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The effect of initial values on extinction or persistence in degenerate diffusion competition systems

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Abstract

When the asymptotic spreading for classical monostable Lotka–Volterra competition diffusion systems is concerned, extinction or persistence of the two competitive species is completely determined by the dynamics of the corresponding kinetic systems, while the size of initial values does not affect the final states. The purpose of this paper is to demonstrate the rich dynamics induced by the initial values in a class of degenerate competition diffusion systems with weak Allee effect. We present various extinction or persistence results by selecting different initial values although the corresponding kinetic system is fixed, which also implies the existence of balance between degenerate nonlinear reaction and diffusion. For example, even if the positive steady state of the corresponding kinetic system is globally asymptotically stable, we observe four different spreading–vanishing phenomena by selecting different initial values. In addition, the interspecific competition of one species may be harmful to the persistence of the other species by taking proper initial values. Our results show that the superior competitor in the sense of the corresponding kinetic system is not always unbeatable, it can be wiped out by the inferior competitor in the sense of the corresponding kinetic system depending on the size of initial habitats as well as the intensity of Allee effect.

Keywords Degenerate competition diffusion system · Extinction · Persistence · Comparison principle · Weak Allee effect

Mathematics Subject Classification 35K45 · 35K57 · 92D25

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1 Introduction

In population dynamics, there are many models describing interspecific and intraspecific competitions. One of the most important models is the following Lotka–Volterra competition diffusion system

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \Delta u(t,x) + r_1 u(t,x)\left[1 - u(t,x) - av(t,x)\right], \\ \frac{\partial v(t,x)}{\partial t} = d_2 \Delta v(t,x) + r_2 v(t,x)\left[1 - bu(t,x) - v(t,x)\right], \end{cases} \tag{1.1}$$

where $u(t,x)$, $v(t,x)$ denote the population densities at time $t > 0$ and location $x \in \mathbb{R}$ of two competitors, respectively, and all parameters are positive. In the last two decades, there is a vast body of literature on the asymptotic spreading of the corresponding initial value problem of (1.1), see Lewis et al. (2002), Weinberger et al. (2002), Lin and Li (2012), Carrère (2018), Girardin and Lam (2019), in which only the nonnegative initial values are considered due to the biological background. The corresponding kinetic system of (1.1) is

$$\begin{cases} \frac{du(t)}{dt} = r_1 u(t)\left[1 - u(t) - av(t)\right], \\ \frac{dv(t)}{dt} = r_2 v(t)\left[1 - bu(t) - v(t)\right], \end{cases} \tag{1.2}$$

and the existence and stability of steady states of (1.2) can be obtained by direct analysis, which will be presented in Sect. 2.

When (1.2) admits only one stable steady state (monostable case), the stability of the steady state is crucial in determining the extinction or persistence of the two competitive species. More precisely, if $0 \le a < 1 < b$, then (1.2) has a stable steady state $(1, 0)$. With this assumption, once the initial value of u has nonempty support, the species u will successfully occupy the habitat while v will be extinct in any compact subset of the habitat (Girardin and Lam 2019; Lewis et al. 2002; Weinberger et al. 2002). If $0 \le a, b < 1$, then (1.2) has a positive stable steady state, which implies the coexistence of both species. In this case, if both initial values of u and v have nonempty supports, then u and v will coexist in any compact subset of the habitat (Lin and Li 2012). Therefore, when the monostable case is involved, the nonlinear reaction plays an important role on spreading or vanishing, while the diffusion only affects the spreading speed. Of course, when $a, b > 1$ such that (1.2) has two locally stable steady states $(0, 1)$ and $(1, 0)$, then the dynamics of (1.1) are rich and depend on the initial values (Carrère 2018).

Will these phenomena occur in more general competition systems? In this paper, we consider the following degenerate competition diffusion system

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \Delta u(t,x) + r_1 u^p(t,x)\left[1 - u(t,x) - av(t,x)\right], \\ \frac{\partial v(t,x)}{\partial t} = d_2 \Delta v(t,x) + r_2 v^q(t,x)\left[1 - bu(t,x) - v(t,x)\right], \end{cases} \tag{1.3}$$

where $u(t,x)$ and $v(t,x)$ denote the population densities of two competitors at time t and location x, respectively, $t > 0$, $x \in \mathbb{R}$, $d_i, r_i > 0$, $i = 1, 2$ and $p, q \ge 1$. The factors $u^p(x,t)$ and $v^q(x,t)$ with $p > 1$ or $q > 1$ describe the situation that one or both competitors exhibit weak Allee effect in the absence of competition (Allee 1931). The interspecific term appears analogously as the self-regulation term and is proportional to $u^p(v^q)$. In addition, $a \geq 0$ ($b \geq 0$) is interpreted as measuring the extent of competitor $v(u)$ consuming resources needed by $u(v)$ and thus reducing the population growth rate for $u(v)$ (Murray 2002, 2003; Cantrell and Cosner 2004). In what follows, we use p and q to describe the degeneracy of u and v, respectively, and (1.3) is called a *non-degenerate system* if $p = q = 1$, while it is *degenerate* if $p > 1$ or $q > 1$. The corresponding kinetic system of (1.3) is

$$\begin{cases} \frac{du(t)}{\partial t} = r_1 u^p(t)\,[1 - u(t) - av(t)]\,, \\ \frac{dv(t)}{\partial t} = r_2 v^q(t)\,[1 - bu(t) - v(t)]\,, \end{cases} \tag{1.4}$$

which has a trivial equilibrium $(0, 0)$ and two spatially homogeneous equilibria $(1, 0)$ and $(0, 1)$. Furthermore, if $a, b \in (0, 1)$ or $a, b \in (1, +\infty)$, then (1.4) has an extra spatially homogeneous equilibrium $K = (k_1, k_2)$ defined by

$$(k_1, k_2) = \left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab} \right).$$

When $p, q \geq 1$, the existence and stability of steady states of (1.4) are similar to that in (1.2), which will be presented in Sect. 2. As we have mentioned, the dynamics of (1.1) are determined by the nonlinear reaction in the monostable case. Moreover, the interspecific competition could increase the local extinction rate in a metapopulation system (Bengtsson 1989). The purpose of this paper is to investigate the long time behavior of (1.3). We want to study whether the dynamics of (1.3) are fully determined by a and b if (1.4) is monostable and consider the effect of degeneracy as well as interspecific competition on the asymptotic spreading of (1.3) in both monostable and bistable cases.

In the degenerate case of (1.3), when the interspecific competition vanishes, we can obtain a Fisher equation with degenerate nonlinearity. Its propagation properties have been investigated intensively, see Aronson and Weinberger (1978), Bebernes et al. (1997), Berestycki and Nirenberg (1992), Wu et al. (2006), Zlatoš (2006), Chen and Qi (2007, 2009, 2019), Liang and Zhao (2007), Wu and Xing (2008), Du and Matano (2010), Alfaro (2017), Chen et al. (2017), He et al. (2017). In particular, Aronson and Weinberger (1978) pointed out that the successful propagation may depend on the degeneracy of nonlinearity as well as the size of the support of initial value. Based on the results by Aronson and Weinberger (1978), we will show that (1.3) may exhibit very rich dynamics depending on the initial values.

It should be noted that in population dynamics, the nonlinear reaction plays a crucial role on spreading or vanishing in the non-degenerate monostable case, while the diffusion only affects the spreading speed. However, our results show that the diffusion may also affect the spreading or vanishing of (1.3) in the degenerate monostable case, and some sufficient conditions on the balance between degenerate nonlinear reaction and diffusion are given in this work. Under proper conditions of the initial values and degeneracy of nonlinearity, we find that the superior competitor could be washed out by the inferior one in the sense of the corresponding kinetic system. Moreover, different from that in the non-degenerate case, by selecting different initial values in the degenerate case with $0 \le a,b < 1$, four different spreading phenomena may be observed and the interspecific competition of one species may be harmful to the persistence of the other species. To illustrate our results, some numerical simulations are presented.

The rest of this paper is organized as follows. In Sect. 2, we give some preliminaries including the dynamics of (1.4), comparison principle of (1.3), and some conclusions from Aronson and Weinberger (1978) and Du and Matano (2010). The main results are presented in Sect. 3, which include some sufficient conditions for the persistence or extinction of u and v. We then provide some numerical simulations in Sect. 4, and the paper ends with a compendious discussion on the topic in Sect. 5.

2 Preliminaries

In this section, we introduce some concepts and review some relevant results. Firstly, we present the dynamics of (1.4) with positive initial values. If $0 \le a, b < 1$, then (1.4) has a unique positive equilibrium (k_1, k_2) which is a stable node. In fact, define

$$
\begin{aligned}
&a_i(s) = sk_i,\ \ b_i(s) = sk_i + (1-s)(1+\epsilon), \quad i = 1, 2,\\
&a(s) = (a_1(s), a_2(s)), \quad b(s) = (b_1(s), b_2(s))
\end{aligned}
$$

with $\epsilon > 0$ small enough and $s \in (0, 1]$. By Lemma 5.7.4 of Smith (2008), we obtain a strictly contracting rectangle of (1.4). Therefore, (k_1, k_2) is globally asymptotically stable, that is, the solution $(u(t), v(t))$ of (1.4) satisfies

$$
\lim_{t\to+\infty} (|u(t) - k_1| + |v(t) - k_2|) = 0
$$

with the initial values $u(0) > 0$ and $v(0) > 0$, which is similar to that in (1.2). For other cases, we can also analyze the stability. Particularly, $p, q \ge 1$ do not affect the existence and stability of steady states in (1.4), and we list the conclusions as follows:

(1) $(u(t), v(t)) \to (0, 1)$ as $t \to +\infty$ if $0 \le b \le 1 \le a$ and $a \ne b$;
(2) $(u(t), v(t)) \to (1, 0)$ as $t \to +\infty$ if $0 \le a \le 1 \le b$ and $a \ne b$;
(3) $(u(t), v(t)) \to \{(0, 1), (1, 0), (k_1, k_2)\}$ as $t \to +\infty$ if $a, b > 1$, which depends on the initial values;
(4) $(u(t), v(t)) \to (k_1, k_2)$ as $t \to +\infty$ if $0 \le a, b < 1$.

Let

$$
X = \{u(x) | u(x) : \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\}
$$

be equipped with the compact open topology and maximum norm $\|\cdot\|$. Moreover, denote

$$
X^+ = \{u : u \in X \text{ and } u(x) \ge 0 \text{ for all } x \in \mathbb{R}\}.
$$

Furthermore, if $m \leq n \in \mathbb{R}$, then define

$$X_{[m,n]} = \{u : u \in X \text{ and } m \leq u(x) \leq n \text{ for all } x \in \mathbb{R}\}$$

and

$$X^2_{[m,n]} = \{(u_1, u_2) : u_i \in X \text{ and } m \leq u_i(x) \leq n \text{ for all } x \in \mathbb{R}, i = 1, 2\}.$$

In order to obtain the spreading speed of the competition system with degenerate nonlinearity described by (1.3), we consider the following initial value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \Delta u(t,x) + r_1 u^p(t,x)\left[1 - u(t,x) - av(t,x)\right], \\ \frac{\partial v(t,x)}{\partial t} = d_2 \Delta v(t,x) + r_2 v^q(t,x)\left[1 - bu(t,x) - v(t,x)\right], \\ (u(0,x), v(0,x)) = (u(x), v(x)) \in X^2_{[0,1]}, \end{cases} \tag{2.1}$$

in which $t > 0$, $x \in \mathbb{R}$, $a, b \geq 0$, $d_i, r_i > 0$, $i = 1, 2$ and $p, q \geq 1$.

Following Definition 4 and Remark 2 of Fife and Tang (1981), we introduce an admissible pair of super- and sub-solutions of (2.1) as follows.

