then obtain all possible bifurcations of the versal unfoldings with phase portraits, including transcritical bifurcation, Hopf bifurcation, and heteroclinic bifurcation [23].

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