DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **18**, Number **8**, October **2013**

doi:10.3934/dcdsb.2013.18.2101

pp. **2101–2121**

BIFURCATION ANALYSIS IN A PREDATOR-PREY MODEL WITH CONSTANT-YIELD PREDATOR HARVESTING

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(Communicated by Yuan Lou)

Abstract. In this paper we study the effect of constant-yield predator harvesting on the dynamics of a Leslie-Gower type predator-prey model. It is shown that the model has a Bogdanov-Takens singularity (cusp case) of codimension 3 or a weak focus of multiplicity two for some parameter values, respectively. Saddle-node bifurcation, repelling and attracting Bogdanov-Takens bifurcations, supercritical and subcritical Hopf bifurcations, and degenerate Hopf bifurcation are shown as the values of parameters vary. Hence, there are different parameter values for which the model has a homoclinic loop or two limit cycles. It is also proven that there exists a critical harvesting value such that the predator specie goes extinct for all admissible initial densities of both species when the harvest rate is greater than the critical value. These results indicate that the dynamical behavior of the model is very sensitive to the constant-yield predator harvesting and the initial densities of both species and it requires careful management in the applied conservation and renewable resource contexts. Numerical simulations, including the repelling and attracting Bogdanov-Takens bifurcation diagrams and corresponding phase portraits, two limit cycles, the coexistence of a stable homoclinic loop and an unstable limit cycle, and a stable limit cycle enclosing an unstable multiple focus with multiplicity one, are presented which not only support the theoretical analysis but also indicate the existence of Bogdanov-Takens bifurcation (cusp case) of codimension 3. These results reveal far richer and much more complex dynamics compared to the model without harvesting or with only constant-yield prey harvesting.

1. **Introduction.** The exploitation of biological resources and the harvesting of populations are commonly practiced in fishery, forestry, and wildlife management. Mathematical models have been used extensively and successfully to gain insight into the scientific management of renewable resources like fisheries and forestries (Clark [10]). The optimal management of renewable resources is based on the

2010 *Mathematics Subject Classification.* Primary: 34C25, 34C23; Secondary: 37N25, 92D25.

Key words and phrases. Predator-prey model, constant-yield predator harvesting, Bogdanov-Takens singularity of codimension 3, Bogdanov-Takens bifurcation, degenerate Hopf bifurcation.

Research of this paper was partially supported by the NSF (DMS-1022728), NSFC (No.11101170, No.11228104), Research Project of CCNU (No.CCNU12A01007), and the State Scholarship Fund of the CSC (No.2011842509).

notion of maximum sustainable yield (MSY) of harvesting, which is the maximum harvesting compatibility with survival. A population will go extinction if it is harvested more than its MSY (i.e. it is over-exploited). We must determine the MSY for the harvesting (when harvesting is allowed) of each population and assure the preservation of all species.

Predator-prey models play an important role in studying the management of renewable resources (Clark [10], Chrstensen [9], Hill et al. [17]). The effect of harvesting on the dynamics of predator-prey systems and the role of harvesting in the management of renewable resources have attracted great attention, see for example, Beddington and Cooke [1], Beddington and May [2], Brauer and Soudack [5, 6, 7], Dai and Tang [12], Etoua and Rousseau [14], Hogarth et al. [18], Leard et al. [23], May et al. [24], Myerscough et al. [27], Xiao and Jennings [33], Xiao and Ruan [34], etc. In order to investigate the interaction between the krill (prey) and whale (predator) populations in the Southern Ocean, May et al. [24] proposed the following model to describe the interaction of predators and their prey subject to various harvesting regimes:

$$\begin{aligned} \dot{x} &= r_1 x(1-\tfrac{x}{K}) - axy - H_1, \\ \dot{y} &= r_2 y(1-\tfrac{y}{bx}) - H_2, \end{aligned} \tag{1}$$

where $x(t) > 0$ and $y(t) \geq 0$ represent the population densities of the prey and predators at time $t \geq 0$, respectively; r_1 and K describe the intrinsic growth rate and the carrying capacity of the prey in the absence of predators, respectively; a is the maximum value at which per capita reduction rate of the prey x can attain; r_2 is the intrinsic growth rate of predators; bx takes on the role of a prey-dependent carrying capacity for predators and b is a measure of the quality of the food for predators.

H_1 and H_2 describe the effect of harvesting on the prey and predators, respectively. Two types of harvesting have been proposed (see May et al. [24]): *constant-effort harvesting*, described by a constant multiplication of the size of the population under harvest, and *constant-yield harvesting*, described by a constant independent of the size of the population under harvest.

Various special cases of system (1) have been studied by different researchers.

(a) Unharvested system. When there is no harvesting, that is, $H_1 = H_2 = 0$, system (1) becomes the so-called Leslie-Gower type predator-prey model

$$\begin{aligned} \dot{x} &= r_1 x(1-\tfrac{x}{K}) - axy, \\ \dot{y} &= r_2 y(1-\tfrac{y}{bx}), \end{aligned} \tag{2}$$

which has been studied extensively, for example, Hsu and Huang [21]. In particular, they showed that the unique positive equilibrium of system (2) is globally asymptotically stable under all biologically admissible parameters.

(b) Constant-effort harvesting on both the prey and predators. Beddington and May [2] analyzed system (1) when both the prey and predators were harvested with constant-effort, $H_1 = r_1 h_1 x$ and $H_2 = r_2 h_2 y$; namely,

$$\begin{aligned} \dot{x} &= r_1 x(1-\tfrac{x}{K}) - axy - r_1 h_1 x = r_1 x[(1-h_1) - \tfrac{x}{K}] - axy, \\ \dot{y} &= r_2 y(1-\tfrac{y}{bx}) - r_2 h_2 y = r_2 y[(1-h_2) - \tfrac{y}{bx})], \end{aligned} \tag{3}$$

they discussed both the maximization of the sustainable yield of predators (whales), for a specified level of fishing effort on the prey (krill), and conversely, the maximization of the sustainable yield of the prey, for a specified level of fishing effort on predators. Notice that H_1 and H_2 are linear functions of x and y, respectively, and the harvesting terms can be combined into the growth/death terms as in system (3), so the dynamics of system (3) are very similar to that of the unharvested system (2).

(c) Constant-yield harvesting on the prey and constant-effort harvesting on predators. The case when the prey are harvested at a constant-yield rate ($H_1 = h_1$) and predators are harvested with constant-effort ($H_2 = r_2 h_2 y$); that is, the model

$$\begin{aligned} \dot{x} &= r_1 x(1 - \tfrac{x}{K}) - axy - h_1, \\ \dot{y} &= r_2 y[(1 - h_2) - \tfrac{y}{bx})] \end{aligned} \tag{4}$$

was considered in Beddington and Cooke [1]. Compared with the case of no predator harvesting their results showed that the equilibrium is closer to the upper boundary of the domain of attraction for same percentage of the stable MSY in the case of constant-effort predator harvesting. Thus, if the same degree of stability is required as in the case of no predator harvesting, the increase in potential prey MSY brought about by predator harvesting cannot be fully exploited.

(d) Constant-yield harvesting on both the prey and predators. Beddington and Cooke [1] also studied system (1) with constant-yield harvesting on both the prey and predators, i.e., $H_1 = h_1$ and $H_2 = h_2$ both are constant:

$$\begin{aligned} \dot{x} &= r_1 x(1 - \tfrac{x}{K}) - axy - h_1, \\ \dot{y} &= r_2 y(1 - \tfrac{y}{bx}) - h_2. \end{aligned} \tag{5}$$

They found that if the predators are in a state of heavy depletion the effect of the prey harvesting on predator replacement yields or recovery rates is not very great, but as the predator rise to higher levels the effect becomes substantial.

(e) Constant-yield harvesting on the prey only. Zhu and Lan [37] and Gong and Huang [16] considered system (1) when only the prey population is harvested at a constant-yield rate, i.e., $H_1 = h_1$ and $H_2 = 0:$

$$\begin{aligned} \dot{x} &= r_1 x(1 - \tfrac{x}{K}) - axy - h_1, \\ \dot{y} &= r_2 y(1 - \tfrac{y}{bx}). \end{aligned} \tag{6}$$

They obtained various bifurcations including saddle-node bifurcation, supercritical and subcritical Hopf bifurcations of codimension 1, and Bogdanov-Takens bifurcation. Notice that the dynamics of model (4) are similar to that of system (6).

To develop a fishery in an unexploited area, the initial target species are usually the larger and higher-priced predators (Chrstensen [9]). One exploitation pattern can be described as the "tuna strategy" (Pauly [28], Chrstensen [9]): if well managed, the strategy may result in a sustainable biomass of predators around half of their maximal level and at MSY in total catches. However, if the predators are fished so heavily that their biomass and production decline to almost zero which results in increasing the biomass of their prey to a maximal level. This is the so-called "whale strategy" (Pauly [28], Chrstensen [9]) and has been debated and become controversial when predators like whales and seals are harvested (May et al. [24], Flaaten [15], Yodzis [35]).

