Lecture Series by Phillip Griffiths

Spring Semester 2018

Where: Ungar Bldg., Room 506 at 5:00 pm

When: Thursday, March 8th
Thursday, March 22nd
Thursday, March 29th
Thursday, April 5th

What is complex algebraic geometry?

Outline

1. Origins; algebraic functions and their integrals (modern algebraic geometry began with a result in calculus)

2. Analytic methods; PDEs and differential geometry (most of the deepest results about complex algebraic varieties such as Kodiara vanishing and Hard Lefschetz require analysis for their proofs)

3. Topology and Hodge Theory (the basic invariant of a complex algebraic variety is the Hodge structure on its cohomology; from this flows the extraordinary properties of the topology of algebraic varieties)

4. What is the Hodge conjecture and why hasn't it been proved? (the Hodge conjecture has an arithmetic aspect that is not yet understood)

Abstract

Algebraic geometry is the study of the geometry of algebraic varieties, defined as the solutions of a system of polynomial equations over a field k. When $k = C$ the earliest deep results in the subject were discovered using analysis, and analytic methods (complex function theory, PDEs and differential geometry) continue to play a central and pioneering role in algebraic geometry. The objective of these talks is to present an informal and illustrative account of some answers to the question in the title. Every attempt will be made to have the talks accessible to an audience of graduate students and post docs.

The purpose of these lectures is to discuss the question

What is complex algebraic geometry?

The question will be addressed mainly by *illustrating* how different perspectives and techniques from complex and real analysis, geometry and topology can be used to study algebraic geometry.

Lecture I: calculus and classical complex analysis will be used to study the integrals of algebraic functions of 1-variable — this is where modern algebraic geometry began.

Lecture II: use of PDEs (the Cauchy-Riemann, or $\overline{\partial}$-operator, and differential geometry (curvature) will be utilized to prove existence and uniqueness results — we will also illustrate Hodge theory for algebraic functions of 1-variable (compact Riemann surfaces))

Lecture III: topology and some Hodge theory — in significant part modern topology began in the study of single and double integrals of algebraic functions of 2-variables (algebraic surfaces, which are topological 4-manifolds) — the way this happened and how Hodge theory and the beginnings of the Hodge conjecture entered into the story will be discussed from a historical perspective.

Lecture IV: in a return to classical complex analysis, this time with a post-modern twist, we will illustrate how classical geometric questions lead to arithmetic issues, this time involving Chow groups and algebraic K-theory leading to an extension of the necessary part of the classical Abel-Jacobi theory — the sufficiency part involves constructing something that has both a geometric and an arithmetic aspect and may suggest part of what is lacking in attacking the Hodge conjecture.

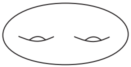
Prerequisites:

These are mainly function theory in 1 complex variable, especially integration theory — some familiarity with the Cauchy-Riemann equations, elementary aspects of differential forms in 1- and 2-variables (Stokes' theorem) and elementary topology will be useful — the subject will be presented from an intuitive, largely historical perspective.

I. Integrals of algebraic functions

Today's lecture will basically be about calculus, specifically the integrals of functions that are defined algebraically. This will lead to

- the analytic definition of the *genus* of an algebraic curve C (this is the basic algebro-geometric invariant)
- topological picture of C as a closed, oriented surface



One punch line will be

$$g(C) = \frac{1}{2} b_1(C)$$

which is the beginning of Hodge theory.

Origins: elliptic integrals

Modern algebraic geometry began with the question

How does one evaluate

$$\int \frac{q(x)dx}{\sqrt{p(x)}}$$

where $p(x), q(x)$ *are polynomials?*

These integrals arose in geometry (arclength)

$$\int ds = \int \sqrt{dx^2 + dy^2}$$

and in physics

$$\dot{y}(t)^2 = p(t) \rightsquigarrow \int \sqrt{p(t)}dt$$

(e.g., motion of a pendulum)

Books of tables have formulas for the integrals when $\deg p(x) = 1, 2$, but you were told that when $\deg p(x) \geqq 3$ they could not be evaluated. However, at a deep level they can be understood, and this story is the beginning of modern algebraic geometry. Today's lecture will try to explain how they can be understood.

