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Hopf bifurcation in a size-structured population dynamic model with random growth

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ABSTRACT

This paper is devoted to the study of a size-structured model with Ricker type birth function as well as random fluctuation in the growth process. The complete model takes the form of a reaction–diffusion equation with a nonlinear and nonlocal boundary condition. We study some dynamical properties of the model by using the theory of integrated semigroups. It is shown that Hopf bifurcation occurs at a positive steady state of the model. This problem is new and is related to the center manifold theory developed recently in [P. Magal, S. Ruan, Center manifold theorem for semilinear equations with non-dense domain and applications to Hopf bifurcation in age-structured models, Mem. Amer. Math. Soc., in press] for semilinear equation with non-densely defined operators.

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1. Introduction

In this work we investigate the Hopf bifurcation for the following system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \underbrace{\dfrac{\partial (gu(t,x))}{\partial x}}_{\text{growth in size}} = \underbrace{\varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2}}_{\text{random noise}} \underbrace{- \mu u(t,x)}_{\text{death}}, \\ -\varepsilon^2 \dfrac{\partial u(t,0)}{\partial x} + gu(t,0) = \alpha h\left(\displaystyle\int_0^{+\infty} \gamma(x) u(t,x)\,dx \right), \\ u(0,.) = u_0 \in L^1_+(0,+\infty), \end{cases} \tag{1.1}$$

where $u(t,x)$ represents the population density of certain species at time t with size x, $g > 0$, $\varepsilon \geqslant 0$, $\mu > 0$, $\alpha > 0$, $\gamma \in L^\infty_+(0,+\infty) \setminus \{0\}$, and the map $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = x \exp(-\xi x), \quad \forall x \geqslant 0.$$

The model (1.1) is viewed as a size-structured model, for example for the growth of trees or fish population, where $x = 0$ is the minimal size. The growth of individuals is described by two terms. First, the term $\frac{\partial (gu(t,x))}{\partial x}$ represents the average growth rate of individuals, and the diffusion term $\varepsilon^2 \frac{\partial^2 u(t,x)}{\partial x^2}$ describes the stochastic fluctuations around the tendency to growth. So $\varepsilon^2 \frac{\partial^2 u(t,x)}{\partial x^2} - \frac{\partial (gu(t,x))}{\partial x}$ describes the fact that given a group of individuals located in some small neighborhood of a given size $x_0 \in (0,+\infty)$, after a period of time this group of individuals will disperse due to the diffusion, and the mean value of the distribution increases due to the convection term. The term $-\mu u(t,x)$ is classical and describes the mortality process of individuals following an exponential law with mean $1/\mu$. The birth function given by $\alpha h(\int_0^{+\infty} \gamma(x) u(t,x)\,dx)$ is a Ricker [38,39] type birth function. This type of birth function has been commonly used in the literature, to take into account some limitation of births when the population increases. In particular, the birth rate function is $\alpha\gamma(x)$ when the total population is close to zero. We refer to Arino [5], Arino and Sanchez [7], Calsina and Saldana [9], Calsina and Sanchón [10], Webb [47], and Ackleh and Deng [1] (and references therein) for studies on size-structured models in the context of ecology and cell population dynamics. As far as we know such a model has not been considered in the context of population dynamics, while it seems very natural to introduce a stochastic random noise to describe the growth of individuals with respect to their size.

In the special case when $\varepsilon = 0$, by making a simple change of variable we can assume that $g = 1$. Then the system (1.1) becomes

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial u(t,x)}{\partial x} = -\mu u(t,x), \\ u(t,0) = \alpha h\left(\displaystyle\int_0^{+\infty} \gamma(x) u(t,x)\,dx\right), \\ u(0,.) = u_0 \in L^1_+(0,+\infty). \end{cases} \tag{1.2}$$

Now if we assume that

$$\gamma(x) = e^{-\beta x} 1_{[\tau,+\infty)}(x),$$

for some $\tau \geqslant 0$ and set

$$\widehat{U}(t) := \int_\tau^{+\infty} e^{-\beta x} u(t,x)\,dx,$$

then for $t \geqslant \tau$, we obtain the following delay differential equation

$$\begin{cases} \dfrac{d\widehat{U}(t)}{dt} = \alpha e^{-\mu\tau} h\big(\widehat{U}(t-\tau)\big) - \mu \widehat{U}(t), \quad \forall t \geqslant \tau, \\ \widehat{U}(t) := e^{-\mu t} \displaystyle\int_\tau^{+\infty} e^{-\beta x} u_0(x-t)\,dx = e^{-(\mu+\beta)t} \int_{\tau-t}^{+\infty} e^{-\beta l} u_0(l)\,dl, \quad \forall t \in [0,\tau]. \end{cases}$$

We refer to Hale and Verduyn Lunel [22], Wu [48], Diekmann et al. [17], and Arino, Hbid, and Ait Dads [6] for detailed results on delay differential equations. System (1.1) can also be viewed as a stochastic perturbation in the transport term of the above delay differential equation. If we consider the special case

$$\gamma(x) = 1_{[0,+\infty)}(x),$$

then for any value of $\varepsilon \geqslant 0$, the total number of individuals $U(t) := \int_0^{+\infty} u(t,x)\,dx$ satisfies the following scalar ordinary differential equation

$$\begin{cases} \dfrac{dU(t)}{dt} = \alpha h\big(U(t)\big) - \mu U(t), \quad \forall t \geqslant 0, \\ U(0) = U_0 \geqslant 0. \end{cases}$$

Hence, there is no bifurcation at the positive equilibrium in this case. In fact the positive equilibrium (when it exists) is globally asymptotically stable.

The main question addressed in the paper is to understand how the diffusion rate ε^2 influences the stability and the Hopf bifurcation of the positive equilibrium of system (1.1). When $\varepsilon = 0$ and $g = 1$, the model becomes a size-structured model which is very similar to the age-structured models studied by Webb [45], Iannelli [24], Cushing [16], and Magal and Ruan [33]. In particular, when $\varepsilon = 0$, it was first observed by Thieme in [42] that this kind of problems can be regarded as abstract non-densely defined Cauchy problems. Even in the case $\varepsilon = 0$, the existence of non-trivial periodic solutions in such age/size-structured models is a very interesting and difficult problem, however, there are very few results (Prüss [36], Cushing [15], Bertoni [8]). It is believed that such periodic solutions in age/size-structured models are induced by Hopf bifurcation (Inaba [25,26], Calsina and Ripoll [11]).

Recently a center manifold theory has been developed for non-densely defined Cauchy problems in Magal and Ruan [35]. This center manifold theory allows us to obtain an abstract Hopf bifurcation theorem (see Liu, Magal and Ruan [29]). This Hopf bifurcation theorem has been successfully applied in [35] to the system (1.2) when

$$\gamma(x) = (x-\tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x). \tag{1.3}$$

It turns out that we can also establish such a Hopf bifurcation theorem when $\varepsilon > 0$ since the problem can also be formulated as an abstract non-densely defined Cauchy problem. This leads to the existence for a Hopf bifurcation when the parameter $\alpha > 0$ increases. But our goal here is to study the effect of the diffusion rate $\varepsilon^2 > 0$ on the existence of Hopf bifurcation. So we investigate the bifurcation by regarding α and ε as parameters of the semiflow. The problem turns to be delicate.

We would like to mention that Amann [2], Crandall and Rabinowitz [14], Da Prato and Lunardi [18], Guidotti and Merino [21], Koch and Antman [28], Sandstede and Scheel [40], and Simonett [41] investigated Hopf bifurcation in various partial differential equations including advection–reaction–diffusion equations. However, their results and techniques do not apply to our model (1.1) as there is a nonlinear and nonlocal boundary condition. Instead, we expect that our techniques might be used to study Hopf bifurcation in the viscous conservation law (Sandstede and Scheel [40]) and other advection–reaction–diffusion equations (for example, Cantrell et al. [12] and Chen et al. [13]).

The paper is organized as follows. In Section 2, we reformulate (1.1) as a non-densely defined Cauchy problem, list the notation, and show the existence and uniqueness of solutions to this non-densely defined Cauchy problem. The positive equilibrium of the system is studied in Section 3. In Section 4, we linearize the system at the positive equilibrium and study the spectral properties of the linearized equation. The characteristic equation is also given in this section. In Section 5, stability of the system is considered. In particular, when α is proportional to ε, we obtain that the positive equilibrium is locally asymptotically stable for ε large enough. In Section 6, we fix $\varepsilon \geqslant 0$ and consider α as a parameter. In Section 6.1 we study the existence of purely imaginary eigenvalues when γ is defined by (6.3). In particular, bifurcation diagrams are given for various cases. The transversality condition is verified in Section 6.2 and Hopf bifurcation theorems are presented in Section 6.3. Finally, in Section 7 we summarize the results of the paper and present some numerical simulations of the model.

2. Preliminary

From here on, we will always assume that $g = 1$ and consider the system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial u(t,x)}{\partial x} = \varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} - \mu u(t,x), \quad t \geqslant 0,\ x \geqslant 0, \\ -\varepsilon^2 \dfrac{\partial u(t,0)}{\partial x} + u(t,0) = \alpha h\left(\displaystyle\int_0^{+\infty} \gamma(x) u(t,x)\,dx \right), \\ u(0,.) = u_0 \in L^1_+(0,+\infty). \end{cases} \tag{2.1}$$

Assumption 2.1. Assume that $\varepsilon > 0$, $\mu > 0$, $\alpha > 0$, $\gamma \in L^\infty_+(0,+\infty) \setminus \{0\}$, and $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = x \exp(-\xi x), \quad \forall x \in \mathbb{R},$$

where $\xi > 0$.

In the sequel, we will use the integrated semigroup theory to study such a PDE. We refer to Arendt [3], Thieme [43], Kellermann and Hieber [27], and the book of Arendt et al. [4] for details on this subject. We also refer to Magal and Ruan [33] for some recent results and update references.

Consider the space

$$X := \mathbb{R} \times L^1(0, +\infty)$$

endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0,+\infty)}.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varepsilon^2 \varphi'(0) - \varphi(0) \\ \varepsilon^2 \varphi'' - \varphi' - \mu\varphi \end{pmatrix}$$

with

$$D(A) = \{0\} \times W^{2,1}(0, +\infty).$$

Then

$$X_0 := \overline{D(A)} = \{0\} \times L^1(0, +\infty).$$

Define $H : X_0 \to X$ by

$$H \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \alpha h(\int_0^{+\infty} \gamma(x)\varphi(x)\, dx) \\ 0 \end{pmatrix}.$$

By identifying $u(t)$ to $v(t) = \binom{0}{u(t)}$ the partial differential equation (2.1) can be rewritten as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H\big(v(t)\big), \quad \text{for } t \geqslant 0, \quad \text{and} \quad v(0) = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in \overline{D(A)}. \tag{2.2}$$

In the sequel, for $z \in \mathbb{C}$, $\sqrt{z}$ denotes the principal branch of the general multi-valued function $z^{\frac{1}{2}}$. The branch cut is the negative real axis and the argument of z, denoted by $\arg z$, is π on the upper margin of the branch cut. Then $z = \rho e^{i\theta}$, $\theta \in (-\pi, \pi)$, $\rho > 0$, and $\sqrt{z} = \sqrt{\rho} e^{i\frac{\theta}{2}}$. In the sequel, we will use the following notation:

$$\Omega := \big\{\lambda \in \mathbb{C}:\ \mathrm{Re}(\lambda) > -\mu\big\}, \tag{2.3}$$

and for $\lambda \in \Omega$,

$$\Lambda := 1 + 4\varepsilon^2(\lambda + \mu). \tag{2.4}$$

Since $\lambda \in \Omega$, $\mathrm{Re}(\Lambda) > 0$, we can use the above definition to define $\sqrt{\Lambda}$. Set

$$\sigma^{\pm} := \frac{1 \pm \sqrt{\Lambda}}{2\varepsilon^2}, \tag{2.5}$$

$$\Lambda_0 = 1 + 4\varepsilon^2\mu := \Lambda \quad \text{for } \lambda = 0, \tag{2.6}$$

and

$$\sigma_0^- = \frac{1 - \sqrt{\Lambda_0}}{2\varepsilon^2} := \sigma^- \quad \text{for } \lambda = 0. \tag{2.7}$$

So $\sigma^{\pm}$ are solutions of the equation

$$\varepsilon^2\sigma^2 - \sigma - (\lambda + \mu) = 0. \tag{2.8}$$

Observe that

$$\mathrm{Re}(\sigma^+) > 0 \quad \text{and} \quad \mathrm{Re}(\sigma^-) < 0.$$

Besides these, later on we will also use the following notation:

$$R_0 := \frac{2\alpha\chi}{1 + \sqrt{\Lambda_0}}, \tag{2.9}$$

$$\chi := \int_0^{+\infty} \gamma(x) \exp\bigl(\sigma_0^- x\bigr)\, dx, \tag{2.10}$$

$$\chi_0 := \lim_{\varepsilon \to 0} \chi = \int_0^{+\infty} \gamma(x) \exp(-\mu x)\, dx, \tag{2.11}$$

and

$$\eta(\varepsilon, \alpha) := \frac{1 + \sqrt{\Lambda_0}}{2\chi}\left(1 - \ln \frac{2\alpha\chi}{1 + \sqrt{\Lambda_0}}\right) = \frac{\alpha}{R_0}(1 - \ln R_0). \tag{2.12}$$

If $\gamma(x) \in L^1_+(0, +\infty)$ and $\alpha = c\varepsilon$ with $c > 0$, set

$$\lim_{\varepsilon \to +\infty} R_0 = \frac{c}{\sqrt{\mu}} \int_0^{+\infty} \gamma(x)\, dx := R_0^\infty, \tag{2.13}$$

$$\lim_{\varepsilon \to +\infty} \frac{\eta(\varepsilon, \alpha)}{\varepsilon} = \frac{\sqrt{\mu}}{\int_0^{+\infty} \gamma(x)\, dx}\bigl(1 - \ln R_0^\infty\bigr) := \eta^\infty. \tag{2.14}$$

To study the characteristic equation, for $\lambda \in \Omega$, define

$$\Delta(\varepsilon, \alpha, \lambda) := 1 - \frac{2\eta(\varepsilon, \alpha)}{1 + \sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx. \tag{2.15}$$

Moreover, if we consider

$$\widetilde{\Delta}(\varepsilon,\alpha,\lambda):=\frac{1+\sqrt{\Lambda}}{2}\Delta(\varepsilon,\alpha,\lambda)=-\varepsilon^2\sigma^-+1-\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^-x}\,dx, \tag{2.16}$$

when ε tends to infinity, and take $\alpha=c\varepsilon$, then $\frac{\widetilde{\Delta}(\varepsilon,\alpha,\lambda)}{\varepsilon}$ goes to

$$\widehat{\Delta}(+\infty,c,\lambda):=\sqrt{\lambda+\mu}-\sqrt{\mu}\left(1-\ln\frac{c\int_0^{+\infty}\gamma(x)\,dx}{\sqrt{\mu}}\right). \tag{2.17}$$

Let $L:D(L)\subset X\to X$ be a linear operator on a Banach space X. Denote by $\rho(L)$ the resolvent set of L. The spectrum of L is $\sigma(L)=\mathbb{C}\setminus\rho(L)$. The point spectrum of L is the set

$$\sigma_P(L):=\bigl\{\lambda\in\mathbb{C}\colon\ N(\lambda I-L)\neq\{0\}\bigr\}.$$

Let Y be a subspace of X. Then we denote by $L_Y:D(L_Y)\subset Y\to Y$ the part of L on Y, which is defined by

$$L_Yx=Lx,\quad\forall x\in D(L_Y):=\bigl\{x\in D(L)\cap Y\colon\ Lx\in Y\bigr\}.$$

In particular, we denote A_0 the part of A in $\overline{D(A)}$. So

$$A_0x=Ax\quad\text{for } x\in D(A_0)=\bigl\{x\in D(A)\colon\ Ax\in\overline{D(A)}\bigr\}.$$

We consider the linear operator $\widehat{A}_0:D(\widehat{A}_0)\subset L^1(0,+\infty)\to L^1(0,+\infty)$ defined by

$$\widehat{A}_0(\varphi)=\varepsilon^2\varphi''-\varphi'-\mu\varphi \tag{2.18}$$

with

$$D(\widehat{A}_0)=\bigl\{\varphi\in W^{2,1}\bigl((0,+\infty),\mathbb{R}\bigr)\colon\ \varepsilon^2\varphi'(0)-\varphi(0)=0\bigr\}.$$

We have the following relationship between A_0 and $\widehat{A}_0$:

$$D(A_0)=\{0\}\times D(\widehat{A}_0)$$

and

$$A_0\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}0\\ \widehat{A}_0\varphi\end{pmatrix}.$$

First we have the following lemma about the representation of the resolvent of A.

