III. Topology and Hodge theory

- These two topics are closely intertwined and constitute a major aspect of complex algebraic geometry, beginning in the later part of the 19^{th} century (Picard, Poincaré, ...) into the 1^{st} half of the 20^{th} century (Lefschetz, Hodge, ...) and continuing through today
- In fact questions about integrals on algebraic surfaces (which are real 4-manifolds) were instrumental in the beginnings of topology — one knew (Darboux, Picard, Poincaré, E. Cartan, ...) what differential forms

$$\varphi = a\,dx + b\,dy + c\,dz$$
$$\psi = A\,dx \wedge dy + B\,dx \wedge dz + C\,dy \wedge dz$$
$$\eta = D\,dx \wedge dy \wedge dz$$

were, and Stokes' theorem

$$\int_{\mathfrak{U}} d\,\omega = \int_{\partial\mathfrak{U}} \omega$$

shows then when $d\omega = 0$ that $\int_\Gamma \omega$ was not only invariant under deformation or homotopy of Γ but also under homology.[1] This led to the notion of *periods*

$$\int_\Gamma \omega, \quad d\omega = 0 \text{ and } \Gamma \in H_p(X, \mathbb{Z}).$$

[1]The exterior derivative d is uniquely determined (i) $df = f_x\,dx + f_y\,dy + f_z\,dz$ for a function f, (ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$ and (iii) $dx \wedge dy = -dy \wedge dx$ etc.

In the complex case when X has local holomorphic coordinates $z = (z_1, \ldots, z_n)$

$$\omega = \sum_{I,J} f_{I\bar{J}}\, dz^I \wedge d\bar{z}^J$$

where $I = (i_1, \ldots, i_p)$, $dz^J = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ etc. and as we saw for algebraic curves the periods reflect the complex structure — this is the start of Hodge theory.

Outline for the remainder of this lecture

- Introductory discussion of what an algebraic variety is
- Statements of the Lefschetz theorems
- How they arose historically from the study of algebraic functions of two variables (Picard-Lefschetz or PL theory)
- Origin of the Hodge conjecture (HC)

$$\text{———} < \circ > \text{———}$$

- *Complex projective space* $\mathbb{P}^N$
  - lines through origin in $\mathbb{C}^{N+1}$
  - $\mathbb{P}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1} \qquad (\mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$
  - homogeneous coordinates $[z] = [z_0, \ldots, z_N]$
- $\mathbb{P}^1 =$ Riemann sphere
- $\mathbb{P}^2 = \mathbb{C}^2 \cup \{\text{lines through the origin}\}$ where $[z] \leftrightarrow$ line with slope z_2/z_1
- $\mathbb{P}^N =$ compact complex manifold

Proof

$$\begin{array}{ccc} \mathcal{U}_i = \{[z] : z_i \neq 0\} & \ni & [z] \\ \downarrow & & \downarrow \\ \mathbb{C}^N & \ni & (z_0/z_i, \ldots \overset{i}{\wedge} \ldots, z_N/z_i) \end{array}$$

- *Algebraic variety* $X \subset \mathbb{P}^N$ given by $F_1(z) = \cdots F_m(z) = 0$ where $F_\alpha(z) =$ homogeneous polynomial.
- Note that $\dim_{\mathbb{R}} X = 2 \dim_{\mathbb{C}} X$ and X is oriented.

Example

C defined by $f(x, y) = 0$ in $\mathbb{C}^2$. Set

$$x = z_1/z_0, \; y = z_2/z_0$$

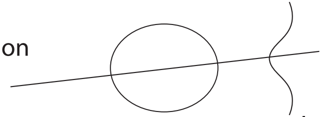
and clear denominators to get

$$\overline{C} = \{F(z) = 0\} \subset \mathbb{P}^2$$

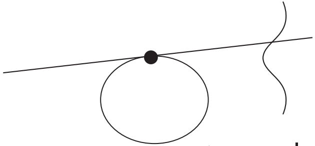
$$\text{where } \overline{C} = \left\{\begin{matrix}\text{our old} \\ C \subset \mathbb{C}^2\end{matrix}\right\} \cup \left\{\begin{matrix}\text{points} \\ \text{at } \infty\end{matrix}\right\}.$$

asymptotes

- suppose $X^n =$ smooth algebraic variety and $Y = \mathbb{P}^{N-1} \cap Y$ is a general hyperplane section hyperplane section