Definition 2.1 Assume that

$$\begin{aligned} &\overline{u}(t,x) = \min\{\overline{u}_1(t,x), \ldots, \overline{u}_n(t,x)\}, \ \overline{v}(t,x) = \min\{\overline{v}_1(t,x), \ldots, \overline{v}_n(t,x)\}, \\ &\underline{u}(t,x) = \max\{\underline{u}_1(t,x), \ldots, \underline{u}_n(t,x)\}, \ \underline{v}(t,x) = \max\{\underline{v}_1(t,x), \ldots, \underline{v}_n(t,x)\} \end{aligned}$$

are continuous functions for some integer n, $x \in \mathbb{R}$, $t \in [0, T^*)$ with some $T^* > 0$. Then $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are said to be a pair of *super-* and *sub-solutions* of (2.1) if they satisfy the following statements:

(i) $(0,0) \leq (\underline{u}(0,x), \underline{v}(0,x)) \leq (u(x), v(x)) \leq (\overline{u}(0,x), \overline{v}(0,x)) \leq (1,1)$ for $x \in \mathbb{R}$ and $(0,0) \leq (\underline{u}(t,x), \underline{v}(t,x)), (\overline{u}(t,x), \overline{v}(t,x)) \leq (1,1)$ for $t > 0$, $x \in \mathbb{R}$;

(ii) if $\overline{u}(t,x) = \overline{u}_i(t,x)$ for some $i \in \{1, \ldots, n\}$, then

$$\frac{\partial \overline{u}_i(t,x)}{\partial t} \geq d_1 \Delta \overline{u}_i(t,x) + r_1 \overline{u}_i^p(t,x)\left[1 - \overline{u}_i(t,x) - a\underline{v}(t,x)\right]$$

and $\overline{v}(t,x)$ satisfies a similar inequality;

(iii) if $\underline{u}(t,x) = \underline{u}_i(t,x)$ for some $i \in \{1, \ldots, n\}$, then

$$\frac{\partial \underline{u}_i(t,x)}{\partial t} \leq d_1 \Delta \underline{u}_i(t,x) + r_1 \underline{u}_i^p(t,x)\left[1 - \underline{u}_i(t,x) - a\overline{v}(t,x)\right]$$

and $\underline{v}(t,x)$ satisfies a similar inequality.

By virtue of the comparison principle, we have the following lemma.

Lemma 2.1 *Assume that* $(\overline{u}(t,x), \overline{v}(t,x))$ *and* $(\underline{u}(t,x), \underline{v}(t,x))$ *are a pair of super- and sub-solutions of (2.1) with* $t \in [0, T^*)$ *and* $x \in \mathbb{R}$. *Then there is a unique solution* $(u(t,x), v(t,x))$ *of (2.1) satisfying*

$(\underline{u}(t,x), \underline{v}(t,x)) \leq (u(t,x), v(t,x)) \leq (\overline{u}(t,x), \overline{v}(t,x))$ *for all* $(t,x) \in [0, T^*) \times \mathbb{R}$.

To describe the extinction or persistence of (1.3), for any $p \geq 1$, we first recall the following initial value problem involving p-degree Fisher nonlinearity

$$\begin{cases} \frac{\partial z(t,x)}{\partial t} = d\Delta z(t,x) + rz^p(t,x)\left[A - Mz(t,x)\right], t > 0, x \in \mathbb{R}, \\ z(0,x) = z(x), x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where all coefficients are positive and $z(x) \in X_{[0,A/M]}$ has nonempty compact support. For the first equation in (2.2), there are many results on traveling wave solutions, see e.g., Berestycki and Nirenberg (1992), Chen and Qi (2007, 2009, 2019), Liang and Zhao (2007), Chen et al. (2017). Furthermore, the upper and lower bounds of the critical speed were investigated by Chen and Qi (2007, 2009, 2019); Chen et al. (2017).

Let

$$w = \frac{M}{A}z, \quad t_1 = \frac{rA^p}{M^{p-1}}t, \quad x_1 = \left(\frac{rA^p}{dM^{p-1}}\right)^{1/2} x.$$

Then the first equation in (2.2) can be transformed into

$$\frac{\partial w(t_1,x_1)}{\partial t_1} = \Delta w(t_1,x_1) + w^p(t_1,x_1)\left[1 - w(t_1,x_1)\right], \ t_1 > 0, \ x_1 \in \mathbb{R}.$$

It should be noted that there is a sharp speed $c^* > 0$ such that (2.2) has a monotone increasing traveling wave solution if and only if $c \geq c^*$ (Berestycki and Nirenberg 1992; Liang and Zhao 2007). Moreover, in view of Theorem 4 of Chen and Qi (2007), we have

$$c^* = \sqrt{\frac{drA^p}{M^{p-1}K(p)}}, \tag{2.3}$$

where $K(p)$ is a strictly monotone increasing function of p and $K(1)=1/4$, $K(2)=2$.

For positive constants d, r, A, M given above, define a continuous function

$$\psi_{(A,M)}(t,x;\xi,p,r,d) = \frac{A}{M}\left[\left(\frac{1}{2} - \frac{1}{p-1}\right)/(st+\xi)\right]^{\frac{1}{p-1}} e^{-sx^2/4d(st+\xi)} \tag{2.4}$$

with some $p > 3$, $\xi > 0$ and $s = \frac{rA^p}{M^{p-1}}$. A straightforward calculation yields that

$$\psi_t - d\Delta\psi - rA\psi^p \geq 0 \text{ for all } (t,x) \in (0,+\infty) \times \mathbb{R},$$

which implies that ψ is a super-solution of (2.2) if $z(x) \leq \psi_{(A,M)}(0, x; \xi, p, r, d)$ for $x \in \mathbb{R}$. By virtue of the stability results of (2.2) investigated in Sect. 3 of Aronson and Weinberger (1978), Theorem 1.1 and Remark 4.12 of Du and Matano (2010), and the spreading theory established by Theorem 2.17 of Liang and Zhao (2007), we can obtain the following spreading properties.

Lemma 2.2 *Assume that $z(x) \in X_{[0,A/M]}$. Then the solution $z(t,x)$ of (2.2) is well defined in $(0,+\infty)\times\mathbb{R}$ and the following properties hold.*

(i) For any fixed $p \in [1,3]$, if $\epsilon \in (0, c^*)$ is given and $z(x) \not\equiv 0$, then

$$\lim_{t\to+\infty,\ |x|<(c^*-\epsilon)t} z(t,x) = \frac{A}{M}.$$

(ii) For any fixed $p > 3$,

(a) suppose further that $\epsilon \in (0, c^*)$ is given and $g(x) \in X_{[0,A/M]}$ has nonempty compact support. Then there exists a positive constant $\sigma := \sigma_{(f,d)}(g(x))$ independent of ϵ such that

$$\lim_{t\to+\infty,|x|<(c^*-\epsilon)t} z(t,x) = \frac{A}{M} \tag{2.5}$$

if $z(x) \geq \lambda g(x)$ for every $\lambda > \sigma$, while

$$\lim_{t\to+\infty} z(t,x) = 0 \text{ uniformly in } \mathbb{R}$$

for $z(x) \leq \sigma g(x)$;

(b) if $z(x) \leq \psi_{(A,M)}(0, x; \xi, p, r, d)$ for $x \in \mathbb{R}$ and some $\xi > 0$, then

$$\lim_{t\to+\infty} z(t,x) = 0 \text{ uniformly in } \mathbb{R}.$$

(iii) For any given $\epsilon > 0$ and $p \geq 1$, if $z(x)$ has nonempty compact support, then

$$\lim_{t\to+\infty,\ |x|>(c^*+\epsilon)t} z(t,x) = 0.$$

Remark 2.1 Actually, for any given $p > 3$, define a continuous function

$$\psi_A(t, x; p, r, d, \alpha, \beta) = g(t)\varphi(t,x)$$

with

$$g(t) := \left(1 - \frac{(p-1)rA\alpha^{p-1}}{2\beta(p-3)}\left(1 - \frac{1}{(1+4\beta dt)^{\frac{p-3}{2}}}\right)\right)^{-\frac{1}{p-1}},$$
$$\varphi(t,x) := \frac{\alpha}{\sqrt{1+4\beta dt}} e^{-\frac{\beta}{1+4\beta dt}x^2}$$

for some positive $\alpha, \beta > 0$ such that $\beta > \frac{(p-1)rA\alpha^{p-1}}{2(p-3)}$. Then it is also a super-solution of (2.2) with $z(x) \leq \psi_A(0, x; p, r, d, \alpha, \beta) = \alpha e^{-\beta x^2}$ and satisfies

$$\lim_{t \to +\infty} \psi_A(t, x; p, r, d, \alpha, \beta) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

3 Main results

In this section, we investigate the asymptotic spreading of (1.3) with $a, b \geq 0$, which involves the cases of strong and weak interspecific competition. Before stating our results, for any given $p, q \geq 1$, we first define positive constants

$$c_1 = \sqrt{\frac{d_1 r_1}{K(p)}},\ c_2 = \sqrt{\frac{d_2 r_2}{K(q)}},$$

where $K(p)$ and $K(q)$ are strictly monotone increasing functions given by (2.3). Further define a continuous function $\overline{v}(t, x)$ as a solution of the following initial value problem

$$\begin{cases} \frac{\partial \overline{v}(t,x)}{\partial t} = d_2 \Delta \overline{v}(t, x) + r_2 \overline{v}^q(t, x) \left[1 - \overline{v}(t, x)\right], t > 0, x \in \mathbb{R}, \\ \overline{v}(0, x) = v(x), x \in \mathbb{R}. \end{cases}$$

In what follows, to simplify the notation, for any given $g(x) \in X^+$ having nonempty compact support, we write $u(x) > \sigma g(x)$ if $u(x) \geq \lambda g(x)$ for every $\lambda > \sigma$. For any given $q > 3$, let $\psi(t, x; \xi_2, q, r_2, d_2)$ be a continuous function defined in (2.4) with $A = M = 1$ and some $\xi_2 > 0$. Moreover, let $g_1(x) \in X_{[0,1]}$ be a continuous function with nonempty compact support. Then Lemma 2.2 implies that there exists a positive constant $\sigma_1 := \sigma_1(g_1(x)) > 0$ such that if $v(x) \leq \psi(0, x; \xi_2, q, r_2, d_2)$ or $v(x) \leq \sigma_1 g_1(x)$, then

$$\lim_{t \to +\infty} \overline{v}(t, x) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

Similarly, we can define $\overline{u}(t, x)$ and $\psi(t, x; \xi_1, p, r_1, d_1)$ for any $p > 3$. In addition, for any given $g_3(x) \in X_{[0,1]}$ with nonempty compact support, there exists a positive constant $\sigma_3 := \sigma_3(g_3(x)) > 0$ such that if $u(x) \leq \psi(0, x; \xi_1, p, r_1, d_1)$ or $u(x) \leq \sigma_3 g_3(x)$, then

$$\lim_{t \to +\infty} \overline{u}(t, x) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

Note that for any given functions $g_1(x), g_3(x) \in X_{[0,1]}$ with nonempty compact support, we could fix constants σ_1 and σ_3 depending on parameters in (2.1). Moreover, from the property of autonomous equations/systems, the above results remain true if

$$u(x) \leq \psi(t', x; \xi_1, p, r_1, d_1)$$

or

$$v(x) \le \psi(t', x; \xi_2, q, r_2, d_2)$$

for some given $t' \in [0, \infty)$. This feature holds in the following discussion on (2.1), but we only take $t' = 0$ for the sake of simplicity.

Using the comparison principle, we have the following theorem.

Theorem 3.1 *Assume that* $(u(x), v(x)) \in X^2_{[0,1]}$ *has nonempty compact support. Then the solution* $(u(t,x), v(t,x))$ *of (2.1) is well defined in* $(0, +\infty) \times \mathbb{R}$ *and satisfies the following statements.*

(i) For any given $p > 3$, *if* $u(x) \le \sigma_3 g_3(x)$ *or* $u(x) \le \psi(0, x; \xi_1, p, r_1, d_1)$, *then*

$$\lim_{t\to+\infty} u(t,x) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

(ii) For any given $q > 3$, *if* $v(x) \le \sigma_1 g_1(x)$ *or* $v(x) \le \psi(0, x; \xi_2, q, r_2, d_2)$, *then*

$$\lim_{t\to+\infty} v(t,x) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

When $a, b \in [0, 1)$, we further define positive constants

$$c_3 = \sqrt{\frac{d_1 r_1 (1-a)^p}{K(p)}},\ c_4 = \sqrt{\frac{d_2 r_2 (1-b)^q}{K(q)}},$$
$$c_5 = \sqrt{\frac{d_1 r_1 (1-a)^{p-1}}{K(p)}},\ c_6 = \sqrt{\frac{d_2 r_2 [1-b(1-a)]}{K(q)}}.$$

Let $\underline{u}(t,x)$ and $\underline{v}(t,x)$ be defined by

$$\begin{cases} \frac{\partial \underline{u}(t,x)}{\partial t} = d_1 \Delta \underline{u}(t,x) + r_1 \underline{u}^p(t,x)\left[1 - a - \underline{u}(t,x)\right], t > 0, x \in \mathbb{R}, \\ \underline{u}(0,x) = u(x), x \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} \frac{\partial \underline{v}(t,x)}{\partial t} = d_2 \Delta \underline{v}(t,x) + r_2 \underline{v}^q(t,x)\left[1 - b - \underline{v}(t,x)\right], t > 0, x \in \mathbb{R}, \\ \underline{v}(0,x) = v(x), x \in \mathbb{R}, \end{cases}$$

respectively. From the comparison principle and Lemma 2.2, for any given $p, q > 3$ and $g_2(x) \in X_{[0,1-b]}$, $g_4(x) \in X_{[0,1-a]}$ with nonempty compact support, there exist constants $\sigma_i := \sigma_i(g_i(x)) > 0 (i = 2, 4)$ such that if $u(x) > \sigma_4 g_4(x)$, then

$$\liminf_{t\to+\infty} \inf_{|x|<(c_3-\epsilon)t} \underline{u}(t,x) \ge 1 - a$$

for any given $\epsilon \in (0, c_3)$, while

$$\liminf_{t\to+\infty} \inf_{|x|<(c_4-\epsilon)t} \underline{v}(t,x) \geq 1-b$$

if $v(x) > \sigma_2 g_2(x)$ and $\epsilon \in (0, c_4)$ is given. In what follows, we assume that $g_i(x)$ are given functions with $i \in \{1, 2, 3, 4\}$, so we may fix the corresponding constants σ_i depending on parameters in (2.1).

