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Generalized traveling waves for time-dependent reaction–diffusion systems

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Abstract

Traveling wave solutions in general time-dependent (including time-periodic) reaction –diffusion equations and systems of equations have attracted great attention in the last two decades. The aim of this paper is to study the propagation phenomenon in a general time-heterogeneous environment. More specifically, we investigate generalized traveling wave solutions for a two-component time-dependent non-cooperative reaction–diffusion system which has applications in epidemiology and ecology. Sufficient conditions on the existence and nonexistence of generalized traveling wave solutions are established. In the susceptible-infectious epidemic model setting, generalized traveling waves describe the spatio-temporal invasion of a disease into a totally susceptible population. In the context of predator–prey systems, the generalized traveling waves describe the spatial invasion of predators introduced into a new environment where the prey population is at its carrying capacity.

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1 Introduction

In this paper we study the existence of generalized traveling wave solutions for the following two-component time-dependent reaction–diffusion system

$$\begin{cases} \partial_t u - d(t)\partial_x^2 u = F_1(t,u,v), \\ \partial_t v - \partial_x^2 v = F_2(t,u,v), \end{cases} \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1.1}$$

where the functions F_1 and F_2 are defined by

$$\begin{aligned} F_1(t,u,v) &:= \Lambda(t) - \mu(t)u - \beta(t)uv, \\ F_2(t,u,v) &:= \beta(t)uv - \gamma(t)v, \end{aligned} \tag{1.2}$$

respectively. Here the time-dependent coefficients $d(t)$, $\Lambda(t)$, $\mu(t)$, $\beta(t)$ and $\gamma(t)$ are all continuous and bounded functions defined on $\mathbb{R}$ and, for any function $m \in \{d, \Lambda, \mu, \beta, \gamma\}$, we assume that

$$\underline{m} := \inf_{t\in\mathbb{R}} m(t) > 0. \tag{1.3}$$

Furthermore, the diffusion coefficient $d(t)$ is assumed to be uniformly continuous on $\mathbb{R}$.

System (1.1) arises in epidemiology and population dynamics. In the context of epidemiology, the state variables $u = u(t,x)$ and $v = v(t,x)$ denote the density of susceptible and infectious individuals, respectively, located at the spatial position $x \in \mathbb{R}$ at time t. In the absence of disease, namely when $v = 0$, the system reduces to the vital dynamics for u that reads as

$$\partial_t u - d(t)\partial_x^2 u = \Lambda(t) - \mu(t)u, \tag{1.4}$$

This means that, in the absence of disease, there is a time-dependent influx $\Lambda(t)$ entering the population as well as a natural—time-dependent—exit rate $\mu(t)$ for the population. When the disease is introduced, $\beta(t)$ denotes the time-dependent infection rate while $\gamma(t)$ represents the recovering or removal rate for the infected individuals. When all coefficient functions are constants, the existence of traveling wave solutions in (1.1) has been studied extensively, see for example, Hosono and Ilyas [12,13], Källen [17], Murray [20], etc. When all coefficient functions are time-dependent, there are very few results on the existence of traveling wave solutions. Recently, Wang et al. [28] studied a periodic SIR epidemic model with diffusion and standard incidence and established the existence of periodic traveling waves by investigating the fixed points of a nonlinear operator defined on an appropriate set of periodic functions. They also proved the nonexistence of periodic traveling waves via the comparison arguments combined with the properties of the spreading speed of an associated subsystem.

System (1.1) can also be used to describe problems in population dynamics, in particular a predator–prey system. In that setting, $u(t,x)$ and $v(t,x)$ represent the density of the prey and that of the predators, respectively, at time t and spatial location

$x \in \mathbb{R}$. Similarly, traveling wave solutions in diffusive predator–prey models with constant coefficients have been investigated by many researchers, see for example Dunbar [9], Gardner [10], Huang et al. [14], Huang [15,16], Lin [18], Mischaikow and Reineck [19], etc. There are some studies on traveling waves in periodic Lotka–Volterra competition models (Bao and Wang [3], Bo et al. [4], Zhao and Ruan [29,30]) and periodic Lotka–Volterra predator–prey models (Wang and Lin [27]), but once again there are very few results on traveling waves in time-dependent diffusive predator–prey models.

In the above time-dependent system (1.1), we assumed that the diffusion coefficient for the v-component does not depend on time and equals 1. However, as it will be discussed below (see Sect. 2), using a suitable change of variable, the results obtained in this work concerning the existence and non-existence of generalized traveling wave solutions for (1.1)–(1.2) also allow us to cover the case when the diffusion coefficient for the v-component also depends on the time variable t.

Traveling wave solutions in general time-dependent (including time-periodic) reaction–diffusion systems have also attracted great attention in the last two decades, we refer to Alikakos et al. [2], Contri [6], Hamel and Rossi [11], Nadin [21], Sheng and Guo [24], Shen [25] and the references cited therein. For general time-dependent reaction–diffusion equations, we use the terminology of *generalized traveling waves* for general time-heterogeneous environments as in Nadin and Rossi [22] and general heterogeneous environments as in Nadin and Rossi [23]. The aim of this paper is to study the propagation phenomenon for (1.1)–(1.2) posed in a general time-heterogeneous environment. We will particularly study the existence and nonexistence of generalized traveling waves for this problem.

In order to precisely define what we mean by a generalized traveling wave solution for (1.1)–(1.2), note that when $v = 0$, this problem, namely (1.4), admits a unique bounded and time global solution $u^* = u^*(t)$ which is explicitly given by the formula

$$u^*(t) = \int_{-\infty}^{t} e^{-\int_s^t \mu(l)dl} \Lambda(s)\mathrm{d}s, \quad t \in \mathbb{R}. \tag{1.5}$$

Note the improper integral converges since $\underline{\mu} > 0$ as mentioned in (1.3). This result directly follows from the parabolic comparison principle. Indeed setting $m^- \le m^+$ the lower and upper bounds of a bounded and time global solution, $u = u(t, x)$, of (1.4), it readily follows that for all $s \le t$ and $x \in \mathbb{R}$

$$m^- e^{-\int_s^t \mu(l)\mathrm{d}l} \le u(t, x) - \int_s^t \Lambda(\sigma) e^{-\int_\sigma^t \mu(l)\mathrm{d}l} \mathrm{d}\sigma \le m^+ e^{-\int_s^t \mu(l)\mathrm{d}l}.$$

Since $\inf\{\mu(t),\ t \in \mathbb{R}\} > 0$, one obtains letting $s \to -\infty$ that $u(t, x) = u^*(t)$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

Equipped with this notation, we introduce the definition of a generalized traveling wave for (1.1)–(1.2).

Definition 1.1 A three-tuple function $(U, V, c) : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to [0, \infty) \times [0, \infty) \times \mathbb{R}$ is said to be a *generalized traveling wave* of Problem (1.1)–(1.2) with the time varying wave speed $t \mapsto c(t)$ if

(i) the function $c(t)$ is measurable and bounded and, the function pair (U, V) is positive and bounded;

(ii) the function

$$(u, v)(t, x) = (U, V)(t, \xi) \text{ with } \xi = x - \int_0^t c(s)ds$$

is an entire solution of Problem (1.1)–(1.2), that is, a solution defined for $(t, x) \in \mathbb{R} \times \mathbb{R}$;

(iii) the pair (U, V) satisfies the following behaviour at $\xi = \infty$

$$\lim_{\xi \to \infty} (U, V)(t, \xi) = \left(u^*(t), 0\right) \text{ uniformly for } t \in \mathbb{R}, \tag{1.6}$$

and a persistence condition at $\xi = -\infty$, that reads as

$$\liminf_{\xi \to -\infty} \inf_{t \in \mathbb{R}} V(t, \xi) > 0 \quad \text{and} \quad \liminf_{\xi \to -\infty} \inf_{t \in \mathbb{R}} \left|u^*(t) - U(t, \xi)\right| > 0. \tag{1.7}$$

In the above definition, (U, V) corresponds to the so-called *wave profile* while the function $t \mapsto c(t)$ is referred to as the *time varying wave speed function*.

The aim of this work is to investigate the existence and nonexistence of generalized traveling wave solutions for System (1.1) with the functions F_1 and F_2 defined in (1.2). Coming back to (1.1) and using Definition 1.1, we observe that a generalized wave profile (U, V, c) satisfies the following parabolic system

$$\begin{cases} \partial_t U = d(t)\partial_\xi^2 U + c(t)\partial_\xi U + F_1(t, U, V), \\ \partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + F_2(t, U, V), \end{cases} \quad \text{for } (t, \xi) \in \mathbb{R} \times \mathbb{R}, \tag{1.8}$$

together with (1.6) and (1.6). In this paper we present sufficient conditions to ensure the existence and nonexistence of solutions for the above system (1.8) under (1.6) and (1.6).

2 Main results

In this section we present the main results that will be proved later. To do so we need to introduce some notations that will be used throughout this paper.

As in [22] or [23], we introduce the so-called *least mean value* for a function $g \in L^\infty(\mathbb{R})$ as follows

$$\mathcal{M}^-(g) := \sup_{T>0} \inf_{s \in \mathbb{R}} \frac{1}{T} \int_0^T g(l + s)\mathrm{d}l. \tag{2.1}$$

Let us also recall (see [22]) that, for each function $g \in L^\infty(\mathbb{R})$, the least mean value $\mathcal{M}^-(g)$ also has the following important reformulations:

$$\mathcal{M}^-(g) = \lim_{T\to\infty} \inf_{s\in\mathbb{R}} \frac{1}{T}\int_0^T g(l+s)\mathrm{d}l = \sup_{a\in W^{1,\infty}(\mathbb{R})} \inf_{t\in\mathbb{R}} \left(a'(t) + g(t)\right). \quad (2.2)$$

This alternative variational reformulation will be used throughout this work.