The purpose of this paper is to study the effect of constant-yield predator harvesting in system (1), that is to consider

$$\begin{aligned} \dot{x} &= r_1 x(1 - \tfrac{x}{K}) - axy, \\ \dot{y} &= r_2 y(1 - \tfrac{y}{bx}) - h_2, \end{aligned} \tag{7}$$

where $h_2 > 0$ denotes the constant-yield predator harvesting. Let us simplify model (7) with the following scaling

$$t \to r_1 t, x \to \frac{x}{K}, y \to \frac{ay}{r_1},$$

then model (7) takes the form

$$\begin{aligned} \dot{x} &= x(1-x) - xy, \\ \dot{y} &= y(\delta - \tfrac{\beta y}{x}) - h, \end{aligned} \tag{8}$$

here $\delta = \frac{r_2}{r_1}$, $\beta = \frac{r_2}{abK}$, $h = \frac{ah_2}{r_1}$ are positive constants. Saddle-node bifurcation, repelling and attracting Bogdanov-Takens bifurcations of codimension 2, supercritical and subcritical Hopf bifurcations, and degenerate Hopf bifurcation are shown in model (8) as the values of parameters vary, and there exists a critical predator harvesting rate such that predators go extinct when the harvest rate is greater than the critical value. It is shown that the model has a Bogdanov-Takens singularity (cusp) of codimension 3 or a weak focus of multiplicity two for some various parameter values. Numerical simulations, including the repelling and attracting Bogdanov-Takens bifurcation diagrams and corresponding phase portraits, two limit cycles, the coexistence of a stable homoclinic loop and an unstable limit cycle, or a stable limit cycle enclosing an unstable multiple focus with multiplicity one, are presented to not only support the theoretical analysis but also indicate the existence of Bogdanov-Takens bifurcation (cusp case) of codimension 3. These complex dynamics cannot occur in the unharvested system (2).

This paper is organized as follows. In section 2, we study the existence of equilibria and various types of dynamical behavior in the small neighborhood of each equilibrium for model (8). In section 3, we discuss bifurcations of model (8) depending on all parameters. We show that the model has a Bogdanov-Takens singularity (cusp) of codimension 3 or a weak focus of multiplicity two for some parameter values and exhibits saddle-node bifurcation, Hopf bifurcation of codimension 1, degenerate Hopf bifurcation, repelling and attracting Bogdanov-Takens bifurcations of codimension 2 in terms of the original parameters. The paper ends with a brief discussion about the effect of constant-yield predator harvesting on system (8) and a brief comparison about different dynamics between systems (2), (6), and (7).

2. **Equilibria and their stability.**

In this section we discuss the existence and stability of equilibria in system (8) which is rewritten as

$$\begin{aligned} \dot{x} &= x(1-x) - xy := f_1(x,y), \\ \dot{y} &= y(\delta - \tfrac{\beta y}{x}) - h := f_2(x,y). \end{aligned} \tag{9}$$

By the biological meaning of the model variables, we only consider system (9) in the region $\Omega = \{(x,y) : x > 0, y \geq 0\}$ in the (x,y)-plane. Notice that unlike the classical predator-prey systems, $(0,0)$ is not an equilibrium of system (9), and the system is not even well defined at $(0,0)$. We can see that the region Ω is not invariant under the flow and all solutions once touching the x-axis will leave the first quadrant since $h > 0$, which makes the analysis of system (9) more challenging.

First, we determine the location and number of equilibria of system (9). To find the equilibria of system (9) in Ω, we consider the algebraic equations in x and y,

$$\begin{aligned} &x(1-x) - xy = 0, \\ &y(\delta - \tfrac{\beta y}{x}) - h = 0. \end{aligned} \tag{10}$$

By the nonnegativeness of solutions of (10), we only need to consider the nonnegative solutions of the following equations:

$$\begin{aligned} &1-x-y=0,\\ &y(\delta-\tfrac{\beta y}{x})-h=0. \end{aligned} \tag{11}$$

Since the x coordinate of any positive equilibrium must satisfy $0<x<1$ and $y=1-x$ (from the first equation of (11)), the second equation of (11) takes the form

$$(\beta+\delta)x^2-(2\beta+\delta-h)x+\beta=0. \tag{12}$$

Using the discriminant $\Delta := (\delta-h)^2-4\beta h$ of (12) and letting

$$h_1=\delta+2\beta-2\sqrt{\beta^2+\beta\delta}, h_2=\delta+2\beta+2\sqrt{\beta^2+\beta\delta}, \tag{13}$$

we obtain the following results:

Lemma 2.1. *The equilibria of system (9) are as follows:*

(i) System (9) has no equilibria in Ω if $h>h_1$. The phase portrait is shown in Fig. 1;

(ii) System (9) has a unique equilibrium (x_1,y_1) in Ω if $h=h_1$, where $x_1=\frac{\delta-h}{\delta+h}, y_1=\frac{2h}{\delta+h}$;

(iii) System (9) has two distinct equilibria (x_2,y_2) and (x_3,y_3) in Ω if $h<h_1$, where

$$\begin{aligned} x_2&=\tfrac{2\beta+\delta-h+\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)}, y_2=\tfrac{\delta+h-\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)},\\ x_3&=\tfrac{2\beta+\delta-h-\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)}, y_3=\tfrac{\delta+h+\sqrt{(\delta-h)^2-4\beta h}}{2(\beta+\delta)}. \end{aligned} \tag{14}$$

Proof. Notice that equation (12) has no real roots if and only if $\Delta<0$, that is $h_1<h<h_2$; equation (12) has only one positive real root if and only if $\Delta=0$ and $h<\delta$, that is $h=h_1$ (because $0<h_1<\delta<h_2$); equation (12) has two distinct positive real roots if and only if $\Delta>0$ and $h<\delta+2\beta$, that is, $h<h_1$ (because $\delta+2\beta<h_2$). We can also check that the real roots of equation (12) are not positive if $\Delta=0, h>\delta$ or $\Delta>0, h>\delta+2\beta$, that is, $h\geq h_2$. □

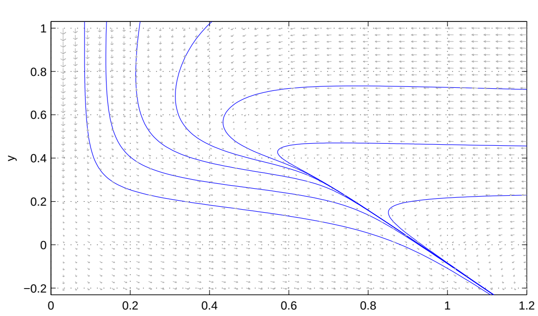

FIGURE 1. The phase portrait of system (9) when it has no equilibria.

Next we study the dynamics of system (9) in the neighborhood of each equilibrium. The Jacobian matrix of system (9) at these equilibria is given by

$$Df(x,y) = \begin{pmatrix} -x & -x \\ \beta(\frac{1}{x}-1)^2 & \delta - 2\beta(\frac{1}{x}-1) \end{pmatrix},$$

where x is the positive real root of equation (12).

Theorem 2.2. *If $h = h_1$, then system (9) has a unique equilibrium $(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$ and no closed orbits in Ω. More precisely,*

(i) if $\delta \neq \frac{h+h^2}{1-h}$, then the equilibrium $(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$ is a saddle-node and it is attracting (repelling) if $\delta > \frac{h+h^2}{1-h}$ $(\delta < \frac{h+h^2}{1-h})$;

(ii) if $\delta = \frac{h+h^2}{1-h}$, then the equilibrium $(h, 1-h)$ is a cusp. The phase portrait is shown in Fig. 2.

Proof. The number of equilibria of system (9) can be obtained straightforward from Lemma 2.1. Next we determine the type of the unique positive equilibrium $(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$.

We calculate the determinant and trace of the Jacobian matrix of system (9) at $(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$ and obtain

$$\mathrm{Det}\big(Df(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})\big) = 0$$

and

$$\mathrm{Tr}\big(Df(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})\big) = \frac{h-\delta}{h+\delta} + h.$$

If $\delta = \frac{h+h^2}{1-h}$, then $\mathrm{Tr}(Df(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})) = 0$, so both eigenvalues of $\mathrm{Det}(Df(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h}))$ are zero. Otherwise, one of the eigenvalues is zero and the other is nonzero. The type of $(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$ can be directly proved by checking the conditions in Zhang et al. [36, Theorems 7.1 - 7.3].

(a) (b) (c)

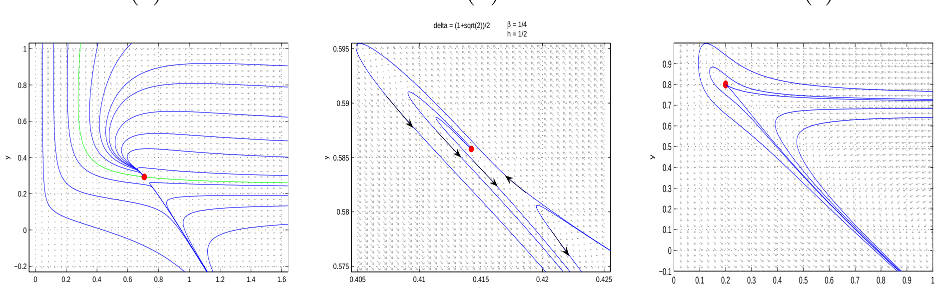

FIGURE 2. The phase portraits of system (2.1) with one equilibrium. (a) Attracting saddle-node with $\delta = 0.4, \beta = 0.4, h = 0.0686$; (b) Repelling saddle-node with $\delta = \frac{1+\sqrt{2}}{2}, \beta = 0.25, h = 0.5$; (c) Cusp with $\delta = 0.3, \beta = 0.0125, h = 0.2$.

