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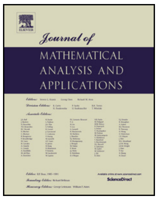

Global dynamics and complex patterns in Lotka-Volterra systems: The effects of both local and nonlocal intraspecific and interspecific competitions

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ABSTRACT

We consider a Lotka-Volterra system with both local and nonlocal intraspecific and interspecific competitions, where nonlocal competitions depend on both spatial and temporal effects in a general form. Firstly, global stability of two constant semi-trivial equilibria and global convergence of the coexistence equilibrium are derived by using the functional and energy method, which implies that strengths of nonlocal intraspecific competitions have great effects on these global dynamics but the nonlocal interspecific competitions not and extends global results of Gourley and Ruan (2003) [11]. Secondly, global attracting region of each constant semi-trivial equilibrium is limited by its environment capacity regardless of the distinction of local and nonlocal intraspecific competitions. Thirdly, in the weak competition case, the coexistence equilibrium becomes Turing unstable when the kernels are chosen as generally distributed delay functions in temporal and the nonlocal intraspecific competitions are suitably strong. Additionally, spatially homogeneous and inhomogeneous periodic solutions are found numerically.

1. Introduction

In modeling population dynamics and interactions of biological species, it is important to consider the biological processes and interactions happened in the past (time delay) and at different locations (nonlocality). In fact, biological models with temporal and spatial delays have been proposed and studied extensively in the literature, see for example [1,2,4,10–12,14–16,18,19,24,28]. Britton firstly proposed a single population model with spatio-temporal weighted kernel [2] as follows

$$\frac{\partial u}{\partial t} = \Delta u + u(1 + \alpha u - (1 + \alpha) g ** u), \tag{1.1}$$

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where $(g**u)(x,t) = \int_{-\infty}^{t}\int_{\mathbb{R}^n} g(x-y,t-s)u(y,s)\mathrm{d}y\mathrm{d}s$ with $x \in \mathbb{R}^n$ and $t > 0$. Britton showed that the nonlocal term $g**u$ could bring different and interesting dynamics by using linear stability analysis and bifurcation theory. Subsequently, for the finite domain case, Gourley and So [13] analyzed a single species model with nonlocal spatial effects induced by the time delay, which expressed another reason for incorporating nonlocality into the time delay. This model takes the following form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u\left(\frac{1-au-b\mathcal{K}*u}{1+au+bc\mathcal{K}*u}\right) \tag{1.2}$$

for $x \in [0,\pi]$ and $\mathcal{K}*u$ being defined by

$$(\mathcal{K}*u)(x,t) = \int_{-\infty}^{t}\int_{0}^{\pi} G(x,y,t-s)f(t-s)u(y,s)\mathrm{d}y\mathrm{d}s, \tag{1.3}$$

where $f(t) \geq 0$ is a delay kernel with $\int_0^\infty f(s)\mathrm{d}s = 1$ and $G(x,y,t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-dn^2t}\cos nx\cos ny$ or $G(x,y,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-dn^2t}\cos nx\cos ny$ is the solution of

$$\begin{cases} \frac{\partial G}{\partial t} = d\frac{\partial^2 G}{\partial y^2}, & y \in (0,\pi), t > 0, \\ G(x,y,0) = \delta(x-y), & y \in [0,\pi] \end{cases} \tag{1.4}$$

subject to homogeneous Neumann boundary condition

$$\frac{\partial G}{\partial y} = 0, \quad \text{at} \quad y = 0,\pi \tag{1.5}$$

or homogeneous Dirichlet condition

$$G(x,y,0) = 0, \quad \text{at} \quad y = 0,\pi, \tag{1.6}$$

where $\delta(\cdot)$ is the Dirac delta function with $\delta(0) = 1$, depending on the boundary condition of (1.2). The descriptions on the kernel G above carry over to the case of n-dimensional spatial domains. Moreover, the global convergence and bifurcations are derived for model (1.2). For more results about biological models with this kernel, we refer to [11,12,14].

Gourley and Ruan [11] incorporated the above kernel $\mathcal{K}$ for the interspecific competition in a two-species Lotka-Volterra competition system as follows

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(r_1 - a_{11}u_1 - b_{12}\mathcal{K}_{12}*u_2), & x \in \Omega, t > 0, \\ \frac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(r_2 - a_{22}u_2 - b_{21}\mathcal{K}_{21}*u_1), & x \in \Omega, t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u_1(x,t) = \phi_1(x,t) \geq (\not\equiv) 0, \quad u_2(x,t) = \phi_2(x,t) \geq (\not\equiv) 0, & (x,t) \in \Omega \times (-\infty, 0], \end{cases} \tag{1.7}$$

where $\Omega \subseteq \mathbb{R}^n (n \leq 3)$ is a bounded domain with sufficient smooth boundary and

$$(\mathcal{K}_{ij}*u)(x,t) = \int_{-\infty}^{t}\int_{\Omega} G(x,y,t-s)k_{ij}(t-s)u_j(y,s)\mathrm{d}y\mathrm{d}s \tag{1.8}$$

with $G(x,y,t)$ satisfying (1.4) and (1.5) for the n-dimensional case and nonnegative functions $k_{ij}(t)$ satisfying (2.2), $i,j = 1,2, i \neq j$. The global convergence of constant semi-trivial and coexistence equilibria was derived by employing the energy function method, which shows similar conditions for the global stability of two constant semi-trivial equilibria and the coexistence equilibrium compared with the classical Lotka-Volterra competition model although the local interspecific competitions were replaced by the nonlocal ones.

The two-species Lotka-Volterra competition systems have also been investigated by many researchers from different perspectives, including the effect of spatial heterogeneity [5,17,20], the effect of boundary conditions [21], the effect of nonlocal terms with spatially heterogeneous kernels [23], the effect of nonlocal terms with spatially heterogeneous kernels and discrete time delay under Dirichlet boundary condition [16] and so on.

In reality, although considering populations competes for resources at the same location and in the entire spatial domain is reasonable [7,8,23,26] since they are moving by diffusion, the regeneration time of resources [10] or delay-induced spatial averaging [13,14] cannot be neglected. Thus intraspecific and interspecific competitions not only simply depend on the local and nonlocal positions but also on previous times.

Shukla [25] studied global stability in a two-species Lotka-Voltera competition model with instantaneous and delayed interactions (that is, local intraspecific and interspecific competitions). Ni et al. [23] investigated global stability and pattern formation in a diffusive Lotka-Volterra model with both local and nonlocal intraspecific and interspecific competitions. Following the modeling setting in [23,25], in this paper we consider a general two-species Lotka-Volterra competition model with both local and nonlocal (double convolutions in space and time) intraspecific and interspecific competitions, which takes the following form:

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(r_1 - a_{11}u_1 - a_{12}u_2 - b_{11}\mathcal{K}_{11} * u_1 - b_{12}\mathcal{K}_{12} * u_2), & x \in \Omega, t > 0, \\ \frac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(r_2 - a_{21}u_1 - a_{22}u_2 - b_{21}\mathcal{K}_{21} * u_1 - b_{22}\mathcal{K}_{22} * u_2), & x \in \Omega, t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u_1(x,\theta) = \phi_1(x,\theta) \geq 0 (\not\equiv 0), \quad u_2(x,\theta) = \phi_2(x,\theta) \geq 0(\not\equiv 0), & (x,\theta) \in \Omega \times (-\infty, 0], \end{cases} \tag{1.9}$$

where

$$\begin{aligned} &\mathcal{K}_{ij}(x,y,t) = G_{ij}(x,y,t)k_{ij}(t), \\ &(\mathcal{K}_{ij} * u_j)(x,t) = \int_\Omega \int_{-\infty}^{t} G_{ij}(x,y,t-s)k_{ij}(t-s)u_j(y,s)\mathrm{d}s\mathrm{d}y, \ i,j = 1,2. \end{aligned} \tag{1.10}$$

Here we emphasize that $G_{ij}(x,y,t)$ need not to satisfy (1.4) and (1.5) and are given in next section. And the assumption is weaker than the one in [11] and will be presented in next section. Evidently, model (1.9) is more general since the kernel function can be chosen in a more general way and the local and nonlocal intraspecific and interspecific competitions are considered.