It should be noted that one important measure of invasion of a species is the speed at which it spreads into the other competitor's habitat. In what follows, we consider the invasion speeds of the diffusion competition model (1.3), in which the competitors occupy a common habitat with different competitive ability. For the sake of clarity, this section is split into two subsections.

3.1 Weak competition case $0 \leq a, b < 1$

Firstly, we consider the asymptotic spreading of (1.3) with $0 \leq a, b < 1$, which is the case of the so-called weak competition. From Sect. 2, when $0 \leq a, b < 1$, the unique positive equilibrium (k_1, k_2) of (1.4) is globally asymptotically stable. If $p = q = 1$, Lin and Li (2012) established some results on the asymptotic spreading of (1.3) with coinvasion-coexistence process, where both u and v are invaders. There are several interesting phenomena modeled by this process, see e.g., Davis (1981), Chesson (2000). In what follows, we only focus on the degenerate case and consider the balance between degenerate nonlinear reaction and diffusion. In particular, we introduce the following condition

(F1): $\frac{d_2 r_2}{K(q)} < \frac{d_1 r_1 (1-a)^p}{K(p)}$.

Theorem 3.2 *Assume that* $(u(x), v(x)) \in X^2_{[0,1]}$ *and* $u(x)$ *has nonempty compact support. If* $0 \leq a, b < 1$, *then the solution* $(u(t,x), v(t,x))$ *of* (2.1) *is well defined for all* $(t,x) \in (0, +\infty) \times \mathbb{R}$ *and the following properties hold:*

(i) For any given $\epsilon > 0$,

$$\lim_{t\to+\infty} \sup_{|x|>(c_1+\epsilon)t} u(t,x) = 0.$$

(ii) Suppose further that $p > 3$ *is given. If* $u(x) \leq \sigma_3 g_3(x)$ *or* $u(x) \leq \psi(0, x; \xi_1, p, r_1, d_1)$, *then*

$$\lim_{t\to+\infty} u(t,x) = 0 \text{ uniformly in } \mathbb{R}.$$

Moreover,

(a) assume that $\epsilon \in (0, c_2)$ *is given and* $v(x) \not\equiv 0$, *if* $q \leq 3$ *or* $q > 3$ *and* $v(x) > \sigma_2 g_2(x)$, *then*

$$\lim_{t\to+\infty} \inf_{|x|<(c_2-\epsilon)t} v(t,x) = 1;$$

(b) *if* $\liminf_{|x|\to+\infty} v(x) > 0$, *then*

$$\lim_{t\to+\infty} v(t,x) = 1 \textit{ uniformly in } \mathbb{R}.$$

(iii) *For any* $v(x) \geq 1-b$ *with* $x \in \mathbb{R}$, *let* $L_1, L_2 \in \mathbb{R}$ *be given constants and* $w(t,x)$ *be the solution of*

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} = d_1 \Delta w(t,x) + r_1 w^p(t,x)\left[1-k-w(t,x)\right], t>0, x\in\mathbb{R}, \\ w(0,x) = u(x), x\in\mathbb{R} \end{cases}$$

with $k \in [0,1)$, *denote*

$$\Sigma_0^k := \{u(x) \in \overline{B} : w(t,x) \to 0 \textit{ as } t \to +\infty \textit{ uniformly in } x \in \mathbb{R}\},$$

where

$$B = \{u(x) : u(x) \in X_{[0,1]} \textit{ with } u(x) > 0 \textit{ in } (L_1, L_2) \\ \textit{and } u(x) = 0 \textit{ in } \mathbb{R}\backslash(L_1, L_2)\}$$

and $\overline{B}$ *is the closure of* B, *if* $u(x) \in \Sigma_0^{1-a(1-b)}$, *then*

$$\lim_{t\to+\infty} (u(t,x) + |v(t,x)-1|) = 0 \textit{ uniformly in } \mathbb{R}.$$

In addition, Σ_0^k *is nonempty, closed in* $\overline{B}$ *and* $\Sigma_0^0 \subsetneq \Sigma_0^{1-a(1-b)}$ *if* $\Sigma_0^0 \neq \{0\}$ *and* $\overline{B}$.

(iv) *Suppose further that (F1) holds and* $v(x)$ *has nonempty compact support. If* $p \leq 3$ *or* $p > 3$ *and* $u(x) > \sigma_4 g_4(x)$, *then we can obtain the following properties:*

(a) *for any given* $\epsilon \in (0, c_5)$,

$$\liminf_{t\to+\infty} \inf_{|x|<(c_5-\epsilon)t} u(t,x) \geq 1-a;$$

(b) *for any given* $q > 3$, *if* $v(x) \leq \sigma_1 g_1(x)$ *or* $v(x) \leq \psi(0, x; \xi_2, q, r_2, d_2)$, *then*

$$\lim_{t\to+\infty} v(t,x) = 0 \textit{ uniformly in } \mathbb{R}$$

and

$$\lim_{t\to+\infty} \inf_{|x|<(c_1-\epsilon)t} u(t,x) = 1$$

for any given $\epsilon \in (0, c_1)$;

(c) for any given $\epsilon > 0$*, if* $c_5 > c_2 + c_6$*, then*

$$\lim_{t\to+\infty} \sup_{|x|>(c_6+\epsilon)t} v(t,x) = 0;$$

(d) for any given $\epsilon \in (0, \frac{c_5-c_2}{2})$*, then*

$$\lim_{t\to+\infty, (c_2+\epsilon)t<|x|<(c_5-\epsilon)t} u(t,x) = 1;$$

(e) for any given $\epsilon \in (0, c_4)$*, if* $q \leq 3$ *or* $q > 3$ *and* $v(x) > \sigma_2 g_2(x)$*, then*

$$\lim_{t\to+\infty} \sup_{|x|<(c_4-\epsilon)t} (|u(t,x) - k_1| + |v(t,x) - k_2|) = 0.$$

For the sake of simplicity, the proof of Theorem 3.2 is divided into several lemmas, through which $0 \leq a, b < 1$, $p, q \geq 1$, $(u(x), v(x)) \in X^2_{[0,1]}$ and $u(x)$ has nonempty compact support.

Lemma 3.1 *For any given* $p > 3$*, if* $u(x) \leq \sigma_3 g_3(x)$ *or* $u(x) \leq \psi(0, x; \xi_1, p, r_1, d_1)$*, then*

$$\lim_{t\to+\infty} u(t,x) = 0 \textit{ uniformly in } \mathbb{R}.$$

Moreover, the following properties hold.

(i) Assume that $\epsilon \in (0, c_2)$ *is given and* $v(x) \not\equiv 0$*. If* $q \leq 3$ *or* $q > 3$ *and* $v(x) > \sigma_2 g_2(x)$*, then*

$$\liminf_{t\to+\infty} \inf_{|x|<(c_2-\epsilon)t} v(t,x) = 1.$$

(ii) If $\liminf_{|x|\to+\infty} v(x) > 0$*, then*

$$\lim_{t\to+\infty} v(t,x) = 1 \textit{ uniformly in } \mathbb{R}.$$

Proof Since $v(t,x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$, we have

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} \leq d_1 \Delta u(t,x) + r_1 u^p(t,x) [1 - u(t,x)], t > 0, x \in \mathbb{R}, \\ u(0,x) = u(x), x \in \mathbb{R}. \end{cases}$$

For any given $p > 3$, if $u(x) \leq \sigma_3 g_3(x)$ or $u(x) \leq \psi(0, x; \xi_1, p, r_1, d_1)$, then $\overline{u}(t,x)$ is a super-solution of $u(t,x)$, the comparison principle implies that

$$\lim_{t\to+\infty} u(t,x) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

It then follows that for any $\epsilon' > 0$, there exists a sufficient large number $T_1 > 0$ such that $u(t,x) \leq \epsilon'$ for all $t > T_1$ and $x \in \mathbb{R}$. Hence,

$$\frac{\partial v(t,x)}{\partial t} \geq d_2 \Delta v(t,x) + r_2 v^q(t,x) \left[1 - b\epsilon' - v(t,x)\right] \tag{3.1}$$

with $t > T_1$ and $x \in \mathbb{R}$. We prove the conclusions in two steps.

Step 1. $v(x) \not\equiv 0$. Since $u(t,x) \leq 1$ and $b < 1$, it follows that

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} \geq d_2 \Delta v(t,x) + r_2 v^q(t,x) \left[1 - b - v(t,x)\right], t > 0, x \in \mathbb{R}, \\ v(0,x) = v(x), x \in \mathbb{R}. \end{cases}$$

So $\underline{v}(t,x)$ is a sub-solution of $v(t,x)$. Thus, if $q \leq 3$ or $q > 3$ and $v(x) > \sigma_2 g_2(x)$, then

$$\liminf_{t \to +\infty} \inf_{|x| < (c_4 - \epsilon_0)t} v(t,x) \geq 1 - b$$

for any given $\epsilon_0 \in (0, c_4)$.

When $\epsilon \in (0, c_2)$ is given, we can choose $\epsilon' > 0$ small enough such that

$$\sqrt{\frac{d_2 r_2}{K(q)}} - \epsilon < \sqrt{\frac{d_2 r_2 (1 - b\epsilon')^q}{K(q)}},$$

then Theorem 2.17 of Liang and Zhao (2007) and (3.1) imply that

$$\liminf_{t \to +\infty} \inf_{|x| < (c_2 - \epsilon)t} v(t,x) \geq 1 - b\epsilon'. \tag{3.2}$$

Due to the arbitrariness of ϵ', we deduce the assertion of (i).

Step 2. $\liminf\limits_{|x| \to +\infty} v(x) > 0$. Let $\beta = 2br_2$. Then

$$\frac{\partial v(t,x)}{\partial t} \geq d_2 \Delta v(t,x) - \beta v(t,x).$$

It then follows that $v(t,x) \geq e^{-\beta t} \widetilde{v}(t,x)$ with $t > 0, x \in \mathbb{R}$, in which $\widetilde{v}(t,x)$ is the solution of

$$\begin{cases} \frac{\partial \widetilde{v}(t,x)}{\partial t} = d_2 \Delta \widetilde{v}(t,x), t > 0, x \in \mathbb{R}, \\ \widetilde{v}(0,x) = v(x), x \in \mathbb{R}. \end{cases}$$

Since $\liminf_{|x| \to +\infty} v(x) > 0$, there exists $\theta \in (0,1)$ such that $\widetilde{v}(t,x) > \theta > 0$ as $|x| \to +\infty$. Furthermore, by virtue of the strong maximum principle, for any fixed $T_2 \geq T_1 > 0$, there exists $\theta_1 := \theta_1(T_2)$ such that $v(T_2, x) > \theta_1 > 0$ for all $x \in \mathbb{R}$. The asymptotic stability of the steady state in (3.1) further implies that

$$\liminf_{t \to +\infty} v(t,x) \geq 1 - (1+b)\epsilon' \text{ uniformly in } x \in \mathbb{R}.$$

Since $\epsilon' > 0$ is arbitrary, we complete the proof. □

Lemma 3.2 *Assume that $q > 3$ is given. If $v(x) \leq \psi(0, x; \xi_2, q, r_2, d_2)$ or $v(x) \leq \sigma_1 g_1(x)$, then*

$$\lim_{t\to+\infty} v(t,x) = 0 \text{ uniformly in } \mathbb{R}. \tag{3.3}$$

Moreover, suppose that $p \leq 3$ or $p > 3$ and $u(x) > \sigma_4 g_4(x)$. Then

$$\lim_{t\to+\infty, |x|<(c_1-\epsilon)t} u(t,x) = 1$$

for any given $\epsilon \in (0, c_1)$.

Proof Following exactly the same arguments as that in Lemma 3.1, we complete the proof. □

By selecting different initial values in (2.1), the above two lemmas reflect that one species may vanish if the Allee effect of this species is strong enough and the initial habitat size is sufficient small, while the invasion front of the other species is similar to that of single species described by the Fisher equation. Comparing with Lemma 3.1, we shall present that the interspecific competition may be harmful to the persistence in the following two lemmas.

