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BIFURCATIONS OF AN SIRS EPIDEMIC MODEL WITH NONLINEAR INCIDENCE RATE

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Abstract. The main purpose of this paper is to explore the dynamics of an epidemic model with a general nonlinear incidence $\beta SI^p/(1+\alpha I^q)$. The existence and stability of multiple endemic equilibria of the epidemic model are analyzed. Local bifurcation theory is applied to explore the rich dynamical behavior of the model. Normal forms of the model are derived for different types of bifurcations, including Hopf and Bogdanov-Takens bifurcations. Concretely speaking, the first Lyapunov coefficient is computed to determine various types of Hopf bifurcations. Next, with the help of the Bogdanov-Takens normal form, a family of homoclinic orbits is arising when a Hopf and a saddle-node bifurcation merge. Finally, some numerical results and simulations are presented to illustrate these theoretical results.

1. **Introduction.** The regular pattern of periodic occurrences/outbreaks has been observed in the epidemiology of many infectious diseases, such as chickenpox, influenza, measles, etc. (see Hethcote [8], Hethcote and Levin [9], and Hethcote and van den Driessche [10]). Understanding such periodic patterns and identifying the specific factors that underlie such periodic outbreaks is very important to predict and control the spread of infectious diseases. Recent studies have demonstrated that the nonlinear incidence rate is one of the key factors that induce periodic oscillations in epidemic models (see Alexander and Moghadas [1, 2], Derrick and van den Driessche [4], Feng and Thieme [5], Li and Wang [12], Liu et al. [13, 14], Lizana

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and Rivero [15], Moghadas [16], Moghadas and Alexander [17], Ruan and Wang [18], Tang et al. [19], Wang [20], and the references cited therein).

Let $S(t)$ and $I(t)$ denote the numbers of susceptible and infectious individuals at time t, respectively. In order to incorporate the effect of behavioral changes, Liu et al. [14] introduced a nonlinear incidence rate of the form

$$f(I)S = \frac{\beta I^p S}{1 + \alpha I^q}, \tag{1}$$

where βI^p measures the infection force of the disease, $1/(1 + \alpha I^q)$ describes the inhibition effect from the behavioral change of the susceptible individuals when the number of infectious individuals increases, β and p are all positive constants, and q and α are nonnegative constants. Notice that the bilinear incidence rate βSI is a special case of (1) with $p = 1$ and $\alpha = 0$ or $q = 0$.

According to Tang et al. [19], the nonlinear function $f(I)$ given by (1) includes three types:

(i) Unbounded incidence function: $p > q$. When $p = q + 1$, it was considered in Hethcote and Levin [9].

(ii) Saturated incidence function: $p = q$. When $p = q = 1$, i.e., $f(I) = \beta I/(1 + \alpha I)$, it was proposed by Capasso and Serio [3] to describe a "crowding effect" or "protection measures" in modeling the cholera epidemics in Bari in 1973. The global dynamics of an SIRS model with $p = q = 2$ was studied in Ruan and Wang [18] and Tang et al. [19]. The global dynamics of an SIRS model with $p = q > 0$ was discussed in Li and Wang [12].

(iii) Nonmonotone incidence function: $p < q$. Such functions can be used to interpret the "psychological effects" (see Capasso and Serio [3]): for a very large number of infectious individuals the infection force may decrease as the number of infectious individuals increases, because in the presence of a large number of infectious individuals the population may tend to reduce the number of contacts per unit time, as seen with the spread of SARS (see Wang [20]), the case $p = 1$ and $q = 2$ was considered in Xiao and Ruan [22].

Most researchers consider the nonlinear incidence rate (1) for special p and q. The bifurcations of SIRS models with general p and q have not been studied in the literature. In this paper, we consider an SIRS model with the nonlinear incidence rate (1). Namely, we consider the following SIRS model

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = A - dS - \dfrac{\beta I^p S}{1 + \alpha I^q} + vR, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta I^p S}{1 + \alpha I^q} - (d + \mu)I, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \mu I - (d + v)R, \end{cases} \tag{2}$$

where $S(t), I(t)$ and $R(t)$ denote the numbers of susceptible, infective, and recovered individuals at time t, respectively. $A > 0$ is the recruitment rate of the population, $d > 0$ is the nature death rate of the population, $\mu > 0$ is the natural recovery rate of infective individuals, $v \geq 0$ is the rate at which recovered individuals lose immunity and return to the susceptible class, $\alpha \geq 0$ is the parameter measures the psychological or inhibitory effect, $\beta > 0$ is the proportionality constant. Summing up the three equations in (2) and denoting the number of total population by $N(t)$, that is,

$$N(t) = S(t) + I(t) + R(t), \tag{3}$$

we obtain

$$\frac{\mathrm{d}N}{\mathrm{d}t} = A - dN.$$

$N(t)$ tends to a constant $N_0 = A/d$ as t tends to infinity. Following Liu et al. [14] and Lizana and Rivero [15], we assume that the population is at equilibrium and investigate the behavior of the system on the plane $S + I + R = N_0$. Thus, we consider the reduced system

$$\begin{cases} \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta I^p}{1+\alpha I^q}(N_0 - I - R) - (d+\mu)I, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \mu I - (d+v)R. \end{cases} \tag{4}$$

It is easy to know that the positive invariant set of system (4) is

$$D = \left\{ (I,R) | I \geq 0, R \geq 0, I + R \leq \frac{A}{d} \right\}.$$

The paper is organized as follows. The existence of multiple equilibria is discussed in section 2. In section 3, the stability of the equilibria for system (4) is analyzed. Bifurcation behavior of the system is studied in section 4. In section 5, some numerical simulations are presented to illustrate the main results. The paper ends with a brief discussion in section 6.

2. **The analysis of equilibria.** The aim of this section is to perform an elaborative analysis of equilibria for system (4).

Obviously, $O(0,0)$ is the disease-free equilibrium of system (4). In order to obtain the positive equilibria of system (4), let the right sides of (4) be equal to zero:

$$\begin{cases} \dfrac{\beta I^p}{1+\alpha I^q}(N_0 - I - R) - (d+\mu)I = 0, \\ \mu I - (d+v)R = 0, \end{cases} \tag{5}$$

which yields

$$\frac{I^{p-1}}{1+\alpha I^q}\left(1 - \frac{I}{K}\right) = \frac{1}{\sigma}, \tag{6}$$

where

$$K = \frac{A(d+v)}{d(d+v+\mu)}, \quad \sigma = \frac{\beta A}{d(\mu+d)}.$$

Note that σ is called the *basic reproduction number* of the disease.

Let

$$\varphi(I) = \frac{I^{p-1}}{1+\alpha I^q}\left(1 - \frac{I}{K}\right), \quad I \in (0, K]. \tag{7}$$

Then

$$\varphi'(I) = \frac{I^{p-2}}{(1+\alpha I^q)^2}\psi(I), \quad I \in (0, K], \tag{8}$$

where

$$\psi(I) = p - 1 - p\frac{I}{K} + \alpha(p-1-q)I^q - \alpha(p-q)\frac{I^{q+1}}{K}, \quad I \in (0, K]. \tag{9}$$

We now discuss the positive roots of equation (6).

(i) When $0 < p < 1$, if $p \geq q$, then $\psi(I) < 0$; if $p < q$, then

$$\begin{aligned}\psi(I) &\leq p - 1 - p\frac{I}{K} + \alpha(p - 1 - q)I^q + \alpha(q - p)I^q \\ &= p - 1 - p\frac{I}{K} - \alpha I^q < 0, \quad I \in (0, K].\end{aligned} \tag{10}$$

Therefore $\varphi'(I) < 0$. As $\varphi(K) = 0$ and $\lim\limits_{I\to 0^+} \varphi(I) = +\infty$, for any $\sigma > 0$ equation (6) has a unique positive solution $I_e (0 < I_e < K)$.

(ii) When $p = 1$, it follows from (9) that if $q \leq 1$, then $\psi(I) < 0$ for $I \in (0, K)$, and if $q > 1$, then

$$\begin{aligned}\psi(I) &\leq -\frac{I}{K} - \alpha q I^q + \alpha(q - 1)I^q \\ &= -\frac{I}{K} - \alpha I^q < 0, \quad I \in (0, K].\end{aligned} \tag{11}$$

Therefore, $\varphi'(I) < 0$ for any $I \in (0, K)$.

(a) If $q = 0$, then $\varphi(0) = \dfrac{1}{1 + \alpha}$ and $\varphi(K) = 0$. Therefore, equation (6) has no positive solution when $\sigma \leq 1 + \alpha$ and has a unique positive solution when $\sigma > 1 + \alpha$.

(b) If $q > 0$, then $\varphi(0) = 1$ and $\varphi(K) = 0$. Therefore, equation (6) has no positive solution when $\sigma \leq 1$ and has a unique positive solution when $\sigma > 1$.

