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QUALITATIVE ANALYSIS OF A NEURAL NETWORK MODEL WITH MULTIPLE TIME DELAYS

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We consider a simplified neural network model for a ring of four neurons where each neuron receives two time delayed inputs: One from itself and another from the previous neuron. Local stability analysis of the positive equilibrium leads to a characteristic equation containing products of four transcendental functions. By analyzing the equivalent system of four scalar transcendental equations, we obtain sufficient conditions for the linear stability of the positive equilibrium. Furthermore, we show that a Hopf bifurcation can occur when the positive equilibrium loses stability.

1. Introduction

In 1984, Hopfield [1984] proposed a simplified neural network model based on the assumption that the elements in the network respond and communicate instantaneously with each other. In reality, neural networks often have time delays, for example due to the finite switching speed of amplifiers in electronic neural networks, or due to finite signal propagation time in biological networks. In a first attempt to study this effect, Marcus and Westervelt [1989] incorporated a single time delay into the connection terms of Hopfield's model and observed sustained oscillations resulting from this time delay. Further detailed investigation on Marcus and Westervelt's neural network model with a single delay can be found in [Bélair, 1993; Bélair *et al.*, 1996; Gopalsamy & Leung, 1996; Liao & Liao, 1997; Ye *et al.*, 1994] and the references therein.

In 1994, Baldi and Atiya [1994] constructed a network that consists of a ring of neurons connected cyclically with delayed interactions. Different delays are introduced for the communication between the adjacent neurons. In the last few years, such neural network models with multiple delays have been studied extensively, we refer to [Gopalsamy & He, 1994; van den Driessche & Zou, 1998; Ye *et al.*, 1995] for stability analysis by constructing Liapunov functions. However, the local stability and bifurcation analysis of neural network models with multiple delays are very complicated. In order to obtain a deep and clear understanding of the dynamics of such models, researchers have focused on two-neuron network models with two delays, see, for example, [Babcock & Westervelt, 1987; Majee & Roy, 1997; Olien & Bélair, 1997; Wei & Ruan, 1999], etc.

Recently, Campbell [1999] generalized Baldi and Atiya's model to a network that consists of a ring of neurons where the jth element receives two time delayed inputs: One from itself and another from the previous element. She studied not only the stability of the fixed points of the network but also the bifurcation of new solutions when stability is lost. Due to the high dimension of the model and the complexity of the analysis, the sufficient conditions for local stability and bifurcations are very general.

In this paper, we simplify Campbell's model in two ways. First, we assume that the time delays in the communication between each pair of adjacent neurons are identical. The model then takes the form:

$$C_j\dot{u}_j(t) = -\frac{1}{R_j}u_j(t) + F_j(u_j(t-\sigma)) + G_j(u_{j-1}(t-\tau)), \quad j = 1, 2, \ldots, n, \tag{1}$$

where $C_j > 0$ and $R_j > 0$ represent the capacitance and resistance of each neuron, respectively, F_j and G_j are nonlinear functions representing, respectively, the feedback from neuron j to itself and the connection from j to $j-1$, and the index 0 is taken equal to n. We should point out that a similar linear neural network model was considered by Kharitonov and Paice [1997].

Normalizing system (1) leads to

$$\dot{u}_j(t) = -d_ju_j(t) + f_j(u_j(t-\sigma)) + g_j(u_{j-1}(t-\tau)), \quad j = 1, 2, \ldots, n, \tag{2}$$

where $d_j > 0$. As shown in [Campbell, 1999], the linear stability analysis of system (2) will lead to a characteristic equation involving products of n terms, where n is the number of neurons in the ring. Clearly, solving the characteristic equation becomes more difficult if n is large. Also, the numbers of the roots of the characteristic equation depends on the oddness or evenness of n. In fact, when $\sigma = 0$, Baldi and Atiya [1994] showed that it was impossible for (2) with the g_j decreasing functions and n even to exhibit self-sustained oscillation. This lack of stable oscillatory behavior for rings with an even number of elements has been the subject of a number of papers (see e.g. [Mallet-Paret & Sell, 1996]). By contrast, Campbell [1999] has shown that (2) with n even can exhibit self-sustained oscillations and thus merits further investigation. The case $n = 2$ has been studied in detail in [Shayer & Campbell, 1999], so our second simplification of Campbell's model is to restrict our attention to the case $n = 4$. By doing so, we are able to derive detailed and easy-to-check conditions on the local stability and Hopf bifurcation of the network.

2. Stability and Hopf Bifurcation

First of all, we assume that system (2) has a fixed point

$$u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T, \tag{3}$$

where u_j^* satisfies

$$d_ju_j^* = f_j(u_j^*) + g_j(u_{j-1}^*).$$

The existence of such solutions depends, of course, on the particular functions f_j and g_j used in the model. Assuming that such a fixed point exists, one can translate it to the origin via the transformation

$$x(t) = u(t) - u^*, \tag{4}$$

where $x = (x_1, x_2, \ldots, x_n)^T$. Then (1) becomes

$$\dot{x}_j(t) = -d_jx_j(t) + F_j(x_j(t-\sigma)) + G_j(x_{j-1}(t-\tau)), \quad j = 1, 2, \ldots, n, \tag{5}$$

where $F_j(x_j(t-\sigma)) + G_j(x_{j-1}(t-\tau)) = -d_ju_j^* + f_j(x_j(t-\sigma)+u_j^*) + g_j(x_{j-1}(t-\tau)+u_{j-1}^*)$. If f_j and g_j are sufficiently smooth, one can expand F_j and G_j in the Taylor series about $x(t) = 0$. Hence, we have the linearization of (5) at $x = 0$.

$$\dot{x}_j(t) = -d_jx_j(t) + a_jx_j(t-\sigma) + b_jx_{j-1}(t-\tau), \quad j = 1, \ldots, n, \tag{6}$$

where $a_j = f_j'(u_j^*)$, $b_j = g_j'(u_{j-1}^*)$. Physically, a_j and b_j can be thought of as strengths of the connections between neurons.