$$\text{———} < \circ > \text{———}$$

- $\int r(x)dx = ?$ where $r(x) = p(x)/q(x)$ is a *rational function* — use partial fractions

$$r(x) = \sum \frac{b_i}{x - a_i} + \sum_{j=-m}^{n} c_j x^j, \ \textit{for simplicity } c_{-1} = 0$$

$$\rightsquigarrow \int r(x)dx = \sum b_i \log(x - a_i) + s(x)$$

$$\text{where } s(x) = \text{rational } f^n$$

$$= \text{“elementary function”}$$

- an *algebraic "function"* is $y(x)$ where

$$f(x, y(x)) = 0, \qquad f(x,y) = \text{ polynomial}$$

$y(x)$ not single-valued — our main example is

$$y^2 = p(x), \qquad y = \sqrt{p(x)}$$

where $p(x)$ = polynomial with distinct roots.

Example

$x^2 + y^2 = 1$, $y = \sqrt{1 - x^2}$

$$\int \sqrt{dx^2 + dy^2} = \int \frac{dx}{y(x)}$$

because $xdx + ydy = 0$ gives

$$\sqrt{dx^2 + dy^2} = \sqrt{dx^2 \left(1 + \frac{x^2}{y^2}\right)} = \sqrt{\frac{dx^2}{y^2}} = \frac{dx}{y(x)}.$$

Example

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where} \quad a > b$$

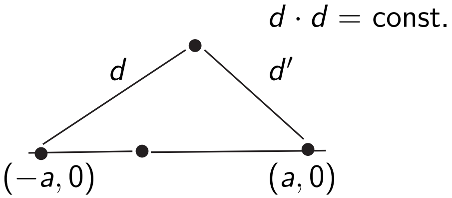
$$\rightsquigarrow x = \sin\theta, \quad y = \cos\theta, \quad k^2 = (a^2 - b^2)/a^2$$

$$\begin{aligned}\int \sqrt{dx^2 + dy^2} &= a \int \frac{(1 - k^2x^2)dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} \\ &= a \int \frac{q(x)dy}{y(x)}\end{aligned}$$

where $f(x, y) = y^2 - (1 - x^2)(1 - k^2x^2)$ and $q(x) =$ polynomial.

Example

lemniscate



$$\rightsquigarrow \int \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{1 - x^4}}$$

(where $2a^2 = 1$)

- singularity of $\frac{dx}{y}$ at $y = 0$

$$\begin{aligned} y &= \sqrt{x}\sqrt{u(x)} \text{ where } u(x) \neq 0 \\ \implies \int \frac{dx}{y} &= \int \frac{d}{x^{1/2}\sqrt{u(x)}} \\ &\sim \int \frac{dx}{x^{1/2}} \end{aligned}$$

- more interesting is the singularity of dx/y at $x = \infty$.
  - $p(x) = x^{2g+2} + a_1 x^{2g+1} + \cdots + a_{2g+2}$, $g \geqq 0$
  - $x = \frac{1}{x'} \implies dx = -\frac{dx'}{(x')^2}$

$$y(x) = \frac{1}{(x')^{g+1}} \cdot u(x'),\ u(0) \neq 0$$

$$\implies \frac{dx}{y(x)} = -\frac{(x')^{g-1}dx'}{\sqrt{u(x')}}$$

$\implies \int \frac{dx}{y(x)}$ converges for $g \geqq 1$.

- More generally

$$\int \frac{q(x)dx}{y(x)} < \infty \ \text{ for } \ \deg q(x) \leqq g - 1$$

- $\deg p(x)$

$$\begin{cases} 1, 2 \implies & \text{can evaluate but} \\ & \int dx/y \text{ diverges as } x \to \infty \end{cases}$$

$$\begin{cases} \geqq 3 \implies & \text{cannot evaluate but} \\ & \int dx/y \text{ converges everywhere} \end{cases}$$

Definition

$f(x,y)=0$ is an *algebraic curve* C (assume f irreducible).