(iii) When $p > 1$, it can be proved that $\psi(I)$ has a unique zero point I^* in the interval $(0, K)$. This result will be proved in the following seven cases.

(a) If $\alpha = 0$ or $q = 0$, then the conclusion is evident.

(b) If $\alpha > 0$, $0 < q \leq 1$ and $1 < p \leq q + 1$, it is obtained from (9) that

$$\psi'(I) = -\frac{p}{K} + \alpha q(p - 1 - q)I^{q-1} - \alpha(p - q)(q + 1)\frac{I^q}{K} < 0, \quad I \in (0, K]. \tag{12}$$

As $\psi(0) = p - 1 > 0$ and $\psi(K) = -1 - \alpha K^q < 0$, $\psi(I)$ has a unique zero I^* in $(0, K)$.

(c) If $\alpha > 0$, $0 < q \leq 1$ and $p > 1 + q$, then

$$\begin{aligned}\psi''(I) &= \alpha q(q - 1)(p - 1 - q)I^{q-2} - \alpha(p - q)q(q + 1)\frac{I^{q-1}}{K} \\ &= \alpha q I^{q-2}\left[(q - 1)(p - q - 1) - (p - q)(q + 1)\frac{I}{K}\right] < 0, \quad I \in (0, K].\end{aligned} \tag{13}$$

Therefore, $\psi(I)$ is a hump-shaped function. As $\psi(0) = p - 1 > 0$ and $\psi(K) = -1 - \alpha K^q < 0$, $\psi(I)$ has a unique zero in $(0, K)$.

(d) If $\alpha > 0, q > 1$ and $q \leq p \leq q + 1$, then it follows from (12) that $\psi'(I) < 0$ in $(0, K)$. As $\psi(0) = p - 1 > 0$ and $\psi(K) < 0$, $\psi(I)$ has a unique zero I^* in $(0, K)$.

(e) If $\alpha > 0$, $q > 1$ and $\dfrac{1 + q}{2} \leq p < q$, then it follows from (13) that

$$\begin{aligned}\psi''(I) &< \alpha q I^{q-2}\left[(q - 1)(p - q - 1) + (q - p)(q + 1)\right] \\ &= \alpha q I^{q-2}(-2p + q + 1) \leq 0, \quad I \in (0, K].\end{aligned} \tag{14}$$

Therefore, $\psi(I)$ is a hump-shaped function. As $\psi(0) = p - 1 > 0$ and $\psi(K) = -1 - \alpha K^q < 0$, $\psi(I)$ has a unique zero I^* in the interval $(0, K)$.

(f) If $\alpha > 0, q > 1$ and $1 < p < \dfrac{1 + q}{2}$, it follows from (13) that on the interval $(0, K)$, $\psi''(I)$ has a unique zero

$$\bar{I} = \frac{(q - 1)(1 + q - p)}{(q - p)(q + 1)}K,$$

Furthermore, $\psi''(I) < 0$ when $0 < I < \bar{I}$ and $\psi''(I) > 0$ when $\bar{I} < I < K$. As $\psi'(0) = -\frac{p}{K}$, $\psi'(I) < 0$ when $0 < I < \bar{I}$. As $\psi'(K) = -\frac{p}{K} - \alpha p K^{q-1} < 0$, $\psi'(I) < \psi'(K) < 0$ when $\bar{I} < I < K$. So, $\psi'(I) < 0$ in the interval $(0, K)$ except $\bar{I}$, i.e. $\psi(I)$ is a monotonically decreasing function on the interval $(0, K)$. As $\psi(0) = p - 1 > 0$ and $\psi(K) < 0$, $\psi(I)$ has a unique zero I^* in the interval $(0, K)$.

(g) If $\alpha > 0$, $q > 1$ and $p > q+1$, then in the interval $(0, K)$, $\psi''(I)$ has a unique zero

$$\bar{I} = \frac{(q-1)(p-1-q)}{(p-q)(q+1)}K.$$

Furthermore, $\psi''(I) > 0$ when $0 < I < \bar{I}$ and $\psi''(I) < 0$ when $\bar{I} < I < K$. So, $\bar{I}$ is a unique extreme point and the maximum point of $\psi'(I)$ on the interval $[0, K]$. Then we will prove the rest in the following two subcases.

(g1) If $\psi'(\bar{I}) \le 0$, then $\psi'(I) \le \psi'(\bar{I}) < 0$ when $I \in (0, K)$ and $I \ne \bar{I}$. So $\psi(I)$ is a monotonically decreasing function in the interval $(0, K)$. As $\psi(0) > 0$ and $\psi(K) < 0$, $\psi(I)$ has a unique zero I^* in the interval $(0, K)$.

(g2) If $\psi'(\bar{I}) > 0$, as $\psi'(0) < 0$ and $\psi'(K) < 0$, therefore, $\psi'(I)$ has two points I_{10} and I_{20} such that

$$0 < I_{10} < \bar{I} < I_{20} < K,$$

and $\psi'(I) < 0$ when $0 < I < I_{10}$, $\psi'(I) > 0$ when $I_{10} < I < I_{20}$ and $\psi'(I) < 0$ when $I_{20} < I < K$. So I_{10} and I_{20} are the minimum point and maximum point of the function $\psi(I)$ on the interval $[0, K]$, respectively. By means of $\psi'(I_{10}) = 0$ and (9), we have

$$\begin{aligned}
\psi(I_{10}) &= p - 1 - p\frac{I_{10}}{K} + \alpha(p-1-q)I_{10}^q + \alpha(q-p)\frac{I_{10}^{q+1}}{K} \\
&= p - 1 - p\frac{I_{10}}{K} + \frac{pI_{10}}{qK} + \alpha(p-q)(q+1)\frac{I_{10}^{q+1}}{qK} + \alpha(q-p)\frac{I_{10}^{q+1}}{K} \\
&> (p-1)\frac{I_{10}}{K} - p\frac{I_{10}}{K} + \frac{pI_{10}}{qK} + \frac{\alpha(p-q)}{qK}I_{10}^{q+1} \\
&= \frac{(p-q)I_{10}}{qK} + \frac{\alpha(p-q)}{qK}I_{10}^{q+1} > 0.
\end{aligned} \tag{15}$$

It follows from $\psi(K) < 0$ that $\psi(I)$ has a unique zero I^* in the interval $(0, K)$.

It is known from the above discussion that if $p > 1$, then $\psi(I)$ has a unique zero I^* in the interval $(0, K)$ such that $\psi(I) > 0$ as $I \in (0, I^*)$, and $\psi(I) < 0$ as $I \in (I^*, K)$. Therefore, it is obtained from (8) that $\varphi'(I) > 0$ as $I \in (0, I^*)$, and $\varphi'(I) < 0$ as $I \in (I^*, K)$. So I^* is a unique extreme point and the maximum point of $\phi(I)$ in the interval $(0, K)$. As $\varphi(0) = \varphi(K) = 0$, let

$$\sigma^* = \frac{1}{\varphi(I^*)} = \frac{1}{\max\limits_{0 \le I \le K} \varphi(I)}. \tag{16}$$

If $\sigma < \sigma^*$, then equation (6) has no positive solution; if $\sigma = \sigma^*$, then equation (6) has a unique positive solution I^*; and if $\sigma > \sigma^*$, then equation (6) has two positive solutions I_1 and I_2, where $0 < I_1 < I^* < I_2 < K$.

By the above discussion, we have the following conclusions.

Theorem 2.1. *System (4) always has a disease-free equilibrium $O(0, 0)$. Furthermore,*

(i) If $0 < p < 1$, then system (4) has a unique endemic equilibrium $E_e(I_e, R_e)$.

(ii) If $p = 1$ and $q = 0$, then system (4) does not have endemic equilibrium when $\sigma \le 1 + \alpha$; and it has a unique endemic equilibrium $E_e(I_e, R_e)$ when $\sigma > 1 + \alpha$.

(iii) If $p = 1$ and $q > 0$, then system (4) does not have endemic equilibrium when $\sigma \le 1$; and it has a unique endemic equilibrium $E_e(I_e, R_e)$ when $\sigma > 1$.

(iv) If $p > 1$, then system (4) does not have endemic equilibrium when $\sigma < \sigma^*$; it has a unique endemic equilibrium $E^*(I^*, R^*)$ when $\sigma = \sigma^*$, and two endemic equilibria $E_1(I_1, R_1)$ and $E_2(I_2, R_2)$ when $\sigma > \sigma^*$.