Nonexistence of limit cycles in Ω comes from the following argument. If there exists a limit cycle in Ω, then the limit cycle must contain some equilibria in its interior and the sum of indices of these equilibria is one. However, $(\frac{\delta-h}{\delta+h}, \frac{2h}{\delta+h})$ is a unique equilibrium of system (9) in Ω, and it is a saddle-node or a cusp whose index is not one. Hence, it is impossible to have any limit cycle in Ω if system (9) has a unique equilibrium in Ω. This completes the proof of the theorem. $\square$

Theorem 2.3. *If $h < h_1$, then system (9) has two distinct equilibria (x_2, y_2) and (x_3, y_3) in Ω, which are given by (14). Furthermore, (x_2, y_2) is a hyperbolic saddle of system (9) and (x_3, y_3) is an anti-saddle of system (9). More precisely,*

(a) (x_3, y_3) is an unstable focus (or a node) if $h > h_3$;
(b) (x_3, y_3) is a weak focus (or a center) if $h = h_3$;
(c) (x_3, y_3) is a stable focus (or a node) if $h < h_3$;

here $h_3 = \frac{1}{4}(6\beta - 4\beta^2 + 3\delta - 6\beta\delta - 2\delta^2 + (-1 + 2\beta + 2\delta)\sqrt{-8\beta + 4\beta^2 + 4\beta\delta + \delta^2})$. The phase portraits are shown in Fig. 3.

Proof. The existence of two equilibria follows from Lemma 2.1. By some simple calculations, we obtain

$$\mathrm{Det}(Df(x_2, y_2)) = -\frac{\Delta + (2\beta + \delta - h)\sqrt{\Delta}}{2\beta + \delta - h + \sqrt{\Delta}} = -\sqrt{\Delta} < 0,$$

$$\mathrm{Tr}(Df(x_3, y_3)) = -\frac{x_3^2 - (2\beta + \delta)x_3 + 2\beta}{x_3} = 0 \ \text{ if } \ h = h_3,$$

and

$$\mathrm{Det}(Df(x_3, y_3)) = \tfrac{\beta}{x_3} - x_3(\beta + \delta) = -\tfrac{\Delta - (2\beta + \delta - h)\sqrt{\Delta}}{2\beta + \delta - h - \sqrt{\Delta}} = \sqrt{\Delta} > 0. \tag{15}$$

Then it is easy to check that these three cases hold. $\square$

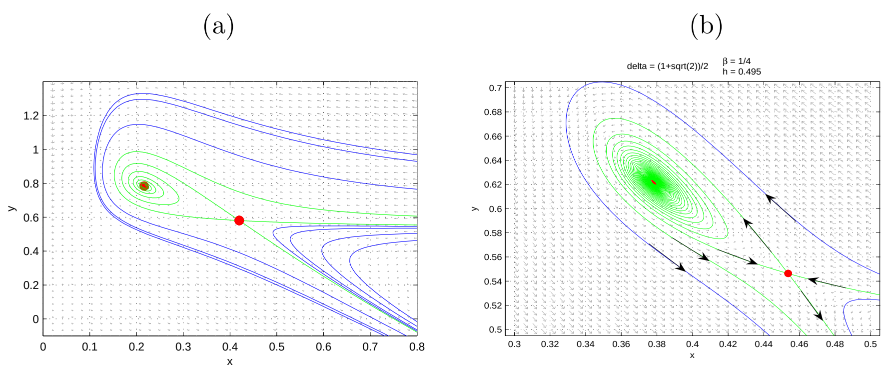

FIGURE 3. The phase portrait of system (9) with two equilibria. (a) A stable focus and a saddle; (b) An unstable focus and a saddle.

3. **Bifurcations.** In this section, we are interested in various possible bifurcations in system (9) including saddle-node, Bogdanov-Takens, and Hopf bifurcations.

3.1. **Saddle-node bifurcation.** It follows from Lemma 2.1, Theorems 2.2 and 2.3 that

$$SN = \{(\beta, \delta, h) : h = h_1, \delta \neq \frac{h+h^2}{1-h}\}$$

is a saddle-node bifurcation surface. When parameters pass from one side of the surface SN to the other side, the number of positive equilibria of system (9) changes from zero to two. This indicates that there exists a critical harvesting rate h_1 such that the predator species goes extinct when the harvesting rate $h > h_1$ and the two species coexist in the form of a positive equilibrium for certain choices of initial values when $h = h_1$.

3.2. **Bogdanov-Takens bifurcation.** First, we investigate the Bogdanov-Takens bifurcation in system (9). The following Lemma 3.1 is from Perko [30], and Lemma 3.2 is Proposition 5.3 in Lamontage et al. [22].

Lemma 3.1. *The system*

$$\begin{aligned} \dot{x} &= y + Ax^2 + Bxy + Cy^2 + o(|x,y|^2), \\ \dot{y} &= Dx^2 + Exy + Fy^2 + o(|x,y|^2) \end{aligned} \tag{16}$$

is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= Dx^2 + (E+2A)xy + o(|x,y|^2) \end{aligned} \tag{17}$$

in some small neighborhood of (0,0) after changes of coordinates.

Lemma 3.2. *The system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^2 + a_{30}x^3 + a_{40}x^4 + y(a_{21}x^2 + a_{31}x^3) + y^2(a_{12}x + a_{22}x^2) + o(|x,y|^4) \end{aligned} \tag{18}$$

is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^2 + Gx^3y + o(|x,y|^4) \end{aligned} \tag{19}$$

in some small neighborhood of (0,0) after changes of coordinates and a rescaling of time, where

$$G = a_{31} - a_{30}a_{21}. \tag{20}$$

Now we state and prove one of the main theorems of this section.

Theorem 3.3. *When* $\beta = \frac{h^3}{(1-h)^2}$, $\delta = \frac{h+h^2}{1-h}$, *and* $0 < h < 1$, *system (9) has an interior equilibrium* $(h, 1-h)$ *which is a cusp. Moreover,*

(i) if $h \neq 2 - \sqrt{3}$, *then* $(h, 1-h)$ *is a cusp of codimension* 2*;*
(ii) if $h = 2 - \sqrt{3}$, *then* $(h, 1-h)$ *is a cusp of codimension* 3.

Proof. From the conditions $h = h_1$ and $\delta = \frac{h+h^2}{1-h}$ in Theorem 2.2, we can express β and δ in terms of h: $\beta = \frac{h^3}{(1-h)^2}$, $\delta = \frac{h+h^2}{1-h}$, $0 < h < 1$.

First of all, we translate the interior equilibrium $(h, 1-h)$ of system (9) into the origin and expand system (9) in power series around the origin. Let $X = x - h$, $Y = y - 1 + h$. Then system (9) can be rewritten as

$$\begin{aligned} \dot{X} &= -hX - hY - X^2 - XY, \\ \dot{Y} &= hX + hY - X^2 + \tfrac{2h}{1-h}XY - \tfrac{h^2}{(1-h)^2}Y^2 + o(|X,Y|^2), \end{aligned} \tag{21}$$

where β and δ are eliminated by the conditions of Theorem 3.3.

Making the affine transformation

$$x = (1+h)X + hY, y = -hX - hY,$$

we obtain

$$\begin{aligned} \dot{x} &= y - \tfrac{h}{(1-h)^2}x^2 + \tfrac{1-h-5h^2+h^3}{h(1-h)^2}xy + \tfrac{1-h-5h^2+h^3}{h(1-h)^2}y^2 + o(|x,y|^2), \\ \dot{y} &= \tfrac{h}{(1-h)^2}x^2 - \tfrac{1-6h+h^2}{(1-h)^2}xy - \tfrac{1-6h+h^2}{(1-h)^2}y^2 + o(|x,y|^2). \end{aligned} \tag{22}$$

By Lemma 3.1 we obtain an equivalent system of (22) as follows

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \tfrac{h}{(1-h)^2}x^2 - \tfrac{1-4h+h^2}{(1-h)^2}xy + o(|x,y|^2), \end{aligned} \tag{23}$$

where $\frac{h}{(1-h)^2} > 0$. When $\frac{1-4h+h^2}{(1-h)^2} \neq 0$, i.e. $h \neq 2-\sqrt{3}$, $(h, 1-h)$ is a cusp of codimension 2 by the results in Perko [30]. Moreover, $-\frac{1-4h+h^2}{(1-h)^2} < 0$ (or > 0) if $0 < h < 2-\sqrt{3}$ (or $2-\sqrt{3} < h < 1$). On the other hand, if $\frac{1-4h+h^2}{(1-h)^2} = 0$, i.e., $h = 2-\sqrt{3}$ (since $0 < h < 1$), then $(h, 1-h)$ is a cusp of codimension at least 3.