Next, define the bounded function $\delta : \mathbb{R} \to \mathbb{R}$ as the growth function and the parameter $\mathcal{T}$, respectively, by

$$\delta(t) = \beta(t)u^*(t) - \gamma(t),\ t \in \mathbb{R}, \quad \text{and} \quad \mathcal{T} = \mathcal{M}^-\left(\delta(\cdot)\right). \quad (2.3)$$

Using the above notations, the main result is the following theorem.

Theorem 2.1 *Assume that $\mathcal{T} > 0$. For each $\lambda > 0$ and $a = a(t) \in W^{1,\infty}(\mathbb{R})$, consider the bounded function $c_{\lambda,a} : \mathbb{R} \to [0,\infty)$ defined by*

$$c_{\lambda,a}(t) = \lambda + \lambda^{-1}\delta(t) + a'(t), \quad t \in \mathbb{R}. \quad (2.4)$$

Then, for each $\lambda \in (0, \sqrt{\mathcal{T}})$ and $a \in W^{1,\infty}(\mathbb{R})$, Problem (1.1)–(1.2) admits a generalized traveling wave, according to Definition 1.1, for the wave speed $t \mapsto c_{\lambda,a}(t)$.

Let us observe that, for each $a \in W^{1,\infty}(\mathbb{R})$ and $\lambda > 0$, one has

$$\mathcal{M}^-(c_{\lambda,a}) = \lambda + \lambda^{-1}\mathcal{T}.$$

Moreover, define the function $\Gamma : (0,\infty) \to (0,\infty)$ by

$$\Gamma(\lambda) := \lambda + \lambda^{-1}\mathcal{T}.$$

Note that it reaches its minimum for $\lambda > 0$ at $\lambda = \sqrt{\mathcal{T}}$. Hence the above result ensures that, for each $\underline{c} > \Gamma\left(\sqrt{\mathcal{T}}\right) = 2\sqrt{\mathcal{T}}$, Problem (1.1)–(1.2) admits a generalized traveling wave solution with a wave speed function $c = c(t)$ such that $\mathcal{M}^-(c) = \underline{c}$. Furthermore, for each $\underline{c} > 2\sqrt{\mathcal{T}}$, there are infinity many admissible functions $c = c(t)$ with $\mathcal{M}^-(c) = \underline{c}$.

Next we show that our above existence result is somehow sharp and provides a necessary condition on the least mean value of the wave speed function for Problem (1.1)–(1.2) to admit generalized traveling waves. Our next result is the following theorem.

Theorem 2.2 *Assume that $\mathcal{T} > 0$. Let (U, V, c) be a generalized traveling wave solution of Problem (1.1)–(1.2) according to Definition 1.1. Then the wave speed function $c = c(t)$ satisfies*

$$\mathcal{M}^-(c) \geq 2\sqrt{\mathcal{T}}.$$

The combination of the above two results provides somehow sharp information on admissible wave speed functions. Indeed, let us consider the set of admissible wave speed functions, denoted by $\mathcal{C}$, that is the set of wave speeds $c = c(t)$ such that Problem (1.1)–(1.2) admits a generalized traveling wave with the wave speed $c = c(t)$. Then the above theorems indicate that

$$\left(2\sqrt{\mathcal{T}}, \infty\right) \subset \left\{\mathcal{M}^-(c) : \ c \in \mathcal{C}\right\} \subset \left[2\sqrt{\mathcal{T}}, \infty\right).$$

Remark 2.3 One may notice that the methodology developed in this work can be applied to obtain the existence of generalized traveling wave solutions for System (1.1)–(1.2) when $\Lambda(t) \equiv \mu(t) \equiv 0$. However as far as non-existence is concerned, our arguments do not apply to this situation. Indeed, our proof for Theorem 2.2 is mostly based on the persistence of the V-component beyond the front, namely for $\xi \to -\infty$. This latter property does not correspond to the expected behaviour of generalized traveling waves in the limit case $\Lambda(t) \equiv \mu(t) \equiv 0$.

In the remaining part of this section, we explore how the above results also allow us to treat systems similar to (1.1)–(1.2) with two time-varying diffusion coefficients, namely a system of the following form

$$\begin{cases} \partial_t u - d_u(t)\partial_x^2 u = \Lambda(t) - \mu(t)u - \beta(t)uv, \\ \partial_t v - d_v(t)\partial_x^2 v = \beta(t)uv - \gamma(t)v, \end{cases} \quad t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{2.5}$$

wherein the functions $d_u(t)$ and $d_v(t)$ are both uniformly continuous on $\mathbb{R}$ and uniformly positive while all other coefficient functions are uniformly positive and bounded. To that aim, one may observe that the above problem reduces to (1.1)–(1.2) by using a suitable change of the time variable. Set

$$\tau(t) = \int_0^t d_v(\sigma)\mathrm{d}\sigma, \quad t \in \mathbb{R}. \tag{2.6}$$

Then this map τ is a strictly increasing bijection of $\mathbb{R}$. Denote

$$(\widehat{u}, \widehat{v})(\tau, x) = (u, v)(t, x),$$

then we have for $w = u, v$ that

$$\partial_t w(t, x) = \partial_\tau \widehat{w}(\tau, x) d_v(t).$$

So the function pair $(\widehat{u}, \widehat{v})$ becomes a solution of the following system of equations

$$\begin{cases} d_v(t)\partial_\tau \widehat{u} = d_u(t)\partial_x^2 \widehat{u} + \Lambda(t) - \mu(t)\widehat{u} - \beta(t)\widehat{u}\widehat{v}, \\ d_v(t)\partial_\tau \widehat{v} = d_v(t)\partial_x^2 \widehat{v} + \beta(t)\widehat{u}\widehat{v} - \gamma(t)\widehat{v}, \end{cases} \quad \text{for } (\tau, x) \in \mathbb{R} \times \mathbb{R},$$

Next, recalling that $\tau = \tau(t)$ is given in (2.6), we set

$$
\begin{aligned}
&d(\tau)=\frac{d_u(t)}{d_v(t)},\ \tilde{\beta}(\tau)=\frac{\beta(t)}{d_v(t)},\quad \tilde{\mu}(\tau)=\frac{\mu(t)}{d_v(t)}\\
&\tilde{\Lambda}(\tau)=\frac{\Lambda(t)}{d_v(t)},\ \tilde{\gamma}(\tau)=\frac{\gamma(t)}{d_v(t)}.
\end{aligned}
\tag{2.7}
$$

Then $(\widehat{u},\widehat{v})$ becomes a solution of the following parabolic problem

$$
\begin{cases}\partial_\tau\widehat{u}=d(\tau)\partial_x^2\widehat{u}+\tilde{\Lambda}(\tau)-\tilde{\mu}(\tau)\widehat{u}-\tilde{\beta}(\tau)\widehat{u}\widehat{v},\\ \partial_\tau\widehat{v}=\partial_x^2\widehat{v}+\tilde{\beta}(\tau)\widehat{u}\widehat{v}-\tilde{\gamma}(\tau)\widehat{v},\end{cases}\qquad \text{for } (\tau,x)\in\mathbb{R}\times\mathbb{R}, \tag{2.8}
$$

that corresponds to Problem (1.1)–(1.2) with different τ-dependent parameter functions.

Note that applying Theorems 2.1 and 2.2 relies on the parameter $\mathcal{T}$. Denote by $\tilde{\delta}(\tau)$ the corresponding growth function defining the parameter $\mathcal{T}$ for System (2.8). The growth function $\delta(t)$ for the original System (2.5) is given by $\delta(t)=d_v(t)\tilde{\delta}(\tau(t))$. Hence, the parameter $\mathcal{T}$ can be rewritten as follows

$$
\begin{aligned}
\mathcal{T}=\mathcal{M}^-\left(\tilde{\delta}\right)&=\lim_{T\to\infty}\inf_{\tau\in\mathbb{R}}\frac{1}{T}\int_\tau^{\tau+T}\tilde{\delta}(s)\mathrm{d}s\\
&=\lim_{T\to\infty}\inf_{l\in\mathbb{R}}\frac{1}{T}\int_{\tau^{-1}(l)}^{\tau^{-1}(l+T)}\tilde{\delta}(\tau(t))d_v(t)\mathrm{d}t\\
&=\lim_{T\to\infty}\inf_{l\in\mathbb{R}}\frac{1}{T}\int_{\tau^{-1}(l)}^{\tau^{-1}(l+T)}\delta(\sigma)\mathrm{d}\sigma.
\end{aligned}
$$

Hence, in terms of the original parameters in System (2.5) the condition $\mathcal{T}>0$ becomes

$$
\mathcal{T}=\lim_{T\to\infty}\inf_{l\in\mathbb{R}}T^{-1}\int_{\tau^{-1}(l)}^{\tau^{-1}(l+T)}\delta(\sigma)\mathrm{d}\sigma>0,
$$

and the latter condition explicitly depends upon the diffusion coefficient $d_v(t)$ and another notion of the least mean value than the one given in (2.1).

The rest of this paper is organized as follows. Section 3 is devoted to the proof of the existence of suitable bounded solutions for system (1.8). In Sect. 4 we complete the proofs of both Theorems 2.1 and 2.2. To this end we develop arguments in Proposition 4.1 that allow us to prove (1.6) as well as the nonexistence result as stated in Theorem 2.2.

3 Existence of solutions

Throughout this section we assume that

$$
\mathcal{T}=\mathcal{M}^-(\delta)>0. \tag{3.1}
$$

Consider the function $\Gamma : (0, \infty) \to \mathbb{R}$ defined by

$$\Gamma(\lambda) := \frac{1}{\lambda}\left[\lambda^2 + \mathcal{T}\right].$$

Next define the quantity $\underline{c}^\star > 0$ by

$$\underline{c}^\star = 2\sqrt{\mathcal{T}} = \min_{\lambda>0}\Gamma(\lambda) = \Gamma\left(\sqrt{\mathcal{T}}\right).$$

We have the following theorem on the existence of bounded solutions.

