HODGE THEORY AND MODULI OF H-SURFACES

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Note: The drafts of I, 0.A, 0.B, III.C and the first part of III.D are included here.

0. Introduction

0.A. **General introduction.** This work brings two of the major areas in algebraic geometry, namely Hodge theory and moduli, to bear on the study of a particular very beautiful algebraic surface. As will now be explained, underlying the choice to focus on a particular surface is that as an example it provides a means to experimentally explore the general relationship between moduli and Hodge theory in a first non-classical case. The surface we will study is an H-*surface*, which is by definition a smooth minimal algebraic surface X of general type satisfying

$$\begin{cases} K_X^2 = 2 \\ p_g(X) = 2,\ q(X) = 0. \end{cases}$$

The "H" stands for Hoikawa, who analyzed them in [Ho79]. For us among the salient aspects of this surface are

- X is of general type, so its KSBA moduli space [Ko13] $\mathcal{M}_H$ is defined and is a projective variety.

It follows from [Ho79] that $\mathcal{M}_H$ is reduced, irreducible and of dimension equal to 26.

- The Hodge numbers are $h^{2,0}(X) = 2$, $h^{1,0}(X) = 0$, and consequently the corresponding period domain D is non-classical; i.e., it is not a Hermitian symmetric domain.

This latter property implies that the period mapping

$$\Phi : \mathcal{M}_H \to \Gamma\backslash D$$

satisfies a differential constraint, the infinitesimal period relation (IPR). In this case $\dim D = 55$ and the IPR is a contact distribution, so the maximal local integrals have dimension 27.

- The numbers K_X^2 and $h^{p,q}(X)$ are small and are close to extremal in terms of Noether's inequality. Thus the geometry of X is particularly rich, and one may explore the relationship between moduli and Hodge theory without the technical complications of a more general case.

Although for general KSBA moduli spaces $\mathcal{M}^{\rm KSBA}$ the local singularity structure of degenerate surfaces X_0 corresponding to boundary points in $\partial\mathcal{M}^{\rm KSBA} = \overline{\mathcal{M}}^{\rm KSBA}\backslash\mathcal{M}^{\rm KSBA}$ is understood, to our knowledge there is no example where the global structure of the components of $\partial\mathcal{M}^{\rm KSBA}$ and structure of a general X_0 corresponding to a particular boundary component has been worked out.

In the cases of curves, abelian varieties and polarized K3's, Hodge theory serves as a guide to suggest the global structure of the singular varieties corresponding to boundary points in the moduli space $\mathcal{M}$. In these cases the period domain is classical and one may use the Satake-Bailey-Borel (SBB) compactification $\Gamma\backslash D^*$ and extension of the period map to

$$\Phi_e : \overline{\mathcal{M}} \to \Gamma\backslash D^*$$

together with known, Lie-theoretic structure of $\partial(\Gamma\backslash D)$ to infer properties of $\partial\mathcal{M}$.

For general period domains D, from the works of Schmid [Sc72], Cattani-Kaplan-Schmid [CKS86], Kato-Usui [KU08] and others, one has an understanding of which Hodge-theoretic objects may be used to provide toroidal-type extensions of $\Gamma\backslash D$'s. More recently, using the still developing theory of naïve or reduced limit period mappings, one is gaining a picture of how the orbit $G_{\mathbb{R}}$-structure of $D \subset \check{D}$ may lead to a SBB-type completion $\Gamma\backslash D^*$ of $\Gamma\backslash D$'s. A guiding principle for this work is

- the analogue of the SBB compactification may be expected to be of significant use in analyzing the algebro-geometric aspects of boundaries of moduli spaces.

Here we shall carry out this program for weight $n = 2$ period domains and H-surfaces. One main new point is that the IPR forces several phenomena in the boundary component structure of $\Gamma\backslash D^*$ that are not present in the classical HSD case. These include (terms to be explained below)

— non-linearity in the partial ordering of the boundary components;

— non-convexity.

One of the main results in this paper, stated informally here, is

THEOREM A: *Let $\mathcal{M}$ be a KSBA-moduli space for algebraic surfaces of general type. For the corresponding period domain D and global monodromy group Γ, the completion $\Gamma\backslash D^*$ may be constructed and the period mapping $\Phi : \mathcal{M} \to \Gamma\backslash D$ extends to*

$$\Phi_e : \overline{\mathcal{M}} \to \Gamma\backslash D^*.$$

The image $\Phi_e(\overline{\mathcal{M}})$ is a projective algebraic variety.

The Hodge-theoretic description of the boundary components is somewhat subtle: Roughly speaking, in the classical case of the SBB compactification, set-theoretically the boundary components consist of the associated graded's to the polarized limiting mixed Hodge structures (PLMHS's) that arise when a family of polarized Hodge structures degenerates. [However, in our non-classical case the boundary components have the information of the associated graded plus some extension data, which is however constant on the images under Φ_e of the boundary components of $\overline{\mathcal{M}}$. **This will have to be modified pending what will emerge from Colleen's work.**]

For H-surfaces the boundary component structure is described by

$$0 \longrightarrow \mathrm{I} \begin{array}{c} \nearrow \mathrm{II} \searrow \\ \searrow \mathrm{III} \nearrow \end{array} \mathrm{IV} \longrightarrow \mathrm{V},$$

where the Roman numerals depict the boundary components and the arrows signify "contained in the closure of "("0" corresponds to D itself). We will let R run through the index set $\{0, \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{IV}\}$ and will denote the corresponding boundary component of $\Gamma\backslash D^*$ by $(\Gamma\backslash D)_R$. Our second main result, again informally stated, is

Theorem B: *For each R there is a KSBA boundary component $\partial\mathcal{M}_{H,R}$ and a non-constant map*

$$\Phi_e : \partial\mathcal{M}_{H,R} \to \Gamma\backslash D_R.$$

Moreover, the incidence relations among the $\Gamma\backslash D_R$ are realized by incidence relations among the $\partial\mathcal{M}_{H,R}$ and the maps Φ_e.

Intuitively, all of the possible Hodge-theoretic degenerations of the Hodge structure of an H-surface are realized algebro-geometrically. In this way, Hodge theory serves as a guide to understand the global geometry of degenerate H-surfaces that appear in the boundary of the KSBA moduli space $\mathcal{M}_H$. In particular, the non-classical properties of $\partial(\Gamma\backslash D)$ will then have implication for the boundary structure of $\partial\mathcal{M}$.

We shall actually show that there are two types of boundary components of $\mathcal{M}_H$ that map to each boundary component of $\Gamma\backslash D^*$. One are boundary components where the surfaces have as singularities double curves with pinch points on degenerations of such. The other is where the surfaces have isolated singularities.

0.B. **Introduction to H-surfaces.** An *H-surface* is a smooth minimal algebraic surface of general type that satisfies

$$K_X^2 = 2$$
$$p_g(X) = 2,\ q(X) = 0.$$

Using standard results from the theory of algebraic surfaces [BPVdV84], the first condition is equivalent to giving the Hilbert polynomial $\sum_{m\geqq 0}\chi(mK_X)$. The second of the above conditions are Hodge theoretic.

Among our reasons for choosing to focus on H-surfaces are

- H-surfaces are close to extremal in terms of Noether's inequality (loc. cit.)
$$p_g(X) \leqq \frac{K_X^2}{2} + 2;$$
- their polarized Hodge structure is the simplest of non-classical type;

- their structure was analyzed by Horikawa [Ho79], whence the name we have given them.

Being of general type, H-surfaces have a KSBA moduli space $\mathcal{M}_H$ [Ko13], and from [Ho79] one sees that this moduli space fails to exhibit the pathologies that frequently occur for algebraic surfaces [Va]. And because the numbers $h^{2,0}(X)$ and K_X^2 are relatively small, the structure of the Hodge-theoretic boundary components of $\Gamma\backslash D$ is relatively simple and the potential for the use of GIT methods is favorable.

Following some algebro-geometric and Hodge-theoretic preliminaries in §I.A, in §I.B we shall reprove somewhat extended versions of the results we need from [Ho79]. In particular, we determine the equations that define the bi-canonical model $\varphi_{2K_X}(X) \subset \mathbb{P}^4$ of a general H-surface, as well as we also determine its pluri-canonical ring $\sum_{m\geqq 0} H^0(mK_X)$.

The equations that define the bi-canonical model suggest an alternate birational realization of X as a hypersurface $X^\flat \in |\mathcal{O}_{\mathbb{P}E}(4)|$ in $\mathbb{P}E$ where $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. Denoting by $\mathbb{P}^2_t$ the fibre of $\mathbb{P}E \to \mathbb{P}^1$ over $t \in \mathbb{P}^*$, for $t \neq 0$ the fibres $C_t = X^\flat \cap \mathbb{P}^2_t$ are the curves in the canonical pencil $|K_X|$; they have $p_a(C_t) = 3$. Moreover, the hypersurface $X^\flat$ is smooth outside of the intersection $X^\flat \cap P^2_0$, which is a double conic D_0. This double conic has eight pinch points, and for a general H-surface the corresponding branched cover $C_0 \to D_0$ is the *unique* hyperelliptic curve in the canonical pencil $|K_X|$. The presence of C_0 is perhaps the distinguishing feature of the geometry of an H-surface.

A very pleasing feature of H-surfaces is that one may see them quite explicitly. In fact, a number of the results about H-surfaces are in part established by geometric arguments based on the pictures given in §I.D, of both H- and $H^{\#}$-surfaces. An illustration of this is the following Torelli-type result: As is the case for any pencil of curves on a smooth surface, the canonical pencil $|K_X|$ defines a variation of polarized Hodge structure (VHS) $(V, \mathcal{F})$ over $\mathbb{P}^1$ (the notations are explained in §I.D).

Theorem: $(V, \mathcal{F})$ *determines the H-surface X.*

The realization of X as $X^\flat \subset \mathbb{P}E$ suggests the consideration of smooth members $X^\# \in |\mathcal{O}_{\mathbb{P}E}(4)|$ of the complete linear system $\mathbb{P}H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(4))$. Then $X^\# \to X^\flat$ is a KSBA degeneration. Although the $H^\#$-surfaces $X^\#$ have a less interesting intrinsic geometry than do H-surfaces, being smooth hypersurfaces implies that various cohomological calculations are simpler than those for H-surfaces and help to suggest the way that the latter should go. $H^\#$-surfaces and their properties are discussed §I.E.

In §I.F in terms of the equation of $X^\flat \subset \mathbb{P}E$ we shall compute the tangent space to moduli of a general H-surface as

$$T_X\mathcal{M}_H \cong \frac{H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J})}{\operatorname{aut}(\mathbb{P}E)}.$$

Here we have set $\xi = \mathcal{O}_{\mathbb{P}E}(1)$, and $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}E}$ is an ideal that contains the information of both the double conic D_0 and the pinch points. This result exhibits clearly the singular role that the unique hyperelliptic curve $C_0 \in |K_X|$ plays in the geometry of H-surfaces.

In §I.G we shall establish the generic local Torelli theorem for both $H^\#$- and H-surfaces. The method is to express in cohomological terms the differential

$$\Phi_* : T_X\mathcal{M}_H \to T_{\Phi(X)}D$$

of the period mapping and carry out a cohomological computation to verify the injectivity of Φ_*. Although this is done classically in generality for varieties that are sections of sufficiently ample line bundles in an ambient variety, the situation here is quite a bit more subtle and we are only able to verify the injectivity of Φ_* for particular “Fermat-like” $H^\#$- and H-surfaces. We note that for H-surfaces this differential is the natural map

$$\frac{H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J})}{\operatorname{aut}(\mathbb{P}E)} \otimes H^0(\mathbb{P}E, \xi \otimes \mathfrak{h}^{-1}) \to \frac{H^0(\mathbb{P}E, \xi^5 \otimes \mathfrak{h}^{-1} \otimes \mathcal{J})}{\operatorname{aut}(\mathbb{P}E)}.$$

Here, $\xi \otimes \mathfrak{h}^{-1}$ restricts to the canonical bundle on $X^\flat$, the $\mathfrak{h}^{-1}$ reflecting the adjunction conditions imposed by a double curve on the canonical series. The term on the RHS is a subspace of $H^1(\Omega^1_X)_{\text{prim}}$, and one notes the interesting point that whereas the pinch points do not impose conditions on $H^0(\Omega^2_X)$, they do impose them on $H^1(\Omega^1_X)$.

In §I.G we show that the global monodromy representation

$$\pi_1(\mathcal{M}_H) \to \mathrm{Aut}(H^2(X,\mathbb{Z})_{\mathrm{prim}})$$

has as image an arithmetic group Γ. The method here is to first use Lefschetz-style geometric arguments ([Le24]) to produce generators for Γ, and to show that the cycles giving the Picard-Lefschetz transformation for the generators of Γ span $H_2(X,\mathbb{Q})_{\mathrm{prim}}$. Here the issues are (i) the line bundle $\xi \to \mathbb{P}E$ is not very ample, so that the dual variety $\check{Q}_0$ of the image Q_0 of $\mathbb{P}E$ under $|\xi^4|$ will have singularities in codimension 1, and (ii) $X^\flat$ is not a general member of $|\xi^4|$, but rather has singularities arising from the unique hyperelliptic curve $C_0 \in |K_X|$. Once again, it is this C_0 that plays a central role in the geometry of an H-surface. In any case, by using the pictures of an H-surface from §I.D we are able to analyze the effect of both issues (i) and (ii) which, when coupled with purely group-theoretic considerations of the type given in [Be84], allows us to give an argument, if not a modern proof, of the arithmeticity of Γ.

To conclude this part of the introduction we note a little interesting numerology:

- $\dim \mathcal{M}_H = 26$;
- the period mapping is $\Phi : \mathcal{M}_H \to \Gamma\backslash D$ where $\dim = 55$;
- the image $\Phi(\mathcal{M}_H)$ is an integral variety of dimension 26 (by generic local Torelli) of the infinitesimal period relation (IPR);
- in the situation when $h^{2,0} = 2$ the IPR is a constant distribution whose maximal integrals have dimension 27.

Thus the image $\Phi(\mathcal{M}_H)$ is of codimension 1 in a maximal integral of the IPR, and history suggests one may expect some as yet undiscovered geometry as a result.

I. H-SURFACES

I.A. **Algebro-geometric and Hodge-theoretic preliminaries.** We shall generally follow the terminology and notation in [BPVdV84] for the theory of algebraic surfaces. Our algebraic surfaces X will either be smooth or will have singularities of KSBA type [Ko13]. In case X is smooth and $C \subset X$ is a reduced curve, we denote by ω_C the dualizing sheaf. Locally, if C is given by an equation $f(x,y)=0$, then the sections of ω_C are Poincaré residues

$$\operatorname{Res}\left(\frac{g(x,y)dx\wedge dy}{f(x,y)}\right)=\frac{g(x,y)dx}{f_y(x,y)}.$$

We also use the standard notation $p_a(C)$ for the arithmetic genus of a curve on a surface.

The surfaces we consider will either be smooth, have hypersurface singularities or will be quotients of the latter by a finite group. If the hypersurface is locally $f(x,y,z)=0$ where all factors of f have multiplicity one, then *adjunction conditions* will mean those conditions on $g(x,y,z)$ that the the Poincaré residue

$$\operatorname{Res}\left(\frac{g(x,y,z)dx\wedge dy\wedge dz}{f(x,y,z)}\right)=\left.\frac{g(x,y,z)dx\wedge dy}{f_z(x,y,z)}\right|_{f=0}$$

pull back to a holomorphic form on one, and hence on any, resolution of singularities of $f(x,y,z)=0$. For an H-surface X with birational model $X^\flat$ as above, these adjunction conditions are given by the vanishing of $g(x,y,z)$ on D_0.

The vanishing theorems we shall use will also be the standard ones from [BPVdV84]. Specifically, if X is a surface that is minimal and of general type, then

$$h^q(mK_X)=0 \text{ for } q>0, m\geqq 2.$$

We shall use the terminology and notations from the classical theory of linear systems. Thus if $L\to Y$ is a holomorphic line bundle over an irreducible variety Y, then

$$|L|=\mathbb{P}H^0(Y,L)$$

will denote the projective space of divisors of non-zero sections of L. We will think of complete linear systems $|L|$ as defining a rational mapping

$$|L| : Y \dashrightarrow \mathbb{P}H^0(Y, L)^*.$$

The notations

$$mL = L^m = L^{\otimes m}$$

will be used interchangeably. For line bundles L, L' we will set $|mL + m'L'| = |L^m \otimes L'^{m'}|$.

We shall identify the group Pic of line bundles with the group divisors modulo linear equivalence. In case Y is regular ($h^{1,0}(Y) = 0$) via the Lefschetz (1,1) theorem we shall also identify $\operatorname{Pic} Y$ with the group of Hodge classes $\operatorname{Hg}^1(Y, \mathbb{Z}) = H^2(Y, \mathbb{Z}) \cap H^{1,1}(Y)$.

Among the specific algebro-geometric notations we shall use are

- $X = H$-surface (§I.B);
- $|mK_X| = \mathbb{P}H^0(K_X^m)$ are the pluri-canonical systems (§I.B);
- $\varphi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(mK_X)^*$ are the pluri-canonical maps;
- $\mathcal{M}_H$ = KSBA moduli space for H-surfaces [Ko13];
- $\overline{\mathcal{M}}_H$ = canonical compactification of $\mathcal{M}_H$ and $\partial\mathcal{M}_H = \overline{\mathcal{M}}_H \backslash \mathcal{M}_H$;
- $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and $\mathbb{P}E \xrightarrow{\pi} \mathbb{P}^1$ is the corresponding projective bundle with fibres $(\mathbb{P}E)_t = \mathbb{P}E_t^*$ over $t \in \mathbb{P}^1$ (§I.C);
- $\xi = \mathcal{O}_{\mathbb{P}E}(1)$ and $\mathfrak{h} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ (§I.C);
- $f : \mathbb{P}E \to \mathbb{P}^4$ is the map given by $|\xi| = \mathbb{P}H^0(\mathbb{P}E, \xi) \cong \mathbb{P}H^0(\mathbb{P}^1, E)$ (§I.C);
- $Q_0 = \{x_0x_2 = x_1^2\} \subset \mathbb{P}^4$ is the image $f(\mathbb{P}E)$ (§I.C);
- $\mathbb{P}(1, 1, 2, 2)$ is the weighted projective space (§I.C);
- $X^\# \in |\xi^4|$ is an $H^\#$-surfaces (§I.E);
- $\hat{X}$ = blow up of X at the base points of $|K_X|$;
- $g : \hat{X} \to \mathbb{P}E$ is the canonical map with image $X^\flat = g(\hat{X}) \in |\xi^4|$ (§I.C).

We shall now briefly review some of the definitions and notations from Hodge theory that will be used.

Definition: A *polarized Hodge structure* (PHS) (V, Q, F) of weight n is given by

- a $\mathbb{Q}$-vector space V and a non-degenerate bilinear form
$$Q : V \otimes V \to Q, \qquad Q(u,v) = (-1)^n Q(v,u);$$
- a decreasing *Hodge filtration* $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$ that satisfies
$$F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 1 \leqq p \leqq n;$$
and where the two *Hodge-Riemann bilinear relations* HRI and HRII given below are satisfied.

Given a PHS, if we set
$$V^{p,q} = F^p \cap \overline{F}^q$$
then we have the usual Hodge decomposition
$$V_{\mathbb{C}} = \underset{p+q=m}{\oplus} V^{p,q}, \qquad V^{q,p} = \overline{V^{p,q}}.$$
Conversely, given such a Hodge decomposition we may define a Hodge filtration by
$$F^p = \underset{p' \geqq p}{\oplus} V^{p',n-p'}.$$
The *Hodge numbers* are defined by $h^{p,q} = \dim V^{p,q}$, and we set $f^p = \sum_{p' \geqq p} h^{p',n-p'}$. The *Weil operator*
$$C : V_{\mathbb{R}} \to V_{\mathbb{R}}$$
is defined by
$$C(v) = i^{p-q} v, \quad v \in V^{p,q}.$$
In terms of it, HRI and HRII are given by

(HRI) $Q(F^p, F^{n+p+1}) = 0$

(HRII) $Q(v, C\overline{v}) > 0$ for $v \neq 0$.

We remark that in all the geometric situations that will arise in this work there will be a lattice $V_{\mathbb{Z}} \subset V$; the bilinear form $Q\big|_{V_{\mathbb{Z}}}$ will in general be $\mathbb{Q}$, but not $\mathbb{Z}$, valued.

Example: Let C be a smooth, connected algebraic curve. Then taking

- $V_{\mathbb{Z}} = H^1(C, \mathbb{Z})$;
- $Q =$ cup-product;
- $F^1 = H^0(\Omega^1_C)$

gives a PHS of weight $n = 1$.

Example: Let X be a smooth algebraic surface of general type. Then taking

- $V = \{v \in H^2(X, \mathbb{Q}) : Q(v, c_1(K_X)) = 0\}$;
- $Q =$ cup-product restricted to V;
- $F^2 = H^0(\Omega^2_X)$

gives a PHS of weight $n = 2$.

We note that for weight $n = 2$, F^2 determines F^1 by $F^1 = F^{2\perp}$.

In practice, we will sometimes describe HS's by classical period matrices and Hodge norms by

$$\|\omega\|^2 = \left(\frac{i}{2}\right)^{\frac{n(n-1)}{2}} \int_Y \omega \wedge \overline{\omega}, \quad \omega \in H^0(\Omega^n_Y).$$

This will be especially the case when we discuss the curvature of the Hodge bundles along boundary components in moduli.

When only $\mathcal{F}(V, F)$ is given satisfying the second property $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$, $V^{q,p} = \overline{V^{p,q}}$ in the definition of a PHS, we speak of a *pure Hodge structure of weight* n.

Definition: (i) A *period domain* D is the set of PHS's (V, Q, F) with given Hodge numbers $h^{p,q}$. (ii) The compact dual $\check{D}$ is the set of filtrations F with given $f^p = \dim F^p$ and satisfying HRI.

Setting $G = \mathrm{Aut}(V, Q)$, it is known (see §II below) that

$$\begin{array}{ll} \check{D} = G_{\mathbb{C}}/P & \text{where } P \subset G_{\mathbb{C}} \text{ is a parabolic subgroup} \\ \cup & \\ D = G_{\mathbb{R}}/H & \text{, where } H = P \cap G_{\mathbb{R}} \text{ and } D \text{ is an open } G_{\mathbb{R}}\text{-orbit in } \check{D}. \end{array}$$

Examples: For weights $n = 1, 2$ we have

- $n = 1$, $D = \mathrm{Sp}(2g, \mathbb{R})/\mathcal{U}(g) \cong \mathcal{H}_g$ (Siegel's generalized upper-half-plane) where $h^{1,0} = g$;
- $n = 2$, $D = \mathrm{SO}(2a, b)/\mathcal{U}(a) \times \mathrm{SO}(b)$ where $h^{2,0} = a$, $h^{1,1} = b$.

We note that in the $n=1$ case D is classical, and that in the $n=2$ case D is classical if, and only if, $a=1$. The main interest in this work is the first non-classical case

$$n=2,\ h^{2,0}=2.$$

Definition: We define the *infinitesimal period relation* (IPR) to be the $G_{\mathbb{C}}$-invariant distribution $I \subset T\check{D}$ given by

$$I=\{\dot{F},\dot{F}^p \subseteq F^{p-1}\}.$$

Here we are thinking of a tangent vector $\dot{F}$ to a curve $\{F^p_t\}$ in $\check{D}$ to be what one gets by

$$\dot{F}^p=\left\{\left.\frac{dF^p_t}{dt}\right|_{t=0} \mod F^p_0\right\}.$$

Example: When $n=2$ and $h^{2,0}=2$, $\dim D=2h^{1,1}+1$ and I is a contact distribution whose maximal local integral manifolds have dimension $h^{1,1}$.

In the following definition we will have a complex manifold S and a monodromy representation

$$\rho:\pi_1(S)\to G_{\mathbb{Z}}$$

with image Γ.

Definition: A *variation of Hodge structure* (VHS) $(V,\mathcal{F})$ is given by a locally liftable, holomorphic mapping

$$\Phi: S\to \Gamma\backslash D$$

that satisfies the IPR

$$\Phi_*: T_sS\to I_{\Phi(s)}.$$

The notation suggests that we may think of a VHS as equivalently given by a local system $\mathbb{V}\to S$ together with a filtration $\mathcal{F}$ on $\mathbb{V}\otimes_{\mathbb{Q}}\mathcal{O}_S$ which at each point gives a PHS that satisfies

$$dF^p_s\subseteq F^{p-1}_s.$$

We observe that the usual notation is to give a flat connection

$$\nabla:\mathcal{F}\to\mathcal{F}\otimes\Omega^1_S$$

that preserves Q and that satisfies

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_S.$$

Always assumed to exist is the form

$$Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$$

that induces the polarization at each fibre.

Example: Let Y be a smooth algebraic surface with a fibering $Y \xrightarrow{\pi} \mathbb{P}^1$ by curves $C_t = \pi^{-1}(t)$. Then all but finitely many C_{t_i} are smooth, and for $S = \mathbb{P}^1 \backslash \{t_i\}$ we obtain a VHS whose general fibre is the PHS on $H^1(C_t)$.

For us the most important case will be when the C_{t_i} are nodal. Then a canonical extension of $\mathcal{F}$ to all of $\mathbb{P}^1$ is given by the direct image $\pi_* \omega_{Y/\mathbb{P}^1}$ of the relative dualizing sheaf [BPVdV84].

Given a VHS there are defined the Hodge filtration bundles $\mathcal{F}^p \to S$ and *Hodge bundles*

$$\mathcal{V}^{p,q} = \mathcal{F}^p / \mathcal{F}^{p+1}.$$

Of particular importance is $\mathcal{F}^n$, and what is generally referred to as *the* Hodge bundle

$$\lambda =: \det \mathcal{F}^n.$$

By abuse of notation we shall not distinguish between the Hodge bundle and its degree when the latter is defined.

The polarizing forms induce metrics in the Hodge bundles, and the associated curvatures have special properties [CM-SP]. For us the most important is the positivity of the Hodge bundle, as expressed by

$$\Omega_\lambda(v) = \|\Phi_*(v)\|^2, \quad v \in T_s S.$$

In §IV.C we will give a general discussion of the curvature properties of $\mathcal{F}^n$, which have the positivity of the Hodge bundle as a particular consequence.

Definition: A *mixed Hodge structure* (MHS) (V, W, F) is given by

- a $\mathbb{Q}$ vector space;

- an increasing *weight filtration* $W_0 \subset \cdots \subset W_m = V$ defined over $\mathbb{Q}$; and
- a decreasing *Hodge filtration* $F_n \subset F_n \subset \cdots \subset F_0 = V_{\mathbb{C}}$

such that the induced filtration

$$F^p(\mathrm{Gr}^W_m) = \frac{F^p \cap W_m + W_{m-1}}{W_{m-1}}$$

defines a pure Hodge structure of weight m on the associated graded's.

A mixed Hodge structure is a special kind of successive extensions of pure Hodge structures. These extensions are generally not split. However, over $\mathbb{R}$ there is a canonical $I^{p,q}$-decomposition of $V_{\mathbb{C}}$ that is as close as possible to giving an $\mathbb{R}$-splitting. It is characterized by

- $V_{\mathbb{C}} = \oplus I^{p,q}$;
- $W_m = \underset{p+q \leqq m}{\oplus} I^{p,q}$;
- $F^p = \underset{p' \geqq p}{\oplus} I^{p',q}$;
- $\overline{I}^{p,q} \equiv I^{q,p} \mod W_{p+q-2}$.

In this work we shall be especially interested in the $I^{p,0}$ terms associated to the PLMHS (defined below) arising from a KSBA degeneration of a smooth surface of general type. In examples these spaces will generally be fairly easily describable by residues, as is of course the case for algebraic curves and where again in the surface case the involution on double curves with pinch points will play a central role.

Example: If X_0 is a possibly singular algebraic surface, then $H^2(X_0, \mathbb{Q})$ has a MHS where the weights are $W_0 \subset W_1 \subset W_2$.