Lemma 3.3 *Suppose that $p \geq 1$ is given. For the following initial value problem*

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} = d_1 \Delta w(t,x) + r_1 w^p(t,x)\left[1 - k - w(t,x)\right], t > 0, x \in \mathbb{R}, \\ w(0,x) = u(x), x \in \mathbb{R} \end{cases} \tag{3.4}$$

with $k \in [0, 1)$, let $L_1, L_2 \in \mathbb{R}$ be given constants and

$$\Sigma_0^k := \{u(x) \in \overline{B} : w(t,x) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}\},$$

where

$$\begin{aligned} B = \{u(x) : u(x) \in X_{[0,1]} \text{ with } u(x) > 0 \text{ in } (L_1, L_2) \\ \text{and } u(x) = 0 \text{ in } \mathbb{R}\backslash(L_1, L_2)\} \end{aligned}$$

and $\overline{B}$ denotes the closure of B, if $\Sigma_0^k \neq \{0\}$ and $\overline{B}$, then Σ_0^k is strictly monotone increasing in k.

Proof Assume that $1 > k_1 > k_2$ are two nonnegative constants, $\Sigma_0^{k_1}$ and $\Sigma_0^{k_2}$ are the corresponding set, w_1 and w_2 are the corresponding solutions of (3.4), then it suffices to prove that $\Sigma_0^{k_2} \subsetneq \Sigma_0^{k_1}$. Form the comparison principle, $\Sigma_0^{k_1}$ and $\Sigma_0^{k_2}$ are connected and $w_2 \geq w_1$ for all $t > 0, x \in \mathbb{R}$, then $\Sigma_0^{k_2} \subset \Sigma_0^{k_1}$. Now we show $\Sigma_0^{k_2} \neq \Sigma_0^{k_1}$.

Firstly, for any given $k \in [0, 1)$, define

$$\Sigma_{1-k}^{k} := \{u(x) \in \overline{B} : w(t, x) \to 1 - k \text{ as } t \to +\infty \text{ locally uniformly in } x \in \mathbb{R}\}.$$

Based on Theorem 1 and Lemma 2.1 of Du and Matano (2010), we have $\Sigma_{1-k}^{k} \cup \Sigma_{0}^{k} = \overline{B}$ and $\Sigma_{1-k}^{k} \cap \Sigma_{0}^{k} = \emptyset$. Since $\Sigma_{0}^{k} \neq \{0\}$ and $\overline{B}$, then Σ_{1-k}^{k} is nonempty. The rest of the proof is divided into two steps.

Step 1. Σ_{1-k}^{k} is open in $\overline{B}$. For any given $\alpha \in (0, 1-k)$, Theorem 2.17 of Liang and Zhao (2007) implies that there exists $L_\alpha > 0$ such that if $u(x) > \alpha$ with $x \in [-L_\alpha, L_\alpha]$, then $w(t, x) \to 1 - k$ as $t \to +\infty$ locally uniformly in $\mathbb{R}$. Hence,

$$u(x) \in \Sigma_{1-k}^{k} \iff \min_{-L_\alpha \le x \le L_\alpha} w(t_0, x) > \alpha \text{ for some } t_0 > 0. \tag{3.5}$$

If $u(x) \in \Sigma_{1-k}^{k}$, then the right side of (3.5) holds. By virtue of the continuous dependence on initial values of the solution, the right side of (3.5) also holds for $|w(x) - u(x)| < \delta$ with $w(x) \in \overline{B}$ if $\delta > 0$ is small enough, so $w(x) \in \Sigma_{1-k}^{k}$. Thus, Σ_{1-k}^{k} is an open set in $\overline{B}$.

Step 2. $\Sigma_{0}^{k_2} \neq \Sigma_{0}^{k_1}$. Since $\Sigma_{1-k_1}^{k_1}$ is nonempty and open in $\overline{B}$ and $0 \in \Sigma_{0}^{k_1}$, it follows that $\Sigma_{0}^{k_1}$ is nonempty and closed in $\overline{B}$, so $\Sigma_{0}^{k_1} \cap \overline{\Sigma}_{1-k_1}^{k_1}$ is nonempty. Taking $u(x) \in \Sigma_{0}^{k_1} \cap \overline{\Sigma}_{1-k_1}^{k_1}$, if $\Sigma_{0}^{k_2} \neq \Sigma_{0}^{k_1}$ is not true, then $u(x) \in \Sigma_{0}^{k_2}$. On the other hand, since $\Sigma_{0}^{k_1} \neq \{0\}$ and $\overline{B}$, the maximum principle implies that $w_2(t, x) > w_1(t, x) > 0$ for $t > 0$, $x \in \mathbb{R}$.

Note that

$$w_i(t, x) = w_i(t - s, x, w_i(s, \cdot)) \text{ with } i = 1, 2 \text{ for any } 0 \le s \le t,$$

and $w_2(s, x, u(x)) > w_1(s, x, u(x))$ for any fixed $s > 0$. By virtue of the continuous dependence on initial values of the solution, the comparison principle implies that $w_2(t, x) \ge 1 - k_1$ locally uniformly in $\mathbb{R}$ as $t \to +\infty$, which contradicts with $u(x) \in \Sigma_{0}^{k_2}$. Hence, $\Sigma_{0}^{k_2} \neq \Sigma_{0}^{k_1}$. The proof is complete. □

Lemma 3.4 *Assume that $p > 3$ is given and Σ_{0}^{k}, $\overline{B}$ are defined by Lemma 3.3 with $k \in [0, 1)$. For any $v(x) \ge 1 - b$ with $x \in \mathbb{R}$, if $u(x) \in \Sigma_{0}^{1-a(1-b)}$, then*

$$\lim_{t \to +\infty} (u(t, x) + |v(t, x) - 1|) = 0 \textit{ uniformly in } \mathbb{R}. \tag{3.6}$$

In addition, Σ_{0}^{k} is nonempty, closed in $\overline{B}$ and $\Sigma_{0}^{0} \subsetneq \Sigma_{0}^{1-a(1-b)}$ if $\Sigma_{0}^{0} \neq \{0\}$ and $\overline{B}$.

Proof If $v(x) \ge 1 - b$, then the comparison principle implies that $v(t, x) \ge 1 - b$ for all $t > 0$, $x \in \mathbb{R}$, that is

$$\frac{\partial u(t, x)}{\partial t} \le d_1 \Delta u(t, x) + r_1 u^p(t, x) \left[1 - a(1 - b) - u(t, x)\right], t > 0, x \in \mathbb{R}.$$

Hence, by Lemma 3.1, (3.6) holds if $u(x) \in \Sigma_0^{1-a(1-b)}$. Note that $v(t,x) \geq 0$ for all $t > 0$, $x \in \mathbb{R}$, then

$$\frac{\partial u(t,x)}{\partial t} \leq d_1 \Delta u(t,x) + r_1 u^p(t,x)\left[1 - u(t,x)\right], t > 0, x \in \mathbb{R},$$

and so

$$\lim_{t \to +\infty} u(t,x) = 0 \text{ uniformly in } \mathbb{R}$$

if $u(x) \in \Sigma_0^0$. When $\Sigma_0^0 \neq \{0\}$ and $\overline{B}$, then the comparison principle reflects that $\Sigma_0^{1-a(1-b)} \neq \{0\}$, so $\Sigma_0^0 \subsetneq \Sigma_0^{1-a(1-b)}$ if $\Sigma_0^{1-a(1-b)} = \overline{B}$. If $\Sigma_0^{1-a(1-b)} \neq \overline{B}$, then $\Sigma_0^{1-a(1-b)}$, $\Sigma_0^0 \neq \{0\}$ and $\overline{B}$, applying Lemma 3.3, we complete the proof. □

It should be noted that if $p \in [1,3]$, then $\Sigma_0^k = \{0\}$ for any given $k \in [0,1)$. The first four lemmas reveal that the vanishing phenomenon occurs in the degenerate system (1.3), which depends on the size of initial habitats as well as the intensity of Allee effect. Hereafter, we consider the coexistence of these two species.

Lemma 3.5 *Assume that $\epsilon \in (0, c_3)$ is given. If $p \leq 3$ or $p > 3$ and $u(x) > \sigma_4 g_4(x)$, then*

$$\liminf_{t \to +\infty} \inf_{|x| < (c_3 - \epsilon)t} u(t,x) \geq 1 - a.$$

Proof Clearly, we have $v(t,x) \leq 1$ for all $t > 0$ and $x \in \mathbb{R}$, then

$$\frac{\partial u(t,x)}{\partial t} \geq d_1 \Delta u(t,x) + r_1 u^p(t,x)\left[1 - a - u(t,x)\right], \ t > 0, \ x \in \mathbb{R}.$$

From the definition of $\underline{u}(t,x)$, if $p \leq 3$ or $p > 3$ and $u(x) > \sigma_4 g_4(x)$, then Lemma 2.2 together with the comparison principle ensures our results. The proof is complete. □

Lemma 3.6 *Assume that (F1) holds and $v(x)$ has nonempty compact support. If $p \leq 3$ or $p > 3$ and $u(x) > \sigma_4 g_4(x)$, then*

$$\liminf_{t \to +\infty} \inf_{|x| < (c_5 - \epsilon)t} u(t,x) \geq 1 - a$$

for any given $\epsilon \in (0, c_5)$.

Proof Since $u(t,x) \geq 0$ and $v(x) \in X_{[0,1]}$ has nonempty compact support, then

$$\frac{\partial v(t,x)}{\partial t} \leq d_2 \Delta v(t,x) + r_2 v^q(t,x)\left[1 - v(t,x)\right], \ t > 0, \ x \in \mathbb{R},$$

from the definition of $\overline{v}(t,x)$, the comparison principle and Lemma 2.2 imply that

$$\lim_{t \to +\infty} \sup_{|x| > (c_2 + \epsilon_0)t} v(t,x) = 0 \tag{3.7}$$

for any given $\epsilon_0 > 0$. When (F1) holds, from Lemma 3.5 and (3.7), for any $\epsilon' > 0$, there exists a sufficient large number $T_1 > 0$ such that

(i) $\sup_{2|x|<(c_2+c_3)t} \frac{v(t,x)}{u(t,x)} < \frac{1}{1-a-\epsilon'}$;
(ii) $\sup_{2|x|\geq(2c_2+\epsilon_0)t} v(t,x) < \epsilon'$

for all $t > T_1$. Hence, we have

$$\frac{\partial u(t,x)}{\partial t} \geq d_1 \Delta u(t,x) + r_1 u^p(t,x)\left[1 - a\epsilon' - \frac{1-\epsilon'}{1-a-\epsilon'}u(t,x)\right]$$

for $t > T_1$ and $x \in \mathbb{R}$.

For any given $\epsilon \in (0, c_5)$, choosing $\epsilon' > 0$ small enough such that

$$\sqrt{\frac{d_1 r_1 (1-a)^{p-1}}{K(p)}} - \epsilon < \sqrt{\frac{d_1 r_1 (1-a\epsilon')^p (1-a-\epsilon')^{p-1}}{(1-\epsilon')^{p-1} K(p)}},$$

then Theorem 2.17 of Liang and Zhao (2007) and the comparison principle imply that

$$\liminf_{t\to+\infty} \inf_{|x|<(c_5-\epsilon)t} u(t,x) \geq \frac{(1-a\epsilon')(1-a-\epsilon')}{1-\epsilon'}.$$

Due to the arbitrariness of ϵ', the proof is complete. □

Lemma 3.7 *Assume that Lemma 3.6 holds and $\epsilon > 0$ is given. If $c_5 > c_2 + c_6$, then*

$$\lim_{t\to+\infty} \sup_{|x|>(c_6+\epsilon)t} v(t,x) = 0.$$

Proof It suffices to consider the case $c_5 - \epsilon > c_2 + c_6$ with any given small number $\epsilon > 0$. Let $\delta > 0$ such that

$$\left(c_6 + \frac{\epsilon}{2}\right)^2 = \frac{d_2 r_2 [1 - b(1-a-\delta)]}{K(q)}.$$

Lemma 3.6 implies that there exists a positive number $T_5 > 0$ such that

(a) $\inf_{|x|<(c_5-\epsilon/4)} u(t,x) \geq 1 - a - \frac{\delta}{2}, t > T_5$.