Theorem 3.1 (Existence of bounded solutions for (1.8)) *Assume that* (3.1) *is satisfied. Then for each* $\lambda \in \left(0, \sqrt{\mathcal{T}}\right)$ *and* $a \in W^{1,\infty}(\mathbb{R})$*, Problem* (1.8) *admits a positive and bounded solution* $(U, V) = (U, V)(t, \xi)$ *for the wave speed function* $t \mapsto c_{\lambda,a}(t) = \lambda + \lambda^{-1}\delta(t) + a'(t)$ *that furthermore satisfies*

$$\begin{aligned} &\inf_{t\in\mathbb{R}} V(t,\xi) > 0, \quad \forall \xi \in \mathbb{R},\\ &\lim_{\xi\to\infty}(U,V)(t,\xi) = \left(u^*(t), 0\right) \text{ uniformly for } t \in \mathbb{R}, \\ &V(t,\xi) \sim e^{-\lambda(\xi+a(t))} \text{ as } \xi \to \infty \text{ uniformly for } t \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Proof The proof relies on the construction of a suitable sub- and super-solution pair. These constructions are described below in the step. The second step deals with the uniform boundedness of the constructed solutions and completes the proof of the theorem.

Throughout this proof we fix $\lambda \in \left(0, \sqrt{\mathcal{T}}\right)$ and $a \in W^{1,\infty}(\mathbb{R})$. For notational simplicity we write $c = c(t)$ instead of $c_{\lambda,a} = c_{\lambda,a}(t)$.

First step (i) consider the function $\bar{U}(t,\xi) := u^*(t)$ and observe that it satisfies

$$\partial_t\bar{U} - d(t)\partial_\xi^2 U - c(t)\partial_\xi\bar{U} = F_1\left(t, \bar{U}, 0\right), \quad \forall (t,\xi) \in \mathbb{R}\times\mathbb{R}.$$

(ii) Next consider the function $\bar{V}(t,\xi) = e^{-\lambda(\xi+a(t))}$ and note that it satisfies the equation

$$\partial_t\bar{V} = -\lambda a'(t)\bar{V},\ \partial_\xi\bar{V} = -\lambda\bar{V},\ \ \partial_\xi^2\bar{V} = \lambda^2\bar{V},$$

so that one gets

$$\begin{aligned} \partial_t\bar{V} - \partial_\xi^2\bar{V} - c(t)\partial_\xi\bar{V} &= \bar{V}\left[-\lambda a'(t) - \lambda^2 + c(t)\lambda\right] \\ &= \lambda\bar{V}\left[-a'(t) - \lambda + \lambda + \lambda^{-1}\delta(t) + a'(t)\right] = \delta(t)\bar{V}. \end{aligned}$$

Thus, $\overline{V}(t,\xi) = e^{-\lambda(\xi+a(t))}$ satisfies

$$\partial_t\bar{V} - \partial_\xi^2\bar{V} - c(t)\partial_\xi\bar{V} = F_2\left(t, u^*(t), \bar{V}\right),\ (t,\xi) \in \mathbb{R}\times\mathbb{R}.$$

(iii) Now we construct a function $\underline{U}(t,\xi)$ satisfying

$$\partial_t \underline{U} - d(t)\partial_\xi^2 \underline{U} - c(t)\partial_\xi \underline{U} \le F_1(t,\underline{U},\bar{V}) = \Lambda(t) - \mu(t)\underline{U} - \beta(t)\underline{U}\bar{V} \text{ on } \{(t,\xi):\ \underline{U} \ge 0\}.$$

We look for such a function with the form

$$\underline{U}(t,\xi) = u^*(t)g(\xi) \quad \text{with } g(\xi) = 1 - Ke^{-\kappa\xi},$$

where $K > 0$ and $0 < \kappa < \lambda$ are parameters that will be chosen later so that $K > 1$ and $\kappa = \kappa(K) > 0$ small enough. For $t \in \mathbb{R}$ and $\xi \in \mathbb{R}$ such that $1 - Ke^{-\kappa\xi} \ge 0$, namely for $\xi \ge \frac{1}{\kappa}\ln(K)$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\partial_t \underline{U} - d(t)\partial_\xi^2 \underline{U} - c(t)\partial_\xi \underline{U} - F_1(t,\underline{U},\bar{V}) \\
&\quad = K\left[u^*(t)d(t)\kappa^2 - c(t)\kappa - \Lambda(t)\right]e^{-\kappa\xi} + \beta(t)u^*(t)e^{-\lambda(\xi+a(t))}\left[1 - Ke^{-\kappa\xi}\right] \\
&\quad \le e^{-\kappa\xi}\left\{K\left[u^*(t)d(t)\kappa^2 - c(t)\kappa - \Lambda(t)\right] + \beta(t)u^*(t)e^{-\lambda a(t)}e^{\frac{\kappa-\lambda}{\kappa}\ln(K)}\right\}.
\end{aligned}$$

Now fix $K > 1$ and choose $\kappa = \kappa(K) \in (0,\lambda)$ small enough such that

$$K\left[u^*(t)d(t)\kappa^2 - c(t)\kappa - \Lambda(t)\right] + \beta(t)u^*(t)e^{-\lambda a(t)}e^{\frac{\kappa-\lambda}{\kappa}\ln(K)} < 0, \quad \forall t \in \mathbb{R}.$$

Recall that the latter condition can be achieved since $\inf\{\Lambda(t),\ t \in \mathbb{R}\} > 0$ and the different time-dependent functions are bounded. Hence with such a choice for the parameters, one obtains that the function $\underline{U}$ satisfies

$$\partial_t \underline{U} - d(t)\partial_\xi^2 \underline{U} - c(t)\partial_\xi \underline{U} \le F_1(t,\underline{U},\bar{V})$$

for all $t \in \mathbb{R}$ and $\xi \ge \kappa^{-1}\ln(K)$, that is on the set $\{(t,\xi) \in \mathbb{R}^2 :\ \underline{U}(t,\xi) \ge 0\}$.

(iv) Now we set

$$U^-(t,\xi) := \max\left(0, \underline{U}(t,\xi)\right)$$

and look for a constant $h > \lambda$ and a function $b = b(t) \in W^{1,\infty}(\mathbb{R})$ such that the function

$$\underline{V}(t,\xi) = e^{-\lambda(\xi+a(t))} - e^{-\lambda a(t)+b(t)}e^{-h\xi}$$

satisfies the following differential inequality

$$\partial_t \underline{V} - \partial_\xi^2 \underline{V} - c(t)\partial_\xi \underline{V} \le F_2\left(t, U^-, \underline{V}\right) = \left[\beta(t)U^- - \gamma(t)\right]\underline{V} \tag{3.3}$$

on the set $\{(t,\xi)\in\mathbb{R}\times\mathbb{R}:\ \underline{V}(t,\xi)\geq 0\}=\{(t,\xi)\in\mathbb{R}\times\mathbb{R}:\ b(t)-(h-\lambda)\xi\leq 0\}$. To that aim, note that, setting $A(t)=e^{-\lambda a(t)+b(t)}$, one has

$$\partial_t\underline{V}(t,\xi)=-\lambda a'(t)e^{-\lambda(\xi+a(t))}-A'(t)e^{-h\xi},\ \partial_\xi\underline{V}(t,\xi)=-\lambda e^{-\lambda(\xi+a(t))}+hA(t)e^{-h\xi},$$
$$\partial_{\xi\xi}\underline{V}(t,\xi)=\lambda^2e^{-\lambda(\xi+a(t))}-h^2A(t)e^{-h\xi}.$$

Hence, recalling that $c(t)=\lambda+\lambda^{-1}\delta(t)+a'(t)$, we have

$$\begin{aligned}
&\partial_t\underline{V}-\partial_\xi^2\underline{V}-c(t)\partial_\xi\underline{V}-\left[\beta(t)U^--\gamma(t)\right]\underline{V}\\
&\quad=e^{-\lambda(\xi+a(t))}\left[-\lambda a'(t)-\lambda^2+c(t)\lambda-\beta(t)U^-+\gamma(t)\right]\\
&\qquad+e^{-h\xi}\left[-A'(t)+h^2A(t)-c(t)hA(t)+A(t)\left[\beta(t)U^--\gamma(t)\right]\right]\\
&\quad=e^{-\lambda(\xi+a(t))}\beta(t)\left[u^*(t)-U^-\right]\\
&\qquad+e^{-h\xi}\left[-A'(t)+h^2A(t)-c(t)hA(t)+A(t)\left[\beta(t)U^--\gamma(t)\right]\right].
\end{aligned}$$

In order to satisfy (3.3), setting $\eta=h-\lambda>0$, it is sufficient to have for all $(t,\xi)\in\mathbb{R}\times\mathbb{R}$ with $b(t)<\eta\xi$ that

$$\beta(t)u^*(t)Ke^{-\kappa\xi}+e^{-\eta\xi}e^{b(t)}\left[\lambda a'(t)-b'(t)+h^2-c(t)h+\delta(t)\right]\leq 0.\quad(3.4)$$

To handle this problem and to find a suitable function $b\in W^{1,\infty}(\mathbb{R})$ and a constant $h=\lambda+\eta>\lambda$ such that the above inequality is satisfied, let us fix $\eta\in(0,\kappa)$ small enough and $b_0\in W^{1,\infty}(\mathbb{R})$ such that there exists $\varepsilon>0$ with

$$\lambda a'(t)-b_0'(t)+h^2-c(t)h+\delta(t)\leq-\varepsilon,\quad\forall t\in\mathbb{R}.$$