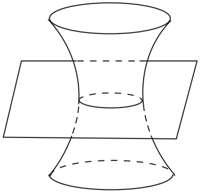


general



not general

quadric surface;
real picture



Note: Equation of the quadric in $\mathbb{C}^3$ is $x^2 + y^2 = z^2 + 1$; equation in $\mathbb{P}^3$ is $z_1^2 + z_2^2 = z_3^2 + z_0^2$; over $\mathbb{C}$ this is equivalent to $z_1' z_2' = z_3' z_0'$ where $z_1' = z_1 + iz_2$, $z_2' = z_1 - iz_2$ etc.

Lefschetz theorem I

- $b_{2p+1}(X) \equiv 0\ (2)$ (odd Betti numbers are even)
- $b_{2p}(X) \geqq 1$ (even Betti numbers are positive).

In the second, if $\dim_{\mathbb{C}} X = n$ and $H \in H_{2n-2}(Y, \mathbb{Z})$ is the class of the cycle given by Y then (non-trivially)

$$\underbrace{H \cap \cdots \cap H}_{n-p} \neq 0 \text{ in } H_{2p}(X, \mathbb{Z})$$

Lefschetz theorem II

$$H_p(Y, \mathbb{Z}) \to H_p(X, \mathbb{Z}) \text{ is } \begin{cases} \text{isomorphism for } p \leqq n-2 \\ \text{onto for } p = n-1 \end{cases}$$

Corollary

Y is connected if $\dim_{\mathbb{C}} X \geqq 2$

Exercise: $f(x, y) =$ irreducible polynomial and $\{f(x, y) = 0\} = C \subset \mathbb{C}^2$. Show that C is connected.

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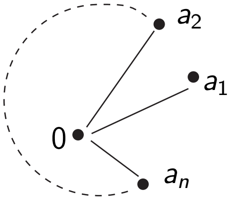
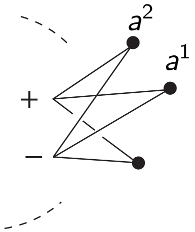
Geometric idea to study topology of an algebraic variety (idea is one of the most basic in algebraic geometry) — *use induction by dimension*.

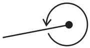
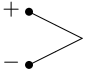
Example

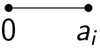
For $y^2 = p(x)$ where $p(x) = \prod_{i=1}^{2g+2}(x - a_i)$

- first take out the two points over $x = \infty$
- next use the picture of the complex x-plane



- retract the slit x-plane and the part of C lying over it onto the part lying over the segments
- 

$= \left\{ \begin{array}{l} \text{1-dimensional} \\ \text{complex} \end{array} \right\}$
- on  as we turn around the branch point the two points interchange (local *monodromy* T_i around a_i)
- $\prod_i T_i = \mathrm{Id}$
- $\Delta_i =$ 

lying over 

- C retracts onto the real 1-dimensional complex given by attaching the $2g+2$ 1-cells Δ_i to the two points $\pm$ lying over 0.
- Δ_i generate the relative homology group $H_1(C, \{+,-\}; \mathbb{Z})$

$\rightsquigarrow$ $H_1(\overline{C}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

This case is too simple to suggest the general pattern. The next dimension up is due to Picard (1880–2000)