In order to study the linearized stability of the fixed point $x = 0$ and the Hopf bifurcation of (5), we must investigate the characteristic equation associated with (6). Denote I as the identity matrix, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, and

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 \\ b_2 & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \\ 0 & 0 & \cdots & b_n & 0 \end{pmatrix}.$$

Then the characteristic equation associated with (6) is

$$\det(\lambda I + D - Ae^{-\lambda\sigma} - Be^{-\lambda\tau}) = 0\,,$$

which leads to the following

$$\prod_{j=1}^{n} (\lambda + d_j - a_j e^{-\lambda\sigma}) = \prod_{j=1}^{n} (b_j e^{-\lambda\tau})\,. \qquad (7)$$

Usually, it is difficult to analyze the distribution of zeros for (7) since there may be complex coefficients. For example, when $n = 3$, $d_j = d$, $a_j = a$, $b_j = b$, (7) is equivalent to

$$\begin{cases} \lambda + d - ae^{-\lambda\sigma} = be^{-\lambda\tau}, \\ \lambda + d - ae^{-\lambda\sigma} = \dfrac{-1+\sqrt{3}i}{2} be^{-\lambda\tau}, \\ \lambda + d - ae^{-\lambda\sigma} = \dfrac{-1-\sqrt{3}i}{2} be^{-\lambda\tau}, \end{cases}$$

It is well known that analyzing a transcendental equation involving both $e^{-\lambda\tau}$ and $e^{-\lambda\sigma}$ could be very complicated (see [Hale & Huang, 1993; Olien & Bélair, 1997] and the references therein). We shall concentrate our study on system (1) or (5) with

$$d_j = d\,, \quad a_j = a\,, \quad b_j = b\,, \quad n = 4\,,$$

that is, we shall consider a network consisting of a ring of four identical neurons (see Fig. 1). The corresponding characteristic equation takes the form

$$(\lambda + d - ae^{-\lambda\sigma})^4 = (be^{-\lambda\tau})^4\,. \qquad (8)$$

In the following, we employ a result from [Ruan & Wei, 1999] and the idea used in [Campbell, 1999] to analyze Eq. (8). For convenience, we first state a result of the authors [Ruan & Wei, 1999] as follows.

Theorem 2.1. *For the transcendental equation*

$$\begin{aligned} &\lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} \\ &\quad + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda \\ &\quad + p_n^{(m)}]e^{-\lambda\tau_m} = 0\,, \end{aligned} \qquad (9)$$

as $(p_1^{(0)}, \ldots, p_n^{(0)}, \ldots, p_n^{(1)}, \ldots, p_n^{(m)};\ \tau_1, \ldots, \tau_m)$ *varies, the sum of the orders of the zeros of (9) in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

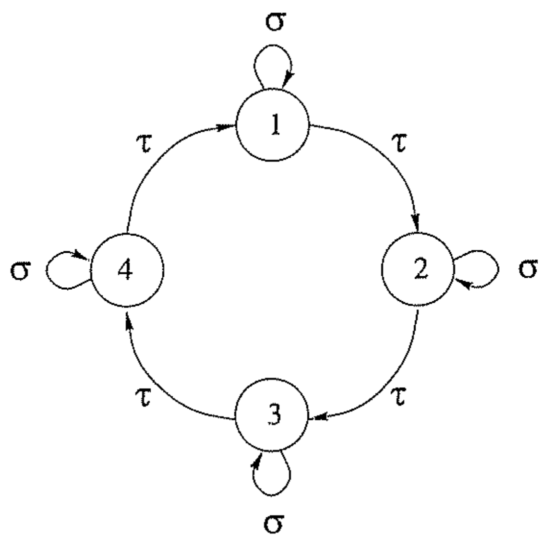

Fig. 1. A ring of four identical neurons.

Now, we consider (8). We know that (8) is equivalent to

$$\begin{aligned} \lambda - ae^{-\lambda\sigma} - be^{-\lambda\tau} + d = 0\,, \\ \lambda - ae^{-\lambda\sigma} + be^{-\lambda\tau} + d = 0\,, \\ \lambda - ae^{-\lambda\sigma} + ibe^{-\lambda\tau} + d = 0\,, \\ \lambda - ae^{-\lambda\sigma} - ibe^{-\lambda\tau} + d = 0\,. \end{aligned} \qquad (10)$$

When $\tau = 0$, (10) becomes

$$\begin{aligned} \lambda - ae^{-\lambda\sigma} + (d-b) = 0\,, \\ \lambda - ae^{-\lambda\sigma} + (d+b) = 0\,, \\ \lambda - ae^{-\lambda\sigma} + (d+ib) = 0\,, \\ \lambda - ae^{-\lambda\sigma} + (d-ib) = 0\,. \end{aligned} \qquad (11)$$

For convenience, we make the following assumptions

(P1) $a^2 - (d-b)^2 \le 0$ and $a^2 - d^2 < 0$;
(P2) (P1) does not hold;
(P3) $a^2 - (d+b)^2 \le 0$ and $a^2 - d^2 < 0$;
(P4) (P3) does not hold.

The following lemma tells us when all roots of Eq. (8) with $\tau = 0$ have negative real parts.

Lemma 2.2. *Suppose that $d > 0$ and $d - a > |b|$.*

(i) *If either $b > 0$ and (P1) or $b < 0$ and (P3) hold, then all roots of (8) with $\tau = 0$ have negative real parts for every $\sigma \in [0, \infty)$;*

(ii) *if either $b > 0$ and (P2) or $b < 0$ and (P4) hold, then there exists a $\sigma_0 > 0$ such that all roots of (8) with $\tau = 0$ have negative real parts for $\sigma \in [0, \sigma_0)$.*

Proof. For (8) with $\tau = 0$, when $\sigma = 0$, its roots can be expressed as

$$\lambda_{1,2} = -(d-a) \pm b \quad \text{and} \quad \lambda_{3,4} = -(d-a) \pm ib .$$

Clearly, $d - a > |b|$ implies that $\lambda_{1,2} < 0$ and $\operatorname{Re}\lambda_{3,4} = -(d-a) < 0$, this shows that all roots of (8) have negative real parts for $\sigma = 0$ and $\tau = 0$.