Definition

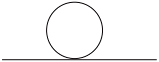
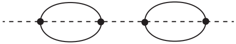
the dimension of the space of $\omega = r(x, y(x))dx$ such that $\int \omega < \infty$ is the *genus* $g(C)$ of the algebraic curve.

Example

$$y^2 = p(x) \text{ where } \deg p(x) = \begin{cases} 2g+2 \\ 2g+1 \end{cases} \quad \text{has } g(C) = g.$$

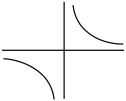
Pictures

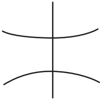
We will use the complex solutions to $f(x, y) = 0$ and will use two types of pictures. The first is the real solutions

- $x^2 + (y-1)^2 = 1 \longleftrightarrow$ 
- $y^2 = \prod_{i=1}^{4}(x - a_i) \longleftrightarrow$ 

Later on we will use complex pictures.

Second are the points at infinity — one adds to the locus $f(x, y) = 0$ in $\mathbb{C}^2$ the asymptotes





meets line at infinity in 2 points

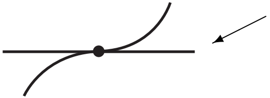
In coordinates set $x = \frac{1}{x'}$, $y = \frac{1}{y'}$ and clear denominators in $f(\frac{1}{x'}, \frac{y'}{x'}) = 0$.

Example

$$y^2 = x^3 + ax + b \quad \text{becomes}$$
$$y' = x'^3 + ax'y'^2 + by'^2$$

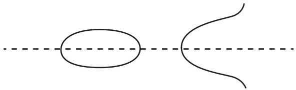
which taking, e.g., $a = 0$, $b = -1$ is

$$y' = x'^3 - y'^3$$



line at infinity
is a flex tangent

The real picture is



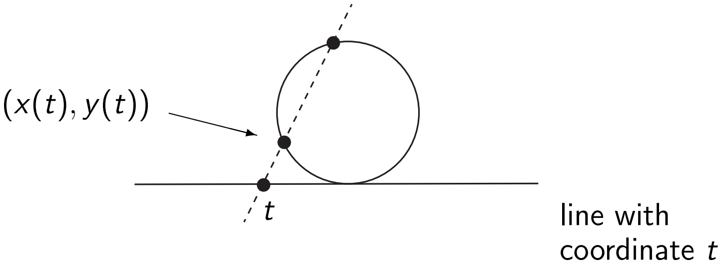
How to understand

$$u = \int r(x, y(x))dx?$$

As a first clue: Why is the integral an elementary function when $\deg f(x, y) \leqq 2$? By a linear change of coordinates we may assume that

$$f(x, y) = x^2 + (y - 1)^2 - 1$$

so that the picture is



Then

$$\begin{cases} x(t) = 4\left(\frac{t}{t^2+4}\right) \\ y(t) = \frac{2t^2}{t^2+4} \end{cases}$$

so that

rational function of t

$$\int r(x, y(x))dx = \int s(t)dt$$

$$= \text{elementary } f^{\text{n}} \text{ of } t$$

Abel's idea (c1820):

Consider the intersection of C with a rational family of other curves

$$\{f(x,y)=0\}\cap\{g(x,y,t)=0\}=\{x_i(t),y_i(t)\}$$

and the *abelian sum*

$$A(t)=\sum_i\int_{(x_0,y_0)}^{(x_i(t),y_i(t))} r(x,y(x))dx.$$

Abel's Theorem

The abelian sum is an elementary function of t; i.e.,

$$A'(t)=\text{rational } f^{\mathfrak{n}} \text{ of } t.$$

Abel's proof was an ingenious explicit computation. For the case $y^2 = p(x)$

$$\frac{g(x)dx}{y}, \qquad y = tx + c$$

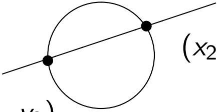
one uses the Lagrange interpolation formula to get

$$A'(t) = -\left\{ \frac{x^2 g(x)}{(tx+c)^2 - p(x)} \right\}$$

where the bracket is the constant term in the Laurent series expansion at $x = \infty$ (set $x' = \frac{1}{x}$ and take the constant term).