For the next definition we recall that a nilpotent endomorphism $N = \mathrm{End}(V)$, $N^{m+1} = 0$ but $N^m \neq 0$, defines a unique weight filtration $W_m(N)$

$$W_m(N) \subset \cdots \subset W_0(N) \subset \cdots \subset W_m(N)$$

characterized by

$$\begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^k : \mathrm{Gr}_k^{W(N)} V \xrightarrow{\sim} \mathrm{Gr}_{-k}^{W(N)} V, \quad k \geqq 0. \end{cases}$$

For $k \geqq 0$ we set

$$\mathrm{Gr}^{W(N)}_{k,\mathrm{prim}} V = \ker\{N^{k+1} : \mathrm{Gr}^{W(N)}_k V \to \mathrm{Gr}^{W(N)}_{-k-2} V\}.$$

In practice we will have $Q : V \otimes V \to \mathbb{Q}$ and $N \in \mathrm{End}_Q(V)$. Then

$$W_k(N) = W_{m-k+1}(N)^\perp,$$

and using Q the spaces $\mathrm{Gr}^{W(N)}_k V$ and $\mathrm{Gr}^{W(N)}_{m-k} V$ are in duality.

Definition: A *limiting mixed Hodge structure* (LMHS) is a mixed Hodge structure $(V, W(N), F_{\lim})$.

Again we will always have a $Q : V \otimes V \to \mathbb{Q}$, and it will be assumed that for $k \geqq 0$ the bilinear forms

$$\begin{aligned} &\widetilde{Q}_k : \mathrm{Gr}^{W(N)}_{k,\mathrm{prim}} V \otimes \mathrm{Gr}^{W(N)}_{k,\mathrm{prim}} \to \mathbb{Q} \\ &\widetilde{Q}_k(u, v) = Q(u, N^k v) \end{aligned}$$

polarize $\mathrm{Gr}^{W(N)}_{k,\mathrm{prim}} V$. This structure is referred to as a *polarized limiting mixed Hodge structure* (PLMHS).

Example: Let $\mathcal{X}^* \xrightarrow{\pi} \Delta^*$ be a family of smooth, projective varieties $X_t = \pi^{-1}(t)$. This means that $\mathcal{X}^*$ is smooth, we have $\mathcal{X}^* \subset \mathbb{P}^N$ and π has everywhere maximal rank. Then it is known that the monodromy

$$T : H^m(X_{t_0})_{\mathrm{prim}} \to H^m(X_{t_0})_{\mathrm{prim}}$$

is quasi-unipotent; i.e., $(T^\ell - I)^{m+1} = 0$ for some $\ell > 0$. Replacing t by t'^ℓ and pulling back (base change), we may assume that T is unipotent with logarithm

$$N = (T - I) - \frac{(T-I)^2}{2} + \frac{(T-I)^3}{3} - \cdots .$$

Then in a manner that will be made precise below, for $V = H^m(X_{t_0})_{\mathrm{prim}}$

$$\lim_{t \to 0} H^m(X_t)_{\mathrm{prim}} = (V, W(N), F_{\lim})$$

defines a PLMHS ([Sc72], [CKS86], [PS08]).

I.B. **Definition of H-surfaces; the canonical series and bicanonical map.**

Definition: An *H-surface* is a smooth, minimal algebraic surface X that satisfies the conditions

- X is of general type,
- $p_g(X) = q(X) = 0$,
- $K_X^2 = 2$.

The "H" stands for Horikawa, in whose paper [Ho79] these surfaces were introduced and described. Our interest in them is that they are in a sense the first surfaces that one encounters whose Hodge structure is not of classical type; i.e., the corresponding period domain is not a Hermitian symmetric domain.

Theorem [Ho79]: *For a general H-surface X,*

(i) *the bicanonical map*

$$\varphi_{2K_X} : X \to X' \subset \mathbb{P}^4$$

is a birational morphism whose image is given by the equations

$$\begin{cases} x_0x_2 = x_1^2 \\ x_0G(x) = F(x)^2 \end{cases}$$

where $G(x), F(x)$ are a general cubic and quadric respectively;

(ii) *a general $C \in |K_X|$ is a smooth, non-hyperelliptic curve of genus $g(C) = 3$;*

(iii) *there exists exactly one hyperelliptic curve $C_0 \in |K_X|$;*

(iv) *the singular locus X'_{sing} is a double conic D_0 given by*

$$\begin{cases} x_0 = x_1 = x_2 = 0 \\ F(x)^2 = 0. \end{cases}$$

There are eight pinch points given by $\{x_0 = x_1 = x_2 = 0\}$, $F(x) = 0$, $G(x) = 0$ together with the image of the base points of the canonical pencil $|K_X|$, and the restriction $\varphi_{2K_X} : C_0 \to D_0$ is the corresponding branched cover;

(v) *X is the normalization of X', and for any X' given by the equations in* (i) *where F and G are general the normalization is an H-surface.*

The definition of what it means for X to be general will be given during the course of the proof below.

Elsewhere in this work we shall need to extend this result to log H-surfaces, and with this in mind we shall give a proof of the above theorem beginning with a heuristic geometric argument that suggests where the particular form of the equations comes from.

Proof. We will begin with showing that for a general H-surface X

(a) the bicanonical map $\varphi_{2K_X} : X \to \mathbb{P}^4$ is regular and birational with image a surface X' lying on the singular quadric $Q_0 = \{x_0 x_2 = x_1^2\}$;
(b) for $C \in |K_X|$ the restriction
$$\varphi_{2K_X}\big|_C = \varphi_{K_C}$$
is the canonical map, and there is exactly one hyperelliptic $C_0 \in |K_X|$ for which $\varphi_{K_{C_0}}$ is a 2:1 branched covering over a double conic.

Taking the double conic to lie in the plane $x_0 = x_1 = x_2 = 0$, it follows that the equations of $X' \subset Q_0$ may be taken to be $x_0 G = F^2$.

Since $p_g(X) = 2$ the canonical series $|K_X|$ is a pencil, and one of the conditions that X be general is

$$|K_X| \textit{ has no fixed component.}$$

Then the canonical pencil has two base points, and by Bertini's theorem a general $C \in |K_X|$ will be smooth away from them. Since $K_X^2 = 2$, a general $C \in |K_X|$ cannot be singular at a base point, and it is therefore a smooth curve of genus
$$g(C) = \frac{1}{2}(K_C \cdot C + C^2) + 1 = 3.$$
Letting $t_0 \in H^0(K_X)$ define C, the condition $q(X) = h^1(K_X) = 0$ of regularity together with the exact cohomology sequence of
$$0 \to K_X \xrightarrow{t_0} 2K_X \to K_C \to 0$$
gives $h^0(2K_X) = 5$. In fact, this remains true for *any* $C \in |K_X|$ and with ω_C replacing K_C. The reasons for this are that C has no multiple components, $p_a(C) = 3$, and due to the assumption that X be minimal C has no exceptional -1 curves as components.

It follows that φ_{2K_X} has no base points, and our second generality assumption is

$$\varphi_{2K_X} : X \to X' \subset \mathbb{P}^4 \textit{ is birational.}^1$$

If $t_0, t_1 \in H^0(K_X)$ are a basis, then from the cohomology sequence

$$0 \to H^0(K_X) \xrightarrow{t_0} H^0(2K_X) \to H^0(K_C) \to 0$$

we will have a basis $t_0^2, t_0t_1, t_1^2, x_3, x_4$ for $H^0(2K_X)$. It follows that X' lies in the quadric Q_0 defined by $x_0x_2 = x_1^2$, and since $(2K_X)^2 = 8$

$$X' = Q_0 \cap Y$$

where Y is a surface of degree 4. Since φ_{2K_X} is birational, this surface is reduced.

This completes the proof of (a), and we now turn to (b) which will initially be a heuristic enumerative argument whose steps will be justified later. The key point is to show that there is exactly one hyperelliptic curve $C_0 \in |K_X|$, and for this we shall use Esteves formula [Est13]

$$h = 9\lambda - \delta_0 - 3\delta_1,$$

where

$h = \#$ hyperelliptics in $|K_X|$

$\lambda =$ degree of the Hodge bundle

$\delta_1 = \#$ of nodal reducible curves C_i with $g(\widetilde{C}_i) = 3$

$\delta_0 = \#$ of nodal irreducible curves (which for a general X will turn out to have one node, and therefore $g(\widetilde{C}_i) = 2$)

to compute that $h = 1$. Along the way we shall derive several formulas that will be used later.

We denote by $\hat{X}$ the blow up of X at the base points of $|K_X|$, so that we have a fibration

$$\hat{X} \xrightarrow{\pi} \mathbb{P}^1$$

[1] In [Ho79] the H-surfaces for which $|K_X|$ has a fixed component or φ_{2K_X} fails to be birational are classified. They give proper sub-varieties in the KSBA moduli space $\mathcal{M}_H$.

with fibres the curves in the canonical pencil. We then use

$$\chi(\mathcal{O}_{\hat{X}}) = \frac{1}{12}\left(K^2_{\hat{X}} + \chi(\hat{X})\right) \qquad \text{(Noether's formula)}$$
$$p_g(\hat{X}) = 2, q(\hat{X}) = 0$$
$$K^2_X = 2 \Rightarrow K^2_{\hat{X}} = 0$$

to give for the topological Euler characteristic

$$\chi(\hat{X}) = 30.$$

We now use that the singular fibres C_i of $\hat{X} \to \mathbb{P}^1$ are nodal with no node being a base point of $|K_X|$ (to be justified in Section H below), together with the general topological formula

$$\chi(\hat{X}) = \chi(\mathbb{P}^1)\chi(C) + \#(\text{nodal } C_i)$$

to infer that

$$\delta_1 + \delta_0 = 44.$$

Next, we shall use that if one of C_i were a compact curve, i.e., where $g(\widetilde{C}_i) = 3$, then the canonical series would have a base point at the node (the residues of $H^0(\omega_{C_i})$ there would be zero). We shall see later that a general X has two distinct base points, which rules out the possibility that a base point can be a singular point of any $C \in |K_X|$.

Finally, we shall see below that

$$\pi_*\omega_{\hat{X}/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3),$$

which gives $\lambda = 5$. Plugging this into Esteves formula gives $h = 1$.

We now turn to the proof of (i) in the theorem. Setting $K_{X}\big|_C = K_C^{1/2}$, from the exact cohomology sequences of

$$0 \to mK_X \to (m+1)K_X \to K_C^{\left(\frac{m+1}{2}\right)} \to 0, \qquad m \geqq 0$$

and $h^1(mK_X) = 0$ for $m \geqq 0$, we have for the plurigenera

$$P_m = h^0(mK_X) = m(m-1) + 3, \quad m \geqq 2.$$

Since φ_{2K_X} is birational, a general $C \in |K_X|$ is non-hyperelliptic so that the canonical curve $\varphi_{K_C}(C) \subset \mathbb{P}H^1(\mathcal{O}_C)$ is a smooth plane quartic and we have

$$\begin{cases} S^2H^0(K_C) \hookrightarrow H^0(2K_C) & \text{(actually an } \cong) \\ S^3H^0(K_C) \hookrightarrow H^0(3K_C) & \text{(actually an } \cong). \end{cases}$$

From this and the exact cohomology sequences

$$0 \to H^0(mK_X) \xrightarrow{t_0} H^0((m+1)K_X) \to H^0\left(K_C^{\binom{m+1}{2}}\right) \to 0$$

we may inductively build up generators and relations for the pluricanonical ring of X. Recalling our notation

- $t_0, t_1 =$ basis for $H^0(K_X)$,
- $t_0^2, t_0t_1, t_1^2, x_3, x_4 =$ basis for $H^0(2K_X)$, where t_1^2, x_3, x_4 is a basis for $H^0(K_C)$,

what is suggested is that we use weighted homogeneous polynomials $P(t_0, t_1, x_3, x_4)$ where t_0, t_1 have weight 1 and x_3, x_4 have weight 2. We shall use the following two lemmas:

Lemma: *If $P(t_0, t_1, x_3, x_4)$ has weighted degree $2m$, then only terms containing $t_1^a t_2^b$ with $a + b \equiv 0(2)$ can occur.*

From weight considerations, this lemma is clear.

Lemma: *If $\deg P = m \leqq 6$, then $P(t_0, t_1, x_3, x_4) \neq 0$.*

Proof. The notation means that if we substitute t_0, t_1, x_3, x_4 as sections of line bundles in a non-zero P, then the result is non-zero. Indeed, if $P(t_0, t_1, x_3, x_4) = 0$, then the restriction

$$P\big|_C = P(0, t_1, x_3, x_4) = 0,$$

and by the above properties $S^\ell H^0(K_C) \hookrightarrow H^0(\ell K_C)$ for $\ell \leqq 3$, we have

$$P = t_0 R$$

where R has weighted degree m–1. Inductively we then have $R = 0$. □

In the following table we shall give the steps used to inductively build up the canonical ring. With each step we shall give the assumptions on

X which, together with the formulas for the plurigenera and the above two lemmas, justify that step.

$H^0(K_X); \dim = 2$ because $p_g(X) = 2$	t_0, t_1
$H^0(2K_X); \dim = 5$ because $q(X) = 0$	quadratic polynomials in t_0, t_1 plus two new weight 2 generators x_3, x_4
$H^0(3K_X); \dim = 9$ and X is minimal of general type	weighted cubic polynomials in t_0, t_1, x_3, x_4 (dimension $= 8$), plus one new generator Φ of weighted degree 3; we note that $\Phi\big\|_C \in H^0(K_C^{3/2})$ is non-zero
$H^0(4K_X); \dim = 15$ and φ_{2K_X} is birational	weighted quartic polynomials in t_0, t_1, x_3, x_4 (dimension $= 14$), which when we add in $t_0\Phi, t_1\Phi$ leads to one linear relation of these modulo $P(t_0, t_1, x_2, x_4)$'s. We may take this relation to be $$t_1\Phi = F,$$ where $F = F(t_0^2, t_0, t_1, t_1^2, x_3, x_4)$ is quadratic in these variables (1st lemma)
$H^0(6K_3); \dim = 33$ and φ_{2K_X} birational	weighted sextic polynomials in t_0, t_1, x_3, x_4, Φ where we use $t_1\Phi = F$; by dimension count there is one linear relation among $t_0^3\Phi, t_0x_3\Phi, t_0x_4\Phi, \Phi^2$ modulo $P(t_0, t_1, x_3, x_4)$'s.

If the coefficient of Φ^2 is zero, then we have a linear relation among $t_0^3\Phi, t_0x_3\Phi$, $t_0x_4\Phi$ modulo $P(t_0, t_1, x_3, x_4)$'s. Restricting to $t_0 = 0$ gives that $\varphi_{K_C}(C)$ lies on a cubic curve which is a contradiction. Thus the coefficient of Φ^2 is non-zero, from which we obtain a relation

$$\Phi^2 = G,$$

where $G = G(t_0^2, t_0t_1, t_3^2, x_3, x_4)$ is a cubic curve in the indicated variables.

Relabelling by replacing t_1 by t_0 and squaring $t_0\Phi = F$ we conclude the proofs of (i), (ii), (iii) in the theorem. The proofs of (iv), (v) will be given in the following section, where we shall give an alternate way of looking at an H surface. □

Here we note the following

COROLLARY: *The canonical ring $R(X)$ of a general H-surface is*

$$R(X) \cong \mathbb{C}[t_0, t_1, x_3, x_4, \Phi]/(t_0\Phi - F, \Phi^2 - G)$$

where t_0, t_1, x_3, x_4, Φ have weights $1, 1, 2, 2, 3$, and where

$$F = F(t_0^2, t_0t_1, t_2^2, x_3, x_4) \qquad \text{has total weight } 4$$
$$G = G(t_0^2, t_0t_1, t_2^2, x_3, x_4) \qquad \text{has total weight } 6.$$

We also note that for the hyperelliptic fibre $C_0 \to D_0$ viewed as a 2:1 covering of the double conic D_0, we have

$$\Phi \in H^0(K_{C_0}^{3/2})^-$$

where the $^-$ refers to the (-1)-eigenspace under the sheet interchange involution on C_0.

I.C. **Alternate realizations of H-surfaces.** There are three related ways of realizing H-surfaces X:

(i) as the normalization of its bicanonical image $X' \subset \mathbb{P}^4$ where the equations are given in the theorem in the preceding section;

(ii) in terms of the fibration $\hat{X} \to \mathbb{P}^1$ obtained by blowing up the base points of the canonical pencil $|K_X|$;

(iii) as the normalization of the image $g(\hat{X}) = X^\flat$ in $\mathbb{P}E$, where $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and the map $g : \hat{X} \to \mathbb{P}E$ will be described below.

Each of these will provide a different perspective; it is (ii) that will provide the most useful computational framework. The three realizations will be related via the commutative diagram, which also will be defined below

$$\begin{array}{ccccc} X & & & & \\ \uparrow & \searrow^{\varphi_{2K_X}} & & & \\ \hat{X} & \xrightarrow{g} & \mathbb{P}E & \xrightarrow{f} & \mathbb{P}^4, \\ \downarrow & & \downarrow & & \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & & \end{array} \qquad f = |\mathcal{O}_{\mathbb{P}E}(1)|.$$

Definition and properties of $\mathbb{P}E$.

The motivation for the construction in (iii) is the observation that the bicanonical map is most naturally described by the factorization

$$X \to \mathbb{P}(1,1,2,2) \hookrightarrow \mathbb{P}^4.$$

Here $\mathbb{P}(1,1,2,2)$ is the weighted projective space described by

$$\mathbb{P}(1,1,2,2) = \mathbb{C}^4 \backslash \{0\} / \mathbb{C}^*$$

where $\lambda \in \mathbb{C}^*$ acts by

$$\lambda(t_0, t_1, x_3, x_4) = (\lambda t_0, \lambda t_1, \lambda^2 x_3, \lambda^2 x_4).$$

The inclusion $\mathbb{P}^1(1,1,2,2) \hookrightarrow \mathbb{P}^4$ is defined by

$$(t_0, t_1, x_3, x_4) \to (t_0^2, t_0 t_1, t_1^2, x_3, x_4).$$

Its image is the singular quadric

$$Q_0 = \{x_0 x_2 = x_1^2\} \subset \mathbb{P}^4,$$

and as described in the preceding section we have the above factorization of the bicanonical map $\varphi_{2K_X} : X \to \mathbb{P}^4$.

PROPOSITION: (i) *The desingularization* $\mathbb{P}(\widetilde{1,1,2},2)$ *of* $\mathbb{P}(1,1,2,2)$ *is canonically isomorphic to* $\mathbb{P}E$ *where* $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$*;*

(ii) *There is a commutative diagram of maps*

$$\begin{array}{ccccc} \hat{X} & \xrightarrow{g} & \mathbb{P}E & \xrightarrow{f} & Q_0 \subset \mathbb{P}^4 \\ \downarrow & & & \nearrow_{\varphi_{2K_X}} & \\ X & & & & \end{array}$$

where $f = |\mathcal{O}_{\mathbb{P}E}(1)|$ *and, setting* $X^\flat = g(\hat{X})$, $\hat{X} \xrightarrow{g} X^\flat$ *is the normalization.*

Proof. We are using the Grothendieck convention

$$(\mathbb{P}E)_t = \mathbb{P}E_t^*$$

for the fibre of $\mathbb{P}E \xrightarrow{\pi} \mathbb{P}^1$ over $t \in \mathbb{P}^1$. Setting

$$\xi = \mathcal{O}_{\mathbb{P}E}(1),\ \mathfrak{h} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$$

we have

$$\pi_*(m\xi + \ell\mathfrak{h}) = S^m E \otimes \mathcal{O}_{\mathbb{P}^1}(\ell) =: S^m E(\ell).$$

In particular this gives

$$H^0(\mathbb{P}E, \xi) \cong H^0(\mathbb{P}^1, E) = H^0(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}^5$$
$$H^0(\mathbb{P}E, \xi - 2\mathfrak{h}) \cong H^0(\mathbb{P}^1, (E(-2))) = H^0(\mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}.$$

Let x be a generator of $H^0(\mathbb{P}E, \xi - 2\mathfrak{h})$. Then the divisor

$$(x) =: S \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Denoting by t_0, t_1 a basis for $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, a basis for $H^0(\mathbb{P}E, \xi)$ is given by

$$\underbrace{xt_0^2, xt_0t_1, xt_1^2}_{H^0(\mathcal{O}_{\mathbb{P}^1}(2))},\quad \underbrace{x_3, x_4}_{H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})}$$

where the brackets refer to the terms on the right in the above identification of $H^0(\mathbb{P}E, \xi)$. The mapping $f : \mathbb{P}E \to \mathbb{P}^4$ is given by taking this basis as homogeneous coordinates. We note that

- $f(\mathbb{P}E) = Q_0 \subset \mathbb{P}^4$;
- $f(S) = Q_{0,\mathrm{sing}} = \{x_0 = x_1 = x_2 = 0\}$,

where $S = (x)$ as above. Denoting by $\widetilde{Q}_0$ the proper transform of the blow up of $\mathbb{P}^4$ along $Q_{0,\mathrm{sing}}$, it follows that we have the identification Q_0 under

$$\widetilde{Q}_0 = \mathbb{P}E;$$

in particular, $f : \mathbb{P}E \to Q_0$ is the standard resolution of the singularities of Q_0.

Turning to (ii) and identifying Q_0 with $\mathbb{P}(1,1,2,2)$, the rational map

$$\mathbb{P}(1,1,2,2) \dashrightarrow \mathbb{P}^1$$

given by $[t_0, t_1, x_3, x_4] \to [t_0, t_1]$ is undefined exactly along $Q_{0,\text{sing}}$. Resolving this indetermination by blowing up leads to $\mathbb{P}E \to \mathbb{P}^1$. If we then take the inverse image $\Gamma(g)$ of the graph $\Gamma(\varphi_{2K_X})$ of φ_{2K_X} in the diagram

$$\begin{array}{ccc} \Gamma(g) & \subset & \hat{X} \times \mathbb{P}E \\ \downarrow & & \downarrow \\ \Gamma(\varphi_{2K_X}) & \subset & X \times Q_0 \end{array}$$

we may define $g : \hat{X} \to \mathbb{P}E$ by the condition that $\Gamma(g)$ be its graph. $\square$

We remark that in the diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{g} & \mathbb{P}E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array}$$

for a general H-surface X, the mapping g is an isomorphism outside of the unique hyperelliptic curve $C_0 \subset \hat{X}$, and

$$g : C_0 \to D_0$$

is a 2:1 covering of the double conic. This will be verified below.

At this point we have described the spaces and maps in the diagram at the beginning of this section.

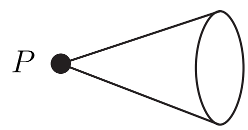
Some further properties of $\mathbb{P}E$

PROPOSITION: *The group* $\mathrm{Aut}(\mathbb{P}E)$ *acts on* $\mathbb{P}^1$ *and there are two orbits for this action:*

- *the closed orbit* S,
- *the open orbit* $\mathbb{P}E \backslash S$.

Proof. We may see this geometrically as follows: The group $\mathrm{Aut}\, Q_0 \subset \mathrm{Aut}(\mathbb{P}^4)$ has two orbits; namely, the singular locus $Q_{0,\text{sing}}$ is a closed orbit, and the complement $Q_0 \backslash Q_{0,\text{sing}}$ is an open orbit. The group $\mathrm{Aut}\, Q_0$ also acts on the proper transform $\widetilde{Q}_0 \cong \mathbb{P}E$ of Q_0 under the blowup of $\mathbb{P}^4$ along the singular locus $\{x_0 = x_1 = x_2 = 0\}$, and the induced action on $\mathbb{P}E$ is equal to that of $\mathrm{Aut}\, \mathbb{P}E$. On $Q_0 \backslash Q_{0,\text{sing}} \cong \mathbb{P}E \backslash S$ the action of $\mathrm{Aut}\, Q_0$ is transitive and gives an open orbit of the action of $\mathrm{Aut}(\mathbb{P}E)$ on $\mathbb{P}E$. The fibres of $\widetilde{Q}_0 \to Q_0$ over points p of $Q_{0,\mathrm{sing}}$ are plane conics C_p, and the induced action of $\mathrm{Aut}(Q_0)$ that fixes p acts transitively on C_p. $\square$

The picture is easier to visualize one dimension down for a singular quadric $Q_0' \subset \mathbb{P}^3$



.

This is a cone with vertex P over a conic C_P in $\mathbb{P}^2$. Then $Q'_{0,\mathrm{sing}} = P$, and the sub-group of $\mathrm{Aut}(Q_0')$ that fixes P acts transitively on the fibre over P of the proper transform $\widetilde{Q}_0' \to Q_0'$. This fibre is isomorphic to C_P.

For later use we shall determine the Lie algebra $H^0(\mathrm{End}(E))$ of the group $\mathrm{Aut}(E)$ of fibre preserving automorphisms of $E \to \mathbb{P}^1$. From

$$h^0(E(-2)) = 1, \;\; h^0(E(-k)) = 0 \;\text{ for }\; k > 2$$

we infer that $\mathrm{Aut}(E)$ preserves the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to E \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to 0.$$

(The last term really should be $\mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^2$.) Denoting by $\mathrm{Sym}^2(t)$ the homogeneous polynomials of degree 2 in t_0 and t_1, we then have

$$H^0(\mathrm{End}(E)) \cong \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathrm{Sym}^2(t) \\ \mathbb{C} & \mathbb{C} & \mathrm{Sym}^2(t) \\ 0 & 0 & \mathbb{C} \end{pmatrix}.$$

More intrinsically

$$H^0(\mathrm{End}(E)) \cong \begin{pmatrix} \mathrm{gl}(2,\mathbb{C}) & \mathcal{U} \\ 0 & \mathrm{gl}(1,\mathbb{C}) \end{pmatrix}$$

where $\mathcal{U} \cong \mathbb{C}^2 \oplus \mathrm{Sym}^2(t)$ is the unipotent radical of the Lie algebra $H^0(\mathrm{End}(E))$.

Further properties of the basic diagram and the equation that defines $X^\flat \subset \mathbb{P}E$.

We recall the part

$$\begin{array}{ccccc} \hat{X} & \xrightarrow{g} & \mathbb{P}E & \xrightarrow{f} & Q_0 \subset \mathbb{P}^4, \qquad\qquad f = |\xi| \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \\ \mathbb{P}^1 & = & \mathbb{P}^1 & & \end{array}$$

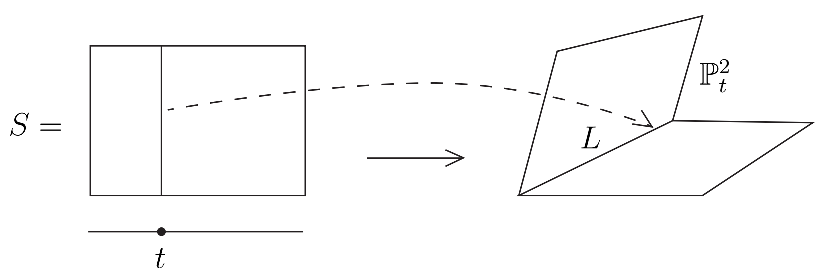
of the basic diagram. As noted above we have on $\mathbb{P}E$ a unique up to scaling section $x \in |\xi - 2\mathfrak{h}|$ with divisor

$$(x) = S \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

where $\pi\big|_S$ is the projection on the first factor. In fact, we may identify

$$S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)).$$

Below in §I.D we will use the picture below of Q_0 as a quadratic pencil of $\mathbb{P}^2_t$'s in $\mathbb{P}^4$ rotating about fixed line $L = Q_{0,\text{sing}}$. Then we have the picture of $f\big|_S$



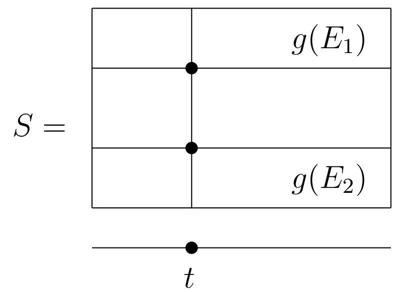
where f maps each vertical $\mathbb{P}^1$ isomorphically to L.

It follows from the theorem in Section B that the equation of $X^\flat = |4\xi|$ is

$$\boxed{x t_0^2 G = F^2}$$

where $G \in |3\xi|$, $F \in |2\xi|$ are the pullbacks under f of a general cubic, quadric in $\mathbb{P}^4$. From this equation we infer that for the blowups

$E_1, E_2 \subset \hat{X}$ of the base points of $|K_X|$ we have the picture



What this means is the divisor (F) meets S in two horizontal $\mathbb{P}^1$'s, which are then the images under g of E_1, E_2. When we intersect S with the fibres of $X^\flat \to \mathbb{P}^1$, which are the curves $C_t \in |K_{X_t}|$, then the intersections are the two marked points, which are the base points. Using this we shall show

PROPOSITION: *In* $\operatorname{Pic}\hat{X}$ *we have* $g^*(\xi) - 2\mathfrak{h} = 2(E_1 + E_2)$.