Now we investigate the exact codimension of the cusp $(h, 1-h)$ when $h = 2-\sqrt{3}$. Rewrite (21) as

$$\begin{aligned} \dot{X} &= (\sqrt{3}-2)X + (\sqrt{3}-2)Y - X^2 - XY, \\ \dot{Y} &= (2-\sqrt{3})X + (2-\sqrt{3})Y - X^2 + (\sqrt{3}-1)XY - \tfrac{7-4\sqrt{3}}{4-2\sqrt{3}}Y^2 \\ &\quad + \tfrac{1}{2-\sqrt{3}}X^3 + \tfrac{2}{1-\sqrt{3}}X^2Y + \tfrac{1}{2}XY^2 + \tfrac{1}{-7+4\sqrt{3}}X^4 \\ &\quad + \tfrac{2}{-5+3\sqrt{3}}X^3Y - \tfrac{1}{4-2\sqrt{3}}X^2Y^2 + o(|X,Y|^4). \end{aligned} \tag{24}$$

Let

$$x = X, y = (\sqrt{3}-2)X + (\sqrt{3}-2)Y - X^2 - XY.$$

Then system (24) is transformed into

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \tfrac{x^2}{2} + \tfrac{(5+2\sqrt{3})y^2}{2} + \tfrac{(19-11\sqrt{3})x^2y}{-71+41\sqrt{3}} + \tfrac{(33-19\sqrt{3})xy^2}{-97+56\sqrt{3}} + \tfrac{(13775-7953\sqrt{3})x^3y}{191861-110771\sqrt{3}} \\ &\quad + \tfrac{(343-198\sqrt{3})x^2y^2}{2702-1560\sqrt{3}} + o(|x,y|^4). \end{aligned} \tag{25}$$

Next, introducing a new time variable τ by $dt = (1-(\frac{5}{2}+\sqrt{3})x)d\tau$ to (25) and rewriting τ as t, we obtain

$$\begin{aligned} \dot{x} &= y(1-(\tfrac{5}{2}+\sqrt{3})x), \\ \dot{y} &= (1-(\tfrac{5}{2}+\sqrt{3})x)(\tfrac{1}{2}x^2 + (\tfrac{5}{2}+\sqrt{3})y^2 + \tfrac{19-11\sqrt{3}}{-71+41\sqrt{3}}x^2y + \tfrac{33-19\sqrt{3}}{-97+56\sqrt{3}}xy^2 \\ &\quad + \tfrac{13775-7953\sqrt{3}}{191861-110771\sqrt{3}}x^3y + \tfrac{343-198\sqrt{3}}{2702-1560\sqrt{3}}x^2y^2 + o(|x,y|^4)). \end{aligned} \tag{26}$$

The transformation $X = x, Y = y(1-(\frac{5}{2}+\sqrt{3})x)$ brings (26) into

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= \tfrac{1}{2}X^2 - (\tfrac{5}{2}+\sqrt{3})X^3 + \tfrac{1}{2}(\tfrac{5}{2}+\sqrt{3})^2X^4 + \tfrac{19-11\sqrt{3}}{-71+41\sqrt{3}}X^2Y \\ &\quad + (\tfrac{13775-7953\sqrt{3}}{191861-110771\sqrt{3}} - \tfrac{29-17\sqrt{3}}{-142+82\sqrt{3}})X^3Y + (\tfrac{33-19\sqrt{3}}{-97+56\sqrt{3}} - (\tfrac{5}{2}+\sqrt{3})^2)XY^2 \\ &\quad + (\tfrac{343-198\sqrt{3}}{2702-1560\sqrt{3}} - (\tfrac{5}{2}+\sqrt{3})^3)X^2Y^2 + o(|X,Y|^4)). \end{aligned} \tag{27}$$

Make the following change of variables

$$x = X, y = \sqrt{2}Y, \tau = \frac{1}{\sqrt{2}}t,$$

then system (27) can be rewritten as

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x^2 - (5+2\sqrt{3})x^3 + (\tfrac{5}{2}+\sqrt{3})^2x^4 + y(-\sqrt{2}(2+\sqrt{3})x^2 + \tfrac{30+17\sqrt{3}}{\sqrt{2}}x^3)\\ &\quad + y^2((-\tfrac{73}{4}-10\sqrt{3})x + \tfrac{-13-6\sqrt{3}}{8}x^2) + o(|x,y|^4).\end{aligned} \tag{28}$$

By Lemma 3.2 we obtain an equivalent system of (28) as

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x^2 + Mx^3y + o(|x,y|^4),\end{aligned} \tag{29}$$

where

$$M = -\frac{2+\sqrt{3}}{\sqrt{2}} \doteq -2.63896.$$

By results in Dumortier et al. [13], $(h, 1-h)$ is a cusp of codimension 3 when $h = 2-\sqrt{3}$. □

In the following, we discuss if system (9) can undergo Bogdanov-Taken bifurcation under a small parameter perturbation if the bifurcation parameters are chosen suitably. Actually, we have the following theorem.

Theorem 3.4. *When $\beta = \frac{h^3}{(1-h)^2}$, $\delta = \frac{h+h^2}{1-h}$, $0 < h < 1$ and $h \neq 2-\sqrt{3}$, system (9) has an interior equilibrium $(h, 1-h)$ which is a cusp of codimension 2 (i.e., B-T singularity). If we choose δ and h as bifurcation parameters, then system (9) undergoes Bogdanov-Taken bifurcation in a small neighborhood of the interior equilibrium $(h, 1-h)$ as (h, δ) varies near (h_0, δ_0), where δ_0 and h_0 satisfy $\beta_0 = \frac{h_0^3}{(1-h_0)^2}$, $\delta_0 = \frac{h_0+h_0^2}{1-h_0}$, $0 < h_0 < 1$ and $h_0 \neq 2-\sqrt{3}$. Moreover,*

(i) if $2-\sqrt{3} < h < 1$, then there exists a repelling Bogdanov-Takens bifurcation of codimension 2. Hence, there exist some parameter values such that system (9) has an unstable limit cycle, and there exist some other parameter values such that system (9) has an unstable homoclinic loop;

(ii) if $0 < h < 2-\sqrt{3}$, then there exists an attracting Bogdanov-Takens bifurcation of codimension 2. Hence, there exist some parameter values such that system (9) has a stable limit cycle, and there exist some other parameter values such that system (9) has a stable homoclinic loop.

Proof. We choose δ and h as bifurcation parameters. Consider

$$\begin{aligned}\dot{x} &= x(1-x) - xy,\\ \dot{y} &= y(\delta_0 + \lambda_1 - \beta_0\tfrac{y}{x}) - (h_0 + \lambda_2),\end{aligned} \tag{30}$$

where the constants β_0, δ_0 and h_0 satisfy $\beta_0 = \frac{h_0^3}{(1-h_0)^2}$, $\delta_0 = \frac{h_0+h_0^2}{1-h_0}$, $0 < h_0 < 1$ and $h_0 \neq 2-\sqrt{3}$, (λ_1, λ_2) is a parameter vector in a small neighborhood of $(0, 0)$. We are only interested in the phase portraits of system (30) when x and y are in a small neighborhood of the interior equilibrium $(h_0, 1-h_0)$.

In the following we will make a series of variable changes to obtain the versal unfolding of system (30). We first translate $(h_0, 1-h_0)$ to the origin and expand system (30) in power series around the origin. Let $X = x-h_0, Y = y-1+h_0$, then we have

$$\begin{aligned}\dot{X} &= -h_0X - h_0Y - X^2 - XY,\\ \dot{Y} &= \lambda_1(1-h_0) - \lambda_2 + h_0X + (h_0+\lambda_1)Y - X^2 + \tfrac{2h_0}{1-h_0}XY\\ &\quad - \tfrac{h_0^2}{(1-h_0)^2}Y^2 + R_1(X,Y),\end{aligned} \tag{31}$$

where R_1 is a C^∞ function at least of the third order with respect to (X,Y). Let

$$x = X, y = -h_0X - h_0Y - X^2 - XY,$$

then system (31) can be written as

$$\begin{aligned} \dot{x} &= y,\\ \dot{y} &= h_0(h_0-1)\lambda_1 + h_0\lambda_2 + ((2h_0-1)\lambda_1+\lambda_2)x + \lambda_1 y + (\tfrac{h_0}{(1-h_0)^2}+\lambda_1)x^2\\ &\quad +(\tfrac{2h_0}{(1-h_0)^2}-1)xy + (\tfrac{h_0}{(1-h_0)^2}+\tfrac{1}{h_0})y^2 + R_2(x,y), \end{aligned} \tag{32}$$

where $R_2(x,y)$ is a C^∞ function at least of the third order with respect to (x,y).