To that aim we make use of the second expression for the least mean value given in (2.2). Recalling that $c(t)=\lambda+\lambda^{-1}\delta(t)+a'(t)$, one has

$$\begin{aligned}
&\lambda a'(t)-b_0'(t)+h^2-c(t)h+\delta(t)\\
&\quad=-h\left[\left(1-\frac{\lambda}{h}\right)a'(t)+\frac{b_0'(t)}{h}-h+\lambda+\left(\lambda^{-1}-h^{-1}\right)\delta(t)\right].
\end{aligned}$$

Since $\lambda^{-1}>h^{-1}$, one gets

$$\begin{aligned}
\mathcal{M}^-\left[-h+\lambda+\left(\lambda^{-1}-h^{-1}\right)\delta(t)\right]&=-h+\lambda+\left(\lambda^{-1}-h^{-1}\right)\mathcal{T}\\
&=\Gamma(\lambda)-\Gamma(h).
\end{aligned}$$

Since $\lambda\in\left(0,\sqrt{\mathcal{T}}\right)$, choose $\eta\in(0,\kappa)$ small enough such that

$$\Gamma(\lambda)-\Gamma(\lambda+\eta)>0.$$

Next letting $\varepsilon = \frac{h}{2}\left[\Gamma(\lambda) - \Gamma(\lambda+\eta)\right]$ and recalling (2.2), we know that there exists $B = B(t) \in W^{1,\infty}(\mathbb{R})$ such that

$$B'(t) - \left(h^2 - c(t)h + \delta(t)\right) \geq \varepsilon, \quad \forall t \in \mathbb{R}.$$

Set $b_0(t) := \lambda a(t) + B(t)$ that satisfies

$$\lambda a'(t) - b_0'(t) + h^2 - c(t)h + \delta(t) \leq -\varepsilon, \quad \forall t \in \mathbb{R}.$$

We now come back to (3.4) and look for the function $b = b(t)$ under the form $b(t) = b_0(t) + \ln \rho$ for some constant $\rho > 0$ sufficiently large. In order to find the constant $\rho > 0$ such that (3.4) holds, recalling that $\eta = h - \lambda < \kappa$, it is sufficient to have

$$\beta(t)u^*(t)Ke^{-b_0(t)} \leq \varepsilon\rho e^{(\kappa\eta^{-1}-1)b_0(t)+(\kappa\eta^{-1}-1)\ln\rho}, \quad \forall t \in \mathbb{R},$$

that is

$$\beta(t)u^*(t)Ke^{-\kappa\eta^{-1}b_0(t)} \leq \varepsilon e^{\kappa\eta^{-1}\ln\rho}, \quad \forall t \in \mathbb{R},$$

which is satisfied as soon as $\rho > 0$ is large enough.

In conclusion, we have chosen the constant $h = \lambda + \eta > \lambda$ and the function $b(t) = b_0(t) + \ln \rho$ such that the corresponding function $\underline{V}$ satisfies (3.3) on the set $\left\{(t,\xi) \in \mathbb{R}\times\mathbb{R} : \underline{V}(t,\xi) \geq 0\right\}$.

Remark 3.2 If the functions $\delta(t)$ and $a(t)$ are T-periodic for some $T > 0$, then the function $b(t)$ can also be chosen to be T-periodic.

Second step In the second step we complete the proof of the theorem. To that aim, for $n \geq 1$ we consider the reaction–diffusion system

$$\begin{cases} \partial_t U = d(t)\partial_\xi^2 U + c(t)\partial_\xi U + F_1(t,U,V), \\ \partial_t V = \partial_\xi^2 V + c(t)\partial_\xi V + F_2(t,U,V), \end{cases} \qquad \text{for } t \geq -n,\ \xi \in \mathbb{R},$$

with the condition at $t = -n$:

$$U(-n,\xi) = \max\left(0, \underline{U}(-n,\xi)\right) \text{ and } V(-n,\xi) = \max\left(0, \underline{V}(-n,\xi)\right),\ \xi \in \mathbb{R}.$$

The above Cauchy problem admits a globally defined solution. We denote it by (U^n, V^n) and recall that it is defined for $t \geq -n$ and for $\xi \in \mathbb{R}$. Observe also that, from the construction of the sub- and super-solutions in the step, successive applications of the parabolic comparison principle ensures that the sequence $\{(U^n, V^n)\}$ satisfies the following super and lower estimates

$$\max\left(0, \underline{U}\right) \leq U^n \leq \overline{U} \quad \text{and} \quad \max\left(0, \underline{V}\right) \leq V^n \leq \overline{V}$$

for any $n \geq 1$ and for all $t \geq -n$ and $\xi \in \mathbb{R}$.

Next in order to take the limit as $n \to \infty$, we need to derive new estimates for the sequence of functions $\{(U^n, V^n)\}$ to ensure that it is uniformly bounded. Notice that since $U^n(t,\xi) \leq u^*(t)$ for all $t \geq -n$ and $\xi \in \mathbb{R}$, we know that this solution component U^n is bounded. We investigate the other solution component V^n in the next lemma.

Lemma 3.3 *The function $V^n(t,\xi)$ is uniformly bounded on $[-n,\infty) \times \mathbb{R}$, namely there exists some constant $M > 0$ such that, for any n large enough, one has*

$$V^n(t,\xi) \leq M, \quad \forall t \geq -n, \quad \xi \in \mathbb{R}.$$

To prove this lemma, consider the sequence of functions $\{(U_n, V_n)\}$ defined for $t \geq 0$ and $\xi \in \mathbb{R}$ by

$$(U_n, V_n)(t,\xi) = \left(U^n, V^n\right)(t-n,\xi),$$

that satisfies the following initial value parabolic system

$$\begin{cases} \partial_t U_n = d(t-n)\partial_\xi^2 U_n + c(t-n)\partial_\xi U_n + F_1(t-n, U_n, V_n), \\ \partial_t V_n = \partial_\xi^2 V_n + c(t-n)\partial_\xi V_n + F_2(t-n, U_n, V_n), \end{cases} \qquad \text{for } t > 0,\ \xi \in \mathbb{R},$$

together with

$$U_n(0,\xi) = \max\left(0, \underline{U}(-n,\xi)\right) \quad \text{and} \quad V_n(0,\xi) = \max\left(0, \underline{V}(-n,\xi)\right), \quad \xi \in \mathbb{R}.$$

To handle this problem we remove the convection term by defining the function $\{(u_n(t,\xi), v_n(t,\xi)\}$ by

$$(u_n, v_n)(t,\xi) = (U_n, V_n)\left(t, \xi - \int_0^t c(l-n)dl\right), \quad \forall t \geq 0,\ \forall \xi \in \mathbb{R}.$$

Note that it satisfies the following reaction–diffusion system without the drift term

$$\begin{cases} \partial_t u_n = d(t-n)\partial_\xi^2 u_n + F_1(t-n, u_n, v_n), \\ \partial_t v_n = \partial_\xi^2 v_n + F_2(t-n, u_n, v_n), \end{cases} \qquad \text{for } t > 0,\ \xi \in \mathbb{R}, \tag{3.5}$$

together with

$$u_n(0,\xi) = U_n(0,\xi) \quad \text{and} \quad v_n(0,\xi) = V_n(0,\xi), \quad \xi \in \mathbb{R}. \tag{3.6}$$

Now to prove the above boundedness lemma, it is sufficient to show that $\{v_n\}$ is uniformly bounded on $[0,\infty) \times \mathbb{R}$ for n large enough. To that aim we follow and extend some ideas taken from Ducrot et al. [8].

Proof As mentioned above, in order to prove Lemma 3.3 we will prove that the solutions of (3.5)–(3.6) are uniformly bounded for $t \geq 0$, $\xi \in \mathbb{R}$ and for n large enough.

To that aim let us observe that the sequences $\{u_n(0,\cdot)\}$ and $\{v_n(0,\cdot)\}$ are uniformly bounded and let us also recall that $u_n(t,\xi) \leq u^*(t-n)$ is uniformly bounded with respect to $t \geq 0$, $\xi \in \mathbb{R}$ and $n \geq 0$. Now in order to prove that the sequence $(t,\xi) \mapsto v_n(t,\xi)$ enjoys the same property, we argue by contradiction by assuming that there exists a sequence $\{(t_n,\xi_n)\}$ with $t_n > 0$, $\xi_n \in \mathbb{R}$ such that

$$v_n(t_n,\xi_n) \to \infty \quad \text{as } n \to \infty.$$

Since $u_n \leq \|u^*\|_{L^\infty(\mathbb{R})}$ is uniformly bounded, fix $K > 0$ large enough such that

$$\beta(t-n)u_n(t,\xi) + v_n(0,\xi) \leq K, \quad \forall t \geq 0,\ \xi \in \mathbb{R},\ n \geq 0.$$

Hence, the comparison principle ensures that

$$v_n(t,\xi) \leq Ke^{Kt}, \quad \forall t \geq 0,\ \xi \in \mathbb{R},\ n \geq 0,$$

so that $t_n \to \infty$ as $n \to \infty$.