Proof. We have noted that $g(E_1)$, $g(E_2)$ lie in $S \in |\xi - 2\mathfrak{h}|$, and from this we may infer the above relation, where the 2 is there because of the F^2 in the equation of $X^\flat$. □

Remark: One important reason for the proposition is this: For the map $\hat{X} \xrightarrow{p} X$ contracting the E_i, we have

$$H^2(\hat{X}, \mathbb{Z}) = p^* H^2(X, \mathbb{Z}) \oplus \underbrace{\mathbb{Z}E_1 + \mathbb{Z}E_2}.$$

If $g^*(\xi), \mathfrak{h}, E_1, E_2$ were independent in $\operatorname{Pic}\hat{X}$, this would imply that for a general X

$$\dim \operatorname{Hg}^1(X) \geqq 2.$$

Thus locally the period mapping would not be to D where the Hodge numbers are $(2, 27, 2)$, but to a sub-domain $D' \subset D$ where the Hodge numbers are $(2, 26, 2)$. In this case, using local Torelli from Section I.G and the fact that the IPR for weight $n = 2$ period domains is a contact distribution, from

$$\dim D' = 53$$
$$\dim(H^{1,1}(X)/\operatorname{Hg}^1(X)) \leqq 2$$

we would conclude that equality holds in the second relation and that the image of the period mapping is a *maximal* integral manifold of a contact distribution. This would certainly be interesting, but it is not what happens.

Proof that there are $8 = 6 + 2$ *pinch points along the double conic* D_0.

We will describe $\mathbb{P}E$ as the proper transform of the singular quadric Q_0 under the blow up $\widetilde{\mathbb{P}(1,1,2,2)}$ of the singular locus $x_0 = x_1 = x_2 = 0$ of $\mathbb{P}(1,1,2,2)$ realized as the singular quadric $Q_0 \subset \mathbb{P}^4$.[2] For this we use the classical description of a quadratic transform given by introducing s_0, s_1, s_2 with

$$s_i x_j = s_j x_i,$$

which then gives

$$(x_0, x_1, x_2) = \lambda(s_0, s_1, s_2).$$

Since we are taking the proper transform $\widetilde{Q}_0$ of $Q_0 = \{x_0 x_2 = x_1^2\}$, we have

$$s_0 s_2 = s_1^2,$$

from which it follows that we have r_0, r_1 with

$$s_0 = r_0^2,\ s_1 = r_0 r_1,\ s_2 = r_1^2,$$

which gives

$$x_0 = \lambda r_0^2,\ x_1 = \lambda r_0 r_1,\ x_2 = \lambda r_1^2.$$

Then up on $\widetilde{\mathbb{P}(1,1,2,2)}$ the equation $x_0 G = F^2$ of $\varphi_{2K_X}(X) \subset \mathbb{P}^4$ becomes

$$\lambda r_0^2 G = F^2.$$

The pinch points of the double conic under the proper transform of $\varphi_{2K_X}(X)$ are

$$r_0 = G = F = 0 \qquad \text{6 pinch points}$$

and

$$r_0 = \lambda = F = 0 \qquad \text{2 pinch points.}$$

[2]This is also the locus where the rational map $\mathbb{P}(1,1,2,2) \hookrightarrow \mathbb{P}^1$ given by $[t_0, t_1, x_3, x_4] \to [t_0, t_1]$ fails to be well defined.

The latter are the base points of the canonical pencil $|K_X|$ mapped to the proper transform of $\varphi_{2K_X}(X)$. □

Note: We have given the computation in coordinates for later purposes where similar, but more complicated ones will be given. In terms of the equation

$$xt_0^2 G = F^2$$

of $X^\flat \subset \mathbb{P}E$, $t_0 = 0$ is the fibre $\mathbb{P}^2_{t_0}$ of $\mathbb{P}E \to \mathbb{P}^1$ and $x = 0$ is the $S' \cong \mathbb{P}^1 \times \mathbb{P}^1$ introduced above. The pinch points are

- $t_0 = G = F = 0$;
- $t_0 = x = F = 0$.

Proof that a general H-surface X contains a unique hyperelliptic curve $C_0 \in |K_X|$.

In $\mathbb{P}E$ we consider the pencil $|X^\flat_\lambda|$ defined by

$$xt_0^2(\lambda_0 G_0 + \lambda_1 G_1) = (\lambda_0 + \lambda_1)F^2$$

where each of $F = |2\xi|$ and $G_0, G_1 \in |3\xi|$ are general and $\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1$. By a slight extension of Bertini's theorem to be given below, the general member of this pencil is smooth outside the base locus

$$xt_0^2 = 0, \quad F = 0.$$

As noted above these equations separate into the parts

$$t_0^2 = 0 = F, \quad x \neq 0$$

which are the points of the double conic outside of the base points of $|K_X|$, and a second part

$$x = 0 = F$$

which are the blown up base points of the canonical pencil. Since the fibres of $g(X^\flat_\lambda) \to \mathbb{P}^1$ are just the canonical images in $\mathbb{P}^2_t = \pi^{-1}(t)$ of the curves in the pencil $|K_{X_\lambda}|$, we may conclude that for a general λ only the fibre over $t_0 = 0$ is hyperelliptic. □

The role of the hyperelliptic curve in the basic diagram.

From the basic diagram we have

$$\begin{array}{ccc} \hat{X} & \xrightarrow{g} & X^\flat \subset \mathbb{P}E \\ \cup & & \cup \\ C_0 & \longrightarrow & D_0 \end{array}$$

where D_0 is the double conic and

$$g : C_0 \to D_0$$

is the 2:1 map branched over D_0. The following formulas will be used below:

(i) $K_{C_0} = \left(g\big|_{C_0}\right)^*(\xi)$;
(ii) $g_*\mathcal{O}_{C_0} \cong \mathcal{O}_{D_0} \oplus \mathcal{O}_{D_0}(L)$ where $L^2 = \mathcal{O}_{D_0}(B_0)$ with $B_0 \subset D_0$ being the branch points;
(iii) $g_*\mathcal{O}_{C_0} \cong \ker\left\{\mathcal{O}_{X^\flat} \otimes \mathfrak{h} \to \mathcal{O}_{C_0} \otimes \mathfrak{h})\right\}$.

Proof. The first follows from $K_{\hat{X}} = g^*(\xi - \mathfrak{h})$ and adjunction for $C_0 \subset \hat{X}$ using that $\mathfrak{h}\big|_{C_0} \cong \mathcal{O}_{C_0}$. The second is the standard relation for a branched double covering. For the third we have the exact sequence

$$0 \to \mathcal{O}_{X^\flat} \to g_*(\mathcal{O}_{\hat{X}}) \to \mathcal{O}_{D_0}(L) \to 0$$

which fits in the diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \uparrow & & & & & & \\ & & \mathcal{O}_{D_0} & & & & 0 & & \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_{X^\flat} & \longrightarrow & g_*\mathcal{O}_{\hat{X}} & \longrightarrow & \mathcal{O}_{D_0}(L) & \longrightarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & g_*\mathcal{O}_{\hat{X}} \otimes \mathfrak{h}^{-1} & \longrightarrow & g_*\mathcal{O}_{\hat{X}} & \longrightarrow & \mathcal{O}_{D_0} \otimes \mathcal{O}_{D_0}(L) & \longrightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \\ & & 0 & & & & \mathcal{O}_{D_0} & & \\ & & & & & & \uparrow & & \\ & & & & & & 0 & & \end{array}$$

which leads to (iii). $\square$

This relation will allow us to compute the $H^q(\hat{X}, g^*(\xi^a)\otimes\mathfrak{h}^b)$ for all q, a, b.

Further comments on H-surfaces as degenerations of $H^{\#}$-surfaces.

In this work there are basically two types of surface degenerations we shall consider:

(i) KSBA degenerations $Y \to Y_0$;
(ii) semi-stable-reduction (SSR) degenerations $Y \to \widetilde{Y}_0$.

It is (i) that arises in the construction of KSBA moduli spaces, while it is (ii) that is the most useful traditionally in analyzing the LMHS associated to a degeneration. We have observed that the birational model $X^\flat \subset \mathbb{P}E$ of an H-surface may be considered as a KSBA degeneration of a smooth $H^{\#}$-surface $X^{\#}$. Here we want to describe a special case of the corresponding SSR family (ii).

More precisely, we will start by discussing a general family of surfaces

$$\mathcal{Y} \xrightarrow{\pi} \Delta$$

where the total space is smooth, the fibres $Y_t = \pi^{-1}(t)$ are smooth for $t \neq 0$, and where over the origin in the disc π is locally given by

$$x^2z - y^2 = t.$$

We will describe the corresponding family

$$\hat{\mathcal{Y}} \to \Delta$$

obtained by SSR from the previous family. In this case no base change will be necessary. For this we denote by

$$\widetilde{C} \xrightarrow{p} C$$

the double cover with branch locus B associated to the double curve with pinch points on Y_0. It is standard that there is a line bundle $L \to C$ with $L^2 = \mathcal{O}_C(B)$. If $s_B \in H^0(\mathcal{O}_C(B))$ is a section with divisor B, then

$$\widetilde{C} = \{x \in C, \ell \in L_x \text{ with } \ell^2 = s_B(x)\} \subset L.$$

Proposition: *The fibre $\hat{Y}_0$ in the SSR reduction family is*

$$\hat{Y}_0 = \widetilde{Y}_0 \cup Z$$

where $\widetilde{Y}_0$ is the normalization of Y_0,

$$Z = \mathbb{P}(\mathcal{O}_C \oplus L^{-1})$$

and $\widetilde{Y}_0 \cap Z = \widetilde{C}$.

Proof. Each $x \in \widetilde{C}$ gives an evaluation map

$$e_x : p_* \mathcal{O}_{\widetilde{C}} \to \mathbb{C}$$

which then defines a map

$$\widetilde{C} \to \mathbb{P}(p_* \mathcal{O}_{\widetilde{C}}).$$

The action of the involution on $\widetilde{C}$ decomposes $p_* \mathcal{O}_{\widetilde{C}}$ into $\mathcal{O}_C \oplus L^{-1}$ where L^{-1} is the -1 eigenspace. Recalling the convention that $\mathbb{P}(\mathcal{O}_C \oplus L^{-1})$ is given by the 1-dimensional quotients of $\mathcal{O}_C \oplus L$, we have $\widetilde{C} \subset L \subset \mathbb{P}(\mathcal{O}_C \oplus L^{-1})$. In fact, $\widetilde{C} \in |\mathcal{O}_{\mathbb{P}(\mathcal{O}_C \oplus L^{-1})}(2)|$, since $H^0(C, p_* \mathcal{O}_C) \cong H^0(C, \mathcal{O}_C \oplus L \oplus L^2)$ and $\widetilde{C}$ corresponds to $1 \oplus 0 \oplus S_B$.

Then the $\hat{Y}_0$ in the statement of the proposition gives a normal crossing divisor $\widetilde{Y}_0 \cup Z$ with $\widetilde{Y}_0 \cap Z = \widetilde{C}$ and that maps to Y_0 where $Z \to C$. □

We note that

$$p_* K_{\widetilde{Y}_0} \cong K_{Y_0} \oplus K_{Y_0}(L).$$

The way to check the signs here is that making L more ample increases $H^0(K_{\widetilde{Y}_0})$.

We shall use the sheaf $K_{\widetilde{Y}_0,\log}$ where the sections of $K_{\widetilde{Y}_0,\log}$ are by definition given by forms on

$$\widetilde{Y}_0 \amalg Z$$

that have log poles with opposite residues along $\widetilde{C}$.

Example: For the degeneration $X^\# \to X^\flat$ we have

$$C = \mathbb{P}^1,\ \deg B = 8,\ g(\widetilde{C}) = 3.^3$$

Turning to the canonical bundles, we have in general the

PROPOSITION: *There is an exact sequence*

$$0 \to H^0(K_{\widetilde{Y}_0}) \to H^0\left(K_{\hat{Y}_{0,\log}}\right) \to H^0(K_{\widetilde{C}})^- \to 0$$

where $K_{\widetilde{Y}_{0,\log}}$ are the logarithmic 2-forms on the NCD $\hat{Y}_0$ and $H^0(K_{\widetilde{C}})^-$ is the -1 eigenspace of action on $H^0(K_{\widetilde{C}})$ of the involution $\widetilde{C} \to \widetilde{C}$.

Proof. We will note below that the pullback to $\widetilde{Y}_0$ of the Poincaré residues of top degree forms with simple poles on the double curve with pinch points are anti-symmetric under the action of the involution. The result follows from this on $H^0(K_Z) = 0$. □

Example: For the degeneration $X^\# \to X^\flat$ where $\widetilde{X}^\flat = \hat{X}$ we have

$$H^0(K_{\widetilde{Y}_0}) \cong H^0(K_{\hat{X}}) \qquad \text{(dimension} = 2)$$
$$H^0(K_{\widetilde{C}})^- \cong H^0(K_{\widetilde{C}}) \qquad \text{(dimension} = 3).$$

Remark: Given a family of smooth surfaces $\mathcal{Y}^* \to \Delta^*$ over the punctured disc, there are (i) a well-defined LMHS, and (ii) a well-defined central fibre Y_0 such that we have a KSBA degeneration. One may ask:

Can one compute the LMHS from Y_0?

According to [Shah] one may compute the $I^{p,0}$ and $I^{0,p}$ terms from Y_0. For a double curve with pinch points the exact sequence in the proposition gives the result. Namely

- $I^{2,0} = H^0(K_{\widetilde{Y}_0})$;
- $I^{1,0} = H^0(K_{\widetilde{C}})^-$.

The proof follows from the description of the LMHS in terms of the cohomology of log-complexes on the normalization of the central fibre in a SSR [PS08]. In the case at hand that prescription reduces essentially to the exact sequence in the proposition so far as the $I^{p,0}$-terms in the LMHS are concerned.

[3]In our earlier notation, C was the double conic D_0 and $\widetilde{C}$ was the hyperelliptic curve C_0 in the pencil $|K_X|$.

In [Ko13] Kollár lists the singularities that may arise from KSBA degenerations of surfaces of general type. Following his notation, the singularities in 3.2 and (3.3.1)–(3.3.3) are the basic cases from which the general situation (3.3.4) is built by taking several cases of the basic cases and gluing them according to certain rules.

In Hodge theory one frequently works modulo the finite automorphism groups of a PHS. For example for a degenerating family of PHS's

$$\Phi : \Delta^* \to \Gamma\backslash D, \qquad \Gamma = \{T^k\}$$

one generally replaces the quasi-unipotent monodromy transformation T by a power so as to be able to take the logarithm N of a unipotent operator. Doing this does not effect the boundary component of $\Gamma\backslash D^*$ to which the origin maps. With this understood, we note that in Kollár's list

Only simple elliptic singularities (3.2.4a) and cusps (3.2.4b) effect the LMHS.

If $p \in X_0$ is the isolated singularity, then if p is a base point K_{X_0} the LMHS is not affected. Otherwise the LMHS's have

$$\dim \begin{cases} I^{2,0} = 1 \\ I^{1,0} = 1 \\ I^{0,0} = 0 \end{cases} \qquad \text{for the simple elliptic case}$$

$$\dim \begin{cases} I^{2,0} = 1 \\ I^{1,0} = 0 \\ I^{0,0} = 1 \end{cases} \qquad \text{for the cusp case.}$$

In the case of several isolated singularities p_i of these types, the LMHS is determined by the extent to which the p_i impose independent conditions on K_{X_0}. The situation is analogous to that for curves with ω_{X_0} replacing K_{X_0}, but where there are the two possibilities $\dim I^{1,0} \neq 0$, $\dim I^{0,0} \neq 0$ depending on whether we are in the isolated elliptic or cusp cases.

Some numerical properties of $\mathbb{P}E$.

We first note that

$$\operatorname{Pic}(\mathbb{P}E) \cong Z\xi \oplus \mathbb{Z}\mathfrak{h}.$$

Identifying line bundles with their Chern classes, from Grothendieck's general formula for a rank r vector bundle $E \to \mathbb{P}^1$

$$\sum_{i=0}^{r} (-1)^i c_i(E) \xi^{r-i} = 0$$

we obtain

$$\xi^3 = 2\xi^2\mathfrak{h} = 2.$$

Next, we note that $X^\flat \in |4\xi|$, and we shall show that

$$K_{\hat{X}} = g^*(\xi - \mathfrak{h}).$$

Assuming this and for notational simplicity dropping the g^*'s, we have

$$K_{\hat{X}}^2 = (\xi - \mathfrak{h})^2 \cdot 4\xi = 4\xi^3 - 8\xi^2\mathfrak{h} = 0,$$

as should be the case since $\hat{X}$ is obtained from X by blowing up two points and $K_X^2 = 2$.

Proof that $K_{\hat{X}} = g^*(\xi - \mathfrak{h})$. A double curve with pinch points is given locally by

$$zx^2 = y^2.$$

The normalization of this singularity is

$$(u, v) \to (u, uv, v^2).$$

The pullback of the Poincaré residue of $\omega = dx \wedge dy \wedge dz/(zx^2 - y^2)$ is

$$\operatorname{Res} \omega = \frac{du \wedge dv}{u},$$

which shows that the adjunction conditions given by the singularity are just the vanishing on the double curve.

The fibres of $\hat{X} \to \mathbb{P}^1$ are canonical divisors. On $X^\flat \subset \mathbb{P}E$ the fibre of $X^\flat \to \mathbb{P}E$ over $t_0 = 0$ is the divisor $(xt_0) \in |\xi - \mathfrak{h}|$. Since g is biregular outside the fibre over $t_0 = 0$, using the above adjunction argument we see that the pinch points do not effect $K_{\hat{X}}$, which implies the result. □

We will see in Section G that although the pinch points do not effect $H^0(\Omega_X^2)$, they do effect $H^1(\Omega_X^1)$. Some sort of "global adjunction conditions" on $H^1(\Omega^1)$ imposed by the pinch points is occurring.

Some cohomological computations.

For later use we shall give conditions under which the groups $H^q(\mathbb{P}E, a\xi+b\xi)$ can be non-zero. These are established using the Leray spectral sequence and

$$\Omega^2_{\mathbb{P}E/\mathbb{P}^1} \cong \det(\pi^* E) \otimes \xi^{-3} = \xi^{-3} \otimes \mathfrak{h}^2.$$

They are

$$\begin{aligned}
H^0(\mathbb{P}E, a\xi + b\mathfrak{h}) \neq 0 &\iff a \geqq 0, b \geqq -2a \\
H^1(\mathbb{P}E, a\xi + b\mathfrak{h}) = 0 &\iff a \geqq 0, b \leqq -2 \\
H^2(\mathbb{P}E, a\xi + b\mathfrak{h}) \neq 0 &\iff a \leqq -3, b \geqq 2a + 8 \\
H^3(\mathbb{P}E, a\xi + b\mathfrak{h}) \neq 0 &\iff a \leqq -3, b \leqq -2a - 6.
\end{aligned}$$

Hodge bundles.

For a general H-surface X we have the fibration

$$\hat{X} \xrightarrow{\pi} \mathbb{P}^1$$

whose fibres are nodal curves. Thus $\pi_* \omega_{X/\mathbb{P}^1}$ is a rank 3 vector bundle over $\mathbb{P}^1$.

Proposition: $\pi_* \omega_{\hat{X}/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) = E(1)$.

Proof. By the theorem of Birkhoff-Grothendieck

$$\pi_* \omega_{\hat{X}/\mathbb{P}^1} \cong \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(k_i).$$

We shall use two general results from Hodge theory:

(i) We have

$$k_i \geqq 0$$

(non-negativity of the Hodge bundles);

(ii) the trivial sub-bundle $\underset{k_i=0}{\oplus} \mathcal{O}_{\mathbb{P}^1}(k_i)$ of $\pi_*\omega_{\hat{X}/\mathbb{P}^1}$ corresponds to the image of the injective map

$$H^0(\Omega^1_{\hat{X}}) \to H^0(\mathbb{P}^1, \pi_*\omega_{\hat{X}/\mathbb{P}^1}).$$

The second result pertains to a pencil of at most nodal curves on any surface X where $\hat{X}$ is the blow up of the base points of the pencil. In our case this implies that all

$$k_i > 0.$$

Both of these results are classical; a reference is [BPVdV84].

Since $\hat{X} \to \mathbb{P}^1$ is locally given by $t = xy$ and $\omega_{\hat{X}/\mathbb{P}^1}$ is generated by $\frac{dx}{x} = -\frac{dy}{dy}$, we infer from $\frac{dx}{x} \wedge dt = dx \wedge dy$ that

$$K_{\hat{X}} \cong \omega_{\hat{X}/\mathbb{P}^1} \otimes \pi^* K_{\mathbb{P}^1},$$

which gives

$$\pi_* K_{\hat{X}} \cong \pi_*\omega_{\hat{X}/\mathbb{P}^1}(-2) \cong \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(k_i - 2).$$

From

$$H^0(K_{\hat{X}}) \cong H^0(\mathbb{P}^1, \pi_*\omega_{\hat{X}/\mathbb{P}^1}(-2)) \cong \bigoplus_{i=1}^{3} H^0\left(\mathcal{O}_{\mathbb{P}^1}(k_i - 2)\right)$$

and $p_X^\flat = 2$, $k_i > 0$ we have

$$\sum_{i=1}^{3} (k_i - 1) = 2.$$

The possibilities for (k_1, k_2, k_3) are

$$(1,1,3) \text{ and } (1,2,2).$$

Since the divisors in $|K_{\hat{X}}|$ consist of the fibres of $\hat{X} \to \mathbb{P}^1$, so that in particular sections of K_X will have a zero, the second possibility cannot occur. □

Remark: The space $H^0(\Omega^2_X)$ enters in two ways:

- via Hodge theory using $H^0(\Omega^2_X) = H^{2,0}(X)$;

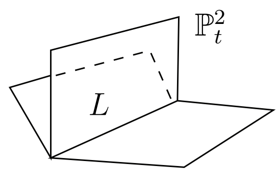
- via the geometry of $|K_X|$, which leads to the vector bundle $E \to \mathbb{P}^1$ and the bicanonical map as the composition

$$\hat{X} \xrightarrow{g} \mathbb{P}E \xrightarrow{f} \mathbb{P}^4, \qquad f = |\mathcal{O}_{\mathbb{P}E}(1)|,$$

which induces the canonical map on the fibres of $\hat{X} \to \mathbb{P}^1$.

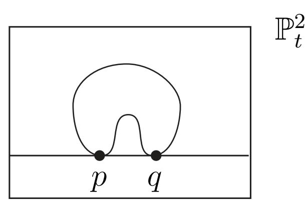
I.D. **Pictures, and a Torelli-type result.** It is convenient to have pictures to describe H-surfaces.[4] We shall use the following ones:

Q_0



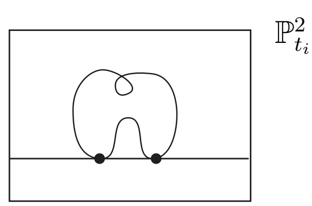
Q_0 is a quadratic pencil of $\mathbb{P}^2$'s in $\mathbb{P}^4$ rotating about the fixed line $L = Q_{0,\text{sing}}$ (we are only able to draw the picture in 3-space)

C_t



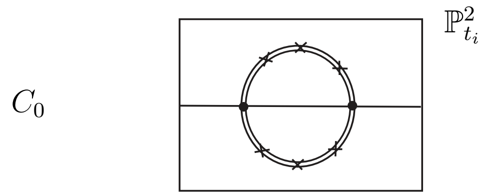
A general C_t is a genus 3 curve canonically embedded in the $\mathbb{P}^2_t$ of the quadric pencil giving Q_0; here p, q are the base points of the pencil $|K_X|$, and they all have L as a common bi-tangent where $K^{1/2}_{C_t} = [p+q]$.

C_{t_i}



There are finitely many singular $C_{t_i} \in |K_X|$, which for general H-surfaces are irreducible plane quartics with a node away from L

[4]We are here dropping the notation $\varphi_{K_{C_t}}(C_t)$ for the canonical curve, and are simply using $C_t \subset \mathbb{P}^2_t$.



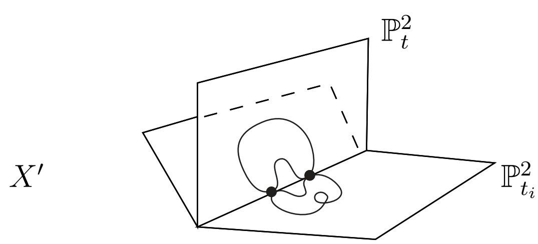
The unique hyperelliptic C_0 in $|K_X|$ has as canonical image a 2:1 branched covering $C_0 \to D_0$ over a double conic.

The eight branch points of $C_0 \to D_0$ are given by the two base points (the dots) of the pencil $|K_X|$ together with the six points of intersection (the x's)

$$\{x_0 = x_1 = x_2 = 0\} \cap \{G(x) = 0\} \cap \{F(x) = 0\}$$

where $x_0 G(x) = F(x)^2$ is the equation of $\varphi_{2K_X}(X) = X' \subset \mathbb{P}^4$.

Putting everything together gives a picture of X as being something like



where we have not drawn in $\mathbb{P}^2_0$ and D_0. These pictures will be useful on several occasions, especially when we discuss the generic degenerations of H-surfaces that arise when X varies in moduli.

A Torelli-type theorem: Associated to a general H-surface X with $\hat{X} \xrightarrow{\pi} \mathbb{P}^1$ obtained by blowing up the base points of $|K_X|$ is a variation of Hodge structure $(V, \mathcal{F})$ where

- V is a direct summand of the local system $R^1_\pi \mathbb{Z}$, and
- $\mathcal{F}$ is a filtration of $\mathcal{V} =: V \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}$ with

$$\mathcal{F}_t \cong \pi_* \left(\omega_{\hat{X}/\mathbb{P}^1} \right)_t .$$

In more detail, the general fibre of the local system is

$$V_t = H^1(C_t)$$

where the fibres of $\mathcal{V}$ for the nodal C_{t_i} are filled in the usual way.[5] Then for a general $t \in \mathbb{P}^1$, $\mathcal{F} \subset \mathcal{V}$ is given by

$$\pi_* \left(\omega_{\hat{X}/\mathbb{P}^1} \right)_t \cong H^0\left(\omega_{C_t}\right) \hookrightarrow V_t \otimes \mathbb{C}.$$

V_{t_i} may be thought of as $H_1(C^*_{t_i}, \mathbb{Z})^*$ where $C^*_{t_i} = C_{t_i} \backslash \{\text{node}\}$, and we obtain the usual logarithmic differentials with opposite residues on the two branches of the node.

Proposition: (i) *The VHS* $(V, \mathcal{F})$ *uniquely determines the polarized Hodge structure on* $H^2(X, \mathbb{Z})_{\text{prim}}$; (ii) *this VHS also uniquely determines the bicanonical model* $\varphi_{2K_X}(X)$.

Proof of (i): We shall show that

(a) $H^2(X, \mathbb{Z})_{\text{prim}} \cong H^1(R^1_\pi \mathbb{Z}) \cap (E_1 - E_2)^\perp$;
(b) $H^0(\Omega^2_X) \cong H^0\left(\pi_* \left(\omega_{\hat{X}/\mathbb{P}^1}\right)(-2)\right)$;
(c) $H^2(\mathcal{O}_X) \cong H^1(R^1_\pi \mathcal{O}_{\hat{X}})$.

In (a), we recall that E_1 and E_2 are the blow ups of the base points of $|K_X|$, and $(E_1 - E_2)^\perp$ is the orthogonal of $E_1 - E_2 \in H^1(R^1_\pi \mathbb{Z})$ under the non-degenerate symmetric pairing

$$H^1(R^1_\pi \mathbb{Z}) \otimes H^1(R^1_\pi \mathbb{Z}) \to H^2(R^2_\pi \mathbb{Z}) \cong \mathbb{Z}.$$

This will be further explained below.

(a) In the diagram

$$\begin{array}{ccc} & \hat{X} & \\ {}^{\pi}\swarrow & & \searrow^{\beta} \\ \mathbb{P}^1 & & X \end{array}$$

[5] In the standard notation,

$$\mathcal{V} = \mathbb{R}^0_\pi \omega_{\hat{X}/\mathbb{P}^1}(\log Z)$$

where Z is the union of the nodal fibres.

we denote by $C \subset X$ a general curve in $|K_X|$ and by $\hat{C}$ the corresponding fibre in $\hat{X} \to \mathbb{P}^1$. Then

$$\beta^*[C] = [\hat{C}] + [E_1] + [E_2].$$

Note that although $\hat{C} \xrightarrow{\sim} C$, we have $\hat{C}^2 = 0$ and $(\beta^*[C])^2 = 2$, as should be the case.

