HODGE THEORY AND MODULI OF H-SURFACES

MARK GREEN, PHILLIP GRIFFITHS,
RADU LAZA AND COLLEEN ROBLES

Appendix to §I: I-surfaces

Definition: An *I-surface* is a smooth, regular minimal surface X of general type that satisfies

$$\begin{cases} K_X^2 = 1 \\ p_g(X) = 2. \end{cases}$$

These surfaces are well known classically (cf. Chapter VII in [BPVdV84]). In a sense they should come before H-surfaces in relating algebro-geometric and Hodge-theoretic moduli. They are structurally simpler than H-surfaces and may be thought of as "toy models" for H-surfaces. We shall derive some of their properties, including the

Theorem: (i) *The local Torelli property is valid for any I-surface.* (ii) *The period mapping is*

$$\Phi : \mathcal{M}_I \to \Gamma\backslash D_I$$

where

$$\dim \mathcal{M}_I = \left(\frac{1}{2}\right)(\dim D_I - 1) = 28,$$

and where $\Phi(\mathcal{M}_I)$ is a maximal integral manifold of the IPR on D_I, which is a contact system.

The only other similar example we are aware of where the IPR may locally be explicitly "integrated" is Calai-Yau 3-folds (cf. [BG].)

(i) *Projective realization of an I-surface*

We denote by $Q_0 \subset \mathbb{P}^3$ the quadric $\{x_0x_2 = x_1^2\}$ with singular point $p = [0,0,0,1]$.

Proposition: *A general I-surface X is realized via the bi-canonical map as a 2:1 covering of Q_0 branched over p and $V \cap Q_0$ where $V \subset \mathbb{P}^3$ is a general quintic surface not passing through p. Via its 5-canonical map it is realized as a hypersurface*

$$z^2 = F_5(t_0,t_1,y)z + F_{10}(t_0,t_1,y)$$

in $\mathbb{P}(1,1,2,5)$ with coordinates $[t_0,t_1,y,z]$ and where F_k is a weighted homogeneous polynomial of degree k.

Proof. The pencil $|K_X|$ has no fixed component so by Bertini a general $C \in |K_X|$ is a smooth curve of genus $g = \frac{1}{2}(K_X \cdot C + C^2) + 1 = 2$. We choose a basis t_0, t_1 for $H^0(K_X)$ such that $C = \{t_0 = 0\}$. From the general formula

$$\begin{aligned} h^0(mK_X) &= \left(\frac{m(m-1)}{2}\right) K_X^2 + \chi(\mathcal{O}_X), \qquad m \geqq 2 \\ &= \frac{m(m-1)}{2} + 3 \end{aligned}$$

we have $h^0(2K_X) = 4$. Setting $K_C^{1/2} = K_X\big|_C$, fom the exact cohomology sequence of

$$0 \to (m-1)K_X \xrightarrow{t_0} mK_X \to K_C^{m/2} \to 0$$

we may choose a basis t_0^2, t_0t_1, t_1^2, y for $H^0(2K_X)$ where the restrictions of t_1^2, y to C give a basis for $H^0(K_C)$. It follows that $(2K_X)$ is base point free and that using the above basis as homogeneous coordinates we have

$$\varphi_{2K_X} : X \to Q_0 \subset \mathbb{P}^3.$$

Since $t_0(p) = t_1(p) = 0$, it follows that $y(p) \neq 0$, so that near p

$$\varphi_{K_C} = t_1^2/y$$

vanishes to 2[nd] order at p. Thus φ_{K_C} is a 2:1 mapping to one of the rulings of Q_0 which is branched at the vertex p and at 5 residual points on the ruling. It follows that

$$\varphi_{2K_X} : X \to Q_0$$

is a 2:1 map branched over $p + V$ when $V \in |Q_0(5)|$ does not pass through p.

For the second part of the theorem, using the above formula for the $h^0(mK_X)$ and the exact cohomology sequences arising from the above exact sheaf sequence we have

- $H^0(2K_X)$ has dimension 4 with basis given by the weighted degree 3 monomials in t_0, t_1, y where t_0, t_1 have weight 1 and y has weight 2;
- $H^0(4K_X)$ has basis the degree 4 weighted monomials in t_0, t_1, y;

- $H^0(5K_X)$ has basis the degree 5 weighted monomials in t_0, t_1, y plus one additional weight 5 generator z.