Lemma 2.2. *We have*

$$\Omega\subset\rho(\widehat{A}_0)=\rho(A_0)=\rho(A),$$

and for each $\lambda\in\Omega$ we obtain the following explicit formula for the resolvent of A:

$$(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix} \quad\Leftrightarrow\quad \varphi(x) = (\lambda I - \widehat{A}_0)^{-1}(\psi)(x) + \alpha\frac{2\exp(\sigma^- x)}{1+\sqrt{\Lambda}}, \tag{2.19}$$

where $(\lambda I - \widehat{A}_0)^{-1}$ *is defined by*

$$\begin{aligned}(\lambda I - \widehat{A}_0)^{-1}(\psi)(x) &= \frac{1}{\sqrt{\Lambda}}\left[\int_0^x \exp\bigl(\sigma^-(x-t)\bigr)\psi(t)\,dt + \int_x^{+\infty}\exp\bigl(\sigma^+(x-t)\bigr)\psi(t)\,dt\right]\\ &\quad + \frac{\sqrt{\Lambda}-1}{(\sqrt{\Lambda}+1)\sqrt{\Lambda}}\left[\int_0^{+\infty}\exp(-\sigma^+ t)\psi(t)\,dt\right]\exp(\sigma^- x).\end{aligned} \tag{2.20}$$

Next we prove the following proposition.

Proposition 2.3. *The following two assertions are satisfied*:

(a) A_0 *the part of* A *in* $\overline{D(A)}$ *is the infinitesimal generator of an analytic semigroup of bounded linear operators* $\{T_{A_0}(t)\}_{t\geqslant 0}$ *on* $\overline{D(A)}$.
(b) A *is a Hille–Yosida operator on* X.

Proof. It is well known that $\widehat{A}_0$ is the infinitesimal generator of an analytic semigroup. In fact, we first consider the linear operator $A_1 : D(A_1) \subset L^1(0,+\infty) \to L^1(0,+\infty)$ defined by $A_1(\varphi) = \varepsilon^2\varphi''$ and $D(A_1) = D(\widehat{A}_0)$. It is well known that A_1 is the infinitesimal generator of an analytic semigroup [20,30]. Consider the linear operator $A_2 : D(A_2) \subset L^1(0,+\infty) \to L^1(0,+\infty)$, $A_2(\varphi) = -\varphi' - \mu\varphi$ with $D(A_2) = W^{2,1}((0,+\infty),\mathbb{R})$. Define $\widehat{A}_0 = A_1 + A_2$. From [37, Theorem 7.3.10], we deduce that $\widehat{A}_0$ is sectorial. Furthermore, for $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\left\|(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ 0\end{pmatrix}\right\| &= |\alpha|\frac{2\int_0^{+\infty}\exp(\sigma^- x)\,dx}{1+\sqrt{\Lambda}}\\ &= |\alpha|\frac{2}{1+\sqrt{\Lambda}} \times \frac{1}{-\sigma^-}\\ &= |\alpha|\frac{2}{1+\sqrt{\Lambda}} \times \frac{2\varepsilon^2}{-1+\sqrt{\Lambda}}\\ &= |\alpha|\frac{4\varepsilon^2}{-1+\Lambda} = |\alpha|\frac{4\varepsilon^2}{4\varepsilon^2(\lambda+\mu)},\end{aligned}$$

so we obtain for $\lambda \in \mathbb{R}$ that

$$\left\|(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ 0\end{pmatrix}\right\| \leqslant \frac{|\alpha|}{\lambda+\mu}, \quad \forall\lambda > -\mu.$$

Finally, it is readily checked that (see the proof of Lemma 4.7)

$$\|T_{\widehat{A}_0}(t)\| \leqslant e^{-\mu t},$$

which implies that

$$\|(\lambda I - \widehat{A}_0)^{-1}\| \leqslant \frac{1}{\lambda+\mu}, \quad \forall\lambda > -\mu.$$

So A is a Hille–Yosida operator. $\square$

Set

$$X_+ := \mathbb{R}_+ \times L^1_+(0,+\infty), \qquad X_{0+} := X_0 \cap X_+.$$

Lemma 2.4. *For $\lambda > 0$ large enough, $(\lambda I - A)^{-1} X_+ \subset X_+$.*

Proof. This lemma follows directly from the explicit formula (2.19) of the resolvent of A. $\square$

By using the results in Thieme [42], Magal [31], and Magal and Ruan [34], we have the following theorem.

Theorem 2.5 *(Existence). There exists a unique continuous semiflow $\{U(t)\}_{t\geqslant 0}$ on X_{0+} such that $\forall x \in X_{0+}$, $t \to U(t)x$ is the unique integrated solution of*

$$\frac{dU(t)x}{dt} = AU(t)x + H\big(U(t)x\big), \qquad U(0)x = x,$$

or equivalently,

$$U(t)x = x + A\int_0^t U(l)x\,dl + \int_0^t H\big(U(l)x\big)\,dl, \quad \forall t \geqslant 0.$$

3. The semiflow and its equilibrium

Now we consider the positive equilibrium solutions of Eq. (2.2).

Lemma 3.1 *(Equilibrium). There exists a unique positive equilibrium of the system* (2.1) *(or Eq.* (2.2)*) if and only if*

$$R_0 := \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}} > 1, \tag{3.1}$$

where χ and Λ_0 are defined in Eq. (2.10) *and Eq.* (2.6)*, respectively. Moreover, when it exists, the positive equilibrium $\bar{v} = \binom{0}{\bar{u}}$ is given by the following formula*

$$\bar{u}(x) = \bar{C}\exp\big(\sigma_0^- x\big), \tag{3.2}$$

where

$$\bar{C} := \frac{1}{\xi\chi}\ln\frac{2\alpha\chi}{1+\sqrt{\Lambda_0}} = \frac{1}{\xi\chi}\ln R_0. \tag{3.3}$$

Proof. We have

$$A\begin{pmatrix}0\\ \bar{u}\end{pmatrix} + H\begin{pmatrix}0\\ \bar{u}\end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{pmatrix}0\\ \bar{u}\end{pmatrix} = (-A)^{-1}\begin{pmatrix}\alpha h(\int_0^{+\infty}\gamma(x)\bar{u}(x)\,dx)\\ 0\end{pmatrix}.$$

According to the explicit formula of the resolvent of A, taking $\lambda = 0$, we have

$$\begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} = (-A)^{-1} \begin{pmatrix} \alpha h(\int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx) \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \bar{u}(x) = \alpha h\left(\int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx\right) \frac{2\exp(\sigma_0^- x)}{1+\sqrt{\Lambda_0}}. \tag{3.4}$$

So

$$\bar{u} \neq 0 \quad \text{iff} \quad \int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx \neq 0.$$

Integrating both sides of Eq. (3.4) by multiplying $\gamma(x)$, then we have

$$\int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx = \alpha h\left(\int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx\right) \int_0^{+\infty} \gamma(x) \frac{2\exp(\sigma_0^- x)}{1+\sqrt{\Lambda_0}}\,dx.$$

In order to have $\bar{u}(x) > 0$, we have

$$\begin{aligned} 1 &= \alpha \exp\left(-\xi \int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx\right) \frac{2\int_0^{+\infty} \gamma(x)\exp(\sigma_0^- x)\,dx}{1+\sqrt{\Lambda_0}} \\ &= \exp\left(-\xi \int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx\right) R_0 \quad \Leftrightarrow \quad \int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx = \frac{1}{\xi}\ln R_0, \end{aligned} \tag{3.5}$$

and the result follows. □

4. Linearized equation and spectral properties

From now on, we set

$$\bar{v} = \begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} \quad \text{with } \bar{u}(x) = \bar{C}\exp(\sigma_0^- x), \ \forall R_0 > 1, \tag{4.1}$$

where $\bar{C} = \frac{1}{\xi\chi}\ln R_0$.

The linearized system of (2.2) around $\bar{v}$ is

$$\frac{dv(t)}{dt} = Av(t) + DH(\bar{v})v(t) \quad \text{for } t \geqslant 0, \ v(t) \in X_0, \tag{4.2}$$

where

$$\begin{aligned} DH(\bar{v})\begin{pmatrix} 0 \\ \varphi \end{pmatrix} &= \begin{pmatrix} \alpha h'(\int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx)\int_0^{+\infty} \gamma(x)\varphi(x)\,dx \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \eta(\varepsilon,\alpha)\int_0^{+\infty} \gamma(x)\varphi(x)\,dx \\ 0 \end{pmatrix} \end{aligned}$$

with

$$\eta(\varepsilon,\alpha) = \alpha h'\left(\int_0^{+\infty} \gamma(x)\bar{u}(x)\,dx\right)$$

and

$$h'(x) = e^{-\xi x}(1 - \xi x).$$

Using (3.5) we obtain

$$\begin{aligned}\eta(\varepsilon,\alpha) &= \frac{\alpha}{R_0}(1 - \ln R_0) \\ &= \frac{1+\sqrt{\Lambda_0}}{2\chi}\left(1 - \ln\frac{2\alpha\chi}{1+\sqrt{\Lambda_0}}\right).\end{aligned}$$

The Cauchy problem (4.2) corresponds to the following linear parabolic differential equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial u(t,x)}{\partial x} = \varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} - \mu u(t,x), \quad t \geqslant 0,\ x \geqslant 0, \\ -\varepsilon^2 \dfrac{\partial u(t,0)}{\partial x} + u(t,0) = \eta(\varepsilon,\alpha) \displaystyle\int_0^{+\infty} \gamma(x) u(t,x)\,dx, \\ u(0,.) = u_0 \in L^1(0,+\infty). \end{cases} \tag{4.3}$$

Next we study the spectral properties of the linearized equation (4.2).

Definition 4.1. Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}_{t\geqslant 0}$ on a Banach space X. We define the *growth bound* $\omega_0(L) \in [-\infty,+\infty)$ of L by

$$\omega_0(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\mathcal{L}(X)})}{t}.$$

The *essential growth bound* $\omega_{0,\mathrm{ess}}(L) \in [-\infty,+\infty)$ of L is defined by

$$\omega_{0,\mathrm{ess}}(L) := \lim_{t\to+\infty} \frac{\ln(\|T_L(t)\|_{\mathrm{ess}})}{t},$$

where $\|T_L(t)\|_{\mathrm{ess}}$ is the essential norm of $T_L(t)$ defined by

$$\|T_L(t)\|_{\mathrm{ess}} = \kappa\big(T_L(t) B_X(0,1)\big),$$

here $B_X(0,1) = \{x \in X:\ \|x\|_X \leqslant 1\}$, and for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0:\ B \text{ can be covered by a finite number of balls of radius} \leqslant \varepsilon\}$$

is the Kuratovsky measure of non-compactness.

In the following result, the existence of the projector was first proved by Webb [45,46] and the fact that there is a finite number of points of the spectrum is proved by Engel and Nagel [20].

Theorem 4.2. *Let $L : D(L) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T_L(t)\}$ on a Banach space X. Then*

$$\omega_0(L) = \max\Big(\omega_{0,\mathrm{ess}}(L), \max_{\lambda\in\sigma(L)\setminus\sigma_{\mathrm{ess}}(L)} \mathrm{Re}(\lambda)\Big).$$

Assume in addition that $\omega_{0,\mathrm{ess}}(L) < \omega_0(L)$. Then for each $\gamma \in (\omega_{0,\mathrm{ess}}(L), \omega_0(L)]$, $\{\lambda \in \sigma(L)\colon \mathrm{Re}(\lambda) \geqslant \gamma\} \subset \sigma_p(L)$ is nonempty, finite and contains only poles of the resolvent of L. Moreover, there exists a finite rank bounded linear projector $\Pi : X \to X$ satisfying the following properties:

(a) $\Pi(\lambda - L)^{-1} = (\lambda - L)^{-1}\Pi$, $\forall \lambda \in \rho(L)$;
(b) $\sigma(L_{\Pi(X)}) = \{\lambda \in \sigma(L)\colon \mathrm{Re}(\lambda) \geqslant \gamma\}$;
(c) $\sigma(L_{(I-\Pi)(X)}) = \sigma(L) \setminus \sigma(L_{\Pi(X)})$.

To simplify the notation, we define $B_\alpha : D(B_\alpha) \subset X \to X$ as

$$B_\alpha x = Ax + DH(\bar{v})x \quad \text{with } D(B_\alpha) = D(A), \tag{4.4}$$

and denote by $(B_\alpha)_0$ the part of B_α on $\overline{D(A)}$.

Lemma 4.3. *For each $\lambda \in \Omega = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) > -\mu\}$, we have*

$$\mathrm{Re}(1 + \sqrt{\Lambda}) > 1,$$

$$\lambda \in \rho(B_\alpha) \quad \Leftrightarrow \quad \Delta(\varepsilon, \alpha, \lambda) \neq 0,$$

and the following explicit formula:

$$\begin{aligned}
&(\lambda I - B_\alpha)^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}\\
&\Leftrightarrow \quad \varphi(x) = (\lambda I - \widehat{A}_0)^{-1}(\psi)(x)\\
&\qquad\qquad + \Delta(\varepsilon, \alpha, \lambda)^{-1}\left[\beta + \eta(\varepsilon, \alpha)\int_0^{+\infty} \gamma(x)(\lambda - \widehat{A}_0)^{-1}(\psi)(x)\,dx\right]\frac{2\exp(\sigma^- x)}{1 + \sqrt{\Lambda}},
\end{aligned}$$

where

$$\Delta(\varepsilon, \alpha, \lambda) := 1 - \frac{2\eta(\varepsilon, \alpha)}{1 + \sqrt{\Lambda}}\int_0^{+\infty} \gamma(x) e^{\sigma^- x}\,dx, \tag{4.5}$$

where $\eta(\varepsilon, \alpha)$, Λ, and σ^- are defined respectively in Eqs. (2.12), (2.4) *and* (2.5).

Remark 4.4. Since by definition of $\sqrt{\cdot}$, $\mathrm{Re}(\sqrt{\Lambda}) > 0$, $\forall \lambda \in \Omega$, we deduce that $\mathrm{Re}(1 + \sqrt{\Lambda}) > 1$.