The Leray spectral sequence for $\hat{X} \to \mathbb{P}^1$ gives a filtration on $H^2(\hat{X}, \mathbb{Z})$ with graded pieces

$$\begin{aligned} &H^0(R^2_\pi \mathbb{Z}) \cong \text{image}\{H^2(\hat{X}, \mathbb{Z}) \xrightarrow{r} H^2(\hat{C}, \mathbb{Z})\} \\ &H^1(R^1_\pi \mathbb{Z}) \cong \ker r/\text{image}\{\pi^* : H^2(\mathbb{P}^*, \mathbb{Z}) \to H^2(\hat{X}, \mathbb{Z})\} \\ &H^2(R^0_\pi \mathbb{Z}) \cong \text{image}\{H^2(\mathbb{P}^1, \mathbb{Z}) \to H^2(\hat{X}, \mathbb{Z})\}. \end{aligned}$$

Then under the inclusion $H^2(X, \mathbb{Z}) \overset{\beta^*}{\hookrightarrow} H^2(\hat{X}, \mathbb{Z})$ and noting that $E_1 - E_2 \in \ker r$, we have

$$H^1(R^1_\pi \mathbb{Z}) = \beta^* H^2(X, \mathbb{Z})_{\text{prim}} \oplus (E_1 - E_2)^\perp,$$

which is (a).[6]

The identification (b) follows from $K_{\hat{X}} \cong \omega_{\hat{X}/\mathbb{P}^1} \otimes \pi^* \Omega^1_{\mathbb{P}^1}$. In detail, if $\psi(t)$ is a section of $\pi_*\big(\omega_{\hat{X}/\mathbb{P}1}\big)$ that vanishes to 2[nd] order at $t = \infty$, then

$$\Psi = \psi(t) \wedge dt \in H^0\big(\Omega^2_X\big)$$

is holomorphic over $t = \infty$ on $\mathbb{P}^1$, and the map $\psi \to \Psi$ gives (b).

For (c), by relative duality we have

$$R^1_\pi \mathcal{O}_{\hat{X}} \cong \pi_*\big(\omega_{\hat{X}/\mathbb{P}^1}\big)^* \cong E^*(-1)$$

and

$$H^1(R^1_\pi \mathcal{O}_{\hat{X}}) \cong H^2(\mathcal{O}_X) \cong H^0(\Omega^2_X)^*.$$

The map $\mathcal{V} \to H^2(\mathcal{O}_X)$ is then given by the map on cohomology

$$H^1(R^1_\pi \mathbb{Z}) \to H^1(R^1_\pi \mathcal{O}_{\hat{X}})$$

induced by the inclusion $R^1_\pi \mathbb{Z} \to R^1_\pi \mathcal{O}_{\hat{X}}$.

[6]The expression for the cup product on $H^2(\hat{X}, \mathbb{Z})$ in terms of the Leray filtration will be discussed below.

Remark: The dual of this map is

$$H^0\big(\pi_*\omega_{\hat{X}/\mathbb{P}^1}(-2)\big) \to H^1(R^1_\pi\mathbb{Z})^* \cong H^1(R^1_\pi\mathbb{Z})$$

as discussed just before the statement of the proposition.

The properties of the cup product in terms of the map β and the Leray spectral sequence may be summarized as follows:

- Denoting by $Q_{\hat{X}}$ and Q_X the cup product induced symmetric bilinear forms on $H^2(X,\mathbb{Z}) \subset H^2(\hat{X},\mathbb{Z})$, under the inclusion $\beta^* H^2(X,\mathbb{Z}) \subset H^2(\hat{X},\mathbb{Z})$ we have
$$Q_{\hat{X}} = \begin{pmatrix} Q_X & & \\ & -1 & \\ & & -1 \end{pmatrix};$$
- The expression for $Q_{\hat{X}}$ in terms of the Leray filtration on $H^2(\hat{X},\mathbb{Z})$ is more subtle (cf. [Le24], and for a treatment that puts Lefschetz's geometric reasoning in a modern setting [Ka]). For us the basic fact is that the symmetric pairing
$$Q_P : H^1(R^1_\pi\mathbb{Z}) \otimes H^1(R^1_\pi\mathbb{Z}) \to \mathbb{Z}$$
induced by $R^1_\pi\mathbb{Z} \otimes R^1_\pi\mathbb{Z} \to \mathbb{Z}$ is non-degenerate and restricts to the pairing Q_X on $H^2(X,\mathbb{Z})_{\text{prim}}$.
- In homology we may describe $H^1(R^1_\pi\mathbb{Z})$ as represented by a quotient of the topological 2-cycles Γ in general position and with $\Gamma\cdot\hat{C} = 0$. Since $\hat{C}^2 = 0$ it follows that with an appropriate choice of a basis adapted to the Leray filtration we will have
$$Q_{\hat{X}} = \begin{pmatrix} 0 & * & 1 \\ * & Q_P & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
For our purposes this is all that is required.

Proof of (ii): The idea is that the VHS enables us to construct first the quadric Q_0 as pictured above, and then to describe the curves $C_t \subset \mathbb{P}^2_t$ which will trace out the bicanonical image X' of the H-surface X.

For the first step we have that from the VHS we may construct

$$E = \pi_*\big(\omega_{\hat{X}/\mathbb{P}^1}\big)(-1)$$

together with the map

$$\mathbb{P}E \xrightarrow{f} Q_0 \subset \mathbb{P}^4, \qquad f = |\mathcal{O}_{\mathbb{P}E}(1)|.$$

Since for $t \in \mathbb{P}^1$

$$(\mathbb{P}E)_t = \mathbb{P}(E_t^*) = \mathbb{P}(R^1_\pi \mathcal{O}_{\hat{X}})_t$$

we see that $\mathbb{P}^2_t$ is projectively the space where the canonical curves $\varphi_{K_{C_t}}(C_t)$ live. For simplicity of notation we shall again drop the $\varphi_{K_{C_t}}$'s.

Next we have from the VHS

$$0 \to R^1_\pi \mathbb{Z} \to R^1_\pi \mathcal{O}_{\hat{X}} \to \mathcal{J} \to 0$$

where $\mathcal{J}$ is the sheaf of normal functions, viewed as holomorphic sections of the family $\mathcal{J}(C_t)$ of Jacobian varieties of the C, with the generalized Jacobian $J(C_{t_i})$ being inserted at the critical values [Zu76]. Using that the pairing

$$R^1_\pi \mathbb{Z} \otimes R^1_\pi \mathbb{Z} \to \mathbb{Z}$$

induces principal polarizations on the $J(C_t)$, since $g = 3$ and X is general there is a unique plane quartic with Jacobian $J(C_t)$. We then have

$$C_t \subset \mathbb{P}T_e J(C_t) = \mathbb{P}^2_t \subset Q_0,$$

and in this way have reconstructed X' from the VHS $(V, \mathcal{F})$. □

Remark: We have noted that

> *p, q are the base points of $|K_X|$, and L is a common bitangent to the $C_t \subset \mathbb{P}^2_t$.*

In fact we may construct p, q directly from the VHS as follows: In the family of Jacobians $J(C_t)$, there is a unique $J(C_0)$ that is the Jacobian of a hyperelliptic curve. Then the canonical mapping

$$\varphi_{K_{C_0}} : C_0 \to \mathbb{P}^2_0$$

is a 2:1 covering over a conic D_0 with eight branch points. Of these branch points, exactly two, namely p and q, are on L.

I.E. $H^{\#}$**-surfaces.**

Definition: An $H^{\#}$-surface is a general $X^{\#} \in |\xi^4|$.

The motivation for introducing these surfaces is that they represent natural smoothings of the birational model $X^{\flat} \subset \mathbb{P}E$ for a general H-surface X. Remark that, as will be seen below, a general deformation of an $H^{\#}$-surface as an abstract surface will not be an $H^{\#}$-surface, in contrast to the analogous situation for H-surfaces. Equivalently, a general deformation of $X^{\#}$ as an abstract surface will not remain as a hypersurface in $\mathbb{P}E$.

PROPOSITION: (i) *A general $X^{\#}$ is smooth.* (ii) *The canonical mapping is given by*

$$\varphi_{K_{X^{\#}}} = f : X^{\#} \to \mathbb{P}^4.$$

The image is $Q_0 \cap Y$ where $Y \in |\mathcal{O}_{\mathbb{P}^4}(4)|$ is a general quartic hypersurface.

Proof. Since $\xi = \mathcal{O}_{\mathbb{P}E}(1)$ is not very ample, or even ample, applied to $X^{\#} \in |\xi^4|$ we cannot apply the usual version of Bertini's theorem. Here the situation is that $|\xi|$ gives the map

$$\mathbb{P}E \xrightarrow{f} Q_0 \subset \mathbb{P}^4$$

where $\mathbb{P}E = \widetilde{Q}_0$ is the standard desingularization of Q_0. In general, if we have a very ample line $L \to Y$ over a singular variety Y and a desingularization $f : \widetilde{Y} \to Y$ with $\widetilde{L} = f^*(L)$ and where $H^0(\widetilde{Y}, \widetilde{L}) \cong H^0(Y, L)$, then we have the following simple result whose proof will be given below.

Extended Bertini I: *A general $Z \in |L|$ is smooth oustide of* Y_{sing}.

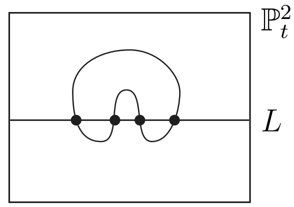
To prove the above proposition we shall examine the picture of

$$f(X^{\#}) = Q_0 \cap Y$$

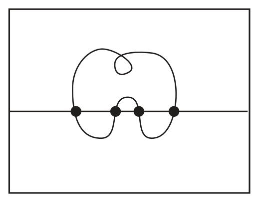
where $Y \subset |\mathcal{O}_{\mathbb{P}^4}(4)|$ is a general quartic. Setting

$$C_t^{\#} = f(X^{\#}) \cap \mathbb{P}_t^2$$

we have for $C_t^{\#}$ the picture



which is a smooth plane quartic. For finitely many t_i the picture of C_{t_i} is



where the node is away from $L = Q_{0,\text{sing}}$. This is intuitively plausible, and will be proved in Section H below. The common points $p_\alpha \in f(X^{\#})$ of intersections of all the $C_t^{\#}$ with L are ordinary nodes, whose resolution gives -2 curves $E_\alpha \subset X^{\#}$. This last statement follows from $\widetilde{Q}_0 = \mathbb{P}E$.

This establishes (i) in the proposition, and for (ii) since on a surface nodes do not impose adjunction conditions we have

$$K_{X^{\#}} = \xi\big|_{X^{\#}}.$$

Then $f\big|_{X^{\#}} = \varphi_{K_{X^{\#}}}$ is biregular aside from contracting the -2 curves E_α. $\square$

We note that the argument also gives that

$$X^{\#} \textit{ contains no } -1 \textit{ curves.}$$

Thus $X^{\#}$ is a minimal surface of general type.

Remark: Referring to (ii) in the theorem in Section I.B and whose proof was begun in Section I.C, the argument may now be completed by using the

Extended Bertini II: *In the situation of $L \to X$ described above, for a general pencil $|Z_\lambda| \subset |L|$ the Z_λ are smooth outside of $Y_{\rm sing} \cup$ (Base locus of the pencil).*

Proof of the extended Bertini statements. Working in a neighborhood of a smooth point of Y with local coordinates $y_1, \dots, y_n$, a 1-parameter family of hypersurfaces $Z_\lambda \subset Y$ is given by

$$f(y_0, \dots, y_n, \lambda) = 0.$$

If $p(\lambda) = (y_1(\lambda), \dots, y_(\lambda))$ is a moving singular point of Z_λ, then using $f(y_1(\lambda), \dots, y_n(\lambda), \lambda) = 0$ and $f_{y_i}(y_1(\lambda), \dots, y_n(\lambda), \lambda) = 0$ we have from the first equation

$$\sum_i f_{y_i}(p(\lambda), \lambda) \frac{\partial y_i(\lambda)}{\partial \lambda} + f_\lambda(p(\lambda), \lambda) = 0,$$

while the second equation gives

$$f_\lambda(p(\lambda), \lambda) = 0.$$

It will suffice to consider the case of a pencil

$$f(y_1, \dots, y_n, \lambda) = g(y_1, \dots, y_n) + \lambda h(y_1, \dots, y_n).$$

Then from $f(p(\lambda), \lambda) = 0$, $f_\lambda(p(\lambda), \lambda) = 0$ we have

$$g(p(\lambda), \lambda) = 0 = h(p(\lambda), \lambda);$$

i.e., we are in the base locus of the pencil. □

The point is that the classical Bertini theorem is really a local result.

Returning to the geometry of $H^\#$-surfaces, we note that

> *A general $H^\#$-surface comes with a canonical, base-point-free pencil $|C_t^\#|$.*

These are the fibres of $X^\# \xrightarrow{\pi} \mathbb{P}^1$. The -2 curves E_α each give cross-sections of the fibration and

$$C_t^\# - 2(E_1 + \dots + E_4) \in |K_{X^\#}|.$$

This gives $K_{X^\#}^2 = 8$, which also follows from

$$K_{X^\#} = \xi\big|_{X^\#}$$

and $X^{\#} \in |4\xi|$, $\xi^2[X^{\#}] = 4\xi^3 = 8\xi^2\mathfrak{h} = 8$.

We note that $|K_{X^{\#}}|$ has the divisor $S \cap X^{\#} = 2(C_1 + \cdots C_4)$ as a fixed component.

H-surfaces as degenerations of $H^{\#}$-surfaces.

The main interest in this work is H-surfaces. However,

> *a general H-surfaces is canonically a degeneneration of an $H^{\#}$-surface.*

More precisely, a general H-surface X has for the birational model $X^{\flat} \subset |4\xi|$ an equation

$$xt_0^2 G - F^2 = 0$$

where $G \in |3\xi|$, $F \in |2\xi|$. Putting this equation in a general pencil in $|4\xi|$ gives a smoothing of $X^{\flat}$ to an $H^{\#}$-surface. A consequence will be

> *The polarized Hodge structure on $H^2(X)$ appears as a subquotient in the limiting mixed Hodge structure associated to the degeneration $X^{\#} \to X^{\flat}$.*

An example of this is the following: Under a general degeneration $X^{\#} \to X^{\flat}$, the limit $X^{\flat} = X_0$ acquires a double curve D_0 with eight pinch points, and a general $\psi \in H^0(\Omega^2_{X^{\#}})$ will tend to a 2-form $\psi_0 \in H^0(\Omega^2_{X_0}(\log D_0))$ with logarithmic poles on D_0 and with residue

$$\operatorname{Res} \psi_0 \in H^0\big(\Omega^1_{D_0}\big)^{-}$$

belonging to the -1 eigenspace of the action of the sheet interchange involution $j \cdot D_0 \to D_0$ where $C_0 \to D_0$ is the branched double cover. This is a general property of the limits of holomorphic 2-forms on a family of surfaces acquiring a double curve with pinch points;[7] it follows from the above local parametrization $(u, v) \to (u, uv, u^2)$ and $\operatorname{Res}\left(\frac{dx\wedge dy\wedge dz}{zx^2-y^2}\right) = \frac{du\wedge dv}{u}$ where the sheet interchange is $u \to -u$, $v \to v$.

In the case at hand, since $D_0 \cong \mathbb{P}^1$

$$H^0\big(\Omega^1_{D_0}\big)^{-} = H^0\big(\Omega^1_{D_0}\big)$$

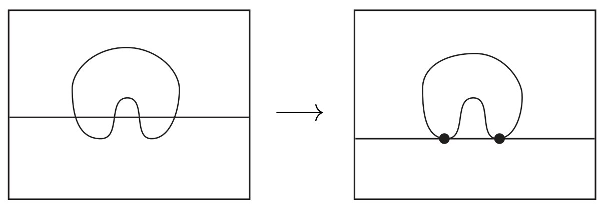
[7] Cf. the discussion in the subsection "Further comments on H-surfaces as degenerations of $H^{\#}$-surfaces" in §I.D above.

and $g(D_0) = 3$. Choosing appropriately the $h^{2,0}(X^{\#}) = 5$ linearly independent forms in $H^0\big(\Omega^2_{X^{\#}}\big)$, via residues these will give all of $H^0\big(\Omega^1_{D_0}\big)$, and the remaining two with zero residues will give a basis for $H^0\big(\Omega^2_X\big) \cong H^0\big(\Omega_{\hat{X}}\big)$.

A further geometric point to note is

> *As $X^{\#} \to X^{\flat}$, the four -2 curves E_α tend pairwise to the two -1 curves that arise from the base points of $|K_X|$.*

This is clear from the picture



$C_t^{\#}$ $\qquad\qquad$ C_t

Finally, when we turn in a subsequent section to generic local Torelli results, it will be convenient to do some of the calculations first for $X^{\#}$-surfaces, where they are in some ways simpler, before turning to the case of H-surfaces.

Further properties of $H^{\#}$-surfaces.

$H^{\#}$-surfaces were introduced as the natural smoothings of the birational model $X^{\flat}$ of an H-surface X. Although not essential for the rest of this work, here we shall explain some aspects of their intrinsic structure. The outcome is that *$H^{\#}$-surfaces are useful in the study of H-surfaces, but in and of themselves are not as interesting as H-surfaces.*

Definition: An *abstract $H^{\#}$-surface*, or $AH^{\#}$-surface, is a smooth minimal algebraic surface $X^{\#}$ of general type that satisfies

$$K^2_{X^{\#}} = 8$$
$$p_g(X^{\#}) = 5 \text{ and } q(X^{\#}) = 0.$$

Proposition: *A general $AH^{\#}$-surface is biholomorphic to a complete intersection*

$$Q \cap Y \subset \mathbb{P}^4$$

where Q, Y are a general quadric, quartic respectively.

Proof. We shall take general to mean that the canonical map

$$\varphi_{K_{X^{\#}}} : X^{\#} \to \mathbb{P}^4$$

is a biholomorphic morphism to its image.[8] This implies that a general $C^{\#} \in |K_{X^{\#}}|$ is a smooth curve, which by adjunction has genus $g(C^{\#}) = 9$. From the exact cohomology sequence of

$$0 \to K_{X^{\#}} \to 2K_{X^{\#}} \to K_{C^{\#}} \to 0$$

we obtain $h^0(2K_{X^{\#}}) = 14$. Since $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 15$ we conclude that the birational image of $X^{\#}$ lies on a quadric Q.

We now assume that Q is smooth; the remaining possibilities for Q will be discussed below. Since $\deg \varphi_{K_{X^{\#}}}(X^{\#}) = 8$ and the hypersurfaces in $\mathbb{P}^4$ cut out complete linear series on Q, it follows that

$$\varphi_{K_{X^{\#}}}(X^{\#}) = Q \cap V. \qquad \square$$

Recall that an $H^{\#}$-surface is defined to be a smooth $X^{\#} \in |\xi^4| \subset \mathbb{P}E$. Thus $X^{\#}$ comes equipped with a mapping $X^{\#} \to \mathbb{P}^1$ whose general fibres are genus 3 curves. On the other hand a general $AH^{\#}$-surface $X^{\#}$ does not have such a map; one may in fact show that $\rho(X^{\#}) = 1$. There is a natural class of $AH^{\#}$-surfaces that are not $H^{\#}$-surfaces but that are intermediate between $H^{\#}$- and $AH^{\#}$-surfaces and which do come equipped with a map to $\mathbb{P}^1$; these arise as follows:

From the split exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to E \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0,$$

we obtain a 2-parameter family of non-split exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to E' \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0.$$

[8]As usual the various degenerate cases can be explicitly analyzed, but for our purposes we shall have no need for this.

A non-split E' has

$$E' \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

We may picture a 2-parameter family $\mathcal{E} \to \Delta^2$ of bundles with fibre E over the origin and E''s as fibres away from the origin. This is of course a special case of jumping phenomena of Hirzebruch surfaces jacked up one dimension. For $\xi' = \mathcal{O}_{\mathbb{P}E'}(1)$ we have using $H^0(\xi')$ a map

$$\mathbb{P}E' \xrightarrow{f'} Q' \subset \mathbb{P}^4$$

whose image is a quadric of rank 3 having a unique singular point $P' \in Q'$. A smooth complete intersection

$$Q' \cap Y$$

where $Y \in |\mathcal{O}_{\mathbb{P}^4}(4)|$ is an $AH^{\#}$-surface which is not an $H^{\#}$-surface. This is the general situation where the quadric Q in the proposition is not smooth.

Summary. There is a hierarchy

$$\{H^{\#}\text{-surfaces}\} \subset \overline{\left\{\begin{matrix}\text{smooth members}\\ \text{of } |\xi'^4| \subset \mathbb{P}E'\end{matrix}\right\}} \subset \overline{AH^{\#}\text{-surfaces}}.$$

Each of these spaces is of codimension 2 in the succeeding one. For the general members in each space we have for the Picard numbers

$$\rho = 4, \quad \rho = 2, \quad \rho = 1$$

respectively.

A final comment is that what motivated this discussion was the issue of proving the local Torelli theorem for $H^{\#}$-surfaces, the computations here of it being similar on the face to but simpler than those for H-surfaces. As the above discussion shows, what is really involved for local Torelli for $H^{\#}$-surfaces is local Torelli for $AH^{\#}$-surfaces *that are constrained to have four* -2 *curves.* Whereas local Torelli for general $AH^{\#}$-surfaces is standard and classical, for $H^{\#}$-surfaces it is a more complex question.

I.F. **Tangent space to moduli for H-surfaces.** Let $\mathcal{M}_H$ be the KSBA moduli space for H-surfaces and X an H surface. We denote by

$$\mathcal{J} \subset \mathcal{O}_{\mathbb{P}E}$$

the Jacobian ideal (defined below) defined by the double curve and pinch points of $X^\flat \subset \mathbb{P}E$. In this section we will prove the

THEOREM: $T_X\mathcal{M}_H \cong H^1(\Theta_X) \cong \frac{H^0(\mathbb{P}E,\xi^4\otimes\mathcal{J})/\mathbb{C}}{\mathrm{aut}(\mathbb{P}E)}$, *and these spaces have dimension 26.*

The nominator in the term on the right is the quotient of $H^0(\mathbb{P}E,\xi^4\otimes\mathcal{J})$ by a scaling action; think of $T\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^N}(k))$ as the tangent space to smooth hypersurfaces of degree k in $\mathbb{P}^N$.

COROLLARY [Ho79]: $\mathcal{M}_H$ *is smooth and connected.*

The essential point in the proof of the theorem is to

express $H^1(\Theta_X)$ in terms of the equation of $X^\flat \in |\xi^4|$.

Proof of the theorem. We shall use the following notations:

- X = smooth H-surface;
- $X' \subset \mathbb{P}^4$ is the image of φ_{2K_X};
- $\hat{X}$ = blow up of X at the base points of $|K_X|$;
- $X^\flat = g(X) \subset \mathbb{P}E$, which we may also view as the proper transform of X' under the map $f : \mathbb{P}E \to Q_0$ that resolves the singularities of Q_0.

We note that even though $\mathbb{P}E$ is a smooth resolution of singularities of Q_0, $X^\flat$ is not smooth but has a double curve with pinch points.

All of these surfaces are birationally equivalent, and although it would be nice to be able to just denote them all by X we found that this leads to confusion.

We write the equation of $X^\flat \subset \mathbb{P}E$ as

$$R =: L^2G - F^2 = 0$$

where

$$\begin{cases} L \in |\mathfrak{h}| \\ G \in |4\xi - 2\mathfrak{h}| \\ F \in |2\xi|. \end{cases}$$

Next we define $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}E}$ to be the sheaf of ideals generated by $\{LG, L^2, F\}$. Since a pinch point is locally given by

$$zx^2 - y^2 = 0$$

we see that *away from the pinch points* $\mathcal{J}$ *is the ideal of the double curve, and that at the pinch points* $\mathcal{J}$ *contains in addition the ideal sheaf of the pinch points on* $X^\flat$.

Proposition: T_X(*space of* $X^\flat \in |\xi^4|$ *arising from* H*-surfaces*) $\cong H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J})$.

Proof. Differentiation of the above equation for $X^\flat$ gives

$$\dot{R} = 2LG\dot{L} + L^2\dot{G} - 2F\dot{F} \in H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J}).$$

This gives the inclusion LHS $\subseteq$ RHS. The dimension counts to be given below will establish equality.

Remark: The pinch points are given by

$$L = G = F = 0.$$

Using $\mathfrak{h}^2 = 0$ and $\xi^2\mathfrak{h} = 1$, the number of these is $\mathfrak{h} \cdot (4\xi - 2\mathfrak{h}) \cdot (2\mathfrak{h}) = 8$. The previous computation identified these as the six points given in $\mathbb{P}^4$ by $\{x_0 = x_1 = x_2 = 0\} \cap \{G = 0\} \cap \{F = 0\}$, plus the two base points of the pencil $|K_X|$. Up on $\mathbb{P}E$ this distinction is not readily apparent.

Assuming the equality in the above proposition and denoting by $\mathrm{aut}(\mathbb{P}E)$ the Lie algebra of $\mathrm{Aut}(\mathbb{P}E)$, we have an identification

$$T_X\mathcal{M}_H \cong \frac{H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J})}{\begin{pmatrix} \text{subspace given by} \\ \text{the action of } \mathrm{Aut}(\mathbb{P}E) \\ \text{on } R \in H^0(\mathbb{P}E, \xi^4) \end{pmatrix}} \cong \frac{H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J})}{\mathrm{aut}(\mathbb{P}E)}.$$

To justify the first identification we note that any vector field V on $\mathbb{P}E$ preserves $\mathrm{Pic}(\mathbb{P}E)$ and therefore acts on $\xi = \mathcal{O}_{\mathbb{P}E}(1)$ and $\mathfrak{h} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$.

Assuming that action satisfies

$$V \cdot R = 0, \quad V \cdot \mathcal{J} \subseteq \mathcal{J}$$

the vector field induces one on X. This is because V induces a vector field on $X^\flat$, which in turn induces one on the normalization $\hat{X}$ of $X^\flat$. On $\hat{X}$ the two -1 curves E_1, E_2 are preserved, and therefore V induces a vector field on X. Since X is of general type, $H^0(\Theta_X) = 0$ and this justifies the first indentification. The second is clear. The crucial computation is given by the

Proposition: *$h^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J}) = 40$.*

Proof. $\{LG, L^2, F^\bullet\}$ is not a regular sequence, but it does have a relatively simple free resolution (not a Koszul resolution, of course). We may describe it by the table

	LG	L^2	F
rel_1	L	$-G$	0
rel_2	F	0	$-LG$
rel_3	0	F	$-L^2$

meaning that each row is a relation among the generators at the top, these relations generate the module of relations, and there is the single generating syzygy

$$F\ \mathrm{rel}_1 - L\ \mathrm{rel}_2 + G\ \mathrm{rel}_3 = 0.$$

Pictorially, there is a resolution

$$0 \to \xi^{-6} \xrightarrow{\begin{pmatrix} F \\ -L \\ G \end{pmatrix}} \begin{matrix} \xi^{-4} \\ \oplus \\ \xi^{-6} \otimes \mathfrak{h} \\ \oplus \\ \xi^{-1} \otimes \mathfrak{h}^{-2} \end{matrix} \xrightarrow{\begin{pmatrix} L & F & 0 \\ -G & 0 & F \\ 0 & -LG & -L^2 \end{pmatrix}} \begin{matrix} \xi^{-4} \otimes \mathfrak{h} \\ \oplus \\ \mathfrak{h}^{-2} \\ \oplus \\ \xi^{-2} \end{matrix} \xrightarrow{(LG, L^2, F)} \begin{matrix} \mathcal{J} \\ \cap \\ \mathcal{O}_{\mathbb{P}E} \end{matrix} \to 0.$$

Here the matrices go the wrong way for multiplication; it is nicer to write

$$0 \leftarrow \mathfrak{J} \xleftarrow{(LG, L^2, F)} \begin{matrix} \xi^{-4}\otimes\mathfrak{h} \\ \oplus \\ \mathfrak{h}^{-2} \\ \oplus \\ \xi^{-2} \end{matrix} \xleftarrow{\begin{pmatrix} L & F & 0 \\ -G & 0 & F \\ 0 & -LG & -L^2 \end{pmatrix}} \begin{matrix} \xi^{-4} \\ \oplus \\ \xi^{-6}\otimes\mathfrak{h} \\ \oplus \\ \xi^{-1}\otimes\mathfrak{h}^{-2} \end{matrix} \xleftarrow{\begin{pmatrix} F \\ -L \\ G \end{pmatrix}} \xi^{-6} \leftarrow 0.$$

We now tensor the resolution of $\mathfrak{J}$ with ξ^4. Then using the cohomological computations in §I.C one may verify that the hypercohomology spectral sequence degenerates and

$$\begin{aligned} h^0(\xi^4\otimes\mathfrak{J}) &= h^0(\xi^2) + h^0(\xi^4\otimes\mathfrak{h}^{-2}) + h^0(\mathfrak{h}) - h^0(\xi^2\otimes\mathfrak{h}^{-2}) - h^0(\mathcal{O}_{\mathbb{P}E}) \\ &= 14 + 30 + 2 - 5 - 1 = 40. \end{aligned}$$ □

Proposition: $\dim \operatorname{aut}(\mathbb{P}E) = 13$.