For the pluri-canonical ring $R_X = H^0(mK_X)$ we have

$$R_X \supset \mathbb{C}[t_0, t_1, y] \oplus z\mathbb{C}[t_0, t_1, y].$$

The two summands on the right are the ± 1 eigenspaces for the action of the involution $\tau : X \to X$ induced by the sheet interchange associated to the branched covering $\varphi_{2K_X} : X \to Q_0$. Computing dimensions we see that equality holds in the above inclusion, from which it follows that for R_X there is a generating relation

$$z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y). \qquad \square$$

For later reference we note that since

$$X \cong \operatorname{Proj} R_X$$

it follows that the image $\varphi_{2K_X}(X) \subset \mathbb{P}(1, 1, 2, 5)$ is a smooth surface biregularity equivalent to X.

(ii) *Alternate realization of an I-surface*

We set

$$F = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \quad \xi = \mathcal{O}_{\mathbb{P}F}(1).$$

Then the linear system $|\xi|$ gives the desingularization map

$$f : \mathbb{P}F \to Q_0.$$

With our usual notations, we have a unique up to scaling section $x \in |\xi - 2\mathfrak{h}|$ with divisor $S \cong \mathbb{P}^1$; then the self-intersection $S^2 = -2$ and f contracts this -2 curve to the node $p \in Q_0$.

Proposition: *Denoting by $\hat{X}$ the blow up of X at the base point p of $|K_X|$, we have a mapping*

$$g : \hat{X} \to \mathbb{P}F$$

which is a 2:1 covering branched over $\hat{B} = S + \hat{V}$ where $\hat{V} \in |5\xi|$.

Proof. This is very similar to the corresponding result for H-surfaces. We note that if a section $\beta \in H^0(\mathbb{P}F, [\hat{B}])$ defines $\hat{B}$, then $[\hat{B}] = 6\xi - 2k\mathfrak{h}$ and for $L = 3\xi - k\mathfrak{h}$ with $L^2 = [\hat{B}]$ we may construct the embedding

$$\hat{X} \to \mathcal{O}_{\mathbb{P}F} \oplus L$$

where $\hat{X} = \{1 \oplus \lambda(q) : q \in \mathbb{P}F \text{ and } \lambda(q)^2 = \beta(q) \in L_q\}$. Using

$$\begin{cases} K_{\hat{X}} = \pi^*(K_{\mathbb{P}F} \otimes L) \\ K_{\mathbb{P}F} = -2\xi \text{ and } L = 3\xi - k\mathfrak{h} \\ h^0(K_{\hat{X}}) = 2 \end{cases}$$

we find that $k = 1$ and $\hat{B} \in |6\xi - 2\mathfrak{h}|$. Using that p is a branch point of all $C \in |K_X|$, writing

$$\hat{B} = S + \hat{V}$$

where $S = (x)$ for $x \in |\xi - 2\mathfrak{h}|$ and $\hat{V} \in |5\xi|$ gives the proposition. □

We note that

$$F_5^2 - 4F_{10} \in |\mathcal{O}_{\mathbb{P}^3}(5)|$$

gives the section with divisor $\hat{V}$.

Computation of moduli

PROPOSITION: *The KSBA moduli space $\mathcal{M}_I$ is smooth of dimension 28.*

Proof. We shall give two arguments. For the first,

$$\hat{V} \in H^0(\mathbb{P}F, 5\xi) \cong H^0(\mathbb{P}^1, S^5F).$$

Using

$$S^5F \cong \mathcal{O}_{\mathbb{P}^1}(10) \oplus \mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$$

we have

$$h^0(\mathbb{P}^1, S^5F) = 11 + 9 + 7 + 5 + 3 + 1 = 36.$$

On the other hand, over $\mathbb{P}F$ we have the Euler sequence

$$0 \to F^* \otimes \xi \to \Sigma_{\mathbb{P}F,\xi} \to \mathfrak{h}^2 \to 0$$

which gives

$$h^0\left(\Sigma_{\mathbb{P}F,\xi}\right) = h^0(\mathfrak{h}^2) + h^0(F^* \otimes F) = 8.$$

The argument is now similar to that for H-surfaces.