For the following equation

$$\frac{\partial \omega(t,x)}{\partial t} = d_2 \Delta \omega(t,x) + r_2 \omega^q(t,x)\left[1 - \omega(t,x)\right], \tag{3.8}$$

there exists a traveling wave solution connecting 0 and 1 if and only if $c \geq c_2$. Let (c_2, ϕ) be the traveling wave solution of (3.8) with critical speed, then $\phi(z)$ is monotone increasing in z and satisfies

$$\begin{cases} d_2\phi''(z) - c_2\phi'(z) + r_2\phi^q(z)(1-\phi(z)) = 0, z \in \mathbb{R}, \\ \phi(-\infty) = 0, \phi(+\infty) = 1. \end{cases} \tag{3.9}$$

Now we define continuous functions

$$\overline{V}(t,x)=\min\{\phi(x+c_2t+s_1),\phi(-x+c_2t+s_1)\}$$

and

$$\overline{v}(t,x)=\min\left\{\phi(\pm x+c_2t+s_1),\phi\left(\pm\frac{c_6+\epsilon/2}{c_2}(x+(c_6+\epsilon/2)t)+s_1\right)\right\}$$

with some $s_1>0$ large enough such that $\overline{v}(T_5,x)\geq v(T_5,x)$ for all $x\in\mathbb{R}$. Note that for any $s_1>0$ such that $v(x)\leq\overline{V}(0,x)$, $\overline{V}(t,x)$ is a super-solution of $\omega(t,x)$ with $\omega(0,x)=v(x)$, $x\in\mathbb{R}$, which implies $v(t,x)\leq\overline{V}(t,x)$ for all $t>0$, $x\in\mathbb{R}$. Moreover, for each fixed $t>0$, since $c_6<c_2$, the monotonicity of ϕ implies that

$$\overline{v}(t,x)=\phi(\mp x+c_2t+s_1)\text{ as }x\to\pm\infty.$$

Hence, s_1 is admissible.

Further construct continuous functions

$$\overline{u}(t,x)=1,\ \underline{u}(t,x)=w(t,x),\ \underline{v}(t,x)=0,$$

in which $w(t,x)$ is given by

$$\begin{cases}\frac{\partial w(t,x)}{\partial t}=d_1\Delta w(t,x)+r_1w^p(t,x)\left[1-w(x,t)-a\overline{V}(t,x)\right],\\ w(x,0)=u(x).\end{cases}$$

Lemmas 3.5 and 3.6 imply that $w(t,x)$ satisfies the inequality (**a**) if $t>T_5$.

In what follows, we verify that $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ are a pair of super- and sub-solutions of (2.1) with $t>T_5$, $x\in\mathbb{R}$. By the construction of $\overline{V}(t,x)$, it is clear that $\overline{u}(t,x)$, $\underline{u}(t,x)$, $\underline{v}(t,x)$ satisfy Definition 2.1. Therefore, we only need to prove that $\overline{v}(t,x)$ also satisfies Definition 2.1.

If $t>T_5$ and $\overline{v}(t,x)=\phi(x+c_2t+s_1)$, then the positivity of $\underline{u}(t,x)$ implies that

$$\begin{aligned}&\overline{v}_t-d_2\overline{v}_{xx}-r_2\overline{v}^q\left[1-b\underline{u}-\overline{v}\right]\\ &\quad\geq\ \overline{v}_t-d_2\overline{v}_{xx}-r_2\overline{v}^q\left[1-\overline{v}\right]\\ &\quad=\ c_2\phi_z-d_2\phi_{zz}-r_2\phi^q\left[1-\phi\right]\\ &\quad=\ 0\end{aligned}$$

with $z=x+c_2t+s_1$.

When $\overline{v}(t,x)=\phi\left(\frac{c_6+\epsilon/2}{c_2}(x+(c_6+\epsilon/2)t)+s_1\right)$, we have

$$\phi\left(\frac{c_6+\epsilon/2}{c_2}(x+(c_6+\epsilon/2)t)+s_1\right)<\phi(x+c_2t+s_1).$$

It then follows from the monotonicity of ϕ that

$$\frac{c_6+\epsilon/2}{c_2}(x+(c_6+\epsilon/2)t)+s_1 < x+c_2t+s_1,$$

that is

$$-x < (c_2+c_6+\epsilon/2)t < (c_5-\epsilon/2)t.$$

Let $\xi = \frac{c_6+\epsilon/2}{c_2}(x+(c_6+\epsilon/2)t)+s_1$, then $\phi(\xi)$ satisfies (3.9). Hence, for any $t > T_5$, (a) holds, so

$$\begin{aligned}
&\overline{v}_t - d_2\overline{v}_{xx} - r_2\overline{v}^q\left[1-b\underline{u}-\overline{v}\right]\\
&\quad\geq \overline{v}_t - d_2\overline{v}_{xx} - r_2\overline{v}^q\left[1-b(1-a-\delta/2)-\overline{v}\right]\\
&\quad= \frac{(c_6+\epsilon/2)^2}{c_2}\phi_\xi - d_2\frac{(c_6+\epsilon/2)^2}{c_2^2}\phi_{\xi\xi} - r_2\phi^q\left[1-b(1-a-\delta/2)-\phi\right]\\
&\quad= \left[\left(\frac{(c_6+\epsilon/2)^2}{c_2^2}-1\right)(1-\phi)+b(1-a-\delta/2)\right]r_2\phi^q\\
&\quad\geq b\delta r_2\phi^q/2 \geq 0.
\end{aligned}$$

Lemma 2.1 further implies that

$$(\underline{u}(t,x),\underline{v}(t,x)) \leq (u(t,x),v(t,x)) \leq (\overline{u}(t,x),\overline{v}(t,x))$$

for all $t \geq T_5$ and $x \in \mathbb{R}$. The proof is complete. □

Lemma 3.8 *Assume that Lemma 3.6 holds and* $\epsilon \in (0, \frac{c_5-c_2}{2})$ *is given. Then*

$$\lim_{t\to+\infty,(c_2+\epsilon)t<|x|<(c_5-\epsilon)t} u(t,x) = 1.$$

Proof Since (F1) holds and $a, b \in [0,1)$, $p, q \geq 1$, then $c_2 < c_5$. From Lemma 3.6, there exists $u_* > 0$ such that

$$\liminf_{t\to+\infty}\inf_{(c_2+\epsilon/2)t<|x|<(c_5-\epsilon/2)t} u(t,x) = u_*,$$
$$\limsup_{t\to+\infty}\sup_{(c_2+\epsilon/2)t<|x|<(c_5-\epsilon/2)t} v(t,x) = 0.$$

Let $\{\epsilon_k\}_{k=1}^{+\infty}$ be a sequence such that

$$\epsilon/2 = \epsilon_1 < \epsilon_2 < \cdots, \lim_{k\to+\infty}\epsilon_k = \epsilon.$$

Further define $\{u_*^k\}_{k=1}^{+\infty}$ by

$$\liminf_{t\to+\infty} \inf_{(c_2+\epsilon_k)t<|x|<(c_5-\epsilon_k)t} u(t,x) = u_*^k.$$

Then $u_* \le u_*^{k-1} \le u_*^k \le 1$ for any positive integer $k \ge 2$. Hence, we have $\lim_{k\to+\infty} u_*^k = u^*$ and $0 < u_* \le u^* \le 1$. In addition,

$$u^* \le \liminf_{t\to+\infty} \inf_{(c_2+\epsilon)t<|x|<(c_5-\epsilon)t} u(t,x).$$

Let $\kappa > 0$ be a constant such that $2\kappa \in (0, u_*)$, then there exist positive constants T and N such that

$$\int_0^T \int_{-N}^N \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{y^2}{4d_1 s}} dyds > 1-\kappa.$$

By the definition of lim inf, we can find a sequence $\{t_n^k\}_{n=1}^{+\infty}$ with $\lim_{n\to+\infty} t_n^k = +\infty$ such that

$$u(t_n^k, x_n^k) \le u_*^k + \kappa$$

for $(c_2+\epsilon_k)t_n^k < |x_n^k| < (c_5-\epsilon_k)t_n^k$. On the other hand, the monotonicity of ϵ_k implies that

$$\inf_{(c_2+\epsilon_{k-1})t<|x|<(c_5-\epsilon_{k-1})t} u(t,x) \ge u_*^{k-1} - \kappa$$

and

$$\sup_{(c_2+\epsilon_{k-1})t<|x|<(c_5-\epsilon_{k-1})t} v(t,x) \le \kappa$$

with t_n^k sufficient large and $t \in [t_n^k - T, t_n^k]$. Further choose t_n^k large enough such that

$$(\epsilon_k - \epsilon_{k-1})(t_n^k - T) > N + c_5 T,$$

then $u(t,y) \ge u_*^{k-1} - \kappa$ and $v(t,y) \le \kappa$ for $t \in [t_n^k - T, t_n^k]$, $y \in [x_n^k - N, x_n^k + N]$.

Denote $\beta = 2(p+q)(r_1+r_2)$ and define continuous functions

$$\begin{cases} F_1(u,v)(t,x) = \beta u(t,x) + r_1 u^p(t,x)[1-u(t,x)-av(t,x)], \\ F_2(u,v)(t,x) = \beta v(t,x) + r_2 v^q(t,x)[1-bu(t,x)-v(t,x)] \end{cases}$$

for $(0,0) \le (u(t,x), v(t,x)) \le (1,1)$. Then F_1 is monotone increasing in u and decreasing in v, while F_2 is monotone increasing in v and decreasing in u. For any given $(u(x), v(x)) \in X_{[0,1]}^2$, define $(T_1(t), T_2(t))$ by

$$\begin{cases} T_1(t)u(x) = \frac{e^{-\beta t}}{\sqrt{4\pi d_1 t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4d_1 t}} u(y)dy, \\ T_2(t)v(x) = \frac{e^{-\beta t}}{\sqrt{4\pi d_2 t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4d_2 t}} v(y)dy. \end{cases}$$

In view of the basic solution of the heat equation, we have

$$u(t_n^k, x_n^k) = T_1(T)u(t_n^k - T, x_n^k) + \int_{t_n^k - T}^{t_n^k} T_1(t_n^k - s)F_1(u, v)(s, x_n^k)ds.$$

Setting $n \to +\infty$, the monotonicity of F_1 implies that

$$u_*^k + \kappa \geq \frac{\beta(u_*^{k-1} - \kappa) + r_1(1-\kappa)(u_*^{k-1} - \kappa)^p \left[1 - (u_*^{k-1} - \kappa) - a\kappa\right]}{\beta}.$$

It then follows from the arbitrariness of κ that

$$u_*^k \geq \frac{\beta u_*^{k-1} + r_1(u_*^{k-1})^p(1 - u_*^{k-1})}{\beta}.$$

Further let $k \to +\infty$, we have $u^* \geq 1$. Note that $u^* \leq 1$, then $u^* = 1$, which completes the proof. □

Lemma 3.9 *Assume that Lemma 3.6 holds and $\epsilon \in (0, c_4)$ is given. If $q \leq 3$ or $q > 3$ and $v(x) > \sigma_2 g_2(x)$, then*

$$\lim_{t\to+\infty} \sup_{|x|<(c_4-\epsilon)t} (|u(t,x) - k_1| + |v(t,x) - k_2|) = 0.$$

Proof Since $u(t,x) \leq 1$, if $q \leq 3$ or $q > 3$ and $v(x) > \sigma_2 g_2(x)$, the comparison principle implies that

$$\liminf_{t\to+\infty} \inf_{|x|<(c_4-\epsilon/2)t} v(t,x) > 0.$$

When (F1) holds, then $c_4 < c_5$, and Lemma 3.6 implies that

$$\liminf_{t\to+\infty} \inf_{|x|<(c_4-\epsilon/2)t} u(t,x) > 0.$$

Hence, there exist positive constants $u_{*,+} \geq u_{*,-} > 0$ and $v_{*,+} \geq v_{*,-} > 0$ such that

$$\liminf_{t\to+\infty} \inf_{|x|<(c_4-\epsilon/2)t} u(t,x) = u_{*,-}, \quad \limsup_{t\to+\infty} \sup_{|x|<(c_4-\epsilon/2)t} u(t,x) = u_{*,+},$$
$$\liminf_{t\to+\infty} \inf_{|x|<(c_4-\epsilon/2)t} v(t,x) = v_{*,-}, \quad \limsup_{t\to+\infty} \sup_{|x|<(c_4-\epsilon/2)t} v(t,x) = v_{*,+}.$$

From the definitions of lim inf and lim sup, applying similar arguments as that in Lemma 3.8, we complete the proof. □

Remark 3.1 Compared with the non-degenerate case, when the positive steady state of the corresponding kinetic system (1.4) is globally asymptotically stable, Theorems 3.1 and 3.2 imply that four different spreading phenomena may occur by selecting different initial values in the degenerate case of (1.3).

Remark 3.2 In the weak competition case of (1.3) with degenerate nonlinearity, our results show that the interspecific competition may play a nontrivial role from the viewpoint of persistence. In fact, Σ_1^0 is a threshold set on the persistence of u when the interspecific competition vanishes while the threshold set may be reduced due to interspecific competition. This phenomenon is significantly different from the case of $p, q \in [1, 3]$.

3.2 Other cases $a \geq 0$, $b \geq 1$

Different from the above subsection, we investigate the asymptotic spreading of (1.3) with $a \geq 0, b \geq 1$ in this subsection, which implies that the competition of u is strong. When $p = q = 1$ and $b > 1$, Lewis et al. (2002) obtained some results on the dynamics of (2.1) with interspecific exclusive process for $0 < a < 1$ while Carrère (2018) showed that the system may form a propagating terrace if $a > 1$, which are different from the weak competition case.