Proof. This follows from the remark "Some further properties of $\mathbb{P}E$" at the end of the section in §I.C. From there we have

$$h^0(\operatorname{End}(E)) = 5h^0(\mathcal{O}_{\mathbb{P}^1}) + 2h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 11.$$

We shall use this together with an alternate argument that will be used in the next section to complete the proof of the earlier statement that $h^0(\mathbb{P}E, \xi^4\otimes\mathfrak{J}) = 40$.

The relative tangent bundle sequence of $\mathbb{P}E \to \mathbb{P}^1$ is

$$0 \to \Theta_{\mathbb{P}E/\mathbb{P}^1} \to \Theta_{\mathbb{P}E} \to \pi^*\Theta_{\mathbb{P}^1} \to 0$$

where $\Theta_{\mathbb{P}E/\mathbb{P}^1}$ are the vertical vector fields. From the Leray spectral sequence and

$$R^q_\pi(\pi^*\Theta_{\mathbb{P}^1}) = R^q_\pi\mathcal{O}_{\mathbb{P}E}\otimes\Theta_{\mathbb{P}^1} = 0 \text{ for } q > 0$$

we have

$$H^0(\pi^*\Theta_{\mathbb{P}^1}) \cong H^0(\Theta_{\mathbb{P}^1}).$$

Since $E \to \mathbb{P}^1$ is a homogeneous vector bundle, $H^0(\Theta_{\mathbb{P}^1})$ lifts to $\operatorname{aut}(\mathbb{P}E)$ giving

$$h^0(\Theta_{\mathbb{P}E}) = h^0(\Theta_{\mathbb{P}E/\mathbb{P}^1}) + h^0(\Theta_{\mathbb{P}^1}) = h^0(\Theta_{\mathbb{P}E/\mathbb{P}^1}) + 3.$$

We next use the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}E} \to \pi^* E^* \otimes \xi \to \Theta_{\mathbb{P}E/\mathbb{P}^1} \to 0$$

to infer that

$$\begin{aligned} h^0(\Theta_{\mathbb{P}E/\mathbb{P}^1}) &= h^0(\pi^* E^* \otimes \xi) - h^0(\mathcal{O}_{\mathbb{P}E}) \\ &= h^0(E^* \otimes E) - 1 = 10. \end{aligned}$$

□

Returning to the proof of the theorem, the two propositions give

$$\begin{aligned} T_X \mathcal{M}_H &= \frac{H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J})/\mathbb{C}}{\mathrm{aut}(\mathbb{P}E)} \\ \dim T_X \mathcal{M}_H &= h^0(\xi^4 \otimes \mathcal{J}) - 1 - h^0(\mathrm{aut}(\mathbb{P}E)) \\ &= 40 - 1 - 13 = 26. \end{aligned}$$

It remains to show that

$$h^1(\Theta_X) = 26.$$

This is proved in [Ho79], and an argument given in §I.G below. An almost precise argument is given by using the Riemann-Roch theorem and $h^0(\Theta_X) = 0$ to show that

$$\begin{aligned} h^1(\Theta_X) &= -\chi(\Theta_X) + h^2(\Theta_X) \\ &= 26 + h^2(\Theta_X) \end{aligned}$$

which gives $h^1(\Theta_X) \leqq 26$. If we know that the map

$$T_X \mathcal{M}_H \to H^1(\Theta_X)$$

is injective, then we are done. This will be verified in the next section.

I.G. **Generic local Torelli theorems for $H^\#$ and H-surfaces.** This section will be in three parts:

(i) Generalities on the computation of $H^1(\Omega^1_Y)_{\mathrm{prim}}$ for a surface Y with an ample line bundle $L \to Y$; introduction and use of the Atiyah class.
(ii) Generic local Torelli for $H^\#$-surfaces.
(iii) Generic local Torelli for H-surfaces.

The structure of the discussion is: (i) will give a general formalism for the computation of the differential of the period mapping, and (ii), (iii) will respectively verify the injectivity of the differential for a Fermat $H^{\#}$-surface and Fermat-like H-surface, both of these to be defined below.

(i) *Generalities on the computation of $H^1(\Omega^1_Y)_{\mathrm{prim}}$ when we have an ample line bundle $L \to Y$ over a smooth surface Y; introduction and use of the Atiyah class.*

Let Y be a compact, complex manifold and $L \to Y$ a holomorphic line bundle.

Definition: $\Sigma_{Y,L}$ is the sheaf of $\mathbb{C}$-linear, 1^{st}-order differential operators $\mathcal{O}_Y(L) \to \mathcal{O}_Y(L)$.

Later on we shall use the obvious extension of this to vector bundles of arbitrary rank. We note the identifications

- $H^0(\Sigma_{Y,L}) =$ Lie algebra of the automorphisms of $L \to Y$;
- $H^1(\Sigma_{Y,L}) = T\ (\mathrm{Def}(Y,L))$.

The $\Sigma_{Y,L}$'s and their extension to vector bundles arose in the original work of Kodaira-Spencer.

Note: The theorem of Birkhoff-Grothendieck states that any holomorphic vector bundle $F \to \mathbb{P}^1$ is uniquely a direct sum $F = \oplus \mathcal{O}_{\mathbb{P}^1}(k_i)$ where $k_1 \geqq k_2 \geqq \cdots$. The corresponding filtration given by lumping together the $\mathcal{O}_{\mathbb{P}^1}(k_i)$ terms with equal k_i's is preserved by $\mathrm{Aut}(F)$.

Proposition: *For Y a surface and $L \to Y$ ample, there is a natural identification*

$$H^1(\Omega^1_Y)_{\mathrm{prim}} \cong H^1(\Sigma_{Y,L} \otimes K_Y).$$

Proof. We have

$$0 \to \mathcal{O}_Y \to \Sigma_{Y,L} \to \Theta_Y \to 0$$

with extension class, or intrinsic curvature, $\lambda = c_1(L) \in H^1(\Omega^1_Y)$. We also have canonically

$$H^1(\Sigma_{Y,L} \otimes K_Y) \cong H^1(\Sigma^*_{Y,L})^*.$$

Dualizing the above exact sheaf sequence and taking cohomology gives

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(\mathcal{O}_Y) & \longrightarrow & H^1(\Omega^1_Y) & \longrightarrow H^1(\Sigma^*_{Y,L}) \longrightarrow H^1(\mathcal{O}_Y) \xrightarrow{\sim} H^2(\Omega^1_Y) \\ & \cup & & \cup & \\ & 1 & \longrightarrow & \lambda & \end{array}$$

where the isomorphism on the right results from "hard Lefschetz." Thus

$$\begin{array}{c} H^1(\Omega^1_Y)/\mathbb{C}\lambda \cong H^1(\Sigma^*_{Y,L}) \\ \wr\| \\ H^1(\Omega^1_Y)^*_{\text{prim}} \end{array}$$

from which the proposition follows. □

We note the result remains true under the assumption

$$H^1(\mathcal{O}_Y) = 0,$$

which will be the case for $H^\#$ and H-surfaces.

It is natural that $\Sigma^*_{Y,L}$ in

$$0 \to \Omega^1_Y \to \Sigma^*_{Y,L} \to \mathcal{O}_Y \to 0$$

should enter into $H^1(\Omega^1)_{\text{prim}}$, since the very definition of primitive involves $c_1(L)$.

We now turn to the second topic. The above proposition and more complex related computations will be used first when $Y = X^\# \subset \mathbb{P}E$ and $L = \mathcal{O}_{X^\#}(\xi)$, and then later for $g(X) \subset \mathbb{P}E$. Then L is not very ample and classical arguments based on vanishing theorems will not be applicable. Rather than having vanishing of the relevant cohomology groups, what we will need is these together with the vanishing of *the connecting map is zero in certain cohomology sequences*. This is a more subtle issue than just making L sufficiently ample to get vanishing. These connecting maps will occur on $X^\#$ and on $\mathbb{P}E$, and for this we will need to push the computations down to $\mathbb{P}^1$ using $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. The technique necessary to carry this out leads to the Atiyah class $a(E)$, which for a vector bundle

$$E \to Z$$

over a complex manifold Z is defined as follows: Identifying vector bundles with locally free sheaves and denoting by $J^1(E)$ the bundle of 1-jets $j^1(s)$ of sections of $E \to Z$, we have

$$0 \to E \otimes \Omega^1_Z \to J^1(E) \to E \to 0.$$

The second map here is the involution

$$j^1(s) \to s(z), \qquad z \in Z$$

while if $s(z) = 0$ then the first order term

$$j^1(s)_z \in E_z \otimes T^*_z Z$$

is well defined and this leads to the first map. The extension class of this sequence is by definition the *Atiyah class*

$$a(E) \in H^1(\Omega^1_Y \otimes \operatorname{End}(E)).$$

Using $a(E)$ as an extension class we may construct the exact sequence

$$0 \to \operatorname{End}(E) \to \Sigma_{Z,E} \to \Theta_Z \to 0,$$

which for line bundles reduces to the one in the 1st bullet in the proof of the above proposition. Also, the dimension count $h^0(\operatorname{aut}(\mathbb{P}E)) = 14$ above is then a result of the following

PROPOSITION: *For $Z = \mathbb{P}^1$ and for any holomorphic vector bundle $E \to \mathbb{P}^1$ we have*

$$0 \to H^0(\pi^* E^* \otimes \xi) \to H^0(\Sigma_{\mathbb{P}E}) \to H^0(\pi^* \Theta_{\mathbb{P}^1}) \to 0.$$

Proof. We first note that we have the exact sequence

$$0 \to \pi^* E^* \otimes \xi \to \Sigma_{\mathbb{P}E,\xi} \to \pi^* \Theta_{\mathbb{P}^1} \to 0.$$

The right-hand map is the composition of

$$\Sigma_{\mathbb{P}E,\xi} \to \Theta_{\mathbb{P}E} \to \pi^* \Theta_{\mathbb{P}^1} \to 0.$$

The kernel may then be identified with $\pi^* E \otimes \xi$ by the previous Euler sequence argument.

Using the Leray spectral sequence, the connecting map in the exact cohomology sequence of the above exact sheaf sequence is

$$\begin{array}{ccc} H^0(\mathbb{P}E, \pi^*\Theta_{\mathbb{P}^1}) & \xrightarrow{\delta} & H^1(\mathbb{P}E, \pi^*E^* \otimes \xi) \\ \wr\| & & \wr\| \\ H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1}) & \xrightarrow{a(E)} & H^1(\mathbb{P}^1, \operatorname{End} E) \end{array}$$

where the bottom arrow is cup product with the Atiyah class. This map is in turn the connecting map in the exact cohomology sequence associated to

$$0 \to \operatorname{End}(E) \to \Sigma_{\mathbb{P}^1,E} \to \Theta_{\mathbb{P}^1} \to 0.$$

What we are saying is that even though this sequence does not split (far from it, as $a(E) \neq 0$) on the H^0-level it is exact. The computation, which serves as a model for the more intricate ones later, goes as follows:

Since $E \cong \oplus \mathcal{O}_{\mathbb{P}^1}(k_i)$ is a direct sum of line bundles, the Atiyah class

$$a(E) = \bigoplus_i a(\mathcal{O}_{\mathbb{P}^1}(k_i))$$

is also a direct sum of Atiyah classes of the $\mathcal{O}_{\mathbb{P}^1}(k_i)$. Therefore, under the image of $a(E)$ in $H^1(\mathbb{P}^1, \operatorname{End} E)$ there are no cross-terms in the terms $H^1(\mathbb{P}^1, \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(k_i), \mathcal{O}_{\mathbb{P}^1}(k_j))) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k_j - k_i))$ $(i \neq j)$ that appear. In other words, the potentially non-zero terms in $H^1(\mathbb{P}^1, \operatorname{End} E)$ are not in the image of $a(E)$, and thus $\delta = 0$.

Note: The non-zero terms in $H^1(\mathbb{P}^1, \operatorname{End} E)$ inject into $H^1(\mathbb{P}^1, \Sigma_{\mathbb{P}^1,E}) = T_E \operatorname{Def}(E)$. Such a deformation arises, e.g., when $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. The non-zero term in $H^1(\mathbb{P}^1, \operatorname{End} E)$ reflects deforming the extension class to be non-zero, which deforms $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. This is the well-known "jump phenomenon," where one has a family E_t over the disc where all the E_t are isomorphic for $t \neq 0$ but the structure jumps at $t = 0$ (e.g., $\lim \mathbb{F}_2 = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$).

I.H. **The generic local Torelli theorem for $H^\#$-surfaces.** An $H^\#$-surface is given by an equation $\mathbb{R}^\# = 0$ where $R^\# \in H^0(\mathbb{P}E, \xi^4)$. There is a natural map $\Sigma_{\mathbb{P}E,\xi} \otimes \xi^4 \to \xi^4$ which induces a map

$$H^0(\mathbb{P}E, \Sigma_{\mathbb{P}E,\xi}) \xrightarrow{dS^\#} H^0(\mathbb{P}E, \xi^4).$$

We denote by $\mathcal{M}_{(H^\#,\mathbb{P}E)}$ the space of smooth $H^\#$-surfaces $X^\# \subset \mathbb{P}E$, modulo the action of $\mathrm{Aut}(\mathbb{P}E)$. As will be explained below, the natural map $\mathcal{M}_{(H^\#,\mathbb{P}E)} \to \mathcal{M}_{H^\#}$ has image a *proper* subvariety, reflecting the fact that a deformation of an $H^\#$-surface may not remain an $H^\#$-surface.

PROPOSITION: *We have the identifications*

(i) $T_{X^\#}\mathcal{M}_{(H^\#,\mathbb{P}E)} \cong \dfrac{H^0(\mathbb{P}E,\xi^4)}{H^0(\mathbb{P}E,\Sigma_{\mathbb{P}E,\xi})}$;

(ii) $H^0(\Omega^2_{X^\#}) \cong H^0(\mathbb{P}E,\xi)$;

and the inclusion

(iii) $H^1(\Omega^1_{X^\#})_{\mathrm{prim}} \supset \dfrac{H^0(\mathbb{P}E,\xi^5)}{\mathrm{Im}\{H^0(\mathbb{P}E,\Sigma_{\mathbb{P}E,\xi}\otimes\xi) \xrightarrow{dS^*} H^0(\mathbb{P}E,\xi^5)\}}$.

Before giving the proof we want to discuss what is behind the proposition.

Discussion: Suppose we are given the situation

- Z is a smooth $(n+1)$-dimensional compact, complex manifold and $L \to Z$ is a holomorphic line bundle with $h^1(\mathcal{O}_Z)=0$;
- $Y \in |L|$ is a smooth hypersurface defined by $s \in H^0(Z,L)$ and where $h^1(\mathcal{O}_Y)=0$.

We may define

$$\mathcal{M}_{(Y,Z)} = \frac{\{\text{space of divisors of smooth sections of } L \to Z\}}{\{\text{automorphisms of } L \to Z\}}$$

where the denominator is the image of $H^0(\Sigma_{Z,L})$ under the map induced by $\Sigma_{Z,L} \xrightarrow{\lrcorner ds} L$ where $Y=(s)$. Thus $\mathcal{M}_{(Y,Z)}$ has as tangent space

$$T_Y\mathcal{M}_{(Y,Z)} = \frac{H^0(Z,L)}{\mathrm{Im}\{H^0(\Sigma_{Z,L}) \to H^0(Z,L)\}}.$$

With the case where $\dim Y = 2$ in mind, we are interested in how the subspace $H^0(\Omega^n_Y)$ varies in $H^n(Y,\mathbb{C})$. From the exact cohomology sequence of

$$0 \to \Omega^{n+1}_Z \to \Omega^{n+1}_Z(L) \xrightarrow{\mathrm{Res}} \Omega^n_Y \to 0$$

we have

$$0 \to H^0(\Omega^{n+1}_Z) \to H^0(\Omega^{n+1}_Z(Y)) \xrightarrow{\mathrm{Res}} H^0(\Omega^n_Y) \to H^1(\Omega^{n+1}_Z) \to \cdots.$$

The cokernel of Res is a piece of the fixed part of the VHS of $H^n(Y,\mathbb{C})$ as Y varies in $\mathcal{M}_{(Y,Z)}$. Therefore, we are interested in the image of

$$\mathrm{Res} : H^0(\Omega_Z^{n+1}(Y)) \to H^0(\Omega_Y^n),$$

i.e., the VHS arising from residues of forms with simple poles along Y. A natural question is

What is the differential of this part of the period map?

Cohomologically the natural pairing is

$$\frac{H^0(Z,L)}{\mathrm{Im}\{H^0(\Sigma_{Z,L}) \to H^0(Z,L)\}} \otimes H^0(Z, K_Z \otimes L) \to \frac{H^0(Z, K_Z \otimes L^2)}{\mathrm{Im}\{H^0(\Sigma_{Z,L} \otimes L) \to H^0(K_Z \otimes L)\}}.$$

Thus we expect a diagram

$$\begin{array}{ccc} \frac{H^0(Z,L)}{H^0(\Sigma_{Z,L}) \to H^0(Z,L)} \otimes H^0(K_Z \otimes L) & \longrightarrow & \frac{H^0(K_Z \otimes L^2)}{H^0(\Sigma_{Z,L} \otimes L) \to H^0(K_Z \otimes L^2)} \\ \downarrow{\scriptstyle \rho \otimes \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}_2} \\ H^1(\Theta_Y) \otimes H^0(\Omega_Y^n) & \longrightarrow & H^1(\Omega_Y^{n-1}) \end{array}$$

where

$$\rho : \frac{H^0(Z,L)}{H^0(\Sigma_{Z,L}) \to H^0(Z,L)} \to H^1(\Theta_Y)$$

is the Kodaira-Spencer map, and where the map Res_2 takes an $(n+1)$-form with a 2^{nd} order pole along Y to a class in $H^1(\Omega_Y^{n-1})$. The differential of that part of the period mapping we are interested in will be injective if the three conditions

(i) ρ is injective;
(ii) Res_2 is injective;
(iii) the mapping

$$\frac{H^0(Z,L)}{\mathrm{Im}\{H^0(\Sigma_{Z,L}) \to H^0(Z,L)\}} \otimes H^0(K_Z, L) \to \frac{H^0(K_Z \otimes L^2)}{\mathrm{Im}\{H^0(\Sigma_{Z,L} \otimes L) \to H^0(K_Z \otimes L^2)\}}$$

is non-degenerate in the first factor

are satisfied.

Here we shall verify (i) and (ii) for smooth $H^{\#}$-surfaces and shall check (iii) for a particular "Fermat" one.

In the next sub-section we shall extend the above discussion to the case of H-surfaces, the point here being that we have a normalization map

$$Y \to Y^\flat \in (L)$$

where $Y^\flat$ will have a double curve with pinch points.

To do (i) for $H^{\#}$-surfaces we shall use the following general

Lemma: *The mapping ρ is injective if*

$$H^1(\Sigma_{Z,L} \otimes L^{-1}) = 0.$$

Proof. We shall use the exact sequence

$$0 \to \mathcal{O}_Y \to \Sigma_{Y,L} \to \Theta_Y \to 0$$

and the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Sigma_{Y,L} & \longrightarrow & \Sigma_{Z,L}\big|_Y & \xrightarrow{\lrcorner ds} & \mathcal{O}_Y(L) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Sigma_{Z,L} & \longrightarrow & \mathcal{O}_Z(L) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Sigma_{Z,L} \otimes L^{-1} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the cohomology sequence of the first exact sequence and $h^1(\mathcal{O}_Y) = 0$ we have

$$\text{(a)} \qquad H^1(\Sigma_{Y,L}) \hookrightarrow H^1(\Theta_Y).$$

From the cohomology sequence of the top row in the diagram we obtain

$$\text{(b)} \qquad \frac{H^0(Y,L)}{H^0\big(\Sigma_{Z,L}\big|_Y\big)} \hookrightarrow H^1(\Sigma_{Y,L}).$$

Again from the cohomology of the diagram and $h^1(\mathcal{O}_Z) = 0$ we have

$$\text{(c)} \qquad \frac{H^0(Z,L)}{H^0(\Sigma_{Z,L})} \hookrightarrow \frac{H^0(Y,L)}{H^0\big(\Sigma_{Z,L}\big|_Y\big)}.$$

Combining (a), (b), (c) gives the injectivity of ρ. □

Application to $H^{\#}$-surfaces: Taking $Z = \mathbb{P}E$ and $Y = X^{\#} \in |\xi^4|$ we have

$$H^1(\Sigma_Z \otimes L^{-1}) = H^1(\Sigma_{\mathbb{P}E} \otimes \xi^{-4}).$$

From the Euler sequence

$$0 \to E^* \otimes \xi^{-3} \to \Sigma_{\mathbb{P}E} \otimes \xi^{-4} \to \mathfrak{h}^2 \otimes \xi^{-4} \to 0,$$

noting that the restriction of ξ to a fibre of $\mathbb{P}E \to \mathbb{P}^1$ is $\mathcal{O}(1)$ we have

$$\begin{cases} R^q_\pi E^* \otimes \xi^{-3} = 0 \\ R^q_\pi \mathfrak{h} \otimes \xi^{-4} = 0 \end{cases} \quad \text{for } q = 0, 1.$$

The desired vanishing then follows from the Leray spectral sequence.

Turning to (iii) above, we have

$$0 \to \Sigma_{X^{\#},\xi} \otimes K_{X^{\#}} \to \Sigma_{\mathbb{P}E,\xi}\big|_{X^{\#}} \otimes K_{X^{\#}} \to \xi \otimes K_{X^{\#}} \to 0$$

which using $K_{X^{\#}} = \xi$ gives

$$\frac{H^0(X^{\#}, \xi^5)}{H^0(X^{\#}, \Sigma_{\mathbb{P}E} \otimes \xi)} \hookrightarrow H^1(\Omega^1_{X^{\#}})_{\text{prim}}.$$

Next we use the cohomology sequences associated to

$$0 \longrightarrow \Sigma_{\mathbb{P}E,\xi} \otimes \xi^{-3} \longrightarrow \Sigma_{\mathbb{P}E} \otimes \xi \longrightarrow \Sigma_{\mathbb{P}E}\big|_{X^{\#}} \otimes \xi \longrightarrow 0$$
$$0 \longrightarrow E^* \otimes \xi^{-2} \longrightarrow \Sigma_{\mathbb{P}E} \otimes \xi^{-3} \longrightarrow \xi^{-3} \longrightarrow 0$$

to conclude that we have

$$\frac{H^0(\mathbb{P}E, \xi^5)}{H^0(\mathbb{P}E, \Sigma_{\mathbb{P}E} \otimes R)} \hookrightarrow \frac{H^0(X^{\#}, \xi^5)}{H^0(X^{\#}, \Sigma_{\mathbb{P}E} \otimes \xi)}$$

which implies (iii).

An important point is that in all of these, the RHS's take place on $\mathbb{P}E$.

Proof.

Using the proposition, the differential of the period map is

$$\frac{H^0(\xi^4)}{H^0(\Sigma_{\mathbb{P}E,\xi})} \to \operatorname{Hom}\left(H^0(\xi), \frac{H^0(\xi^5)}{\operatorname{Im}\{H^0(\Sigma_{\mathbb{P}E,\xi} \otimes \xi) \to H^0(\xi^5)\}} \right)$$

where all of the cohomology groups take place on $\mathbb{P}E$.

THEOREM: *This map is injective for a general $H^{\#}$-surface.*

Proof. It will suffice to choose one $S^{\#} \in |\xi^4|$ and show that

- $S^{\#} = 0$ defines an $H^{\#}$-surface $X^{\#}$ (i.e., $X^{\#}$ is smooth);
- the above map is injective for $S^{\#}$.

Here, for notational convenience we shall use $xt_0^2, xt_0t_1, xt_1^2, r_0, r_1$ as a basis for $H^0(\mathbb{P}E, \xi)$; i.e., we replace x_3, x_4 with r_0, r_1. For $X^{\#}$ we take the *Fermat $H^{\#}$-surface* where

$$S^{\#} = x^4(t_0^8 + t_1^8) + r_0^4 + r_1^4.$$

We begin by showing that

$$X^{\#} \textit{ is smooth.}$$

For $A \in |\xi^m|$ we shall use the *weighted Euler's formula*

$$mA = \frac{1}{2}\left(t_0\frac{\partial A}{\partial t_0} + t_1\frac{\partial A}{\partial t_1}\right) + r_0\frac{\partial A}{\partial r_0} + r_1\frac{\partial A}{\partial r_1}$$

then

$$\{A = 0\} \text{ is smooth} \Leftrightarrow \left\{\frac{\partial A}{\partial t_0} = \frac{\partial A}{\partial t_1} = \frac{\partial A}{\partial r_0} = \frac{\partial A}{\partial r_1} = 0\right\} = \emptyset.$$

Away from $x = 0$ we can solve for $\partial A/\partial x$ in terms of $\partial A/\partial t_0$, $\partial A/\partial t_1$. Now $\{x = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1$ where the homogeneous coordinates are $[t_0, t_1]$ and $[r_0, r_1]$. Thus local coordinates on $\mathbb{P}E$ are $x, [t_0, t_1], [r_0, r_1]$.

We have

$$\mathfrak{h} \cdot \mathfrak{h} = 0 \Rightarrow \{t_0 = t_1 = 0\} = \emptyset$$
$$(\xi - \partial\mathfrak{h}) \cdot \xi \cdot \xi = \xi^2 - 2\xi^2\mathfrak{h} = 0 \Rightarrow \{x = r_0 = r_1 = 0\} = \emptyset.$$

For the $S^{\#}$ above

$$\frac{\partial S^{\#}}{\partial r_0} = \frac{\partial S^{\#}}{\partial r_1} = 0 \Rightarrow r_0 = r_1 = 0 \Rightarrow x \neq 0, \text{ which gives}$$
$$\frac{\partial S^{\#}}{\partial t_0} = \frac{\partial S^{\#}}{\partial t_1} = 0 \Rightarrow t_0 = t_1 = 0, \text{ which can't happen.}$$ ∎

Proof that the differential of the period map is injective.

Definition: $J^{\#}$ = *Jacobian ideal generated by the partials of $S^{\#}$.*

Recall the standard notation: For A = weighted homogeneous form

$$[J^{\#} : A] = \{B : AB \in J^{\#}\}.$$

The injectivity of the differential of the period map is equivalent to

$$\{A \in H^0(\xi^4) : A \cdot H^0(\xi)\} \subseteq \operatorname{Im}\{H^0(\Sigma_{\mathbb{P}E} \otimes \xi) \xrightarrow{dS^\#} H^0(\xi^5)\}$$
$$\text{is equal to } H^0(\Sigma_{\mathbb{P}E}) \cap H^0(\xi^4).$$

The LHS of the above inclusion is

$$\{A : xt_0^2A, xt_0t_1A, xt_1^2A, r_0A, r_1A\} \in J^\#.$$

Since

$$[J^\# : r_0, r_1, xt_0^2, xt_0t_1, xt_1^2] = [J^\# : r_0] \cap [J^\# : r_1] \cap [J^\# : xt_0^2] \cap [J^\# : t_0t_1] \cap [J^\# : t_1^2]$$

we have to show that

The LHS of the expression just above is equal to $J^\#$.

In other words, the injectivity of the period mapping becomes translated into a statement about divisibility of ideals in a weighted homogeneous coordinate ring.