Next, we introduce a new time variable τ by $dt = (1-(\frac{h_0}{(1-h_0)^2}+\frac{1}{h_0})x)d\tau$. Rewriting τ as t, we have from (32) that

$$\begin{aligned} \dot{x} &= y(1-(\tfrac{h_0}{(1-h_0)^2}+\tfrac{1}{h_0})x),\\ \dot{y} &= (1-(\tfrac{h_0}{(1-h_0)^2}+\tfrac{1}{h_0})x)(h_0(h_0-1)\lambda_1 + h_0\lambda_2\\ &\quad +((2h_0-1)\lambda_1+\lambda_2)x + \lambda_1 y + (\tfrac{h_0}{(1-h_0)^2}+\lambda_1)x^2\\ &\quad +(\tfrac{2h_0}{(1-h_0)^2}-1)xy + (\tfrac{h_0}{(1-h_0)^2}+\tfrac{1}{h_0})y^2 + R_2(x,y)). \end{aligned} \tag{33}$$

Let $X = x, Y = y(1-(\frac{h_0}{(1-h_0)^2}+\frac{1}{h_0})x)$, then (33) can be rewritten as

$$\begin{aligned} \dot{X} &= Y,\\ \dot{Y} &= \psi_1 + \psi_2 X + \psi_3 Y + \psi_4 X^2 + \psi_5 XY + R_3(X,Y,\lambda_1,\lambda_2), \end{aligned} \tag{34}$$

where $R_3(X,Y,\lambda_1,\lambda_2)$ is a C^∞ function at least of the third order with respect to (X,Y), whose coefficients depend smoothly on λ_1 and λ_2, and

$$\psi_1 = h_0(h_0-1)\lambda_1 + h_0\lambda_2, \psi_2 = \frac{-1+h_0-2h_0^2}{h_0-1}\lambda_1 - (1+\frac{2h_0^2}{(1-h_0)^2})\lambda_2, \psi_3 = \lambda_1,$$

$$\psi_4 = \frac{h_0}{(1-h_0)^2} + \frac{-1+5h_0-9h_0^2+9h_0^3-3h_0^4}{h_0(h_0-1)^3}\lambda_1 + (\frac{h_0^3}{(h_0-1)^4}-\frac{1}{h_0})\lambda_2,$$

$$\psi_5 = \frac{-1+4h_0-h_0^2}{(1-h_0)^2} - (\frac{1}{h_0}+\frac{h_0}{(1-h_0)^2})\lambda_1.$$

Notice that $\psi_4 > 0$ when λ_i are small. Make the following change of variables

$$x = X, y = \frac{Y}{\sqrt{\psi_4}}, \tau = \sqrt{\psi_4}t,$$

then system (34) becomes

$$\begin{aligned} \dot{x} &= y,\\ \dot{y} &= \tfrac{\psi_1}{\psi_4} + \tfrac{\psi_2}{\psi_4}x + \tfrac{\psi_3}{\sqrt{\psi_4}}y + x^2 + \tfrac{\psi_5}{\sqrt{\psi_4}}xy + R_4(x,y,\lambda_1,\lambda_2), \end{aligned} \tag{35}$$

where $R_4(x,y,\lambda_1,\lambda_2)$ is a C^∞ function at least of the third order with respect to (x,y), and the coefficients depend smoothly on λ_1 and λ_2. Let

$$X = x + \frac{\psi_2}{2\psi_4}, Y = y.$$

Then (35) can be written as

$$\begin{aligned} \dot{X} &= Y,\\ \dot{Y} &= \tfrac{\psi_1}{\psi_4} - \tfrac{\psi_2^2}{4\psi_4^2} + (\tfrac{\psi_3}{\sqrt{\psi_4}} - \tfrac{\psi_2\psi_5}{2\psi_4\sqrt{\psi_4}})Y + X^2 + \tfrac{\psi_5}{\sqrt{\psi_4}}XY + R_5(X,Y,\lambda_1,\lambda_2), \end{aligned} \tag{36}$$

where $R_5(X,Y,\lambda_1,\lambda_2)$ is a C^∞ function at least of the third order with respect to (X,Y), and the coefficients depend smoothly on λ_1 and λ_2.

Note that $\psi_5 \neq 0$ (since $h_0 \neq 2-\sqrt{3}$ and $0<h_0<1$) when λ_i are small. Make the change of variables one more time by setting

$$x = \frac{\psi_5^2}{\psi_4}X,\ y = \frac{\psi_5^3}{\psi_4\sqrt{\psi_4}}Y,\ \tau = \frac{\sqrt{\psi_4}}{\psi_5}t.$$

Then we obtain the versal unfolding of system (30)

$$\begin{aligned} &\dot{x} = y,\\ &\dot{y} = \xi_1 + \xi_2 y + x^2 + xy + R_6(x,y,\lambda_1,\lambda_2), \end{aligned} \tag{37}$$

where $R_6(x,y,\lambda_1,\lambda_2)$ is a C^∞ function at least of the third order with respect to (x,y), whose coefficients depend smoothly on λ_1 and λ_2, and

$$\xi_1 = \frac{\psi_1\psi_5^4}{\psi_4^3} - \frac{\psi_2^2\psi_5^4}{4\psi_4^4},\ \xi_2 = \frac{\psi_3\psi_5}{\psi_4} - \frac{\psi_2\psi_5^2}{2\psi_4^2}.$$

By some simple computation, we obtain that

$$\begin{aligned} \xi_1 =& -\tfrac{1}{Q^2(\lambda)}(-1+h_0)^6[h_0(1-4h_0+h_0^2)+(1-2h_0+2h_0^2)\lambda_1]^4[(5-22h_0+41h_0^2-40h_0^3+16h_0^4)\lambda_1^2\\ &+(5-10h_0+9h_0^2)\lambda_2^2-4h_0^2\lambda_2+4(1-h_0)h_0^2\lambda_1+2(-5+16h_0-21h_0^2+12h_0^3)\lambda_1\lambda_2],\\ \xi_2 =& \tfrac{1}{Q(\lambda)}(-1+h_0)^2[h_0(1-4h_0+h_0^2)+(1-2h_0+2h_0^2)\lambda_1][(-1+h_0)^2(-3+10h_0-14h_0^2+10h_0^3)\lambda_1^2\\ &+h_0(1-6h_0+12h_0^2-14h_0^3+3h_0^4)\lambda_2-h_0(1-h_0)(1-3h_0+5h_0^2-9h_0^3+2h_0^4)\lambda_1\\ &-3(1-h_0)(-1+3h_0-4h_0^2+2h_0^3)\lambda_1\lambda_2],\\ Q(\lambda) =& 2[-h_0^2(1-h_0)^2+(-1+6h_0-14h_0^2+18h_0^3-12h_0^4+3h_0^5)\lambda_1+(1-4h_0+6h_0^2-4h_0^3)\lambda_2)]^2. \end{aligned}$$

Since

$$\left|\frac{\partial(\xi_1,\xi_2)}{\partial(\lambda_1,\lambda_2)}\right|_{\lambda=0} = \frac{(3-h_0)(1-4h_0+h_0^2)^5}{2h_0^3(1-h_0)} \neq 0$$

when $0<h_0<1$ and $h_0\neq 2-\sqrt{3}$, the above parameter transformation is a homeomorphism in a small neighborhood of the origin, and ξ_1 and ξ_2 are independent parameters.

By the results in Bogdanov [3, 4] and Takens [31] or Perko [30], we obtain the following local representations of the bifurcation curves up to second-order approximations:

(1) The saddle-node bifurcation curve $SN = \{(\xi_1,\xi_2) : \xi_1 = 0, \xi_2 \neq 0\}$, i.e.,

$$SN = \{(\lambda_1,\lambda_2) : \lambda_2 = (1-h_0)\lambda_1\}.$$

(2) The Hopf bifurcation curve $H = \{(\xi_1,\xi_2) : \xi_2 = \sqrt{-\xi_1}, \xi_1 < 0\}$, i.e.,

$$\begin{aligned} H = \{(\lambda_1,\lambda_2) :\ & \tfrac{(1-4h_0+h_0^2)^4}{h_0^2(-1+h_0)}\lambda_1 + \tfrac{(1-4h_0+h_0^2)^4}{h_0^2(-1+h_0)^2}\lambda_2 - \tfrac{3(1-4h_0+h_0^2)^4(-1+4h_0-6h_0^2+4h_0^3)}{h_0^4(-1+h_0)^4}\lambda_2^2\\ &+\tfrac{(1-4h_0+h_0^2)^3(6-54h_0+168h_0^2-257h_0^3+215h_0^4-75h_0^5+9h_0^6)}{h_0^4(-1+h_0)^3}\lambda_1\lambda_2\\ &+\tfrac{(1-4h_0+h_0^2)^3(3-42h_0+24h_0^2-595h_0^3+903h_0^4-826h_0^5+411h_0^6-99h_0^7+9h_0^8)}{h_0^4(-1+h_0)^2}\lambda_1^2 = 0\}. \end{aligned}$$