Example

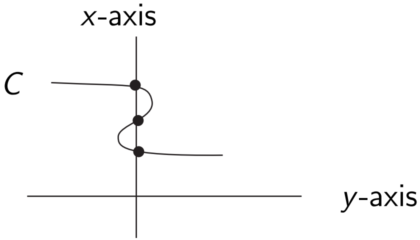
X is the algebraic surface

$$z^2 = f(x, y)$$

where $C = \{f(x, y) = 0\}$ is a non-singular plane curve. For a general y we let

$$X_y = \text{ curve } z^2 = f(x, y), \quad y \text{ fixed}$$

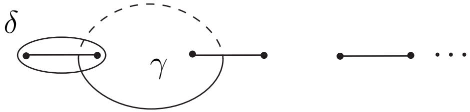
The picture is



X_y is the algebraic curve of the type we have been considering; it is 2:1 covering of the line $y =$ constant branched at the points of $C \cap \{y = \text{ constant}\}$

- smooth for general y
- singular when the line $y =$ constant becomes tangent to C

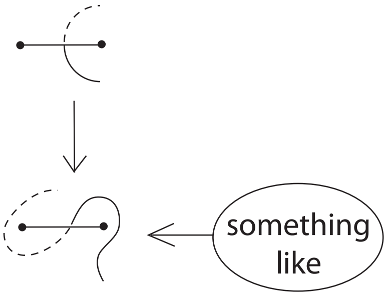
- the picture of X_y is



where the branch points and slits will vary with y

- at a point of tangency two branch points come together and interchange.

- $\delta \to \delta$
- $\gamma \to ?$



Picard-Lefschetz formula

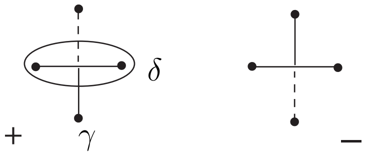
$$\text{(PL)} \qquad \gamma \to \gamma + \delta$$

How to show PL? The original argument was analytic and in outline went as follows:

- locally analytically change coordinates so that the picture is a neighborhood of the curves

$$C_t = \{u^2 + v^2 = t\}$$

  of the origin in $\mathbb{C}^3$ with coordinates (u, v, t)
- the local picture is



- set $t = \sigma^2$ and consider the integrals

$$I_t(\delta) = \int_\delta \frac{du}{\sqrt{t - u^2}} = \int_\delta \frac{du}{\sqrt{\sigma^2 - u^2}} = \int_\delta \frac{du}{\sigma\sqrt{1 - (u/\sigma)^2}}$$

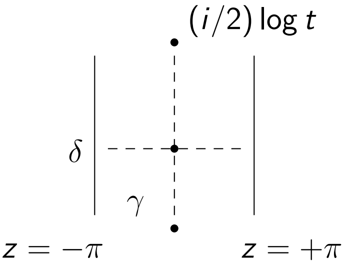
$$I_t(\gamma) = \int_\gamma \frac{du}{\sqrt{t - u^2}} = \int_\gamma \frac{du}{\sqrt{\sigma^2 - u^2}} = \int_\gamma \frac{du}{\sigma\sqrt{1 - (u/\sigma)^2}}$$

- the curves C_t are parametrized by

$$z \to (\sigma \sin z, \sigma \cos z),$$

and a calculation gives

$$\begin{cases} I_t(\delta) = 2\pi \\ I_t(\gamma) = i \log t \end{cases}$$



Conclusion

$$\begin{cases} I_{e^{2\pi i}t}(\delta) = I_t(\delta) \\ I_{e^{2\pi i}t}(\gamma) = I_t(\gamma) + I_t(\delta) \end{cases}$$

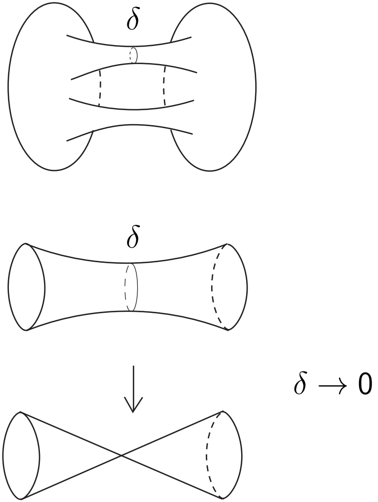
$$\implies T(\gamma) = \gamma + \delta.$$