Assume, without loss of generality, that

$$t_n = \min\{t > 0 : \ \|v_n(t,\cdot)\|_{L^\infty(\mathbb{R})} = n\} \quad \text{and} \quad v_n(t_n,\xi_n) \in [n/2, n].$$

Define $\{\tilde{v}_n(t,\xi)\}$ by

$$\tilde{v}_n(t,\xi) = \frac{v_n(t+t_n,\xi+\xi_n)}{v_n(t_n,\xi_n)}.$$

It becomes a solution of the equation

$$\partial_t \tilde{v}_n(t,\xi) = \partial_\xi^2 \tilde{v}_n(t,\xi) + a_n(t,\xi)\tilde{v}_n(t,\xi), \quad t > -t_n,\ \xi \in \mathbb{R},$$

wherein the sequence of functions $\{a_n(t,\xi)\}$ is defined by

$$a_n(t,\xi) = \beta(t+t_n-n)u_n(t+t_n,\xi_n+\xi) - \gamma(t+t_n-n), \quad \forall t \geq -t_n,\ \forall \xi \in \mathbb{R}. \tag{3.7}$$

Now, since $\{u_n(t,\xi)\}$ is uniformly bounded, the sequence of functions $\{a_n(t,\xi)\}$ is uniformly bounded for $t \geq -t_n$, $\xi \in \mathbb{R}$ and $n \geq 0$. From our construction, one may observe that, for all $n \geq 1$, all $t \in [-t_n, 0]$ and all $\xi \in \mathbb{R}$, one has

$$\tilde{v}_n(t,\xi) \leq \max_{t \in [0,t_n]} \|v_n(t,\cdot)\|_{L^\infty(\mathbb{R})} \times \frac{2}{n} \leq 2.$$

On the other hand, since $a_n(t,\xi) \leq K$ one obtains from the parabolic comparison principle that

$$\tilde{v}_n(t,\xi) \leq 2e^{Kt}, \quad \forall n \geq 1,\ t \geq 0,\ \xi \in \mathbb{R}.$$

As a consequence, the sequence of functions $\{\tilde{v}_n(t,\xi)\}$ is uniformly bounded with respect to $n \geq 1$ on each domain of the form $[-t_n, T] \times \mathbb{R}$ with $T > 0$. Due to parabolic estimates, one may assume, possibly along a subsequence still denoted by n, that

$$\tilde{v}_n(t,\xi) \to \tilde{v}(t,\xi) \text{ locally uniformly for } (t,\xi) \in \mathbb{R} \times \mathbb{R} \text{ as } n \to \infty.$$

Furthermore, the function $\tilde{v}(t,\xi)$ satisfies

$$\begin{cases} \tilde{v}(0,0) = 1, \\ \left(\partial_t - \partial_\xi^2\right) \tilde{v}(t,\xi) + a(t,\xi)\tilde{v}(t,\xi) = 0, \quad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}, \end{cases}$$

wherein the bounded function $a(t,\xi)$ denotes an L^∞_{loc}-weak star limit of the bounded sequence of functions $\{a_n(t,\xi)\}$. As a consequence we obtain that $\tilde{v}(t,\xi) > 0$ for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}$ and, using the definition of the sequence $\{\tilde{v}_n(t,\xi)\}$, we conclude that

$$v_n(t+t_n, \xi_n+\xi) \to \infty \text{ locally uniformly for } (t,\xi) \in \mathbb{R}\times\mathbb{R} \text{ as } n \to \infty. \tag{3.8}$$

Now we make the following claim.

Claim 3.4 *The sequence of functions* $\{u_n(t,\xi)\}$ *satisfies*

$$\lim_{n\to\infty} u_n(t+t_n, \xi+\xi_n) = 0, \quad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}.$$

Proof of Claim 3.4 Let $M > 0$, $R > 0$ and $T > 0$ be given. Consider the function $u = u^n_{T,R}$, the solution of the problem

$$\begin{cases} \partial_t u = d(t_n+t)\partial_\xi^2 u + \overline{\Lambda} - \left(\underline{\mu} + \underline{\beta} M\right) u, \ t \in (-T,T), \ |\xi| \leq R, \\ u(-T,\xi) = \overline{u^*}, \ \xi \in [-R,R], \\ u\,(t, \pm R) = \overline{u^*}, \ t \in (-T,T), \end{cases}$$

wherein we have set

$$\overline{\Lambda} = \sup_{t\in\mathbb{R}} \Lambda(t), \ \overline{u^*} = \sup_{t\in\mathbb{R}} u^*(t), \quad \underline{\mu} = \inf_{t\in\mathbb{R}} \mu(t), \underline{\beta} = \inf_{t\in\mathbb{R}} \beta(t).$$

Due to (3.8), choose $n_0 \geq 1$ large enough (depending on M, R and T) such that

$$v_n(t+t_n, \xi+\xi_n) \geq M, \quad \forall n \geq n_0, \ t \in [-T,T], \ \xi \in [-R,R].$$

Hence, because of the above definition, the parabolic comparison principle ensures that

$$u_n(t+t_n, \xi+\xi_n) \leq u^n_{T,R}(t,\xi) \quad \text{for any } n \geq n_0, \ |\xi| \leq R, \ |t| \leq T. \tag{3.9}$$

Next observe that one has

$$0 \leq u^n_{T,R}(t,\xi) \leq \frac{\overline{\Lambda}}{\underline{\mu}} \quad \text{for any } n \geq n_0,\ |\xi| \leq R,\ |t| \leq T.$$

As a consequence of this uniform boundedness and parabolic regularity, for any positive constants T, R, M and any sequence $\{n_k\}_{k\geq 0}$ tending to ∞ as $k \to \infty$, one may extract a subsequence, still denoted by $\{n_k\}$, such that

$$u^{n_k}_{T,R}(t,\xi) \to \hat{u}_{T,R}(t,\xi),$$

uniformly for $|t| \leq \frac{T}{2}$ and $|\xi| \leq \frac{R}{2}$, wherein the function $\hat{u}_{T,R}$ satisfies

$$\partial_t u = d^*(t)\partial^2_\xi u + \overline{\Lambda} - \left(\underline{\mu} + \underline{\beta} M\right) u, \quad |t| \leq \frac{T}{2},\ |\xi| \leq \frac{R}{2}.$$

Herein d^* is a L^∞-weak star limit of the bounded sequence $\{d(t_{n_k} + \cdot)\}_{k\geq 0}$. Here again, because of the uniform boundedness with respect to T and R of $\hat{u}_{T,R}$, using parabolic regularity, for any sequence $\{T_k\}$ and $\{R_k\}$ tending to infinity, one may extract subsequences, still denoted by $\{T_k\}$ and $\{R_k\}$ such that

$$\hat{u}_{T_k,R_k}(t,\xi) \to \hat{u}(t,\xi) \text{ locally uniformly for } (t,\xi) \in \mathbb{R} \times \mathbb{R},$$

where the limit function $\hat{u}$ becomes a bounded entire solution of the problem

$$\partial_t u = d^*(t)\partial^2_\xi u + \overline{\Lambda} - \left(\underline{\mu} + \underline{\beta} M\right) u, \quad (t,\xi) \in \mathbb{R} \times \mathbb{R},$$

so that $\hat{u}(t,\xi) = \frac{\overline{\Lambda}}{\underline{\mu}+M\underline{\beta}}$ for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}$.

As a consequence one gets

$$\limsup_{\substack{T\to\infty \\ R\to\infty}} \limsup_{n\to\infty} u^n_{T,R}(t,\xi) \leq \frac{\overline{\Lambda}}{\underline{\mu} + M\underline{\beta}}, \quad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}.$$

Next we infer from (3.9) that, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}$,

$$\limsup_{n\to\infty} u_n(t + t_n, \xi + \xi_n) \leq \frac{\overline{\Lambda}}{\underline{\mu} + M\underline{\beta}}, \quad \forall M > 0,$$

which yields, letting $M \to \infty$,

$$\lim_{n\to\infty} u_n(t + t_n, \xi + \xi_n) = 0, \quad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}.$$

This completes the proof of Claim 3.4. □

We are now able to complete the proof of Lemma 3.3. To that aim, recall that $\underline{\gamma} > 0$ is given by

$$\underline{\gamma} = \inf_{t\in\mathbb{R}} \gamma(t).$$

Using Claim 3.4 and recalling the definition of the sequence $\{a_n(t,\xi)\}$ in (3.7), its $L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$-weak limit $a(t,\xi)$ satisfies

$$a(t,\xi) \le -\underline{\gamma}, \quad \forall (t,\xi) \in \mathbb{R}\times\mathbb{R}.$$

Hence, from the above construction, the function $\tilde{v}(t,\xi)$ satisfies the following properties

$$\begin{aligned}
&\tilde{v}(0,0) = 1,\\
&\tilde{v}(t,\xi) \le 2, \quad \forall t \in (-\infty, 0], \ \forall \xi \in \mathbb{R},\\
&\partial_t \tilde{v}(t,\xi) \le \partial_\xi^2 \tilde{v}(t,\xi) - \underline{\gamma}\tilde{v}(t,\xi), \quad \forall (t,\xi) \in \mathbb{R}\times\mathbb{R}.
\end{aligned}$$

Observe that $\overline{v}(t,\xi) := e^{-\underline{\gamma}t}$ is a super-solution of the above equation. Hence, fix $T > 0$, so that we get

$$\tilde{v}(-T,\xi) \le 2 \le 2e^{-\underline{\gamma}T}\overline{v}(-T,\xi), \quad \forall \xi \in \mathbb{R},$$

and thus, using the parabolic comparison, we obtain

$$\tilde{v}(t,\xi) \le 2e^{-\underline{\gamma}T}\overline{v}(t,\xi), \quad \forall t \ge -T, \ \xi \in \mathbb{R}.$$

This implies, with $(t,\xi) = (0,0)$, that

$$\tilde{v}(0,0) \le 2e^{-\underline{\gamma}T}, \quad \forall T > 0.$$

Finally, since $\underline{\gamma} > 0$, this contradicts the normalization condition $\tilde{v}(0,0) = 1$ and completes the proof of the lemma. □

Now using parabolic regularity coupled with the above estimates, there exists a sub-sequence of $\{(U^n, V^n)\}$ still denoted by the same notation, and a bounded limit function pair (U, V) such that

$$\left(U^n, V^n\right)(t,\xi) \to (U,V)(t,\xi) \text{ locally uniformly for } (t,\xi) \in \mathbb{R}\times\mathbb{R}.$$