Under this weak assumption for the kernel $\mathcal{K}_{ij}, i,j = 1,2$, the estimates in [11, Lemma 2.1] cannot be applied, and not only more complex dynamics but also global stability or convergence of constant equilibria will occur. Therefore, we give some estimates for the term $\mathcal{K}_{ij} * u_j$ and then apply the functional and energy function method (see [11,29]) to prove the global asymptotic stability of constant semi-trivial equilibria and global convergence of the coexistence equilibrium, which shows that the property of global stability and convergence is independent of the strength of local interspecific competition and will not change if the strength of nonlocal intraspecific competition is weak compared with results in [11] and cannot be affected by the strength of nonlocal interspecific competitions compared with the result of classical two-species competition system. Moreover, the global attracting region of each semi-trivial equilibrium point limited by its environmental capacity is obtained when we do not distinguish local and nonlocal competitions.

Furthermore, many researches have introduced spatio-temporal kernels and discrete time delays into biological models [3,16]. To investigate the complex patterns induced by kernels defined in (1.10), we take a spatially uniform kernel $G_{ij}(x,y,t)=\frac{1}{\pi}$ which is simple but important in biological problems [8]. We prove that Turing instability would occur as the strength of nonlocal intraspecific competitions are suitably strong from linear stability analysis near the coexistence equilibrium under the weak competition case, which presents a new phenomenon in this model. Additionally, numerical results show that there exist more complex patterns, like the existence of spatially homogeneous and inhomogeneous periodic solutions when $\mathcal{K}_{ij}(x,y,t)=\frac{1}{\pi}\delta(t-\tau),\ i,j=1,2$.

In section 2, we will discuss the global stability of constant equilibria. In section 3, Turing instability will be studied for system (1.9) with the kernel $\mathcal{K}_{ij}(x,y,t)=\frac{1}{\pi}k_{ij}(t), i,j=1,2$, and the numerical results for specific kernel functions will be presented. Finally, conclusions and discussions will be given in section 4.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. $\|\cdot\|_2$ denotes the usual norm in the Banach space $L^2(\Omega)$, which can also be defined by the standard inner product $\langle\cdot,\cdot\rangle$. $C^\sigma((-\infty,0];C(\bar{\Omega},\mathbb{R}))$ with $\sigma\in(0,1)$ denotes the space endowed with the norm satisfying

$$\sup_{t\le 0}\|\phi(t)\|_{C(\bar{\Omega},\mathbb{R})}+\sup_{t,s\le 0,t\neq s}\frac{\|\phi(t)-\phi(s)\|_{C(\bar{\Omega},\mathbb{R})}}{|\,t-s\,|^\sigma}<+\infty.$$

Moreover, $|||\cdot|||_2$ denotes the norm in $L^2((0,T);L^2(\Omega,\mathbb{R}))$(or $L^2((0,T);W^{1,2}(\Omega,\mathbb{R}))$) with $|||u|||_2=(\int_0^T\|u(s)\|_2^2\mathrm{d}s)^{1/2}$(or $|||\,u|||_2^2=(\int_0^T\|u(s)\|^2_{W^{1,2}(\Omega,\mathbb{R})}\mathrm{d}s)^{1/2}$), where $W^{1,2}(\Omega,\mathbb{R})$ and $\|\cdot\|_{W^{1,2}(\Omega,\mathbb{R})}$ are defined in the usual sense.

2. Global stability

For model (1.9), $\phi_i\in C^\sigma((-\infty,0];C(\bar{\Omega},\mathbb{R}))$ with $\phi_i(0)\in H^2(\Omega)$ and $\frac{\partial\phi_i(0)}{\partial\nu}=0$. $\Omega\subseteq\mathbb{R}$ is a bounded domain with sufficiently smooth boundary; $\partial\nu$ denote the outward normal derivative on $\partial\Omega$; $u_i(x,t)$ is the population density of species u_i, and $d_i>0, r_i>0$ are the diffusion coefficient and the intrinsic growth rate of the species u_i respectively, $i=1,2$. $b_{ij}\ge 0$ and $a_{ij}\ge 0(a_{ij}b_{ij}\neq 0, a_{ii}\neq 0)$ represent the nonlocal and local competition strength of the species u_j to the species u_i respectively, $i,j=1,2$.

We make the assumption for the kernels $\mathcal{K}_{ij}(x,y,t)$ defined by (1.10) as follows

(H): $G_{ij}(x,y,t),k_{ij}(t),\ x,y\in\Omega,\ t\ge 0$, are nonnegative functions satisfying

$$\int_\Omega G_{ij}(x,y,t)k_{ij}(t)\mathrm{d}x=\int_\Omega G_{ij}(x,y,t)k_{ij}(t)\mathrm{d}y=k_{ij}(t), \tag{2.1}$$

and

$$\int_0^\infty k_{ij}(t)\mathrm{d}t=1,\quad \int_0^\infty tk_{ij}(t)\mathrm{d}t<\infty, \tag{2.2}$$

where $i,j=1,2$.

For convenience, denote $m_{ij}=a_{ij}+b_{ij}$, which describes the combining strength of local and nonlocal competition of the species u_j to the species u_i and is positive since $a_{ij}b_{ij}\neq 0$. Also, we define $\frac{r_i}{m_{ii}}$ the value of environment capacity for species u_i. Following to the definition in [23], $m_{11}m_{22}>m_{12}m_{21}(m_{11}m_{22}<m_{12}m_{21})$ is the weak competition case (strong competition case). Here, $i,j=1,2$.

The local existence of solution $(u_1(x,t),u_2(x,t))$ to system (1.9) follows from [29]. And from the comparison theorem for parabolic equations, $(u_1(x,t),u_2(x,t))$ exists globally and satisfies

$$0\le u_i(x,t)\le max\{\frac{r_i}{a_{ii}},\sup_{\theta\le 0}\|\phi_i(\cdot,\theta)\|_{C(\bar{\Omega};\mathbb{R})}\}. \tag{2.3}$$

Moreover, $u_i(x,t) > 0$ for $x \in \bar{\Omega}, t > 0$ if $\phi_i(x,\theta) \not\equiv 0$ by the strong maximum principle, $i = 1,2$.

Clearly, system (1.9) has a trivial equilibrium $E_0 = (0,0)$, two semitrivial constant equilibria $E_1 = (\frac{r_1}{m_{11}}, 0)$, $E_2 = (0, \frac{r_2}{m_{22}})$, and a coexistence constant equilibrium $E_3 = (\frac{r_1 m_{22} - r_2 m_{12}}{m_{11}m_{22} - m_{12}m_{21}}, \frac{r_2 m_{11} - r_1 m_{21}}{m_{11}m_{22} - m_{12}m_{21}})$ if $\frac{m_{12}}{m_{22}} < \frac{r_1}{r_2} < \frac{m_{11}}{m_{21}}$ or $\frac{m_{11}}{m_{21}} < \frac{r_1}{r_2} < \frac{m_{12}}{m_{22}}$. Moreover, by the method of linearization, we can derive the following local results for two semi-trivial equilibria E_1 and E_2.

Theorem 2.1. *Assume that assumption **(H)** holds. Then*

(1) E_1 is locally asymptotically stable if $\frac{r_1}{r_2} > \frac{m_{11}}{m_{21}}$.

(2) E_2 is locally asymptotically stable if $\frac{r_1}{r_2} < \frac{m_{12}}{m_{22}}$.

Proof. We only perform the proof of (1) since (2) can be derived similarly. By linearizing system (1.9) at E_1 and assumption **(H)**, we have

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + \frac{r_1}{m_{11}}(-a_{11}u_1 - a_{12}u_2 - b_{11}\mathcal{K}_{11} * u_1 - b_{12}\mathcal{K}_{12} * u_2), \\ \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + (r_2 - \frac{m_{21}}{m_{11}} r_1) u_2. \end{cases} \tag{2.4}$$

Clearly, the second equation in (2.4) is uncoupled with the first one. Thus we obtain the following characteristic equations of the second equation in (2.4)

$$\lambda = -d_2 c_i + r_2 - \frac{m_{21}}{m_{11}} r_1, \quad i \in \mathbb{N}_0, \tag{2.5}$$

where $0 = c_0 < c_1 \leq c_2 \leq c_3 \leq \cdots$ are sequence of eigenvalues for elliptic operator $-\Delta$ subject to the Neumann boundary condition on Ω. All λ satisfied (2.5) must be less than zero if $\frac{r_1}{r_2} > \frac{m_{11}}{m_{21}}$. And the set of eigenvalues of the characteristic equations for system (2.4) must be a subset of the set of λ which is satisfied (2.5). This completes the proof. □

For the global stability analysis, we firstly give some useful estimates in the following lemma.