Example

$$x^2 + y^2 = 1, \quad g(x, y, t_1, t_2) = 0 \ \text{ is given by } \ y = t_1 x - t_2$$



Use the formulae for $\partial_{t_i} A(t_1, t_2)$ to obtain

$$\begin{cases} \partial_{t_1} A = \left\{ \frac{-2x^2}{f(x_1 t_1 x + t_2)} \right\} = -\frac{2}{1+t_2^2} \\ \partial_{t_2} A = \left\{ \frac{-2x}{f(x_1 t_2 x + t_2)} \right\} = 0 \end{cases}$$

$$\Longrightarrow \ u(t_1, t_2) = -2 \arctan t_2 = \arcsin\left(\frac{-2t_2}{1+t_2^2}\right)$$

Solving the equations for the intersection points leads to

$$\frac{-2t_1}{1+t_2^2} = x_1y_2 + x_2y_1.$$

In classical notation this gives

$$\int_0^{x_1} \frac{dx}{\sqrt{1-x^2}} + \int_0^{x_2} \frac{dx}{\sqrt{1-x^2}} = \int^{x_1y_2+x_2y_1} \frac{dx}{\sqrt{1-x^2}}$$

Suppose now we *define* $\sin u, \cos u$ by

$$u = \int_{(1,0)}^{(\sin u, \cos u)} \frac{dx}{\sqrt{1-x^2}}.$$

Then Abel's theorem yields the *addition theorem*

$$\sin(u_1+u_2) = \sin u_1 \cos u_2 + \sin u_2 \cos u_1.$$

The one for cos is similar, or we can use the general relation

$$du = \frac{x'(u)}{y(u)}du \implies y(u) = x'(u).$$

The point here is not the specific formulas but rather it is the conceptual principle given by the

Interpretation of Abel's theorem:
Define $x(u), y(u)$ by inversion of the integral; i.e.,

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} r(x, y(x))dx.$$

Then the $x(u), y(u)$ satisfy addition theorems.

What **is** *meant by the expression*

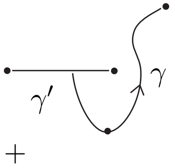
$$\int r(x, y(x))dx \ ?$$

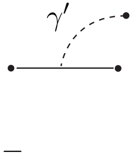
- integral takes place in complex plane

Example: Take two copies of $\mathbb{C}$ corresponding to $y = \pm\sqrt{1-x^2}$

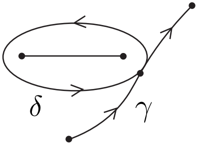
$$x^2 + y^2 = 1, \qquad y = \sqrt{1-x^2}, \qquad w = \frac{dx}{y}$$

integral depends on choice of path





γ, γ' end at different points integral is multi-valued



for $\gamma' = \gamma + \delta$

$$\int_{\gamma'} \omega = \int_{\gamma} \omega + \underset{\substack{\| \\ 2\pi}}{\int_{\delta} \omega}$$

$$\Longrightarrow \int \omega \in \mathbb{C}/2\pi\mathbb{Z} \cong \mathbb{C}^*$$

As $x \to \infty$ the integral $\int \to \infty$ so we don't include the points $x = \infty$, $y = \pm\infty$.

From the relation

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \frac{dx}{y}$$

since $(x(u+2\pi), y(u+2\pi))$ are the coordinates of the same point we have

$$\begin{cases} x(u+2\pi) = x(u) \\ y(u+2\pi) = y(u). \end{cases}$$

Taking the derivative as above

$$du = \frac{x'(u)du}{y(u)} \implies y(u) = x'(u).$$

In this way one has

- defined $(\sin u, \cos u)$ analytically
- shown they are periodic
- shown that $\sin' u = \cos u$
- shown they have an addition theorem