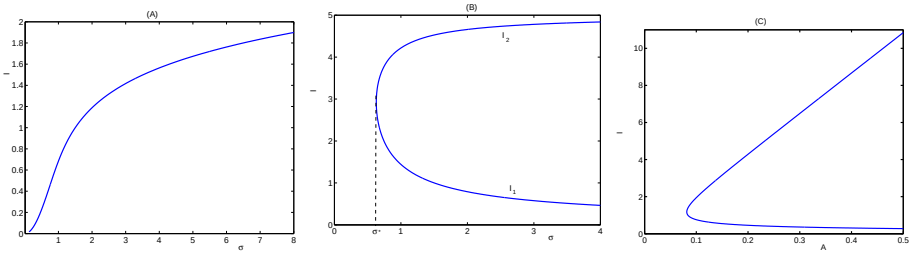

FIGURE 1. The solutions of equation (6) versus σ or A. (A) $p = 0.5 > 1$, $q = 4$, $\alpha = 0.2$ and $K = 5$; (B) $p = 2.5 > 1$, $q = 0.4$, $\alpha = 0.2$ and $K = 5$; (C) $p = 3, q = 2, \beta = 0.1, \alpha = 0.02, \mu = 0.36, v = 0.09$ and $d = 0.01$.

For system (4), Fig. 1 show that the solutions of equation (6) change with parameters σ or A.

3. **The stability of equilibria.** In this section, we study the stability of the disease-free and endemic equilibria. For the disease-free equilibrium $O(0, 0)$ of system (4), we have the following conclusions.

Theorem 3.1. *(i) If $0 < p < 1$, then $O(0, 0)$ is unstable.*

(ii) If $p = 1$ and $q = 0$, then O is globally asymptotically stable in D when $\sigma \le 1 + \alpha$, and unstable when $\sigma > 1 + \alpha$.

(iii) If $p = 1$ and $q > 0$, then O is globally asymptotically stable in D when $\sigma \le 1$, and unstable when $\sigma > 1$.

(iv) If $p > 1$, then O is locally asymptotically stable. Furthermore, O is globally asymptotically stable in D when $\sigma < \sigma^*$.

Proof. (i) Suppose that when $0 < p < 1$, the equilibrium $O(0, 0)$ of system (4) is stable. We select a sufficiently small positive number ε_0 such that

$$\varepsilon_0^{1-p} - \beta \frac{N_0 - 2\varepsilon_0}{2(d + \mu)(1 + \alpha\varepsilon_0^q)} < 0. \tag{17}$$

For the above ε_0, there exists a positive number δ_0 such that the solution $(I(t), R(t))$ of system (4) with initial conditions $I(0) = I_0 > 0$ and $R(0) = R_0 > 0$ satisfies that for all $t > 0$,

$$0 < I(t) < \varepsilon_0, \quad 0 < R(t) < \varepsilon_0. \tag{18}$$

Let $U = I^{1-p}$, then the first equation of system (4) is changed into

$$\frac{\mathrm{d}U}{\mathrm{d}t} = (1 - p)\beta \frac{N_0 - U^{\frac{1}{1-p}} - R}{1 + \alpha U^{\frac{q}{1-p}}} - (d + \mu)(1 - p)U. \tag{19}$$

Noting (18), it follows that

$$U(t) = I(t)^{1-p} < \varepsilon_0^{1-p}. \tag{20}$$

Recalling (18) and (19) yields

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} > (1-p)\beta\frac{N_0-2\varepsilon_0}{1+\alpha\varepsilon_0^q} - (d+\mu)(1-p)U(t). \tag{21}$$

It follows that

$$\begin{aligned} U(t) &\geq U(0)\mathrm{e}^{-(1-p)(d+\mu)t} + \int_0^t (1-p)\frac{\beta(N_0-2\varepsilon_0)}{1+\alpha\varepsilon_0^q}\mathrm{e}^{-(1-p)(d+\mu)(t-y)}\mathrm{d}y \\ &> \beta\frac{(N_0-2\varepsilon_0)}{(d+\mu)(1+\alpha\varepsilon_0^q)}\left(1-\mathrm{e}^{-(1-p)(d+\mu)t}\right). \end{aligned} \tag{22}$$

When t is sufficient large, we have

$$U(t) > \beta\frac{N_0-2\varepsilon_0}{2(d+\mu)(1+\alpha\varepsilon_0^q)}. \tag{23}$$

From (20) and (23), it follows that

$$\varepsilon_0^{1-p} > \beta\frac{N_0-2\varepsilon_0}{2(d+\mu)(1+\alpha\varepsilon_0^q)}, \tag{24}$$

which is a contradiction with the selection of ε. So $O(0,0)$ is unstable when $0 < p < 1$.

(ii) If $p = 1$ and $q = 0$, the Jacobian matrix of system (4) at $O(0,0)$ is

$$J_0 = \begin{pmatrix} \beta\dfrac{A}{d(1+\alpha)} - (d+\mu) & 0 \\ \mu & -(d+v) \end{pmatrix}.$$

The eigenvalues of the matrix J_0 are $\lambda_1 = -(d+v) < 0$ and $\lambda_2 = (d+\mu)\left(\dfrac{\sigma}{1+\alpha} - 1\right)$. So $O(0,0)$ is locally asymptotically stable when $\sigma < 1+\alpha$, and unstable when $\sigma > 1+\alpha$.

In order to prove the global stability of the equilibrium O, we construct a Lyapunov function $V = I$. Then the derivative of V along the solutions of system (4) is

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &\leq \frac{\beta I}{1+\alpha}\left(\frac{A}{d} - I\right) - (d+\mu)I \\ &= (d+\mu)\left(\frac{\sigma}{1+\alpha} - 1\right)I - \beta\frac{I^2}{1+\alpha}. \end{aligned} \tag{25}$$

If $\sigma \leq 1$, it is obtained from (25) that $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ is negative definite. So $\lim\limits_{t\to+\infty} I(t) = 0$. By means of the limit equation of (4), it is easy to prove that $\lim\limits_{t\to+\infty} R(t) = 0$. Therefore, O is globally asymptotically stable in D as $\sigma \leq 1+\alpha$.

(iii) Similarly, we can prove that if $p = 1$ and $q > 0$, then O is globally asymptotically stable when $\sigma \leq 1$, and unstable when $\sigma > 1$.

(iv) If $p > 1$, then the eigenvalues of the Jacobian matrix of system (4) at $O(0,0)$ are $\lambda_1 = -(d+v) < 0$ and $\lambda_2 = -(d+\mu) < 0$. So O is locally asymptotically stable.

If $\sigma < \sigma^*$, it is known from Theorem 2.1 that system (4) does not have positive equilibrium in the invariant set D. So O is globally asymptotically stable in D. □

Theorem 3.2. *Suppose $0 < p \leq 1$, if the endemic equilibrium $E_e(I_e, R_e)$ of system (4) exists, then it is locally asymptotically stable.*

Proof. If the endemic equilibrium $E_e(I_e, R_e)$ of system (4) exists, then it follows from (5) and (6) that the Jacobian matrix of system (4) at $E_e(I_e, R_e)$ is

$$J_e = \begin{pmatrix} (p-1)(d+\mu) - \alpha q(d+\mu)\dfrac{I_e^q}{1+\alpha I_e^q} - \beta\dfrac{I_e^p}{1+\alpha I_e^q} & -\beta\dfrac{I_e^p}{1+\alpha I_e^q} \\ \mu & -(d+v) \end{pmatrix}. \tag{26}$$

The determinant of the matrix J_e is

$$\begin{aligned} \det(J_e) = (d+\mu)(d+v)\left[1-p+\alpha q\frac{I_e^q}{1+\alpha I_e^q}\right] \\ +\beta(d+\mu+v)\frac{I_e^p}{1+\alpha I_e^q} > 0. \end{aligned} \tag{27}$$

The trace of the matrix J_e is

$$\operatorname{tr}(J_e) = (p-1)(d+\mu) - (d+v) - \alpha q(d+\mu)\frac{I_e^q}{1+\alpha I_e^q} - \beta\frac{I_e^p}{1+\alpha I_e^q} < 0. \tag{28}$$

So E_e is locally asymptotically stable. □

Theorem 3.3. *If $p > 1$ and $\sigma > \sigma^*$, then the endemic equilibrium $E_1(I_1, R_1)$ of system (4) with lower number of infected individuals is a saddle point.*

Proof. When $p > 1$ and $\sigma > \sigma^*$, the endemic equilibria $E_1(I_1, R_1)$ and $E_2(I_2, R_2)$ of system (4) exist.