(3) The homoclinic bifurcation curve $HL = \{(\xi_1,\xi_2) : \xi_2 = \frac{5}{7}\sqrt{-\xi_1}, \xi_1 < 0\}$, i.e.,

$$\begin{aligned} HL = \{(\lambda_1,\lambda_2) :\ & \tfrac{25(1-4h_0+h_0^2)^4}{49h_0^2(-1+h_0)}(\lambda_1 + \tfrac{\lambda_2}{-1+h_0}) - \tfrac{3(1-4h_0+h_0^2)^4(27-108h_0+170h_0^2-124h_0^3+18h_0^4)}{49h_0^4(-1+h_0)^4}\lambda_2^2\\ &+\tfrac{(1-4h_0+h_0^2)^3(162-1410h_0+4368h_0^2-6761h_0^3+5795h_0^4-2247h_0^5+297h_0^6)}{49h_0^4(-1+h_0)^3}\lambda_1\lambda_2\\ &+\tfrac{(1-4h_0+h_0^2)^2(81-1086h_0+5714h_0^2-15163h_0^3+23073h_0^4-21262h_0^5+10881h_0^6-2691h_0^7+249h_0^8)}{49h_0^4(-1+h_0)^2}\lambda_1^2 = 0\}. \end{aligned}$$

From the expression of ψ_5 and the transformation $\tau = \frac{\sqrt{\psi_4}}{\psi_5}t$, which was used to obtain system (37), we also have the following results:

(i) If $2-\sqrt{3} < h < 1$, then $\psi_5 > 0$. Therefore, there exists a repelling Bogdanov-Takens bifurcation of codimension 2. Hence, there exist some parameter values such that system (9) has an unstable limit cycle, and there exist some other parameter values such that system (9) has an unstable homoclinic loop.

(ii) If $0 < h < 2-\sqrt{3}$, then $\psi_5 < 0$. Therefore, there exists an attracting Bogdanov-Takens bifurcation of codimension 2. Hence, there exist some parameter values such that system (9) has a stable limit cycle, and there exist some other parameter values such that system (9) has a stable homoclinic loop. $\square$

The repelling B-T bifurcation diagram and phase portraits of system (30) with $h = h_0 = \frac{3}{10} > 2-\sqrt{3}$ are given in Fig. 4. These bifurcation curves H, HL, and SN divide the small neighborhood of the origin in the parameter (λ_1, λ_2)-plane into four regions (see Fig. 4(a)).

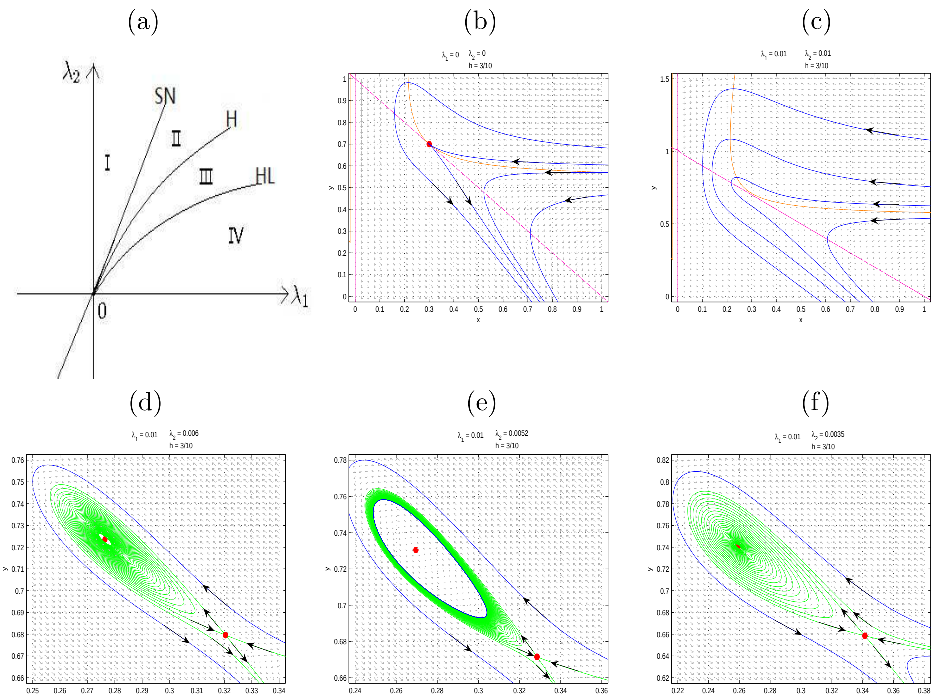

FIGURE 4. The repelling B-T bifurcation diagram and corresponding phase portraits of system (30) with $h = h_0 = \frac{3}{10}$. (a) Bifurcation diagram; (b) A cusp of codimension 2 when $(\lambda_1, \lambda_2) = (0, 0)$; (c) No equilibria when $(\lambda_1, \lambda_2) = (0.01, 0.01)$ lies in region I; (d) An unstable focus when $(\lambda_1, \lambda_2) = (0.01, 0.006)$ lies in region II; (e) An unstable limit cycle when $(\lambda_1, \lambda_2) = (0.01, 0.0052)$ lies in region III; (f) A stable focus when $(\lambda_1, \lambda_2) = (0.01, 0.0035)$ lies in region IV.

(a) When $(\lambda_1, \lambda_2) = (0, 0)$, the unique positive equilibrium is a cusp of codimension 2 (see Fig. 4(b)).

(b) There are no equilibria when the parameters lie in region I (see Fig. 4(c)), all solutions will pass through the x-axis and go out of the first quadrant.

(c) When the parameters lie on the curve SN, there is a positive equilibrium, which is a saddle-node.

(d) Two positive equilibria, one is an unstable focus and the other is a saddle, will occur through the saddle-node bifurcation when the parameters cross SN into region II (see Fig. 4(d)).

(e) An unstable limit cycle will appear through the subcritical Hopf bifurcation when the parameters cross H into region III (see Fig. 4(e)), where the focus is stable, whereas the focus is an unstable one with multiplicity one when the parameters lie on the curve H.

(f) An unstable homoclinic cycle will occur through the repelling homoclinic bifurcation when the parameters pass region III and lie on the curve HL.

(g) The relative location of one stable and one unstable manifold of the saddle (x_2, y_2) will be reverse when the parameters cross III into region IV (compare Fig. 4(e) and Fig. 4(f)).

The attracting B-T bifurcation diagram of codimension 2 and phase portraits in system (30) with $h = h_0 = \frac{1}{5} < 2 - \sqrt{3}$ are given in Fig. 5, where a stable homoclinic loop arises from attracting homoclinic bifurcation is given in Fig. 5(d), and a stable limit cycle arises from supercritical Hopf bifurcation is given in Fig. 5(e).

When δ tends to δ_0, the above results indicate that if the harvesting rate h tends to h_0, then there exist some parameter values such that the prey and predators coexist in the form of a positive equilibrium or a periodic orbit with a finite period for different initial values, respectively. There exist some other parameter values such that the prey and predator coexist in the form of a positive equilibrium or a periodic orbit with an infinite period for different initial values, respectively.

3.3. **Hopf bifurcation.** From Theorem 2.3(b) we know that (x_3, y_3) is a weak focus or a center. Hence Hopf bifurcation may occur at this equilibrium. In this subsection we present conditions under which the stability of (x_3, y_3) changes such that system (9) undergoes Hopf bifurcation and degenerate Hopf bifurcation and exhibits two limit cycles.

We first make a transformation of $u = x - x_3, v = y - y_3$, and then rewrite u, v as x and y, respectively, system (9) can be changed into

$$\begin{aligned} \dot{x} &= \textstyle\sum_{i+j=1}^{3} a_{ij} x^i y^j + O_1(|(x,y)|^4), \\ \dot{y} &= \textstyle\sum_{i+j=1}^{3} b_{ij} x^i y^j + O_2(|(x,y)|^4), \end{aligned} \tag{38}$$

where a_{ij} and b_{ij} are the coefficients of the power series expansions of $f_1(x,y)$ and $f_2(x,y)$ at (x_3, y_3), respectively, $i, j = 0, 1, 2, 3$. $O_k(|(x,y)|)^4$ is the same order infinity, $k = 1, 2$. Hence, using the formula of the first Liapunov number σ from Perko [30] at the origin of (38), we have

$$\sigma = \frac{3\pi\beta Q}{2x_3^5 \Delta_1^{\frac{3}{2}}},$$

where $\Delta_1 = \frac{\beta}{x_3} - x_3(\beta + \delta)$, and

$$Q = x_3^5 - (\beta - 4)x_3^4 - 2(\beta + 1)x_3^3 + \beta(2\beta + 3)x_3^2 - 4\beta^2 x_3 + 2\beta^2.$$

From (15), we can see that $\Delta_1 > 0$, therefore the sign of σ is determined by Q. By some numerical calculations, we know that there exist parameter values $(\beta, \delta, h) = (\frac{37}{1000}, \frac{3}{5}, \frac{1}{4}(\frac{290831+137\sqrt{39569}}{250000}) \doteq 0.318083)$ which satisfy the conditions of