Theorem 3.3 *Assume that* $(u(x), v(x)) \in X^2_{[0,1]}$ *and* $u(x)$ *has nonempty compact support. For any* $a \geq 0, b \geq 1$ *and* $p > 3$, *if* $u(x) \leq \sigma_3 g_3(x)$ *or* $u(x) \leq \psi(0, x; \xi_1, p, r_1, d_1)$, *then the classical solution* $(u(t, x), v(t, x))$ *of (2.1) is well defined in* $(0, +\infty) \times \mathbb{R}$ *and*

$$\lim_{t \to +\infty} u(t, x) = 0 \text{ uniformly in } \mathbb{R}.$$

In addition, the following properties hold:

(i) Suppose further that $\epsilon \in (0, c_2)$ *is given and* $v(x) \not\equiv 0$. *If* $q \leq 3$ *or* $q > 3$ *and*

$$\liminf_{t \to +\infty} v(t, x) > 0 \text{ uniformly in any compact subset with } x \in \mathbb{R}, \tag{3.10}$$

then

$$\liminf_{t \to +\infty} \inf_{|x| < (c_2 - \epsilon)t} v(t, x) = 1.$$

(ii) If $\liminf_{|x| \to +\infty} v(x) > 0$, *then*

$$\lim_{t \to +\infty} v(t, x) = 1 \text{ uniformly in } \mathbb{R}.$$

Proof According to the comparison principle and Lemma 3.1, it suffices to consider the properties of $v(t, x)$. Since

$$\lim_{t\to+\infty} u(t, x) = 0 \text{ uniformly in } \mathbb{R},$$

for any $\epsilon' > 0$, there exists $T' > 0$ such that

$$\frac{\partial v(t, x)}{\partial t} \geq d_2 \Delta v(t, x) + r_2 v^q(t, x)\left[1 - b\epsilon' - v(t, x)\right] \tag{3.11}$$

for all $t > T'$ and $x \in \mathbb{R}$.

When $v(x) \not\equiv 0$, if $q \leq 3$ or $q > 3$ and

$$\liminf_{t\to+\infty} v(t, x) > 0 \text{ uniformly in any compact subset with } x \in \mathbb{R},$$

from Theorem 2.17 of Liang and Zhao (2007) and Lemma 2.2, for any given $\epsilon \in (0, c_2)$, choosing $\epsilon' > 0$ small enough such that

$$\sqrt{\frac{d_2 r_2}{K(q)}} - \epsilon < \sqrt{\frac{d_2 r_2 (1 - b\epsilon')^q}{K(q)}},$$

then

$$\liminf_{t\to+\infty} \inf_{|x|<(c_2-\epsilon)t} v(t, x) \geq 1 - b\epsilon',$$

which implies the assertion of (i). When $\liminf_{|x|\to+\infty} v(x) > 0$, applying the same argument as that in Step 2 of Lemma 3.1, the proof is complete, □

Now, we give some sufficient conditions to ensure (3.10).

Proposition 3.1 *Assume that* $p > 3$ *and* $u(x) \leq \psi(0, x; \xi_1, p, r_1, d_1)$ *with some* $\xi_1 > \left(\frac{1}{2} - \frac{1}{p-1}\right) b^{p-1}$. *Let* $q > 3$ *and*

$$m := \left(\left(\frac{1}{2} - \frac{1}{p-1}\right)/\xi_1\right)^{1/(p-1)},$$

suppose further that $g_5(x) \in X_{[0,1-bm]}$ *is given with nonempty compact support. Then there exists a positive constant* $\sigma_5 := \sigma_5(g_5(x))$ *such that if* $v(x) > \sigma_5 g_5(x)$, *then (3.10) holds.*

Proof If $u(x) \le \psi(0, x; \xi_1, p, r_1, d_1)$, then the comparison principle implies that $u(t, x) \le \psi(t, x; \xi_1, p, r_1, d_1)$ for all $t > 0, x \in \mathbb{R}$, so

$$\begin{aligned} u(t,x) &\le \left[\left(\frac{1}{2} - \frac{1}{p-1}\right)/(r_1 t + \xi_1)\right]^{1/(p-1)} e^{-r_1 x^2/4d_1(r_1 t + \xi_1)} \\ &\le \left(\left(\frac{1}{2} - \frac{1}{p-1}\right)/\xi_1\right)^{1/(p-1)} = m. \end{aligned}$$

For any $\xi_1 > \left(\frac{1}{2} - \frac{1}{p-1}\right) b^{p-1}$, we have $1 - bu(t, x) \ge 1 - bm > 0$. Thus,

$$\frac{\partial v(t,x)}{\partial t} \ge d_2 \Delta v(t,x) + r_2 v^q(t,x)\left[1 - bm - v(t,x)\right]$$

with $1 - bm > 0$ and $t > 0, x \in \mathbb{R}$. If $g_5(x) \in X_{[0,1-bm]}$ with nonempty compact support, then the comparison principle and Lemma 2.2 imply the existence of σ_5. The proof is complete. □

In what follows, we summarize some of implications of the above results, which will be divided into several cases.

3.2.1 Case 1: *a* > 1, *b* > 1

It should be noted that $a > 1$ and $b > 1$ ensure that the equilibria $(0, 1)$ and $(1, 0)$ are locally stable and the coexistence state (k_1, k_2) is unstable for (1.4), which is the case of the so-called strong competition. When $p = q = 1$, as stated by Carrère (2018), the dynamics of the diffusion competition system depend crucially on the sign of traveling wave solutions of (1.3) with proper initial conditions. However, due to the degeneracy, Theorems 3.1 and 3.3 imply that one competitor may be vanishing once the degeneracy of nonlinearity is strong enough and the size of the support of initial value is sufficiently small, which is independent of the sign of traveling wave solutions. Moreover, if $p = 1$ or $q = 1$, which shows that the per capita growth of one competitor is maximal at small densities, in light of constructing appropriate super- and sub-solutions as that in Lemma 2 of Carrère (2018), the following results could also be obtained.

Lemma 3.10 *Assume that* $(u(x), v(x)) \in X^2_{[0,1]}$ *has nonempty compact support and* $a > 1, b > 1$. *Then the solution* $(u(t, x), v(t, x))$ *of (2.1) is well defined for all* $(t, x) \in (0, +\infty) \times \mathbb{R}$ *and satisfies the following statements:*

(i) For any given $\epsilon > 0$,

$$\lim_{t\to+\infty} \sup_{|x|>(\overline{c}+\epsilon)t} (u(t,x) + v(t,x)) = 0, \ \overline{c} = \max\{c_1, c_2\}. \tag{3.12}$$

(ii) Suppose further that $\epsilon \in (0, \frac{|c_1 - c_2|}{2})$ *is given,*

(a) if $c_2 < c_1$ *and* $p = 1$*, then*

$$\lim_{t\to+\infty} \sup_{(c_2+\epsilon)t<|x|<(c_1-\epsilon)t} (|u(t,x)-1| + v(t,x)) = 0;$$

(b) if $c_1 < c_2$ *and* $q = 1$*, then*

$$\lim_{t\to+\infty} \sup_{(c_1+\epsilon)t<|x|<(c_2-\epsilon)t} (u(t,x) + |v(t,x)-1|) = 0.$$

3.2.2 Case 2: $0 < a < 1$, $b > 1$

Firstly, we recall some classical results on the asymptotic spreading of (1.3) with $0 < a < 1, b > 1$. Let $w = 1 - v$ in (1.3), then a straightforward calculation yields

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \Delta u(t,x) + r_1 u^p(t,x)\left[1 - a - u(t,x) + aw(t,x)\right], \\ \frac{\partial w(t,x)}{\partial t} = d_2 \Delta w(t,x) + r_2(1 - w(t,x))^q \left[bu(t,x) - w\right] \end{cases} \tag{3.13}$$

for $(t,x) \in (0,+\infty)\times\mathbb{R}$, which is cooperative in the range $(0,0) \le (u,w) \le (1,1)$. This change of variables maps $(0,1)$ to $(0,0)$, $(1,0)$ to $(1,1)$, and $(0,0)$ to $(0,1)$. Assume that $(0,0) \le (u(x), v(x)) \le (1,1)$, $(u(x), v(x)) = (0,1)$ outside a bounded interval, and $u(x) \not\equiv 0$ in $\mathbb{R}$, the spreading speed has been well established, see e.g., Lewis et al. (2002), Weinberger et al. (2002), Li et al. (2005), Liang and Zhao (2007), Fang and Zhao (2014), which can be described by the following lemma.

Lemma 3.11 *Assume that* $(u(x), v(x)) \in X^2_{[0,1]}$ *and* $0 < a < 1, b > 1$*. Then for any* $p, q \ge 1$*, the classical solution* $(u(t,x), v(t,x))$ *of (2.1) is well defined in* $(0,+\infty)\times\mathbb{R}$ *and satisfies the following statements:*

(i) If for any $\sigma > 0$*, there is a positive number* r_σ *such that* $u(x) > \sigma$, $1 - v(x) > \sigma$ *with* x *on an interval of length* $2r_\sigma$*, then there exists a constant* $c_* > 0$ *such that*

$$\lim_{t\to+\infty} \sup_{|x|<(c_*-\epsilon)t} (|u(t,x)-1| + v(t,x)) = 0$$

for any given $\epsilon \in (0, c_*)$.

(ii) For any given $\epsilon > 0$*, if* $(u(x), v(x)) = (0,1)$ *outside a bounded interval and* $u(x) \not\equiv 0$ *in* $\mathbb{R}$*, then there exists a constant* $c_f^* \ge c_*$ *such that*

$$\lim_{t\to+\infty} \sup_{|x|>(c_f^*+\epsilon)t} (u(t,x) + |v(t,x)-1|) = 0.$$

The classical results reveal the interspecific competitive exclusive process between the resident and the invader, in which the invader is superior than the resident. There are many historical records reflect this process, such as the competition between gray and red squirrels in United Kingdom (Okubo et al. 1989). It should be noted that if $p = q = 1$, then $c_f^* = c_*$ and r_σ is independent of σ (Lewis et al. 2002), which implies that the invasion is successful no matter how small the size of the support of initial values is. When p or $q > 1$, Theorems 3.1 and 3.3 reflect that although the invader is strong enough, the invasion may also be failure.

3.2.3 Case 3: $a = 1$, $b = 1$

When $a = b = 1$, due to the appearance of infinitely many steady states $u+v = 1$, the dynamics of (1.3) are rather rich. As we know that if the competition system modeled by (1.3) differs only in their diffusion rates, that is, $r_1 = r_2 = a = b = 1$, then the existence and stability of the classical solutions with $p = q = 1$ in a bounded domain were considered by Dockery et al. (1998), Hutson et al. (2001, 2002), Lou (2006) and a very recent paper by Lou et al. (2019). In reaction-diffusion systems, it is normally expected that the reaction terms will play a central role in the dynamics. However, in the present case the per capita growth represented by these terms are identical. Thus, the dynamics are principally driven by the differences in the diffusion. Their results show that the slower diffusion competitor has stronger competition. When p or $q > 1$, Theorems 3.1 and 3.3 imply that either the slow or fast diffusion could be selected, which depends on the degeneracy of nonlinearity as well as the size of the support of initial values.

4 Numerical simulations

In this section, we present some numerical simulations to illustrate our main results obtained in Sect. 3. We consider the following degenerate competition diffusion system

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \Delta u(t,x) + u^p(t,x)\left[1 - u(t,x) - av(t,x)\right], \\ \frac{\partial v(t,x)}{\partial t} = d_2 \Delta v(t,x) + v^q(t,x)\left[1 - bu(t,x) - v(t,x)\right] \end{cases} \tag{4.1}$$

with constants $a, b \geq 0$ and $p, q \geq 1$, through which we define

$$u(x) = v(x) = \cos x, |x| \leq \pi/2; \quad u(x) = v(x) = 0, |x| \geq \pi/2.$$

To simulate u and v, we cutoff the domain $\mathbb{R}$ by letting $u(t,x) = v(t,x) = 0$ if x is large, which is due to the fact

$$\lim_{x\to\infty} u(t,x) = \lim_{x\to\infty} v(t,x) = 0 \text{ for any fixed } t > 0.$$

4.1 Weak competition

In this subsection, we consider (4.1) with

$$d_1 = d_2 = 1,\ p = 3.1,\ q = 3.2,\ a = 1/10,\ b = 3/10, \tag{4.2}$$

which implies the weak competition case in the corresponding kinetic system. With the initial value $(u(x), v(x))$, Fig. 1a, b imply the coexistence of both species. Figure 1c, d are obtained if the initial value becomes $(u(x)/10, v(x))$, and Fig. 1e, f are accomplished by taking the initial value $(u(x)/10, v(x)/10)$. Hence, by selecting different initial values, we may observe some different spreading–vanishing phenomena in (4.1), see Fig. 1 and Theorems 3.1 and 3.2.