Under the conditions of this lemma, it is clear that (8) can have no zero roots. Further, $i\omega(\omega > 0)$ is a root of (8) with $\tau = 0$ if and only if $i\omega$ is a root of one of (11). If one can prove that (11) has no purely imaginary roots, then applying Theorem 2.1 one obtains that all roots of (8) have negative real parts.

In fact, if $i\omega(\omega > 0)$ is a root of (11), one of the following must hold

$$\begin{cases} d - b = a \cos \omega\sigma \\ \omega = -a \sin \omega\sigma, \end{cases} \\ \begin{cases} d + b = a \cos \omega\sigma \\ \omega = -a \sin \omega\sigma, \end{cases} \\ \begin{cases} d = a \cos \omega\sigma \\ \omega + b = -a \sin \omega\sigma, \end{cases} \\ \begin{cases} d = a \cos \omega\sigma \\ \omega - b = -a \sin \omega\sigma. \end{cases} \tag{12}$$

By (12), we have

$$\begin{aligned} \omega^2 &= a^2 - (d-b)^2 \\ \omega^2 &= a^2 - (d+b)^2 \\ (\omega + b)^2 &= a^2 - d^2 \\ (\omega - b)^2 &= a^2 - d^2 . \end{aligned} \tag{13}$$

If $b > 0$ and (P1) hold, then each of (13) has no positive real root for ω. This proves the first part of (i).

Similarly, one can prove the same conclusion under the conditions $b < 0$ and (P3).

Now, we prove (ii). Assumptions (P2) and $d - a > b$ mean that either $a^2 - (d-b)^2 > 0$ or $a^2 - d^2 \geq 0$ holds. Without loss of generality, assume that

$$a^2 - (d-b)^2 > 0 , \quad a^2 - (d+b)^2 > 0 , \\ a^2 - d^2 > 0 \quad \text{and} \quad \sqrt{a^2 - d^2} > b .$$

From (13), we then have

$$\omega_0^{(1)} = \sqrt{a^2 - (d-b)^2} , \quad \omega_0^{(2)} = \sqrt{a^2 - (d+b)^2} , \\ \omega_0^{(3)} = -b + \sqrt{a^2 - d^2} , \quad \omega_0^{(4)} = b + \sqrt{a^2 - d^2} .$$

Let

$$\begin{aligned} \sigma_0^{(1)} &= \frac{1}{\omega_0^{(1)}} \arccos \frac{d-b}{a} , \\ \sigma_0^{(2)} &= \frac{1}{\omega_0^{(2)}} \arccos \frac{d+b}{a} , \\ \sigma_0^{(3)} &= \frac{1}{\omega_0^{(3)}} \arccos \frac{d}{a} , \\ \sigma_0^{(4)} &= \frac{1}{\omega_0^{(4)}} \arccos \frac{d}{a} \end{aligned}$$

and denote

$$\sigma_0 = \min\{\sigma_0^{(1)}, \sigma_0^{(2)}, \sigma_0^{(3)}, \sigma_0^{(4)}\} . \tag{14}$$

Clearly, σ_0 is the first value of $\sigma \geq 0$ such that (8) with $\tau = 0$ has purely imaginary roots.

Notice that all roots of (8) with $\tau = 0$ have negative real parts for $\sigma = 0$, so by Theorem 2.1, we know that all roots of (8) with $\tau = 0$ have negative real parts for $\sigma \in [0, \sigma_0)$.

The proof of the second part of (ii) is similar to the above discussion. This completes the proof. ■

Now, we return to Eq. (8). $i\omega(\omega > 0)$ is a root of (8) if and only if $i\omega$ is a root of one of (10), which means that ω satisfies one of the following set of equations

$$\begin{cases} d - a \cos \omega\sigma = b \cos \omega\tau \\ \omega + a \sin \omega\sigma = -b \sin \omega\tau, \end{cases} \\ \begin{cases} d - a \cos \omega\sigma = -b \cos \omega\tau \\ \omega + a \sin \omega\sigma = b \sin \omega\tau, \end{cases} \\ \begin{cases} d - a \cos \omega\sigma = -b \sin \omega\tau \\ \omega + a \sin \omega\sigma = -b \cos \omega\tau, \end{cases} \\ \begin{cases} d - a \cos \omega\sigma = b \sin \omega\tau \\ \omega + a \sin \omega\sigma = b \cos \omega\tau. \end{cases} \tag{15}$$

From these equations we see that ω must satisfy

$$\frac{\omega^2 + (d^2 + a^2 - b^2) + 2a\omega \sin \omega\sigma}{2ad} = \cos \omega\sigma . \tag{16}$$

Clearly, if (16) has roots, then the number of roots is finite, denoted by $\omega_1, \ldots, \omega_n$. Then, from (15) we can define that ($j = 1, 2, \ldots, n$)

$$\tau_j^{(1)} = \frac{1}{\omega_j} \arccos\left(\frac{d - a\cos\omega_j\sigma}{b}\right),$$

$$\tau_j^{(2)} = \frac{1}{\omega_j} \arccos\left(-\frac{d - a\cos\omega_j\sigma}{b}\right),$$

$$\tau_j^{(3)} = \frac{1}{\omega_j} \left[\pi - \arcsin\left(-\frac{d - a\cos\omega_j\sigma}{b}\right)\right],$$

$$\tau_j^{(4)} = \frac{1}{\omega_j} \arcsin\left(\frac{d - a\cos\omega_j\sigma}{b}\right).$$

Let

$$\tau_0 = \min_{1\le j\le n}\{\tau_j^{(1)}, \tau_j^{(2)}, \tau_j^{(3)}, \tau_j^{(4)}\}. \qquad (17)$$

Thus, applying Theorem 2.1 and Lemma 2.2, we obtain the following.

