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Bifurcation of Codimension 3 in a Predator–Prey System of Leslie Type with Simplified Holling Type IV Functional Response

Jicai Huang*, Xiaojing Xia and Xinan Zhang
School of Mathematics and Statistics,
Central China Normal University, Wuhan,
Hubei 430079, P. R. China
*hjc@mail.ccnu.edu.cn

Shigui Ruan
Department of Mathematics, University of Miami,
Coral Gables, FL 33124-4250, USA

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It was shown in [Li & Xiao, 2007] that in a predator–prey model of Leslie type with simplified Holling type IV functional response some complex bifurcations can occur simultaneously for some values of parameters, such as codimension 1 subcritical Hopf bifurcation and codimension 2 Bogdanov–Takens bifurcation. In this paper, we show that for the same model there exists a unique degenerate positive equilibrium which is a degenerate Bogdanov–Takens singularity (focus case) of codimension 3 for other values of parameters. We prove that the model exhibits degenerate focus type Bogdanov–Takens bifurcation of codimension 3 around the unique degenerate positive equilibrium. Numerical simulations, including the coexistence of three hyperbolic positive equilibria, two limit cycles, bistability states (one stable equilibrium and one stable limit cycle, or two stable equilibria), tristability states (two stable equilibria and one stable limit cycle), a stable limit cycle enclosing a homoclinic loop, a homoclinic loop enclosing an unstable limit cycle, or a stable limit cycle enclosing three unstable hyperbolic positive equilibria for various parameter values, confirm the theoretical results.

Keywords: Predator–prey model of Leslie type; simplified Holling type IV functional response; degenerate focus type Bogdanov–Takens bifurcation of codimension 3.

1. Introduction

Due to the complexity of biological interactions, the mathematical models that describe population dynamics usually involve many meaningful parameters. If the qualitative or topological structure of a given model with parameters changes as the parameters vary in the neighborhood of a special value of parameters, then the model is said to undergo a bifurcation and the special value of parameters is referred to as the bifurcation value. In order to understand the nonlinear dynamics of a model, it is important to determine which parameters are important for the dynamical change, how to find the bifurcation values, how do the dynamics change when the parameters vary in the neighborhood of the bifurcation values, etc. These problems are very important for species conservation and management or disease control (see for example [Cai *et al.*, 2013; Chen *et al.*, 2013; Etoua & Rousseau, 2010; Hu *et al.*, 2011; Huang *et al.*, 2013a; Huang *et al.*, 2013b; Tang *et al.*, 2008; Xiao & Ruan, 1999; Xiao & Jennings, 2005], and references cited therein). These studies indicate that the nonlinear dynamics of such biological and epidemiological models not only depend on more bifurcation parameters but are also very sensitive to parameter perturbations (see for example [Baer *et al.*, 2006; Huang *et al.*, 2014; Xiao & Ruan, 2001; Xiao & Zhang, 2007; Zhu *et al.*, 2002]).

The predator–prey interaction is one of the most fundamental interactions in ecology and mathematical ecology. Based on a predator–prey system first proposed by May [1973], Caughley [1976] used the following Holling–Tanner model with Holling type II functional response

$$\begin{aligned}\dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{mxy}{a+x},\\ \dot{y} &= sy\left(1-\frac{y}{hx}\right)\end{aligned}\tag{1}$$

to model the biological control of the prickly-pear cactus by the moth *Cactoblastis cactorum*. Wollkind, Collings and Logan [Wollkind *et al.*, 1988] also employed model (1) to study the temperature-mediated stability of the predator–prey mite interaction between *Metaseiulus occidentalis* and the phytophagous spider mite *Tetranychus mcdanieli* on apple trees. They showed that the model exhibits a stable low population density equilibrium, population cycles, or population outbreaks in response to population perturbations for different parameter values.

In order to study how robust the model (1) is with respect to the functional response and to determine how the type of functional response influences bifurcation and stability behavior, Collings [1997] further considered system (1) that incorporates the Holling types III and IV function responses. The Holling type IV or nonmonotonic functional response incorporates prey interference with predation in that the per capita predation rate increases with prey density to a maximum at a critical prey density beyond which it decreases. When the prey species is a spider mite, such as *T. mcdanieli*, one possible source of interference is the webbing produced by these mites which is known to interfere with predators by decreasing their walking speed and reducing their chances of contacting the prey. Thus, predatory mites that are not adapted to walking on webbing can starve in the presence of spider mite prey. Notice that this phenomenon is also known as group defence in population dynamics (see [Ruan & Xiao, 2001; Xiao & Ruan, 2001], and the references cited therein). By numerical simulations, Collings [1997] showed that the model (1) with types I–III functional responses exhibit qualitatively similar bifurcation and stability behavior over the interval of definition of the temperature parameter. Similar behavior is observed in the system (1) with type IV functional response at low levels of prey interference, while the prevalence of bistability and the presence of three attractors for some values of the model parameters demonstrate that higher levels of interference are destabilizing the system.

Recently, we [Huang *et al.*, 2014] provided detailed analysis on the nonlinear dynamics of model (1) with generalized Holling type III functional response. Li and Xiao [2007] considered model (1) with simplified Holling type IV functional response

$$\begin{aligned}\dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{mxy}{x^2+b},\\ \dot{y} &= sy\left(1-\frac{y}{hx}\right),\end{aligned}\tag{2}$$

where r, K, m, b, s and h are all positive parameters, and $\frac{mx}{x^2+b}$ is the simplified Holling type IV functional response. For the sake of simplicity, they scaled x, y, t and parameters in (2) by letting

$$\bar{t} = rt,\quad \bar{x} = \frac{x}{K},\quad \bar{y} = \frac{my}{rK^2},$$

$$a = \frac{b}{K^2},\quad \delta = \frac{s}{r},\quad \beta = \frac{sK}{hm}.$$

Dropping the bars, model (2) becomes

$$\begin{aligned}\dot{x} &= x(1-x)-\frac{xy}{x^2+a},\\ \dot{y} &= y\left(\delta-\frac{\beta y}{x}\right),\end{aligned}\tag{3}$$

where a, δ, β are all positive parameters. It was shown in [Li & Xiao, 2007] that the model (3) can have simultaneously two nonhyperbolic positive equilibria for some values of parameters, one is a cusp of codimension 2 and the other is a multiple focus of multiplicity one. Bogdanov–Takens bifurcation and subcritical Hopf bifurcation can occur in the small neighborhoods of these two equilibria, respectively. By theoretical analysis and numerical simulations, it was shown that the model (3) can have a stable limit cycle enclosing two positive equilibria, an unstable limit cycle enclosing a hyperbolic positive equilibrium, or two limit cycles enclosing a hyperbolic positive equilibrium by choosing different values of parameters.