Lemma 2.2. *Assume that assumption **(H)** holds. Then*

$$\|(\mathcal{K}_{ij} * v)(x,t)\|_2 \leq \int_{-\infty}^{t} k_{ij}(t-s)\|v(s)\|_2 \mathrm{d}s, t > 0 \tag{2.6}$$

and for any $T > 0$,

$$|\int_0^T \langle (\mathcal{K}_{ij} * v)(x,t), w(t) \rangle \mathrm{d}t \,| \leq C \sup_{s \leq 0} \|v(s)\|_2 \sup_{s \in [0,T]} \|w(s)\|_2 + |||v\,|||_2\, |||w|||_2, \quad i,j = 1,2, \tag{2.7}$$

where $v, w \in C((-\infty, T]; L^2(\Omega))$ and C is a constant independent of T.

Proof. By (1.4) of **(H)**, we obtain that

$$\begin{aligned} \|(\mathcal{K}_{ij} * v)(x,t)\|_2 &= \| \int_\Omega \int_{-\infty}^{t} \mathcal{K}_{ij}(x,y,t-s) v(y,s) \mathrm{d}s \mathrm{d}y \|_2 \\ &= \| \int_{-\infty}^{t} \int_\Omega \mathcal{K}_{ij}(x,y,t-s) v(y,s) \mathrm{d}y \mathrm{d}s \|_2 \end{aligned}$$

$$
\begin{aligned}
&\le \int_{-\infty}^{t} \|\int_{\Omega} \mathcal{K}_{ij}(x,y,t-s)v(y,s)\mathrm{d}y\|_2 \mathrm{d}s \\
&\le \int_{-\infty}^{t} \|\int_{\Omega} \sqrt{\mathcal{K}_{ij}(x,y,t-s)}\sqrt{\mathcal{K}_{ij}(x,y,t-s)} \mid v(y,s) \mid \mathrm{d}y\|_2 \mathrm{d}s \\
&\le \int_{-\infty}^{t} \|(\int_{\Omega} \mathcal{K}_{ij}(x,y,t-s)\mathrm{d}y)^{1/2}(\int_{\Omega} \mathcal{K}_{ij}(x,y,t-s) \mid v(y,s) \mid^2 \mathrm{d}y)^{1/2}\|_2 \mathrm{d}s \\
&\le \int_{-\infty}^{t} k_{ij}(t-s)(\int_{\Omega} \mid v(y,s) \mid^2 \mathrm{d}y)^{1/2}\mathrm{d}s.
\end{aligned}
$$

Then inequality (2.6) holds. Moreover, we have

$$
\int_{-\infty}^{t} k_{ij}(t-s)(\int_{\Omega} \mid v(y,s) \mid^2 \mathrm{d}y)^{1/2}\mathrm{d}s \le \sup_{s\le 0}\|v(s)\|_2 \int_{t}^{\infty} k_{ij}(s)\mathrm{d}s + \int_{0}^{t} k_{ij}(t-s)\|v(s)\|_2 \mathrm{d}s. \tag{2.8}
$$

Next, we prove inequality (2.7). For any $T > 0$, we know that $v, w \in L^2((0,T); L^2(\Omega))$, and

$$
\begin{aligned}
\mid \int_0^T \langle(\mathcal{K}_{ij} * v)(x,t), w(t)\rangle \mathrm{d}t \mid &\le \int_0^T \|(\mathcal{K}_{ij} * v)(x,t)\|_2 \|w(t)\|_2 \mathrm{d}t \\
&\le \sup_{s\le 0}\|v(s)\|_2 \sup_{t\in[0,T]}\|w(t)\|_2 \int_0^T\int_t^\infty k_{ij}(s)\mathrm{d}s\mathrm{d}t \\
&\quad + \int_0^T \|w(t)\|_2 \int_0^t k_{ij}(t-s)\|v(s)\|_2 \mathrm{d}s\mathrm{d}t \\
&\le \sup_{s\le 0}\|v(s)\|_2 \sup_{t\in[0,T]}\|w(t)\|_2 \int_0^\infty\int_t^\infty k_{ij}(s)\mathrm{d}s\mathrm{d}t \\
&\quad + \int_0^T \|w(t)\|_2 \int_0^t k_{ij}(t-s)\|v(s)\|_2 \mathrm{d}s\mathrm{d}t \\
&\le \sup_{s\le 0}\|v(s)\|_2 \sup_{t\in[0,T]}\|w(t)\|_2 \int_0^\infty s k_{ij}(s)\mathrm{d}s \\
&\quad + \int_0^T \|w(t)\|_2 \int_0^t k_{ij}(t-s)\|v(s)\|_2 \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

For the second term in the above inequality we have

$$
\int_0^T \|w(t)\|_2 \int_0^t k_{ij}(t-s)\|v(s)\|_2 \mathrm{d}s\mathrm{d}t \le \||w\||_2 \,(\int_0^T(\int_0^t k_{ij}(t-s)\|v(s)\|_2\mathrm{d}s)^2\mathrm{d}t)^{1/2}
$$

$$
\begin{aligned}
&= |||w|||_2 \Big(\int_0^T \Big(\int_0^t \sqrt{k_{ij}(t-s)}\sqrt{k_{ij}(t-s)}\|v(s)\|_2 \mathrm{d}s\Big)^2 \mathrm{d}t\Big)^{1/2} \\
&\le |||w|||_2 \Big(\int_0^T \Big(\int_0^t k_{ij}(t-s)\mathrm{d}s\Big)\Big(\int_0^t k_{ij}(t-s)\|v(s)\|_2^2 \mathrm{d}s\Big)\mathrm{d}t\Big)^{1/2} \\
&\le |||w|||_2 \Big(\int_0^T \int_0^t k_{ij}(t-s)\|v(s)\|_2^2 \mathrm{d}s\mathrm{d}t\Big)^{1/2} \\
&= |||w|||_2 \Big(\int_0^T \int_s^T k_{ij}(t-s)\|v(s)\|_2^2 \mathrm{d}t\mathrm{d}s\Big)^{1/2} \\
&\le |||w|||_2 \Big(\int_0^T \|v(s)\|_2^2 \mathrm{d}s\Big)^{1/2}.
\end{aligned}
$$

Clearly, inequality (2.7) holds if we combine the above two estimates. This completes the proof. □

Remark 2.3. If $\|v(t)\|_2, \|w(t)\|_2$ $(t \le T)$ are bounded independent of t and T, then it is clear that the first term on the right side of inequality (2.7) is bounded by a positive constant, which will be useful to prove the following global results.

Then we give the following theorem on the global stability of E_1, E_2 and E_3.

Theorem 2.4. *Assume that assumption* ***(H)*** *holds. Let* $(u_1(x,t), u_2(x,t))$ *satisfy* (1.9) *with* $\phi_i(x,\theta) \not\equiv 0, i = 1,2$.

(1) If $\frac{a_{11}-b_{11}}{m_{21}} > \frac{m_{12}}{a_{22}}$ *and* $\frac{r_1}{r_2} > \frac{m_{11}}{m_{21}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t), u_2(x,t)) = (\frac{r_1}{m_{11}}, 0)$$

uniformly for $x \in \bar{\Omega}$.

(2) If $0 < \frac{m_{12}}{a_{22}-b_{22}} < \frac{a_{11}}{m_{21}}$ *and* $\frac{r_1}{r_2} < \frac{m_{12}}{m_{22}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t), u_2(x,t)) = (0, \frac{r_2}{m_{22}})$$

uniformly for $x \in \bar{\Omega}$.

(3) If $\frac{a_{11}-b_{11}}{m_{21}} > \frac{m_{12}}{a_{22}-b_{22}} > 0$ *and* $\frac{m_{12}}{m_{22}} < \frac{r_1}{r_2} < \frac{m_{11}}{m_{21}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t), u_2(x,t)) = (\frac{r_1 m_{22} - r_2 m_{12}}{m_{11}m_{22} - m_{12}m_{21}}, \frac{r_2 m_{11} - r_1 m_{21}}{m_{11}m_{22} - m_{12}m_{21}})$$

uniformly for $x \in \bar{\Omega}$.