For the alternate proof, denoting by $P_k(t_0,t_1)$ a homogeneous polynomial of degree k in t_0,t_1, we have for the weight 10 homogeneous polynomials in t_0,t_1,y,z

$$\begin{array}{cccccc}
 & z^2 & & & y^2P_6(t_0,t_1) & \\
 & z\cdot P_5(t_0,t_1) & z\cdot P_3(t_0,t_1) & zy^2P_1(t_0,t_1) & yP_8(t_0,t_1) & \\
 & y^5 & y^4\cdot P_2(t_0,t_1) & y^3P_4(t_0,t_1) & P_{10}(t_0,t_1) & \\
\text{dimension} & =8 & +7 & +7 & +27 & =49
\end{array}$$

The automorphisms of $\mathbb{P}(1,1,2,5)$ are

$$\begin{cases} z\to zP_0+y^2P_1+yP_3+P_5 & 13\\ y\to yP_0+P_2 & 4\\ t_0,t_1\to at_0+bt_1+ct_0+dt_1 & \underline{4}\\ & 21\end{cases}$$

which gives the result. □

From Noether's formula

$$\chi(\mathcal{O}_X)=\frac{1}{12}(K_X^2+\chi_{\text{top}}(X))=\frac{1}{12}(1+2h^0+2h^{2,0}+h^{1,1})$$

we have

$$36=7+h^{1,1},$$

which gives

$$h^{1,1}_{\text{prim}}=28.$$

Thus

$$\dim D=2h^{1,1}_{\text{prim}}+1=57,$$

and the maximal integral manifolds of the IPR, which is a contact system, have dimension 28=number of moduli.

(iii) *The local Torelli theorem for smooth I-surfaces*

The argument will follow along the general lines of that for smooth surfaces in ordinary $\mathbb{P}^3$, but with an interesting wrinkle. We begin by collecting a few general facts about weighted projective spaces (cf. [Dol]). For positive interers $a_0,\dots,a_r$ with $\gcd(a_i)=1$ we denote by

$$\mathbb{P}=:\mathbb{P}(a_0,\dots a_r)$$

the corresponding weighted projective space, defined as the quotient

$$\mathbb{C}^{r+1}\backslash\{0\}/\mathbb{C}^*$$

by the action

$$\lambda \cdot (x_0, \dots, x_r) = (\lambda^{a_0} x_0, \dots \lambda^{a_r} x_r)$$

of the 1-parameter group generated by the Euler vector field

$$\mathbf{e} = \sum_{i=0}^{r} a_i x_i \partial_{x_i}.$$

For the standard sheaf $\mathcal{O}_{\mathbb{P}}(1)$ we have the weighted projective version of the consequences of Bott vanishing and Kodaira-Serre duality

- $H^0(\mathcal{O}_{\mathbb{P}}(d)) \cong \{\text{weighted homogeneous polynomials of degree } d\}$
- $H^q(\mathcal{O}_{\mathbb{P}}(d)) = 0$ for all d and $0 < q < r$
- $H^r(\mathcal{O}_P(d)) \cong H^0\left(\mathcal{O}_{\mathbb{P}}\left(\sum_i a_i - d\right)\right)^*$.

We identify the sheaf $\Sigma_{\mathbb{P},\xi} =: \Sigma_{\mathbb{P}}$ of differential operators of order $\leqq 1$ of $\mathcal{O}_{\mathbb{P}}(d)$ as

$$\Sigma_{\mathbb{P}} \cong \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}}(-(d-a_i))$$

via the map

$$F \to \sum_{i=0}^{r} G_i \partial_{x_i} F, \qquad G_i \in \mathcal{O}_{\mathbb{P}}(-(d-a_i)).$$

Assuming that the Jacobi ideal has no base locus, i.e., that the intersection

$$\bigcap_{i=0}^{r} \{\partial_{x_i} F = 0\} = \emptyset,$$

we have

$$\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}}(-(d-a_i)) \xrightarrow{dF} \mathcal{O}_{\mathbb{P}} \to 0.$$

This sequence completes to a Koszul complex

$$0 \to \bigwedge^{r+1} \big((\oplus \mathcal{O}_{\mathbb{P}}(-(d-a_i))\big) \to \bigwedge^{r} \big(\oplus \mathcal{O}_{\mathbb{P}}(-(d-a_i))\big)$$
$$\to \cdots \to \mathcal{O}_{\mathbb{P}}(-(d-a_i)) \to \mathcal{O}_{\mathbb{P}} \to 0.$$