Theorem 2.3. *Suppose $d > 0$ and $d - a > |b|$.*

(i) *If $b > 0$ and (P1) (or $b < 0$ and (P3)) hold and (16) has positive roots, then there exists a $\tau_0 > 0$ such that all roots of (8) have negative real part for $\tau \in [0, \tau_0)$, where τ_0 is defined by (17);*
(ii) *if $b > 0$ and (P1) (or $b < 0$ and (P3)) hold and (16) has no positive root, then all roots of (8) have negative real parts for every $\tau \ge 0$;*
(iii) *if $b > 0$ and (P2) (or $b < 0$ and (P4)) hold and for $\sigma \in [0, \sigma_0)$ Eq. (16) has positive roots, then there exists a $\tau_0 > 0$ such that all roots of (8) with $\sigma \in [0, \sigma_0)$ have negative real parts for $\tau \in [0, \tau_0)$, where τ_0 is defined by (17);*
(iv) *if $b > 0$ and (P2) (or $b < 0$ and (P4)) hold and for $\sigma \in [0, \sigma_0)$ (16) has no positive roots, then all roots of (8) with $\sigma \in [0, \sigma_0)$ have negative real parts for every $\tau \ge 0$.*

It is easy to find conditions which guarantee that ((16)) has no positive root. For instance,

$$\left|\frac{\omega^2 + (d^2 + a^2 - b^2) + 2a\omega\sin\omega\sigma}{2ad}\right| > 1 \qquad (18)$$

is such a condition.

In fact, from the inequality $d - a > |b|$, we know that $(d^2 + a^2 - b^2)/2ad > 1$, if $ad > 0$. On the other hand, as long as $2|a| \le 1$ and $\sigma \le 1$, we have $\omega^2 + 2a\omega\sin\omega\sigma \ge 0$. Hence, under the conditions $d - a > |b|$, $ad > 0$, $2|a| \le 1$ and $\sigma \le 1$, we know that (18) has no positive root. Summarizing the above discussion, we have the following result.

Corollary 2.4. *Suppose that $d > 0$, $d - a > |b|$, $0 < 2a \le 1$ and $\sigma \le 1$.*

(i) *If $b > 0$ and (P1) (or $b < 0$ and (P3)) hold, then all roots of (8) have negative real parts for $\tau \ge 0$;*
(ii) *if $b > 0$ and (P2) (or $b < 0$ and (P4)) hold, then all roots of (8) have negative real parts for $\sigma \in [0, 1] \cap [0, \sigma_0)$ and every $\tau \ge 0$.*

By the above discussion, we know that under the condition (i) or (ii) of Theorem 2.3, τ_0 may be a Hopf bifurcation value for the system (5). To verify this, we study the transversality condition. Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$$

be the root of Eq. (8) satisfying

$$\alpha(\tau_0) = 0, \quad \omega(\tau_0) = \omega_0,$$

where τ_0 is defined by (17), ω_0 is ω_j related to τ_0.

Theorem 2.5. *Suppose that $d > 0$, $d - a > |b|$ and either the condition (i) or (ii) in Theorem 2.3 holds, then $\alpha'(\tau_0) > 0$ or $\alpha'(\tau_0) = 0$, and when $\alpha'(\tau_0) > 0$, τ_0 is the Hopf bifurcation value of system (5).*

Proof. From the proof of Theorem 2.3, we know that all roots of (8) have negative real parts for $\tau \in [0, \tau_0)$, and when $\tau = \tau_0$, (8) has a pair of purely imaginary roots $\pm i\omega_0$. For a contradiction, we assume that $\alpha'(\tau_0) < 0$. Then by the continuity of $\alpha(\tau)$, we have that $\alpha(\tau) > 0$ as $\tau < \tau_0$ and close to τ_0. This contradicts Theorem 2.3 and completes the proof. ∎

Finally, we regard b and τ as parameters and use the method from [Campbell, 1999] to investigate the system (5).

When $b = 0$, (8) becomes

$$\lambda + d - ae^{-\lambda\sigma} = 0. \qquad (19)$$

Lemma 2.6. *If*

$$|a| < d \quad and \quad \sigma \ge 0$$

or

$$a < -d \quad and \quad \sigma < \frac{1}{\sqrt{a^2 - d^2}} \arccos\left(\frac{d}{a}\right),$$

then all roots of (19) have negative real parts.

Proof If $|a| < d$ or $a < -d$, then the root of (19) for $\sigma = 0$ is

$$\lambda = -d + a < 0\,.$$

$i\omega(\omega > 0)$ is a root of (19) if and only if ω satisfies

$$\begin{aligned} d &= a \cos \omega\sigma\,, \\ \omega &= -a \sin \omega\sigma\,, \end{aligned} \tag{20}$$

which implies that (19) has no purely imaginary roots for every $\sigma \geq 0$ as $|a| < d$. This completes the proof of the first part of the lemma.

By (20) we have

$$\omega = \sqrt{a^2 - d^2}\,.$$

Thus, ω is well-defined for $a < -d < 0$. Let

$$\sigma_0 = \frac{1}{\sqrt{a^2 - d^2}} \arccos\left(\frac{d}{a}\right)\,, \tag{21}$$

we know that $i\sqrt{a^2 - d^2}$ is a purely imaginary root of (19) with $\sigma = \sigma_0$. Clearly, for $\sigma \in [0, \sigma_0)$, Eq. (19) has no root on the imaginary-axis. By Theorem 2.1, we obtain the second conclusion of the lemma. This completes the proof. ■