Proof. We only prove (1) since the proofs of (2) and (3) are similar. Define the nonnegative functionals as follows

$$V(u_i) = \int_\Omega (u_i - \frac{r_i}{m_{ii}} - \frac{r_i}{m_{ii}} \log \frac{u_i}{r_i/m_{ii}})\mathrm{d}x, \quad W(u_i) = \int_\Omega u_i \mathrm{d}x, i = 1,2. \tag{2.9}$$

To investigate the stability of equilibrium E_1, let $\alpha > 0$ be determined later. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\alpha V(u_1) + W(u_2)]\\
=&\alpha \int\limits_{\Omega} \frac{\partial u_1}{\partial t}(1 - \frac{r_1/m_{11}}{u_1})\mathrm{d}x + \int\limits_{\Omega} \frac{\partial u_2}{\partial t}\mathrm{d}x\\
=&-\alpha d_1 \frac{r_1}{m_{11}} \int\limits_{\Omega} \frac{\mid \nabla u_1 \mid^2}{u_1^2}\mathrm{d}x - \alpha a_{11} \int\limits_{\Omega} (u_1 - \frac{r_1}{m_{11}})^2 \mathrm{d}x - \alpha a_{12} \int\limits_{\Omega} (u_1 - \frac{r_1}{m_{11}}) u_2 \mathrm{d}x\\
&- \alpha b_{11} \int\limits_{\Omega} (\mathcal{K}_{11} * (u_1 - \frac{r_1}{m_{11}}))(x,t)(u_1 - \frac{r_1}{m_{11}})\mathrm{d}x - \alpha b_{12} \int\limits_{\Omega} (\mathcal{K}_{12} * u_2)(x,t)(u_1 - \frac{r_1}{m_{11}})\mathrm{d}x - a_{22} \int\limits_{\Omega} u_2^2 \mathrm{d}x\\
&- a_{21} \int\limits_{\Omega} (u_1 - \frac{r_2}{m_{21}}) u_2 \mathrm{d}x - b_{21} \int\limits_{\Omega} (\mathcal{K}_{21} * (u_1 - \frac{r_1}{m_{11}}))(x,t) u_2 \mathrm{d}x - b_{22} \int\limits_{\Omega} (\mathcal{K}_{22} * u_2)(x,t) u_2 \mathrm{d}x.
\end{aligned}$$

Since $\frac{r_1}{m_{11}} > \frac{r_2}{m_{21}}$, we derive that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\alpha V(u_1) + W(u_2)]\\
=&\alpha \int\limits_{\Omega} \frac{\partial u_1}{\partial t}(1 - \frac{r_1/m_{11}}{u_1})\mathrm{d}x + \int\limits_{\Omega} \frac{\partial u_2}{\partial t}\mathrm{d}x\\
\leq&-\alpha d_1 \frac{r_1}{m_{11}} \int\limits_{\Omega} \frac{\mid \nabla u_1 \mid^2}{u_1^2}\mathrm{d}x - \alpha a_{11} \int\limits_{\Omega} (u_1 - \frac{r_1}{m_{11}})^2 \mathrm{d}x - \alpha a_{12} \int\limits_{\Omega} (u_1 - \frac{r_1}{m_{11}}) u_2 \mathrm{d}x\\
&- \alpha b_{11} \int\limits_{\Omega} (\mathcal{K}_{11} * (u_1 - \frac{r_1}{m_{11}}))(x,t)(u_1 - \frac{r_1}{m_{11}})\mathrm{d}x - \alpha b_{12} \int\limits_{\Omega} (\mathcal{K}_{12} * u_2)(x,t)(u_1 - \frac{r_1}{m_{11}})\mathrm{d}x - a_{22} \int\limits_{\Omega} u_2^2 \mathrm{d}x\\
&- a_{21} \int\limits_{\Omega} (u_1 - \frac{r_1}{m_{11}}) u_2 \mathrm{d}x - b_{21} \int\limits_{\Omega} (\mathcal{K}_{21} * (u_1 - \frac{r_1}{m_{11}}))(x,t) u_2 \mathrm{d}x - b_{22} \int\limits_{\Omega} (\mathcal{K}_{22} * u_2)(x,t) u_2 \mathrm{d}x.
\end{aligned} \tag{2.10}$$

For any $T > 0$, by using Hölder inequality twice, we have

$$\mid \int\limits_0^T \int\limits_{\Omega} (u_1 - \frac{r_1}{m_{11}}) u_2 \mathrm{d}x\mathrm{d}t \mid \leq |||u_1 - \frac{r_1}{m_{11}} |||_2 \, |||u_2 |||_2 \,. \tag{2.11}$$

Then by (2.11) and (2.3), and integrating (2.10) over $[0, T]$, we obtain that

$$\begin{aligned}
&\alpha d_1 \frac{r_1}{m_{11}} ||| \frac{\nabla u_1}{u_1} |||_2^2 + \alpha a_{11} ||| u_1 - \frac{r_1}{m_{11}} |||_2^2 + a_{22} ||| u_2 |||_2^2\\
\leq& (\alpha a_{12} + a_{21}) ||| u_1 - \frac{r_1}{m_{11}} |||_2 \, |||u_2 |||_2 + \alpha b_{11} \mid \int\limits_0^T \langle \mathcal{K}_{11} * (u_1 - \frac{r_1}{m_{11}}))(x,t), (u_1 - \frac{r_1}{m_{11}})\rangle \mathrm{d}t \mid\\
&+ \alpha b_{12} \mid \int\limits_0^T \langle (\mathcal{K}_{12} * u_2)(x,t), u_1 - \frac{r_1}{m_{11}} \rangle \mathrm{d}t \mid + b_{21} \mid \int\limits_0^T \langle (\mathcal{K}_{21} * (u_1 - \frac{r_1}{m_{11}}))(x,t), u_2 \rangle \mathrm{d}t \mid .
\end{aligned}$$

Thus, by Lemma 2.2 and (2.3), we derive that

$$\begin{aligned}&\alpha d_1\frac{r_1}{m_{11}}\,|||\,\frac{\nabla u_1}{u_1}\,|||_2^2+\alpha(a_{11}-b_{11})\,|||\,u_1-\frac{r_1}{m_{11}}\,|||_2^2+a_{22}\,|||\,u_2|||_2^2\\ &\leq C+(\alpha a_{12}+a_{21}+\alpha b_{12}+b_{21})\,|||\,u_1-\frac{r_1}{m_{11}}\,|||_2\,|||u_2|||_2\\ &=C+(\alpha m_{12}+m_{21})\,|||\,u_1-\frac{r_1}{m_{11}}\,|||_2\,|||u_2|||_2,\end{aligned}$$

where C is a positive constant independent of T.

From Young's inequality, it follows that

$$\alpha d_1\frac{r_1}{m_{11}}\,|||\,\frac{\nabla u_1}{u_1}\,|||_2^2+\alpha(a_{11}-b_{11})\,|||\,u_1-\frac{r_1}{m_{11}}\,|||_2^2+a_{22}\,|||\,u_2\,|||_2^2\leq C+(\alpha m_{12}+m_{21})(\frac{\epsilon}{2}\,|||\,u_1-\frac{r_1}{m_{11}}\,|||_2^2+\frac{1}{2\epsilon}\,|||\,u_2\,|||_2^2),\tag{2.12}$$

where $\epsilon>0$. Without loss of generality, we choose $\epsilon=\frac{\alpha m_{12}+m_{21}}{2a_{22}}$. Therefore, (2.12) becomes

$$\alpha d_1\frac{r_1}{m_{11}}\,|||\,\frac{\nabla u_1}{u_1}\,|||_2^2+\alpha(a_{11}-b_{11})\,|||\,u_1-\frac{r_1}{m_{11}}|||_2^2\leq C+\frac{(\alpha m_{12}+m_{21})^2}{4a_{22}}\,|||\,u_1-\frac{r_1}{m_{11}}\,|||_2^2\,.\tag{2.13}$$

It can be easily shown that there must exist $\alpha>0$ such that $\frac{(\alpha m_{12}+m_{21})^2}{4a_{22}}<\alpha(a_{11}-b_{11})$ if $\frac{a_{11}-b_{11}}{m_{21}}>\frac{m_{12}}{a_{22}}$. Then we can conclude that

$$|||\nabla u_1|||_2\leq C,\quad |||u_1-\frac{r_1}{m_{11}}|||_2\leq C\tag{2.14}$$

for some constant C independent of T. This indicates that $u_1-\frac{r_1}{m_{11}}\in L^2((0,\infty);W^{1,2}(\Omega;\mathbb{R}))$ and

$$\lim_{t\to\infty}\|u_1(t)-\frac{r_1}{m_{11}}\|_{W^{1,2}(\Omega;\mathbb{R})}=0.\tag{2.15}$$