For some other values of parameters, model (3) can have a unique degenerate positive equilibrium, and may exhibit higher codimension and more complex bifurcation phenomena, which was not discussed in [Li & Xiao, 2007]. We will show that this unique degenerate positive equilibrium is indeed a degenerate Bogdanov–Takens singularity (focus case) of codimension 3 for other values of parameters and prove that the model undergoes a degenerate focus type Bogdanov–Takens bifurcation of codimension 3. Hence, the model can exhibit more complex and new bifurcation phenomena, such as saddle-node loop bifurcation and multiple limit cycle bifurcation. Numerical simulations, including a stable limit cycle enclosing a homoclinic loop, a homoclinic loop enclosing an unstable limit cycle, a stable limit cycle enclosing three unstable hyperbolic positive equilibria, confirm the theoretical results.

This paper is organized as follows. In Sec. 2, we show that the unique degenerate positive equilibrium of model (3) is a degenerate Bogdanov–Takens singularity (focus case) of codimension 3 for some values of parameters. In Sec. 3, we prove that the model (3) exhibits degenerate focus type Bogdanov–Takens bifurcation of codimension 3 around the unique degenerate positive equilibrium. Numerical simulations about phase portraits are also given to confirm the theoretical results. This paper ends with a discussion.

2. Degenerate Bogdanov–Takens Singularity (Focus Case) of Codimension 3

In order to present our analysis about system (3), we first quote Lemma 2.1(ii) in [Li & Xiao, 2007] as our first lemma.

Lemma 1. *Let* $A = 1 - 3(a + \frac{\delta}{\beta})$ *and* $\Delta = -4A^3 + (1-27a-3A)^2$. *If both* $\Delta = 0$ *and* $A = 0$, *i.e.* $a = \frac{1}{27}$, $\frac{\delta}{\beta} = \frac{8}{27}$, *then system* (3) *has a unique positive equilibrium* $E^*(x^*, y^*) = (\frac{1}{3}, \frac{8}{81})$, *which is a degenerated equilibrium.*

Li and Xiao [2007] have not discussed the type of the unique degenerate positive equilibrium $E^*(\frac{1}{3}, \frac{8}{81})$, we now consider in this case.

Theorem 1

(i) *If* $(a, \beta) = (\frac{1}{27}, \frac{27}{8}\delta)$ *and* $\delta \neq \frac{2}{3}$, *then system* (3) *has a unique degenerate positive equilibrium* $E^*(\frac{1}{3}, \frac{8}{81})$, *which is a stable* (*an unstable*) *degenerate node if* $\delta > \frac{2}{3}$ ($\delta < \frac{2}{3}$, *respectively*);

(ii) *If* $(a, \beta, \delta) = (\frac{1}{27}, \frac{9}{4}, \frac{2}{3})$, *then the unique degenerate positive equilibrium* $E^*(\frac{1}{3}, \frac{8}{81})$ *of system* (3) *is a codimension 3 degenerate Bogdanov–Takens singularity* (*focus case*). *The phase portraits are given in Fig. 1.*

Proof. (i) When $(a, \beta) = (\frac{1}{27}, \frac{27}{8}\delta)$ and $\delta \neq \frac{2}{3}$, then $\mathrm{Det}(J(E^*)) = 0$ and $\mathrm{Tr}(J(E^*)) \neq 0$ (here $J(E^*)$

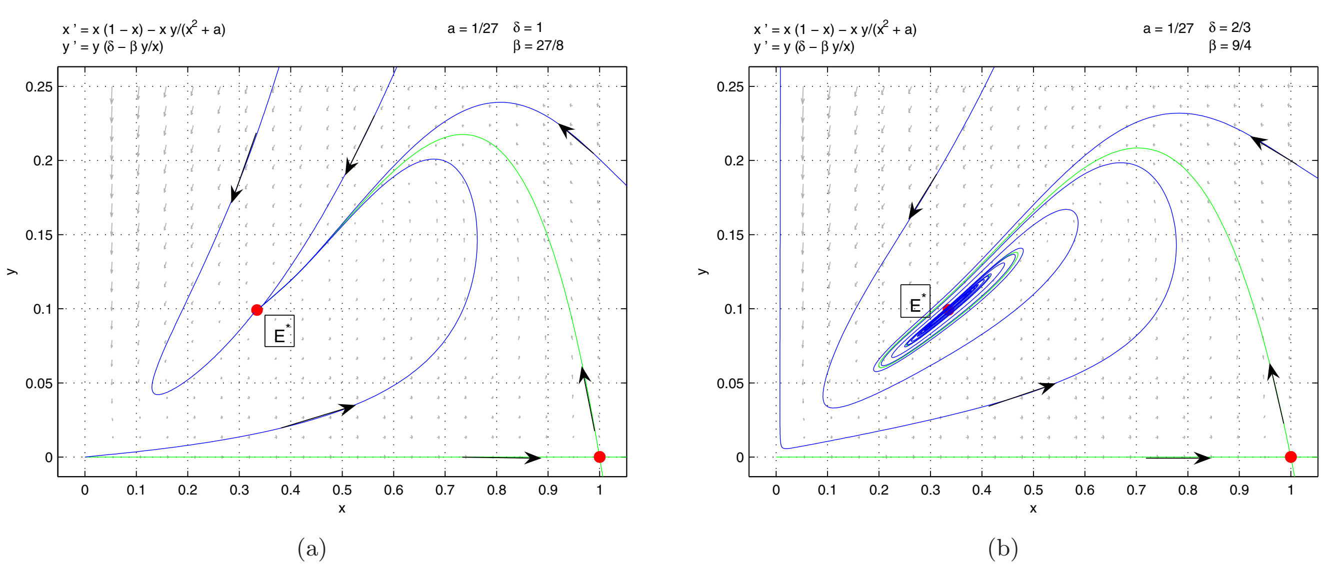


Fig. 1. Phase portrait of the model with a unique degenerate positive equilibrium E^*: (a) a stable degenerate node and (b) a codimension 3 Bogdanov–Takens singularity (focus case).