Proof of Lemma 4.3. Since $\lambda \in \Omega$, from Lemma 2.2, we know that $(\lambda I - A)$ is invertible. Then

$$\lambda I - B_\alpha \text{ is invertible} \quad \Leftrightarrow \quad I - DH(\bar{v})(\lambda I - A)^{-1} \text{ is invertible},$$

and

$$(\lambda I - B_\alpha)^{-1} = (\lambda I - A)^{-1}\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]^{-1}.$$

We also know that $[I - DH(\bar{v})(\lambda I - A)^{-1}]\binom{\hat{\beta}}{\varphi} = \binom{\beta}{\psi}$ is equivalent to $\varphi = \psi$ and

$$\hat{\beta} - \hat{\beta}\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\frac{2\exp(\sigma^- x)}{1+\sqrt{\Lambda}}\,dx = \beta + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda - \widehat{A}_0)^{-1}(\varphi)(x)\,dx.$$

We deduce that $[I - DH(\bar{v})(\lambda I - A)^{-1}]$ is invertible if and only if $\Delta(\varepsilon,\alpha,\lambda) \neq 0$. Moreover,

$$\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix} = \begin{pmatrix}\hat{\beta}\\ \varphi\end{pmatrix}$$

is equivalent to $\varphi = \psi$ and

$$\hat{\beta} = \Delta(\varepsilon,\alpha,\lambda)^{-1}\left[\beta + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda - \widehat{A}_0)^{-1}(\psi)(x)\,dx\right].$$

Therefore,

$$\begin{aligned}
(\lambda I - B_\alpha)^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix} &= (\lambda I - A)^{-1}\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix}\\
&= (\lambda I - A)^{-1}\begin{pmatrix}\hat{\beta}\\ \varphi\end{pmatrix}\\
&= (\lambda I - A)^{-1}\begin{pmatrix}\Delta(\varepsilon,\alpha,\lambda)^{-1}[\beta + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda - \widehat{A}_0)^{-1}(\psi)(x)\,dx]\\ \psi\end{pmatrix}.
\end{aligned}$$

Then by Lemma 2.2, the result follows. $\square$

By using the above explicit formula for the resolvent of B_α we obtain the following lemma.

Lemma 4.5. *If $\lambda_0 \in \sigma(B_\alpha) \cap \Omega$, then we deduce that λ_0 is a simple eigenvalue of B_α if and only if*

$$\frac{d\Delta(\varepsilon,\alpha,\lambda_0)}{d\lambda} \neq 0.$$

Consider a linear operator $(\widehat{B}_\alpha)_0 : D((\widehat{B}_\alpha)_0) \subset L^1(0,+\infty) \to L^1(0,+\infty)$ defined by

$$(\widehat{B}_\alpha)_0(\varphi) = \varepsilon^2\varphi'' - \varphi' - \mu\varphi$$

with

$$D\big((\widehat{B}_\alpha)_0\big) = \left\{\varphi \in W^{2,1}\big((0,+\infty),\mathbb{R}\big):\ \varepsilon^2\varphi'(0) - \varphi(0) + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\varphi(x)\,dx = 0\right\}.$$

Then we have the following lemma.

Lemma 4.6. *For each $\varphi \in L^1(0,+\infty)$ and each $t \geqslant 0$, we have the following equality*:

$$\frac{d}{dt}\int_0^{+\infty}T_{(\widehat{B}_\alpha)_0}(t)(\varphi)(x)\,dx = -\mu\int_0^{+\infty}T_{(\widehat{B}_\alpha)_0}(t)(\varphi)(x)\,dx + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)T_{(\widehat{B}_\alpha)_0}(t)(\varphi)(x)\,dx.$$

Lemma 4.7. *Let Assumption* 2.1 *be satisfied. Then the linear operator B_α is a Hille–Yosida operator and its part $(B_\alpha)_0$ in X_0 satisfies*

$$\omega_{0,\mathrm{ess}}\big((B_\alpha)_0\big) \leqslant -\mu. \tag{4.6}$$

Proof. Since $DH(\bar v)$ is a bounded linear operator and A is a Hille–Yosida operator, it follows that B_α is also a Hille–Yosida operator. By Lemma 4.6 with $\eta(\varepsilon,\alpha)=0$, we have

$$\begin{aligned}\|T_{\widehat A_0}(t)\varphi\|_{L^1(0,+\infty)} &= \big\||T_{\widehat A_0}(t)\varphi|\big\|_{L^1(0,+\infty)} \leqslant \big\|T_{\widehat A_0}(t)|\varphi|\big\|_{L^1(0,+\infty)}\\ &= \int_0^\infty T_{\widehat A_0}(t)|\varphi|(x)\,dx = \int_0^\infty e^{-\mu t}|\varphi|(x)\,dx = e^{-\mu t}\|\varphi\|_{L^1(0,+\infty)},\end{aligned}$$

then

$$\omega_{0,\mathrm{ess}}(\widehat A_0) \leqslant -\mu.$$

By using the result in Thieme [44] or Ducrot, Liu and Magal [19, Theorem 1.2], we deduce that

$$\omega_{0,\mathrm{ess}}\big((\widehat B_\alpha)_0\big) \leqslant \omega_{0,\mathrm{ess}}(\widehat A_0) \leqslant -\mu,$$

and the result follows. □

Lemma 4.8.

$$\sigma\big((B_\alpha)_0\big)\cap\Omega = \sigma_p\big((B_\alpha)_0\big)\cap\Omega = \big\{\lambda\in\Omega\colon\ \Delta(\varepsilon,\alpha,\lambda)=0\big\},$$

where

$$\Delta(\varepsilon,\alpha,\lambda) = 1 - \frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda}}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx.$$

Proof. By Lemma 4.7, we have

$$\sigma\big((B_\alpha)_0\big)\cap\Omega = \sigma_p\big((B_\alpha)_0\big)\cap\Omega,$$

and by Lemma 4.3, the result follows. □

Later on, we will study the eigenvalues of the *characteristic equation*

$$1 = \frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda}}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx \quad \text{with } \lambda\in\Omega, \tag{4.7}$$

or equivalently, the following equation

$$-\varepsilon^2\sigma^- + 1 = \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx, \tag{4.8}$$

where σ^- is the solution of

$$\varepsilon^2\sigma^2 - \sigma = \lambda + \mu, \quad \lambda \in \Omega, \tag{4.9}$$

with $\mathrm{Re}(\sigma^-) < 0$.

5. Local stability

This section is devoted to the local stability of the positive steady state $\bar{v}$. Recall that this positive equilibrium exists and is unique if and only if $R_0 > 1$.

Lemma 5.1. *If* $R_0 = \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}} > 1$, *then* $\lambda = 0$ *is not a root of the characteristic equation* $\Delta(\varepsilon, \alpha, \lambda) = 0$, *where* $\Delta(\varepsilon, \alpha, \lambda)$ *is explicitly defined in* (4.5)*.*

Proof. We have

$$\begin{aligned}
\Delta(\varepsilon, \alpha, 0) &= 1 - \frac{2\eta(\varepsilon, \alpha)}{1 + \sqrt{\Lambda_0}} \int_0^\infty \gamma(x) e^{\sigma_0^- x}\, dx \\
&= 1 - \frac{2\chi}{1 + \sqrt{\Lambda_0}} \eta(\varepsilon, \alpha) \\
&= 1 - \frac{R_0}{\alpha}\left(\frac{\alpha}{R_0}(1 - \ln R_0)\right) \\
&= \ln R_0.
\end{aligned}$$

Since $R_0 > 1$, we obtain that

$$\Delta(\varepsilon, \alpha, 0) > 0$$

and the result follows. $\square$

Lemma 5.2. *If* λ *is a root of the characteristic equation and* $\mathrm{Re}(\lambda) \geqslant 0$, *then we have*

$$\mathrm{Re}(\sigma^-) < \sigma_0^-$$

and

$$\mathrm{Re}(\sqrt{\Lambda}) > \sqrt{\Lambda_0} > 1.$$

Proof. Since σ^- is the root of

$$\varepsilon^2\sigma^2 - \sigma - (\mu + \lambda) = 0$$

with

$$\mathrm{Re}(\sigma^-) < 0,$$

we have the following relationship between $\mathrm{Re}(\sigma^-)$, $\mathrm{Im}(\sigma^-)$, $\mathrm{Re}(\lambda)$, and $\mathrm{Im}(\lambda)$

$$\varepsilon^2 \operatorname{Re}(\sigma^-)^2 - \operatorname{Re}(\sigma^-) - \mu = \operatorname{Re}(\lambda) + \varepsilon^2 \operatorname{Im}(\sigma^-)^2, \tag{5.1}$$

$$2\varepsilon^2 \operatorname{Re}(\sigma^-) \operatorname{Im}(\sigma^-) - \operatorname{Im}(\sigma^-) = \operatorname{Im}(\lambda). \tag{5.2}$$

If $\operatorname{Re}(\lambda) = 0$ and $\operatorname{Im}(\sigma^-) = 0$, then by using (5.2) we have $\operatorname{Im}(\lambda) = 0$. So $\lambda = 0$, which is impossible by Lemma 5.1. Thus if $\operatorname{Re}(\lambda) \geqslant 0$, we have

$$\operatorname{Re}(\lambda) + \varepsilon^2 \operatorname{Im}(\sigma^-)^2 > 0.$$

By using (5.1) we deduce that

$$\varepsilon^2 \operatorname{Re}(\sigma^-)^2 - \operatorname{Re}(\sigma^-) - \mu > 0.$$

Since $\operatorname{Re}(\sigma^-) < 0$, if follows that

$$\operatorname{Re}(\sigma^-) < \frac{1 - \sqrt{1 + 4\varepsilon^2 \mu}}{2\varepsilon^2} = \sigma_0^-.$$

Now since $\sigma^- = \frac{1-\sqrt{\Lambda}}{2\varepsilon^2}$ and $\sigma_0^- = \frac{1-\sqrt{\Lambda_0}}{2\varepsilon^2}$, we deduce that $\operatorname{Re}(\sqrt{\Lambda}) > \sqrt{\Lambda_0}$. $\square$

Theorem 5.3. *Let Assumption* 2.1 *be satisfied. If*

$$1 < R_0 \leqslant e^2,$$

then the positive equilibrium $\bar{v}$ of the system (2.1) *is locally asymptotically stable.*

Proof. We consider the characteristic equation

$$1 - \varepsilon^2 \sigma^- = \eta(\varepsilon, \alpha) \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx. \tag{5.3}$$

By Lemma 5.2, if $\operatorname{Re}(\lambda) \geqslant 0$, we must have

$$\operatorname{Re}(\sigma^-) < \sigma_0^-.$$

Then we derive from Eq. (5.3) that

$$\begin{aligned}
\left|\varepsilon^2 \sigma^- - 1\right| &= \left|\eta(\varepsilon, \alpha) \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx\right| \\
&\leqslant \left|\eta(\varepsilon, \alpha)\right| \int_0^{+\infty} \gamma(x) e^{\operatorname{Re}(\sigma^-) x}\, dx \\
&< \left|\eta(\varepsilon, \alpha)\right| \int_0^{+\infty} \gamma(x) e^{\sigma_0^- x}\, dx = \left|\eta(\varepsilon, \alpha)\right| \chi.
\end{aligned}$$

On the other hand, if $\mathrm{Re}(\lambda) \geqslant 0$, then by Lemma 5.2, we have

$$\left|\varepsilon^2\sigma^- - 1\right| = \left|\frac{1+\sqrt{\Lambda}}{2}\right| > \mathrm{Re}\left(\frac{1+\sqrt{\Lambda}}{2}\right) > \frac{1+\sqrt{\Lambda_0}}{2}.$$

So if

$$\left|\eta(\varepsilon,\alpha)\right|\chi \leqslant \frac{1+\sqrt{\Lambda_0}}{2}, \tag{5.4}$$

then there will be no roots of the characteristic equation with non-negative real part.

By (2.9) and (2.12), the above inequality is equivalent to

$$\frac{\alpha}{R_0}|\ln R_0 - 1| \leqslant \frac{\alpha}{R_0},$$

and the result follows. □

Next let ε go to infinity, we study the characteristic equation

$$-\varepsilon^2\sigma^- + 1 = \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx,$$

where

$$\sigma^- = \frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon^2} \sim O\left(\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to +\infty.$$

In order to obtain the limit characteristic equation when ε tends to infinity, we rewrite the characteristic equation as

$$-\varepsilon\sigma^- + \frac{1}{\varepsilon} - \frac{\eta(\varepsilon,\alpha)}{\varepsilon}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx = 0. \tag{5.5}$$

To simplify the notation, we set

$$\widetilde{\Delta}(\varepsilon,\alpha,\lambda) = -\varepsilon^2\sigma^- + 1 - \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx.$$

Then the rewritten equation (5.5) becomes

$$\frac{\widetilde{\Delta}(\varepsilon,\alpha,\lambda)}{\varepsilon} = 0.$$

Note that

$$\chi = \int_0^{+\infty}\gamma(x)\exp\left(\sigma_0^- x\right)dx$$

and

$$\sigma_0^- = \frac{1 - \sqrt{1 + 4\varepsilon^2 \mu}}{2\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to +\infty.$$

It is important to observe that to obtain a positive equilibrium, by Lemma 3.1 we must have

$$R_0 > 1,$$

or equivalently,

$$\alpha > \frac{1 + \sqrt{\Lambda_0}}{2\chi}, \quad \text{where } \Lambda_0 = 1 + 4\varepsilon^2 \mu. \tag{5.6}$$

We make the following assumption.

Assumption 5.4. Assume that

$$\gamma(x) \in L^1_+(0, +\infty), \qquad \alpha = c\varepsilon,$$

for some $c > 0$.

Under Assumption 5.4, we have

$$\chi \to \int_0^{+\infty} \gamma(x)\, dx,$$

$$\frac{\sqrt{\Lambda_0}}{\alpha} = \frac{\sqrt{1 + 4\varepsilon^2 \mu}}{c\varepsilon} \to \frac{2\sqrt{\mu}}{c} \quad \text{as } \varepsilon \to +\infty,$$

so we obtain

$$R_0 \to \frac{c}{\sqrt{\mu}} \int_0^{+\infty} \gamma(x)\, dx := R_0^\infty \quad \text{as } \varepsilon \to +\infty,$$

$$\frac{\eta(\varepsilon, c\varepsilon)}{\varepsilon} \to \frac{\sqrt{\mu}}{\int_0^{+\infty} \gamma(x)\, dx} \left(1 - \ln R_0^\infty\right) := \eta^\infty \quad \text{as } \varepsilon \to +\infty.$$

Lemma 5.5. *Let Assumptions* 2.1 *and* 5.4 *be satisfied. Then there exists* $\hat{\varepsilon} > 0$ *such that* $\forall \varepsilon > \hat{\varepsilon}$, *if*

$$\lambda \in \Omega \quad \textit{and} \quad \widetilde{\Delta}(\varepsilon, \alpha, \lambda) = 0,$$

then

$$|\lambda| < \mu + \mu\left(1 - \ln R_0^\infty\right)^2 + 1.$$

Proof. If $\lambda \in \Omega$, $\widetilde{\Delta}(\varepsilon, \alpha, \lambda) = 0$, then

$$-\varepsilon^2\sigma^- + 1 = \eta(\varepsilon, \alpha) \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx$$

with

$$\operatorname{Re}(\sigma^-) < 0.$$

So we have

$$\left|\varepsilon^2\sigma^- - 1\right| \leqslant \left|\eta(\varepsilon, \alpha)\right| \int_0^{+\infty} \gamma(x)\, dx,$$

thus

$$|\sigma^-| \leqslant \frac{|\eta(\varepsilon, \alpha)|}{\varepsilon^2} \int_0^{+\infty} \gamma(x)\, dx + \frac{1}{\varepsilon^2}.$$

Observe that σ^- satisfies

$$\varepsilon^2(\sigma^-)^2 - \sigma^- = \lambda + \mu,$$

we have

$$|\lambda| \leqslant |\sigma^-|\left|\varepsilon^2\sigma^- - 1\right| + \mu \leqslant \left(\frac{|\eta(\varepsilon, \alpha)|}{\varepsilon} \int_0^{+\infty} \gamma(x)\, dx\right)^2 + \frac{|\eta(\varepsilon, \alpha)| \int_0^{+\infty} \gamma(x)\, dx}{\varepsilon^2} + \mu. \tag{5.7}$$