Thus,

$$\lim_{t\to\infty}\|u_1(t)-\frac{r_1}{m_{11}}\|_{C(\bar{\Omega};\mathbb{R})}=0,\tag{2.16}$$

since $W^{1,2}(\Omega;\mathbb{R})$ can be continuously embedded into $C(\bar{\Omega},\mathbb{R})$ from [6]. Similarly, we can derive

$$|||u_2|||_2\leq C.\tag{2.17}$$

Then it suffices to show $|||\nabla u_2|||_2\leq C$ by the energy function method, where C is a constant independent of T. Multiplying the second equation of (1.9) by u_2 and integrating over Ω we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_2^2+d_2\|\nabla u_2(t)\|_2^2\\ &=\langle u_2(t)(r_2-a_{21}u_1(t)-a_{22}u_2(t)-b_{21}(\mathcal{K}_{21}*u_1)(x,t)-b_{22}(\mathcal{K}_{22}*u_2)(x,t)),u_2(t)\rangle\\ &\leq r_2\|u_2\|_2^2+\sup_{0\leq t\leq T}u_2(t)\langle a_{21}u_1(t)+a_{22}u_2(t)+b_{21}(\mathcal{K}_{21}*u_1)(x,t)+b_{22}(\mathcal{K}_{22}*u_2)(x,t),u_2(t)\rangle.\end{aligned}\tag{2.18}$$

Therefore, integrating (2.18) over $[0,T]$ yields that

$$|||\nabla u_2|||_2\leq C\tag{2.19}$$

by Lemma 2.2, where C is a constant independent of T. And we can get

$$\lim_{t\to\infty}\|u_2(t)\|_{C(\bar{\Omega};\mathbb{R})}=0 \tag{2.20}$$

in a similar way. This completes the proof. □

Remark 2.5. If $a_{12}=a_{21}=b_{11}=b_{22}=0$, then Theorem 2.4 reduces to the global results for system ((1.7) in [11]). Furthermore, when we only take $b_{11}=b_{22}=0$, in the weak competition case, Theorem 2.4 indicates that nonlocal interspecific competitions have no effect on the global convergence of E_1, E_2 and E_3 compared with the classical two-species competition Lotka-Volterra system.

Remark 2.6. From assumption $(\boldsymbol{H})$, one can see that the kernel $G_{ij}(x,y,t)$ can be chosen in various forms including the spatially average kernel $\frac{1}{|\Omega|}$ or $\delta(x-y)$ independent of time t, the spatio-temporal kernel satisfying (1.4) and (1.5) and so on.

From the proof of Theorem 2.4, we find that the convergence of the equilibrium E_1 (or E_2) still holds if the initial data of species u_1 is smaller than $\frac{r_1}{m_{11}}$ (or $u_2\le\frac{r_2}{m_{22}}$) even though the strength of nonlocal intraspecific competitions may be strong.

Corollary 2.7. *Assume that assumption* $(\boldsymbol{H})$ *holds. Let* $(u_1(x,t),u_2(x,t))$ *satisfy* (1.9) *with* $\phi_i(x,\theta)\not\equiv 0, i=1,2$.

(1) If $\frac{r_1}{r_2}>\frac{m_{11}}{m_{21}}>\frac{m_{12}}{m_{22}}$ *and* $\sup\limits_{\theta\le 0}\|\phi_1(\cdot,\theta)\|_{C(\bar{\Omega};\mathbb{R})}\le\frac{r_1}{m_{11}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t),u_2(x,t))=(\frac{r_1}{m_{11}},0)$$

uniformly for $x\in\bar{\Omega}$.

(2) If $\frac{r_1}{r_2}<\frac{m_{12}}{m_{22}}<\frac{m_{11}}{m_{21}}$ *and* $\sup\limits_{\theta\le 0}\|\phi_2(\cdot,\theta)\|_{C(\bar{\Omega};\mathbb{R})}\le\frac{r_2}{m_{22}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t),u_2(x,t))=(0,\frac{r_2}{m_{22}})$$

uniformly for $x\in\bar{\Omega}$.

Remark 2.8. Corollary 2.7 implies that the convergence of $E_1(E_2)$ may be related to the initial data of species $u_1(u_2)$, but we cannot remove the restrictions on the initial data by the above analysis.

Corollary 2.9. *Assume that assumption* $(\boldsymbol{H})$ *holds and* $b_{11}=b_{22}=0$. *Let* $(u_1(x,t),u_2(x,t))$ *satisfy* (1.9) *with* $\phi_i(x,\theta)\not\equiv 0, i=1,2$.

(1) If $\frac{r_1}{r_2}>\frac{m_{11}}{m_{21}}>\frac{m_{12}}{m_{22}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t),u_2(x,t))=(\frac{r_1}{m_{11}},0)$$

uniformly for $x\in\bar{\Omega}$.

(2) If $\frac{r_1}{r_2}<\frac{m_{12}}{m_{22}}<\frac{m_{11}}{m_{21}}$, *then*

$$\lim_{t\to\infty}(u_1(x,t),u_2(x,t))=(0,\frac{r_2}{m_{22}})$$

uniformly for $x\in\bar{\Omega}$.

(3) If $\frac{m_{12}}{m_{22}} < \frac{r_1}{r_2} < \frac{m_{11}}{m_{21}}$*, then*

$$\lim_{t\to\infty}(u_1(x,t),u_2(x,t)) = (\frac{r_1m_{22}-r_2m_{12}}{m_{11}m_{22}-m_{12}m_{21}}, \frac{r_2m_{11}-r_1m_{21}}{m_{11}m_{22}-m_{12}m_{21}})$$

uniformly for $x \in \bar{\Omega}$*.*

Remark 2.10. From Corollary 2.9, it can be easily seen that the strength of local interspecific competitions does not have any effect on the global stability of E_1, E_2 and E_3 compared with the results in [11] or the classical results of two-species Lotka-Volterra competition systems if the strength of nonlocal intraspecific competition equals to zero. Moreover, from Theorem 2.4 and Corollary 2.9, the global stability of E_1, E_2 and E_3 does not change if the strength of nonlocal intraspecific competitions is weak enough.

Remark 2.11. By the local results in Theorem 2.1, the global convergence of E_1 and E_2 in Theorem 2.4 and Corollary 2.9 also implies that they are globally asymptotically stable.

Based on the above results, we think that the global convergence of the coexistence equilibrium E_3 will change if the strength of nonlocal intraspecific competitions is suitably strong. Thus, we will show that the strength of nonlocal intraspecific competitions is the key factor for the stability of E_3 by linear stability analysis in next section.

3. Effects of nonlocal intraspecific competitions

To investigate the effects of the strength of nonlocal intraspecific competition on the stability of E_3, we make the following assumption

($\boldsymbol{H_1}$): $\frac{m_{12}}{m_{22}} < \frac{r_1}{r_2} < \frac{m_{11}}{m_{21}}$.

Clearly, assumption ($\boldsymbol{H_1}$) is reasonable since it preserves the existence of E_3.

3.1. Local stability analysis

Let $G_{ij}(x,y,t) = \frac{1}{|\Omega|}$ be the spatially uniform average kernel, $i,j = 1,2$. Here, we have to emphasize that the following methods are applicable for other kernels $G_{ij}(x,y,t)$ if assumption ($\boldsymbol{H}$) holds. Without loss generality and for notational simplicity, let $\Omega = (0,\pi)$ and $E_3 = (u_1^*, u_2^*)$. We linearize the system at E_3 and get the following linearized system

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1^*(-a_{11}u_1 - a_{12}u_2 - b_{11}\mathcal{K}_{11} * u_1 - b_{12}\mathcal{K}_{12} * u_2), & x \in (0,\pi), t > 0, \\ \frac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2^*(-a_{21}u_1 - a_{22}u_2 - b_{21}\mathcal{K}_{21} * u_1 - b_{22}\mathcal{K}_{22} * u_2), & x \in (0,\pi), t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, & x = 0, \pi, t > 0, \\ u_1(x,\theta) = \phi_1(x,\theta) \geq 0, \quad u_2(x,\theta) = \phi_2(x,\theta) \geq 0, & (x,\theta) \in (0,\pi) \times (-\infty, 0], \end{cases} \tag{3.1}$$

where

$$(\mathcal{K}_{ij} * u_j)(x,t) = \frac{1}{\pi}\int_0^\pi \int_{-\infty}^t k_{ij}(t-s)u_j(y,s)\mathrm{d}s\mathrm{d}y, i,j = 1,2. \tag{3.2}$$

When $b_{11} = b_{22} = 0$, similarly to the classical Lotka-Volterra system, we can obtain that E_3 is locally asymptotically stable if ($\boldsymbol{H_1}$) is satisfied, and it is also globally asymptotically stable following Corollary 2.9.