Furthermore, (U, V) satisfies the problem

$$\begin{cases} \partial_t U = d_1(t)\partial_\xi^2 U + c(t)\partial_x U + F_1(t,U,V), \\ \partial_t V = \partial_\xi^2 V + c(t)\partial_x V + F_2(t,U,V), \end{cases} \qquad \text{for } t \in \mathbb{R}, \ \xi \in \mathbb{R},$$

and, recalling that (U, V) are bounded, they also enjoy the following estimates

$$\max\left(0, \underline{U}\right) \le U \le \overline{U} \quad \text{and} \quad \max\left(0, \underline{V}\right) \le V \le \overline{V}, \tag{3.10}$$

The lower estimate above for V ensures that there exists ξ_0 large enough such that

$$\inf_{t\in\mathbb{R}} V(t,\xi) > 0, \quad \forall \xi > \xi_0.$$

Therefore, $V > 0$ on $\mathbb{R}\times\mathbb{R}$ due to the maximum principle and one also deduces that

$$\inf_{t\in\mathbb{R}} V(t,\xi) > 0, \quad \forall \xi \in \mathbb{R}.$$

This proves the property in (3.2). Furthermore, it follows from the positivity of V and the maximum principle that $U(t,\xi) < u^*(t)$ for all $(t,\xi)\in\mathbb{R}^2$. Next the above estimates, namely (3.10), also ensure that

$$\lim_{\xi\to\infty} (U(t,\xi), V(t,\xi)) = \left(u^*(t), 0\right) \text{ uniformly for } t \in \mathbb{R},$$

that completes the proof of the second property in (3.2). Finally, the third condition in (3.2) follows from the second estimate in (3.10) since the functions $\overline{V}$ and $\underline{V}$ have the same behavior as $\xi \to \infty$. This completes the proof of Theorem 3.1. □

4 Proofs of Theorems 2.1 and 2.2

In this section, we complete the proofs of Theorems 2.1 and 2.2. To that aim throughout this section we again assume that

$$\mathcal{T} > 0. \tag{4.1}$$

Then Theorems 2.1 and 2.2 will follow from the next key proposition.

Proposition 4.1 *Let (U, V) be a bounded and positive solution of (1.8) for some given bounded wave speed function $t \mapsto c(t)$ and satisfy*

$$\exists \xi_0 \in \mathbb{R}, \quad \inf_{t\in\mathbb{R}} V(t,\xi_0) > 0. \tag{4.2}$$

Then, for any $c \in \left[0, 2\sqrt{\mathcal{T}}\right)$, it holds that

$$\liminf_{t\to\infty} \inf_{\tau\in\mathbb{R}} V\left(t-\tau, \int_0^t [c - c(l-\tau)]\, dl\right) > 0.$$

As already mentioned above this key proposition will allow us to prove both Theorems 2.1 and 2.2. The proof of Proposition 4.1 relies on several steps. Its proof is inspired from dynamical systems ideas and more precisely from uniform persistence theory. We refer to Cantrell and Cosner [5] for the persistence theory in reaction–diffusion equations, Smith and Thieme [26] for persistence theory in general dynamical systems, and Zhao [31] for persistence theory in infinite dimensional dynamical systems. We also refer to [1,7,8] for the study of spatial propagation for reaction–diffusion equations using such ideas. The two last mentioned works [7,8] are concerned with the study of asymptotic speed of spread while the one [1] deals with the study of traveling wave for some reaction–diffusion problems in some homogeneous environment. Here we extend such arguments to the delicate case of problems posed in a time-varying environment.

We start with the following remark and lemma. Let $g \in L^\infty(\mathbb{R})$ be given. Consider the set of translations of g, namely the set $\{g(\cdot + h),\ h \in \mathbb{R}\}$. It is relatively compact with respect to the $L^\infty(\mathbb{R})$ weak$-\star$ topology and we denote by $\mathcal{H}_g$ its closure with respect to $L^\infty(\mathbb{R})$ weak$-\star$ topology. This set is referred to as the ω-limit set of g in [23] and we refer to Proposition 4.4 of the aforementioned paper for further properties on the least mean value on the ω-limit set. Now we present a lemma which will be needed in the following discussion.

Lemma 4.2 *Let $g \in L^\infty(\mathbb{R})$ be given. Then the following holds*

$$\mathcal{M}^-(g) \le \mathcal{M}^-\left(\widehat{g}\right), \quad \forall \widehat{g} \in \mathcal{H}_g.$$

Proof Let $\widehat{g} \in \mathcal{H}_g$ be given. Then there exists a sequence $\{h_n\}$ such that

$$g(t + h_n) \to \widehat{g}(t) \text{ in } L^\infty(\mathbb{R}) \text{ weak-}\star \text{ topology.}$$

Thus, one has for all $T > 0$ and $n \ge 0$ that

$$\inf_{s\in\mathbb{R}} \frac{1}{T}\int_0^T g(l+s)\mathrm{d}l \le \frac{1}{T}\int_\sigma^{\sigma+T} g(l+h_n)\mathrm{d}l, \quad \forall \sigma \in \mathbb{R}.$$

Hence, letting $n \to \infty$ yields

$$\inf_{s\in\mathbb{R}} \frac{1}{T}\int_0^T g(l+s)\mathrm{d}l \le \frac{1}{T}\int_\sigma^{\sigma+T} \widehat{g}(l+\sigma)\mathrm{d}l, \quad \forall \sigma \in \mathbb{R}.$$

Taking the infimum for $\sigma \in \mathbb{R}$ on the right hand side and the limit when $T \to \infty$ yields

$$\mathcal{M}^-(g) \le \mathcal{M}^-\left(\widehat{g}\right).$$

This completes the proof of the lemma. □

Fix $c \in (0, c^\star)$ and consider the τ-parametrized function pair (u, v) defined by

$$(u, v)(t, \xi; \tau) = (U, V)\left(t - \tau, \xi + \int_0^t [c - c(l-\tau)]\,\mathrm{d}l\right) \tag{4.3}$$

for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}$ and for parameter $\tau \in \mathbb{R}$. Next observe that (u, v) satisfies the system of equations

$$\begin{cases} \partial_t u = d(t-\tau)\partial_\xi^2 u + c\partial_\xi u + F_1(t-\tau, u, v), \\ \partial_t v = \partial_\xi^2 v + c\partial_\xi v + F_2(t-\tau, u, v), \end{cases} \qquad \text{for } t \in \mathbb{R},\ \tau \in \mathbb{R},\ \xi \in \mathbb{R}.$$

Consider the set of time translations given by

$$S = \{(u, v)(t+h, \xi; \tau),\ h \in \mathbb{R}, \tau \in \mathbb{R}\}.$$

Due to parabolic regularity this set is relatively compact with respect to the open compact topology on $\mathbb{R} \times \mathbb{R}$. We consider its closure, denoted by $\overline{S}$, with respect to the open compact topology, namely those of $C^0_{\text{loc}}(\mathbb{R} \times \mathbb{R})^2$. In other words, one has $(\tilde{u}, \tilde{v}) \in \overline{S}$ if there exist two sequences $\{h_n\}_{n\geq 0}$ and $\{\tau_n\}_{n\geq 0}$ such that

$$\lim_{n\to\infty} (u, v)(t+h_n, \xi; \tau_n) = (\tilde{u}, \tilde{v})(t, \xi) \text{ in } C^0_{\text{loc}}(\mathbb{R} \times \mathbb{R})^2.$$

One may also observe that the above convergence also holds weakly in $W^{1,2}_{p,\text{loc}}(\mathbb{R} \times \mathbb{R})$ for any $p \in [1, \infty)$ and that, for each $k \leq 1$, one also has

$$\lim_{n\to\infty} \partial_\xi^k (u, v)(t+h_n, \xi; \tau_n) = \partial_\xi^k (\tilde{u}, \tilde{v})(t, \xi) \text{ in } C^0_{\text{loc}}(\mathbb{R} \times \mathbb{R})^2.$$

With this notation, the step to prove Proposition 4.1 consists in the following lemma, that somehow corresponds to a weak persistence property for the set $\overline{S} \setminus \left\{(\tilde{u}, \tilde{v}) \in \overline{S}:\ \tilde{v} \equiv 0\right\}$.

Lemma 4.3 *There exists $\varepsilon > 0$ such that for all $(\tilde{u}, \tilde{v}) \in \overline{S}$ with $\tilde{v} \not\equiv 0$ one has*

$$\limsup_{t\to\infty} \tilde{v}(t, 0) \geq \varepsilon.$$

In order to prove the above lemma, we consider now the hull, denoted by Σ, of the function $\sigma \in \text{BUC}(\mathbb{R}) \times L^\infty(\mathbb{R})^4 \times \text{BUC}(\mathbb{R})$ given by

$$\sigma(t) = \left(d(t), \Lambda(t), \mu(t), \beta(t), \gamma(t), u^*(t)\right), \quad t \in \mathbb{R}.$$

It is defined by

$$\Sigma = \text{cl } \{\sigma(\cdot + h), \quad h \in \mathbb{R}\},$$

wherein cl denotes the closure with respect to the open compact topology for the and the last components and the $L^\infty(\mathbb{R})^4$ weak$-\star$ topology. The set Σ is endowed with this topology. In other words, one has $\tilde{\sigma} = \left(\tilde{d}, \tilde{\Lambda}, \tilde{\mu}, \tilde{\beta}, \tilde{\gamma}, \tilde{u}^*\right) \in \Sigma$ if there exists a sequence $\{h_n\}_{n\geq 0} \subset \mathbb{R}$ such that

$$\left(d(t+h_n), u^*(t+h_n)\right) \to \left(\tilde{d}(t), \tilde{u}^*(t)\right) \text{ in } C^0_{loc}(\mathbb{R}),$$
$$(\Lambda, \mu, \beta, \gamma)(t+t_n) \to \left(\tilde{\Lambda}, \tilde{\mu}, \tilde{\beta}, \tilde{\gamma}\right)(t) \text{ for the } L^\infty(\mathbb{R})^4 \text{ weak-}\star \text{ topology.}$$

Consider also the set

$$\mathcal{S} = \{((u,v)(t+h,\xi;\tau), \sigma(t+h-\tau)),\ (h,\tau) \in \mathbb{R}\times\mathbb{R}\},$$

as well as $\overline{\mathcal{S}}$, the closure of $\mathcal{S}$ with respect to the product topology of $\overline{S} \times \Sigma$. Let us also observe that for each $(\tilde{u}, \tilde{v}) \in \overline{S}$ there exists $\tilde{\sigma} \in \Sigma$ such that $((\tilde{u}, \tilde{v}), \tilde{\sigma}) \in \overline{\mathcal{S}}$. In other words, the projection on the variable (u, v) from $\overline{\mathcal{S}}$ into $\overline{S}$ is onto.