We will compute the RHS of the above expression. For this

$$[J^\# : r_0] = \{x^3(t_0^8 + t_1^8), x^4t_0^7, x^4t_1^7, r_0^3, r_1^3]$$
$$[J^\# : r_1] = \{x^3(t_0^8 + t_1^8), x^4t_0^7, x^4x_1^2, r_0^3, r_1^2\}$$
$$[J^\# : r_0] \cap [J^\# : r_1] = \{x^3(t_0^8 + t_1^8), x^4t_0^7, x^4t_1^7, r_0^3, r_0^2, r_1^2, r_1^3$$
$$[J^\# : xt_0^2] = \{\cancel{x^3(t_0^2 + t_1^8)}, x^3t_0^5, x^3t_1^7, r_0^3, r_0^3\}$$
$$[J^\# : xt_0t_2] = \{\cancel{x^3(t_0^8 + t_1^8)}, x^3t_0^8, x^3t_1^6, r_0^3, r_1^3\}$$
$$[J^\# : xt_1^2] = \{\cancel{x^3(t_0^8 + t_1^8)}, x^3t_0^7, x^3t_1^5, r_0^3, r_1^3\}.$$

In $|\xi^4|$

$$[J^\# : xt_0^2] \cap [J^\# : xt_0t_1] \cap [J^\# : xt_1^2] = \{x^3t_0^7, x^3t_0^7, r_0^3, r_1^3\}.$$

The intersection of this with $[J^\# : r_1] \cap [J^\# : r_0]$ is in $J^\#$. □

(iii) *Local Torelli for a generic H-surface*

THEOREM: (a) *With the understanding that all of the following cohomology groups are computed on $\mathbb{P}E$, for X a smooth H-surface, we have*

(i) $\dfrac{H^0(\xi\otimes\mathcal{J})}{H^0(\Sigma_{\mathbb{P}E,\xi})}\hookrightarrow H^1(\Theta_X)\cong T_X\mathcal{M}_H$;

(ii) $H^0(\Omega^2_X)\cong H^0(\xi\otimes\mathfrak{h}^{-1})$;

(iii) $H^1(\Omega^1_X)_{\mathrm{prim}}\cong\dfrac{H^0(\xi^5\otimes\mathfrak{h}^{-1}\otimes\mathcal{J})}{H^0(\Sigma_{\mathbb{P}E,\xi})}\supseteq H^1(\Omega^1_X)_{\mathrm{prim}}$.

(b) *For a generic X, the differential of the period mapping*

$$T_X\mathcal{M}_H\to\mathrm{Hom}\left(H^0(\xi\otimes\mathfrak{h}^{-1}),\frac{H^0(\xi^5\otimes\mathfrak{h}^{-1}\otimes\mathcal{J})}{H^0(\Sigma_{\mathbb{P}E,\xi})}\right)$$

is injective.

Proof of (a). The argument will parallel that for $H^{\#}$-surfaces: we shall verify (i), (ii), (iii) in (a) for a general smooth H-surface X, and then shall check (b) for a particular "Fermat-like" X.

Proof of (i) *in* (a). We begin with the well-known

LEMMA: *Let $\hat{Y}$ be a smooth surface with $h^1(\mathcal{O}_{\hat{Y}})=0$ and $E\subset\hat{Y}$ a -1 curve that contracts to a point p on a smooth surface Y. Then under any deformation of $\hat{Y}$ the curve E deforms to -1 curves. Thus the natural map*

$$H^1(\Theta_{\hat{Y}})\to H^1(\Theta_Y)$$

is an isomorphism.

Proof (sketch). We first show that the class $\varphi_E=:[E]\in H^1(\Omega^1_{\hat{Y}})$ satisfies

$$\theta\cdot\varphi_E=0\ \text{ in }\ H^2(\mathcal{O}_{\hat{Y}}),\quad\theta\in H^1(\Theta_{\hat{Y}}).$$

This is equivalent to

$$\langle\theta\cdot\hat{\omega},\varphi_E\rangle=0\ \text{ for all }\ \hat{\omega}\in H^0(\Omega^2_{\hat{Y}}).$$

The point is that as a current

$$\varphi_E=\int_E,$$

while $\hat{\omega}$ is the pullback of a unique $\omega\in H^0(\Omega^2_Y)$ which gives that the divisor

$$(\hat{\omega})=E+(\text{proper transform of }(\omega)).$$

This implies that $\int_E\theta\cdot\hat{\omega}=0$.

We next note that the section of the line bundle $[E]$ that defines E moves with the line bundle. This follows from

$$H^1(\hat{Y}, [E]) \cong H^1(Y, I_p) = 0,$$

the second equality resulting from the cohomology sequence of $0 \to I_p \to \mathcal{O}_Y \to \mathbb{C}_p \to 0$ and $h^1(\mathcal{O}_Y) = 0$. $\square$

As a consequence of the lemma, it will suffice to prove (i) in (a) for the blow up $\hat{X}$ of X at the base points of $|K_X|$. With our usual notation we have

$$0 \to \Sigma_{\hat{X},\xi} \to g^*\Sigma_{\mathbb{P}E,\xi} \to g^*(\xi^4 \otimes \mathcal{J}) \to 0,$$

which together with $H^1(\hat{X}, \Sigma_{\hat{X},\xi}) \xrightarrow{\sim} H^1(\Theta_{\hat{X}})$ gives an inclusion

$$\frac{H^0(\hat{X}, g^*(\xi^4 \otimes \mathcal{J}))}{H^0(\hat{X}, g^*\Sigma_{\mathbb{P}E,\xi})} \hookrightarrow H^1(\Theta_{\hat{X}}).$$

From

$$0 \to \Sigma_{\mathbb{P}E,\xi} \otimes \xi^{-4} \to \Sigma_{\mathbb{P}E,\xi} \xrightarrow{g^*} g^*\Sigma_{\mathbb{P}E,\xi} \to 0$$

and, as previously calculated, $H^1(\mathbb{P}E, \Sigma_{\mathbb{P}E,\xi} \otimes \xi^{-4}) = 0$ we obtain

$$\frac{H^0(\mathbb{P}E, \xi^4 \otimes \mathcal{J}))}{H^0(\mathbb{P}E, \Sigma_{\mathbb{P}E,\xi})} \hookrightarrow \frac{H^0(\hat{X}, g^*(\xi^4 \otimes \mathcal{J}))}{H^0(\hat{X}, g^*\Sigma_{\mathbb{P}E,\xi})}.$$

Combining the two inclusions gives (i).

We have already noted (ii), and for (iii) we use

$$0 \to \Sigma_{\hat{X},\xi} \otimes K_{\hat{X}} \to g^*(\Sigma_{\mathbb{P}E} \otimes \xi \otimes \mathfrak{h}^{-1}) \to g^*(\xi^5 \otimes \mathfrak{h}^{-1} \otimes \mathcal{J}) \to 0$$

and

$$0 \to K_{\hat{X}} \to \Sigma_{\hat{X},\xi} \otimes K_{\hat{X}} \to \Omega^1_{\hat{X}} \to 0$$

to give

$$\frac{H^0(\hat{X}, g^*(\xi^5 \otimes \mathfrak{h}^{-1} \otimes \mathcal{J}))}{H^0(\hat{X}, g^*(\Sigma_{\mathbb{P}E,\xi} \otimes \xi \otimes \mathfrak{h}^{-1}))} \hookrightarrow H^1(\Omega^1_{\hat{X}})_{\mathrm{prim}}$$

where $H^1(\Omega^1_{\hat{X}})_{\mathrm{prim}} = \xi^{\perp} \subset H^1(\Omega^1_{\hat{X}})$.

Next we use the cohomology sequence of

$$0 \to \Sigma_{\mathbb{P}E,\xi} \otimes \xi^{-3} \otimes \mathfrak{h}^{-1} \to \Sigma_{\mathbb{P}E,\xi} \otimes \xi \otimes \mathfrak{h}^{-1} \to g^*(\Sigma_{\mathbb{P}E,\xi} \otimes \xi \otimes \mathfrak{h}^{-1}) \to 0$$

and

$$0 \to E^* \otimes \xi^{-2} \otimes \mathfrak{h}^{-1} \to \Sigma_{\mathbb{P}E,\xi} \otimes \xi^{-3} \otimes \mathfrak{h}^{-1} \otimes \mathfrak{h} \to 0$$

together with $H^1(\mathbb{P}E, \Sigma_{\mathbb{P}E,\xi} \otimes \xi^{-3} \otimes \mathfrak{h}^{-1}) = 0$ to give (iii). □

Proof of (b). As for $H^{\#}$-surfaces we will verify the injectivity of the differential of the period map for a particular H-surface. For this we choose the *Fermat-like* H-surface with equation $S = 0$ where

$$S = xt_0^2(x^3(t_0^6 + t_1^6) + r_0^3 + r_1^3) - (r_0^2 + r_1^2)^2.$$

Proof that $S = 0$ defines an H-surface. For H-surfaces we want an equation of the form

$$S = xt_0^2\mathcal{U} - (r_0^2 + r_1^2)^2 = 0.$$

The pinch points are

$$x\mathcal{U} = 0,\ t_0 = 0,\ r_0^2 + r_1^2 = 0,$$

so we need $x\mathcal{U} = 0$ to be eight distinct points on $t_0 = 0$, $r_1^2 + r_2^2 = 0$. We have

$$\begin{aligned}
\partial S/\partial x &= t_0^2\mathcal{U} + xt_0^2\,\partial\mathcal{U}/\partial x\\
\partial S/\partial t_0 &= 2xt_0\mathcal{U} + xt_0^2\,\partial\mathcal{U}/\partial t_0\\
\partial S/\partial t_1 &= xt_0^2\,\partial\mathcal{U}/\partial t_1\\
\partial S/\partial r_0 &= xt_0^2\,\partial\mathcal{U}/\partial r_0 - 4r_0(r_0^2 + r_1^2)\\
\partial S/\partial r_1 &= xt_0^2\,\partial\mathcal{U}/\partial r_1 - 4r_1(r_0^2 + r_1^2).
\end{aligned}$$

When $x = 0$, the vanishing of the partials implies that

$$r_0^2 + r_1^2 = 0,\ \mathcal{U} = 0.$$

We also need that for $t_0 \neq 0$

$$x = 0,\ r_0^2 + r_1^2 = 0,\ \mathcal{U} = 0 \text{ is empty}.$$

Now $\mathcal{U} = x \cdot B +$ (homogeneous cubic in r_0, r_1), and we need that

$$\mathcal{U}\big|_{x=0},\ r_0^2 + r_1^2 = 0 \text{ have no common zeroes.}$$

We may take

$$\mathcal{U} = r_0^3 + r_1^3 + x^3(t_0^6 + t_1^6),$$

which takes care of the $x = 0$ case.

If $x \neq 0$, $t_0 \neq 0$ the vanishing of the partials is

$$\begin{aligned}
&(1) \quad \mathcal{U} + x\,\partial\mathcal{U}/\partial x = 0\\
&(2) \quad 2\mathcal{U} + \partial\mathcal{U}/\partial t_0 = 0\\
&(3) \quad \partial\mathcal{U}/\partial t_1 = 0\\
&(4) \quad 3xt_0^2r_0^2 - 4r_0(r_0^2 + r_1^2) = 0\\
&(5) \quad 3xt_0^2r_1^2 - 4r_1(r_1^2 + r_2^2) = 0.
\end{aligned}$$

Note that

$$\partial\mathcal{U}/\partial t_1 = 6x^3t_1^5 \text{ so } (3) \implies t_1 = 0.$$

Then

$$\mathcal{U} + x\,\partial\mathcal{U}/\partial x = r_0^3 + r_1^3 + x^3(t_0^6 + \cancel{t_1^0}\,) + 3x^3(t_0^6 + \cancel{t_1^0}\,)$$

and thus

$$\begin{aligned}
(1) &\implies r_0^3 + r_1^3 + (xt_0^2)^3 = 0\\
(4) &\implies r_0^2xt_0^2 = (4/3)r_0(r_0^2 + r_1^2)\\
(5) &\implies r_1^2xt_0^2 = (4/3)r_1(r_0^2 + r_1^2)
\end{aligned}$$

$$\Downarrow$$

$$(r_0^2r_1 - r_0r_1^2)xt_0^2 = 0$$

$$\Downarrow$$

$$r_0r_1(r_1 - r_0) = 0.$$

Thus

$$r_0 = 0,\ r_1 = 0, \quad \text{or } r_0 = r_1.$$

If $r_0 = r_1 = 0$ then $r_0^2 + r_1^2 = 0$. If $r_0 = 0$, $r_1 \neq 0$ then

$$xt_0^2 = (4/3)r_1^3$$

$$\Downarrow$$

$$(xt_0^2)^3 = (4/3)^3r_1^3.$$

But by (1), $-r_1^3 = xt_0^2 \implies (xt_0^2)^3 = r_1^9 \implies r_1 = 0$, and by symmetry, $r_1 = 0 \implies r_0 = 0$.

If $r_0 = r_1 \neq 0$, (4) $\implies x_0^t = (4/3)\partial r_0 \implies (xt_0^2)^3 = (8/3)^3r_0^3$.

But (1) $\implies -2r_0^3 = (xt_0^2)^3 \implies r_0 = 0$ and we get $r_0 = r_1 = 0 \implies (xt_0^2)^3 = 0$, contradicting $x_0 \neq 0$, $t_0 \neq 0$.

Conclusion: *The singular locus of* $S = 0$ *is* $t_0 = 0$, $r_0^2 + r^2 + 1$.

This is a double curve. Now

$\{x\mathfrak{U} = 0\} \cap \{t_0 = 0\} \cap \{r_0^2 + r_1^2 = 0\}$ is $\begin{cases} t_0 = 0, r_2/r_0 = \pm i \text{ on } x = 0 \\ x^3 t_1^6 + r_0^3 + r_1^2 = 0. \end{cases}$

The roots are distinct: On $\mathbb{P}^1 \times \mathbb{P}^1$ we have $[t_0, t_1] = [0, 1]$, $r_1 = \pm i r_0$ and

$$-x^3 t_1^6 = r_0^3 + i r_0^3$$
$$\Downarrow$$
$$r_0^3 = \frac{-x^3 t_1^6}{1 \pm i}.$$

Regarding xt_1^2 as fixed we get six distinct solutions. Thus we have $2 + 6 = 8$ distinct pinch points. □

Proof of generic local Torelli for H-surfaces. We want to show that the map

$$\frac{H^0(\xi^4 \otimes \mathfrak{J})}{H^0(\Sigma_{\mathbb{P}E})} \to \operatorname{Hom}\left(H^0(\xi \otimes \mathfrak{h}^{-1}), \frac{H^0(\xi^5 \otimes \mathfrak{h}^{-1} \otimes \mathfrak{J})}{H^0(\Sigma_{\mathbb{P}E} \otimes \xi \otimes \mathfrak{h}^{-1})} \right)$$

is injective. Here, all of these cohomology groups are computed on $\mathbb{P}E$ and the image of $H^0(\Sigma_{\mathbb{P}E})$ under the map $A \to A \lrcorner dS$, $\quad A \in H^0(\Sigma_{\mathbb{P}E})$ has basis

$$\frac{\partial S}{\partial x} = 4x^3(t_0^0 + t_0^2 t_1^6) + t_0^2(r_0^3 + r_1^3) \in |\xi^3 \mathfrak{h}^2|$$
$$\frac{\partial S}{\partial t_0} = 84^4 t_0^7 + 2x^4 t_0 t_1^6 + 2x t_0 (r_0^3 + r_1^3) \in |\xi^4 \mathfrak{h}^{-1}|$$
$$\frac{\partial S}{\partial t_1} = 6x^4 t_0^2 t_1^5 \in |\xi^4 \mathfrak{h}^{-1}|$$
$$\frac{\partial S}{\partial r_0} = 3x t_0^2 r_0^2 - 4r_0(r_0^2 + r_1^2) \in |\xi^3|$$
$$\frac{\partial S}{\partial r_1} = 3x t_0^2 r_1^2 - 4r_1(r_0^2 + r_1^2) \in |\xi^3|.$$

Then $H^0(\Sigma_{\mathbb{P}E} \otimes \xi \otimes \mathfrak{h}^{-1})$ is the image of $H^0(\Sigma_{\mathbb{P}E} \otimes \xi \otimes \mathfrak{h}^{-1})$ under this same map where now $A \in H^0(\Sigma_{\mathbb{P}E} \otimes \xi \otimes \mathfrak{h}^{-1})$. Since $H^0(\xi \otimes \mathfrak{h}^{-1})$ has basis xt_0xt_1 the ideal we want is

$$[J : (xt_0, xt_1)].$$

Initial efforts to verify by hand the injectivity of the differential of the period mapping led to complicated impressions. Turning to Macaulay turned up the following two generators, which were unlikely to have been found by hand, of the above ideal:

$$I_1 = x^6t_0t_1^{10}r_0 + x^6t_0t_1^{10}r_1 + \left(\tfrac{3}{4}\right)x^4t_0^3t_1^4r_0r_1^2 = \left(\tfrac{3}{4}\right)x^4t_0^3t_1^4r_1^3 + 2x^3t_0t_1^4r_1$$
$$I_2 = x^6t_0t_1^{10}r_1 + \left(\tfrac{3}{4}\right)x^4t_0^3t_1^4r_0r_1^2 - \left(\tfrac{3}{4}\right)x^4t_0^3t_1^4r_1^3 - x^3t_0t_1^4r_0r_1^3 + x^3t_0t_1^4r_1^4.$$

By degree considerations this implies the injectivity of

$$\begin{array}{l}\dfrac{H^0(\xi^4)}{H^0(\Sigma_{\mathbb{P}E})} \hookrightarrow \operatorname{Hom}\left(H^0(\xi\otimes\mathfrak{h}^{-1}), \dfrac{H^0(\xi^5\otimes\mathfrak{h}^{-1})}{H^0(\Sigma_{\mathbb{P}E}\otimes\xi\otimes\mathfrak{h}^{-1})}\right)\\ \quad\cup \\ \dfrac{H^0(\xi^4\otimes\mathfrak{J})}{H^0(\Sigma_{\mathbb{P}E})}.\end{array}$$

In fact, it is only when one gets to

$$\frac{H^0(\xi^7\otimes\mathfrak{h}^{-1})}{H^0(\Sigma_{\mathbb{P}E}\otimes\xi^3\otimes\mathfrak{h}^{-1})} \to \operatorname{Hom}\left(H^0(\xi\otimes\mathfrak{h}^{-1}), \frac{H^0(\xi^8\otimes\mathfrak{h}^{-2})}{H^0(\Sigma_{\mathbb{P}E}\otimes\xi^4\otimes\mathfrak{h}^{-2})}\right)$$

that injectivity fails. □

I.I. **Global monodromy.** Let $\mathcal{M}_H$ be the KSBA moduli space for smooth H-surfaces X. The object of this section is to prove the

THEOREM: *Let $\Phi : \mathcal{M}_H \to \Gamma\backslash D$ be the period mapping associated to $H^2(X)_{\text{prim}}$. Then Γ is an arithmetic group.*

The usual method to establish such results for a smooth variety Y of general type is to realize a general Y as a smooth section of a very ample line bundle $L \to Z$ over a smooth variety and use the classical method of Lefschetz [Le24] to produce generators γ_i for Γ, followed by a group theoretic argument [Be] to show that the γ_i generate an arithmetic group. The idea here is to extend, in several significant ways, this method.

Recall that an H-surface is described by a diagram

$$\begin{array}{ccc} \hat{X} \xrightarrow{g} \mathbb{P}E \xrightarrow{f} & Q_0 \subset \mathbb{P}^4 \\ \downarrow & \cup \\ X \xrightarrow{\qquad\varphi_{2K_X}\qquad} & X' \end{array}$$

where $X^\flat \in |\xi^4|$, f is given by $|\xi|$, and $X' = Q_0 \cap V$ is the bicanonical image of X where $V \in |\mathcal{O}_{\mathbb{P}^4}(4)|$. The monodromy representations for X and $\hat{X}$ differ by an inessential factor, and we shall concentrate on the one for $\hat{X}$, noting for the purpose of arguments using pictures that each has the same bicanonical model:

$$\varphi_{2K_{\hat{X}}}(\hat{X}) = \varphi_{2K_X}(X) = X'.$$

There are two issues in trying to apply the Lefschetz method, each of interest in its own right:

(a) the line bundle $\xi \to \mathbb{P}E$ is not very ample; it is generated by global sections and the image

$$f(\mathbb{P}E) = Q_0 \subset \mathbb{P}^4$$

is the singular quadric $x_0 x_2 = x_1^2$;

(b) the normalization map $\hat{X} \to X^\flat \subset \mathbb{P}E$ is not biregular but has image singular along a double conic $D_0 \subset \mathbb{P}^2_0$ with eight pinch points, which divide into two monodromy invariant groups of six plus two.

Each of (a) and (b) presents issues in seeking to extend the Lefschetz method. The heuristic reason why one might hope to extend it is this:

> *The failure of $g^*(\xi) \to \hat{X}$ to be very ample is given by monodromy invariant data along a monodromy invariant curve $C_0 \subset X$. Thus one might hope that the Lefschetz method will apply to $H^2(X \backslash C_0)$ and where the data along C_0 will only contribute a monodromy invariant subspace in* $\mathrm{Hg}^1(X)$.

This section will be divided into three subsections:

(i) The analogue of the theorem for J-surfaces, these being defined as complete intersections

$$Y = Q \cap V$$

where Q is a smooth quadric in $\mathbb{P}^4$ and $V \in |\mathcal{O}_Q(4)|$ is a general quartic; here the Lefschetz method will apply. Here we are using the complete linear system $|\mathcal{O}_Q(4)|$ to embed $Q \hookrightarrow \mathbb{P}^N$, so that elements in $\check{\mathbb{P}}^N$ give sections of $Q \subset \mathbb{P}^4$ by quartic threefolds.

(ii) The result for $H^\#$-surfaces $X^\# \subset \mathbb{P}E$, which are a general member $X^\# \in |4\xi|$; here only the first complication (a) will occur and the above general principle will apply.

(iii) The result for H-surfaces, where both issues (a) and (b) are present.

Finally, a word about the arguments to be given. These will be "pictorial" in the style of Lefschetz where in some ways the essential geometric aspects of the problem can perhaps most readily be seen. Presumably they can be recast in more modern language, as is the case for the usual Lefschetz pencils in [Ka], these arise when $L \to Z$ is very ample and the pencil is constructed from general lines in the dual projective space relative to a projective embedding $Z \hookrightarrow \mathbb{P}H^0(Z, L)^*$.

Referring to the steps $\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3$ and s_4 below, the first three are geometric and are carried out below for $J, H^\#$ and H-surfaces. This brings us to a situation

- $\Lambda \cong \mathbb{Z}^b$ a lattice with a non-degenerate form

$$Q : \Lambda \otimes \Lambda \to \mathbb{Z};$$

- a set $\Delta = \{\delta_i\}$ where $\delta_i \in \Lambda$ with $\delta_i^2 = -2$ and where $\mathrm{span}_{\mathbb{Z}}\{\delta_i\} = \Lambda$;
- Picard-Lefschetz transformations

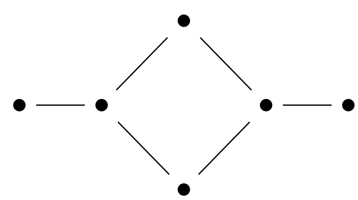
$$T_{\delta_i}(\gamma) = \gamma + Q(\gamma, \delta_i)\delta_i, \qquad \delta_i \in \Delta$$

that generate a subgroup $\Gamma_\Delta \subset \mathrm{Aut}(\Lambda, Q)$ and where Γ_Δ acts transitively on Δ.

I had thought that under the conditions the results in Beauville would give that

$$\Gamma_\Delta \textit{ is arithmetic.}$$

In fact, either $\Gamma_\Delta = \mathrm{Aut}(\Lambda, Q)$ or, in some special cases, the index $[\Gamma_\Delta : \mathrm{Aut}(\Lambda, Q)] = 2$. But it seems that for this strong result one needs that Γ_Δ contains an element T that would be generated by the monodromy of a surface acquiring a particular type of *du Val singularity*; viz., in addition to the δ_i above we should have

$T \longleftrightarrow$ 

Thus it seems that we need a degeneration of an H-surface with an equation (cf. Beauville, p. 10, line 14$\downarrow$)

$$x^3 + y^3 + z^4 + HOT.$$

I believe that Radu has such an example?

(i) *Global monodromy for J-surfaces*

THEOREM: *The global mondromy group acting on $H^2(Y)_{\mathrm{prim}}$ for J-surfaces is an arithmetic group.*[9]

Here there are four steps:

(s_1): to use the Lefschetz pencil $|Y_\lambda|$ to produce generators γ_i for the global monodromy group Γ;

(s_2): by varying the Lefschetz pencil and using the irreducibility of the dual variety $\check{Q} \subset \mathbb{P}^N$ show that monodromy acts simply transitively on the γ_i;

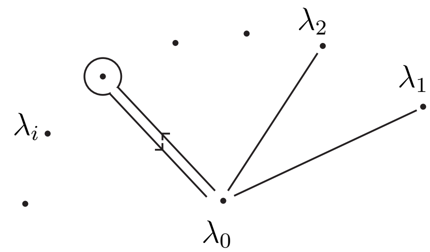
(s_3): show that the vanishing cycles associated to the γ_i generate $H_2(Y)_{\mathrm{prim}}$;

(s_1): For a Lefschetz pencil $|Y_\lambda|$ defined by a general line $\Lambda \subset \check{\mathbb{P}}^n$ we denote by $\hat{Q}$ the blowup of Q at the base points of the pencil. We then have $\hat{Q} \to \mathbb{P}^1$ with fibres the Y_λ; $H_2(\hat{Q})$ and $H_2(Q)$ differ only

[9]See the above note.

by the subspace of $H_2(\hat{Q})$ generated by the fundamental classes of the blownup base points. This is a subspace in $\mathrm{Hg}^1(\hat{Q})$ which is invariant when we vary the Lefschetz pencil. Effectively, it can be ignored in what follows.

We now draw the classic Lefschetz picture



of the λ-plane $\mathbb{C} \subset \mathbb{P}^1$ with a reference point λ_0 corresponding to a smooth Y_{λ_0}, and where the λ_i correspond to nodal Y_{λ_i}'s where the line $\Lambda \subset \mathbb{P}^N$ that defines the Lefschetz pencil meets the dual variety $\check{Q}$ transversely. Corresponding to each λ_i there is a vanishing cycle $\delta_i \in H_2(Y_{\lambda_0})$ which maps to zero under the collapsing map $H_2(Y_{\lambda_0}) \to H_2(Y_{\lambda_i})$ along the path $\overline{\lambda_0\lambda_i}$. The Picard-Lefschetz transformation T_{δ_i} given by the action on $H_2(Y_{\lambda_0})$ induced by going around the path drawn above is

$$T_{\delta_i}(\gamma) = \gamma + (\gamma_1\delta_i)\delta_i.$$

Since $\delta_i^2 = -2$ we have

$$T_{\delta_i}(\delta_i) = -\delta_i.$$

The surface $\hat{Q}^0 =: \hat{Q}\backslash Y_\infty$ retracts onto the part of $\hat{Q}$ over the slits, from which it follows by the classical Lefschetz arguments that

- the T_{δ_i} generate the action of monodromy on $H_2(Y_{\lambda_0})$ for the family of smooth surfaces

$$\hat{Q}\backslash\left(\bigcup_i Y_{\lambda_i}\right) \to \mathbb{P}^1\backslash\{\lambda_1, \dots, \lambda_m\}$$

  obtained by taking out the singular fibres in $\hat{Q} \to \mathbb{P}^1$;
- there is a relation

$$\prod_i T_{\delta_i} = I.$$

Setting $\Lambda^* = \Lambda \backslash \{\lambda_1, \dots, \lambda_m\}$, if $\mathcal{U} \subset \check{\mathbb{P}}^N$ parametrizes the smooth surfaces $Q \cap V$, the map

$$\pi_1(\Lambda^*) \to \pi_1(\mathcal{U})$$

is surjective, from which it follows that the T_{δ_i} generate the global monodromy group for the action of $\pi_1(\mathcal{U})$ on $H_2(Y_{\lambda_0})$.

$\underline{\text{s}_2}$: We now vary the Lefschetz pencil. In this case, since $\check{Q}$ is irreducible, $\check{Q} \backslash \check{Q}_{\text{sing}}$ is connected, from which we conclude that we may choose a closed loop in the space of Lefschetz pencils which interchanges any pair λ_i, λ_j of critical values corresponding to singular surfaces in the Lefschetz pencil.