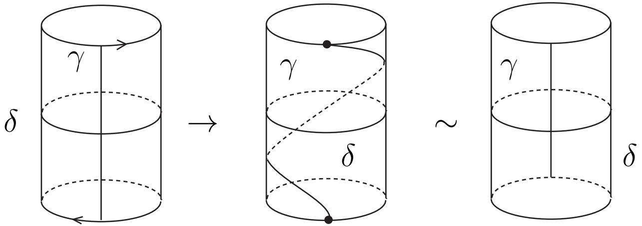
□

Topological pictures

global

local

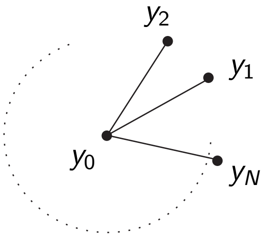




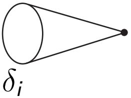
$$\begin{cases} \delta \to \delta \\ \gamma \to \gamma + \delta \end{cases}$$

- few pictures worth 1,000 (10,000?) words
- heuristic analytic reasoning suggests what the answer should be — then know what to prove.

- $X^* = X \backslash X_\infty$
- topological picture of X^*



- along $\overline{y_0 y_i}$ we have the locus of the vanishing cycle



$= \Delta_i$

$\Longrightarrow$ - X^* obtained from X_0 by attaching 2-cells Δ_i

- In general

$$X^* \textit{ obtained from } X_0 \textit{ by attaching}$$
$$n = \tfrac{1}{2}(\dim_{\mathbb{R}} X) \textit{ cells}$$

$\implies$ Lefschetz theorems I, II

- *Single and double integrals*

  Returning to X given by

$$z^2 = f(x, y)$$

  there are single integrals (1-forms)

$$\psi = \frac{p(x, y)\, dx}{z} + \frac{q(x, y)\, dy}{z}$$

  and double integrals (2-forms)

$$\varphi = \frac{r(x, y)\, dx \wedge dy}{z}$$

The story of the ψ's is very interesting but we will only have time to make a few observations. For one such we note that

- $\int \psi < \infty \implies d\psi = 0.$

Proof:

$$\begin{aligned} d\psi &= d\left(\frac{p(x,y)}{z}\right) \wedge dx + d\left(\frac{q(x,y)}{z}\right) \wedge dy \\ &= \frac{r(x,y)\, dx \wedge dy}{z} \end{aligned}$$

$$\implies \frac{1}{4}(d\psi \wedge \overline{d\psi}) = \left|\frac{r(x,y)}{z}\right|^2 \left(\frac{i}{2}\right) dx \wedge d\bar{x} \wedge \left(\frac{i}{2}\right) dy \wedge d\bar{y}$$

$$= \text{volume form on } X$$

$$0 < \int_X d\psi \wedge \overline{d\psi} = \int_X d(\psi \wedge \overline{d\psi}) = 0 \implies d\psi = 0.$$

- The space of single integrals is denoted by $H^{1,0}(X)$ and its dimension $h^{1,0}(X)$ is called the *irregularity* — reason for the name is that in the early days "most" surfaces seemed to be *regular*, i.e., to have $h^{1,0}(X) = 0$.

Example

For $z^2 = f(x, y)$ to be irregular the curve C cannot be smooth, or even have generic singularities, those being where

$$\begin{cases} f_x(x,y) = f_y(x,y) = 0 \\ \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{xx} \end{vmatrix} (x,y) \neq 0 \end{cases}$$

Similarly for a hypersurface

$$F(z_0, z_1, z_2, z_3) = 0$$

in $\mathbb{P}^3$ it is not easy to write down on F where X is irregular.

- Suppose now φ is a regular 2-form; i.e.,

$$\int_\sigma \varphi < \infty$$

for any 2-chain σ. We set

$$H^{2,0}(X) = \left\{ \begin{array}{c} \text{space of} \\ \text{regular 2-forms} \end{array} \right\}.$$

The *periods* of ψ are the

$$\int_\Gamma \psi, \qquad \Gamma \in H_2(X, \mathbb{Z}).$$

Among the Γ's are the fundamental classes of algebraic curves $C \subset X$; i.e., the images of

$$H_2(C, \mathbb{Z}) \to H_2(X, \mathbb{Z}).$$

We will discuss these further below.