Next, since for each pair of sequences $\{h_n\}$ and $\{\tau_n\}$, the function pair $(u_n, v_n)(t,\xi) := (u, v)(t+h_n, \xi; \tau_n)$ satisfies the system of equations

$$\begin{cases} \partial_t u_n = d(t+h_n-\tau_n)\partial_\xi^2 u_n + c\partial_\xi u_n + F_1(t+h_n-\tau_n, u_n, v_n), \\ \partial_t v_n = \partial_\xi^2 v_n + c\partial_\xi v_n + F_2(t+h_n-\tau_n, u_n, v_n), \end{cases} \quad \text{for } (t,\xi)\in\mathbb{R}^2,$$

it follows that for each $(\tilde{u}, \tilde{v}) \in \overline{S}$ there exists $\tilde{\sigma} \in \Sigma$ such that $((\tilde{u}, \tilde{v}), \tilde{\sigma}) \in \overline{\mathcal{S}}$ while $(\tilde{u}, \tilde{v})$ is a solution of the following system of equations

$$(\mathcal{P}_{\tilde{\sigma}}) \quad \begin{cases} \left[\partial_t - \tilde{d}(t)\partial_\xi^2 - c\partial_\xi\right]\tilde{u} = \tilde{\Lambda}(t) - \tilde{\mu}(t)\tilde{u} - \tilde{\beta}(t)\tilde{u}\tilde{v}, \\ \left[\partial_t - \partial_\xi^2 - c\partial_\xi\right]\tilde{v} = \tilde{v}\left[\tilde{\beta}(t)\tilde{u} - \tilde{\gamma}(t)\right], \end{cases} \quad (t,\xi)\in\mathbb{R}^2.$$

in which we have set $\tilde{\sigma} = (\tilde{d}, \tilde{\Lambda}, \tilde{\mu}, \tilde{\beta}, \tilde{\gamma}, \tilde{u}^*)$. One may in addition observe, due to the strong comparison principle and since $\tilde{v} \geq 0$, that

$$(\tilde{u}, \tilde{v}) \in \overline{S} \quad \text{and} \quad \tilde{v} \not\equiv 0 \;\Leftrightarrow\; \tilde{v} > 0,$$

while

$$(\tilde{u}, \tilde{v}) \in \overline{S} \quad \text{and} \quad \tilde{v} \equiv 0 \;\Leftrightarrow\; (\tilde{u}, \tilde{v}) \equiv \left(\tilde{u}^*(t), 0\right).$$

This last point follows from the fact that (1.4) has a unique bounded entire solution, that is spatially homogeneous. Such a property has been discussed in Sect. 1.

We are now in position to prove Lemma 4.3.

Proof of Lemma 4.3 To prove this lemma, let us argue by contradiction by assuming that there exist a sequence $\{(u_n, v_n, \sigma_n)\} \in \overline{\mathcal{S}}$ with $\sigma_n = \left(d_n, \Lambda_n, \mu_n, \beta_n, \gamma_n, u_n^*\right) \in \Sigma$, $v_n > 0$ and a sequence $\{t_n\}$ such that

$$v_n(t+t_n, 0) \leq \frac{1}{n+1}, \quad \forall n \geq 0,\ \forall t \geq 0. \tag{4.4}$$

Here recall that (u_n, v_n) satisfies Problem $\left(\mathcal{P}_{\sigma_n}\right)$ above. Consider now the sequence $\{(U_n, V_n, \sigma_n(\cdot + t_n))\}$ given by

$$(U_n, V_n)(t, \xi) := (u_n, v_n)(t + t_n, \xi). \tag{4.5}$$

Then (U_n, V_n) becomes a solution of $\left(\mathcal{P}_{\sigma_n(\cdot+t_n)}\right)$. Next, since (4.4) implies that

$$V_n(t, 0) \to 0 \text{ as } n \to \infty \text{ uniformly for } t \geq 0, \tag{4.6}$$

it readily follows that, possibly up to a subsequence,

$$\left(U_n(t, \xi) - u_n^*(t + t_n), V_n(t, \xi)\right)(t, \xi) \to (0, 0) \text{ as } n \to \infty \tag{4.7}$$

uniformly on $[0, \infty) \times [-R, R]$, for any $R > 0$. To see this, assume by contradiction that there exists two sequences $\{\tau_n\} \subset [0, \infty)$ and $\{\xi_n\} \subset \mathbb{R}$ with $\xi_n \to \xi_\infty \in \mathbb{R}$ as $n \to \infty$ such that

$$V_n(\tau_n, \xi_n) \to \alpha > 0 \text{ as } n \to \infty.$$

Then due to parabolic regularity, up to a subsequence, $\{(U_n, V_n)(t + \tau_n, \xi)\}$ converges to $(U_\infty, V_\infty)(t, \xi)$ locally uniformly on $\mathbb{R} \times \mathbb{R}$. Here (U_∞, V_∞) is a solution of $(\mathcal{P}_{\widehat{\sigma}})$, for some $\widehat{\sigma} \in \Sigma$, that satisfies $V_\infty(0, \xi_\infty) = \alpha > 0$ and $V_\infty(t, 0) = 0$ for all $t \geq 0$ (see (4.6)). This contradicts the parabolic comparison principle for the V_∞-equation and ensures that $V_n(t, \xi) \to 0$ as $n \to \infty$, uniformly for $t \geq 0$ and locally uniformly for $\xi \in \mathbb{R}$. The limit (4.7) for the sequence $\{U_n\}$ follows from those of $\{V_n\}$ together with the uniqueness of bounded entire solution for (1.4).

Now we make the following claim.

Claim 4.4 *Let $\theta > 0$ be given. Then there exists $R = R(\theta) > 0$ large enough such that for each function $g = g(t) \in L^\infty(\mathbb{R})$ with*

$$\mathcal{M}^-(g) - \frac{c^2}{4} > \theta,$$

any function $z = z(t, \xi) : [0, \infty) \times [-R, R] \to [0, \infty)$ solving the following problem

$$\begin{cases} z(0, \xi) \in C^0([-R, R]) \text{ with } z(0, \xi) \geq 0, \ z(0, \xi) \not\equiv 0, \\ \partial_t z = \partial_\xi^2 z + c\partial_\xi z + g(t)z, \ t > 0, \ \xi \in (-R, R), \\ z(t, \pm R) = 0 \ \forall t \geq 0, \end{cases}$$

becomes unbounded in infinite time, namely it satisfies

$$\lim_{t \to \infty} \sup_{|\xi| \leq R} z(t, \xi) = \infty.$$

Proof of Claim 4.4 Define the parabolic operator L by

$$L = \partial_t - \partial_\xi^2 - c\partial_\xi - g(t).$$

In order to prove this claim, it is sufficient to construct a function $\underline{z} = \underline{z}(t,\xi)$ such that, for $R > 0$ large enough, one has

$$\begin{aligned} &L\left[\underline{z}\right](t,\xi) \leq 0,\ \forall (t,\xi) \in [0,\infty) \times [-R,R], \\ &\underline{z}(t,\pm R) = 0, \quad \forall t \geq 0 \end{aligned}$$

and such that $z(t,0) \to \infty$ as $t \to \infty$.