The Jacobian matrix of system (4) at $E_i(I_i, R_i)(i = 1, 2)$ is

$$J_i = \begin{pmatrix} (p-1)(d+\mu) - \alpha q(d+\mu)\dfrac{I_i^q}{1+\alpha I_i^q} - \beta\dfrac{I_i^p}{1+\alpha I_i^q} & -\beta\dfrac{I_i^p}{1+\alpha I_i^q} \\ \mu & -(d+v) \end{pmatrix}, i = 1, 2. \tag{29}$$

For the small endemic equilibrium $E_1(I_1, R_1)$, by making use of (6) and after some tedious calculations, the determinant of J_1 is obtained:

$$\begin{aligned} \det(J_1) &= (d+\mu)(d+v)\left[1-p+\alpha q\frac{I_1^q}{1+\alpha I_1^q}\right] + \beta(d+\mu+v)\frac{I_1^p}{1+\alpha I_1^q} \\ &= -\frac{(d+\mu)(d+v)}{(1+\alpha I_1^q)\left(1-\dfrac{I_1}{K}\right)}\psi(I_1). \end{aligned} \tag{30}$$

For the small endemic equilibrium $E_1(I_1, R_1)$, as $0 < I_1 < I^*$ and it is known from the proof of Theorem 2.1 that $\psi(I) > 0$ in interval $(0, I^*)$, so $\psi(I_1) > 0$. It follows from (30) that $\det(J_1) < 0$. Therefore, the small endemic equilibrium $E_1(I_1, R_1)$ is a saddle and unstable. □

Theorem 3.4. *If $p > 1$ and $\sigma > \sigma^*$, then the endemic equilibrium $E_2(I_2, R_2)$ of system (4) with higher number of infected individuals is either an attractor or a repeller. Moreover, $E_2(I_2, R_2)$ is locally asymptotically stable as $1 < p \le 1 + \dfrac{d+v}{d+\mu}$.*

Proof. For the large endemic equilibrium $E_2(I_2, R_2)$, by making use of (6), as $I^* < I_2 < K$, we have $\psi(I_2) < 0$. So the determinant of the matrix J_2 is

$$\det(J_2) = -\frac{(d+\mu)(d+v)}{(1+\alpha I_2^q)\left(1-\dfrac{I_2}{K}\right)}\psi(I_2) > 0. \tag{31}$$

So E_2 cannot be a saddle point, it is either an attractor or a repeller.

If $1 < p \le 1 + \dfrac{d+v}{d+\mu}$, the trace of J_2 is

$$\operatorname{tr}(J_2) = (p-1)(d+\mu) - (d+v) - \alpha q(d+\mu)\frac{I_2^q}{1+\alpha I_2^q} - \beta\frac{I_2^p}{1+\alpha I_2^q} < 0. \tag{32}$$

Therefore, E_1 is locally asymptotically stable. □

Now, we will consider the nonexistence of limit cycles in system (4). We construct a Dulac function to obtain some sufficient conditions for the nonexistence of limit cycles for system (4). It is convenient to denote the right-hand sides of (4) by $P(I,R)$ and $Q(I,R)$, respectively.

Theorem 3.5. *For system (4), if the parameter p satisfies $0 < p \le 1$ or $1 < p \le 1 + \dfrac{d+v}{d+\mu}$, then there exists no limit cycle.*

Proof. We consider the Dulac function

$$B(I,R) = \frac{1+\alpha I^q}{\beta I^p}.$$

As

$$\begin{aligned}&\frac{\partial(PB)}{\partial I} + \frac{\partial(QB)}{\partial R}\\ &\quad = -1 - \frac{(d+\mu)(1-p) + (d+v) + [(d+\mu)(1-p+q) + d + v]\,\alpha I^q}{\beta I^p},\end{aligned}$$

when $0 < p \le 1$ or $1 < p \le 1 + \dfrac{d+v}{d+\mu}$, we have $\dfrac{\partial(PB)}{\partial I} + \dfrac{\partial(QB)}{\partial R} < 0$ in D. So it follows form Dulac criterion that a closed orbit of system (4) does not exist. □

From Theorems 3.2 and 3.5, we have the following conclusion.

Corollary 3.6. *Suppose $0 < p \le 1$. If the endemic equilibrium $E_e(I_e, R_e)$ of system (4) exists, then it is globally asymptotically stable.*

4. **Bifurcation analysis.** In this section, different kinds of bifurcations will be discussed. We will undertake the stability analysis of the equilibria to obtain normal forms of the model for Hopf and Bogdanov-Takens bifurcations. We consider σ or A and v as bifurcation parameters and derive normal forms of the system in the vicinity of the bifurcation points.

4.1. **Hopf Bifurcation.** In order to show that system (4) undergoes a Hopf bifurcation at $E_2(I_2, R_2)$, let

$$f(I) = \frac{\beta I^p}{1+\alpha I^q}. \tag{33}$$

Set $I = I_2 + x$ and $R = R_2 + y$ to translate (I_2, R_2) to the origin of the co-ordinates (x, y).

Noting that I_2 and R_2 satisfy (5), after some manipulations, system (4) is transformed into the following system

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = J(A)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} M(x,y;A) \\ 0 \end{pmatrix}, \tag{34}$$

where

$$J(A) = \begin{pmatrix} f'(I_2)\left(\frac{A}{d} - I_2 - R_2\right) - (d+\mu) - f(I_2) & -f(I_2) \\ \mu & -(d+v) \end{pmatrix} \tag{35}$$

and

$$\begin{aligned} M(x,y;A) = {} & \frac{1}{2}f''(I_2)\left(\frac{A}{d} - I_2 - R_2\right)x^2 - f'(I_2)x(x+y) \\ & -\frac{1}{2}f''(I_2)x^2y - \frac{1}{3!}f'''(I_2)\left(\frac{A}{d} - I_2 - R_2\right)x^3 - \frac{1}{2}f''(I_2)x^3 \\ & +\frac{1}{3!}f'''(I_2)x^3y + f_4(x,A)\left(\frac{A}{d} - I_2 - R_2 - x - y\right), \end{aligned} \tag{36}$$

where $f_4(x,A)$ denotes the fourth and higher order terms in x of the expression

$$f(I_2(A)+x;A) = f(I_2) + f'(I_2)x + \frac{1}{2}f''(I_2)x^2 + \frac{1}{3!}f'''(I_2)x^3 + f_4(x;A). \tag{37}$$

Because the trace of the matrix $J(A)$ is

$$T(I_2(A),A) \triangleq \mathrm{tr}(J(A)) = f'(I_2)\left(N_0 - I_2 - R_2\right) - f(I_2) - (2d+v+\mu), \tag{38}$$

by calculation, we have

$$\frac{\mathrm{d}T}{\mathrm{d}A} = \frac{[f(I_2)f''(I_2) - f'^2(I_2)](N_0 - I_2 - R_2) + f'(I_2)(d+\mu-f(I_2))}{d\left[-T(I_2(A),A) + \frac{\mu}{d+v}f(I_2) - (d+v)\right]}. \tag{39}$$

Suppose that $\mathrm{tr}(J(A)) = 0$ at A^c, then it can be seen that

$$J(A^c) = \begin{pmatrix} d+v & -f(I_2) \\ \mu & -(d+v) \end{pmatrix}. \tag{40}$$

We now find the normal form of the system as follows. Let $\mathbf{u}$ be an eigenvector of the matrix $J(A^c)$ corresponding to the eigenvalue $-i\omega_c$, i.e. $J(A^c)\mathbf{u} = -i\omega_c\mathbf{u}$, where $\mathbf{u} = (u_1,u_2)^T \in C^2$ and $\omega_c = \sqrt{\mu f(I_2) - (d+v)^2}$. A simple calculation gives

$$\mathbf{u} = \begin{pmatrix} d+v-i\omega_c \\ \mu \end{pmatrix}.$$

Let $\mathbf{u} = \mathrm{Re}(\mathbf{u}) + i\mathrm{Im}(\mathbf{u})$. Then

$$\begin{cases} J(A^c)\mathrm{Re}(\mathbf{u}) = \omega_c\mathrm{Im}(\mathbf{u}), \\ J(A^c)\mathrm{Im}(\mathbf{u}) = -\omega_c\mathrm{Re}(\mathbf{u}), \end{cases}$$

which implies that

$$J(A^c)\left(\mathrm{Re}(\mathbf{u}),\mathrm{Im}(\mathbf{u})\right) = \left(\mathrm{Re}(\mathbf{u}),\mathrm{Im}(\mathbf{u})\right)\begin{pmatrix} 0 & -\omega_c \\ \omega_c & 0 \end{pmatrix}.$$

Defining matrix $Q = (\mathrm{Re}(\mathbf{u}),\mathrm{Im}(\mathbf{u}))$, it is clear that

$$Q^{-1}J(A^c)Q = \begin{pmatrix} 0 & -\omega_c \\ \omega_c & 0 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} d+v & -\omega_c \\ \mu & 0 \end{pmatrix}. \tag{41}$$

Therefore, if we consider the transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{42}$$

then we obtain the normal form of system (34) as follows

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = -\omega_c\eta, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = \omega_c\xi - \dfrac{1}{\omega_c}\widetilde{M}(\xi,\eta;A^c), \end{cases} \tag{43}$$

where $\widetilde{M}(\xi,\eta;A^c) = M((d+v)\xi - \omega_c\eta, \mu\xi; A^c)$.