From (15), for fixed a, d and σ, one can see that $i\omega(\omega \geq 0)$ is a root of (8) if and only if b, τ and ω satisfy

$$b^+ = \sqrt{\omega^2 + (d^2 + a^2) + 2a\omega \sin \omega\sigma - 2ad \cos \omega\sigma} \tag{22}$$

and one of the following $\tau_i^+ (i = 1, 2, 3, 4)$:

$$\tau_1^+ = \begin{cases} \left.\begin{aligned} &\frac{1}{\omega}\left[\arctan\left(\frac{-(\omega + a \sin \omega\sigma)}{d - a \cos \omega\sigma}\right) + 2\ell\pi\right], \quad \omega + a \sin \omega\sigma < 0 \\ &\frac{1}{\omega}\left[\arctan\left(\frac{-(\omega + a \sin \omega\sigma)}{d - a \cos \omega\sigma}\right) + 2(\ell+1)\pi\right], \quad \omega + a \sin \omega\sigma > 0 \end{aligned}\right\} d - a \cos \omega\sigma > 0\,, \\ \frac{1}{\omega}\left[\arctan\left(\frac{-(\omega + a \sin \omega\sigma)}{d - a \cos \omega\sigma}\right) + (2\ell+1)\pi\right], \quad d - a \cos \omega\sigma < 0\,; \end{cases}$$

$$\tau_2^+ = \begin{cases} \frac{1}{\omega}\left[\arctan\left(\frac{\omega + a \sin \omega\sigma}{-(d - a \cos \omega\sigma)}\right) + (2\ell+1)\pi\right], \quad d - a \cos \omega\sigma > 0\,, \\ \left.\begin{aligned} &\frac{1}{\omega}\left[\arctan\left(\frac{\omega + a \sin \omega\sigma}{-(d - a \cos \omega\sigma)}\right) + 2\ell\pi\right], \quad \omega + a \sin \omega\sigma > 0 \\ &\frac{1}{\omega}\left[\arctan\left(\frac{\omega + a \sin \omega\sigma}{-(d - a \cos \omega\sigma)}\right) + 2(\ell+1)\pi\right], \quad \omega + a \sin \omega\sigma < 0 \end{aligned}\right\} d - a \cos \omega\sigma < 0\,; \end{cases} \tag{23}$$

$$\tau_3^+ = \begin{cases} \frac{1}{\omega}\left[\operatorname{arccot}\left(\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + (2\ell+1)\pi\right], \quad d - \cos \omega\sigma > 0\,, \\ \frac{1}{\omega}\left[\operatorname{arccot}\left(\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + 2\ell\pi\right], \quad d - \cos \omega\sigma < 0\,; \end{cases}$$

$$\tau_4^+ = \left[\begin{aligned} &\frac{1}{\omega}\left[\operatorname{arccot}\left(\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + 2\ell\pi\right], \quad d - a \cos \omega\sigma > 0\,, \\ &\frac{1}{\omega}\left[\operatorname{arccot}\left(\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + (2\ell+1)\pi\right], \quad d - a \cos \omega\sigma < 0\,, \end{aligned}\right.$$

where $\ell = 0, 1, 2, \ldots$, or

$$b^- = -\sqrt{\omega^2 + (d^2 + a^2) + 2a\omega \sin \omega\sigma - 2ad \cos \omega\sigma} \tag{24}$$

and one of the following $\tau_i^-(i = 1, 2, 3, 4)$:

$$\tau_1^- = \begin{cases} \dfrac{1}{\omega}\left[\arctan\left(\dfrac{\omega + a\sin\omega\sigma}{-(d - a\cos\omega\sigma)}\right) + (2\ell+1)\pi\right], & d - a\cos\omega\sigma > 0\,, \\ \left.\begin{array}{ll} \dfrac{1}{\omega}\left[\arctan\left(\dfrac{\omega + a\sin\omega\sigma}{-(d - a\cos\omega\sigma)}\right) + 2\ell\pi\right], & \omega + a\sin\omega\sigma > 0 \\ \dfrac{1}{\omega}\left[\arctan\left(\dfrac{\omega + a\sin\omega\sigma}{-(d - a\cos\omega\sigma)}\right) + 2(\ell+1)\pi)\right], & \omega + a\sin\omega\sigma < 0 \end{array}\right\} & d - a\cos\omega\sigma < 0\,; \end{cases}$$

$$\tau_2^- = \begin{cases} \left.\begin{array}{ll} \dfrac{1}{\omega}\left[\arctan\left(\dfrac{-(\omega + a\sin\omega\sigma)}{d - a\cos\omega\sigma}\right) + 2\ell\pi\right], & \omega + a\sin\omega\sigma < 0 \\ \dfrac{1}{\omega}\left[\arctan\left(\dfrac{-(\omega + a\sin\omega\sigma)}{d - a\cos\omega\sigma}\right) + 2(\ell+1)\pi\right], & \omega + a\sin\omega\sigma > 0 \end{array}\right\} & d - a\cos\omega\sigma > 0\,, \\ \dfrac{1}{\omega}\left[\arctan\left(\dfrac{-(\omega + a\sin\omega\sigma)}{d - a\cos\omega\sigma}\right) + (2\ell+1)\pi\right], & d - \cos\omega\sigma < 0\,; \end{cases} \tag{25}$$

$$\tau_3^- = \begin{cases} \dfrac{1}{\omega}\left[\operatorname{arccot}\left(\dfrac{\omega + a\sin\omega\sigma}{d - a\cos\omega\sigma}\right) + 2\ell\pi\right], & d - \cos\omega\sigma > 0\,, \\ \dfrac{1}{\omega}\left[\operatorname{arccot}\left(\dfrac{\omega + a\sin\omega\sigma}{d - a\cos\omega\sigma}\right) + (2\ell+1)\pi\right], & d - \cos\omega\sigma < 0\,; \end{cases}$$

$$\tau_4^- = \begin{bmatrix} \dfrac{1}{\omega}\left[\operatorname{arccot}\left(\dfrac{\omega + a\sin\omega\sigma}{d - a\cos\omega\sigma}\right) + (2\ell+1)\pi\right], & d - a\cos\omega\sigma > 0\,, \\ \dfrac{1}{\omega}\left[\operatorname{arccot}\left(\dfrac{\omega + a\sin\omega\sigma}{d - a\cos\omega\sigma}\right) + 2\ell\pi\right], & d - a\cos\omega\sigma < 0\,, \end{bmatrix}$$

where $\ell = 0, 1, 2, \ldots$.