is the Jacobian matrix of system (3) at E^*), system (3) becomes

$$\begin{aligned}\dot{x} &= x(1-x) - \frac{xy}{x^2+\frac{1}{27}} := F_1(x,y),\\ \dot{y} &= y\left(\delta - \frac{27\delta y}{8x}\right) := G_1(x,y).\end{aligned} \tag{4}$$

To translate the equilibrium $E^*(\frac{1}{3}, \frac{8}{81})$ to the origin and expand system (4) in power series around the origin, let

$$x_1 = x - \frac{1}{3}, \quad y_1 = y - \frac{8}{81}.$$

Then system (4) can be rewritten as

$$\begin{aligned}\dot{x}_1 &= F_1\left(x_1 + \frac{1}{3}, y_1 + \frac{8}{81}\right),\\ \dot{y}_1 &= G_1\left(x_1 + \frac{1}{3}, y_1 + \frac{8}{81}\right).\end{aligned} \tag{5}$$

Make a change of variables as follows:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 9 \\ \frac{4}{9} & 4\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then system (5) becomes

$$\begin{aligned}\dot{x} &= \frac{27\delta xy}{2} + 81\delta(-1+3\delta)y^2 + \frac{243\delta x^3}{32-48\delta} - \frac{729\delta(17-24\delta+18\delta^2)xy^2}{-8+12\delta} + \frac{2187\delta x^2 y}{16-24\delta}\\ &\quad - \frac{2187\delta(11-24\delta+18\delta^2)y^3}{-4+6\delta} + \frac{2187\delta x^4}{64(-2+3\delta)} + \frac{19683\delta(62-123\delta+72\delta^2)xy^3}{8(-2+3\delta)}\\ &\quad + \frac{6561\delta(52-75\delta+36\delta^2)x^2y^2}{-32+48\delta} - \frac{19683(-2+\delta)\delta x^3 y}{-64+96\delta} + \frac{59049\delta(25-57\delta+36\delta^2)y^4}{-8+12\delta} + O(|x,y|^5),\\ \dot{y} &= \left(\frac{2}{3}-\delta\right)y - \frac{3xy}{2} - \frac{9(2-6\delta+9\delta^2)y^2}{2} + \frac{27}{-32+48\delta}x^3 + \frac{243(3+4\delta-12\delta^2+9\delta^3)xy^2}{-8+12\delta}\\ &\quad + \frac{243x^2y}{-16+24\delta} + \frac{729(1+4\delta-12\delta^2+9\delta^3)y^3}{-4+6\delta} + \frac{243}{64(2-3\delta)}x^4 - \frac{6561(10+7\delta-48\delta^2+36\delta^3)xy^3}{8(-2+3\delta)}\\ &\quad - \frac{2187(12-\delta-24\delta^2+18\delta^3)x^2y^2}{-32+48\delta} + \frac{2187(-2+\delta)x^3y}{-64+96\delta} - \frac{19683(3+5\delta-24\delta^2+18\delta^3)y^4}{-8+12\delta} + O(|x,y|^5).\end{aligned} \tag{6}$$

Finally, make a time transform $\tau = \frac{2-3\delta}{3}t$, then system (6) becomes (still denote time by t)

$$\begin{aligned}\dot{x} &= \frac{81\delta xy}{4-6\delta} + \frac{243\delta(-1+3\delta)y^2}{2-3\delta} + \frac{729\delta x^3}{16(2-3\delta)^2} + \frac{2187\delta(17-24\delta+18\delta^2)xy^2}{4(2-3\delta)^2} + \frac{6561\delta x^2y}{8(2-3\delta)^2}\\ &\quad + \frac{6561\delta(11-24\delta+18\delta^2)y^3}{2(2-3\delta)^2} - \frac{6561\delta x^4}{64(2-3\delta)^2} - \frac{59049\delta(62-123\delta+72\delta^2)xy^3}{8(2-3\delta)^2}\\ &\quad - \frac{19683\delta(52-75\delta+36\delta^2)x^2y^2}{16(2-3\delta)^2} + \frac{59049(-2+\delta)\delta x^3y}{32(2-3\delta)^2} - \frac{177147\delta(25-57\delta+36\delta^2)y^4}{4(2-3\delta)^2} + O(|x,y|^5),\\ \dot{y} &= y + \frac{9xy}{-4+6\delta} + \frac{27(2-6\delta+9\delta^2)y^2}{-4+6\delta} - \frac{81x^3}{16(2-3\delta)^2} - \frac{729(3+4\delta-12\delta^2+9\delta^3)xy^2}{4(2-3\delta)^2} - \frac{729x^2y}{8(2-3\delta)^2}\\ &\quad - \frac{2187(1+4\delta-12\delta^2+9\delta^3)y^3}{2(2-3\delta)^2} + \frac{729x^4}{64(2-3\delta)^2} + \frac{19683(10+7\delta-48\delta^2+36\delta^3)xy^3}{8(2-3\delta)^2}\\ &\quad + \frac{6561(12-\delta-24\delta^2+18\delta^3)x^2y^2}{16(2-3\delta)^2} - \frac{6561(-2+\delta)x^3y}{32(2-3\delta)^2} + \frac{59049(3+5\delta-24\delta^2+18\delta^3)y^4}{4(2-3\delta)^2} + O(|x,y|^5).\end{aligned} \tag{7}$$

Hence Theorem 7.1 in [Zhang *et al.*, 1992] implies that the unique degenerate positive equilibrium $E^*(\frac{1}{3}, \frac{8}{81})$ of system (3) is a stable (an unstable) degenerate node if $\delta > \frac{2}{3}$ ($\delta < \frac{2}{3}$, respectively).

(ii) When $(a, \beta) = (\frac{1}{27}, \frac{27}{8}\delta)$ and $\delta = \frac{2}{3}$, that is $(a, \beta, \delta) = (\frac{1}{27}, \frac{9}{4}, \frac{2}{3})$, we have $\mathrm{Det}(J(E^*)) = \mathrm{Tr}(J(E^*)) = 0$, then $E^*(\frac{1}{3}, \frac{8}{81})$ is a nilpotent (or double-zero eigenvalue) equilibrium. We provide a series of explicitly smooth transformations to derive a normal form to determine the exact type of this equilibrium.