- By restriction

$$\psi \to \psi_y = \frac{p(x, y)\, dx}{z}$$

we will generally have $\psi_y \neq 0$ which gives

$$H^{1,0}(X) \hookrightarrow H^{1,0}(X_y).$$

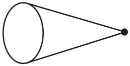
This suggests that we have

$$H^1(X, \mathbb{C}) \hookrightarrow H^1(X_y, \mathbb{C}),$$

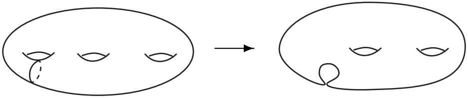
which is true and is what originally suggested the first non-easy case of Lefschetz II — again analysis and topology went hand in hand.

Another example of the use of analysis to suggest topology:

For a vanishing cycle

$\Delta_i =$ 

traced out by $\delta_y \in H_1(X_y, \mathbb{Z})$ along the path from 0 to a_i



we have

$$\int_{\delta_0} \psi = \int_{\delta_{a_i}} \psi = 0, \qquad \psi \in H^{1,0}(X).$$

This led to Picard's argument that

$$\ker\{H_1(X_0,\mathbb{Z}) \to H_1(X,\mathbb{Z})\} = \begin{cases}\text{span of the} \\ \text{space of vanishing cycles.}\end{cases}$$

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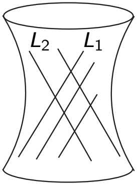
Returning to the discussion of

- Among the classes in $H_2(X,\mathbb{Z})$ are those given by the fundamental classes of the algebraic curves C contained in X.

Example:



$$\longleftarrow \left\{ \begin{array}{c} \text{two families} \\ \text{on lines on a} \\ \text{quadric surface} \\ z_0 z_1 = z_2 z_3 \end{array} \right\}^{2}$$

$$\rightsquigarrow H_2(X, \mathbb{Z}) \cong \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2]$$

- In general C is a component of

$$\begin{cases} z^2 = f(x, y) \\ g(x, y, z) = 0 \end{cases}$$

(may take $g(x, y, z) = g_0(x, y) + g_1(x, y)z$)

[2] The lines are $z_0 = z_2 = 0$, $[z_1, z_3] \in \mathbb{P}^1$ arbitrary and $z_1 = z_3 = 0$, $[z_0, z_2]$ arbitary.

$\implies$ On X

$$0 = dg = g_x\, dx + g_y\, dy + g_z\, dz$$

which using $dz = \left(-\frac{1}{2}\right)(f_x\, dx + f_y\, dy)$ gives a relation

$$a\, dx + b\, dy \,\Big|_C = 0$$

$$\implies \psi\Big|_C = 0$$

$$\implies \int_{[C]} \psi = 0.$$

Conclusion: *The periods of $H^{2,0}(X)$ on the homology classes of algebraic curves are equal to zero.*

- The converse statement is the famous *Lefschetz (1,1) theorem*.
- The converse to the analogous statement for arbitrary X is the *Hodge conjecture*.
- In terms of differential forms of degree 2 on X there are three types:
  - $\frac{p(x,y)\,dx \wedge dy}{z} \leftrightarrow H^{2,0}(X)$
  - conjugates of these $\leftrightarrow \overline{H^{2,0}(X)} = H^{0,2}(X)$
  - those that have a $dx \wedge d\bar{x}, dx \wedge d\bar{y}, d\bar{x} \wedge dy, dy \wedge d\bar{y}$ which are said to be of type (1,1) and contribute $H^{1,1}(X)$ to $H^2(X, \mathbb{C})$; it is these that are Poincaré dual to the homology classes carried by the algebraic curves in X.