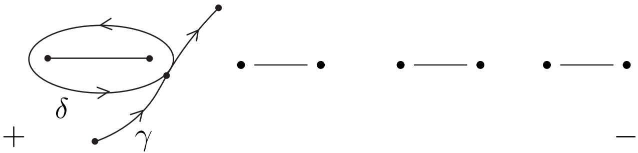
- the map

$$\begin{array}{ccc} \mathbb{C}/2\pi\mathbb{Z} & \xrightarrow{\sim} & C\setminus\{\pm\infty\} \\ \Cup & & \Cup \\ u & \longrightarrow & (x(u), y(u)) \end{array}$$

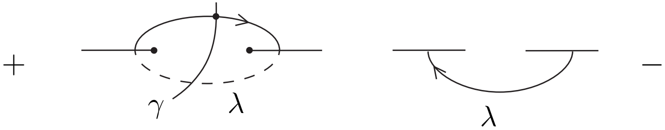
parametrizes the *algebraic curve* (minus 2 points) by *transcendental functions*

Example

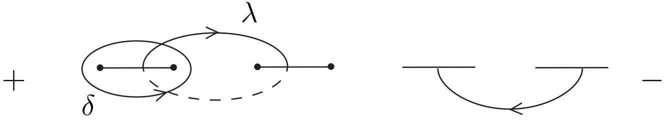
$$y^2 = \prod(x - a_i), \omega = \frac{dx}{y}$$



two copies of $\mathbb{C}$ each with two slits



and combining and leaving out γ



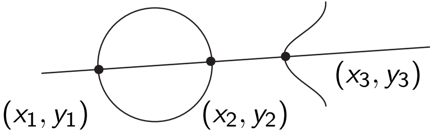
$$(\delta, \lambda) = 1$$

Same discussion applies to

$$y^2 = x^3 + ax + b$$

where the right endpoints are $\pm\infty$.

For the family of lines



Abel's theorem gives

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \omega + \int_{(x_0,y_0)}^{(x_2,y_2)} \omega + \int_{(x_0,y_0)}^{(x_3,y_3)} \omega = 0$$

where now $(x_0, y_0) =$ flex point (∞, ∞).
On the other hand

$$\begin{cases} x_3 = R(x_1, y_1; x_2, y_2) \\ y_3 = S(x_1, y_1; x_2, y_2) \end{cases}$$

where R, S are rational functions of x_i, y_i, a, b. Setting

$$\begin{cases} \pi_1 = \int_\delta \omega \\ \pi_2 = \int_\lambda \omega \end{cases}$$

and defining $x(u), y(u)$ by

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \omega$$

we have

- $x(u+\pi_i) = x(u)$, $y(u+\pi_i) = y(u)$ (doubly periodic)
- $x'(u) = y(u)$
- $\begin{cases} x(u_1+u_2) = R(x(u_1), y(u_1), x(u_2), y(u_2)) \\ y(u_1+u_2) = S(x(u_1), y(u_1), x(u_2), y(u_2)) \end{cases}$

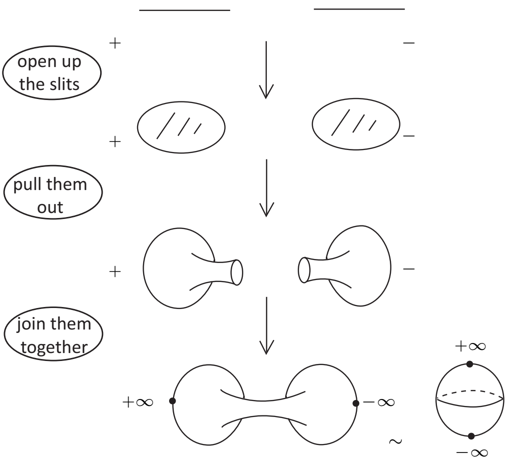
- $$\begin{array}{ccc} \mathbb{C}\Big/\mathbb{Z}\pi_2 + \mathbb{Z}\pi_2 & \longrightarrow & C \\ \Uparrow & & \Uparrow \\ u & \longrightarrow & (x(u), y(u)) \end{array}$$

  parametrizes the algebraic curve C by transcendental functions (these are the Weierstrass p, p' functions)

  What do we get topologically when we attach the two sheets across the slits?