First of all, we translate the unique interior equilibrium $E^*(\frac{1}{3}, \frac{8}{81})$ to the origin and expand system (3) in power series around the origin. Let

$$\text{(i)}\quad x_1 = x - \frac{1}{3}, \quad y_1 = y - \frac{8}{81}.$$

Then system (3) can be rewritten as

$$\begin{aligned}\dot{x}_1 &= F_1\left(x_1 + \frac{1}{3}, y_1 + \frac{8}{81}\right),\\ \dot{y}_1 &= G_1\left(x_1 + \frac{1}{3}, y_1 + \frac{8}{81}\right).\end{aligned} \tag{8}$$

Secondly, in order to transform the linear part of system (8) to the Jordan canonical form, we let

$$\text{(ii)}\quad x_1 = \frac{1}{12}(x_2 + y_2), \quad y_1 = \frac{2}{81}x_2 - \frac{1}{81}y_2,$$

then system (8) can be rewritten as

$$\begin{aligned}\dot{x}_2 &= y_2 - \frac{x_2y_2}{24} - \frac{7y_2^2}{24} - \frac{x_2^3}{192} + \frac{3x_2y_2^2}{64} - \frac{x_2^2y_2}{64}\\ &\quad + \frac{11y_2^3}{192} + \frac{x_2^4}{1536} - \frac{35x_2y_2^3}{1536} - \frac{3x_2^2y_2^2}{512} + \frac{7x_2^3y_2}{1536}\\ &\quad - \frac{5y_2^4}{384} + O(|x_2, y_2|^5),\\ \dot{y}_2 &= -\frac{x_2y_2}{12} + \frac{y_2^2}{6} - \frac{x_2^3}{96} - \frac{3x_2y_2^2}{32} - \frac{x_2^2y_2}{32} - \frac{7y_2^3}{96}\\ &\quad + \frac{x_2^4}{768} + \frac{37x_2y_2^3}{768} + \frac{9x_2^2y_2^2}{256} + \frac{7x_2^3y_2}{768} + \frac{y_2^4}{48}\\ &\quad + O(|x_2, y_2|^5).\end{aligned} \tag{9}$$

We next make the following near identity transformation to eliminate the y^2 terms in system (9),

$$\text{(iii)}\quad x_2 = x_3 + \frac{1}{12}x_3^2, \quad y_2 = y_3 + \frac{1}{6}x_3y_3 + \frac{7}{24}y_3^2,$$

which brings system (9) into

$$\begin{aligned}\dot{x}_3 &= y_3 + a_{30}x_3^3 + a_{12}x_3y_3^2 + a_{21}x_3^2y_3\\ &\quad + a_{03}y_3^3 + a_{40}x_3^4 + a_{13}x_3y_3^3 + a_{31}x_3^3y_3\\ &\quad + a_{22}x_3^2y_3^2 + a_{04}y_3^4 + Q_1(x_3, y_3),\\ \dot{y}_3 &= b_{11}x_3y_3 + b_{30}x_3^3 + b_{12}x_3y_3^2 + b_{21}x_3^2y_3\\ &\quad + b_{03}y_3^3 + b_{40}x_3^4 + b_{13}x_3y_3^3 + b_{31}x_3^3y_3\\ &\quad + b_{22}x_3^2y_3^2 + b_{04}y_3^4 + Q_2(x_3, y_3),\end{aligned} \tag{10}$$

where $Q_1(x_3, y_3)$ and $Q_2(x_3, y_3)$ are smooth functions of their arguments with at least fifth-order terms of (x_3, y_3), and

$$\begin{aligned}&a_{30} = -\frac{1}{192}, \quad a_{12} = -\frac{1}{16}, \quad a_{21} = -\frac{29}{1152},\\ &a_{03} = -\frac{65}{576}, \quad a_{40} = \frac{1}{3072}, \quad a_{13} = \frac{131}{6912},\\ &a_{31} = \frac{25}{9216}, \quad a_{22} = \frac{49}{6912}, \quad a_{04} = \frac{85}{6912},\\ &b_{11} = -\frac{1}{12}, \quad b_{30} = -\frac{1}{96}, \quad b_{12} = -\frac{1}{72},\\ &b_{21} = -\frac{7}{192}, \quad b_{03} = \frac{7}{288}, \quad b_{40} = \frac{5}{4608},\\ &b_{13} = -\frac{85}{6912}, \quad b_{31} = \frac{7}{576}, \quad b_{22} = \frac{137}{6912},\\ &b_{04} = -\frac{167}{6912}.\end{aligned}$$

Since

$$b_{11}b_{30} = \frac{1}{1152} \neq 0,$$

it follows from Lemma 3.1 in [Cai *et al.*, 2013] that there exists a small neighborhood U of $(0, 0)$ such that in this neighborhood U system (10) is locally topologically equivalent to

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= b_{11}xy + b_{30}x^3 + (b_{21} + 3a_{30})x^2y\\ &\quad + (b_{40} - b_{11}a_{30})x^4\\ &\quad + \left(4a_{40} + b_{31} + \frac{1}{3}b_{11}a_{21} + \frac{1}{6}b_{11}b_{12}\right)x^3y\\ &\quad + Q(x, y),\end{aligned} \tag{11}$$

where $Q(x, y)$ is a smooth function of their arguments with at least fifth-order terms of (x, y).