Suppose

$$\begin{aligned} \Phi \triangleq \Big\{ &[f(I_2(A))f''(I_2(A)) - f'^2(I_2(A))]\left(\frac{A}{d} - I_2(A) - R_2(A)\right) \\ &+ f'(I_2(A))[d+\mu - f(I_2(A))]\} \left[\mu f(I_2) - (d+v)^2\right], \end{aligned} \tag{44}$$

then Φ and $\dfrac{\mathrm{d}}{\mathrm{d}A}T(A)$ have the same sign at $A = A^c$. It follows from Glendinning [6] that E_2 is locally asymptotically stable for $A > A^c$ (respectively, $A < A^c$) and unstable for $A < A^c$ (respectively, $A > A^c$) if $\Phi_{A=A^c} < 0$ (respectively, $\Phi|_{A=A^c} > 0$), and the system undergoes a Hopf bifurcation at $A = A^c$.

Evaluating the first Lyapunov coefficient (see Glendinning [6], Hassard et al. [7] and Kuznetsov [11]) of the system at $(0,0,A^c)$ gives (see also Wiggins [21, p. 277])

$$\begin{aligned} \kappa &= -\frac{1}{16\omega_c}\left(\widetilde{M}_{\xi\xi\eta} + \widetilde{M}_{\eta\eta\eta}\right) - \frac{1}{16\omega_c^3}\widetilde{M}_{\xi\eta}\left(\widetilde{M}_{\xi\xi} + \widetilde{M}_{\eta\eta}\right) \\ &= \frac{1}{16}\left(2(d+v)\mu M_{xxy} + ((d+v)^2 + \omega_c^2)M_{xxx}\right) \\ &+ \frac{1}{16\omega_c^2}\left(\mu M_{xy} + (d+v)M_{xx}\right)\left[2(d+v)\mu M_{xy} + ((d+v)^2 + \omega_c^2)M_{xx}\right], \end{aligned} \tag{45}$$

where

$$\begin{aligned} &M_{xx}(0,0,A^c) = f''(I_2)\,(N_0 - I_2 - R_2) - 2f'(I_2), \\ &M_{xy}(0,0,A^c) = -f'(I_2), \\ &M_{xxy}(0,0,A^c) = -f''(I_2), \\ &M_{xxx}(0,0,A^c) = f'''(I_2)\,(N_0 - I_2 - R_2) - 3f''(I_2). \end{aligned}$$

Theorem 4.1. *Suppose $\sigma > \sigma^*$ and there exists $A^c > 0$ such that $T(I_2(A^c);A^c) = 0$ and $\Phi|_{A=A_c} \neq 0$. If $\kappa \neq 0$, then a family of periodic solutions bifurcates from the endemic equilibrium E_2 such that:*

(i) for $\kappa < 0$, the system undergoes a supercritical Hopf bifurcation if $\Phi|_{A=A_c} > 0$ and a backward supercritical Hopf bifurcation if $\Phi|_{A=A_c} < 0$;

(ii) for $\kappa > 0$, the system undergoes a subcritical Hopf bifurcation if $\Phi|_{A=A_c} > 0$ and a backward subcritical Hopf bifurcation if $\Phi|_{A=A_c} < 0$.

4.2. **Bogdanov-Takens bifurcation.** In this section, we consider the Bogdanov-Takens bifurcation, i.e. the bifurcation of a cusp of codimension 2. The normal form of this bifurcation gives the local representation of a homoclinic curve at the Bogdanov-Takens point (v,A), where $\mathrm{tr}(J^*(v,A)) = \det(J^*(v,A)) = 0$. By considering v and A as the bifurcation parameters and applying the transformations $I = I^* + x$ and $R = R^* + y$ at the Bogdanov-Takens point at which $v = v_{BT}, A = A_{BT}$, system (4) transforms to (34) about $E^*(I^*,R^*)$, the unique endemic equilibrium of system. It can be seen that at the Bogdanov-Takens point, we have

$$J_{BT} = \frac{1}{\mu}\begin{pmatrix} (d+v_{BT})\mu & -(d+v_{BT})^2 \\ \mu^2 & -(d+v_{BT})\mu \end{pmatrix}. \tag{46}$$

Thus, it follows from (5) and $\det(J_{BT}) = 0$ that

$$I_{BT} \triangleq I^*(v_{BT}, A_{BT}) = \frac{(d+v_{BT})^2 A_{BT}}{d[\mu(d+\mu)+(d+v_{BT})(d+v_{BT}+\mu)]}. \tag{47}$$

Since $J_{BT} \neq 0$, there exist real linearly independent vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $J_{BT}\mathbf{x}_1 = 0$ and $J_{BT}\mathbf{x}_2 = \mathbf{x}_1$. These vectors are given by

$$\mathbf{x}_1 = -\frac{1}{\sqrt{\mu}}\begin{pmatrix} d+v_{BT} \\ \mu \end{pmatrix}, \quad \mathbf{x}_2 = \frac{\sqrt{\mu}}{d+v_{BT}+\mu}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{48}$$

Similarly, there exist vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ such that $J_{BT}^T\mathbf{y}_1 = 0$ and $J_{BT}^T\mathbf{y}_2 = \mathbf{y}_1$, where J_{BT}^T is the transposed matrix. These vectors may be expressed as

$$\mathbf{y}_1 = \frac{1}{\sqrt{\mu}}\begin{pmatrix} -\mu \\ d+v_{BT} \end{pmatrix}, \quad \mathbf{y}_2 = \frac{\sqrt{\mu}}{d+v_{BT}+\mu}\begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{49}$$

It is easy to verify that $\mathbf{x}_1 \cdot \mathbf{y}_2 = \mathbf{x}_2 \cdot \mathbf{y}_1 = 1$ and $\mathbf{x}_2 \cdot \mathbf{y}_2 = \mathbf{x}_1 \cdot \mathbf{y}_1 = 0$. Defining a matrix $U = [\mathbf{x}_1, \mathbf{x}_2]$ and a transformation $(z_1, z_2)^T = U^{-1}(x, y)^T$, from which the new coordinates (z_1, z_2) are obtained as

$$\begin{aligned} z_1 &= -\frac{\sqrt{\mu}}{d+v_{BT}+\mu}(x+y), \\ z_2 &= -\sqrt{\mu}x + \frac{d+v_{BT}}{\sqrt{\mu}}y. \end{aligned} \tag{50}$$

Then system (34) becomes

$$\begin{pmatrix} \dfrac{\mathrm{d}z_1}{\mathrm{d}t} \\ \dfrac{\mathrm{d}z_2}{\mathrm{d}t} \end{pmatrix} = \widetilde{J}^*(v, A)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \sqrt{\mu}\begin{pmatrix} \dfrac{1}{d+v_{BT}+\mu}\widetilde{M}(z_1, z_2; v, A) \\ \widetilde{M}(z_1, z_2; v, A) \end{pmatrix}, \tag{51}$$

where

$$\widetilde{J}^*(v, A) = U^{-1}J^*U = \begin{pmatrix} \tilde{a}_{11}(v, A) & \tilde{a}_{12}(v, A) \\ \tilde{a}_{21}(v, A) & \tilde{a}_{22}(v, A) \end{pmatrix} \tag{52}$$

with

$$\begin{aligned} &\tilde{a}_{11}(v, A) \\ &= -\frac{(d+v_{BT}+\mu)f(I^*) + \mu(d+v) + (d+v_{BT})[d-(N_0-I^*-R^*)f'(I^*)]}{d+v_{BT}+\mu}, \\ &\tilde{a}_{12}(v, A) = \frac{\mu[v+(N_0-I^*-R^*)f'(I^*)]}{(d+v_{BT}+\mu)^2}, \\ &\tilde{a}_{21}(v, A) = -(d+v_{BT}+\mu)f(I^*) \\ &\qquad - (d+v_{BT})[d+\mu+v_{BT}-v-(N_0-I^*-R^*)f'(I^*)], \\ &\tilde{a}_{22}(v, A) \\ &= -\frac{d^2+\mu^2+\mu v_{BT}+vv_{BT}+d(2\mu+v+v_{BT})-\mu(N_0-I^*-R^*)f'(I^*)}{d+v_{BT}+\mu}. \end{aligned}$$

Noting that $\tilde{a}_{ij}(v, A) = \tilde{b}_{ij}^{(0)} + b_{ij}(v - v_{BT}) + \tilde{b}_{ij}(A - A_{BT}) + O(2)$, $\tilde{b}_{ij}^{(0)} = 0$ for $(i, j) \neq (1, 2)$ and $\tilde{b}_{12}^{(0)} = 1$, it follows that at the Bogdanov-Takens point:

$$\widetilde{J}^*(v_{BT}, A_{BT}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{53}$$

We now introduce a change of variables by denoting the right-hand side of the first equation in (51) by Y and letting $X = z_1$. After some tedious algebra, it can be obtained that system (51) transforms to