$\underline{\text{s}_3}$: Recalling the notation $\hat{Q}^0 = \hat{Q} \backslash Y_\infty$, in the commutative diagram

$$\begin{array}{ccccccccc} H_3(\hat{Q}^0) & \xrightarrow{i^0} & H_3(\hat{Q}^0, Y_{\lambda_0}) & \xrightarrow{\partial^0} & H_2(Y_{\lambda_0}) & \xrightarrow{j^0} & H_2(\hat{Q}^0) & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H_3(\hat{Q}) & \xrightarrow{i} & H_3(\hat{Q}, Y_{\lambda_0}) & \xrightarrow{\partial} & H_2(Y_{\lambda_0}) & \xrightarrow{j} & H_2(\hat{Q}) & \longrightarrow & 0 \end{array}$$

the essential points are:

- the locus of the vanishing cycle δ_i along each path $\overline{\lambda_0 \lambda_i}$ generates a 3-cycle Δ_i with $\partial^0 \Delta_i = \delta_i$, and the Δ_i generate $H_3(\hat{Q}^0, Y_\infty)$ ([Le24]);
- by definition, the vanishing cycles are given by $\ker j$, which in this case is $H_2(Y_{\lambda_0})_{\text{prim}}$;
- the map $H_2(\hat{Q}^0) \to H_2(\hat{Q})$ is a morphism of mixed Hodge structures which induces an injection on the Gr_2-terms.

This last point implies that $\ker j = \ker j^0$, which by the second bullet establishes the desired result.

(s_4): a variant of the group-theoretic argument [Be] will give that Γ is arithmetic.

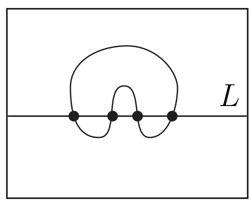
(ii) *Global monodromy for $H^{\#}$-surfaces*

We recall that for an $H^{\#}$-surface, defined as $X^{\#} \subset \mathbb{P}E$ given by a general member $X^{\#} \in |\xi^4|$, the canonical map

$$\varphi_{K_{X^{\#}}} : X^{\#} \to \mathbb{P}^4$$

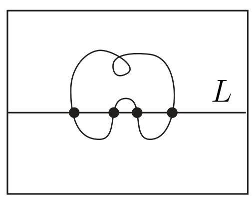
is given by the restriction of $f : \mathbb{P}E \to Q_0 \subset \mathbb{P}^4$, and the fibres $C_t^{\#} = \pi^{-1}(t)$ of $X^{\#} \to \mathbb{P}^1$ are described pictorially below in terms of the picture of Q_0 as a quadratic pencil of $\mathbb{P}^2_t$'s rotating about the line $L = Q_{0,\mathrm{sing}}$

general $C_t^{\#}$



, here omitting $\varphi_{K_{X^{\#}}}$

nodal $C_{t_a}^{\#}$



The base points of this pencil are the images of the four -2 curves C_α that are contracted by the canonical mapping, and whose canonical images are the dots on L. Recalling that $\xi, \mathfrak{h} \in \mathrm{Pic}(X^{\#})$ we have the

Lemma: $[C_1], \dots, [C_4], \xi$ *are linearly independent in* $\mathrm{Pic}(X^{\#})$, *and* $\mathfrak{h} \in \mathrm{span}_{\mathbb{Q}}([C_1], \dots, [C_4], \xi)$. *Thus the Picard number is*

$$\rho(X^{\#}) \geqq 5.$$

Proof. Recalling that the divisor $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ of $x \in H^0(\mathbb{P}E, \xi - 2\mathfrak{h}) \cong \mathbb{C}$ is contracted to L under the mapping $f : \mathbb{P}E \to Q_0$, we have in $\mathrm{Pic}(X^{\#})$

$$C_1 + C_2 + C_3 + C_4 = \xi - 2\mathfrak{h}.$$

Then from $\xi^2 = -8$ and

$$\begin{aligned} C_\alpha^2 &= -2 \\ C_\alpha \cdot \mathfrak{h} &= 1 \\ C_\alpha \cdot \xi &= 0 \end{aligned}$$

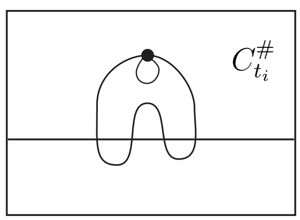
we obtain the lemma. □

The $C_{t_a}^{\#}$ are the nodal curves that appear as singular fibres in the fibration $X^{\#} \to \mathbb{P}^1$; the nodes on the $C_{t_a}^{\#}$ are a smooth points of $X^{\#}$.

THEOREM: *Setting $H^2(X^{\#})_{\text{prim}} = \text{span}\{C_1, \dots, C_4, \xi\}^{\perp}$, the global monodromy group acts on $H^2(X^{\#})_{\text{prim}}$ and there it is an arithmetic group.*

Proof. The idea is to parallel the above argument for J-surfaces with the $\mathbb{P}E$ replacing Q and a general pencil $|X_\lambda^{\#}|$ replacing $|Y_\lambda|$. For this we need first to determine what the degeneracies are for a general $|X_\lambda^{\#}|$. They are of three types.

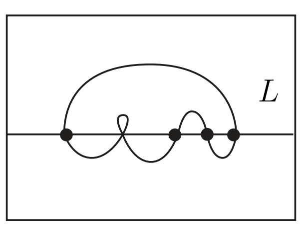
(i)$_{\lambda_i}$



Here a new nodal curve appears in a fibre of $X_{\lambda_i}^{\#} \to \mathbb{P}^1$. This corresponds on Q_0 to a $Y_{\lambda_i} \in |\mathcal{O}_{Q_0}(4)|$ becoming simply tangent at a point of $Q_{0,\text{reg}}$. Back up on $\mathbb{P}E$ the surface X_{λ_i} acquires an ordinary node, as in the standard Lefschetz picture.

The second type is

(ii)



This corresponds to one of the nodes on a $C^{\#}_{t_a}$ moving onto L. Back up on $X^{\#}_{\lambda}$ this does not create a new singular fibre in the pencil. Here one needs to distinguish between new singularities appearing in the family of $H^{\#}$-surfaces as hypersurfaces in $\mathbb{P}E$, as apposed to appearing in their canonical images in $\mathbb{P}^4$, which may be thought of as singularities of the target rather than as singularities of the source.

There remains the possibility that as we vary the pencil $|X^*_\lambda|$, one of the nodes on $X^*_{\lambda_i}$ moves onto the fixed component $S \cap X^{\#}_{\lambda_i}$ of $|K_{X^{\#}_{\lambda_i}}|$ that maps to $L = Q_{0,\mathrm{sing}}$. The picture here is the same as for (ii); again it reflects a singularity of the canonical map rather than a new one that appears back up on $X^{\#}_{\lambda_i} \subset \mathbb{P}E$.

The discussion of the steps s_1 and s_2 now proceeds exactly as for J-surfaces. The point is that the dual variety $\check{Q}_{0,\mathrm{reg}}$ is irreducible, and adding points to its closure in $\check{\mathbb{P}}^N$ does not affect the transitive action of monodromy on the critical values λ_i in case (i) above. In other words, nodes that appear on a $X^{\#}_{\lambda_i}$ as a result of the point of tangency of a hyperplane moving onto $Q_{0,\mathrm{sing}}$ are not monodromy invariant.

For s_3 we denote by $\hat{\mathbb{P}E}$ the blow up of the base locus of the general pencil $|X^{\#}_\lambda|$. Setting $\hat{\mathbb{P}E}^0 = \hat{\mathbb{P}E} \backslash X^{\#}_\infty$ we consider the diagram

$$\begin{array}{ccccccccc}
H_3(\hat{\mathbb{P}E}^0) & \xrightarrow{i^0} & H_3(\hat{\mathbb{P}E}^0, X^{\#}_{\lambda_0}) & \xrightarrow{\partial^0} & H_2(X^{\#}_{\lambda_0}) & \xrightarrow{j^0} & H_2(\hat{\mathbb{P}E}^0) & & \\
\downarrow & & \downarrow{\scriptstyle k} & & \downarrow & & \downarrow & & \\
H_3(\hat{\mathbb{P}E}) & \xrightarrow{i} & H_3(\hat{\mathbb{P}E}, X^{\#}_{\lambda_0}) & \xrightarrow{\partial} & H_2(X^{\#}_{\lambda_0}) & \xrightarrow{j} & H_2(\hat{\mathbb{P}E}) & \longrightarrow & 0.
\end{array}$$

Here the difference between J-surfaces and $H^{\#}$-surfaces appears. Referring to the three bullets following the analogue for J-surfaces of the diagram just above,

- as before the Lefschetz argument applies to show that $\ker j^0$ is generated by the cones Δ_i traced out by the vanishing cycles δ_i along the segments $\overline{\lambda_0\lambda_i}$;
- by definition, $\ker j$ are the vanishing cycles for the map induced on homology by the inclusion $Y_{\lambda_0} \hookrightarrow \mathbb{P}E$;

- however, since the map k is not injective on the Gr_3-part of the mixed Hodge structures on $H_3(\hat{\mathbb{P}E}^0, X^{\#}_{\lambda_0})$ and $H_3(\hat{\mathbb{P}E}, X^{\#}_{\lambda_0})$, we have

$$\operatorname{im}\partial^0 \subsetneqq \operatorname{im}\partial.$$

The discrepancy is generated in $H_2(X^{\#}_{\lambda_0})$ by the differences $C_\alpha - C_\beta$ between pairs of the -2 curves that are contracted by the canonical map. In cohomology the monodromy representation is reducible with one summand being

$$\operatorname{span}\{\xi, \mathfrak{h}, C_1, \dots, C_4\} \subset \mathrm{Hg}^1(X^{\#}_{\lambda_0}),$$

and the other summand being

$$H^2(X^{\#}_{\lambda_0})_{\mathrm{tr}} =: \operatorname{span}\{\xi, \mathfrak{h}, C_1, \dots, C_4\}^{\perp}.$$

For a generic $X^{\#}$ the monodromy representation on $\mathrm{Hg}^1(X^{\#})$ is a finite group, while the above Lefschetz-type argument coupled with the group theoretic result in s_4 gives that the monodromy representation on $H^2(X^{\#})_{\mathrm{tr}}$ is arithmetic.

(iii) *Global monodromy for H-surfaces*

We begin with a few general observations.

- If $\mathcal{U}$ is a smooth, irreducible algebraic variety and $\mathcal{U}_0 \subset \mathcal{U}$ is a Zariski open set, then the induced mapping

$$\pi_1(\mathcal{U}_0) \to \pi_1(\mathcal{U})$$

  is surjective. Thus, to show that Γ is arithmetic we may restrict to the monodromy representation to a Zariski open.
- In the situations we are concerned with, we will have

$$\mathcal{U} \xrightarrow{p} \mathcal{M}_H$$

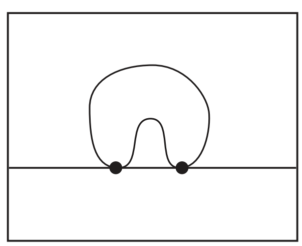
  where $\mathcal{U}$ fibres over a Zariski open in $\mathcal{M}_H$ and where

$$p_* : \pi_1(\mathcal{U}) \to \pi_1(p(\mathcal{U}))$$

  will be surjective.

- We will use the equation of $X^\flat \subset \mathbb{P}E$ to describe the Zariski open $\mathcal{U}$ in the preceding bullet, and here we will take the equation of $X^\flat$ to be
$$xt_0^2 G = F^2$$
where G, F are general elements in $|3\xi|, |2\xi|$ respectively.
- Finally, we recall that $x \in |\xi - 2\mathfrak{h}|$ is a generating section with divisor $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and where
$$\mathcal{O}_S(1,0) = \mathfrak{h}, \quad \mathcal{O}_S(0,1) = \xi.$$
Then the restriction $F\big|_S H^0(\mathcal{O}_S(0,2))$ has divisor $\mathbb{P}^1 \coprod \mathbb{P}^1$, a disjoint union of two $\mathbb{P}^1$'s, and
$$S \cap \{F = 0\} = C_1 + C_2$$
where C_1, C_2 are disjoint curves on $X^\flat$ that satisfy
  - $C_1^2 = C_2^2 = -1$;
  - C_1, C_2 map under $f : \mathbb{P}E \to Q_0$ to the two base points of the pencil $\varphi_{2K_X}(C_t)$.

In the picture



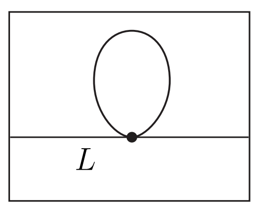
of $\varphi_{2K_X}(C_t)$ for a general t, C_1 and C_2 map to the two marked points. Under the map
$$g : \hat{X} \to X^\flat \subset \mathbb{P}E$$
they are the images of the exceptional -1 curves obtained by blowing up the base points of $|K_X|$.

We shall choose a general $\mathbb{P}^1$ in the space of the above equations (it doesn't have to be a pencil), and shall examine the degeneracies that occur among the $X^\flat_\lambda$'s. In addition to those that occur for a general pencil $|X^\#_\lambda|$ of $H^\#$-surfaces, the new ones will involve those that occur along the double conic. We shall draw pictures of the degeneracies of $X^\flat_\lambda = \varphi_{2K_{X_\lambda}}(X_\lambda)$, and we shall interperate those back up on the X_λ.

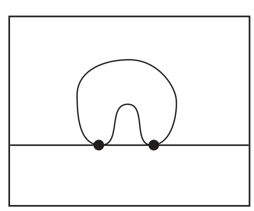
The things that can happen in the codimension 2 in the parameter space for the $X^\flat$'s are

- d1 $x = F = 0$ is one $\mathbb{P}^1$ counted twice, instead of two distinct $\mathbb{P}^1$'s;
- d2 $t_0 = F = 0$ is a singular plane conic $L_1 L_2 = 0$;
- d3 the points of $t_0 = G = F = 0$ are not distinct;
- d4 F, G become tangent at a point not on $t_0 = 0$;
- d5 $G\big|_{t_0=0}$ is a plane cubic that acquires a node.

d1: Here the two curves C_1, C_2 above come together and interchange. The picture is

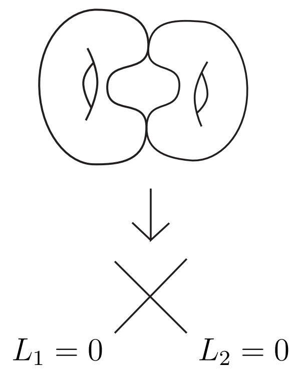


where L becomes a 4^{th}-order flex-tangent to the curves in the pencil. This is the limiting case of the picture



for $H^{\#}$-surfaces, and as was the case there the monodromy action takes place in $\mathrm{Hg}^1(\hat{X})$ for a generic H-surface X.

d2: Here the unique hyperelliptic C_0 on a general X becomes reducible



We now show that this doesn't contribute to a singularity of $X^{\flat}_{\lambda_i}$. If locally we take

$$u = t_0, \quad v = \sqrt{xG}$$

so that

$$(uv)^2 = t_0^2 xG = L_1^2 L_2^2,$$

then after a choice of the square root above we obtain

$$uv = L_1 L_2.$$

This map is 1:1 except when $u = 0$, where it is 2:1, including at $L_1 = L_2 = 0$, and has branch points at the points of $xG = 0$. This does not contribute to the computation of monodromy.

d3: This is the case where two of the branch points of $C_0 \to D_0$ come together, which means that $C_0 \in |K_{X_\lambda}|$ acquires a new node; namely, one that it is not the limit of one of the nodal curves in $|K_{X_\lambda}|$ for a general λ. As was the situation for $H^{\#}$-surfaces, this is the case where the quartic hypersurface in $\mathbb{P}^4$ is the limit of hypersurfaces that are simply tangent to $Q_{0,\mathrm{reg}}$. As such, it falls in the general Lefschetz pencil situation.

d4: This is the general Lefschetz pencil situation; the limiting case here is d3 above.

d5: The node will not be on the double conic, and therefore it does not affect the picture.

At this juncture we see that the critical values in a general family $X^\flat_\lambda$ that are not of types d3 and d4 will only affect the $\mathrm{Hg}^1(\hat{X})$-part of global monodromy.

(s$_3$): The situation for H-surfaces is in one way simpler than that for $H^\#$-surfaces. Namely, the bicanonical map $\varphi_{2K_X} : X \to \mathbb{P}^3$ does not contract any curves. Rather it is the normalization map for the bicanonical image $X' = \varphi_{2K_X}(X) \subset \mathbb{P}^4$. Recall that for $\hat{X}$ given by the blowup of X at the base points of $|K_X|$, the equation we are using to describe a birational model of X is the one that describes the image of $X^\flat \subset \mathbb{P}E$; viz

$$t_0^2 xG = F^2$$

where $G, F \in |\xi^3|, |\xi^2|$. As was the case for $H^\#$-surfaces, we are concerned with the monodromy representation of

$$H^2(X)_{\mathrm{tr}} =: \mathrm{Hg}^1(X)^\perp$$

where X is a very general H-surface. Noting that

$$H^2(X)_{\mathrm{tr}} = H^2(\hat{X})_{\mathrm{tr}} =: \mathrm{Hg}^1(\hat{X})^\perp,$$

it will suffice to work with $\hat{X}$. We then claim that using duality to identify $H^2(\hat{X})$ and $H_2(\hat{X})$, we have

> $\ker\{H_X(\hat{X})_{\mathrm{tr}} \xrightarrow{g_*} H_2(\mathbb{P}E)\}$ *is generated by the vanishing cones* Δ_i *traced out by the locus of vanishing cycles* δ_i *along the paths* $\overline{\lambda_0\lambda_i}$.

The argument is similar to that for $H^\#$-surfaces, although it is simpler in that to work from the pictures it is only the two -1 curves E_1, E_2 that are contracted by $f \circ g\varphi_{2K_X} = \varphi_{2K_{\hat{X}}}$.

(s$_4$): In the case of $J, H^\#$ and H-surfaces in the terminology of [Be] we have a vanishing lattice (Λ, Δ). In order to conclude the arithmeticity of Γ_Δ, we need to know that there is additional degeneration of a special sort. In [Be] the singularity is an isolated singularity of type $\mathcal{U}_-$ (12); this is what is needed to be able to apply the results of [ED]. In the case of H-surfaces we shall use an example of an isolated Dolgacov singularity of type D_{-} (10) (or equivalently an Arnold exceptional uninodal singularity). Such a singularity is given by an equation

to be continued

III.C. **Positivity of the Hodge bundle.** Let $\mathcal{M}$ be a KSBA moduli space for surfaces of general type and

$$\Phi_e : \overline{\mathcal{M}} \to \Gamma\backslash D^*$$

the extended period map. Assuming two technical extensions, stated below, of the proof of the classical Kodaira embedding theorem we shall prove the

Theorem: *The Hodge bundle λ is ample relative to Φ_e in the sense that*

$$\Phi_e(\overline{\mathcal{M}}) = \mathrm{Proj}(\lambda).$$

For the argument we may replace $\Phi_e : \overline{\mathcal{M}} \to \Gamma\backslash D^*$ by the diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \Gamma\backslash D \\ \cap & & \cap \\ \overline{S} & \xrightarrow{\Phi_e} & \Gamma\backslash D^* \end{array}$$

where $\overline{S}$ is a smooth, compact complex manifold, $Z =: \overline{S}\backslash S$ is a divisor with normal crossings and $\Phi : S \to \Gamma\backslash D$ is a VHS whose local monodromies around the branches of Z are unipotent. The reason we may do this is that $\Phi_e(\overline{\mathcal{M}})$ $\Phi_e(\overline{S})$, and these differ only by fixed subspaces in Hg^1 in the associated graded VHS's along the boundary strata. Thus as compact analytic varieties we have $\Phi_e(\overline{\mathcal{M}}) = \Phi_e(\overline{S})$. With the technical assumptions mentioned above, we shall show that

> *The image $\Phi_e(\overline{S})$ is a compact analytic variety and the Hodge bundle $\lambda \to \overline{\Phi}(\overline{S})$ is ample.*

As noted above we will phrase this by saying that

> *λ is ample relative to $\Phi_e : \overline{S} \to \Gamma\backslash D^*$.*

Since we are working in a complex analytic setting it is natural that curvature methods will be used. Before presenting the argument, we begin with a few remarks that are meant to explain its essential aspects.

(i) *It is classical that with the Hodge metric, the Hodge bundle has positive curvature on $\Phi(S)$.*

This is one of the two basic "positive curvature" aspects of the result.

We may assume that $Z = \cup Z_i$ is stratified in such a way that setting $Z_i^0 = Z_i \backslash Z_{i+1}$ the restriction

$$\Phi_e : Z_i^0 \to \Gamma_i \backslash D_i$$

maps to boundary components as descrbed in §II— above.

(ii) *Since this mapping is étalè over a product of period domains, we would like to apply* (i).

There is a subtlety here in that even though the Hodge bundles extend homomorphically to $\Gamma\backslash D^*$, the Hodge metrics do not. Rather, the Hodge lengths of holomorphic sections of the extended Hodge bundles will have logarithmic singularities of the type

$$\left(\log \frac{1}{|t|}\right)^k .$$

Thus even when $\Phi_e(\overline{S})$ is a complex manifold, the natural metric in the Hodge bundle λ has singularities.

(iii) *We will see that the strength of the singularities, as measured by the exponent k above, is in a precise sense proportional to the size of the log of monodromy around the branches of $Z_i \backslash Z_{i+1}$. In particular, the stronger the monodromy the more positive in the distributional sense are the Chern forms of λ.*

This phenomenon, which was classical in the geometric case, is related to but not the same as that in [Sc72] and [CKS86] showing how the monodromy weight filtration may be defined in terms of the Hodge lengths of section of the corresponding local system. Another way of expressing it is that for

$$\Phi_e : Z_i^0 \to (\Gamma_i \backslash D_i)$$

the positivity of the Hodge bundle of $\lambda\big|_{D_i}$ is supplemented by a factor contributed by the action of monodromy around Z_i^0.

Before turning to the arguments we wish to make one more remark: The above result and comments refer to the line bundle

$$\lambda = \det F_e^n .$$

(iv) *The vector bundle F^n_e is semi-positive. The degeneracies of the curvature form reflect the determinantal structure of the Kodaira-Spencer mappings.*

The terms in this remark will be defined below. The point is that there is more geometry in the curvature of F^n_e than just the positivity of its associated determinant line bundle.

We will proceed with a general review of the general curvature properties of positive and semi-positive holomorphic vector bundles. The applications to the above theorem will be given at the end, and the two required extensions of the Kodaira embedding theorem will be presented in the appendix to this section.

III.C.a. *Positive vector bundles.* A *Hermitian vector bundle* is given by a holomorphic vector bundle $S \to M$ over a complex manifold together with a Hermitian metric $h = (\ ,\ )$ in the fibres of S. Such a vector bundle has a canonical connection with associated curvature

$$\Theta_S\, e\, A^{1,1}(\mathrm{Hom}(S,S)),$$

a $\mathrm{Hom}(S,S)$-valued $(1,1)$ form that satisfies ${}^t\overline{\Theta}_S = -\Theta_S$. The Chern forms $c_q(\Theta_S)$ representing the Chern classes $c_q(S)$ of $S \to M$ are given by

$$\det\left(I + \left(\frac{i}{2\pi}\right)\Theta_S\right) = \sum_q c_q(\Theta_S).$$

Upon choice of a frame e_α for S and local coordinates z^i on M, Θ_S is represented by a matrix[10]

$$\Theta_{\alpha\bar\beta i\bar j} e_\alpha \otimes e^*_\beta \otimes dz^i \wedge d\bar z^j$$

where e^*_β is dual to e_β using $S \cong S^*$ via the metric.

Definition: The *curvature form* is defined for $\xi \in T_xS$, $e \in E_x$ by

$$\Theta_S(e,\xi) = \langle(\Theta_S(e), e), \xi \wedge \bar\xi\rangle.$$

In matrices it is given for $e = \sigma^\alpha e_\alpha$, $\xi = \xi^i \partial/\partial z^i$ by

$$\Theta_S(e,\xi) = \Theta_{\alpha\bar\beta i\bar j}\sigma^\alpha\bar\sigma^\beta\xi^i\bar\xi^j.$$

[10]We shall be using summation convention.

Definition: $S \to M$ is *positive* if there exists a metric such that $\Theta_S(e,\xi) > 0$. It is *semi-positive* if there exists a metric such that

$$\begin{cases} \Theta_S(e,\xi) \geqq 0 \\ \operatorname{Tr}\Theta_S(e,\xi) > 0. \end{cases}$$

One geometric interpretation is this. Over $\mathbb{P}S^*$ we have the tautological line bundle $\mathcal{O}_{\mathbb{P}S}(1)$.[11] The Hermitian metric on $S \to M$ induces one in the line bundle $\mathcal{O}_{\mathbb{P}S}(1)$ with curvature form ω_S. At each point $[e^*] \in \mathbb{P}$ the tangent space is a direct sum

$$T_{[e^*]}\mathbb{P}S = V_{[e^*]} \oplus H_{[e^*]}$$

of vertical and horizontal subspaces. Then

- $\omega_S\big|_V > 0$;
- $\omega_S\big|_H \geqq 0$;
- $\omega_S > 0 \iff S \to M$ is positive.

In particular, *positive vector bundles are ample.* It is an old question whether the converse is true.

The last relation above results by identifying $H \cong TM$ and then at (e,ξ) we have

$$\omega_S(\xi) = \|\xi(e)\|^2.$$

III.C.b. *The universal bundle over the Grassmannian. Note on signs:* The Hodge bundle is a holomorphic sub-bundle of a flat vector bundle. However, because of the sign properties in Hodge theory it will be semi-positive whereas over the Grassmannian the standard sub-bundle of the flat bundle is semi-negative; hence the signs in this section.

We begin with a general remark about the curvature of sub-bundles. Let $V \to M$ be a holomorphic vector bundle with an Hermitian metric

[11] Recall that the fibre $(\mathbb{P}S)_s$ at $s \in S$ is $\mathbb{P}S^*_x$. This convention gives for $\mathbb{P}S \xrightarrow{\pi} M$ that

$$\begin{cases} R^o_\pi \mathcal{O}_{\mathbb{P}S}(k) = \operatorname{Sym}^k S \\ R^q_\pi \mathcal{O}_{\mathbb{P}S}(k) = 0, \qquad k \geqq -\operatorname{rank} S. \end{cases}$$

and $S \subset V$ a holomorphic sub-bundle. Then there is a canonical 2^{nd} *fundamental form* of S in V

$$A \in \Omega^1 \otimes \operatorname{Hom}(S, V)$$

such that the curvatures of V and S with the induced metric are related by

$$\Theta_S(e, \xi) = \Theta_V(e, \xi) - (A(e), A(e)).$$

In matrix form

$$\Theta_S = \Theta_V\big|_S - {}^t\bar{A} \wedge A.$$

This formula is usually expressed by: *curvatures decrease on holomorphic sub-bundles.* In particular

- $\Theta_V = 0 \implies \Theta_S \leqq 0$;
- $\operatorname{Tr} \Theta_S < 0$ if $A : T \to \operatorname{Hom}(S, V)$ is injective.

We now let V be an Hermitian vector space and $G(k, V)$ the Grassmannian of k-planes $F \subset V$. With the standard identification

$$T_F G(k, V) = \operatorname{Hom}(F, V),$$

the above, A is the identity and we have over $G(k, V)$

$$\begin{cases} \Theta_S \leqq 0 \\ \operatorname{Tr} \Theta_S < 0. \end{cases}$$

An interesting question is *For which* (e, ξ) *do we have*

$$\Theta_S(e, \xi) = 0.$$

To give the answer we use the above identification and stratify $\mathbb{P}TG(k, V)$ by

$$\mathbb{P}TG(k, V)_\ell = \{\xi \in T_F G(k, V) : \dim\{\ker \xi : F \to V/F\} \geqq \ell\}.$$

Geometrically, we think of ξ as infinitesimally displacing F to F_ξ, and then we have that

$$F \cap F_\xi = \dim \ker\{\xi : F \to V/F\}$$

interpreted as what might be called the *infinitesimal base locus of the family* F_ξ.

Noting that

$$S^* \text{ is positive} \iff \operatorname{rank} S = 1 \iff G(k,V) = \mathbb{P}V$$

we see that the bundles $S^* \to G(k,V)$ are semi-positive but are not ample. Geometrically

The curves in $\mathbb{P}S$ that prevent the ampleness of $S^* \to G(k,V)$ are given by curves of $(k-\ell)$-planes $[F_t] \subset \mathbb{P}V$ that contain a fixed k-plane for some $\ell \geqq 1$.

The Nakai criterion for the ampleness of $\mathcal{O}_{\mathbb{P}S^*}(1)$ fails exactly for such curves.