Moreover, we have that

$$5b_{30}(b_{21}+3a_{30}) - 3b_{11}(b_{40} - b_{11}a_{30}) = \frac{53}{18432} \neq 0$$

and

$$b_{30} = -\frac{1}{96} < 0, \quad b_{11}^2 + 8b_{30} = -\frac{11}{144} < 0.$$

Once again Lemma 3.1 in [Cai *et al.*, 2013] implies that the equilibrium $(0,0)$ of system (11) is a degenerate focus of codimension 3, thus the unique degenerate positive equilibrium $E^*(\frac{1}{3}, \frac{8}{81})$ of system (3) is a codimension 3 degenerate Bogdanov–Takens singularity (focus case). ■

3. Degenerate Focus Type Bogdanov–Takens Bifurcation of Codimension 3

Theorem 1 indicates that if $(a, \beta, \delta) = (\frac{1}{27}, \frac{9}{4}, \frac{2}{3})$, then the unique degenerate positive equilibrium $E^*(\frac{1}{3}, \frac{8}{81})$ of system (3) is a codimension 3 Bogdanov–Takens singularity (focus case). We next discuss the existence of degenerate focus type Bogdanov–Takens bifurcation of codimension 3 for system (3). To do that, we choose a, δ and β as bifurcation parameters. Set

$$a = \frac{1}{27} + r_1, \quad \delta = \frac{2}{3} + r_2,$$
$$\beta = \frac{9}{4} + r_3, \quad r = (r_1, r_2, r_3)$$

and consider the unfolding system of system (3) as follows

$$\begin{aligned} \dot{x} &= x(1-x) - \frac{xy}{\left(\frac{1}{27} + r_1\right) + x^2}, \\ \dot{y} &= y\left(\frac{2}{3} + r_2 - \left(\frac{9}{4} + r_3\right)\frac{y}{x}\right), \end{aligned} \tag{12}$$

where $r = (r_1, r_2, r_3)$ is a parameter vector in a small neighborhood of $(0,0,0)$. Thus, we have the main theorem of this paper on the existence of a degenerate focus type Bogdanov–Takens bifurcation of codimension 3 in the model.

Theorem 2. *When parameters (a, δ, β) vary in a small neighborhood of $(\frac{1}{27}, \frac{2}{3}, \frac{9}{4})$, system (3) undergoes a degenerate focus type Bogdanov–Takens bifurcation of codimension 3 in a small neighborhood of $E^*(\frac{1}{3}, \frac{8}{81})$. Moreover, in a small neighborhood of the point $(a, \delta, \beta) = (\frac{1}{27}, \frac{2}{3}, \frac{9}{4})$ in the (a, δ, β)-parameter space, there exist a Hopf bifurcation surface, two homoclinic bifurcation surfaces, two saddle-node loop bifurcation surfaces, a multiple limit cycle bifurcation surface, and two saddle-node bifurcation surfaces for system (3). Hence, system (3) has three hyperbolic positive equilibria, two limit cycles, bistability states (one stable equilibrium and one stable limit cycle, or two stable equilibria), or tristability states (two stable equilibria and one stable limit cycle) for various parameter values.*

Proof. First of all, we make successively smooth transformations (i)–(iii) for system (12), which have been used in the proof of Theorem 1, to obtain the following system:

$$\begin{aligned} \dot{X} &= Y + a_{00}(r) + a_{10}(r)X + a_{01}(r)Y \\ &\quad + a_{20}(r)X^2 + a_{11}(r)XY + a_{02}(r)Y^2 \\ &\quad + a_{30}(r)X^3 + a_{12}(r)XY^2 + a_{21}(r)X^2Y \\ &\quad + a_{03}(r)Y^3 + O(|X,Y|^4), \\ \dot{Y} &= b_{00}(r) + b_{10}(r)X + b_{01}(r)Y + b_{20}(r)X^2 \\ &\quad + b_{11}(r)XY + b_{02}(r)Y^2 + b_{30}(r)X^3 \\ &\quad + b_{12}(r)XY^2 + b_{21}(r)X^2Y + b_{03}(r)Y^3 \\ &\quad + O(|X,Y|^4), \end{aligned} \tag{13}$$

where $a_{ij}(r)$ and $b_{ij}(r)$ are smooth functions, we omit their long expressions here for the sake of simplicity, $a_{00}(0) = a_{10}(0) = a_{01}(0) = a_{02}(0) = a_{20}(0) = a_{11}(0) = b_{00}(0) = b_{01}(0) = b_{10}(0) = b_{20}(0) = b_{02}(0) = 0$, $a_{03}(0) = a_{03}$, $a_{12}(0) = a_{12}$, $a_{21}(0) = a_{21}$, $a_{30}(0) = a_{30}$, $b_{03}(0) = b_{03}$, $b_{11}(0) = b_{11}$, $b_{12}(0) = b_{12}$, $b_{21}(0) = b_{21}$, $b_{30}(0) = b_{30}$ and $a_{03}, a_{12}, a_{21}, a_{30}, b_{03}, b_{11}, b_{12}, b_{21}, b_{30}$ are given in system (10).

Secondly, we make the following near-identity transformation to simplify the third-order terms when $r = 0$

$$\text{(iv)} \quad X = x_1 + \frac{b_{12}}{6}x_1^3 + \frac{a_{12}+b_{03}}{2}x_1^2y_1 + a_{03}x_1y_1^2,$$
$$Y = y_1 + \frac{b_{12}}{2}x_1^2y_1 + b_{03}x_1y_1^2$$

and rewrite system (13) as follows

$$\dot{x}_1 = y_1 + e_{00}(r) + e_{10}(r)x_1 + e_{01}(r)y_1 + e_{20}(r)x_1^2 + e_{11}(r)x_1y_1 + e_{02}(r)y_1^2 + e_{30}(r)x_1^3$$

$$+ e_{12}(r)x_1y_1^2 + e_{21}(r)x_1^2y_1 + e_{03}(r)y_1^3 + O(|x_1, y_1|^4),$$

$$\begin{aligned}\dot{y}_1 = {} & f_{00}(r) + f_{10}(r)x_1 + f_{01}(r)y_1 + f_{20}(r)x_1^2 \\ & + f_{11}(r)x_1y_1 + f_{02}(r)y_1^2 + f_{30}(r)x_1^3 \\ & + f_{12}(r)x_1y_1^2 + f_{21}(r)x_1^2y_1 + f_{03}(r)y_1^3 \\ & + O(|x_1, y_1|^4),\end{aligned} \tag{14}$$

where $e_{ij}(r)$ and $f_{ij}(r)$ can be expressed by $a_{ij}(r)$, $b_{ij}(r)$, b_{12}, a_{12}, b_{03} and a_{03}, we also omit their expressions here for the sake of brevity.