Since when ε tends to infinity, the right-hand side of the inequality (5.7) goes to

$$\mu + \mu\left(1 - \ln R_0^\infty\right)^2,$$

and the result follows. □

Lemma 5.6 *(Convergence). Let Assumptions* 2.1 *and* 5.4 *be satisfied. Then we have*

$$\lim_{\varepsilon \to +\infty} \lim_{\lambda \to \hat{\lambda}} \frac{\widetilde{\Delta}(\varepsilon, c\varepsilon, \lambda)}{\varepsilon} = \widehat{\Delta}(+\infty, c, \hat{\lambda}),$$

where

$$\widehat{\Delta}(+\infty, c, \hat{\lambda}) := \sqrt{\hat{\lambda} + \mu} - \sqrt{\mu}\left(1 - \ln R_0^\infty\right).$$

Proof. We have

$$\lim_{\varepsilon\to+\infty}\lim_{\lambda\to\hat\lambda}\varepsilon\sigma^- = \lim_{\varepsilon\to+\infty}\lim_{\lambda\to\hat\lambda}\frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon} = -\sqrt{\hat\lambda+\mu},$$

$$\lim_{\varepsilon\to+\infty}\lim_{\lambda\to\hat\lambda}\sigma^- = \lim_{\varepsilon\to+\infty}\lim_{\lambda\to\hat\lambda}\frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon^2} = 0,$$

and we deduce for $\lambda = 0$ that

$$\lim_{\varepsilon\to+\infty}\chi = \lim_{\varepsilon\to+\infty}\int_0^{+\infty}\gamma(x)e^{\sigma_0^- x}\,dx = \int_0^{+\infty}\gamma(x)\,dx.$$

Since

$$\lim_{\varepsilon\to+\infty}\frac{\eta(\varepsilon,c\varepsilon)}{\varepsilon} = \frac{\sqrt{\mu}}{\int_0^{+\infty}\gamma(x)\,dx}\bigl(1-\ln R_0^\infty\bigr),$$

we have

$$\lim_{\varepsilon\to+\infty}\lim_{\lambda\to\hat\lambda}\frac{\widetilde\Delta(\varepsilon,c\varepsilon,\lambda)}{\varepsilon} = \lim_{\varepsilon\to+\infty}\lim_{\lambda\to\hat\lambda}\left(-\varepsilon\sigma^- + \frac{1}{\varepsilon} - \frac{\eta(\varepsilon,\alpha)}{\varepsilon}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx\right)$$
$$= \sqrt{\hat\lambda+\mu} - \sqrt{\mu}\bigl(1-\ln R_0^\infty\bigr).$$

This completes the proof of the lemma. □

Remark 5.7. From Lemma 5.6, the limit equation of the characteristic equation when ε tends to infinity is

$$\sqrt{\lambda+\mu} = \sqrt{\mu}\bigl(1-\ln R_0^\infty\bigr), \quad \lambda\in\Omega. \tag{5.8}$$

Eq. (5.8) has at most one real negative solution. Indeed, if $q := 1-\ln R_0^\infty \in [0,1)$, then

$$\sqrt{\lambda+\mu} = \sqrt{\mu}q,$$

i.e.,

$$\lambda = -\mu\bigl(1-q^2\bigr) < 0;$$

and if $q := 1-\ln R_0^\infty \in (-\infty,0)$, then $\sqrt{\lambda+\mu} < 0$. Since by construction we have $\mathrm{Re}(\sqrt{\lambda+\mu}) \geqslant 0$, $\lambda\in\Omega$, so there is no solution.

Theorem 5.8. *Let Assumptions* 2.1 *and* 5.4 *be satisfied, and assume that* $R_0^\infty > 1$. *Then for each* $\varepsilon > 0$ *large enough the positive equilibrium* $\bar v$ *of system* (2.1) *is locally asymptotically stable.*

Proof. We claim that if Assumption 5.4 is satisfied, then for ε positive and large enough, there are no roots of the characteristic equation with non-negative real part. Otherwise, we can find a sequence $\varepsilon_n \to +\infty$, and a sequence λ_n such that

$$\mathrm{Re}(\lambda_n) \geqslant 0, \qquad \widetilde{\Delta}(\varepsilon_n, c\varepsilon_n, \lambda_n) = 0.$$

By using Lemma 5.5, we know that λ_n is bounded for each ε positive and large enough. Thus we can find a subsequence of λ_n which converges to $\hat{\lambda}$. We also denote this subsequence by λ_n. Obviously, we have

$$\mathrm{Re}(\hat{\lambda}) \geqslant 0, \qquad \widetilde{\Delta}(\varepsilon_n, c\varepsilon_n, \lambda_n) = 0. \tag{5.9}$$

Let n tend to infinity in Eq. (5.9). Then by Lemma 5.6, we have

$$\widehat{\Delta}(+\infty, c, \hat{\lambda}) = 0$$

with

$$\mathrm{Re}(\hat{\lambda}) \geqslant 0,$$

which leads to a contradiction with Remark 5.7, and the result follows. □

Remark 5.9. In order to show that, under Assumptions 2.1 and 5.4, Theorem 5.8 is more general than Theorem 5.3, we observe that

$$R_0 \to R_0^\infty = \frac{c}{\sqrt{\mu}} \int_0^{+\infty} \gamma(x)\, dx, \quad \varepsilon \to +\infty.$$

So when $\varepsilon \to +\infty$, the condition of Theorem 5.3 gives

$$1 < R_0^\infty < e^2.$$

6. Hopf bifurcation

In this section we will study the existence of Hopf bifurcation when α is regarded as the bifurcation parameter of the system. By Theorem 5.3 we already knew that the positive equilibrium $\bar{v}$ of the system (2.1) is locally asymptotically stable if

$$1 < R_0 \leqslant e^2, \qquad R_0 = \frac{2\alpha\chi}{1 + \sqrt{\Lambda_0}},$$

which corresponds to $\alpha \in (\hat{\alpha}_0, \hat{\alpha}_1]$, where

$$\hat{\alpha}_0 := \frac{1 + \sqrt{\Lambda_0}}{2\chi} \quad \text{and} \quad \hat{\alpha}_1 := \frac{1 + \sqrt{\Lambda_0}}{2\chi} e^2.$$

For a fixed value of $\varepsilon > 0$, we will study the existence of a bifurcation value $\alpha^* > \hat{\alpha}_1$. Recall the characteristic equation $\Delta(\varepsilon, \alpha, \lambda) = 0$, where

$$\Delta(\varepsilon,\alpha,\lambda) = 1 - \frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda}}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx$$
$$= 1 - \frac{\eta(\varepsilon,\alpha)}{1-\varepsilon^2\sigma^-}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx,$$

in which

$$\sigma^- = \frac{1-\sqrt{\Lambda}}{2\varepsilon^2}$$

and

$$\eta(\varepsilon,\alpha) = \frac{\alpha}{R_0}(1-\ln R_0) < 0 \quad \text{for } \alpha > \hat{\alpha}_1.$$

6.1. Existence of purely imaginary eigenvalues

We consider the characteristic equation of $\sigma^- \in \mathbb{C}$: $\mathrm{Re}(\sigma^-) < 0$,

$$\varepsilon^2\sigma^- - 1 = -\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}\,dx, \tag{6.1}$$
$$\varepsilon^2(\sigma^-)^2 - \sigma^- = \lambda + \mu \tag{6.2}$$

with

$$\mathrm{Re}(\lambda) > -\mu.$$

Set

$$\sigma^- = -(a+ib).$$

Then $a > 0$, from Eq. (6.2) we obtain

$$\varepsilon^2\big(a^2 - b^2 + 2abi\big) + a + ib = \lambda + \mu,$$

i.e.,

$$\begin{cases} \varepsilon^2\big(a^2-b^2\big) + a = \mathrm{Re}(\lambda) + \mu, \\ 2\varepsilon^2 ab + b = \mathrm{Im}(\lambda). \end{cases} \tag{6.3}$$

It follows that

$$b = \frac{\mathrm{Im}(\lambda)}{2a\varepsilon^2 + 1}. \tag{6.4}$$

Thus, if we look for purely imaginary roots $\lambda = \omega i$ with $\omega > 0$, then from Eq. (6.4) we have $b > 0$. Since $a > 0$, by the first equation in (6.3) we obtain

$$a = \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{2\varepsilon^2},$$

and by the second equation in (6.3) we obtain

$$\omega = b\left(2\varepsilon^2 a + 1\right) = b\sqrt{1 + 4\varepsilon^2\left(\mu + \varepsilon^2 b^2\right)}.$$

On the other hand, from Eq. (6.1), we have

$$\varepsilon^2(a + ib) + 1 = \eta(\varepsilon, \alpha) \int_0^{+\infty} \gamma(x) e^{-(a+ib)x}\, dx.$$

The rest of this subsection is devoted to the existence of purely imaginary roots of the characteristic equation when

$$\gamma(x) = (x - \tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x)$$

with $\tau > 0$, $\beta \geqslant 0$, and $n \in \mathbb{N}$.

Since the function $\gamma(x)$ must be bounded, we study the following two cases:

(a) $\beta = 0$ and $n = 0$.
(b) $\beta > 0$ and $n \geqslant 0$.

6.1.1. Case (a)

In this subsection, we will make the following assumption.

Assumption 6.1. Assume that $\varepsilon > 0$ and $\gamma(x) = 1_{[\tau,+\infty)}(x)$ for some $\tau > 0$.

As we described in the introduction, when $\gamma(x) = 1_{[\tau,+\infty)}(x)$ the original system (1.1) can be viewed as a stochastic perturbation (in the transport term) of a delay differential equation. So in this section we investigate the bifurcation properties of this problem in terms of parameters α and ε.

Under Assumption 6.1, we have

$$\int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx = -\frac{e^{\sigma^- \tau}}{\sigma^-},$$

and the characteristic equation becomes

$$\operatorname{Re}(\sigma^-) < 0$$

and

$$\varepsilon^2(\sigma^-)^2 - \sigma^- = \eta(\varepsilon, \alpha) e^{\sigma^- \tau},$$

i.e.,

$$\varepsilon^2\left[a^2 - b^2 + 2abi\right] + a + ib = \eta(\varepsilon, \alpha) e^{-a\tau}\left[\cos(b\tau) - i \sin(b\tau)\right].$$

If we look at the curve $\lambda = \omega i$ with $\omega > 0$ and set

$$a := \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{2\varepsilon^2},$$
$$\omega := \varepsilon^2 2ab + b = b\sqrt{1 + 4\varepsilon^2\left(\mu + \varepsilon^2 b^2\right)},$$

then

$$\varepsilon^2\left(a^2 - b^2\right) + a = \mu \quad \text{and} \quad a > 0,$$

and we obtain

$$\mu + i\omega = \eta(\varepsilon, \alpha)e^{-a\tau}e^{-ib\tau} = \eta(\varepsilon, \alpha)e^{-a\tau}\left[\cos(b\tau) - i\sin(b\tau)\right].$$

Now we fix

$$\eta(\varepsilon, \alpha) = -ce^{a\tau} = -ce^{\frac{-1+\sqrt{1+4\varepsilon^2(\mu+\varepsilon^2 b^2)}}{2\varepsilon^2}\tau}$$

for some constant $c > 0$. We obtain

$$\mu + i\omega = ce^{-ib\tau} = -c\left[\cos(b\tau) - i\sin(b\tau)\right].$$

We must have

$$c = \sqrt{\mu^2 + \omega^2} = \sqrt{\mu^2 + \left(2\varepsilon^2 ab + b\right)^2},$$
$$\tan(b\tau) = -\frac{\omega}{\mu} = -\frac{\varepsilon^2 2ab + b}{\mu} = -\frac{b\sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{\mu},$$

and impose that

$$\sin(b\tau) = \frac{\omega}{c} > 0 \quad \text{and} \quad \cos(b\tau) = -\frac{\mu}{c} < 0.$$

From the above computation we obtain the following proposition.

Proposition 6.2. *Let $\varepsilon > 0$, $\tau > 0$ and $\mu > 0$ be fixed. Then the characteristic equation has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if there exists $b > 0$ which is a solution of equation*

$$\tan(b\tau) = -\frac{b\sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{\mu} \tag{6.5}$$

with

$$\sin(b\tau) > 0, \tag{6.6}$$

and

$$\omega = b\sqrt{1 + 4\varepsilon^2\left(\mu + \varepsilon^2 b^2\right)}, \qquad \eta(\varepsilon, \alpha) = \hat{\eta}(\varepsilon, a, b),$$

where

$$\hat{\eta}(\varepsilon, a, b) := -ce^{a\tau},$$

$$c = \sqrt{\mu^2 + \left(2\varepsilon^2 ab + b\right)^2}, \qquad a = \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{2\varepsilon^2}.$$

Moreover, for each $k \in \mathbb{N}$, there exists a unique $b_k \in ((2k + \frac{1}{2})\frac{\pi}{\tau}, (2k+1)\frac{\pi}{\tau})$ (which is a function of τ, μ and ε) satisfying (6.5) *and* (6.6)*.*

Proof. Set $\hat{b} = b\tau$. Then Eq. (6.5) becomes

$$\tan(\hat{b}) = -\frac{(\sqrt{1 + 4\varepsilon^2(\mu + (\frac{\varepsilon}{\tau})^2 \hat{b}^2)})\hat{b}}{\mu\tau}. \tag{6.7}$$

We observe that the right-hand side of Eq. (6.7) is a strictly monotone decreasing function of $\hat{b}$, and since the function $\tan(x)$ is increasing, we deduce that Eq. (6.7) has a unique solution $b_m \in ((m - \frac{1}{2})\pi, m\pi)$ for each $m \geqslant 1$, $m \in \mathbb{N}$. But since we need to impose $\sin(b_m\tau) > 0$, the result follows. □

6.1.2. Bifurcation diagrams for case (a)

We obtain a sequence $b_k \in ((2k + \frac{1}{2})\frac{\pi}{\tau}, (2k+1)\frac{\pi}{\tau})$ satisfying (6.5) and (6.6). Moreover, we have

$$\eta(\varepsilon, \alpha_k) = \hat{\eta}(\varepsilon, a_k, b_k),$$

where

$$\hat{\eta}(\varepsilon, a_k, b_k) = -c_k e^{a_k\tau},$$

$$c_k = \sqrt{\mu^2 + \left(2\varepsilon^2 a_k b_k + b_k\right)^2}, \qquad a_k = \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b_k^2)}}{2\varepsilon^2},$$

and obtain the following bifurcation curves

$$\alpha_k \frac{1}{R_0}\left(\ln(R_0) - 1\right) = c_k e^{a_k\tau}.$$

We can rewrite the bifurcation curves as

$$\ln(R_0) = \frac{R_0}{\alpha_k} c_k e^{a_k\tau} + 1.$$

Thus

$$R_0 = e^{[1 + c_k e^{a_k\tau} \frac{R_0}{\alpha_k}]}.$$

But $R_0 = \frac{2\alpha_k\chi}{1+\sqrt{\Lambda_0}}$, so

$$\alpha_k = \frac{1 + \sqrt{\Lambda_0}}{2\chi} \exp\left(1 + c_k e^{a_k\tau} \frac{2\chi}{1 + \sqrt{\Lambda_0}}\right),$$

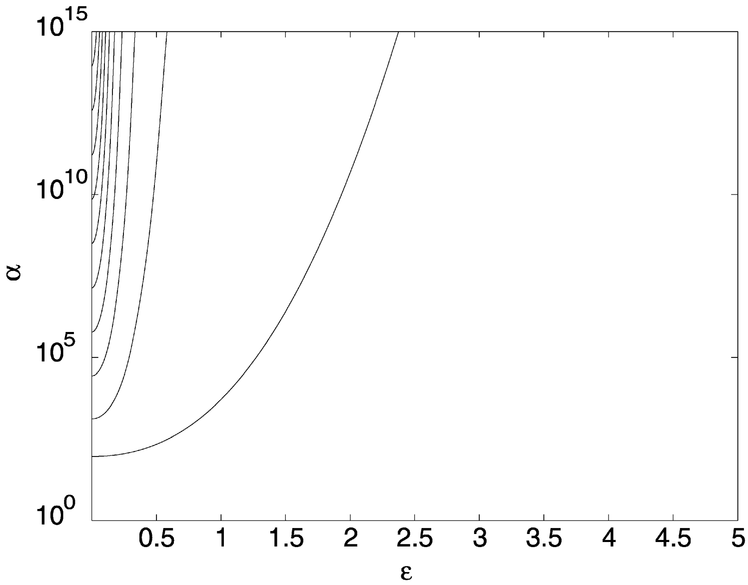

Fig. 6.1. A few bifurcation curves given by (6.5), (6.6) and (6.8) in the (ε, α)-plane for $\tau = 2$ and $\mu = 1$.

where

$$\chi = \int_0^{+\infty} \gamma(x) e^{\sigma_0^- x}\, dx = -\frac{e^{\sigma_0^- \tau}}{\sigma_0^-},$$

$$\sigma_0^- = \frac{1 - \sqrt{\Lambda_0}}{2\varepsilon^2}, \qquad \Lambda_0 = 1 + 4\varepsilon^2 \mu.$$

So

$$\frac{2\chi}{1 + \sqrt{\Lambda_0}} = \frac{4\varepsilon^2 e^{(\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2})\tau}}{\Lambda_0 - 1} = \frac{e^{(\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2})\tau}}{\mu}.$$

Hence, we obtain bifurcation curves

$$\alpha_k = \mu e^{-(\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2})\tau} \exp\left(c_k e^{a_k \tau} \frac{e^{\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2}\tau}}{\mu} + 1 \right). \tag{6.8}$$

See Figs. 6.1 and 6.2.