4.2 Other cases

(i) **Monostable case**. First let

$$d_1 = d_2 = 1, \;\; p = 3.1, \;\; q = 3.2, \;\; a = 2, \;\; b = 3/10, \tag{4.3}$$

then the corresponding kinetic system is the monostable case and v is much stronger than u. Figure 2a, b are derived by taking the initial value $(2u(x), v(x)/3)$. In Fig. 2c, d, we still use the initial value $(u(x), v(x))$. Figure 2 demonstrates that the superior competitor v could be wiped out by the inferior competitor u, which depends on the initial values, also see Theorems 3.1 and 3.3.

(ii) **Bistable case**. Selecting

$$d_1 = d_2 = 1, \;\; p = 3.1, \;\; q = 3.2, \;\; a = 1.2, \;\; b = 1.3, \tag{4.4}$$

then the corresponding kinetic system becomes the bistable case. Figure 3a–d are obtained with the corresponding initial values $(2u(x), v(x)/3)$ and $(u(x)/3, v(x))$, respectively. These figures illustrate that the successful spreading of u and v may depend on the initial values, and the degeneracy may lead to extinction, see Theorems 3.1 and 3.3.

(iii) **Having infinitely many steady states**. Now we consider (4.1) with $a = b = 1$, under which the system admits infinitely many steady states satisfying $u + v = 1$. Even in the corresponding kinetic system, it is difficult to confirm the limit behavior of a solution. By selecting

$$d_1 = d_2 = 1, \;\; p = q = 4, \;\; a = b = 1 \tag{4.5}$$

with initial value $(u(x)/2, v(x)/5)$, we obtain Fig. 4a, b, which implies the effect of initial value. By these figures, it is possible that large initial value leads to stronger ability for persistence.

4.3 The effect of interspecific competition

Now we consider the effect of interspecific competition on the asymptotic spreading in (4.1) by taking $d_1 = d_2 = 1$. Let

$$p = 3.1, \;\; a = 0. \tag{4.6}$$

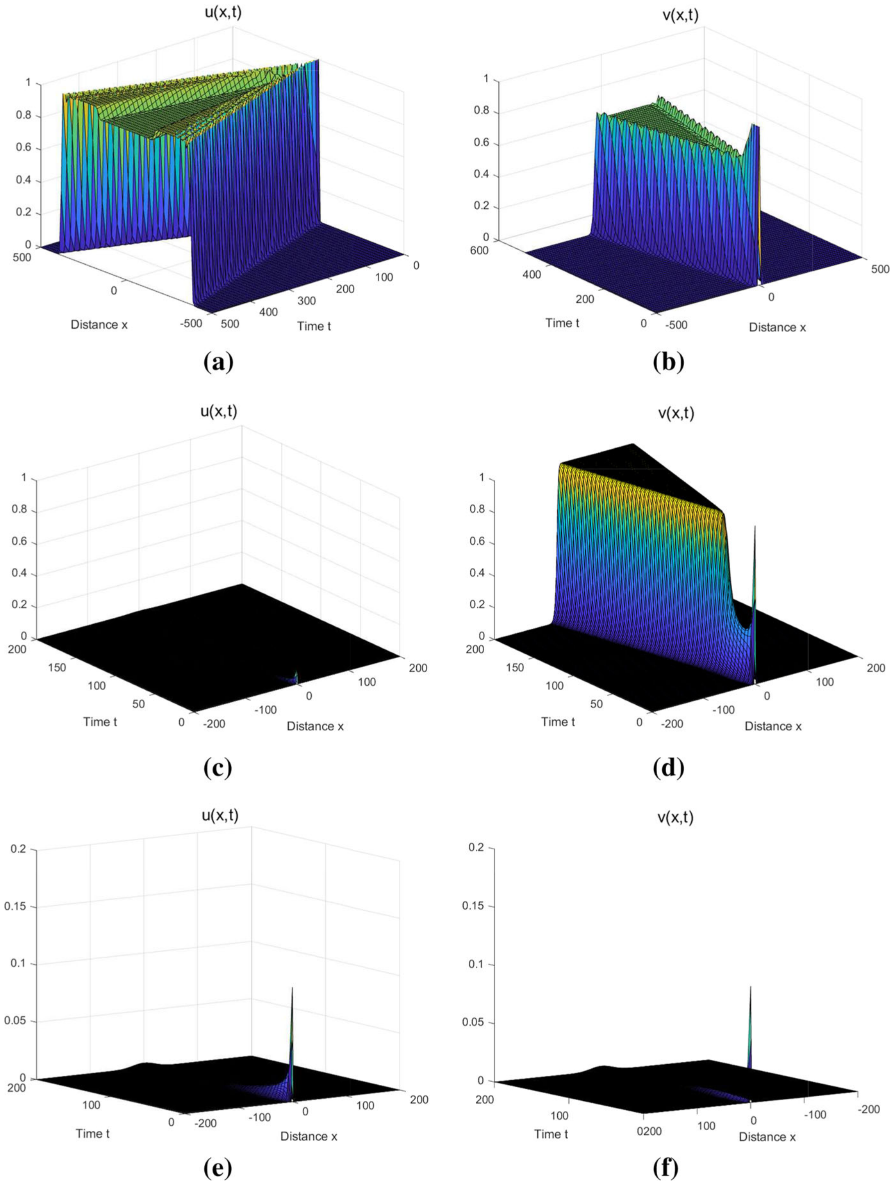


Fig. 1 Weak competition. Numerical simulations of the solution $(u(x,t), v(x,t))$ of (4.1) with parameters given in (4.2). **a**, **b** $u(x,t)$ and $v(x,t)$ with initial value $(u(x), v(x))$; **c**, **d** $u(x,t)$ and $v(x,t)$ with initial value $(u(x)/10, v(x))$; **e**, **f** $u(x,t)$ and $v(x,t)$ with initial value $(u(x)/10, v(x)/10)$

We obtain Fig. 5a for $u(x,t)$ with the initial value $u(x)$. Figure 5b, c are obtained by selecting

$$p = 3.1,\ q = 2,\ a = 9/10,\ b = 3/10 \tag{4.7}$$

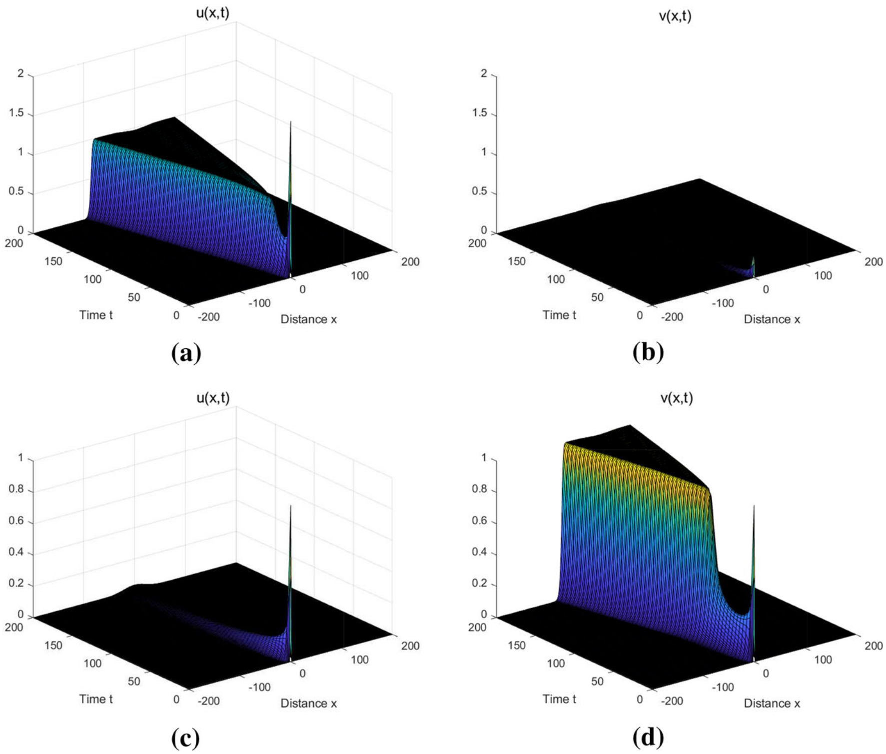


Fig. 2 Monostable case. Numerical simulations of the solution $(u(x,t), v(x,t))$ of (4.1) with parameters given in (4.3). **a**, **b** $u(x,t)$ and $v(x,t)$ with initial value $(2u(x), v(x)/3)$; **c**, **d** $u(x,t)$ and $v(x,t)$ with initial value $(u(x), v(x))$

and taking the initial value $(u(x), v(x))$. These figures reveal that the interspecific competition may lead to extinction, see (iii) of Theorem 3.2.

With the initial value $u(x)$, Fig. 5d for $u(x,t)$ is obtained by choosing

$$p = 2.9, \;\; a = 0. \tag{4.8}$$

That is, when the interspecific competition vanishes, u will spread and almost arrive its capacity 1 in the compact interval. To show the effect of interspecific competition, further letting

$$p = 2.9, \;\; q = 2, \;\; a = 9/10, \;\; b = 3/10, \tag{4.9}$$

we have Fig. 5e, f with the initial value $(u(x), v(x))$. These figures reflect that the interspecific competition may decrease the spreading speed of u significantly. However, we are not able to present a precise spreading speed here and plan to study this in the future.

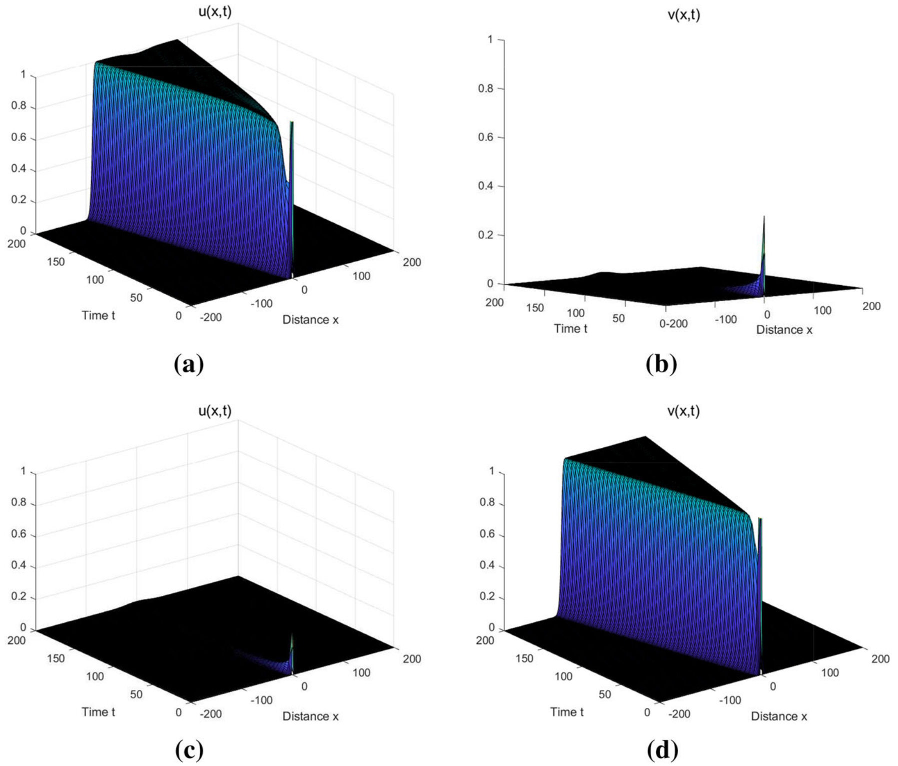


Fig. 3 Bistable case. Numerical simulations of the solution $(u(x,t), v(x,t))$ of (4.1) with parameters given in (4.4). **a**, **b** $u(x,t)$ and $v(x,t)$ with initial value $(2u(x), v(x)/3)$; **c**, **d** $u(x,t)$ and $v(x,t)$ with initial value $(u(x)/3, v(x))$