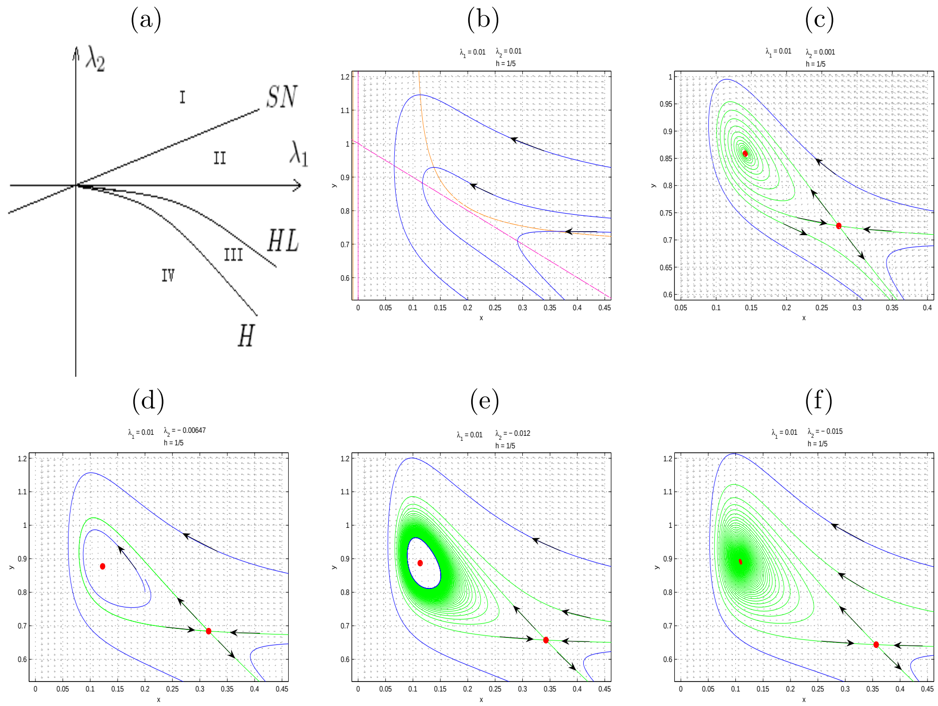

FIGURE 5. The attracting B-T bifurcation diagram and corresponding phase portraits of system (30) with $h = h_0 = \frac{1}{5}$. (a) Bifurcation diagram; (b) No equilibria when $(\lambda_1, \lambda_2) = (0.01, 0.01)$ lies in region I; (c) An unstable focus when $(\lambda_1, \lambda_2) = (0.01, 0.001)$ lies in region II; (d) A stable homoclinic loop when $(\lambda_1, \lambda_2) = (0.01, -0.00647)$ lies on the curve HL; (e) A stable limit cycle when $(\lambda_1, \lambda_2) = (0.01, -0.012)$ lies in region III; (f) A stable focus when $(\lambda_1, \lambda_2) = (0.01, -0.015)$ lies in region IV.

Theorem 2.3(b), such that $\sigma \doteq 8.23692$. On the other hand, there exist parameter values $(\beta, \delta, h) = (\frac{1}{80}, \frac{3}{10}, \frac{119}{640})$ which also satisfy the conditions of Theorem 2.3(b), such that $\sigma \doteq -272.777$. Therefore, there exists an open set V_1 in the parameter space (β, δ, h), such that $\sigma > 0$ and $h < h_1$, i.e.,

$$V_1 = \{(\beta, \delta, h) : h < h_1, \sigma > 0\}.$$

And there exists another open set V_2 in the parameter space (β, δ, h), such that $\sigma < 0$ and $h < h_1$, i.e.,

$$V_2 = \{(\beta, \delta, h) : h < h_1, \sigma < 0\}.$$

Summarizing the above discussion, we have the following results.

Theorem 3.5. *(i) If $h = h_3$ and the parameters (β, δ, h) are in V_1, then the equilibrium (x_3, y_3) of system (9) is a multiple focus of multiplicity one and is unstable. System (9) has an unstable limit cycle arising from the subcritical Hopf bifurcation;*

(ii) If $h = h_3$ and the parameters (β, δ, h) are in V_2, then the equilibrium (x_3, y_3) of system (9) is a multiple focus of multiplicity one and is stable. System (9) has a stable limit cycle arising from the supercritical Hopf bifurcation. The phase portrait for one limit cycle is given in Fig. 6.

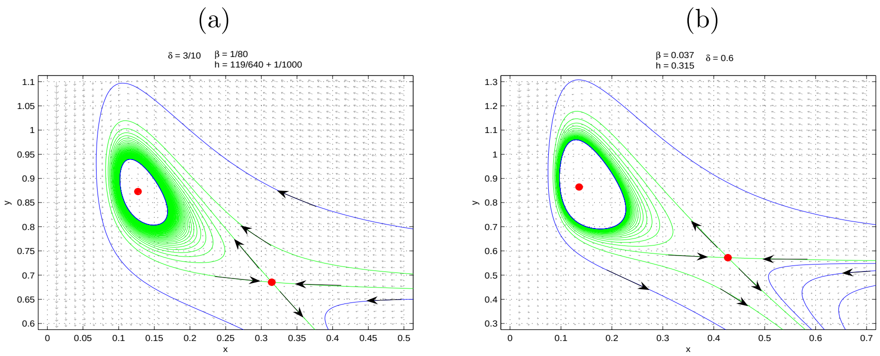

FIGURE 6. (a) A stable limit cycle created by the supercritical Hopf bifurcation; (b) An unstable limit cycle created by the subcritical Hopf bifurcation.

In Fig. 6, by incorporating the above analysis and using numerical simulations we present the supercritical and subcritical Hopf bifurcations of codimension 1. When $(\beta, \delta, h) = (\frac{1}{80}, \frac{3}{10}, \frac{119}{640})$, (x_3, y_3) is a stable multiple fucus with multiplicity one, then we let h increase to $\frac{119}{640} + \frac{1}{1000}$, the existence of one stable limit cycle of system (9) arises from the supercritical Hopf bifurcation is illustrated in Fig. 6(a); When $(\beta, \delta, h) = (\frac{37}{1000}, \frac{3}{5}, \frac{1}{4}(\frac{290831+137\sqrt{39569}}{250000}) \doteq 0.318083)$, (x_3, y_3) is an unstable multiple fucus with multiplicity one, then we let h decrease to 0.315, the existence of one unstable limit cycle of system (9) arises from the subcritical Hopf bifurcation is illustrated in Fig. 6(b).

Next, we present an example to show that system (9) has an interior equilibrium which is a stable weak focus of multiplicity two, and under a small perturbation, system (1.1) undergoes a degenerate Hopf bifurcation and produces two limit cycles.

Theorem 3.6. *When $(\beta, \delta, h) = (\frac{-4+\sqrt{51}}{100}, \frac{-3+2\sqrt{51}}{25}, \frac{4(1+\sqrt{51})}{125})$, system (9) has an interior equilibrium (x_3, y_3) which is a stable weak focus of multiplicity two. System (1.1) has two limit cycles arising from the degenerate Hopf bifurcation, the repelling cycle is surrounded by an attracting cycle. The phase portrait for two limit cycles is given in Fig. 7.*

Proof. From Theorem 2.3(b), we know that $\mathrm{Tr}(Df(x_3, y_3)) = 0$ when $\delta = x_3 + 2\beta(\frac{1}{x_3} - 1)$. We make the following linear transformation of state variables to system (38),

$$X = -a_{01}x, \;\; Y = a_{01}x + \sqrt{d}y,$$

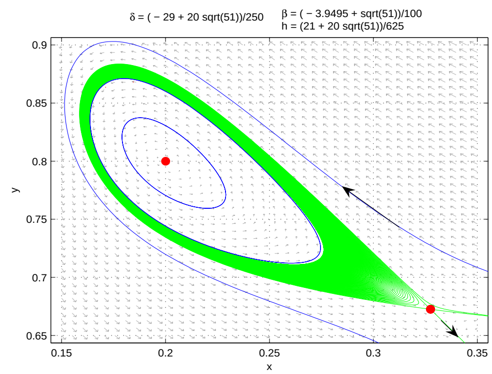

FIGURE 7. Two limit cycles arising from the degenerate Hopf bifurcation.

where $d = \mathrm{Det}(Df(x_3, y_3)) = \beta x_3 - x_3^2 - 2\beta + \frac{\beta}{x_3}$, we still denote X, Y by x, y, respectively, then system (38) becomes

$$\begin{array}{l} \dot{x} = -\sqrt{d}y + O_3(|(x,y)|^2), \\ \dot{y} = \sqrt{d}x + O_4(|(x,y)|^2), \end{array} \tag{39}$$

where $O_k(|(x,y)|^2)$ is the same order infinity, $k = 3, 4$.

Next, we determine the multiplicity of the weak focus $(0,0)$ of system (39) by the successor function method. It is convenient to introduce polar coordinates (r, θ) and rewrite system (39) in polar coordinates by $x = r\cos\theta, y = r\sin\theta$. It is clear that in a small neighborhood of the origin the successor function $D(c_0)$ of system (39) can be expressed by

$$D(c_0) = r(2\pi, c_0) - r(0, c_0),$$

where $r(\theta, c_0)$ is the solution of the following Cauchy problem

$$\begin{array}{l} \frac{dr}{d\theta} = R_2(\theta)r^2 + R_3(\theta)r^3 + R_4(\theta)r^4 + R_5(\theta)r^5 + ..., \\ r(0) = c_0, 0 <| c_0 |\ll 1, \end{array} \tag{40}$$

where $R_i(\theta)$ is a polynomial of $(\sin\theta, \cos\theta), i = 2, 3, ...$, whose coefficients can be expressed by the coefficients of system (39). We omit them here since the expressions are too long.