Remark 6.3. Note that for any fixed $\varepsilon > 0$, α is a strictly increasing function of b, and as for each $k \in \mathbb{N}$, $b_k < b_{k+1}$, so the bifurcation curves cannot cross each other.

6.1.3. Special case $\varepsilon = 0$

In that case, we obtain a characteristic equation which corresponds to delay differential equations. Then the characteristic equation becomes

$$-\sigma^- = \eta(0, \alpha) e^{\sigma^- \tau}, \tag{6.9}$$

where

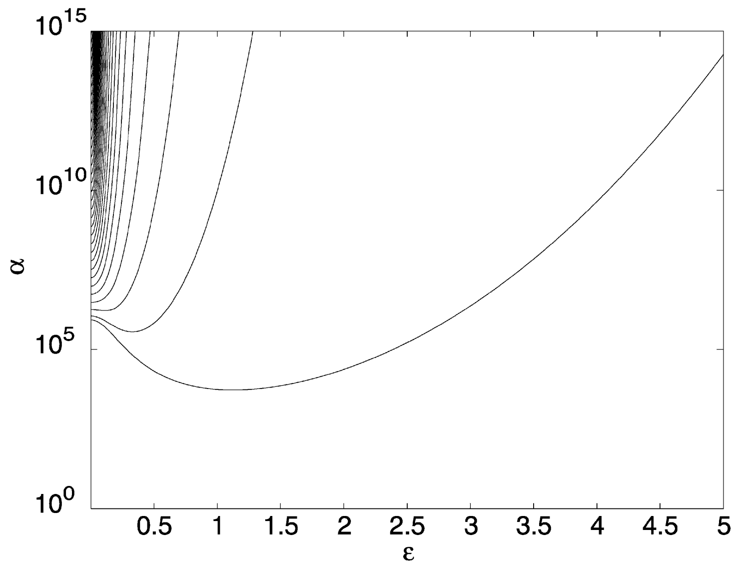

Fig. 6.2. A family of bifurcation curves given by (6.5), (6.6) and (6.8) in the (ε, α)-plane for $\tau = 2$ and $\mu = 5$.

$$\eta(0,\alpha) = \frac{1}{\chi_0}\bigl(1 - \ln(\alpha\chi_0)\bigr) \quad \text{with } \chi_0 = \int_0^\infty \gamma(x) e^{-\mu x}\,dx = \frac{e^{-\mu\tau}}{\mu},$$
$$-\sigma^- = \lambda + \mu.$$

Corresponding to Proposition 6.2, we have the following proposition.

Proposition 6.4. *Let $\tau > 0$ and $\mu > 0$ be fixed. Then the characteristic equation* (6.9) *has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if*

$$\tan(\omega\tau) = -\frac{\omega}{\mu}, \tag{6.10}$$
$$\sin(\omega\tau) > 0, \tag{6.11}$$

and

$$\eta(0,\alpha) = -\sqrt{\mu^2 + \omega^2} e^{\mu\tau}.$$

Moreover, for each $k \in \mathbb{N}$, there exists a unique $\omega_k \in ((2k + \frac{1}{2})\frac{\pi}{\tau}, (2k+1)\frac{\pi}{\tau})$, which satisfies Eqs. (6.10) *and* (6.11) *(and is a function of τ and μ).*

In this case, the bifurcation curves are given by

$$\alpha_k = \mu e^{\mu\tau} \exp\bigl(1 + \sqrt{\mu^2 + \omega_k^2}\,\mu^{-1}\bigr),$$

where ω_k is described in Proposition 6.4, $k \in \mathbb{N}$.

6.1.4. Case (b)

In this subsection, we will make the following assumption.

Assumption 6.5. Assume that $\varepsilon > 0$ and

$$\gamma(x) = (x - \tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x)$$

for some $n \geqslant 0$, $\tau > 0$, and $\beta > 0$.

When $\varepsilon = 0$, this problem corresponds to the example treated in [35]. Nevertheless, the existence of purely imaginary solutions was obtained implicitly in [35]. Here we extend this study to the case when $\varepsilon > 0$ and we specify the bifurcation diagram when $\varepsilon = 0$. Under Assumption 6.5, we have

$$\begin{aligned}
\int_0^{+\infty} \gamma(x)e^{\sigma^- x}\,dx &= e^{\beta\tau} \int_\tau^{+\infty} (x-\tau)^n e^{(\sigma^- - \beta)x}\,dx \\
&= e^{\beta\tau} \int_0^{+\infty} s^n e^{(\sigma^- - \beta)(s+\tau)}\,ds \\
&= -e^{\beta\tau} e^{(\sigma^- - \beta)\tau} \int_{-\infty}^{0} \left(\frac{l}{(\sigma^- - \beta)}\right)^n e^l \frac{1}{(\sigma^- - \beta)}\,dl \\
&= \frac{-e^{\sigma^-\tau}}{(\sigma^- - \beta)^{n+1}} \int_0^{+\infty} (-1)^n x^n e^{-x}\,dx \\
&= \frac{(-1)^{n+1} e^{\sigma^-\tau} n!}{(\sigma^- - \beta)^{n+1}} \\
&= \frac{n! e^{\sigma^-\tau}}{(\beta - \sigma^-)^{n+1}}.
\end{aligned}$$

So the characteristic equation becomes $\Delta(\varepsilon, \alpha, \lambda) = 0$, where

$$\Delta(\varepsilon, \alpha, \lambda) = 1 - \frac{\eta(\varepsilon, \alpha)}{1 - \varepsilon^2 \sigma^-} \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\,dx = 1 - \frac{\eta(\varepsilon, \alpha) n! e^{\sigma^-\tau}}{(1 - \varepsilon^2\sigma^-) \times (-\sigma^- + \beta)^{n+1}}.$$

First we give the following lemma to show that under Assumption 6.5, for any given $\varepsilon > 0$ and $\alpha > 0$, there exists at most one pair of purely imaginary solutions of the characteristic equation.

Lemma 6.6. *Let Assumptions* 2.1 *and* 6.5 *be satisfied. Then for each real number* δ_1, *there exists at most one* $\delta_2 \in (0, +\infty)$ *such that if*

$$\lambda \in \Omega, \quad \mathrm{Re}(\lambda) = \delta_1 \quad \textit{and} \quad \Delta(\varepsilon, \alpha, \lambda) = 0,$$

then

$$\mathrm{Im}(\lambda) = \pm\delta_2.$$

Proof. Since $\Delta(\varepsilon, \alpha, \lambda) = 0$, we obtain

$$1 - \frac{\eta(\varepsilon, \alpha) n! e^{\sigma^-\tau}}{(1 - \varepsilon^2\sigma^-) \times (-\sigma^- + \beta)^{n+1}} = 0, \tag{6.12}$$

where σ^- is the solution of

$$\varepsilon^2\sigma^2 - \sigma = \lambda + \mu, \quad \lambda \in \Omega, \tag{6.13}$$

with $\mathrm{Re}(\sigma^-) < 0$.

From (6.13) we have

$$\mathrm{Im}(\sigma^-)^2 = \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-) + \mathrm{Re}(\lambda) + \mu}{\varepsilon^2} \geqslant 0, \tag{6.14}$$

$$\mathrm{Im}(\lambda) = 2\varepsilon^2 \,\mathrm{Re}(\sigma^-)\mathrm{Im}(\sigma^-) - \mathrm{Im}(\sigma^-), \tag{6.15}$$

and by (6.12) we have

$$\left|1 - \varepsilon^2\sigma^-\right| \times \left|(-\sigma^- + \beta)^{n+1}\right| = \left|\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}\right|,$$

i.e.,

$$\begin{aligned}&\left(\left(1-\varepsilon^2\,\mathrm{Re}(\sigma^-)\right)^2 + \left(\varepsilon^2\,\mathrm{Im}(\sigma^-)\right)^2\right) \times \left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \left(\mathrm{Im}(\sigma^-)\right)^2\right)^{n+1}\\ &\quad= \left|\eta(\varepsilon,\alpha)n!\right|^2 e^{2\,\mathrm{Re}(\sigma^-)\tau}.\end{aligned}$$

By using (6.15), we have

$$\begin{aligned}&\left(\left(1-\varepsilon^2\,\mathrm{Re}(\sigma^-)\right)^2 + \varepsilon^4\left(\mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)\right)\\ &\qquad\times\left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)^{n+1}\\ &\quad= \left|\eta(\varepsilon,\alpha)n!\right|^2 e^{2\,\mathrm{Re}(\sigma^-)\tau}.\end{aligned}$$

Now set

$$\begin{aligned}f\left(\mathrm{Re}(\sigma^-)\right) = {}&\left(\left(1-\varepsilon^2\,\mathrm{Re}(\sigma^-)\right)^2 + \varepsilon^4\left(\mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)\right)\\ &\times\left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)^{n+1}\\ &- \left|\eta(\varepsilon,\alpha)n!\right|^2 e^{2\,\mathrm{Re}(\sigma^-)\tau},\end{aligned}$$

then

$$\begin{aligned}\frac{df}{d\,\mathrm{Re}(\sigma^-)}\left(\mathrm{Re}(\sigma^-)\right) = {}&\left(-2\varepsilon^2\left(1-\varepsilon^2\,\mathrm{Re}(\sigma^-)\right) + 2\varepsilon^4\,\mathrm{Re}(\sigma^-) - \varepsilon^2\right)\\ &\times\left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)^{n+1}\\ &+ (n+1)\times\left(\left(1-\varepsilon^2\,\mathrm{Re}(\sigma^-)\right)^2 + \varepsilon^4\left(\mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)\right)\\ &\times\left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)^{n}\\ &\times\left(-2\left(-\mathrm{Re}(\sigma^-)+\beta\right) + 2\,\mathrm{Re}(\sigma^-) - \frac{1}{\varepsilon^2}\right)\\ &- 2\tau\left|\eta(\varepsilon,\alpha)n!\right|^2 e^{2\,\mathrm{Re}(\sigma^-)\tau}.\end{aligned}$$

By using (6.14) and the above equation we deduce that

$$\mathrm{Re}(\sigma^-)<0 \quad \Rightarrow \quad \frac{df}{d\,\mathrm{Re}(\sigma^-)}\big(\mathrm{Re}(\sigma^-)\big)<0.$$

Thus for any fixed $\mathrm{Re}(\lambda)$, we can find at most one $\mathrm{Re}(\sigma^-)$ satisfying the characteristic equation (6.12). Using (6.14) and (6.15), we obtain the result. □

Now we consider the characteristic equation as the system (6.1) and (6.2) with $\mathrm{Re}(\lambda) > -\mu$, $\mathrm{Re}(\sigma^-)<0$. Under Assumption 6.5, the characteristic equation (6.1) is equivalent to

$$\big(\varepsilon^2\sigma^- - 1\big) = -n!\eta(\varepsilon,\alpha)\frac{e^{\sigma^-\tau}}{(\beta-\sigma^-)^{n+1}}. \tag{6.16}$$

We look for purely imaginary roots $\lambda=\omega i$ with $\omega>0$. As in the discussion of Section 6.1, we set

$$\sigma^- := -a-ib, \qquad \omega := 2\varepsilon^2 ab+b,$$
$$a := \frac{-1+\sqrt{1+4\varepsilon^2(\mu+\varepsilon^2 b^2)}}{2\varepsilon^2}$$

with $b>0$, then Eq. (6.2) is satisfied. Now it remains to find b such that it satisfies Eq. (6.16), or equivalently,

$$\sigma^-\times\big(\varepsilon^2\sigma^- - 1\big) = i\omega+\mu = -\sigma^-\times n!\eta(\varepsilon,\alpha)\frac{e^{\sigma^-\tau}}{(\beta-\sigma^-)^{n+1}}.$$

Now we have

$$\sigma^- = -a-ib = \sqrt{a^2+b^2}\,e^{i\theta},$$
$$\beta-\sigma^- = a+\beta+ib = \sqrt{(a+\beta)^2+b^2}\,e^{i\hat\theta},$$

where

$$\theta := \arctan\left(\frac{b}{a}\right)+\pi, \qquad \hat\theta := \arctan\left(\frac{b}{a+\beta}\right).$$

Then we obtain

$$\mu+i\omega = -\eta(\varepsilon,\alpha)\frac{n!\sqrt{a^2+b^2}\,e^{-a\tau}}{(\sqrt{(a+\beta)^2+b^2}\,)^{n+1}}e^{i(\theta-(n+1)\hat\theta-\tau b)}.$$

Now we fix

$$\eta(\varepsilon,\alpha) = -c\frac{(\sqrt{(a+\beta)^2+b^2}\,)^{n+1}}{n!\sqrt{a^2+b^2}}e^{a\tau}$$

with $c>0$, then we obtain

$$\mu+i\omega = \mu+i\big(2\varepsilon^2 ab+b\big) = ce^{i(\theta-(n+1)\hat\theta-\tau b)}$$

and have

$$c=\sqrt{\mu^2+\omega^2}=\sqrt{\mu^2+\left(2\varepsilon^2ab+b\right)^2},$$
$$\frac{(2\varepsilon^2ab+b)}{\mu}=\tan\left(\theta-(n+1)\hat{\theta}-\tau b\right).$$

We must impose that

$$\sin\left(\theta-(n+1)\hat{\theta}-\tau b\right)=\frac{2\varepsilon^2ab+b}{c}>0.$$

From the above computation we obtain the following proposition.