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = Y, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = r_{10}X + r_{01}Y + r_{11}XY + r_{20}X^2 + r_{02}Y^2 \\ \qquad\quad +O(\|(X, Y, v - v_{BT}, A - A_{BT})\|^3), \end{cases} \tag{54}$$

where

$$\begin{aligned} r_{10} &= b_{21}(v - v_{BT}) + \tilde{b}_{21}(A - A_{BT}), \\ r_{01} &= (b_{11} + b_{22})(v - v_{BT}) + (\tilde{b}_{11} + \tilde{b}_{22})(A - A_{BT}), \\ r_{11} &= \frac{d(2d + 2v_{BT} + \mu)f'(I^*) - [A - d(I^* + R^*)](d + v_{BT})f''(I^*)}{d\sqrt{\mu}}, \\ r_{20} &= \frac{(d + v_{BT})\left[2d(d + v_{BT} + \mu)f'(I^*) - (A - d(I^* + R^*))\,(d + v_{BT})f''(I^*)\right]}{2d\sqrt{\mu}}, \\ r_{02} &= \frac{1}{2d(d + v_{BT} + \mu)^2}\left[2d(d^2 + v_{BT}^2 + 2d(v_{BT} + \sqrt{\mu}) + 2v_{BT}\sqrt{\mu} - \mu^2)f'(I^*)\right. \\ &\quad \left. - (A - d(I^* + R^*))\sqrt{\mu}(2d + 2v_{BT} + \mu)f''(I^*)\right], \\ R^* &= \frac{\mu}{d + v_{BT}}I^*. \end{aligned}$$

Assume that $r_{11} \neq 0$ at the Bogdanov-Takens point. Then there is a neighbourhood of $(I_{BT}, R_{BT}, v_{BT}, A_{BT})$ in which $r_{11} \neq 0$. Letting $\Theta_1 = X - \rho$, where $\rho = r_{01}/r_{11}$, denoting Θ_1 as X and using a time reparametrization $\mathrm{d}t = (1 - r_{02}X)\mathrm{d}\tau$, it is easy to check that system (54) can be written as

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}\tau} = (1 - r_{02}X)Y, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}\tau} = (1 - r_{02}X)\left[(r_{10} + \rho r_{20})\rho + (r_{10} + 2\rho r_{20})X\right. \\ \qquad\quad \left. +r_{11}XY + r_{20}X^2 + r_{02}Y^2 + O(3)\right], \end{cases} \tag{55}$$

where $O(3)$ is a smooth function of $(X, Y, v - v_{BT}, A - A_{BT})$ of at least the third order. Define new variables $\theta_1 = X$ and $\theta_2 = (1 - r_{02}X)Y$, then system (55) transforms to

$$\begin{cases} \dfrac{\mathrm{d}\theta_1}{\mathrm{d}\tau} = \theta_2, \\ \dfrac{\mathrm{d}\theta_2}{\mathrm{d}\tau} = \rho(r_{10} + \rho r_{20}) + \left[r_{10} + 2\rho r_{20} - 2\rho r_{02}(r_{10} + \rho r_{20})\right]\theta_1 + r_{11}\theta_1\theta_2 \\ \qquad\quad + \left[\rho r_{02}^2(r_{10} + \rho r_{20}) - 2r_{02}(r_{10} + 2\rho r_{20}) + r_{20}\right]\theta_1^2 + O(3). \end{cases} \tag{56}$$

Let

$$\Lambda = \rho r_{02}^2(r_{10} + \rho r_{20}) - 2r_{02}(r_{10} + 2\rho r_{20}) + r_{20}. \tag{57}$$

Since $(r_{10}, r_{01}, \rho) \to (0, 0, 0)$ as $(v, A) \to (v_{BT}, A_{BT})$, it follows that

$$\begin{aligned} \Lambda_{BT} &= \lim_{(v,A)\to(v_{BT},A_{BT})} \Lambda = \lim_{(v,A)\to(v_{BT},A_{BT})} r_{20} \\ &= \frac{d + v_{BT}}{2d\sqrt{\mu}}\left[\frac{2d^2(d + v_{BT} + \mu)\left[d^2 + v_{BT}^2 + v_{BT}\mu + \mu^2 + 2d(v_{BT} + \mu)\right]^2}{\mu^2(d + \mu)A_{BT}}\right. \\ &\quad \left. - \frac{\mu(d + v_{BT})(d + \mu)A_{BT}f''(I_{BT})}{d^2 + v_{BT}^2 + v_{BT}\mu + \mu^2 + 2d(v_{BT} + \mu)}\right]. \end{aligned} \tag{58}$$

If $\Lambda_{BT} \neq 0$, then $\Lambda \neq 0$ in a small neighbourhood of the Bogdanov-Takens point. Since $r_{11} \neq 0$, by making the change of variables

$$\Theta_1 = \frac{r_{11}^2 \theta_1}{\Lambda}, \quad \Theta_2 = \frac{r_{11}^3 \theta_2}{\Lambda^2}, \quad t = \frac{\Lambda \tau}{r_{11}}$$

and renaming Θ_1, Θ_2 as θ_1, θ_2, respectively, we have

$$\begin{cases} \dfrac{\mathrm{d}\theta_1}{\mathrm{d}t} = \theta_2, \\ \dfrac{\mathrm{d}\theta_2}{\mathrm{d}t} = \rho r_{11}^4 (r_{10} + \rho r_{20})/\Lambda^3 + [r_{10} + 2\rho r_{20} - 2\rho r_{02}(r_{10} + \rho r_{20})]\, r_{11}^2 \theta_1/\Lambda^2 \\ \qquad\quad + \theta_1^2 + \theta_1 \theta_2 + O(3), \end{cases} \tag{59}$$

Therefore, from Theorem 8.4 and equations (8.52)-(8.54) in Kuznetsov [11], the following theorem is established.

Theorem 4.2. *Suppose $p > 1$ and $\sigma = \sigma^*$. If there exists a Bogdanov-Takens point (I_{BT}, R_{BT}) with parameters $v = v_{BT}, A = A_{BT}$, such that*

(i) $\dfrac{f'(I_{BT})}{f''(I_{BT})} - \dfrac{[A_{BT} - d(I_{BT} + R_{BT})](d + v_{BT})}{d(2d + 2v_{BT} + \mu)} \neq 0.$

(ii) $f''(I_{BT}) \neq \dfrac{2d^2(d + v_{BT} + \mu)\left[d^2 + v_{BT}^2 + v_{BT}\mu + \mu^2 + 2d(v_{BT} + \mu)\right]^3}{\mu^3 (d + \mu)^2 (d + v_{BT}) A_{BT}^2},$

then in a small neighborhood of $E^*(I_{BT}, R_{BT})$, system (4) has the following bifurcation curves:

(a) A saddle-node bifurcation curve

$$SN = \left\{ (v, A) | 4\rho(r_{10} + \rho r_{20})\Lambda = [r_{10}(1 - 2\rho r_{02}) + 2\rho r_{20}(1 - \rho r_{02})]^2 \right\};$$

(b) A non-degenerate Hopf bifurcation curve

$$HP = \{ (v, A) | r_{01} = 0, r_{10} < 0 \};$$

(c) A homoclinic bifurcation curve

$$\begin{aligned} P = \quad & \{ (v, A) | r_{10} + 2\rho r_{20} < 2\rho r_{02}(r_{10} + \rho r_{20}) \,, \\ & 6\left[r_{10} + 2\rho r_{20} - 2\rho r_{02}(r_{10} + \rho r_{20})\right]^2 \\ & + 25\rho(r_{10} + \rho r_{20})\Lambda = o(\| \, (v - v_{BT}, A - A_{BT}) \, \|^2) \}. \end{aligned}$$

5. **Numeric simulations of the model.** In this section, some numerical simulations of system (4) are given to illustrate the main results.

Example 5.1. For system (4), the parameters are chosen as follows: $p = 0.5, q = 0.8, \beta = 0.34, \alpha = 0.4, \mu = 0.1, v = 0.1, d = 0.2$ and $A = 3$. The unique endemic equilibrium is $E_e(6.5265, 2.1755)$, the phase diagram of system (4) is illustrated in Fig. 2. It is known from Fig. 2 that the equilibrium E_e is globally asymptotically stable in the interior of positive invariant set D.

Example 5.2. For system (4), the parameters are chosen as follows: $p = 1, q = 0.8, \beta = 0.34, \alpha = 0.4, \mu = 0.6, v = 0.1$ and $d = 0.6$. If taking $A = 1$, then $\sigma = 0.47 < 1$, the phase diagram of system (4) is illustrated in Fig. 3-A. It is known from Fig. 3-A that the equilibrium O is globally asymptotically stable in D.

If $A = 3$, then $\sigma = 1.41667 > 1$, the unique endemic equilibrium is $E_e(0.4154, 0.3561)$. The phase diagram of system (4) is illustrated in Fig. 3-B. From Fig. 3-B, it is known that the equilibrium E_e is globally asymptotically stable in the interior of D.