Thirdly, under the following near-identity transformation

$$\text{(v)}\quad \begin{aligned} x_2 = {} & x_1, \\ y_2 = {} & y_1 + e_{00}(r) + e_{10}(r)x_1 + e_{01}(r)y_1 \\ & + e_{20}(r)x_1^2 + e_{11}(r)x_1y_1 + e_{02}(r)y_1^2 \\ & + e_{30}(r)x_1^3 + e_{12}(r)x_1y_1^2 + e_{21}(r)x_1^2y_1 \\ & + e_{03}(r)y_1^3 + O(|x_1, y_1|^4),\end{aligned}$$

system (14) becomes

$$\begin{aligned}\dot{x}_2 = {} & y_2, \\ \dot{y}_2 = {} & g_{00}(r) + g_{10}(r)x_2 + g_{01}(r)y_2 \\ & + g_{20}(r)x_2^2 + g_{11}(r)x_2y_2 + g_{02}(r)y_2^2 \\ & + g_{30}(r)x_2^3 + g_{12}(r)x_2y_2^2 + g_{21}(r)x_2^2y_1 \\ & + g_{03}(r)y_2^3 + O(|x_2, y_2|^4),\end{aligned} \tag{15}$$

where $g_{ij}(r)$ can be expressed by $e_{ij}(r)$ and $f_{ij}(r)$, we also omit their expressions here for reasons of space.

Finally, following the steps in [Xiao & Zhang, 2007], we can rewrite system (15) as

$$\begin{aligned}\dot{x}_3 = {} & \frac{\sigma(r)}{\nu(r)}y_3, \\ \dot{y}_3 = {} & \frac{-g_{30}(r)}{\sigma(r)}[\lambda_1(r) + \lambda_2(r)\nu(r)x_3 - \nu^3(r)x_3^3] \\ & + g_{21}(r)y_3[\lambda_3(r) + A(r)\nu(r)x_3 + \nu^2(r)x_3^2] \\ & + y_3^2Q_1(x_3, y_3, r) + O(|x_3, y_3|^4),\end{aligned} \tag{16}$$

where

$$\lambda_1(r) = -\frac{g_{00}(r)}{g_{30}(r)} + \frac{g_{10}(r)g_{20}(r)}{3g_{30}^2(r)} - \frac{g_{20}^3(r)}{9g_{30}^3(r)} + \frac{g_{20}^3(r)}{27g_{30}^3(r)},$$

$$\lambda_2(r) = -\frac{g_{10}(r)}{g_{30}(r)} + \frac{g_{20}^2(r)}{3g_{30}^2(r)},$$

$$\lambda_3(r) = \frac{g_{01}(r)}{g_{21}(r)} - \frac{g_{11}(r)g_{20}(r)}{3g_{21}(r)g_{30}(r)} + \frac{g_{21}(r)g_{20}^2(r)}{9g_{21}(r)g_{30}^2(r)},$$

$$A(r) = \frac{g_{11}(r)}{g_{21}(r)} + \frac{2g_{20}(r)}{3g_{30}(r)},$$

$$Q_1(x_3, y_3, r) = \sigma(r)\left[g_{02}(r) + \frac{g_{12}(r)g_{20}^2(r)}{9g_{30}^2(r)} + \sigma(r)g_{03}(r)y_3 + \nu(r)g_{12}(r)x_3\right].$$

We obtain that $g_{30}(0) = -\frac{1}{96} < 0$ and $g_{21}(0) = -\frac{5}{96} < 0$ by using the computer software *Mathematica*, so we can choose

$$\sigma(r) = -\frac{g_{30}(r)}{g_{21}(r)}\nu(r),$$

$$\nu(r) = \sqrt{-\frac{g_{30}(r)}{g_{21}^2(r)}}$$

in the small neighborhood of $r = (0, 0, 0)$. In order to get the canonical unfolding of the focus type Bogdanov–Takens singularity of codimension 3, we make the final time transformation $\tau = -\frac{g_{30}(r)}{g_{21}(r)}t$, and still denote τ by t, system (16) becomes

$$\begin{aligned}\dot{x}_3 = {} & y_3, \\ \dot{y}_3 = {} & \mu_1(r) + \mu_2(r)x_3 - x_3^3 \\ & + y_3[\mu_3(r) + A_1(r)x_3 + x_3^2] \\ & + y_3^2Q_2(x_3, y_3, r) + O(|x_3, y_3|^4),\end{aligned} \tag{17}$$

where $A_1(r) = \frac{g_{21}(r)\sqrt{-g_{30}(r)}}{g_{30}(r)}A(r)$, $Q_2(x_3, y_3, r) = -\frac{g_{21}(r)}{g_{30}(r)}Q_1(x_3, y_3, r)$, and

$$\mu_1(r) = \frac{g_{21}^3(r)}{g_{30}(r)\sqrt{-g_{30}(r)}}\lambda_1(r),$$

$$\mu_2(r) = -\frac{g_{21}^2(r)}{g_{30}(r)}\lambda_2(r),$$

$$\mu_3(r) = -\frac{g_{21}^2(r)}{g_{30}(r)}\lambda_3(r). \tag{18}$$

Since

$$\left|\frac{\partial(\mu_1(r),\mu_2(r),\mu_3(r))}{\partial(r_1,r_2,r_3)}\right|_{r=0} = \frac{-62500\sqrt{\frac{2}{3}}}{9} \neq 0,$$

the above parameter transformation (18) is a homeomorphism in a small neighborhood of the origin, and μ_1, μ_2 and μ_3 are independent parameters. Furthermore, for system (17), the coefficients of x_3^3 and $x_3^2y_3$ are -1 and 1, respectively, the coefficient of x_3y_3 is $A_1(r)$, which can be calculated as follows when $r = 0$