Proposition 6.7. *Let $\varepsilon>0$, $\tau>0$, $\mu>0$, $\beta>0$, and $n\in\mathbb{N}$ be fixed. Then the characteristic equation has a pair of purely imaginary solutions $\pm i\omega$ with $\omega>0$ if and only if there exists $b>0$ which is a solution of equation*

$$\frac{(2\varepsilon^2ab+b)}{\mu}=-\tan\Theta(b) \tag{6.17}$$

with

$$\sin\left(\Theta(b)\right)<0, \tag{6.18}$$

and we have

$$\omega=b\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2b^2\right)},\qquad \eta(\varepsilon,\alpha)=\tilde{\eta}(\varepsilon,a,b),$$

where

$$\Theta(b)=-\theta+(n+1)\hat{\theta}+\tau b,\qquad \tilde{\eta}(\varepsilon,a,b):=-c\frac{(\sqrt{(a+\beta)^2+b^2}\,)^{n+1}}{n!\sqrt{a^2+b^2}}e^{a\tau},$$
$$\theta=\arctan\left(\frac{b}{a}\right)+\pi,\qquad \hat{\theta}=\arctan\left(\frac{b}{a+\beta}\right),$$
$$c=\sqrt{\mu^2+\left(2\varepsilon^2ab+b\right)^2},\qquad a=\frac{-1+\sqrt{1+4\varepsilon^2(\mu+\varepsilon^2b^2)}}{2\varepsilon^2}.$$

Moreover, there exists a sequence $b_k\to+\infty$ as $k\to+\infty$, $k\in\mathbb{N}$ (which is a function of ε, τ, μ, β, and n) satisfying (6.17) and (6.18). In particular, for each k large enough, there exists a unique $b_k\in(\Theta^{-1}(2k\pi-\frac{\pi}{2}),\Theta^{-1}(2k\pi))$ satisfying (6.17) and (6.18), where Θ^{-1} is the inverse function of $\Theta(b)$ on $[\hat{b},+\infty)$ for $\hat{b}$ large enough.

Proof. Note that for $b>0$,

$$\frac{(2\varepsilon^2ab+b)}{\mu}=-\tan\Theta(b)>0,$$

we have

$$\tan\Theta(b)<0,$$

so

$$\Theta(b) \in \left(m\pi - \frac{\pi}{2}, m\pi\right), \quad m \in \mathbb{Z}.$$

Moreover, in order to ensure

$$\sin\Theta(b) < 0,$$

we must take $m = 2k$, $k \in \mathbb{Z}$. Now since $\Theta(b)$ is a continuous function of b,

$$\Theta(0) = -\pi, \qquad \Theta(+\infty) = +\infty,$$

so for any $k \in \mathbb{N}$, there exist $\hat{b}_{k1}, \hat{b}_{k2} > 0$ such that $\Theta(b_{k1}) = 2k\pi - \frac{\pi}{2}$, $\Theta(b_{k2}) = 2k\pi$. Observe that the left-hand side of Eq. (6.17) is a strictly monotone increasing function of b, and since the function $\tan(\Theta(b))$ can take any value from $-\infty$ to $+\infty$ when $b \in (\hat{b}_{k1}, \hat{b}_{k2})$ or $b \in (\hat{b}_{k2}, \hat{b}_{k1})$, we deduce that Eq. (6.17) has a solution $b_k \in (\hat{b}_{k1}, \hat{b}_{k2})$ or $b_k \in (\hat{b}_{k2}, \hat{b}_{k1})$. Thus there exists a sequence of $b_k \to +\infty$ satisfying (6.17) and (6.18). We denote the derivative of function f with respect to b by f'. Then

$$\begin{aligned}
\Theta'(b) := \frac{d}{db}\Theta(b) &= -\left[\arctan\left(\frac{b}{a}\right) + \pi\right]' + (n+1)\left[\arctan\left(\frac{b}{a+\beta}\right)\right]' + \tau \\
&= -\frac{(\frac{b}{a})'}{1 + (\frac{b}{a})^2} + (n+1)\frac{(\frac{b}{a+\beta})'}{1 + (\frac{b}{a+\beta})^2} + \tau \\
&= -\frac{\frac{a-ba'}{a^2}}{1 + (\frac{b}{a})^2} + (n+1)\frac{\frac{a+\beta-ba'}{(a+\beta)^2}}{1 + (\frac{b}{a+\beta})^2} + \tau \\
&= -\frac{\frac{a-ba'}{b^2}}{(\frac{a}{b})^2 + 1} + (n+1)\frac{\frac{a+\beta-ba'}{b^2}}{(\frac{a+\beta}{b})^2 + 1} + \tau,
\end{aligned}$$

where

$$a' := \frac{d}{db}\left(\frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{2\varepsilon^2}\right) = \frac{2\varepsilon^2 b}{\sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}.$$

Since

$$\begin{gathered}
a' = \frac{2\varepsilon^2 b}{\sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}} \to 1 \quad \text{as } b \to +\infty, \\
\frac{a}{b} = \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b^2)}}{2\varepsilon^2 b} \to 1 \quad \text{as } b \to +\infty, \\
\frac{a+\beta}{b} \to 1 \quad \text{as } b \to +\infty,
\end{gathered}$$

we obtain

$$\Theta'(b) \to \tau \quad \text{as } b \to +\infty.$$

That is, when b is large enough, $\Theta(b)$ is a strictly monotone increasing function of b. Denote by Θ^{-1} the inverse function of $\Theta(b)$ on $[\hat{b}, +\infty)$ for $\hat{b}$ large enough. So for k large enough we have $\hat{b}_{k1} = \Theta^{-1}(2k\pi - \frac{\pi}{2})$, $\hat{b}_{k2} = \Theta^{-1}(2k\pi)$, and the function $\tan(\Theta(b))$ is increasing when $b \in (\hat{b}_{k1}, \hat{b}_{k2})$. Thus there exists a unique $b_k \in (\Theta^{-1}(2k\pi - \frac{\pi}{2}), \Theta^{-1}(2k\pi))$ satisfying (6.17) and (6.18), and the result follows. □

6.1.5. Bifurcation diagrams for case (b)

We find a sequence b_k going to $+\infty$ and satisfying (6.17) and (6.18). By using a similar procedure as before, we can derive a bifurcation diagram. Using our construction, we have

$$\eta(\varepsilon, \alpha_k) = \tilde{\eta}(\varepsilon, a_k, b_k).$$

But

$$\eta(\varepsilon, \alpha) = \alpha \frac{1}{R_0}\bigl(1 - \ln(R_0)\bigr),$$

so we obtain the bifurcation curves

$$\alpha_k \frac{1}{R_0}\bigl(1 - \ln(R_0)\bigr) = \tilde{\eta}(\varepsilon, a_k, b_k).$$

Since

$$\ln(R_0) = 1 - \frac{R_0}{\alpha_k}\tilde{\eta}(\varepsilon, a_k, b_k),$$

it follows that

$$R_0 = e^{[1 - \tilde{\eta}(\varepsilon, a_k, b_k)\frac{R_0}{\alpha_k}]}.$$

Notice that we also have

$$R_0 = \frac{2\alpha_k \chi}{1 + \sqrt{\Lambda_0}},$$

so

$$\alpha_k = \frac{1 + \sqrt{\Lambda_0}}{2\chi} \exp\biggl(1 - \tilde{\eta}(\varepsilon, a_k, b_k)\frac{2\chi}{1 + \sqrt{\Lambda_0}}\biggr).$$

Now since

$$\chi = \int_0^{+\infty} \gamma(x) e^{\sigma_0^- x}\, dx = \frac{n! e^{\sigma_0^- \tau}}{(\beta - \sigma_0^-)^{n+1}},$$

$$\sigma_0^- = \frac{1 - \sqrt{\Lambda_0}}{2\varepsilon^2}, \qquad \Lambda_0 = 1 + 4\varepsilon^2 \mu,$$

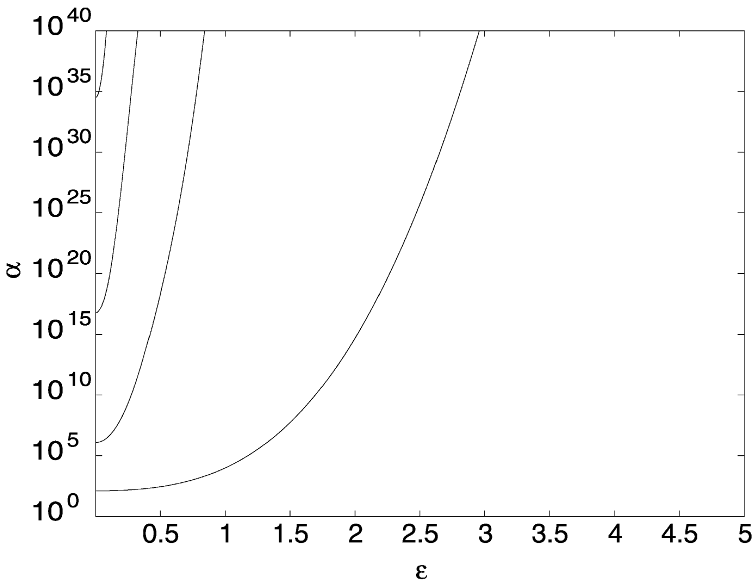

Fig. 6.3. A few bifurcation curves given by (6.17), (6.18), and (6.19), in the (ε, α)-plane for $\tau = 2$, $\mu = 1$, $n = 1$, and $\beta = 0.1$.

we obtain

$$\frac{2\chi}{1+\sqrt{\Lambda_0}} = \frac{2n! e^{\sigma_0^- \tau}}{(1+\sqrt{\Lambda_0})(\beta - \sigma_0^-)^{n+1}}.$$

Thus we obtain the bifurcation curves

$$\alpha_k = \frac{(1+\sqrt{\Lambda_0})(\beta - \sigma_0^-)^{n+1}}{2n! e^{\sigma_0^- \tau}} \exp\left(1 - \tilde{\eta}(\varepsilon, a_k, b_k) \frac{2n! e^{\sigma_0^- \tau}}{(1+\sqrt{\Lambda_0})(\beta - \sigma_0^-)^{n+1}}\right), \tag{6.19}$$

where

$$\tilde{\eta}(\varepsilon, a_k, b_k) = -c_k \frac{(\sqrt{(a_k+\beta)^2 + b_k^2})^{n+1}}{n! \sqrt{a_k^2 + b_k^2}} e^{a_k \tau},$$

$$c_k = \sqrt{\mu^2 + \left(2\varepsilon^2 a_k b_k + b_k\right)^2}, \qquad a_k = \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b_k^2)}}{2\varepsilon^2}.$$

See Figs. 6.3 and 6.4.

Remark 6.8. By Lemma 6.6, for any given $\varepsilon > 0$ and $\alpha > 0$, we obtain at most one pair of purely imaginary solutions of the characteristic equation. So on the bifurcation diagram in Fig. 6.5, the crossing point in fact corresponds to the point where two branches of eigenvalues coincide.

6.1.6. Special case for $\varepsilon = 0$

This special case corresponds to the characteristic equation studied in [35]. Here we improve the description given in [35] by specifying the bifurcation curves. When $\varepsilon = 0$, the characteristic equation becomes

$$1 = n! \eta(0, \alpha) \frac{e^{\sigma^- \tau}}{(\beta - \sigma^-)^{n+1}} \tag{6.20}$$

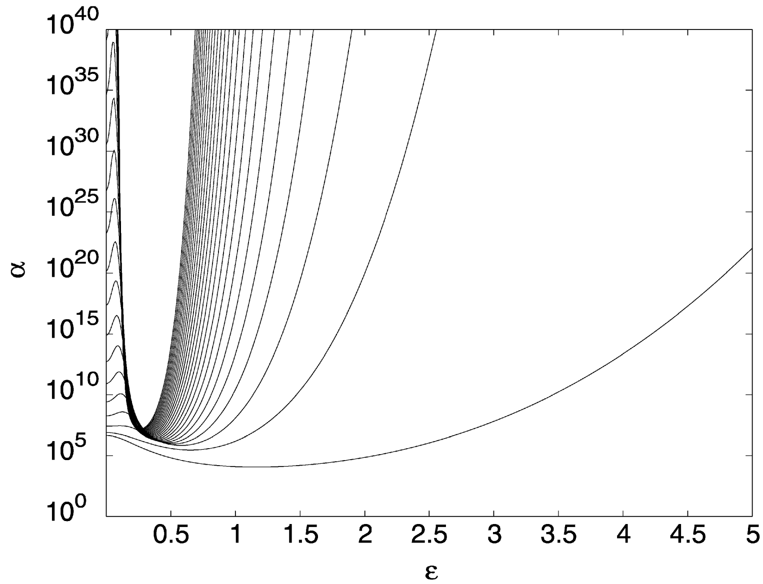

Fig. 6.4. A family of bifurcation curves given by (6.17), (6.18), and (6.19), in the (ε, α)-plane for $\tau = 2$, $\mu = 5$, $n = 1$, and $\beta = 0.1$.

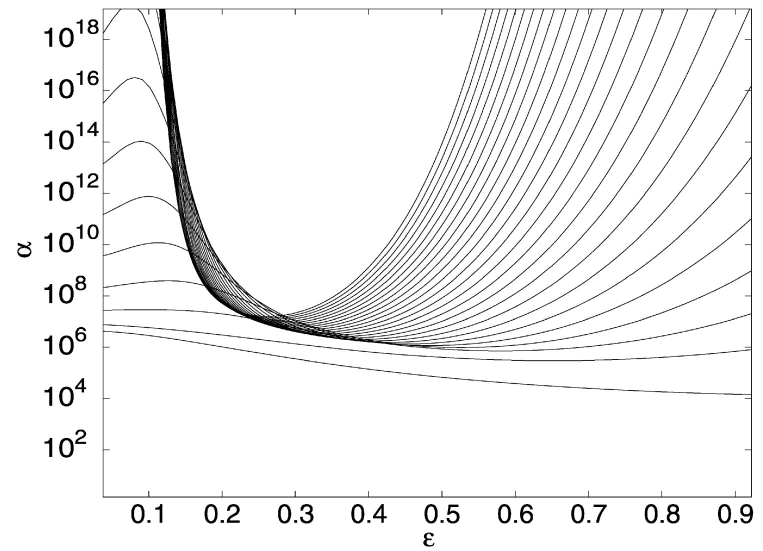

Fig. 6.5. A zoom on the region where the bifurcation curves are crossing each other in Fig. 6.4.

with

$$-\sigma^- = \lambda + \mu,$$

$$\eta(0,\alpha) = \frac{1}{\chi_0}\big(1 - \ln(\alpha\chi_0)\big), \qquad \chi_0 = \int_0^\infty \gamma(x) e^{-\mu x}\, dx = \frac{n! e^{-\mu\tau}}{(\beta+\mu)^{n+1}}.$$

If we now look for $\lambda = i\omega$ with $\omega > 0$ and set $\sigma^- = -a - ib$ with $a > 0$, $b > 0$, then

$$a = \mu, \qquad \omega = b,$$

and b must satisfy

$$1 = n!\eta(0,\alpha)\frac{e^{(-\mu - ib)\tau}}{(\beta+\mu+ib)^{n+1}},$$

i.e.,

$$1 = n!\eta(0,\alpha)\frac{e^{-\mu\tau}}{(\sqrt{(\beta+\mu)^2+b^2}\,)^{n+1}}e^{-i(b\tau+(n+1)\arctan\frac{b}{\beta+\mu})}.$$

Since $\eta(0,\alpha) < 0$ for $\alpha > \hat{\alpha}_1$, we have

$$\eta(0,\alpha) = -\frac{(\sqrt{(\mu+\beta)^2+b^2}\,)^{n+1}}{n!}e^{\mu\tau}$$

and

$$-\left(b\tau + (n+1)\arctan\left(\frac{b}{\beta+\mu}\right)\right) = \pi - 2k\pi$$

for some $k \in \mathbb{Z}$.