In the following, we fix $x_3 = \frac{1}{5}$, then $\beta > \frac{1}{80}$ because $d > 0$. From the method of successor function in Zhang et al. [36], we can obtain the first Liapunov number of the equilibrium (0, 0) of system (39)

$$L_1 = \frac{5\beta(-7 + 160\beta + 2000\beta^2)}{4(80\beta - 1)^{\frac{3}{2}}}.$$

Obviously, $L_1 = 0$ when $\beta = \frac{-4+\sqrt{51}}{100}$. Therefore, the interior equilibrium (x_3, y_3) is an unstable (stable) weak focus of multiplicity one if $\beta > \frac{-4+\sqrt{51}}{100}$ ($\frac{1}{80} < \beta < \frac{-4+\sqrt{51}}{100}$, respectively).

However, if $\beta = \frac{-4+\sqrt{51}}{100}$, then $L_1 = 0$. We further compute the second Liapunov number of equilibrium $(0,0)$ of system (39) as $\beta = \frac{-4+\sqrt{51}}{100}$ and obtain

$$L_2 = -\frac{\sqrt{165405 + 23180\sqrt{51}}}{14400} \doteq -0.0399498.$$

Therefore, the interior equilibrium (x_3, y_3) is a stable weak focus of multiplicity two if $(\beta, \delta, h) = (\frac{-4+\sqrt{51}}{100}, \frac{-3+2\sqrt{51}}{25}, \frac{4(1+\sqrt{51})}{125})$, where we can determine δ from $\mathrm{Tr}(Df(x_3, y_3)) = 0$ and h from the equilibria equation (12). $\square$

In the following, we present additional interesting dynamics to show the existence of attracting Bogdanov-Takens bifurcation (cusp case) of codimension 3 in system (9) by numerical simulations.

In Fig. 8, we fix $(\beta, h) = (\frac{1}{80}, \frac{21}{100})$. The phase portrait about the existence of a stable homoclinic cycle enclosing an unstable hyperbolic focus when $\delta = 0.37507$ is given in Fig. 8(a); When δ increases to 0.378, the existence of a stable limit cycle enclosing an unstable hyperbolic focus and arising from the attracting homoclinic bifurcation is shown in Fig. 8(b); When δ increases to 0.38, the existence of two limit cycles enclosing a stable hyperbolic focus is given in Fig. 8(c), the inner limit cycle is unstable and the outer is stable, the inner one arises from the subcritical Hopf bifurcation. Comparing Fig. 8(b) with Fig. 8(c), we know that there exists a unique δ_0 satisfying $0.378 < \delta_0 < 0.38$ such that system (9) has a stable limit cycle enclosing an unstable multiple focus with multiplicity one.

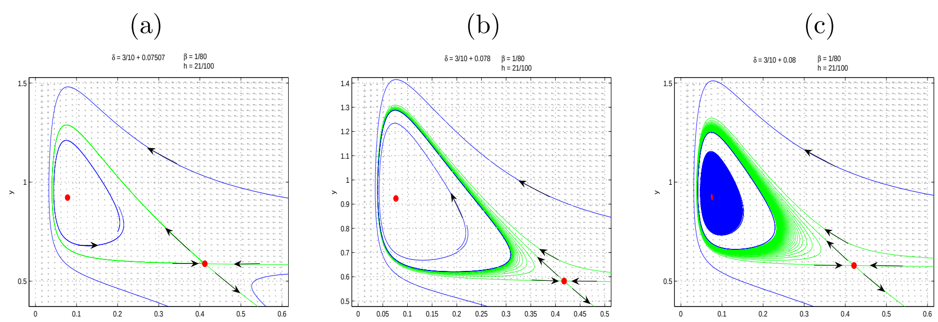

FIGURE 8. Phase portraits of system (9) with $(\beta, h) = (\frac{1}{80}, \frac{21}{100})$, (a) A stable homoclinic cycle enclosing an unstable hyperbolic focus when $\delta = 0.37507$; (b) A stable limit cycle enclosing an unstable hyperbolic focus and arising from the attracting homoclinic bifurcation when $\delta = 0.378$; (c) Two limit cycles enclosing a stable hyperbolic focus when $\delta = 0.38$.

In Fig. 9, we show the coexistence of a stable homoclinic loop and an unstable limit cycle in system (9).

4. **Discussion.** Our qualitative analysis on system (9) reveals that the constant-yield predator harvesting h plays an important role in determining the dynamics and bifurcations of system (9): it can affect the number and type of equilibria (Lemma 2.1 and Theorems 2.2 and 2.3) and the type of bifurcations (saddle-node bifurcation,

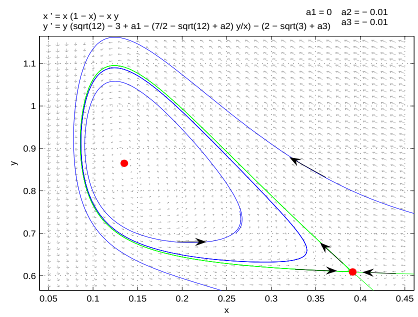

FIGURE 9. The coexistence of a stable homoclinic loop and an unstable limit cycle in system (9).

Hopf bifurcation, and Bogdanov-Takens bifurcation) in the model. The saddle-node bifurcation, repelling and attracting Bogdanov-Takens bifurcations of codimension 2, supercritical and subcritical Hopf bifurcations, and degenerate Hopf bifurcation are shown in model (9) as the values of parameters vary, and there exists a critical predator harvesting rate such that predators go extinct when the harvest rate is greater than the critical value. It is also shown that the model has a Bogdanov-Takens singularity (cusp) of codimension 3 for some parameter values, numerical simulations, including the repelling and attracting Bogdanov-Takens bifurcation diagrams and corresponding phase portraits, two limit cycles, the coexistence of a stable homoclinic loop and an unstable limit cycle, and a stable limit cycle enclosing an unstable multiple focus with multiplicity one, are presented to not only support the theoretical analysis but also indicate the existence of attracting Bogdanov-Takens bifurcation (cusp case) of codimension 3. These complex dynamics cannot occur in the unharvested systems (2) and the case (6) with only constant-yield prey harvesting. System (2) has only one positive equilibrium which is global stable under all admissible parameters (Hsu and Huang [21]), while system (1) with only constant-yield prey harvesting has at most two positive equilibria and exhibits Hopf bifurcation of codimension 1 (Zhu and Lan [37]) and Bogdanov-Takens bifurcation of codimension 2 (Gong and Huang [16]). Thus we can see that the constant-yield predator harvesting in system (7) can cause more complex dynamical behaviors and bifurcation phenomena compared with no harvesting in system (2) or (6) with only constant-yield prey harvesting.

It is obvious that the absolute maximum sustainable yield of predator harvesting is $h_{MSY} = h_1$. From the theoretical analysis (Theorems 3.3-3.6) and numerical simulations (Fig. 4-9), we can choose δ, h and β as bifurcation parameters such that system (9) exhibits Bogdanov-Taken bifurcation of codimension 3 under a small parameter perturbation, which has been studied for general differential equations by Dumortier et al. [13] and by Chow et al. [11]. The bifurcation diagram for such a bifurcation is known ([13]) and the unfolding of system (29) is equivalent to

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \mu_1 + \mu_2 y + \mu_3 xy + x^2 - x^3 y + o(|x, y|^4),
\end{aligned}
$$

system (9) can have at least two limit cycles or the coexistence of a limit cycle and a homoclinic loop for different parameter values. See Zhu et al. [38] for such bifurcations in a predator-prey model, and Tang et al. [32] and Cai et al. [8] for such bifurcations in epidemic models.

The reduction of the predator stock level in predator-prey interactions within fishery systems may increase the surplus production of the prey. Harvesting predators becomes controversial (May et al. [24], Flaaten [15], Yodzis [35]) and is important in maintaining the abundance of fish stocks. The Bogdanov-Takens bifurcation diagrams and the numerical simulations in section 3 demonstrate that there are some parameter regions in which predator species can be driven to extinction by constant-yield harvesting of the predators. These may provide some explanations for the collapse of the Atlantic cod stocks in the Canadian Grand Banks (Hutchings and Myers [20], Myers et al. [25], Hutchings [19]) and may be significant and useful in designing fishing policies for the fishery industry (Pauly et al. [29], Myers and Worm [26]).

Since fishing is a seasonal activity, it will be very interesting to study how seasonal harvesting affects the existing bifurcations in predator-prey systems. We leave these for future consideration.

Acknowledgments. Research of this paper was completed when the first author was visiting the University of Miami, he would like to thank the faculty and staff in the Department of Mathematics there for their kind hospitality and helpful assistance.

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Received April 2013; revised June 2013.

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