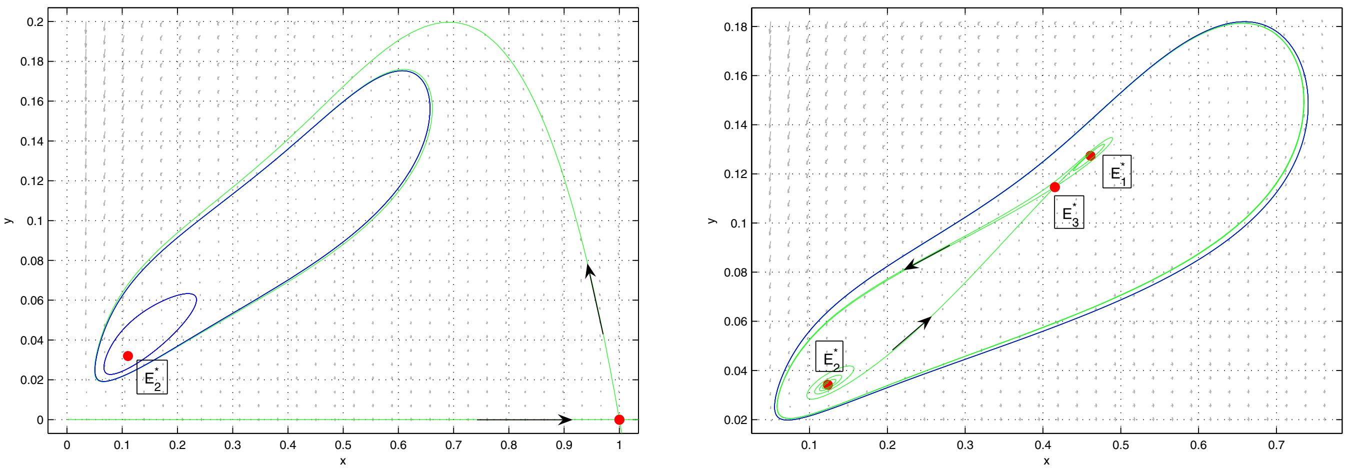
$$A_1(0) = \sqrt{\frac{2}{3}} < 2\sqrt{2}.$$

By the results in [Dumortier *et al.*, 1991] or [Xiao & Zhang, 2007] or [Huang *et al.*, 2014], we know that system (17) is a generic 3-parameters family or standard family of Bogdanov–Takens singularity of codimension 3 (focus case). Thus, system (3) will undergo a degenerate focus type Bogdanov–Takens bifurcation of codimension 3 by choosing a, δ and β as bifurcation parameters in a small neighborhood of $(\frac{1}{27}, \frac{2}{3}, \frac{9}{4})$. ■

Remark 3.1. Because determining the maximum number of limit cycles for the versal unfolding of a focus type Bogdanov–Takens singularity of codimension 3 is still an open problem, we refer the reader to the conjecture bifurcation diagram Fig. 3 on page 7 in [Dumortier *et al.*, 1991] (note that we have made the time reversal transformation $\tau = -\frac{g_{30}(r)}{g_{21}(r)}t$, which transforms stable points and cycles into unstable points and cycles).

Next, we provide a series of phase portraits by numerical simulation to confirm the existence of degenerate focus type Bogdanov–Takens bifurcation of codimension 3 in model (3). Based on the parameter values in [Li & Xiao, 2007], we fix $a = \frac{41\sqrt{17}-169}{2}$, $\delta = \frac{\sqrt{17}-3}{2} + \lambda_1$, $\beta = \frac{\sqrt{17}+4}{4} + \lambda_2$.

(i) When $(\lambda_1, \lambda_2) = (-0.01, -0.12)$, there exist a big stable limit cycle enclosing a little unstable limit cycle and a unique stable hyperbolic positive equilibrium E_2^* [see Fig. 2(a)].
(ii) When $(\lambda_1, \lambda_2) = (-0.056, -0.2)$, there exist a big stable limit cycle enclosing three unstable hyperbolic positive equilibria, where E_3^* is a saddle [see Fig. 2(b)].



(a) (b)

Fig. 2. Phase portraits for model (3) with $a = \frac{41\sqrt{17}-169}{2}$, $\delta = \frac{\sqrt{17}-3}{2} + \lambda_1$, $\beta = \frac{\sqrt{17}+4}{4} + \lambda_2$: (a) $(\lambda_1, \lambda_2) = (-0.01, -0.12)$, a big stable limit cycle enclosing a little unstable limit cycle, (b) $(\lambda_1, \lambda_2) = (-0.056, -0.2)$, a big stable limit cycle enclosing three unstable hyperbolic positive equilibria, (c) $(\lambda_1, \lambda_2) = (-0.0566, -0.2)$, a big stable limit cycle enclosing a little unstable limit cycle and three hyperbolic positive equilibria, bistability states, (d) $(\lambda_1, \lambda_2) = (-0.0568, -0.2)$, a big stable limit cycle enclosing a little homoclinic cycle, (e) $(\lambda_1, \lambda_2) = (-0.06022, -0.2)$, a big stable limit cycle enclosing three hyperbolic positive equilibria, (f) $(\lambda_1, \lambda_2) = (0.052, 0.2)$, three hyperbolic positive equilibria and a little unstable limit cycle enclosing a stable equilibrium E_2^*, (g) $(\lambda_1, \lambda_2) = (0.0537, 0.2)$, a big homoclinic cycle enclosing an unstable limit cycle, bistability states and (h) $(\lambda_1, \lambda_2) = (0.055, 0.2)$, a big stable limit cycle enclosing an unstable limit cycle, tristability states.