For fixed a, d and σ, Eqs. (22) and (23), (24) and (25) describe curves which lie in the right and left of the (b, τ) plane, respectively, and are parameterized by ω. Clearly, τ_i^+ and $\tau_i^-(i = 1, 2, 3, 4)$ may not be continuous at some points ω, this means that the curve (b^+, τ_i^+) or (b^-, τ_i^-) has discontinuities. This shows that these curves may be very complicated. Nevertheless, we can still make some assertions about the parameter values for which the trivial solution of (5) is asymptotically stable.

Lemma 2.7. *For fixed a, d and σ, if $b^\pm$ as defined by (22) and (24) are monotone in ω, then there are no intersection points of the curves defined by (b^+, τ_i^+) or of the curves defined by (b^-, τ_i^-) for every $i \in \{1, 2, 3, 4\}$. If b^+ is increasing in ω then there are no intersection points of the curves defined by (b^+, τ_i^+) or $(b^-, \tau_i^-)(i = 1, 2, 3, 4)$ with the line $b = d - a$ or $b = -(d - a)$, respectively.*

The proof of Lemma 2.7 is similar to that of Lemma 2.3 of [Campbell, 1999], so is omitted.

Theorem 2.8. *Let a, d, σ be fixed and the conditions of Lemma 2.6 be satisfied. If b^+ is increasing in ω, then the stability region of the system (5) is*

$$-(d - a) < b < d - a \quad \textit{and} \quad \tau > 0\,. \tag{26}$$

Proof. Lemma 2.6 implies that all roots of (8) with $b = 0$ have negative real parts. Lemma 2.7 implies that (8) has no root appearing on the imaginary axis when b^+ is increasing in ω and (26) holds. By Theorem 2.1 the conclusion follows. This completes the proof. ∎

Remark 2.9. From (10), we know that $\lambda = 0$ is a root of (8) when $b = d - a$ or $b = -(d - a)$. Denote $\lambda = \lambda(b)$ the root of (8) satisfying $\lambda(d - a) = 0$ or $\lambda(a - d) = 0$. We obtain that

$$\left.\frac{d\lambda}{db}\right|_{b=d-a} = \frac{1}{1 + a\sigma + (d - a)\tau} > 0$$

or

$$\left.\frac{d\lambda}{db}\right|_{b=-(d-a)} = \frac{-1}{1 + a\sigma + (d - a)\tau} < 0\,.$$

This means that (8) has a positive real root when $b > d - a$ or $b < d - a$. So, we can say that Theorem 2.8 gives the best estimate for the stability of system (5).

A natural question is when is b^+ increasing. The following lemma answers this question.

Lemma 2.10. *If $|a| < d$, $2|a|\sigma < 1$ and $|a| < (1/2\sigma(1 + d\sigma))$ for $a < 0$, or $a < (3\pi/4\sigma(1 + d\sigma))$ for $a \geq 0$, then b^+ is increasing in ω and*

$$\begin{aligned}\tau_2^- = \tau_1^+ &= \frac{1}{\omega}\left[\arctan\left(-\frac{\omega + a\ \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + 2(\ell + 1)\pi\right],\\ \tau_1^- = \tau_2^+ &= \frac{1}{\omega}\left[\arctan\left(-\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + (2\ell + 1)\pi\right],\\ \tau_4^- = \tau_3^+ &= \frac{1}{\omega}\left[\operatorname{arccot}\left(\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + (2\ell + 1)\pi\right],\\ \tau_3^- = \tau_4^+ &= \frac{1}{\omega}\left[\operatorname{arccot}\left(\frac{\omega + a \sin \omega\sigma}{d - a \cos \omega\sigma}\right) + 2\ell\pi\right],\end{aligned} \tag{27}$$

where $\ell = 0, 1, 2, \ldots$.

Proof. Differentiating both sides of (22) with ω, we have

$$\begin{aligned}\frac{db^+}{d\omega} &= \frac{1}{b^+}[\omega + a\sin\omega\sigma + a\sigma\omega\cos\omega\sigma + ad\sigma\sin\omega\sigma]\\ &= \frac{1}{b^+}\left\{\omega\left(\frac{1}{2} + a\sigma\cos\omega\sigma\right)\right.\\ &\quad \left. + \left[\frac{1}{2}\omega + a(1 + d\sigma)\sin\omega\sigma\right]\right\}.\end{aligned}$$

From $|a|\sigma < (1/2)$ we have $\omega((1/2) + a\sigma \cos \omega\sigma) > 0$ for $\omega > 0$. Meanwhile, the inequality $|a| < (1/2\sigma(1+d\sigma))$ for $a < 0$ or $a < (3\pi/4\sigma(1+d\sigma))$ for $a \geq 0$ implies that $(1/2)\omega + a(1 + d\sigma) \sin \omega\sigma > 0$ for $\omega > 0$. Thus we obtain $(db^+/d\omega) > 0$ for $\omega > 0$. This completes the proof of the first part.

Also, $|a| < d$ and $2|a|\sigma < 1$ imply that $d - a \cos \omega a > 0$ and $\omega + a \sin \omega\sigma > 0$ for $\omega > 0$, respectively. Hence, (27) follows from (23) and (25). This completes the proof. ■

Applying Theorem 2.8 and Lemma 2.10, we have the following conclusion.

Theorem 2.11. *If $|a| < d$, $2|a|\sigma < 1$ and either $|a| < (1/2\sigma(1 + d\sigma))$ for $a < 0$ or $a < (3\pi/4\sigma(1 + d\sigma))$ for $a > 0$, then the stability region of the system (5) is given by*

$$-(d - a) < b < d - a \quad \textit{and} \quad \tau > 0.$$

Clearly, under the conditions of Lemma 2.10, $(\omega + a \sin \omega\sigma)/(d - a \cos \omega\sigma) > 0$ for all $\omega > 0$. Hence, each function of (27) is continuous in ω and

$$\lim_{\omega\to 0^+} \tau_i^+ = \lim_{\omega\to 0^+} \tau_i^- = \infty, \quad i = 1, 2, 3, 4 \tag{28}$$

and

$$\lim_{w\to\infty} \tau_i^+ = \lim_{w\to\infty} \tau_i^- = 0, \quad i = 1, 2, 3, 4. \tag{29}$$

Denote the curves defined by (22) and (23), and (24) and (25) by $\tau_i^+(b)$ and $\tau_i^-(b)(i = 1, 2, 3, 4)$, respectively. From $\lim_{w\to\infty} b^\pm = \infty$, $\lim_{w\to 0} b^\pm = \pm(d - a)$, (28) and (29), we have

$$\lim_{b\to\infty} \tau_i^+(b) = \lim_{b\to\infty} \tau_i^-(b) = 0$$

and

$$\lim_{b\to(d-a)} \tau_i^+(b) = \lim_{b\to-(d-a)} \tau_i^-(b) = \infty$$

for $i = 1, 2, 3, 4$.