For $M \subset G(k,V)$ a smooth subvariety, we have that

$$S^* \to M \textit{ is ample} \iff \textit{all } \xi \in TM \subset TG(k,V) \textit{ are injective.}$$

Example: Suppose $\operatorname{rank} S = 2$. Then the stratum $\mathbb{P}TG(2,V)_1$ of rank one ξ's is a bundle of rational normal scrolls over $G(2,V)$ whose fibres are $\mathbb{P}^{a-1}$ bundles over $\mathbb{P}^1$ $(\dim V = 2 + a)$.

III.C.c. *Semi-positivity of $F^n \to M$ for a PVHS.* We consider a PVHS $(V, \mathbb{Q}, F^p, \nabla)$ over a complex manifold M. Since we are mainly concerned with algebraic surfaces we will mainly be concerned with the bundle

$$F^2 \subset V \qquad\qquad (\text{think of } F^2 = H^{2,0}).$$

The polarization induces a metric in $F^2 \to M$, and the sign conventions work out to have for $F = F^2$

$$\Theta_F(e,\xi) = \|\widetilde{\nabla}_\xi(e)\|^2$$

where setting $\widetilde{\nabla} = \nabla \mod F^2$

$$\widetilde{\nabla}_\xi(e) = \langle \widetilde{\nabla} e, \xi\rangle \in F^1/F^2.$$

Main example: $\mathcal{X} \xrightarrow{\pi} M$ is a family of smooth algebraic surfaces X_t for $t \in M$. Then $F = \pi_*\omega_{\mathcal{X}/M}$, and a section $\varphi(t)$ of $F \to M$ is

$$\varphi(t) \in H^0(\Omega^2_{X_t})$$
$$\|\varphi(t)\|^2 = \left(\frac{1}{4}\right)\int_{X_t} \omega(t) \wedge \overline{\omega(t)}$$

(note that $\left(\frac{1}{4}\right)dz\wedge dw\wedge d\bar{z}\wedge d\bar{w}=\left(\frac{i}{2}\right)dz\wedge d\bar{z}\wedge\left(\frac{i}{2}\right)dw\wedge d\bar{w}$).

The map $\widetilde{\nabla}$ is given by the *Kodaira-Spencer map*

$$\rho: T_tM\to \operatorname{Hom}\left(H^0(\Omega^2_{X_t}),H^1(\Omega^1_{X_t})_{\rm prim}\right).$$

Thus we have for $\xi\in T_tM$, $\varphi\in H^0(\Omega^2_{X_t})$

$$\boxed{\Theta_F(\varphi,\xi)=\|\rho(\xi)\varphi\|^2}$$

where $\|\ \|^2$ is the Hermitian length in $H^1(\Omega^1_{X_t})_{\rm prim}$.

Conclusion: *Under the assumption that the Kodaira-Spencer maps are injective, the Hodge bundle $F^2\to M$ is semi-positive (in the differential-geometric sense), meaning*

$$\begin{aligned}&\Theta_F(\varphi,\xi)\geqq 0 &&\text{for } \varphi\in F^2,\xi\in TM\\ &\Theta_{\det F}(\xi)>0 &&\text{for } \xi\in TM.\end{aligned}$$

In practice, M will be a Zariski open in a smooth projective variety $\overline{M}$ with $\overline{M}\backslash M=$ a normal crossing divisor with stratification Z_k and with $Z_k^*=Z_k\backslash Z_{k+1}$. One may ask how the metrics, curvature and Chern forms behave along the strata. This will be analyzed below, and the conclusions are

- On each stratum Z_k^* there is a PVMHS and the above curvature results apply to the Hodge bundles $I^{0,0},I^{1,0}$ and $I^{2,0}$ in the associated graded for Gr(PVMHS). Note that

$$\operatorname{rank} I^{0,0}+\operatorname{rank} I^{1,0}+\operatorname{rank} I^{2,0}=\operatorname{rank} F^2$$

  and the VHS associated to $I^{0,0}$ is constant.
- On Z_0^* as we approach Z_1 as $t\to 0$, there is the weight filtration F_ℓ, $0<\ell\leqq 2$ on F and the Hodge metrics for the canonical extensions satisfy

$$\begin{cases}\|\ \|\sim C^\infty f^{\mathfrak{n}} & \\ \|\ \|\sim\log(1/|t|)C^\infty f^{\mathfrak{n}} & \text{on } F_1/F_0\\ \|\ \|\sim\log(1/|t|)^2C^\infty f^{\mathfrak{n}} & \text{on } F_2/F_1.\end{cases}$$

- In the normal direction

$$\begin{cases} \Theta_{F_0}(e,\xi) & \sim C^\infty \\ \Theta_{F_1/F_0}(e,\xi) & \sim \mathrm{PM}(\xi) + C^\infty \quad > 0 \\ \Theta_{F_2/F_1}(e,\xi) & \sim 2\,\mathrm{PM}(\xi) + C^\infty \quad > 0 \end{cases}$$

where PM denotes the Poincaré metric.

In more detail, in the degenerations we consider we will have the Hodge lengths of a canonical extension of the Hodge bundles. For 2-forms on surfaces we will have the local situation

$$\varphi(t) = \mathrm{Res}\left(\frac{g(x,y,z)dx\wedge dy\wedge dz}{xy-t}\right)$$
$$\psi(t) = \mathrm{Res}\left(\frac{d(x,y,z)dx\wedge dy\wedge dz}{xyz-t}\right).$$

The local contributions to the Hodge norms are as above

$$\|\varphi(t)\|^2 = \int \varphi(t)\wedge\overline{\varphi(t)} = \left(\log\frac{1}{|t|}\right)\cdot C^\infty f^{\mathfrak{n}}$$
$$\|\psi(t)\|^2 = \int \psi(t)\wedge\overline{\psi(t)} = \left(\log\frac{1}{|t|}\right)^2\cdot C^\infty f^{\mathfrak{n}}.$$

The critical observation then is that

$$i\partial\overline{\partial}\left(\log\left(\log\frac{1}{|t|}\right)\right) = \mathrm{PM}\,.$$

Thus when we compute the Chern forms as a sum of the *local* contributions of the L^2-norms of holomorphic 2-forms acquiring singularities as above, the singular contributions are all proportional to a *positive* multiple of the Poincaré metric in the normal directions to the strata Z_i^o.

Conclusion (singularities allowed): The same conclusions hold as above, with the modification

- the inequalities are in the sense of currents;
- the Kodaira-Spencer map is injective in the normal directions to the strata where the monodromy logarithm $N=0$.

Appendix to §III.C: The extended Kodaira embedding theorem (EKET's)

There are two types of results that are needed. For the first we let Y be a compact analytic variety and $L \to Y$ a holomorphic line bundle. We assume that L has a smooth Hermitian metric in the sense that locally we may embed Y in an open set $\mathcal{U} \subset \mathbb{C}^N$ and there is over $\mathcal{U}$ a holomorphic line bundle with smooth metric that restricts to that on Y. The curvature form Ω_L is then a C^∞ form on Y that may be evaluated in the Zariski tangent spaces to Y.

EKET I: *If Ω_L is positive then $L \to Y$ is ample.*

For the purposes of this work we may assume that there is a resolution of singularities $\widetilde{Y} \xrightarrow{\pi} Y$ where $\widetilde{Y}$ is a projective variety. Then $\widetilde{L} = \pi^{-1}(L)$ has an Hermitian metric whose curvature $\Omega_{\widetilde{L}}$ satisfies

$$\Omega_{\widetilde{L}}(v) \geqq 0, \quad \text{and } \Omega_{\widetilde{L}}(v) = 0 \iff \pi_*(v) = 0$$

for $v \in T\widetilde{Y}$. If $D \subset \widetilde{Y}$ is a divisor with normal crossings that is partially contracted by π, then the normal bundles to the strata of D are "negative" along the fibres of the contraction. In relatively simple cases when $\widetilde{Y} \to Y$ is a succession of blowups with non-singular centers, the methods of [Gri68] can be used to construct a metric in the line bundle $[D]$ over $\widetilde{Y}$ such that

$$\Omega_{k\widetilde{L}-[D]} > 0 \qquad \text{for } k \gg 0$$

and one may imagine that the classical arguments of Kodaira [Ko] can be extended.

Application: *Let $\widetilde{Y}$ be a compact analytic variety and*

$$\Phi : \widetilde{Y} \to \Gamma\backslash D$$

a VHS. Then λ is ample relative to $\Phi : \widetilde{Y} \to \Gamma\backslash D$.

Briefly, horizontal compact analytic subvarieties $Y \subset \Gamma\backslash D$ are projective algebraic.

For the second extension of the KET, we assume given a smooth projective variety Y and a reduced normal crossing divisor $D \subset Y$. For simplicity we assume that D is locally given by $t = 0$ where $t, y_2, \dots y_n$ are local holomorphic coordinates on Y. We denote by

$$\mathrm{PM} = \frac{dt \wedge d\bar{t}}{|t|^2 \left(\log\left(\frac{1}{|t|}\right)\right)^2}$$

the Poincaré metric in the normal directions to D.

Now assume given a holomorphic line bundle $L \to Y$ that has a Hermitian metric h which is smooth on $Y^* = Y \backslash D$ and that locally near a point of D has the form

$$h = \log \frac{1}{|t|} \cdot h^*$$

where h^* is a smooth positive function.

EKET II: *Assume that the Chern form ω_L is a positive $(1,1)$ current. Then $L \to Y$ is ample.*

We note that

$$\omega_L = \mathrm{PM} + \omega_{h^*}$$

where $\omega_{h^*} = (i/2\pi)\partial\overline{\partial} \log h^*$. Thus, ω_L is positive in the distributional sense in the directions normal to D.

III.D. **Relationship between moduli-theoretic and Hodge-theoretic boundary components.** Given a KSBA moduli space $\mathcal{M}$ for surfaces of general type with period mapping

$$\Phi : \mathcal{M} \to \Gamma\backslash D,$$

we have seen that there is an extension

$$\Phi_e : \overline{\mathcal{M}} \to \Gamma\backslash D^*$$

from the canonical compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ to the SBB-type completion $\Gamma\backslash D^*$ of $\Gamma\backslash D$ constructed above. We may stratify the boundary $\partial\mathcal{M}$ by the condition that the deformations of the degenerate surfaces parametrized by a given component be equisingular. The boundary $\partial(\Gamma\backslash D^*)$ is stratified by the type of limiting mixed Hodge structures that appear. When this is done

> Φ_e *maps boundary components of* $\overline{\mathcal{M}}$ *to boundary components of* $\Gamma\backslash D^*$*.*

Natural questions that arise are

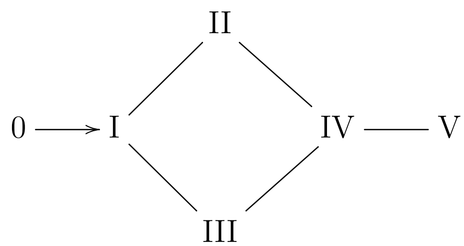
(i) *Do all boundary components of* $\Gamma\backslash D^*$ *have points that appear in the image of boundary components of* $\overline{\mathcal{M}}$*?*

If this is the case then we shall say that the boundary components of $\Gamma\backslash D^*$ are *realized* by boundary components of $\overline{\mathcal{M}}$.

(ii) *Are the incidence relations among boundary components of* $\Gamma\backslash D^*$ *that are realized by boundary components of* $\overline{\mathcal{M}}$ *also realized?*

What this means is the following: The boundary components of $\overline{\mathcal{M}}$ form a partially ordered set, where the ordering relation is "contained in the closure of." We shall also say that a point in a boundary component specializes to one in its closure. The boundary components of $\Gamma\backslash D^*$ form a similar partially ordered set. Here, specialization means that a given type of LMHS degenerates further, in the sense of VMHS, to one in the closure. Realization of incidence relations means that every possible specialization relation among realized boundary components of $\Gamma\backslash D^*$ is itself realized by a specialization relation among the components of $\partial\mathcal{M}$. In other words, if a specialization realtion is Hodge-theoretically possible, then there is an algebro-geometric specialization that covers it via the extended pencil mapping Φ_e.

For H-surfaces we have seen that the boundary component structure of $\Gamma\backslash D^*$ may be pictured by



where 0 corresponds to the least degenerate PHS's $\Gamma\backslash D$ and V to the most degenerate PHS's, which in this case are of Hodge-Tate type.

Theorem: *For H-surfaces the boundary components of $\Gamma\backslash D^*$ and the incidence relations among them are all realized by boundary components and their incidence relations for $\overline{\mathcal{M}}_H$.*

We will in fact show there are components of $\partial\mathcal{M}_H$ corresponding to degenerate H-surfaces having both singularities of the double curve with pinch points type and of the isolated singularity type and that realize the above diagram. The following discussion will be broken into two parts corresponding to the two different singularity types.

Part 1: Double curves with pinch points

The simple basic idea is to degenerate the equation

$$xt_0^2G = F^2$$

of the birational image $X^\flat$ of a smooth H-surface X to an equation such that the corresponding KSBA degeneration X_0 realizes the Hodge-theoretic boundary components and incidence relations in the above diagram. The subtlety is that the above equation describes a KSBA degeneration $X_0^\#$ of a smooth $H^\#$-surface $X^\#$, so that degenerating the equation gives a further degeneration of $X_0^\#$ of $X^\#$ rather than a degeneration of X. The upshot is that we will have to construct from $X_0^{\#}$ a surface X_0 that is a KSBA degeneration of X. Before getting into the details we turn back to Hodge theory and shall give a heuristic discussion of how Hodge theory provides a guide for where to look for type I degenerations of a smooth H-surface X.

Hodge-theoretic interlude: Suppose that $\omega \in H^0(\Omega_X^2)$ has divisor $D = (\omega)$ and that we have a specialization $X \to X_0$ such that ω specializes to a meromorphic differential ω_0 on X_0 and where D specializes to a double curve D_0 on X_0. Whereas a general differential $\varphi \in H^0(\Omega_X^2)$ will specialize to a φ_0 having a log-pole on D_0, because of the assumption that $(\omega) = D$ specializes to D_0 the differential ω_0 will actually be holomorphic on X_0. For the specialization $\varphi \to \varphi_0$ we will have

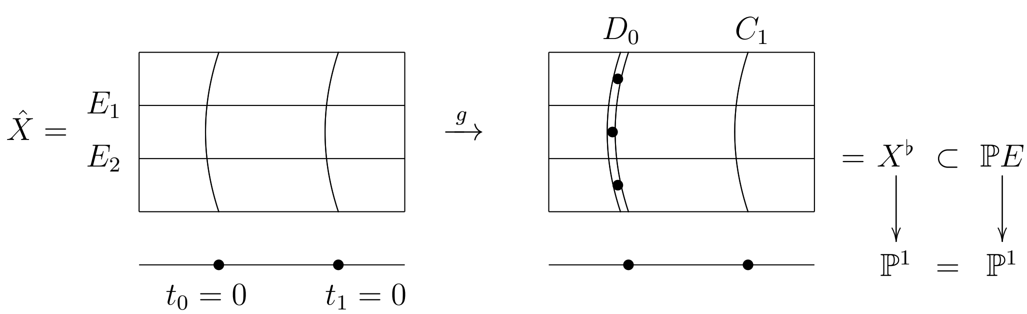
$$\operatorname{Res}_{D_0}(\varphi_0) \in H^0(\Omega^1_{D_0}).$$

Thus for the LMHS we may hope that

$$\begin{cases} \dim I^{2,0} = 1 \\ \dim I^{1,0} = 1. \end{cases}$$

If we are lucky, the double curve D_0 will be an elliptic curve and the normalization $\widetilde{X}_0$ will be a K3. This cannot be quite right, because then $(\widetilde{X}_0, D_0)$ would be of log-general type, which by an easy argument may be seen not to be the case. However the argument does point us in the right direction.

To take the next step, we consider the picture



where $\hat{X}$ is the blow up of X at the base points of $|K_X|$ and E_1, E_2 are the -1 curves. Here, $D_0 \subset X^\flat$ is the double curve $t_0 = 0$ with the eight pinch points (three of which are marked by dots)

- $x = 0$, $F = 0$ = two pinch points given by $(E_1 + E_2) \cdot D_0$,
- $x \neq 0$ and $F = G = 0$, six pinch points.

We let ω in $H^0(\Omega^2_X)$ have divisor C_1, so that the divisor of the pullback $\hat{\omega}$ of ω to $\hat{X}$ is

$$(\hat{\omega}) = C_1 + 2(E_1 + E_2).$$

Note that

$$K^2_{\hat{X}} = 4C_1 \cdot (E_1 + E_2) + 4(E_1^2 + E_1^2) = 8 - 8 = 0,$$

as should be the case.

Now let $X^\flat_0$ have the equation

$$xt_0^2 xt_1^2 Q = F^2$$

where $Q, F \in |2\xi|$ are general. In other words, on $X^\flat_0$ we put another double curve with pinch points by a similar equation that gave D_0 on $X^\flat$. The degeneration

$$X^\flat \to X^\flat_0$$

exactly arises by $X^\flat$ acquiring a double curve on the limit of the divisor $(\hat{\omega})$, as was suggested by the Hodge-theoretic considerations above.

We now turn to the central question:

> *What degeneration of H-surfaces corresponds to the above degeneration $X^\flat \to X^\flat_0$?*

More precisely, for $s \neq 0$ let X_s be the H-surface given as the normalization of the surface

$$X^\flat_s : xt_0^2(sG - xt_1^2 Q) = F^2$$

where G, Q, F are general elements in $|3\xi|, |2\xi|, |2\xi|$ respectively.

Proposition: *The family of smooth H-surfaces associated to $\mathcal{X}^* \xrightarrow{\pi} \Delta^*$ where $\pi^{-1}(s) = X_s$ uniquely extends to a KSBA family $\mathcal{X} \to \Delta$, and the extended period mapping $\Phi_e : \Delta \to \Gamma\backslash D^*$ maps the origin to a* *type I degeneration. Moreover, the natural desingularization of X_0 is a K3 surface.*

Proof. We write the equation of $X_0^\flat$ as

$$x^2 t_0^2 t_1^2 Q = F^2.$$

From this equation we see that $X_0^\flat$ has double curve

$$\{x = 0\} \cup \{t_0 = 0\} \cup \{t_1 = 0\}.$$

We recall that the divisor $S = (x) \subset \mathbb{P}E$ is a $\mathbb{P}^1 \times \mathbb{P}^1$ and $F\big|_S \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 2)|$. Thus

- $S \cap X^\flat$ is a $\mathbb{P}^1 \amalg \mathbb{P}^1$,
- $S \cap X_0^\flat$ is a double curve $2(\mathbb{P}^1 \amalg \mathbb{P}^1)$, and this double curve has no pinch points.

The other double curves on $X_0^\flat$ are $t = 0$ and $t_1 = 0$ with pinch points given by

$$\{t_0 = 0\} \cap \{Q = F = 0\}$$
$$\{t_1 = 0\} \cap \{Q = F = 0\}.$$

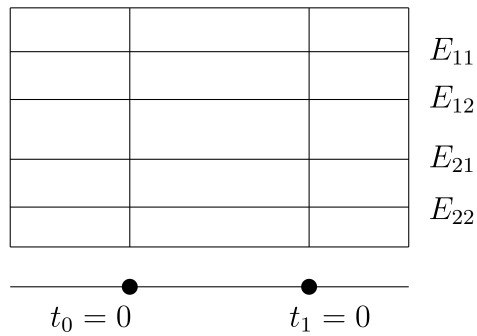
More precisely,

- on $x \neq 0$, $t_0 = 0$ they are given by $F = Q = 0$ on $\mathbb{P}^2 \backslash \mathbb{P}^1$, which is four points;
- on $x = 0$ and $t_0 = 0$, due to the x^2 the equation is locally of the form

  $$u^2 v^2 - w^2 = 0,$$

  which is $(uv - w)(uv + w) = 0$ giving locally a double curve without pinch points;
- on $x \neq 0$ and $t_1 = 0$ on $x = 0, t_1 = 0$ the situation is similar.

The normalization $\widetilde{X}_0^\flat$ of $X_0^\flat$ has the picture



where C_0, C_1 are elliptic curves with maps $C_0 \to \mathbb{P}^1$, $C_1 \to \mathbb{P}^1$ branched at the four pinch points. The E_{ij} are $\mathbb{P}^1$'s that arise from the normalization of the $2(\mathbb{P}^1 \amalg \mathbb{P}^1)$ above. There are involutions

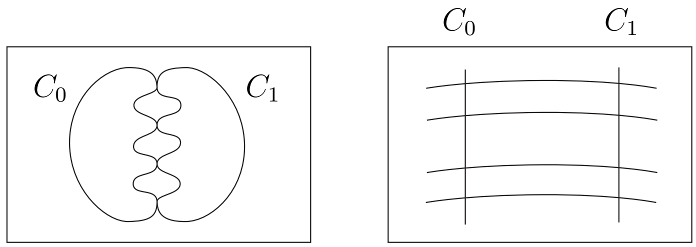
$$i_0 : C_0 \to C_0,\ i_1 : C_1 \to C_1$$

that interchange the intersections $E_{11} \cap C_0$ and $E_{12} \cap C_0$, etc.

The issue is that $\widetilde{X}_0^\flat$ is *not* the normalization $\widetilde{X}_0$ of the degeneration $X \to X_0$ of H-surfaces. The reason is that the double curve $t_0 = 0$ on $X^\flat$ is a singularity of the bi-canonical map of X and not of X itself. The correct interpretation is

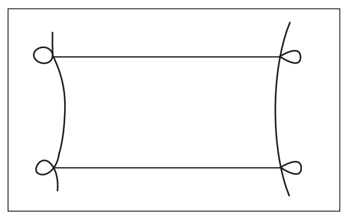
> *$\widetilde{X}_0$ is a K3 surface. The limit of C_0 as $X \to X_0$ is a $\mathbb{P}^1$ on $\widetilde{X}_0$, while the limit of C_1 is an elliptic curve on $\widetilde{X}_0$ with a map $C_1 \to \mathbb{P}^1$ branched at the four pinch points.*

Put another way, the reverse construction is this: Start with a K3 surface having two elliptic curves C_0, C_1 meeting at four points in two pairs where each curve has involution interchanging points in these pairs. Then blow up the four points to obtain a picture

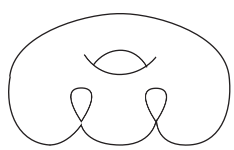


$\widetilde{X}_0^\flat$

where the horizontal lines are the $\mathbb{P}^1$'s obtained by blowing up $C_0 \cap C_1$. Then identifying E_{11}, E_{12} and E_{21}, E_{22} to get a picture



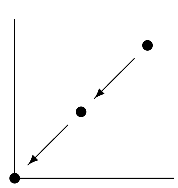
where each of the curves has arithmetic genus 3 and 2 nodes



Now contract C_0 to $\mathbb{P}^1$ using the involution. This is our desired surface X_0.

As $X \to X_0$ one holomorphic 2-form remains holomorphic in the limit and generates $H^0(\Omega^2_{X_0})$. The remaining 2-form acquires a log pole on C_1 and its residue generates $H^0(\Omega^1_{\widetilde{C}_1})$. Thus the LMHS is of type I as desired. $\square$

We now turn to degenerations of type II, which are described by

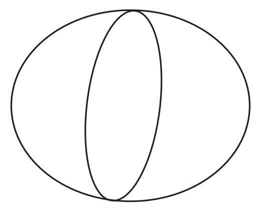


$$\dim \mathrm{I}^{2,0} = 1, \dim \mathrm{I}^{1,0} = 0, \dim \mathrm{I}^{0,0} = 1$$

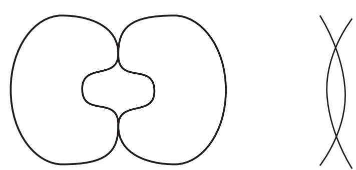
The idea is to degenerate the above equation

$$xt_0^2\, xt_1^2\, Q = F^2$$

of $X_0^\flat$ by letting Q, F become special so that the elliptic curve C_1 acquires a node. This is done by letting the conics $Q = 0$, $F = 0$ in the $\mathbb{P}^2 \subset \mathbb{P}E$ given by $t_1 = 0$ become simply tangent



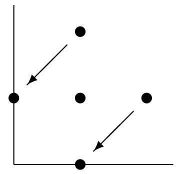
If we think of the equation of C_1 for a type I degeneration as a 2:1 branched covering of $(t_1 = 0) \cap (F = 0)$ given by $y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$, then the degeneration has an equation $y^2 = (x - b_1)^2(x - b_2)^2$, which is a reducible curve consisting of a pair of $\mathbb{P}^1$'s meeting in two points



The holomorphic 2-form with a log-pole on C_1 then specializes to one whose residue is a 1-form with log poles at the double points of the reducible curves above. Up to scaling there is a unique such form.

By a discussion similar to the one given above, we may describe the further degenerations of the type I degeneration of an H-surface to the one given above.

For type III degenerations with the description



$$\dim \mathrm{I}^{2,0} = 0, \dim \mathrm{I}^{1,0} = 2, \dim \mathrm{I}^{0,0} = 0$$

we consider a surface in $\mathbb{P}E$ given by an equation

$$(xt_0^2)(xt_1^2)(x(t_0+t_1)^2)R = F^2$$

where $R \in |\xi|$, $F \in |2\xi|$ are general. Writing the LHS as $x^3t_0^2t_1^2(t_0 + t_1)^2R$ and recalling that $S = \{x = 0\}$ meets a surface with this equation in two disjoint $\mathbb{P}^1$'s, the singular curves over $t_1 = 0$, $t_1 + t_2 = 0$ have an equation

$$y^2 = \underbrace{(x-a_1)^3(x-a_2)^3}_{x=0}\underbrace{(x-a_3)(x-a_4)}_{F=0}.$$

Using the birational transformation

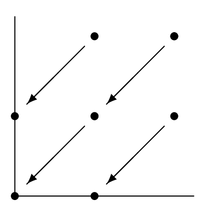
$$\begin{cases} u = x \\ v = y/(x-a_1)(x-a_2) \end{cases}$$

with inverse $x = u, y = v(u-a_1)(u-a_2)$, the above equation is transferred to one of the form

$$u^2 = (v-b_1)(v-b_2)(v-b_3)(v-b_4).$$

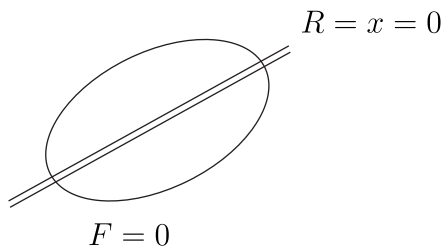
Going through an analysis as before, we may see that the holomorphic 2-forms on the smooth H-surface acquire independent log-poles on the elliptic curves that arise over $t_1 = 0$, $t_1 + t_2 = 0$. This gives a type III degeneration of an H-surface.

Turning to type a IV degeneration with the picture



$\dim \mathrm{I}^{2,0} = 0, \dim \mathrm{I}^{1,0} = 1, \dim \mathrm{I}^{0,0} = 1$

the idea is to further degenerate the above type III degeneration by letting the line $R = 0$ and conic $F = 0$ in the plane $t_1 = 0$ fail to meet transversely. For example, for $x^3t_0^2t_1^2(t_0+t_1)^2R = F^2$ if we have the picture over $t_1 = 0$

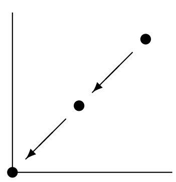


then the curve will have an equation of the form

$$y^2 - (x - a_1)^4 (x - a_2)^4 = 0,$$

which as above factors into $y \pm (x - a_1)^2 (x - a_2)^1 = 0$ and describes the situation where an elliptic curve has acquired 2 nodes.

Finally, for degenerations of type V



$$\dim \mathrm{I}^{2,0} = 1, \dim \mathrm{I}^{1,0} = 0, \dim \mathrm{I}^{0,0} = 2$$

we may carry out the contruction for type IV degenerations over $t_1 = 0$, $t_1 + t_2 = 0$. We note that each of these imposes two conditions on R, and since $R = |\xi|$ where $\dim |\xi| = 4$ such a construction is possible.

Summary: By using the equation

$$x t_0^2 G = F^2$$

of the birational model $X^\flat \subset \mathbb{P}E$ of a general H-surface, it is possible to see that the Hodge-theoretic boundary structure is realized by the algebro-geometric boundary structure. The key points are:

(i) the degeneration of the PHS's may be seen by understanding the residues of the limiting holomorphic 2-forms;

(ii) some care is required to understand the KSBA degeneration of an H-surface that arises from degenerating the equation of a particular birational model.

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