Proposition 6.9. *Let $\tau > 0$, $\mu > 0$, $\beta > 0$, and $n \in \mathbb{N}$ be fixed. Then the characteristic equation* (6.20) *has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if there exists $k \in \mathbb{Z}$ such that ω is a solution of equation*

$$-\left(\omega\tau + (n+1)\arctan\frac{\omega}{\beta+\mu}\right) = \pi - 2k\pi \tag{6.21}$$

and

$$\eta(0,\alpha) = \tilde{\eta}(0,\mu,\omega),$$

where

$$\tilde{\eta}(0,\mu,\omega) := -\frac{(\sqrt{(\mu+\beta)^2+\omega^2}\,)^{n+1}}{n!}e^{\mu\tau}.$$

Moreover, for each $k \in \mathbb{N}^+$, there exists a unique ω_k (which is a function of τ, μ, β, and n) satisfying Eq. (6.21)*.*

In this case, the bifurcation curves are

$$\alpha_k = \frac{(\beta+\mu)^{n+1}}{n!e^{-\mu\tau}}\exp\left(1 - \tilde{\eta}(0,\mu,\omega_k)\frac{n!e^{-\mu\tau}}{(\beta+\mu)^{n+1}}\right), \quad k \in \mathbb{N}^+,$$

where

$$\tilde{\eta}(0,\mu,\omega_k) = -\frac{(\sqrt{(\mu+\beta)^2+\omega_k^2}\,)^{n+1}}{n!}e^{\mu\tau},$$

and ω_k is described in Proposition 6.9.

6.2. Transversality condition

The aim of this section is to prove a transversality condition for the model with Assumption 6.1 or Assumption 6.5. Since Assumption 6.1 is a special case of Assumption 6.5, we start to investigate the transversality condition under Assumption 6.5.

Lemma 6.10. *For fixed $\varepsilon > 0$, if $\alpha > \hat{\alpha}_1$, $\lambda \in \Omega$ and $\Delta(\varepsilon, \alpha, \lambda) = 0$, then*

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} < 0.$$

Proof. Since

$$\Delta(\varepsilon, \alpha, \lambda) = 1 - \frac{2\eta(\varepsilon, \alpha)}{1 + \sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx,$$

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} = -\frac{2\frac{\partial \eta(\varepsilon, \alpha)}{\partial \alpha}}{1 + \sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x}\, dx,$$

and

$$\eta(\varepsilon, \alpha) = \frac{1 + \sqrt{\Lambda_0}}{2\chi} \left(1 - \ln\left(\alpha \frac{2\chi}{1 + \sqrt{\Lambda_0}}\right)\right),$$

where $\frac{1+\sqrt{\Lambda_0}}{2\chi}$ is independent of α, we have

$$\frac{\partial \eta(\varepsilon, \alpha)}{\partial \alpha} = -\frac{1 + \sqrt{\Lambda_0}}{2\alpha\chi}.$$

But $\Delta(\varepsilon, \alpha, \lambda) = 0$, we obtain

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} = -\frac{\frac{\partial \eta(\varepsilon, \alpha)}{\partial \alpha}}{\eta(\varepsilon, \alpha)} = \frac{1}{\alpha(1 - \ln \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}})}.$$

Moreover, if $\alpha > \hat{\alpha}_1$, then $\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} < 0$. $\square$

Lemma 6.11. *Let Assumptions* 2.1 *and* 6.5 *be satisfied. For fixed $\varepsilon > 0$, if $\alpha > \hat{\alpha}_1$, $\lambda \in \Omega$ and $\Delta(\varepsilon, \alpha, \lambda) = 0$, then*

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \lambda} = \frac{2\varepsilon^2}{\sqrt{\Lambda}} \left(\frac{1}{1 + \sqrt{\Lambda}} + \frac{\tau}{2\varepsilon^2} + \frac{1 + n}{(2\varepsilon^2\beta - 1) + \sqrt{\Lambda}}\right) \neq 0.$$

Proof. Under Assumption 6.5 we have

$$\Delta(\varepsilon, \alpha, \lambda) = 1 - \frac{\eta(\varepsilon, \alpha) n! e^{\sigma^- \tau}}{(1 - \varepsilon^2\sigma^-) \times (-\sigma^- + \beta)^{n+1}}$$

and

$$\begin{aligned}
\frac{\partial \Delta(\varepsilon,\alpha,\lambda)}{\partial \lambda} &= \frac{\partial \Delta(\varepsilon,\alpha,\lambda)}{\partial \sigma^-}\frac{\partial \sigma^-}{\partial \lambda} = -\eta(\varepsilon,\alpha)n!\frac{\partial}{\partial \sigma^-}\left(\frac{e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}}\right)\frac{\partial \sigma^-}{\partial \lambda} \\
&= -\frac{\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}}\left(\tau+\frac{\varepsilon^2}{1-\varepsilon^2\sigma^-}+\frac{n+1}{-\sigma^-+\beta}\right)\frac{\partial \sigma^-}{\partial \lambda} \\
&= \big(\Delta(\varepsilon,\alpha,\lambda)-1\big)\times\left(\tau+\frac{2\varepsilon^2}{1+\sqrt{\Lambda}}+\frac{n+1}{-\sigma^-+\beta}\right)\frac{\partial \sigma^-}{\partial \lambda}.
\end{aligned}$$

Since

$$\frac{\partial \sigma^-}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon^2}\right) = -\frac{1}{\sqrt{1+4\varepsilon^2(\lambda+\mu)}} = -\frac{1}{\sqrt{\Lambda}},$$

we obtain

$$\frac{\partial \Delta(\varepsilon,\alpha,\lambda)}{\partial \lambda} = -\frac{\Delta(\varepsilon,\alpha,\lambda)-1}{\sqrt{\Lambda}}\left(\frac{2\varepsilon^2}{1+\sqrt{\Lambda}}+\tau+\frac{n+1}{-\sigma^-+\beta}\right).$$

So if $\Delta(\varepsilon,\alpha,\lambda)=0$, then

$$\frac{\partial \Delta(\varepsilon,\alpha,\lambda)}{\partial \lambda} = \frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}}+\frac{\tau}{2\varepsilon^2}+\frac{1+n}{(2\varepsilon^2\beta-1)+\sqrt{\Lambda}}\right). \tag{6.22}$$

Now note that $\frac{\partial \Delta(\varepsilon,\alpha,\lambda)}{\partial \lambda}=0$ if and only if

$$\frac{\tau}{2\varepsilon^2}\Lambda+(2+n+\tau\beta)\sqrt{\Lambda}+\frac{\tau}{2\varepsilon^2}\big(2\varepsilon^2\beta-1\big)+n+2\varepsilon^2\beta=0. \tag{6.23}$$

As $\eta(\varepsilon,\alpha)<0$ for $\alpha>\hat{\alpha}_1$, we have for $\lambda\in\mathbb{R}$ and $\lambda>-\mu$ that

$$\Delta(\varepsilon,\alpha,\lambda) = 1-\frac{\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}}>0.$$

So the solutions of the characteristic equation in Ω cannot be real numbers. When $\lambda\in\mathbb{C}\setminus\mathbb{R}$, we have $\sqrt{\Lambda}=\sqrt{1+4\varepsilon^2(\lambda+\mu)}\in\mathbb{C}\setminus\mathbb{R}$ with $\mathrm{Re}(\sqrt{\Lambda})>0$. By noting that the sign of the imaginary part of Λ is the same as the sign of the imaginary part of $\sqrt{\Lambda}$ and by noticing that $\frac{\tau}{2\varepsilon^2}>0$ and $2+n+\tau\beta>0$, we deduce that Eq. (6.23) cannot be satisfied. $\square$

Theorem 6.12. *Let Assumptions* 2.1 *and* 6.5 *be satisfied and let* $\varepsilon>0$ *be given. For each* $k\geqslant 0$ *large enough, let* $\lambda_k=i\omega_k$ *be the purely imaginary root of the characteristic equation associated to* $\alpha_k>0$ *(defined in Proposition* 6.7*), then there exists* $\rho_k>0$ *(small enough) and a* C^1*-map* $\hat{\lambda}_k:(\alpha_k-\rho_k,\alpha_k+\rho_k)\to\mathbb{C}$ *such that*

$$\hat{\lambda}_k(\alpha_k)=i\omega_k, \qquad \Delta\big(\varepsilon,\alpha,\hat{\lambda}_k(\alpha)\big)=0, \quad \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k),$$

satisfying the transversality condition

$$\mathrm{Re}\left(\frac{d\hat{\lambda}_k(\alpha_k)}{d\alpha}\right)>0.$$

Proof. By Lemma 6.11 we can use the implicit function theorem around each $(\alpha_k, i\omega_k)$ provided by Proposition 6.7, and obtain that there exists $\rho_k > 0$ and a C^1-map $\hat{\lambda}_k : (\alpha_k - \rho_k, \alpha_k + \rho_k) \to \mathbb{C}$ such that

$$\hat{\lambda}_k(\alpha_k) = i\omega_k, \qquad \Delta\big(\varepsilon, \alpha, \hat{\lambda}_k(\alpha)\big) = 0, \quad \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

Moreover, we have

$$\frac{\partial \Delta(\varepsilon, \alpha, \hat{\lambda}_k(\alpha))}{\partial \alpha} + \frac{\partial \Delta(\varepsilon, \alpha, \hat{\lambda}_k(\alpha))}{\partial \lambda} \frac{d\hat{\lambda}_k(\alpha)}{d\alpha} = 0, \quad \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

So

$$\frac{d\hat{\lambda}_k(\alpha)}{d\alpha} = -\frac{1}{\frac{\partial \Delta(\varepsilon, \alpha, \hat{\lambda}_k(\alpha))}{\partial \lambda}} \frac{\partial \Delta(\varepsilon, \alpha, \hat{\lambda}_k(\alpha))}{\partial \alpha}, \quad \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

By using Lemma 6.10, we deduce $\forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k)$ that

$$\operatorname{Re}\left(\frac{d}{d\alpha}\hat{\lambda}_k(\alpha)\right) > 0 \quad \Leftrightarrow \quad \operatorname{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha, \hat{\lambda}_k(\alpha))}{\partial \lambda}\right) > 0.$$

In particular, we have

$$\operatorname{Re}\left(\frac{d}{d\alpha}\hat{\lambda}_k(\alpha_k)\right) > 0 \quad \Leftrightarrow \quad \operatorname{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda}\right) > 0.$$

Using the notation of Proposition 6.7, we have

$$\sqrt{\Lambda} = 1 - 2\varepsilon^2\sigma^- = 1 + 2\varepsilon^2(a_k + ib_k) := \gamma_k + i\delta_k,$$

where γ_k and δ_k are positive and $\gamma_k^2 - \delta_k^2 = 1 + 4\varepsilon^2\mu$.

Therefore, we get

$$\begin{aligned}
\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda} &= \frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}} + \frac{\tau}{2\varepsilon^2} + \frac{1+n}{(2\varepsilon^2\beta - 1) + \sqrt{\Lambda}}\right) \\
&= \frac{2\varepsilon^2}{\gamma_k + i\delta_k}\left(\frac{1+\gamma_k - i\delta_k}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\tau}{2\varepsilon^2} + (n+1)\frac{2\varepsilon^2\beta - 1 + \gamma_k - i\delta_k}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2}\right) \\
&= \frac{2\varepsilon^2}{\gamma_k^2 + \delta_k^2}(\gamma_k - i\delta_k)\left(\frac{1+\gamma_k - i\delta_k}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\tau}{2\varepsilon^2} + (n+1)\frac{2\varepsilon^2\beta - 1 + \gamma_k - i\delta_k}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda}\right) \\
&= \frac{2\varepsilon^2}{\gamma_k^2 + \delta_k^2}\left(\frac{\gamma_k(1+\gamma_k) - \delta_k^2}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\gamma_k\tau}{2\varepsilon^2} + (n+1)\frac{\gamma_k(2\varepsilon^2\beta - 1 + \gamma_k) - \delta_k^2}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2}\right) \\
&= \frac{2\varepsilon^2}{\gamma_k^2 + \delta_k^2}\left(\frac{\gamma_k + \gamma_k^2 - \delta_k^2}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\gamma_k\tau}{2\varepsilon^2} + (n+1)\frac{\gamma_k(2\varepsilon^2\beta - 1) + \gamma_k^2 - \delta_k^2}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2}\right) \\
&= \frac{2\varepsilon^2}{\gamma_k^2 + \delta_k^2}A_k,
\end{aligned}$$

where

$$A_k = \left(\frac{\gamma_k + 1 + 4\varepsilon^2\mu}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\gamma_k \tau}{2\varepsilon^2} + (n+1)\frac{\gamma_k(2\varepsilon^2\beta - 1) + 1 + 4\varepsilon^2\mu}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2} \right).$$

By Proposition 6.7 we have

$$a_k = \frac{-1 + \sqrt{1 + 4\varepsilon^2(\mu + \varepsilon^2 b_k^2)}}{2\varepsilon^2}, \qquad b_k \to +\infty \quad \text{as } k \to +\infty.$$

Then we obtain

$$\gamma_k = 1 + 2\varepsilon^2 a_k = \sqrt{1 + 4\varepsilon^2\big(\mu + \varepsilon^2 b_k^2\big)} \to +\infty \quad \text{as } k \to +\infty,$$
$$\delta_k = 2\varepsilon^2 b_k \to +\infty \quad \text{as } k \to +\infty,$$

and

$$\lim_{k\to+\infty} \frac{A_k}{\gamma_k} = \lim_{k\to+\infty} \left(\frac{1 + \frac{1+4\varepsilon^2\mu}{\gamma_k}}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\tau}{2\varepsilon^2} + (n+1)\frac{(2\varepsilon^2\beta - 1) + \frac{1+4\varepsilon^2\mu}{\gamma_k}}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2} \right)$$
$$= \frac{\tau}{2\varepsilon^2} > 0.$$

We deduce that $A_k > 0$ for k large enough and $\mathrm{Re}(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda}) > 0$. So $\mathrm{Re}(\frac{d\hat{\lambda}_k(\alpha_k)}{d\alpha}) > 0$ and the result follows. □

In the case (a) we have a full description of the problem in terms of transversality condition and the following result.

Theorem 6.13. *Let Assumptions* 2.1 *and* 6.1 *be satisfied and let $\varepsilon > 0$ be given. For each $k \geqslant 0$, let $\lambda_k = i\omega_k$ be the purely imaginary root of the characteristic equation associated to $\alpha_k > 0$ (defined in Proposition* 6.2*), then there exists $\rho_k > 0$ (small enough) and a C^1-map $\hat{\lambda}_k : (\alpha_k - \rho_k, \alpha_k + \rho_k) \to \mathbb{C}$ such that*

$$\hat{\lambda}_k(\alpha_k) = i\omega_k, \qquad \Delta\big(\varepsilon, \alpha, \hat{\lambda}_k(\alpha)\big) = 0, \quad \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k),$$

satisfying the transversality condition

$$\mathrm{Re}\left(\frac{d\hat{\lambda}_k(\alpha_k)}{d\alpha} \right) > 0.$$

Proof. According to the proof of Theorem 6.12, we have

$$\mathrm{Re}\left(\frac{d}{d\alpha} \hat{\lambda}_k(\alpha_k) \right) > 0 \quad \Leftrightarrow \quad \mathrm{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda} \right) > 0.$$

Taking $n = 0$, $\beta = 0$ in (6.22), we have for each $k \geqslant 0$ that

$$\begin{aligned}\frac{\partial\Delta(\varepsilon,\alpha_k,i\omega_k)}{\partial\lambda}&=\frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}}+\frac{\tau}{2\varepsilon^2}+\frac{1}{-1+\sqrt{\Lambda}}\right)\\&=\frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{2\sqrt{\Lambda}}{4\varepsilon^2(\lambda+\mu)}+\frac{\tau}{2\varepsilon^2}\right)\\&=\frac{1}{i\omega_k+\mu}+\frac{\tau}{\sqrt{\Lambda}}.\end{aligned}$$

Since

$$\mathrm{Re}(\sqrt{\Lambda})>0,$$

we obtain for each $k\geqslant 0$ that

$$\mathrm{Re}\left(\frac{\partial\Delta(\varepsilon,\alpha_k,i\omega_k)}{\partial\lambda}\right)>0,$$

so the result follows. □

6.3. Hopf bifurcations

By combining the results on the essential growth rate of the linearized equations (Eq. (4.6)), the simplicity of the imaginary eigenvalues (Lemmas 4.5 and 6.11), the existence of purely imaginary eigenvalues (Proposition 6.7 or Proposition 6.2), and the transversality condition (Theorem 6.12 or Theorem 6.13), we are in a position to apply the center manifold Theorem 4.21 and Proposition 4.22 in Magal and Ruan [35]. Applying the Hopf bifurcation theorem proved in Hassard et al. [23] to the reduced system, we have the following Hopf bifurcation results.