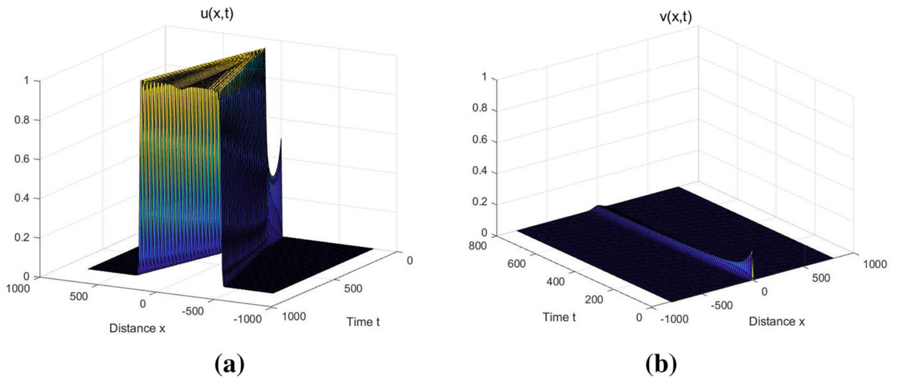


Fig. 4 When $a = b = 1$. Numerical simulations of the solution $(u(x,t), v(x,t))$ of (4.1) with parameters given in (4.5) and initial value $(u(x)/2, v(x)/5)$

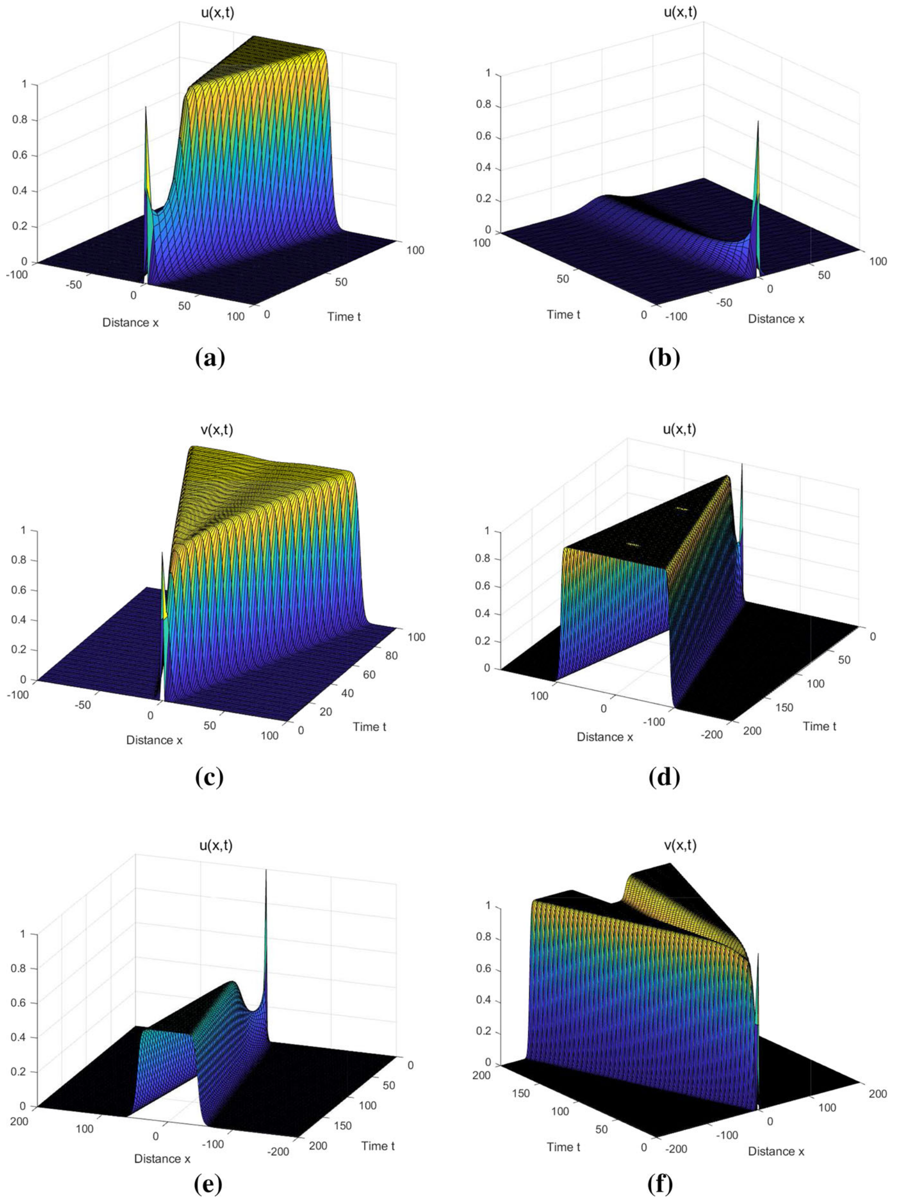
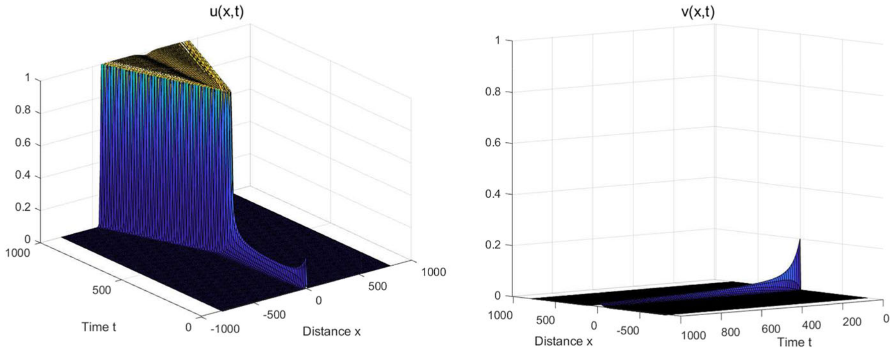


Fig. 5 The effect of interspecific competition. Numerical simulations of the solution $(u(x,t), v(x,t))$ of (4.1) with $d_1 = d_2 = 1$ and different interspecific competition parameter values. **a** $u(x,t)$ with parameters in (4.6) and initial value $u(x)$; **b**, **c** $u(x,t)$ and $v(x,t)$ with parameters in (4.7) and initial value $(u(x), v(x))$; **d** $u(x,t)$ with parameters in (4.8) and initial value $u(x)$; **e**, **f** $u(x,t)$ and $v(x,t)$ with parameters in (4.9) and initial value $(u(x), v(x))$



(a) **(b)**

Fig. 6 The effect of degeneracy. Numerical simulations of the solution $u(x,t)$ and $v(x,t)$ of (4.1) with parameters in (4.10) and initial value $(u(x)/5, v(x)/5)$

4.4 The effect of degeneracy and diffusion

In this subsection, we consider the effect of degeneracy and diffusion in (4.1) with $a = b = 1$, so far these results have not been proven in the present paper. With the same initial value $(u(x)/5, v(x)/5)$, we obtain Fig. 6a, b by letting

$$d_1 = d_2 = 1, p = 3.1, q = 20, a = b = 1, \tag{4.10}$$

which imply that the stronger degeneracy may be harmful to the persistence in (4.1).

To show the effect of diffusion, we consider (4.1) with different diffusion rates. By selecting

$$d_1 = 1,\ d_2 = 8,\ p = q = 4,\ a = b = 1 \tag{4.11}$$

and

$$d_1 = 6,\ d_2 = 1,\ p = q = 2,\ a = b = 1, \tag{4.12}$$

we obtain Fig. 7a–d, which show that the smaller diffusion may be favorable for the persistence in (4.1). Unfortunately we are not able to analyze the effect of diffusion in this paper.

5 Discussion

In population dynamics, it is natural to assume that the per capita growth is maximal at low densities since the competition is sufficiently small (Fisher 1937; Kolmogorov et al. 1937). However, individuals may have trouble to find mates and the genetic diversities are scant at very low densities. In addition, some species engage in group defense, cooperative hunting or other beneficial social behaviors which are not possible if the densities are too low. Hence, this assumption may be unrealistic in some situations, that is, the per capita growth is no longer maximal at low densities, which is the so-called Allee effect (Allee 1931). In addition, it turns out that diffusion can

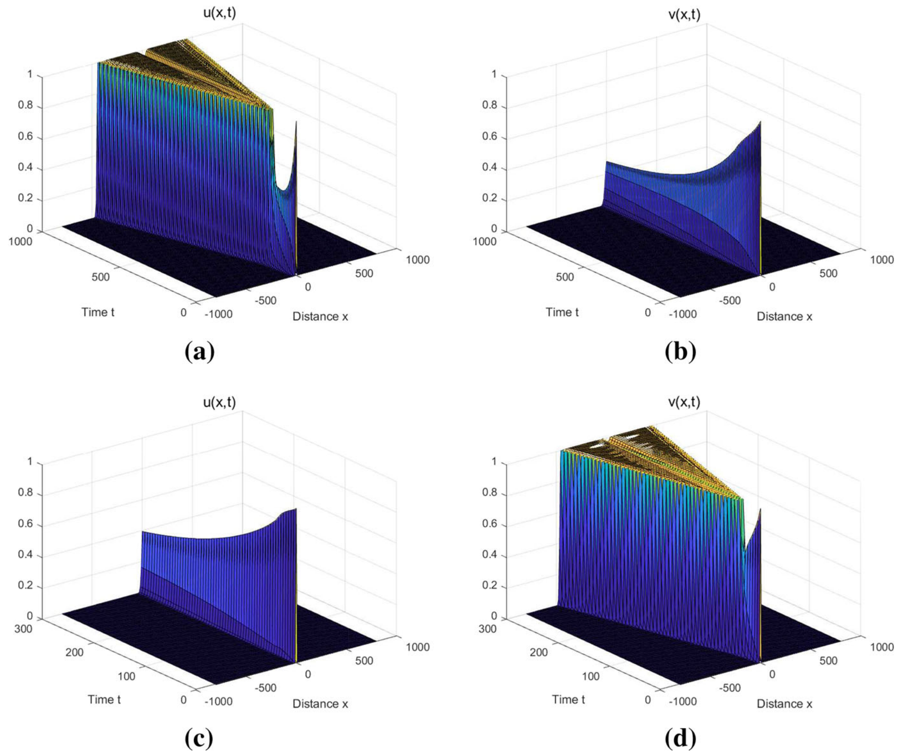


Fig. 7 The effect of diffusion. Numerical simulations of the solution of (4.1). **a**, **b** $u(x,t)$ and $v(x,t)$ with parameters in (4.11) and initial value $(u(x), v(x))$; **c**, **d** $u(x,t)$ and $v(x,t)$ with parameters in (4.12) and initial value $(u(x), v(x))$

amplify Allee effect and in some cases can create Allee effect at the level of the overall population even when such effect is not present locally. With these concerns, we investigated the diffusion competition system (1.3) with degenerate nonlinearity in this work, the degeneracy means that the per capita growth of one or both species is zero at low densities, which is also referred to as weak Allee effect (Allee 1931). Several interesting phenomena ranging from population dynamics to chemical waves are modeled by the weak Allee effect, see Aronson and Weinberger (1978), Alikakos et al. (1999), Kim and Lin (2006), Chen and Qi (2007).

In this paper, we investigated the long time behavior of (1.3) with degenerate nonlinearity, in which the interspecific competition may be strong or weak. To our knowledge, the extinction or persistence of two competitors modeled by (1.1) is principally driven by the dynamics of the corresponding kinetic systems in the monostable case, which is independent of the size of the support of initial values. For the degenerate monostable case of (1.3), various extinction or persistence results may occur by selecting different initial values, which imply that the size of the support of initial values could affect the extinction or persistence of (1.3). Moreover, some sufficient conditions that assure the extinction or persistence of biological invasion were given. These conditions can be interpreted as requiring appropriate sizes of initial habitats as well as suitable intensity of Allee effect. Moreover, some numerical simulations were also presented to illustrate these theoretical results.

To better understand these phenomena, we provided some explanations. In the non-degenerate monostable case of (1.3), the nonlinearity reaction is strong enough such that it plays a crucial role on spreading or vanishing, while the diffusion only affects the spreading speed. On the other hand, the nonlinearity involves a weak Allee effect in the degenerate case, which makes the reaction less competitive. Then the balance between the degenerate nonlinear reaction and diffusion should be taken into consideration. In this paper, we presented some sufficient conditions for the balance of degenerate nonlinear reaction and diffusion. From the biological point of view, due to the influence of diffusion (by selecting different initial values), these conditions reveal that the superior competitor in the sense of the corresponding kinetic system does not always win, it can be washed out by the inferior competitor in the sense of the corresponding kinetic system. Moreover, if the degenerate competition system modeled by (1.3) differs only in their diffusion rates, then either the slow or fast diffusion could be selected.

Different from that in the non-degenerate case, the size of the support of initial values could also affect the extinction or persistence for the degenerate diffusion system (1.3) in the monostable case, and the interspecific competition of one species may be harmful to the persistence of the other species. We also obtained some results on the bistable case, which are independent of the sign of traveling wave solutions. Moreover, numerical simulations provide some illustrations about the effect of interspecific competition, degeneracy as well as diffusion on the asymptotic spreading of (1.3) (see Figs. 5, 6, 7). However, only some sufficient conditions for the success or failure of asymptotic spreading were given in this paper. To obtain more precise results, further investigations are needed.

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