To that aim, fix $R > 0$ that will be chosen latter and consider a function ϕ_R such that

$$\begin{aligned} &-\phi_R''(\xi) = \lambda_R \phi_R(\xi), \quad \xi \in (-R,R), \\ &\phi_R(\pm R) = 0 \quad \text{and} \quad \phi_R(\xi) > 0, \quad \forall \xi \in (-R,R). \end{aligned}$$

Now we construct the function $\underline{z} = \underline{z}(t,\xi)$ with the following form

$$\underline{z}(t,\xi) = e^{\lambda t - a(t)} \phi_R(\xi) e^{-\frac{c}{2}\xi}$$

for some parameter $\lambda > 0$ and some function $a = a(t) \in W^{1,\infty}(\mathbb{R})$ to be determined latter. Note that one has

$$L\left[\underline{z}\right](t,\xi) = e^{\lambda t - a(t)} e^{-\frac{c\xi}{2}} \phi_R(\xi) \left[\lambda - a'(t) - g(t) + \lambda_R + \frac{c^2}{4}\right].$$

Hence, we choose $0 < \lambda < \theta$ and, recalling that $\lambda_R \to 0$ as $R \to \infty$, fix $R = R(\theta) > 0$ large enough such that

$$\lambda + \lambda_R < \theta \leq \mathcal{M}^-(g) - \frac{c^2}{4}.$$

Next, one gets

$$\mathcal{M}^-\left(g - \frac{c^2}{4} - (\lambda + \lambda_R)\right) = \mathcal{M}^-(g) - \frac{c^2}{4} - (\lambda + \lambda_R) \geq \theta - (\lambda + \lambda_R).$$

Thus, due to (2.2), there exists $a \in W^{1,\infty}(\mathbb{R})$ such that

$$a'(t) + g(t) - \frac{c^2}{4} > \lambda + \lambda_R \quad \text{for all } t \in \mathbb{R},$$

and the function $\underline{z}$ satisfies all suitable properties. This completes the proof of the claim. □

Now fix $\varepsilon > 0$ small enough such that

$$\mathcal{T} - \varepsilon\|\beta\|_{L^\infty(\mathbb{R})} - \frac{c^2}{4} > \varepsilon > 0$$

and fix $R = R(\varepsilon) > 0$, the constant provided by Claim 4.4 with $\theta = \varepsilon$. Next, due to (4.7), one has for all n large enough that

$$U_n(t,\xi) \geq u_n^*(t_n + t) - \varepsilon, \quad \forall t \geq 0,\ \xi \in [-R, R].$$

Fix n large enough as above, so that $V_n > 0$ satisfies

$$\partial_t V_n \geq \partial_\xi^2 V_n + c\partial_\xi V_n + g_n(t)V_n(t,\xi), \quad t \geq 0,\ \xi \in [-R, R],$$

with the function $g_n = g_n(t)$ defined by $g_n(t) := \beta_n(t+t_n)u_n^*(t+t_n) - \gamma_n(t+t_n) - \varepsilon\beta_n(t+t_n)$. Observe that

$$\mathcal{M}^-(g_n) \geq \mathcal{T} - \varepsilon\|\beta\|_{L^\infty(\mathbb{R})},$$

so that Claim 4.4 together with the parabolic comparison principle ensures that

$$\lim_{t\to\infty} \sup_{|\xi|\leq R} V_n(t,\xi) = \infty,$$

which contradicts the boundedness of the function V. This completes the proof of Lemma 4.3. ◻

Now to complete the proof of Proposition 4.1, we turn to our second step which corresponds to the following lemma.

Lemma 4.5 *The function $v = v(t,\xi;\tau)$ defined in (4.3) satisfies*

$$\inf_{\substack{t\geq 1\\ \tau\in\mathbb{R}}} v\,(t, 0; \tau) > 0.$$

Remark 4.6 Note that Proposition 4.1 follows directly from the above lemma by recalling the definition of the function v in (4.3).

Proof Assume by contradiction that there exists two sequences $\{l_n\}_{n\geq 0} \subset \mathbb{R}$ and $\{\tau_n\}_{n\geq 0} \subset \mathbb{R}$ such that $l_n \geq 1$ for all $n \geq 0$ and

$$v(l_n, 0; \tau_n) \leq \frac{1}{n+1}, \quad \forall n \geq 0.$$

Recalling the definition of the function v in (4.3) and noticing (see (4.2)) that

$$\inf_{\tau\in\mathbb{R}} v(0,0;\tau) = \inf_{\tau\in\mathbb{R}} V(\tau, 0) := \sigma > 0,$$

for each $n \geq 0$ there exists $s_n \in [0, l_n)$ such that for all n large enough one has

$$\begin{cases} v(s_n, 0; \tau_n) = \sigma_0 := \frac{1}{2} \min(\sigma, \varepsilon), \\ v(t + s_n, 0; \tau_n) \leq \sigma_0 \text{ for } t \in [0, l_n - s_n], \\ v(l_n, 0; \tau_n) \leq \frac{1}{n+1}. \end{cases}$$

In the condition above, namely in the definition of σ_0, $\varepsilon > 0$ corresponds to the constant provided by Lemma 4.3.

First, it is easy to check that $\varrho_n := l_n - s_n \to \infty$ as $n \to \infty$. To see this, consider the sequence of functions $\{(u_n, v_n)(t, \xi)\} := \{(u, v)(s_n + t, \xi; \tau_n)\}$ and recall that it is a bounded solution of Problem $\left(\mathcal{P}_{\theta_n}\right)$ for a suitable sequence $\{\theta_n\} \in \Sigma$. It further satisfies

$$v_n(0, 0) = \sigma_0 \text{ and } v_n(l_n - s_n, 0) \leq \frac{1}{n+1} \text{ for any } n \text{large enough.}$$

Hence, because of parabolic regularity, one may assume that $(u_n, v_n) \to (u_\infty, v_\infty)$ as $n \to \infty$ locally uniformly for $(t, \xi) \in \mathbb{R} \times \mathbb{R}$. And the function pair (u_∞, v_∞) is a bounded solution of $\left(\mathcal{P}_{\theta_\infty}\right)$ for a suitable $\theta_\infty \in \Sigma$ while the function v_∞ satisfies $v_\infty(0, 0) = \sigma_0 > 0$. Hence, if the sequence $\{l_n - s_n\}$ were bounded then the function v_∞ furthermore satisfies $v_\infty(\varrho, 0) = 0$, where $\varrho \geq 0$ denotes a limit point of the bounded sequence $\{\varrho_n\} = \{l_n - s_n\}$. This contradicts the maximum principle and ensures that $\varrho_n \to \infty$.

Equipped with this remark, consider once again the sequence of functions $\{(u_n, v_n)\}$ defined above. Then possibly along a sub-sequence, one has

$$(u_n, v_n)(t, \xi) \to (\tilde{u}, \tilde{v})(t, \xi) \quad \text{in } C^0_{\text{loc}}(\mathbb{R})^2,$$

where $(\tilde{u}, \tilde{v})$ is a bounded solution of $\left(\mathcal{P}_{\tilde{\theta}}\right)$ for some $\tilde{\theta} \in \Sigma$ with $\tilde{v}(0, 0) = \sigma_0 > 0$. And, since $\varrho_n \to \infty$, the function $\tilde{v}$ satisfies

$$\tilde{v}(t, 0) \leq \sigma_0 < \varepsilon \quad \text{for } t \in [0, \infty).$$

Thus $(\tilde{u}, \tilde{v}) \in \overline{S}$ and $\tilde{v} \not\equiv 0$ and Lemma 4.3 applies and implies that

$$\limsup_{t \to \infty} \tilde{v}(t, 0) \geq \varepsilon,$$

which contradicts $\tilde{v}(t, 0) \leq \sigma_0 < \varepsilon$ for all $t \geq 0$. This completes the proof of Lemma 4.5 and also that of Proposition 4.1. □

Equipped with Proposition 4.1 we are now able to complete the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1 Fix $a \in W^{1,\infty}(\mathbb{R})$ and let $\lambda \in (0, \sqrt{\mathcal{T}})$ be given. Set $c(t) = \lambda + \lambda^{-1}\delta(t) + a'(t)$. Let (U, V) be the solution of (1.8) provided by Theorem 3.1 associated to the wave speed function $c = c(t)$. To complete the proof of Theorem 2.1

it remains to verify that the function pair (U, V) satisfies (1.6). This latter property follows from Proposition 4.1.

To see this, note that applying Proposition 4.1, by choosing $\tau = t - s$, yields the existence of $\varepsilon > 0$ and $T > 0$ large enough such that

$$V\,(s, -\gamma(t,s)t) \geq \varepsilon, \quad \forall s \in \mathbb{R},\ t \geq T,$$

wherein we have set

$$\gamma(t,s) = \frac{1}{t}\int_0^t c(l - t + s)dl, \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R}.$$

Next note that

$$\gamma(t,s) \geq \inf_{s\in\mathbb{R}} \frac{1}{t}\int_0^t c(l + s)dl,$$

so that we obtain

$$\liminf_{t\to\infty} \inf_{s\in\mathbb{R}} \gamma(t,s) \geq \mathcal{M}^-(c) \geq 2\sqrt{\mathcal{T}} > 0.$$

This ensures that there exists $\varepsilon > 0$ and $X > 0$ large enough such that

$$V(s,\xi) \geq \varepsilon, \quad \forall s \in \mathbb{R},\ \forall \xi \leq -X,$$

and this completes the proof of (1.6) for the V-component.

To complete the proof of this theorem it remains to check that (1.6) for the U-component also holds. This follows directly from the parabolic maximum principle due to the persistence property of V and the upper estimate $U(t,\xi) \leq u^*(t)$ for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}$. The proof of Theorem 2.1 is completed. □

We shall now complete the proof of Theorem 2.2.

Proof of Theorem 2.2 In order to prove this result, we argue by contradiction by assuming that (1.1)–(1.2) possesess a generalized traveling wave according to Definition 1.1, denoted by (U, V, c), and such that

$$\mathcal{M}^-(c) < 2\sqrt{\mathcal{T}}. \tag{4.8}$$

Now fix c in between, namely

$$\mathcal{M}^-(c) < c < 2\sqrt{\mathcal{T}}, \tag{4.9}$$

and let $\gamma = \gamma(t,s)$ the function be defined by

$$\gamma(t,s) = \frac{1}{t}\int_0^t c(l - t + s)dl - c, \quad \forall (t,s) \in \mathbb{R} \times \mathbb{R}.$$

Next due to (4.9) there exist two sequences $\{t_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ with $t_n \to \infty$ and such that

$$\limsup_{n\to\infty} \gamma(t_n, s_n) < 0.$$

Hence, up to a subsequence, still denoted by n, one has

$$\lim_{n\to\infty} -\gamma(t_n, s_n)t_n = \infty.$$

On the other hand, Proposition 4.1 implies that

$$\liminf_{n\to\infty} V(s_n, -\gamma(t_n, s_n)t_n) > 0.$$

The above two properties contradict the definition of a generalized traveling wave as stated in Definition 1.1 and in particular the property of the V-component arising in (1.6). This completes the proof of Theorem 2.2. □

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