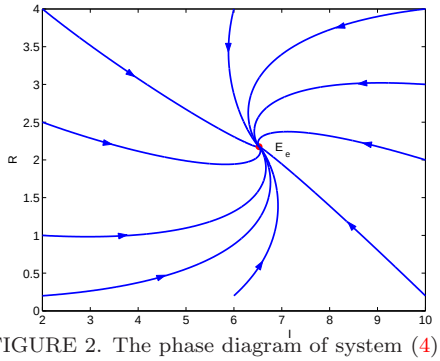

FIGURE 2. The phase diagram of system (4).

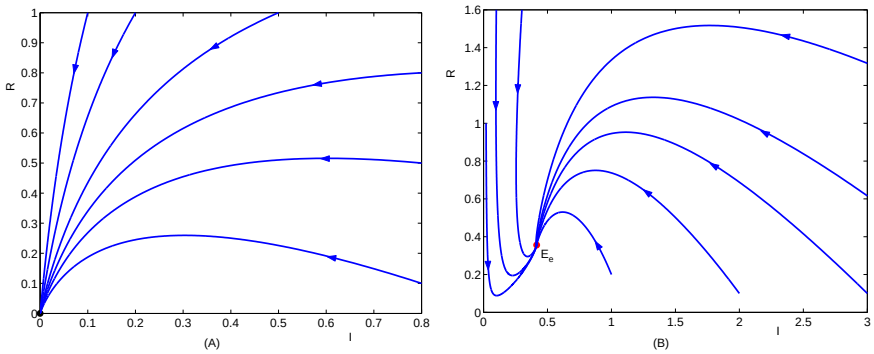

FIGURE 3. The phase diagram of system (4). Left (A): $A = 1$; Right (B): $A = 3$.

Example 5.3. (i) For system (4), we choose the parameters as follows: $p = 2, q = 3, \beta = 0.34, \alpha = 0.4, \mu = 0.6, v = 0.1$ and $d = 0.6$. If taking $A = 5$, then $\sigma^* = 1.79277$ and $\sigma = 2.36111 > \sigma^*$, two endemic equilibria are $E_1(0.50065, 0.429129)$ and $E_2(1.5040, 1.2892)$, respectively. The phase diagram of system (4) is illustrated in Fig. 4, which shows that the equilibria O and E_2 are asymptotically stable in D, and the stable and unstable manifolds of saddle E_1 divide the invariant set D into three regions D_1, D_2 and D_3, the trajectories in D_1 and D_2 tend to the disease-free equilibrium O, and the trajectories in D_3 tend to the endemic equilibrium E_2 as $t \to \infty$.

(ii) For system (4), the parameters are sellected as follows: $p = 3, q = 2, \beta = 0.1, \alpha = 0.02, \mu = 0.36, v = 0.09$, $d = 0.01$ and $A = 0.09$. Then $\sigma^* = 1.82268$ and $\sigma = 2.43243 > \sigma^*$, two endemic equilibria are $E_1(0.8647129, 3.11297)$ and $E_2(1.642164, 5.91179)$, respectively. The phase diagram of system (4) is illustrated in Fig. 5, which shows that the disease-free equilibrium O is asymptotically stable in D, the endemic equilibrium E_2 is a unstable focus, and all trajectories in D except E_1, E_2 and the stable manifolds of saddle E_1, tend to the disease-free equilibrium O as $t \to \infty$.

(iii) For system (4), the parameters are chosen as follows: $p = 2, q = 2, \beta = 0.1, \alpha = 0.02, \mu = 0.36, v = 0.09$ and $d = 0.01$.

If we choose $A = 0.09698$, then $\sigma^* = 1.93857$, $\sigma = 2.62108 > \sigma^*$, $I_2(A^c) = 1.57099$, $T(I_2(A^c), A^c) \approx 7.98482 \times 10^{-7} \approx 0$, $\left.\dfrac{\mathrm{d}T(I_2(A), A)}{\mathrm{d}A}\right|_{A=A^c} = -10.3176 < 0$ and $\kappa = 0.0001463 > 0$. The conditions (ii) of Theorem 4.1 are satisfied, so there exists a unstable limit cycle when $A > A^c$ and A is sufficiently near A^c.

In fact, if take $A = 0.09714$, then $\sigma^* = 1.9355$ and $\sigma = 2.6254 > \sigma^*$, there are two endemic equilibria $E_1(0.5023, 1.80828)$ and $E_2(1.576, 5.6736)$, respectively. The phase diagram of system (4) is illustrated in Fig. 6, where the dots on the curve indicate the terminal points of the trajectories. It is known from Fig. 6 that the disease-free equilibrium O and the endemic equilibrium E_2 are asymptotically stable in D, E_2 is a stable focus, system (4) has a unstable limit cycle Γ circling E_2.

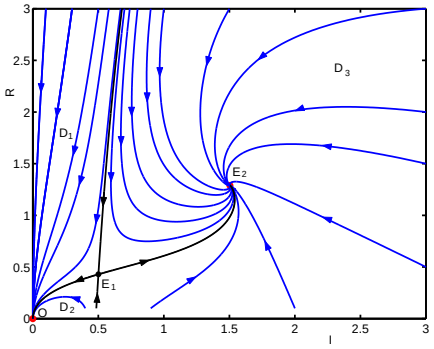

FIGURE 4. The phase diagram of system (4) with $p = 2$.

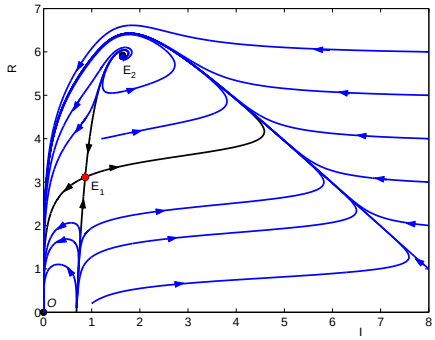

FIGURE 5. The phase diagram of system (4) with $p = 3$.

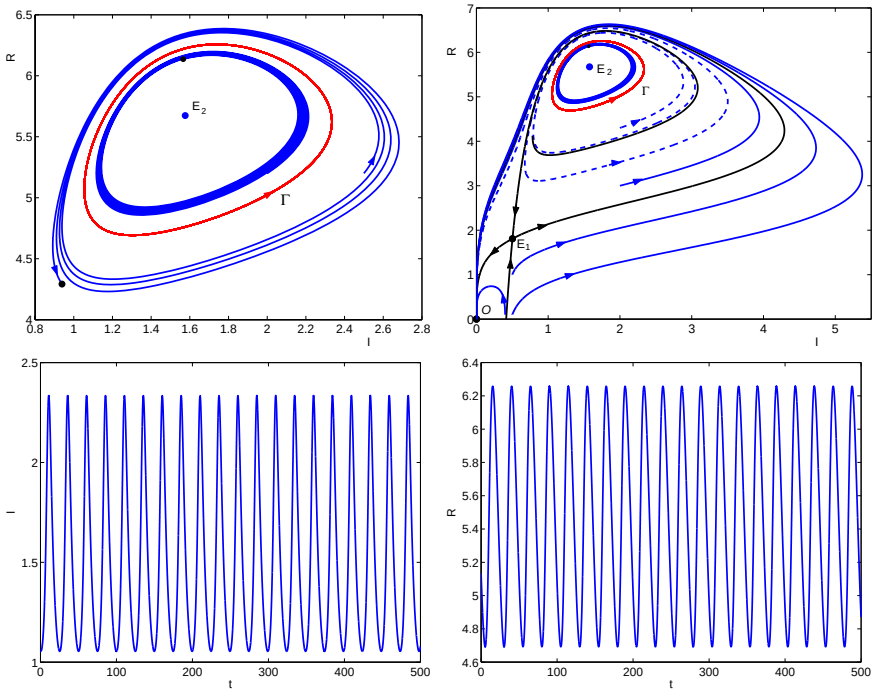

FIGURE 6. The phase diagram and periodic solution of system (4) with $p = 2$, there exists an unstable limit cycle.

Example 5.4. For system (4), we choose the parameters as follows: $p = 3, q = 2, \beta = 0.1, \alpha = 0.02, \mu = 0.36$ and $d = 0.01$. By calculation, the Bogdanov-Takens point is $(I_{BT}, R_{BT}) = (1.543, 1.56285)$, corresponding parameters $v = v_{BT} =$

0.3454271 and $A = A_{BT} = 0.0473392$. $r_{11} = 0.416453 \neq 0$ and $\Lambda_{BT} = 0.144536 \neq 0$ at the Bogdanov-Takens point.

(i) If we choose $v = v_{BT}$ and $A = A_{BT}$, then the unique endemic equilibrium $E^*(I_{BT}, R_{BT})$ is a cusp of codimension 2. The phase portrait of system (4) is illustrated in Fig. 7.