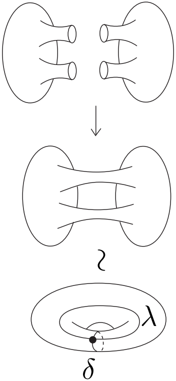
Example

2-slits, one on each sheet

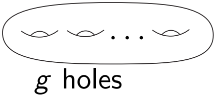


Example

4-slits, two on each sheet



In general for $y^2 = \prod_{i=1}^{2g+2} (x - a_i)$ we obtain



Conclusions:

- topologically the algebraic curve is a surface of genus g
- but g is also equal to the dimension of the space $H^{1,0}(C)$ of differentials $\omega = q(x, y(x))dx$ for which $\int \omega$ is everywhere finite.

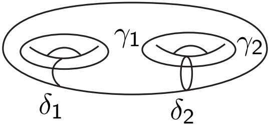
Thus

$$\boxed{\dim H^{1,0}(C) = \frac{1}{2} \dim H^1(C, \mathbb{Z})}$$

This is the first connection between algebraic geometry and topology, and it is also the beginning of Hodge theory.

- for $g = 0$ $\quad C \cong \mathbb{P}^1$ Riemann sphere
- for $g \geqq 1$ choose
  - canonical generator

$$\delta_1, \ldots, \delta_g; \gamma_1, \ldots, \gamma_g \ \text{ for } \ H_1(C, \mathbb{Z})$$



  - basis $\omega_1, \ldots, \omega_g$ for $H^{1,0}(C)$ and form the period matrix

$$\Omega = \underbrace{\begin{pmatrix} \int_{\delta_1} \omega_1 & \cdots & \int_{\delta_g} \omega_1 \\ \vdots & & \vdots \\ \int_{\delta_1} \omega_g & \cdots & \int_{\delta_g} \omega_g \end{pmatrix}}_{A} \underbrace{\begin{matrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_g} \omega_1 \\ \vdots & & \\ \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_g} \omega_g \end{matrix}}_{B}$$

where A, B are $g \times g$ matrices.

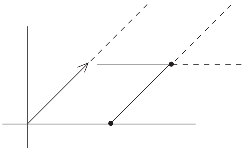
Lemma (to be proved in 2^{nd} lecture)

The columns of Ω *are linearly independent in* $\mathbb{C}^g$
$\implies$ *columns generate a lattice*

$$\begin{array}{l} \Gamma \subset \mathbb{C}^g \\ \wr\| \\ \mathbb{Z}^{2g} \end{array}$$

Example

$g = 1$



Corollary

$J(C) = \mathbb{C}^g/\Lambda$ *is a compact complex torus of dimension* g.

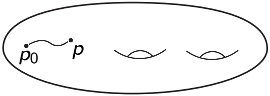
For a base point $p_0 \in C$ define

$$AJ : C \to J(C)$$

by

$$AJ(p) = \begin{pmatrix} \int_{p_0}^{p} \omega_1 \\ \vdots \\ \int_{p_0}^{p} \omega_g \end{pmatrix} \bmod \Gamma$$

where



It can be proved that

- AJ is 1-1, so that $C \subset J(C)$

For $g = 1$ we have $C \cong \mathbb{C}/\Lambda$ as above.

- for $C^{(g)}$ = configurations of g points on C, written

$$D = p_1 + \cdots + p_g,$$

the map

$$\begin{array}{ccc} C^{(g)} & \longrightarrow & J(C) \\ \Cup & & \Cup \\ D & \longrightarrow & \sum AJ(p_i) \end{array}$$

is generically 1-1 (birational)
- We will discuss the image of

$$C^{(g-1)} \longrightarrow J(C)$$

in the next lecture.

Next steps:

The surface C locally looks like open sets in $\mathbb{C}$ — it is a compact, complex manifold[1] on which one may use analysis (PDEs and differential geometry) to study its properties — we will discuss/illustrate this in the next lecture.

[1] This will be defined in the 2^{nd} lecture.