Choosing suitably trial solution as $(u_1(x,t), u_2(x,t)) = (c_1, c_2)e^{\lambda t}\cos nx (n \in \mathbb{N}_0)$, and then the nonlocal terms become

$$(\mathcal{K}_{ij} * u_j)(x,t) = \begin{cases} \hat{k}_{ij}(\lambda), & n = 0, \\ 0, & n \in \mathbb{N}, \end{cases} \tag{3.3}$$

where $\hat{k}_{ij}(\lambda)$ is the Laplace transform of $k_{ij}(t)$ defined by $\hat{k}(\lambda) = \int_0^\infty k_{ij}(s)e^{-\lambda s}\mathrm{d}s, i, j = 1, 2$.

Therefore, by substituting the trial solution to system (3.1), we can derive the eigenvalue problem

$$P(\lambda, n^2) = 0, \qquad n \in \mathbb{N}_0, \tag{3.4}$$

in which

$$P(\lambda, n^2) = \begin{cases} \lambda^2 + [u_1^* s_{11}(\lambda) + u_2^* s_{22}(\lambda)]\lambda + u_1^* u_2^*[s_{11}(\lambda)s_{22}(\lambda) - s_{12}(\lambda)s_{21}(\lambda)], & n = 0, \\ \lambda^2 + [(d_1 + d_2)n^2 + a_{11}u_1^* + a_{22}u_2^*]\lambda + f(n^2), & n \in \mathbb{N}, \end{cases} \tag{3.5}$$

where $s_{ij}(\lambda) = a_{ij} + b_{ij}\hat{k}_{ij}(\lambda)$ and $f(n^2) = d_1 d_2 n^4 + (d_1 a_{22} u_2^* + d_2 a_{11} u_1^*)n^2 + u_1^* u_2^*(a_{11}a_{22} - a_{12}a_{21})$.

Thus, we know that E_3 is linearly stable if all roots λ of (3.4) are strictly on the left half side of the complex plane, and linearly unstable if there exists a root λ of (3.4) on the right half side of the complex plane. Now we give the linear stability analysis of E_3.

Firstly, we investigate the distribution of roots for $P(\lambda, n^2) = 0$ when $n = 0$. For notational simplicity, denote $P(\lambda) = P(\lambda, 0)$. Since $P(\lambda) = 0$ is a transcendental equation, we apply the argument principle in complex analysis to discuss it. It can be easily seen that $P(\lambda)$ is analytical in the right half side of the complex plane since the Laplace transform of $k_{ij}(t)$ converges. By the argument principle, the number of roots of $P(\lambda) = 0$ is equal to the following integral

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{\gamma(R)} \frac{P'(\lambda)}{P(\lambda)}\mathrm{d}\lambda, \tag{3.6}$$

where $R > 0$ and $\gamma(R)$ is composed of $\gamma_1(R) := \{Re^{i\theta} : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$ and $\gamma_2(R) := \{iy : y \in [R, -R]\}$, and the prime denotes the derivation with respect to λ. From the results in [9], we know that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{\gamma(R)} \frac{P'(\lambda)}{P(\lambda)}\mathrm{d}\lambda = \lim_{R\to\infty} (1 - \frac{1}{\pi}\arg P(iR)). \tag{3.7}$$

To calculate the formula (3.7), we need to know the graph of $\Im(P(iR))$ against $\Re(P(iR))$ as R goes from zero to infinity. Note that

$$\begin{aligned} \Re(s_{ij}(iR)) &= a_{ij} + b_{ij}\int\limits_0^\infty k_{ij}(s)\cos Rs\mathrm{d}s, \\ \Im(s_{ij}(iR)) &= -b_{ij}\int\limits_0^\infty k_{ij}(s)\sin Rs\mathrm{d}s. \end{aligned} \tag{3.8}$$

Since $|\sin Rs| \leq Rs$ for $s > 0$, we have

$$\begin{aligned}&\Big|\int_0^\infty k_{ij}(s)\cos Rs\mathrm{d}s\,\Big| \le 1,\\ &\Big|\int_0^\infty k_{ij}(s)\sin Rs\mathrm{d}s\,\Big| \le RT_{ij},\end{aligned} \tag{3.9}$$

where $T_{ij} := \int_0^\infty sk_{ij}(s)\mathrm{d}s$ describes the mean delay, see details in [22], $i,j=1,2$. From the assumption ($\boldsymbol{H}$) we know T_{ij} is bounded.

By direct calculations, we can get

$$\begin{aligned}\Re(P(iR)) =& -R^2 - R[u_1^*\Im(s_{11}(iR)) + u_2^*\Im(s_{22}(iR))] + u_1^*u_2^*[\Re(s_{11}(iR))\Re(s_{22}(iR))\\ &- \Im(s_{11}(iR))\Im(s_{22}(iR)) - \Re(s_{12}(iR))\Re(s_{21}(iR)) + \Im(s_{12}(iR))\Im(s_{21}(iR))],\\ \Im(P(iR)) =& R[u_1^*\Re(s_{11}(iR)) + u_2^*\Re(s_{22}(iR))] + u_1^*u_2^*[\Re(s_{11}(iR))\Im(s_{22}(iR)) + \Im(s_{11}(iR))\Re(s_{22}(iR))\\ &- \Re(s_{12}(iR))\Im(s_{21}(iR)) - \Im(s_{12}(iR))\Re(s_{21}(iR))].\end{aligned} \tag{3.10}$$

By (3.8), (3.9) and (3.10), we derive that

$$\begin{aligned}&\Re(P(iR)) \sim -R^2 \quad (R \text{ large}),\\ &\Im(P(iR)) \sim const.R \quad (R \text{ large}),\end{aligned} \tag{3.11}$$

and

$$P(0) = u_1^*u_2^*(m_{11}m_{22} - m_{12}m_{21}) > 0 \tag{3.12}$$

if assumption ($\boldsymbol{H_1}$) is satisfied.

Therefore, from (3.11) and (3.12), $P(iR)$ starts with the positive real part at $R=0$ and ends up in the second quadrant as R tends to ∞, and $\lim\limits_{R\to\infty} \arg P(iR)$ must be the values of $\pi, -\pi, -3\pi$ and etc. More precisely, the value of $\lim\limits_{R\to\infty} \arg P(iR)$ depends on the total change of $P(iR)$. Certainly, all roots of $P(\lambda)=0$ have negative real parts only when $\lim\limits_{R\to\infty} \arg P(iR) = \pi$ by formula (3.7).

We have the following result on the distribution of roots of $P(\lambda)=0$.

Lemma 3.1. *Assume that assumptions ($\boldsymbol{H}$) and ($\boldsymbol{H_1}$) hold. Then there exist T_{ij} sufficiently small such that all roots of $P(\lambda)=0$ are strictly on the left half side of the complex plane if $a_{ii}-b_{ii}>0$ and other parameters are fixed, $i,j=1,2$.*

Proof. By formulas (3.8), (3.9) and (3.10), we can get the following inequality

$$\begin{aligned}\Im(P(iR)) \ge& [u_1^*(a_{11}-b_{11}) + u_2^*(a_{22}-b_{22})]R\\ &- u_1^*u_2^*(m_{11}b_{22}T_{22} + m_{22}b_{11}T_{11} + m_{12}b_{21}T_{21} + m_{21}b_{12}T_{12})R.\end{aligned}$$

Thus, it can be easily derived that $\Im(P(iR))>0$ for all $R>0$ if $a_{ii}-b_{ii}>0$ and T_{ij} are sufficiently small. Note that $P(0)>0$, we can derive that $P(\lambda)=0$ has no zero root on the boundary of $\gamma(R)$. By (3.7), $\lim\limits_{R\to\infty} \arg P(iR)$ must be equal to π, and the proof is completed. □

To investigate the effect of nonlocal intraspecific competitions, denote

$$S = \{n \in \mathbb{N} : n^2 < \frac{u_2^*(a_{12}a_{21} - a_{11}a_{22})}{d_2a_{11}}\}. \tag{3.13}$$

Next, we study the distribution of the roots of $P(\lambda, n^2) = 0, n \in \mathbb{N}$.