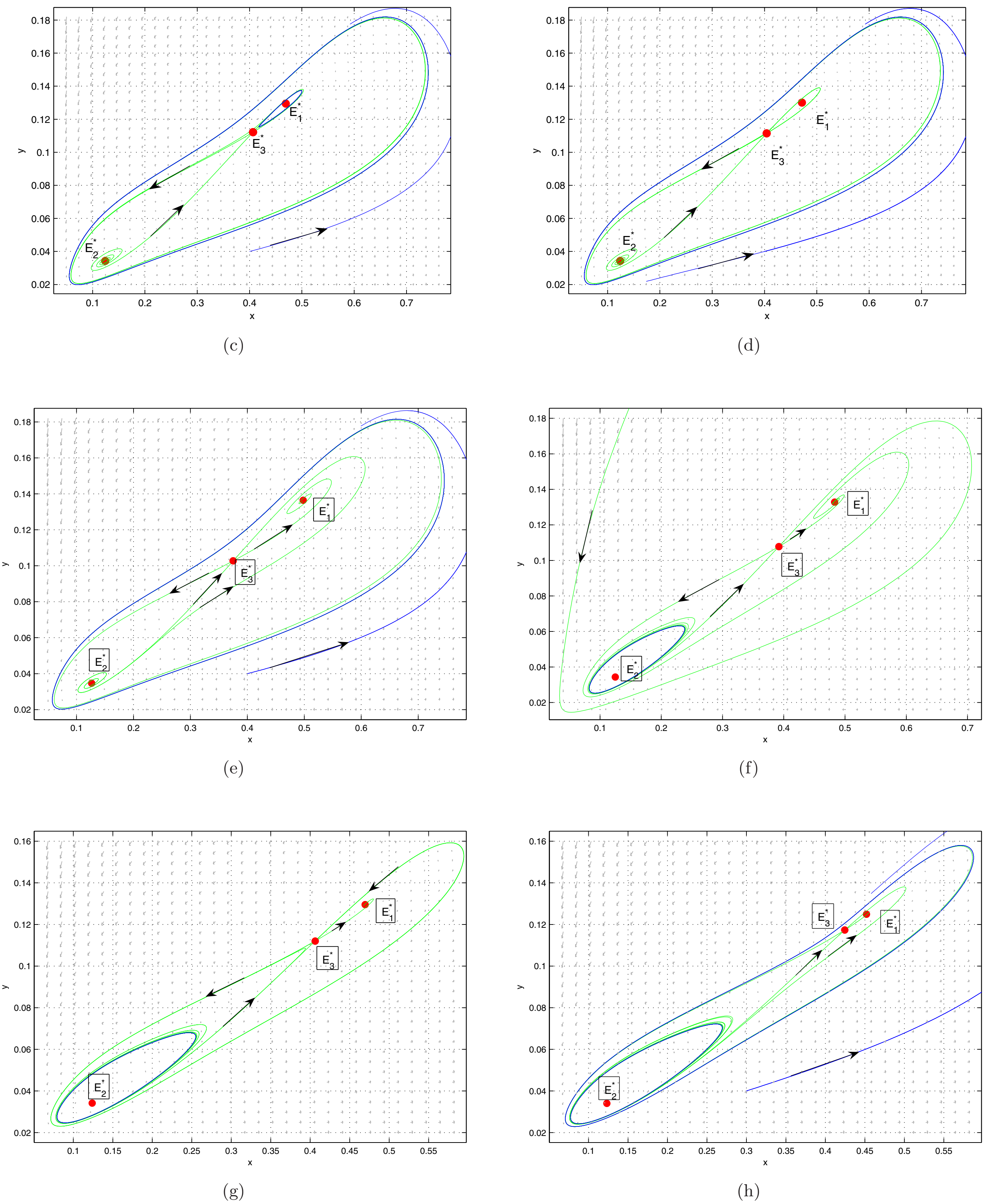


Fig. 2. (*Continued*)

(iii) When $(\lambda_1, \lambda_2) = (-0.0566, -0.2)$, there exist a big stable limit cycle enclosing a little unstable limit cycle and three hyperbolic positive equilibria, which exhibit a kind of bistability states (a big stable limit cycle and one stable equilibrium E_1^*) [see Fig. 2(c)]. From Figs. 2(b) and 2(c), we can see that a subcritical Hopf bifurcation occurs around E_1^*.

(iv) When $(\lambda_1, \lambda_2) = (-0.0568, -0.2)$, there exist a big stable limit cycle enclosing a little homoclinic cycle and three hyperbolic positive equilibria [see Fig. 2(d)].

(v) When $(\lambda_1, \lambda_2) = (-0.06022, -0.2)$, there exist a big stable limit cycle enclosing three hyperbolic positive equilibria [see Fig. 2(e)]. From Figs. 2(c)–2(e), we can see that a little homoclinic bifurcation occurs around E_3^*.

(vi) When $(\lambda_1, \lambda_2) = (0.052, 0.2)$, there exist three hyperbolic positive equilibria and a little unstable limit cycle enclosing a stable equilibrium E_2^* [see Fig. 2(f)]. From Figs. 2(e)–2(f), we can see that a subcritical Hopf bifurcation occurs around E_2^*.

(vii) When $(\lambda_1, \lambda_2) = (0.0537, 0.2)$, there exist a big homoclinic cycle enclosing an unstable limit cycle [see Fig. 2(g)], which exhibits another kind of bistability states (two stable equilibria E_1^* and E_2^*).

(viii) When $(\lambda_1, \lambda_2) = (0.055, 0.2)$, there exist a big stable limit cycle enclosing an unstable limit cycle [see Fig. 2(h)], which exhibits tristability states (two stable equilibria E_1^* and E_2^* and a big stable limit cycle). From Figs. 2(f)–2(h), we can see that a big homoclinic bifurcation occurs around E_3^*.

4. Discussion

In order to better understand the temperature-mediated stability of the predator–prey mite interaction between *Metaseiulus occidentalis* and the phytophagous spider mite *Tetranychus mcdanieli* on apple trees, Collings [1997] extended a predator–prey model of Leslie type with Holling type II functional response used by Wollkind *et al.* [1988] to one with Holling type IV function. On top of the existence of a stable low population density equilibrium, population cycles, or population outbreaks in response to population perturbations were shown for different parameter values, his numerical simulations demonstrated the prevalence of bistability and the presence of three attractors for some values of the model parameters in the case of Holling type IV functional response. Li and Xiao [2007] showed that the model (3) can have simultaneously two nonhyperbolic positive equilibria for some values of parameters, one is a cusp of codimension 2 and the other is a multiple focus of multiplicity one. Bogdanov–Takens bifurcation and subcritical Hopf bifurcation can occur in the small neighborhoods of these two equilibria, respectively. By theoretical analysis and numerical simulations, it was shown that the model (3) can have a stable limit cycle enclosing two positive equilibria, an unstable limit cycle enclosing a hyperbolic positive equilibrium, or two limit cycles enclosing a hyperbolic positive equilibrium by choosing different values of parameters.

In this paper we showed that for the model (3) with simplified Holling type IV functional response there exists a degenerate Bogdanov–Takens singularity (focus case) of codimension 3 for some values of parameters. We proved that the model exhibits a degenerate focus type Bogdanov–Takens bifurcation of codimension 3 around the unique degenerate positive equilibrium. Hence, more complex and new dynamics, such as the existence of three hyperbolic positive equilibria, two limit cycles, two kinds of bistability states (one stable equilibrium and one stable limit cycle, two stable equilibria), or tristability states (two stable equilibria and one stable limit cycle) can occur for various parameter values, which not only support the numerical observations of Collings [1997] that there are different kinds of population oscillations and outbreaks in response to increasing temperature-dependent parameters and population perturbations (initial population density) but also complete the bifurcation analysis of Li and Xiao [2007] on the model, which provides a mechanism of the occurrence of complex dynamics in the model. Therefore, our results are a complement of the results in [Li & Xiao, 2007] and [Collings, 1997] on the model.

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