We know that if $(b(\omega_0), \tau(\omega_0))$ lies on one of the curves mentioned above, then (8) has a pair of purely imaginary roots $\pm i\omega_0$. Denote the root of (8) by

$$\lambda(b) = \alpha(b) + i\omega(b)$$

satisfying $\alpha(b(\omega_0)) = 0$ and $\omega(b(\omega_0)) = \omega_0$.

Lemma 2.12. *Suppose that $|a| < d$, $2|a|\sigma < 1$ and $\tau > \min(\sigma, \sigma|a|/(d - |a|))$.*

(i) *If $(b(\omega_0), \tau(\omega_0))$ lies on one of the curves defined by (22), then for all ω_0*

$$\frac{d\alpha(b(\omega_0))}{db} > 0;$$

(ii) *if $(b(\omega_0), \tau(\omega_0))$ lies on one of the curves defined by (24), then for all ω_0*

$$\frac{d\alpha(b(\omega_0))}{db} < 0.$$

Proof. Consider first the situation when $(b(\omega_0), \tau(\omega_0))$ belongs to one of the curves defined by (22) and (23). Note that along these curves $b(\omega_0) > 0$, and that $|a| < d$ and $2|a|\sigma < 1$ imply $d - a \cos \omega_0\sigma > 0$ and $\omega_0 + a \sin \omega_0\sigma > 0$, respectively.

There are four cases depending on which of the τ_j^+ in (23) gives the value of $\tau(\omega_0)$.

Case 1. In this case, $\lambda(b)$ is a root of the first equation of (11). Differentiating both sides of it with respect to b, we have

$$\frac{d\lambda}{db} = \frac{e^{-\lambda\tau}}{1 + a\sigma e^{-\lambda\sigma} + b\tau e^{-\lambda\tau}}.$$

Hence

$$\left(\frac{d\lambda}{db}\right)^{-1} = b\tau + e^{\lambda\tau} + a\sigma e^{-\lambda(\sigma-\tau)}.$$

Substituting $b(\omega_0)$ into this equation, we obtain

$$\begin{aligned}\left(\frac{d\lambda(b(\omega_0))}{db}\right)^{-1} &= [b(\omega_0)\tau(\omega_0) + \cos\,\omega_0\tau(\omega_0)\\ &\quad + a\sigma\,\cos\,\omega_0(\sigma - \tau(\omega_0))]\\ &\quad + i[\sin\,\omega_0\tau(\omega_0)\\ &\quad - a\sigma\,\sin\,\omega_0(\sigma - \tau(\omega_0))].\end{aligned}$$

From the first equation of (15) we have $\cos\,\omega_0\tau(\omega_0) > 0$ and from the second

$$\begin{aligned}\omega_0 b(\omega_0)\tau(\omega_0) &\geq -b(\omega_0)\,\sin\,\omega_0\tau(\omega_0)\\ &= \omega_0 + a\,\sin\,\omega_0\sigma \geq \omega_0(1 - |a|\sigma),\end{aligned}$$

which leads to $b(\omega_0)\tau(\omega_0) \geq 1 - |a|\sigma$. Thus

$$\begin{aligned}&[b(\omega_0)\tau(\omega_0) + \cos\,\omega_0\tau(\omega_0) + a\sigma\,\cos\,\omega_0(\sigma - \tau(\omega_0))]\\ &\quad > 1 - |a|\sigma - |a|\sigma > 0.\end{aligned}$$

Cases 2 and 3 correspond to $\lambda(b)$ being a root of the second and third equations of (11) and can be proven in a similar manner to Case 1.

Case 4. In this case, $\lambda(b)$ is a root of the fourth equation of (11). In a similar manner to Case 1 we obtain

$$\begin{aligned}\left(\frac{d\lambda(b(\omega_0))}{db}\right)^{-1} &= [b(\omega_0)\tau(\omega_0) + \sin\,\omega_0\tau(\omega_0)\\ &\quad - a\sigma\,\sin\,\omega_0(\sigma - \tau(\omega_0))]\\ &\quad - i[\cos\,\omega_0\tau(\omega_0)\\ &\quad + a\sigma\,\cos\,\omega_0(\sigma - \tau(\omega_0))].\end{aligned}$$

From the seventh equation of (15) we have $\sin\,\omega_0\tau(\omega_0) > 0$ and from the eighth

$$\begin{aligned}b(\omega_0) &\geq b(\omega_0)\,\cos\,\omega_0\tau(\omega_0)\\ &= \omega_0 + a\sin\,\omega_0\sigma \geq \omega_0(1 - |a|\sigma),\end{aligned}$$

which leads to $b(\omega_0)\tau(\omega_0) \geq \omega_0\tau(\omega_0)[1-|a|\sigma]$. Thus if $\sigma < \tau$

$$\begin{aligned}&[b(\omega_0)\tau(\omega_0) + \sin\,\omega_0\tau(\omega_0) + a\sigma\,\sin\,\omega_0(\tau(\omega_0) - \sigma)]\\ &\quad > \omega_0\tau(\omega_0)[1 - |a|\sigma] - |a|\sigma\omega_0[\tau(\omega_0) - \sigma]\\ &\quad = \omega_0\tau(\omega_0)[1 - 2|a|\sigma] + |a|\omega_0\sigma^2 > 0.\end{aligned}$$