Lemma 3.2. *Assume that* $(\boldsymbol{H_1})$ *holds. Then* E_3 *is linearly unstable if*

$$a_{12}a_{21} - a_{11}a_{22} > \frac{d_1 d_2 + d_1 u_2^* a_{22} + d_2 u_1^* a_{11}}{u_1^* u_2^*}. \tag{3.14}$$

Proof. It can be easily verified that $f(1^2) < 0$ if $a_{12}a_{21} - a_{11}a_{22} > \frac{d_1 d_2 + d_1 u_2^* a_{22} + d_2 u_1^* a_{11}}{u_1^* u_2^*}$. And $P(\lambda, 1) = 0$ must have a root λ with positive real part. This completes the proof. □

Remark 3.3. The assumption $(\boldsymbol{H_1})$ implies that $(a_{11} + b_{11})(a_{22} + b_{22}) > (a_{12} + b_{12})(a_{21} + b_{21})$, and it can be easily checked that $a_{11}a_{22} > (a_{12} + b_{12})(a_{21} + b_{21}) \geq a_{12}a_{21}$ if b_{11} and b_{22} are sufficiently small, i.e. the strength of nonlocal intraspecific competition is weak, which is consistent with the global analysis in section 2 and indicates that inequality (3.14) fails. Otherwise, inequality (3.14) may hold if b_{11} and b_{22} are sufficiently large, that is, strong nonlocal intraspecific competition will result in the instability of E_3 possibly.

For the case that b_{11} and b_{22} are suitably large, the set S is probably nonempty. We give the following lemma.

Lemma 3.4. *Assume that* $(\boldsymbol{H_1})$ *holds and* S *is not empty. Then* $P(\lambda, n^2) = 0$ *has a simple zero root and a negative root if* $d_1 = d_1(n^2)$ *for each* $n \in S$. *Moreover, when other system parameters except* d_1 *are fixed, then* $d_1(i^2) \neq d_1(j^2)$ *for* $i, j \in S$ *and* $i \neq j$. *Here,*

$$d_1(n^2) = \frac{u_1^* u_2^* (a_{12}a_{21} - a_{11}a_{22}) - d_2 u_1^* a_{11} n^2}{d_2 n^4 + u_2^* a_{22} n^2}, \quad n \in S.$$

Proof. Note that $f(n^2) = 0$ if and only if $d_1 = d_1(n^2)$ by the definition of $d_1(n^2)$. The first part of the lemma can be easily derived. And the monotonicity of $f(n^2)$ with respect to n^2 and d_1 implies that the rest follows. □

Remark 3.5. Under assumptions of Lemma 3.4, there must exist an unique n^* such that $d_1(n^{*2}) = \max\limits_{j \in S}\{d_1(j^2)\}$ since S is a finite set, and it can be easily verified that all roots of $P(\lambda, n^2) = 0 (n \neq n^*)$ have negative real parts if $d_1 = d_1(n^{*2})$.

Therefore, when other system parameters except d_1 are fixed, we combine the above results into the following theorem, illustrating that Turing instability (Turing bifurcation) will occur when b_{11} and b_{22} are suitably large, in which $d_1(n^2)$ and n^* are defined in the above.

Theorem 3.6. *Assume that* $(\boldsymbol{H})$ *and* $(\boldsymbol{H_1})$ *hold. If* $a_{ii} - b_{ii} > 0$ *and* S *is not empty, then there must exist* T_{ij} *sufficiently small such that the original system* (1.9) *with the kernels defined by* (3.2) *undergoes Turing bifurcation at* $d_1 = d_1(n^2)$, $i, j = 1, 2$ *and* $n \in S$. *Moreover, if* $d_1 = d_1(n^{*2})$ *and* T_{ij} *sufficiently small, then all roots of* $P(\lambda, n^2) = 0 (n \in \mathbb{N}_0)$ *have strictly negative real parts except a simple zero root.*

Proof. The results are obvious by Lemmas 3.1, 3.4 and Remark 3.5, and we omit the proof here. □

Remark 3.7. In Theorem 3.6, the condition that the set S is nonempty is possible when b_{11} and b_{22} are suitably large. Although the original system is under the weak competition case ($m_{11}m_{22} > m_{12}m_{21}$), we find that the spatially inhomogeneous solution will bifurcate from the coexistence equilibrium E_3, which

Table 1
Values of system parameters.

Parameters	d_1	d_2	r_1	r_2	a_{11}	a_{12}	a_{21}	a_{22}	b_{11}	b_{12}	b_{21}	b_{22}	τ
Values	1.45	1	10	10	4	7	6	5	3	1	0.5	4	0.1
	1.45	1	10	10	4	7	6	5	3	1	0.5	4	0.675
	1.45	1	10	10	4	7	6	5	3	1	0.5	4	1

is a new phenomenon compared with the two species competitive Lotka-Volterra system without nonlocal intraspecific competition.

Moreover, we believe that Hopf bifurcation and Turing-Hopf bifurcation will also occur in this system since $P(\lambda) = 0$ may have purely imaginary roots if the mean delays are large enough, which will be shown in simulations for specific temporal kernel $k_{ij}(t)$ in next subsection. Because of the unspecific temporal kernel $k_{ij}(t)$, we do not know whether $P(\lambda) = 0$ will have purely imaginary roots.

3.2. Spatial, temporal and spatio-temporal patterns

For convenient to perform simulations, we choose the kernels $G_{ij}(x, y, t) = \frac{1}{\pi}$ and $k_{ij}(t) = \delta(t - \tau)$ where $\tau > 0, i, j = 1, 2$. Then system (1.9) becomes

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + u_1(r_1 - a_{11}u_1 - a_{12}u_2 - \frac{b_{11}}{\pi}\int_0^\pi u_1(y, t-\tau)\mathrm{d}y - \frac{b_{12}}{\pi}\int_0^\pi u_2(y, t-\tau)\mathrm{d}y), \\ \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + u_2(r_2 - a_{21}u_1 - a_{22}u_2 - \frac{b_{21}}{\pi}\int_0^\pi u_1(y, t-\tau)\mathrm{d}y - \frac{b_{22}}{\pi}\int_0^\pi u_2(y, t-\tau)\mathrm{d}y) \end{cases} \tag{3.15}$$

with $x \in (0, \pi),\ t > 0$.

The mean delays introduced in subsection 3.1 are all equal to the fixed time delay τ by direct calculations.

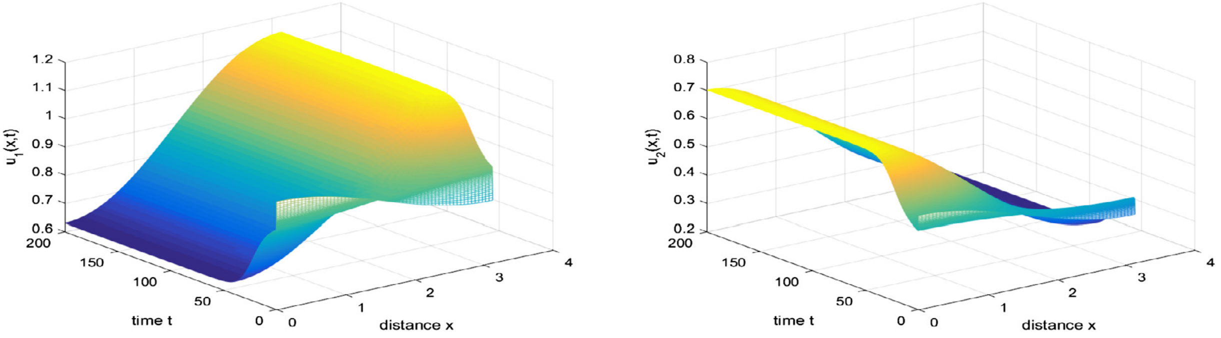
Firstly, we apply the results of Theorem 3.6 and know that the coexistence equilibrium E_3 of system (3.15) becomes Turing unstable by choosing the parameters b_{11}, b_{22} suitably large, the time delay τ small enough and by fixing other parameters properly. The set $S = \{1\}$ can be easily verified if we take the parameter values as Table 1 shows. Thus, when taking the values of parameters as the first row of Table 1, we obtain a pair of spatially inhomogeneous solutions, see Fig. 1.

It is well known that the time delay incorporated in ODE or PDE may result in the occurrence of oscillations when the delay is large enough. Inspired by that, let the value of time delay vary and other parameters fixed, which is shown as the parameter values in second and third rows of Table 1. Therefore, by further simulations, we find spatially inhomogeneous and homogeneous periodic solutions when changing values of the delay τ, see Figs. 2 and 3.