In the case (a), we obtain the following result.

Theorem 6.14 *(Hopf bifurcation). Let Assumptions* 2.1 *and* 6.1 *be satisfied. Then for any given* $\varepsilon>0$ *and any* $k\in\mathbb{N}$*, the number* α_k *(defined in Proposition* 6.2*) is a Hopf bifurcation point for system* (1.2) *parametrized by* α*, and around the positive equilibrium point* $\bar{v}$ *given in* (3.2)*.*

For the case (b), the result is only partial with respect to k.

Theorem 6.15 *(Hopf bifurcation). Let Assumptions* 2.1 *and* 6.5 *be satisfied. Then for any given* $\varepsilon>0$*, there exists* $k_0\in\mathbb{N}$ *(large enough) such that for each* $k\geqslant k_0$*, the number* α_k *(defined in Proposition* 6.7*) is a Hopf bifurcation point for system* (1.2) *parametrized by* α*, around the equilibrium point* $\bar{v}$ *given in* (3.2)*.*

7. Discussion and numerical simulations

We first summarize the main results of this study. They are essentially divided into three parts: (a) the existence of a positive equilibrium; (b) the local stability of this equilibrium; and (c) the Hopf bifurcation at this equilibrium. To be more precise we obtain the following results:

(a) There exists a unique positive equilibrium if and only if

$$R_0:=\frac{2\alpha\chi}{1+\sqrt{\Lambda_0}}>1.$$

(b) The positive equilibrium is locally asymptotic stable:
  (b1) if $1<R_0\leqslant e^2$, or
  (b2) if $\varepsilon>0$ is large enough when we fix $\alpha=c\varepsilon$ with $\gamma\in L^1_+(0,+\infty)$, and $c>\frac{\sqrt{\mu}}{\int_0^{+\infty}\gamma(x)\,dx}$.

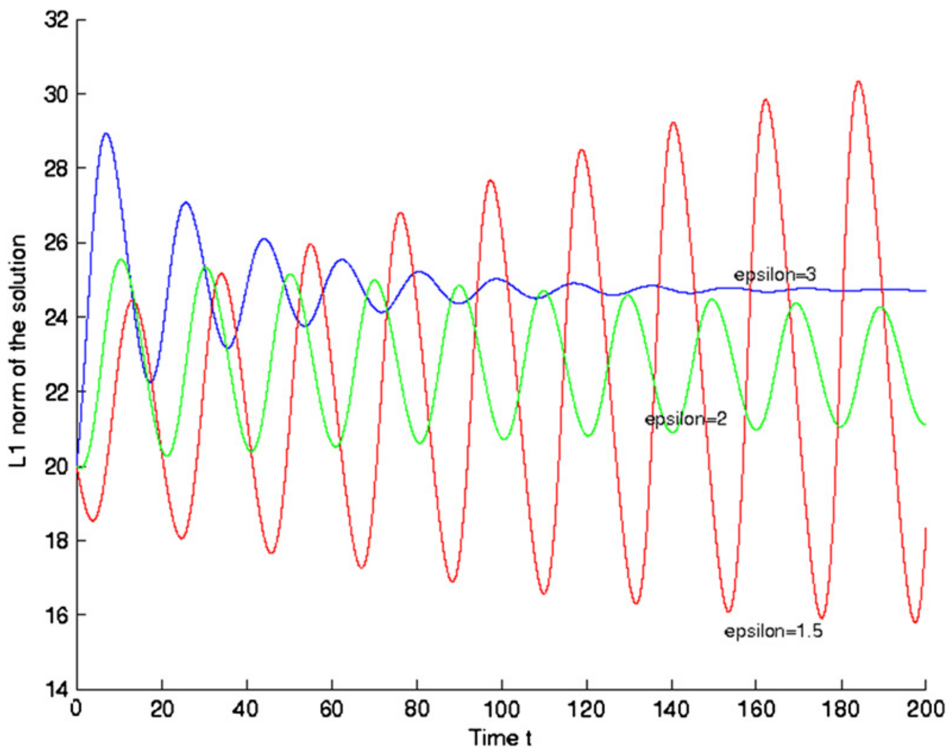

Fig. 7.1. Graph of the evolution of the L^1-norm of the solution in function of time. Here we fix $\alpha = 10$ and ε varies in $\{1.5, 2, 3\}$.

(c) Consider the following special case for $\gamma(x)$:

$$\gamma(x) = \begin{cases} (x-\tau)^n \exp(-\beta(x-\tau)), & \text{if } x \geqslant \tau, \\ 0, & \text{if } 0 \leqslant x < \tau, \end{cases} \tag{7.1}$$

for some integer $n \geqslant 0$, $\tau > 0$, and $\beta > 0$. There is a Hopf bifurcation around the positive equilibrium for any fixed $\varepsilon > 0$. For each $\varepsilon > 0$ there exists an infinity of bifurcating branches $\varepsilon \to \alpha_k(\varepsilon)$.

The result in (b1) is not really surprising since after the first bifurcation (i.e. the bifurcation of the null equilibrium) one may apply the result in Magal [32] to prove the global asymptotic stability of this equilibrium. Nevertheless the result allows to specify a set of the parameters for which the local stability holds.

The local stability result (b2) along the line $\alpha = c\varepsilon$ is more surprising since there is no more local effect (with respect to the parameters), and this result can be summarized by saying that the diffusion part gains when $\varepsilon > 0$ is large enough and α is proportional to ε. So in order to obtain a Hopf bifurcation the parameter α needs to increase faster than any linear map of ε.

Concerning the existence of Hopf bifurcation, the case $\varepsilon > 0$ small corresponds to a small perturbation of an age-structured model which has been studied in Magal and Ruan [35]. Here we have obtained a more precise result by showing the existence of an infinite number of Hopf bifurcating branches. The case $\varepsilon > 0$ is new and was not expected at first.

We now provide some numerical simulations in order to illustrate the Hopf bifurcation for system (1.1). These numerical simulations are fulfilled with the following parameters:

$$\beta = 0.5, \quad \mu = 0.05 \quad \text{and} \quad \gamma(x) = 1_{[7,20]}(x). \tag{7.2}$$

Here we observe that increasing the diffusion coefficient ε^2 with a fixed α tends to stabilize the positive equilibrium (see Fig. 7.1). On the other hand, when ε is fixed, increasing α tends to destabilize the positive equilibrium and leads to undamped oscillating solutions (see Fig. 7.2).

In Figs. 7.3 and 7.4, we look at the surface solutions for a fixed value of α and for different values of ε. We observe that the diffusion in the size variable disperses through the size variable. When

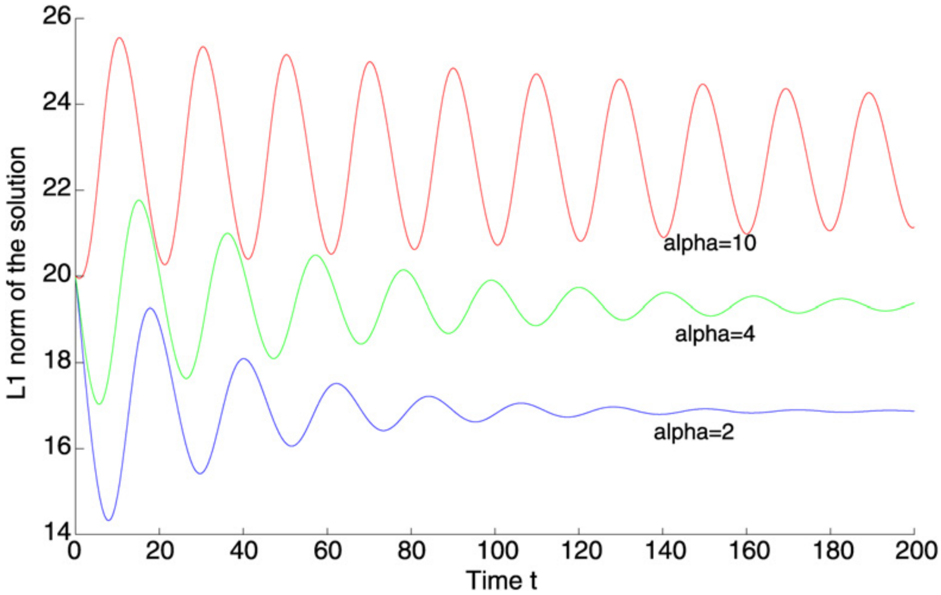

Fig. 7.2. Graph of the evolution of the L^1-norm of the solution in function of time. We fix $\varepsilon = 2$ and α varies in $\{2, 4, 10\}$.

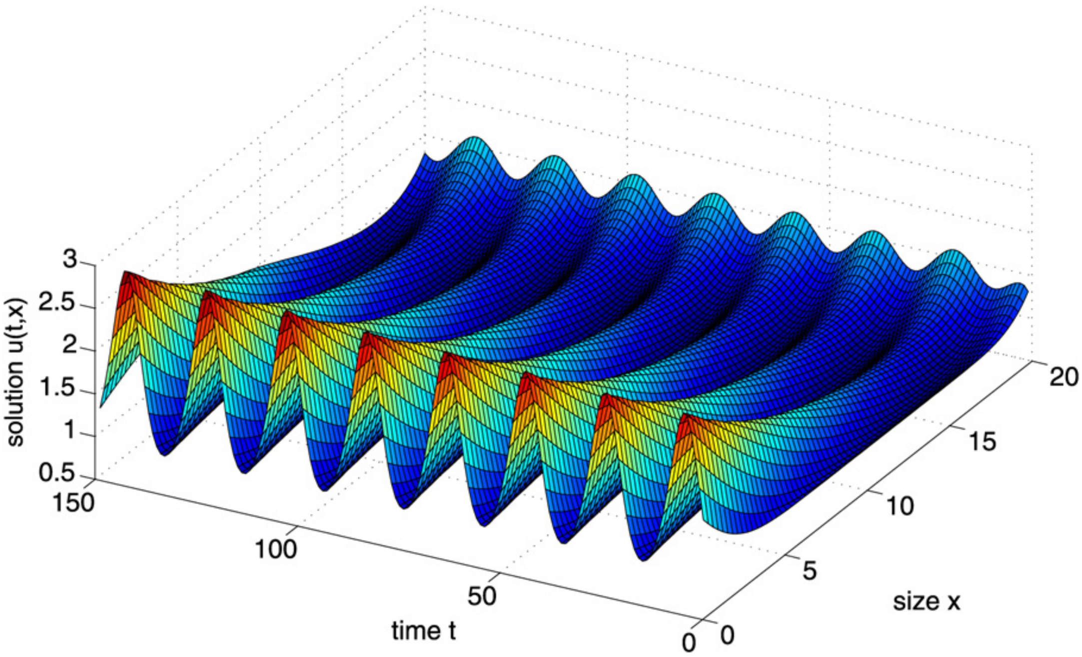

Fig. 7.3. Surface solution for $\alpha = 15$ and $\varepsilon^2 = 2$.

we increase the diffusion coefficient, we also increase the dispersion process. This dispersion, when it becomes sufficiently high, is responsible for the breaking of the self-sustained oscillations of the solutions. As a consequence, the diffusion will reduce the temporal oscillations and then will stabilize the positive equilibrium.

Note that our results depend on the assumption on the function $h(x)$: when h is monotone decreasing near the positive equilibrium and the slope decreases, then Hopf bifurcation occurs at the positive equilibrium. The periodic solutions induced by the Hopf bifurcation indicate that the population density exhibits temporal oscillatory patterns. We expect that the results can be generalized to different and more general types of functions.

As a conclusion, we can say that the effect of the stochastic fluctuations in the size-structured model (1.1), modelled by a simple diffusion term, acts in favor of the stabilization of the populations. Small fluctuations remain in a small perturbation of the classical case $\varepsilon = 0$, but by increasing the

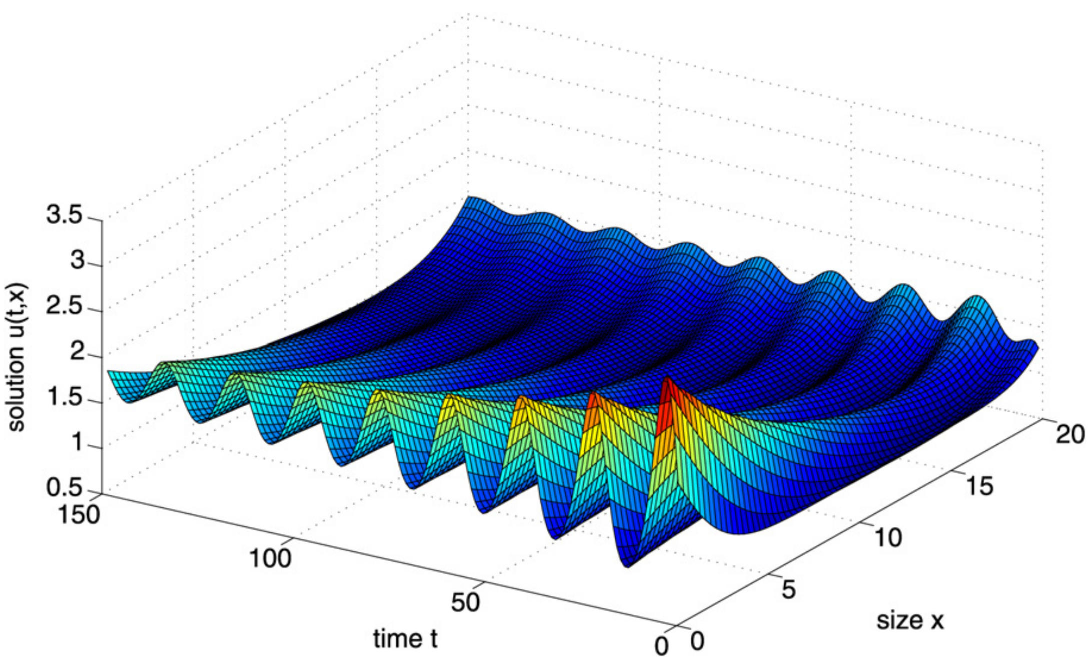

Fig. 7.4. Surface solution for $\alpha = 15$ and $\varepsilon^2 = 2.5$.

value of α the positive steady state can be destabilized. When the stochastic fluctuations are large (i.e. ε is large), then it turns to be very difficult to destabilize the positive equilibrium, because the threshold value of α increases exponentially with respect to ε.

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