Further, from the seventh equation of (15) we have

$$b(\omega_0) \geq b(\omega_0)\,\sin\,\omega_0\tau(\omega_0) = d + a\,\cos\,\omega_0\sigma \geq d - |a|.$$

Thus, if $\tau > |a|\sigma/(d - |a|)$ then

$$\begin{aligned}&[b(\omega_0)\tau(\omega_0) + \sin\,\omega_0\tau(\omega_0) + a\sigma\,\sin\,\omega_0(\tau(\omega_0) - \sigma)]\\ &\quad > \tau(\omega_0)(d - |a|) - |a|\sigma > 0.\end{aligned}$$

In all cases we have shown that

$$\mathrm{Re}\left(\frac{d\lambda(b(\omega_0))}{db}\right)^{-1} > 0.$$

The conclusion follows from the fact that

$$\mathrm{sign}\,\frac{d\mathrm{Re}\,\lambda(b(\omega_0))}{db} = \mathrm{sign}\,\mathrm{Re}\left(\frac{d\lambda(b(\omega_0))}{db}\right)^{-1}.$$

The proof for the situation when $(b(\omega_0), \tau(\omega_0))$ belongs to one of the curves defined by (24) and (25) is similar.

Our final result is about the Hopf bifurcation in system (5).

Theorem 2.13. *Suppose that the hypothesis of Lemma 2.12 are satisfied, then for any $(b(\omega_0), \tau(\omega_0))$ which lies on exactly one of the curves defined by (22) or (24), system (5) undergoes a Hopf bifurcation at $b(\omega_0)$.*

Remark 2.14. Points of intersection of two curves defined by (22) or (24) are excluded from Theorem 2.13 to ensure the nonresonant condition of the Hopf Bifurcation Theorem [Hale & Verduyn Lunel, 1993] is satisfied.

Theorems 2.11 and 2.13 are illustrated in Fig. 2 and Theorems 2.3 and 2.13 are illustrated in Fig. 3. In both cases, the shadowed region is the stability region, and the solid curves are Hopf bifurcation curves.

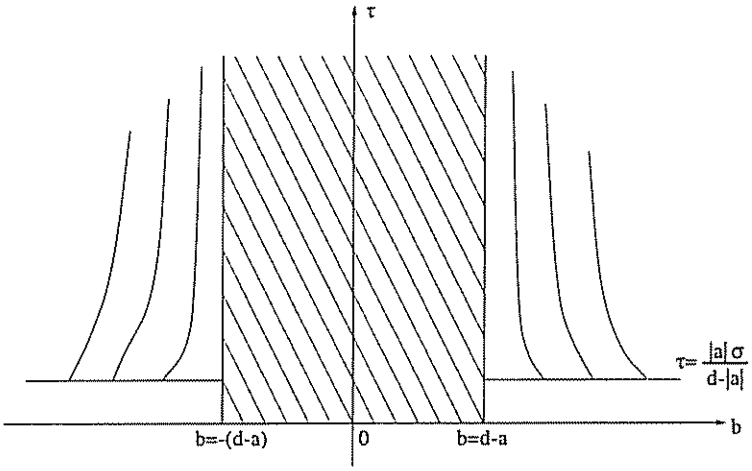

Fig. 2. The stability diagram for system (5) under the conditions of Theorem 2.11.

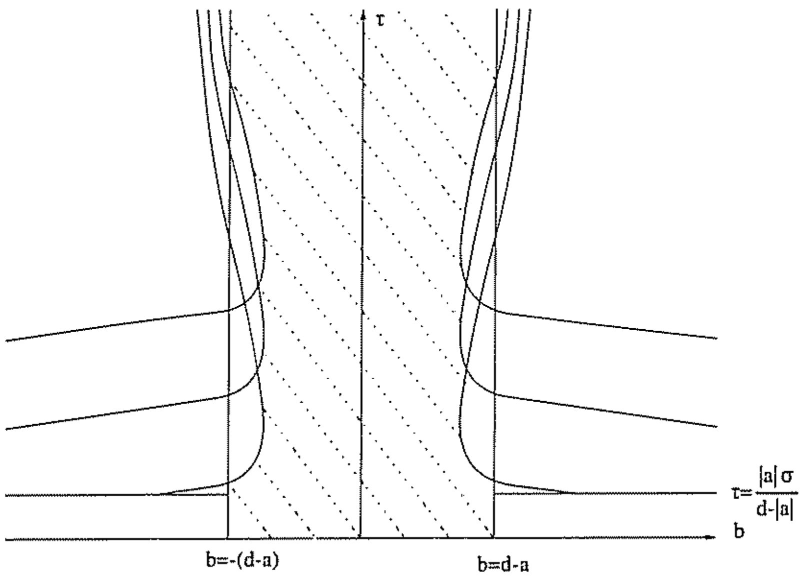

Fig. 3. The stability diagram for system (5) under the conditions (i) of Theorem 2.3.

3. Discussion

It is well known that time delays in response or transmission have a dramatic influence on the dynamics of the neural network models. In particular, a time delay can induce sustained oscillations in convergent networks (see [Marcus & Westervelt, 1989]) and even chaos in three-neuron networks (see [Marcus *et al.*, 1991]). Recently, research has focused on the dynamics of neural networks with multiple delays (see [Babcock & Westervelt, 1987; Baldi & Atiya, 1994; Campbell, 1999; Majee & Roy, 1997; Olien & Bélair, 1997; Wei & Ruan, 1999]) and many interesting dynamical phenomena have been observed.

In this paper, we have considered a simplified version of the neural network model with multiple delays proposed by one of us in [Campbell, 1999]. We have studied a ring of four neurons where each neuron receives two time delayed inputs: one from itself and another from the previous neuron. We have analyzed the local stability of the positive equilibrium and obtained explicit conditions for when this stability is independent of the size of the time delays. We have also characterized when the time delay can cause the equilibrium to lose stability and studied the Hopf bifurcation which results. As shown in [Campbell, 1999] (and suggested by Fig. 3), we expect that this type of neural networks could exhibit more complicated and interesting dynamics such as codimension two and Hopf–Hopf bifurcations. We leave this for future investigation.

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