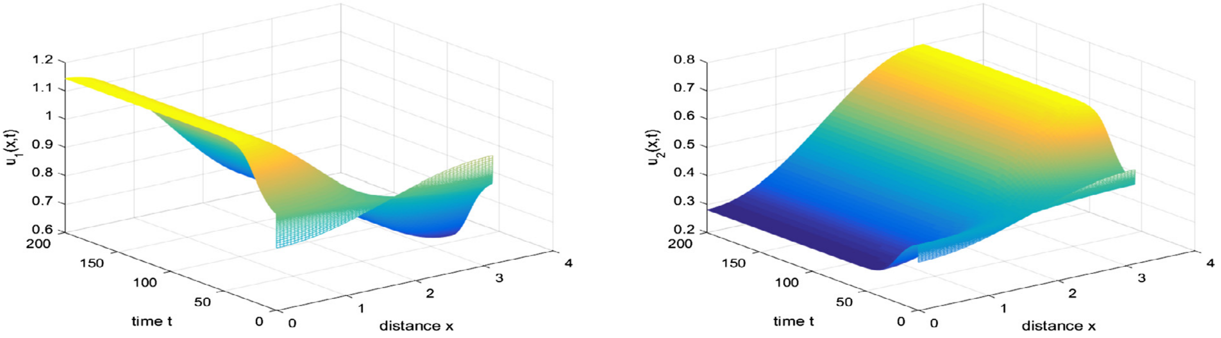
From the numerical results, we know that more complex pattern formations, like the existence of spatially inhomogeneous periodic solutions occur in this two-species Lotka-Volterra competition systems under the weak competition case compared with [23,27] in which spatial or temporal patterns can be formed. We believe that such properties could be theoretically studied by center manifold theory, bifurcation theory and normal form reduction method.

4. Conclusions and discussions

In this paper we studied the global dynamics and complex pattern formations in two-species Lotka-Volterra systems with local as well as nonlocal intraspecific and interspecific competitions by using both mathematical and numerical methods. Our results imply that the strength of nonlocal intraspecific competitions induces complex dynamics of the competitive system as it becomes suitably strong and that the strength of nonlocal interspecific ones has no effect on the global dynamics.

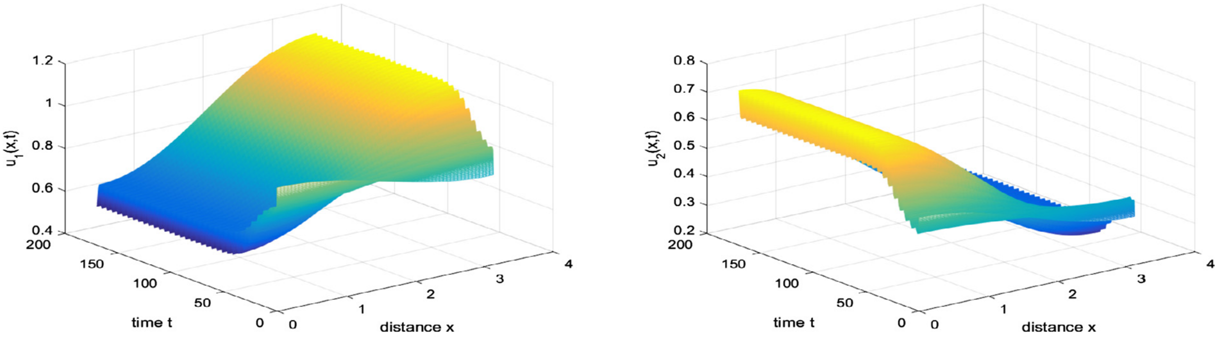


(a) $u_1(x,t)$ (b) $u_2(x,t)$

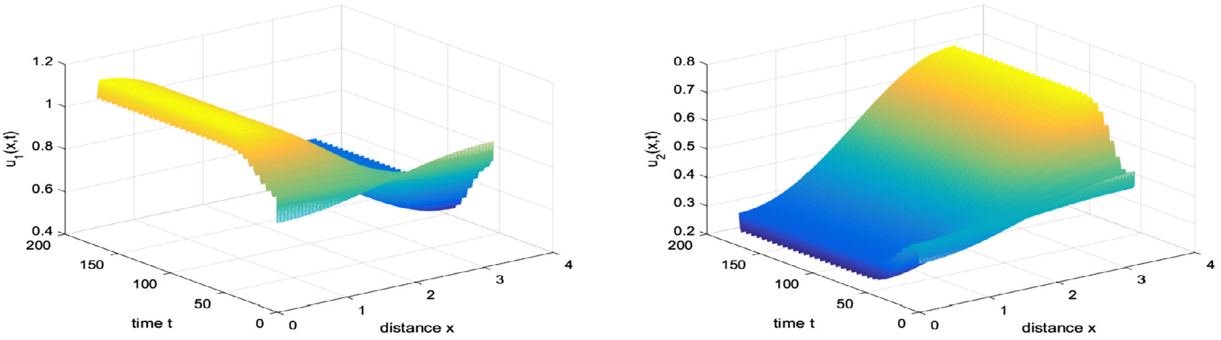


(c) $u_1(x,t)$ (d) $u_2(x,t)$

Fig. 1. $\tau = 0.1$, (a), (b) with initial values $\phi_1(x,\theta) = 0.9 + 0.08\cos x, \phi_2(x,\theta) = 0.45 + 0.08\cos x$ and (c), (d) with $\phi_1(x,\theta) = 0.9 - 0.08\cos x, \phi_2(x,\theta) = 0.45 - 0.08\cos x$.

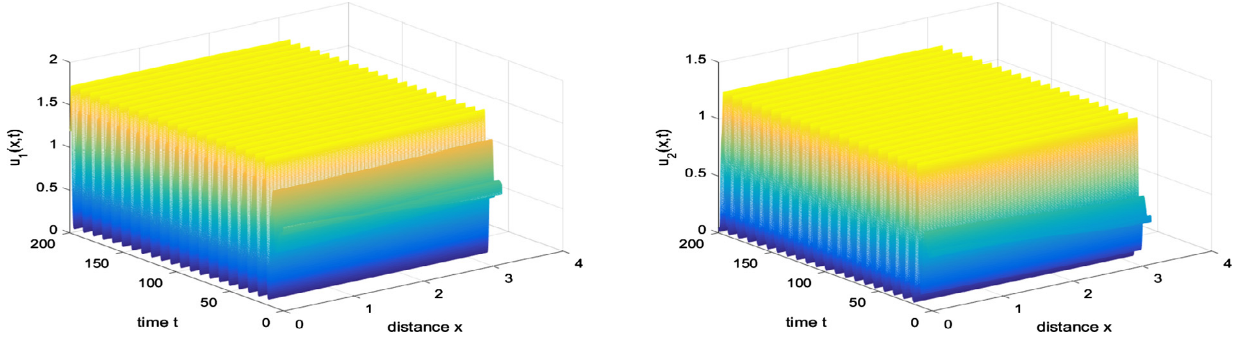


(a) $u_1(x,t)$ (b) $u_2(x,t)$



(c) $u_1(x,t)$ (d) $u_2(x,t)$

Fig. 2. $\tau = 0.675$, (a), (b) with initial values $\phi_1(x,\theta) = 0.9 + 0.08\cos x, \phi_2(x,\theta) = 0.45 + 0.08\cos x$ and (c), (d) with $\phi_1(x,\theta) = 0.9 - 0.08\cos x, \phi_2(x,\theta) = 0.45 - 0.08\cos x$.



(a) $u_1(x,t)$

(b) $u_2(x,t)$

Fig. 3. $\tau = 1$, and the initial values are $\phi_1(x,\theta) = 0.9 + 0.08\cos x$, $\phi_2(x,\theta) = 0.45 + 0.08\cos x$.

Firstly, the nonlocal intraspecific competitions have a great effect on the stability of the coexistence equilibria and whether it will affect global stability of two semi-trivial equilibrium is not clear since only the case with initial values below the corresponding environment capacity value has been proved in the Corollary 2.7. Moreover, the strength of local interspecific competitions will not change global stability of the equilibria compared with the results in [11]. Secondly, the complex pattern formations, like Turing instability, spatially homogeneous and inhomogeneous oscillations will occur when the nonlocal intraspecific competitions are suitably strong, which is new for this model in the literature. Lastly, the method of analyzing global stability or attracting region and the method of the linear stability analysis can be applied to other population dynamical models.

However, the effects of nonlocal intraspecific competitions on global stability of two semi-trivial equilibria under the weak or strong competition case, local stability of the coexistence equilibrium for a general kernel and theoretical proof for the complex pattern formations are not completely clear. These problems will be studied in the future work.

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