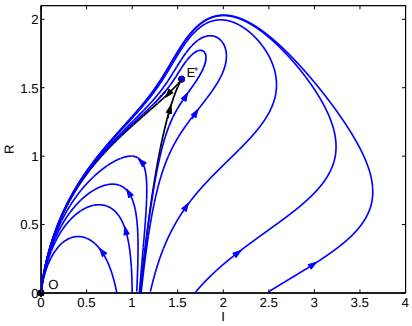

FIGURE 7. The unique endemic equilibrium is a cusp of codimension 2 when $(v, A) = (0.3454271, 0.0473392)$.

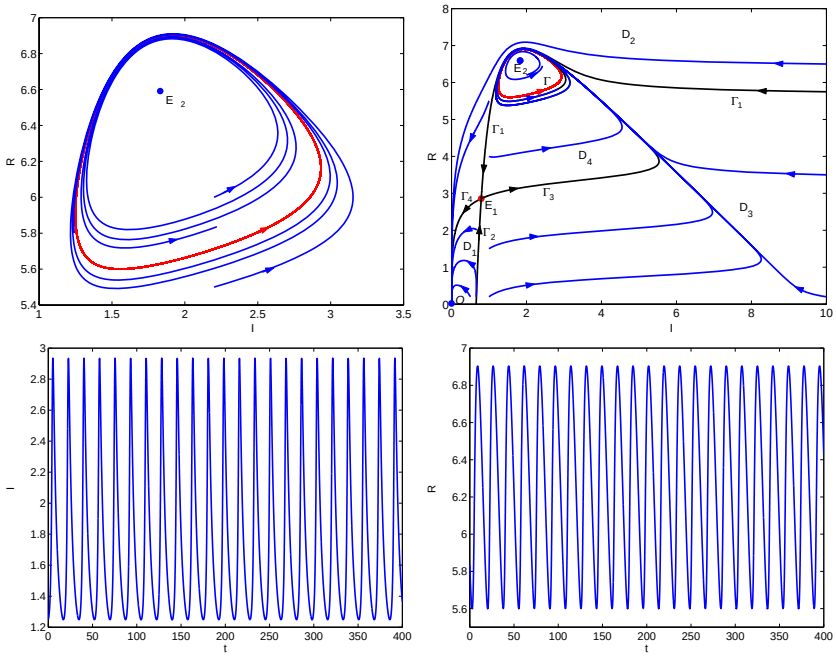

FIGURE 8. When $(v, A) = (0.09, 0.096)$, there exists a stable limit cycle.

(ii) If the bifurcation parameters $v = 0.09$ and $A^c = 0.09665$ are chosen, then $\sigma^* = 1.58828, \sigma = 2.61216 > \sigma^*$, $I_2(A^c) = 1.8499764, T(I_2(A^c), A^c) \approx 0.00001585 \approx 0$, $\mathrm{d}T(I_2(A), A)/\mathrm{d}A|_{A=A^c} = -28.21 < 0$ and $\kappa = -0.00498636 < 0$. The conditions (i) of Theorem 4.1 are satisfied, so there exists a stable limit cycle when $A < A^c$ and A is sufficiently near A^c.

In fact, we choose again $A = 0.096$, other parameters are not changed. By calculation, $\sigma^* = 1.60906$ and $\sigma = 2.59459 > \sigma^*$. The two endemic equilibria are $E_1(0.79354, 2.85676)$ and $E_2(1.83093, 6.59135)$, respectively. The phase diagram of system (4) is illustrated in Fig. 8. It is known from Fig. 8 that the disease-free

equilibrium O is asymptotically stable in D, E_2 is an unstable focus, system (4) has a stable limit cycle Γ circling E_2. The stable and unstable manifolds of the saddle E_2 divide the invariant set D into four regions D_1, D_2, D_3 and D_4. The trajectories in regions D_1 and D_2, which are on the left side of the stable manifolds Γ_2 and the upper left side of the stable manifolds Γ_1 about E_2, tend to the disease-free equilibrium O, and the trajectories in regions D_3 and D_4, which are on the right side of the stable manifolds Γ_2 and the lower right side of the stable manifolds Γ_1 about E_2, tend to the stable limit cycle Γ as $t \to \infty$.

Remark 5.5. In order to carry out numerical simulations on two limit cycles, Tang et al. [19, p. 8] chose parameters $m = 3, p = 0.1, q = 15$ and $A = 21.99$. It seems that these parameters need to be re-chosen since they are incompatible with the condition $m > q$ with m and q defined as $m = \dfrac{\delta + \gamma}{\delta + \nu}$ and $q = \dfrac{\gamma}{\delta + \nu}$, where δ, γ and ν are positive constants.

(iii) If we choose $v = 0.09$ and $A = 0.095806985$, then there is a homoclinic loop Γ. The phase diagram of system (4) is illustrated in Fig. 9, which shows that the stable and unstable manifolds of saddle E_1 and E_1 form a homoclinic loop.

(iv) If we choose $v = 0.09, A = 0.0815889$, then $\sigma^* = 1.58828$, $\sigma = \sigma^* = 2.2051$, there is a unique endemic equilibrium $E^*(1.17284, 4.22223)$. There exists a saddle-node bifurcation, the phase diagram is illustrated in Fig. 10. It is known from Fig. 10 that all trajectories of system (4) except individual trajectories tend to the disease-free equilibrium O as $t \to \infty$.

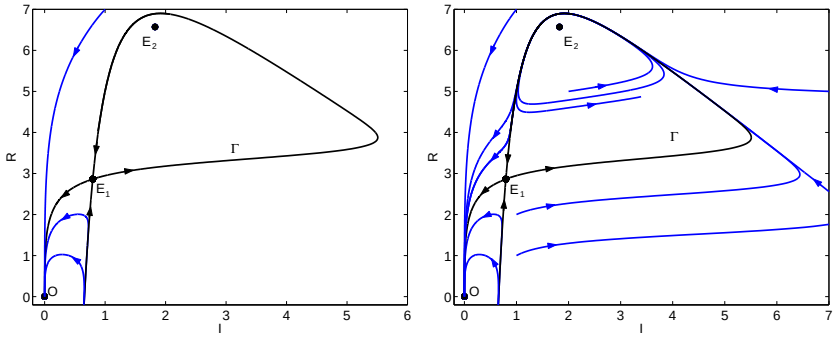

FIGURE 9. There is a homoclinic loop when $(v, A) = (0.09, 0.095806985)$.

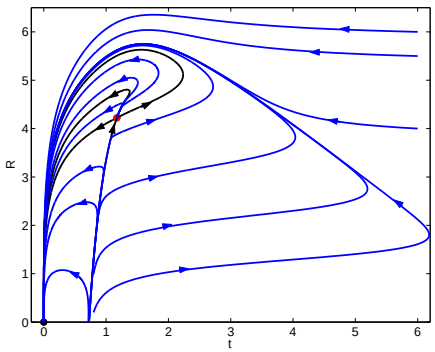

FIGURE 10. There exists a saddle-node bifurcation when $(v, A) = (0.09, 0.0815889)$.

6. **Conclusion.** In this paper, we focused on the equilibrium and bifurcation analysis of an SIRS epidemic model with generalized nonlinear incidence $\beta I^p/(1+\alpha I^q)$. The stability analysis of the model equilibria enabled us to completely analyze their local bifurcation behavior, such as Hopf, saddle-node and Bogdanov-Takens bifurcation. The first Lyapunov coefficient was computed to determine the types of Hopf bifurcations the model undergoes. The normal form of the system at the Bogdanov-Takens bifurcation was derived, from which the local representation of a homoclinic bifurcation curve was determined. Finally, we detailed and numerically illustrated our results with different p. For $p > 1$, the behavior of system (4) relies on not only the basic reproduction number σ but also other parameters in the system. Our epidemic model undergos codimension 2 bifurcations near degenerate equilibria, i.e., a Bogdanov-Takens bifurcation can occur when two major parameters v and A vary near critical values.

Our results indicate that for $0 < p < 1$, the disease cannot be eradicated. For $p = 1$, there exists a threshold, the disease cannot be eradicated when the basic reproduction number σ is more than the threshold, otherwise the disease will die out. For $p > 1$, the threshold concept becomes more complicated since the asymptotic behavior depend on both the threshold and the initial conditions. Below the threshold ($\sigma < \sigma^*$), the disease dies out. Above the threshold ($\sigma > \sigma^*$), there are two endemic equilibria, the smaller equilibrium is always a saddle, and the larger endemic equilibrium is locally attractive or repulsive. Therefore, above the threshold, the disease dies out for some initial conditions; if the larger endemic equilibrium is locally attractive so that disease levels in some nearby region approach it, and a local disease is formed; if the larger endemic equilibrium is repulsive, then the disease may die out or exhibit periodic oscillations with certain conditions.

The model we considered in this paper is an SIRS type epidemic model with a general nonlinear incidence rate which can be employed to study various infectious diseases. It would be very interesting to apply the model and the obtained results to some specific infectious diseases such as measles with reported seasonal data.

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