Further topics

- These involve the *multiplicative structure* on cohomology: For X of dimension n and $H \in H^2(X)$ the class of a hyperplane section

$$(*) \qquad L^k : H^{n-k}(X) \to H^{n+k}(X).$$

Hard Lefschetz theorem: $(*)$ *is an isomorphism*

Lefschetz stated the result but his proof was incomplete. Hodge developed Hodge theory to prove $(*)$.

- Define operators L, H, Λ on $H^*(X)$ by
  - L as above
  - $H = (d-n)\,\mathrm{Id}$ on $H^d(X)$

Then the commutator

$$[H, L] = 2L.$$

There is a unique $\mathrm{sl}_2 = \{L, H, \Lambda\}$ with

$$\begin{cases} [L, \Lambda] = H \\ [L, \Lambda] = -2\Lambda. \end{cases}$$

Decomposing $H^*(X)$ into irreducible sl_2-modules gives the *Lefschetz decomposition* of cohomology into primitive subspaces — every class is a linear combination of powers of L applied to primitive classes

$$\begin{cases} L^k \cdot \eta \\ \Lambda\eta = 0. \end{cases}$$

- Any irreducible sl_2-module is isomorphic to
  - $V = \mathrm{span}\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$
  - $L = \partial_x$, $\Lambda = \partial_y$
  - primitive part is generated by x^n.

Example: X = algebraic surface

$$H^1(X) \xrightarrow{\sim} H^3(X)$$

and

$$H^0(X) \xrightarrow{L} H^2(X) \xrightarrow{L} H^4(X)$$

has

- $H^2(X)_{\mathrm{prim}} = \ker\{H^2(X) \xrightarrow{L} H^4(X)\}$
- $H^2(X) = LH^0(X) \oplus H^2(X)_{\mathrm{prim}}$

- Finally, you may ask: OK, we know a lot about the homology of X — what about its homotopy?

Theorem

The rational homotopy type of X is uniquely determined by $H^*(X)$.

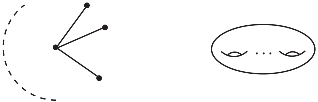
Thus the

- $\pi_i(X) \otimes \mathbb{Q}$
- Massey triple products $/\mathbb{Q}$, etc. are all equal to zero

$\Longrightarrow$ Very strong homotopy-theoretic conditions that X be topologically a smooth algebraic variety.

Appendix: Monodromy

- C_0 = smooth algebraic curve over the origin



- fundamental group $\pi_1 = \pi_1(\mathbb{C}\setminus\{\text{slits}\})$ acts on $H_1(C_0, \mathbb{Z})$
- action of π_1 is generated by PL transformation

$$T_i : \gamma \to \gamma + (\gamma, \delta_i)\delta_i$$

- $\prod T_i$ = identity
- action of π_1 preserves the intersection form

$$Q : H_1(C_0, \mathbb{Z}) \otimes H_1(C_0, \mathbb{Z}) \to \mathbb{Z}$$

- Invariant cycles

$$H_1(C_0, \mathbb{Q})^{\text{inv}} = \text{span}\{\gamma : (\gamma, \delta_i) = 0 \text{ for all } i\}$$

- Vanishing cycles

$$H_1(C,\mathbb{Q})^{\mathrm{van}} = \mathrm{span}\{\delta_i\}$$

- *If* we know that

$$(*) \qquad H_1(C_0,\mathbb{Q})^{\mathrm{van}} \cap H_1(C_0,\mathbb{Q})^{\mathrm{inv}} = (0)$$

  then

$$Q = \begin{pmatrix} * & * = 0 \\ 0 & * \end{pmatrix}$$

  and the monodromy representation is semi-simple
- Lefschetz stated $(*)$ but his proof was incomplete — in fact

$$(*) \text{ is true, but its proof requires analysis}$$

  The analysis was provided by Hodge.

- It is a general fact proved by Deligne in the geometric case and by Schmid in general that *general monodromy representations are always semi-simple*.

The proofs require Hodge theory and are among the most basic properties of the topology of algebraic varieties.

- The reason Lefschetz wanted to have the result is that

$$(*) \iff \text{Hard Lefschetz}$$

  Lefschetz proof of this assertion was correct.