From the Bott-vanishing and duality results listed above we have

$$\frac{H^0(\mathcal{O}_{\mathbb{P}}(k))}{\operatorname{Im} dF} \cong \ker\left\{H^r\left(\mathcal{O}_{\mathbb{P}}(-(r+1)d+\Sigma a_i)\right) \xrightarrow{dF} \oplus H^r\left(\mathcal{O}_{\mathbb{P}}(-(r+1)d+2\Sigma a_i)\right\}\right.$$
$$\cong \left(\frac{H^0(\mathcal{O}_{\mathbb{P}}(r+1)d-2\Sigma a_i-k)}{\operatorname{Im} dF}\right)^*,$$

which then gives a perfect pairing

$$\frac{H^0(\mathcal{O}_{\mathbb{P}}(k))}{\operatorname{Im} dF}\otimes\frac{H^0(\mathcal{O}_{\mathbb{P}}((r+1)d-2\Sigma a_i-k)}{\operatorname{Im} dF} \to \frac{H^0(\mathcal{O}_{\mathbb{P}}((r+1)d-2\Sigma a_i)}{dF} \cong \mathbb{C}$$

for $0 \leqq k \leqq (r+1)d-2\Sigma a_i$. This is the weighted projective space analogue of the usual Macaulay's theorem.

The situation we are interested in is where

- $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2, a_3)$
- $F \in H^0(\mathcal{O}_{\mathbb{P}}(d))$ and $F = 0$ defines a smooth surface $X \subset \mathbb{P}$.

Then

- $K_X \cong \mathcal{O}_X(d - \Sigma a_i)$,

and for the sheaf Σ_X defined above and primitive part of $H^1(\Omega^1_X)$ we have

- $H^1(\Sigma_X) \cong H^0(\mathcal{O}_{\mathbb{P}}(d))/\operatorname{Im} dF$
- $H^1(\Omega^1_X)_{\text{prim}} \cong H^0(\mathcal{O}_{\mathbb{P}}(2d-\Sigma a_i))/\operatorname{Im} dF$.

Here the second identifiaction uses that for any smooth surface

$$H^1\left(\Omega^1_X\right)_{\text{prim}} \cong H^1(\Sigma_X \otimes K_X)$$

(cf. the first proposition in section I.G). Using the identifications

- $\bigoplus_i H^0(\mathcal{O}_{\mathbb{P}}(a_i)) \cong \bigoplus_i H^0(\mathcal{O}_X(a_i))$
- $H^0(K_X) \cong H^0(\mathcal{O}_{\mathbb{P}}(d-\Sigma a_i))$
- $H^1(\Omega^1_X)_{\text{prim}} \cong \frac{H^0(\mathcal{O}_{\mathbb{P}}(2d-\Sigma a_i))}{\operatorname{Im} dF}$
- $H^1(\Sigma_X)^* \cong \left(\frac{H^0(\mathcal{O}_{\mathbb{P}}(3d-\Sigma a_i))}{dF}\right)^*$,

local Torelli will follow if the map

$$H^0\big(\mathcal{O}_{\mathbb{P}}(d-\Sigma a_i)\big)\otimes H^0\big(\mathcal{O}_{\mathbb{P}}(2d-\Sigma a_i)\big) \to H^0\big(\mathcal{O}_{\mathbb{P}}(3d-2\Sigma a_i)\big)$$

is surjective. More precisely, as a consequence of the perfect pairing above

Local Torelli will follow if this map is surjective modulo the image of dF.

For smooth I-surfaces we will see that the above pairing is not surjective, but it is surjective modulo the image of dF. This is in contrast to the case of the usual projective space where all maps

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell)) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k+\ell)), \quad k, \ell \geqq 0,$$

are surjective.

We now turn to the case where we have

$$\mathbb{P} = \mathbb{P}(1,1,2,5), \qquad d = 10.$$

Then $d - \Sigma a_i = 1$ giving

$$K_X \cong \mathcal{O}_X(1),$$

and as noted above $H^0(K_X) \cong H^0(\mathbb{P}, \mathcal{O}(1))$ has dimension 2 with basis t_0, t_1. Using a table to exhibit bases and count dimensions as was done before, for $H^0(\mathcal{O}_{\mathbb{P}}(11))$ we have

$$\begin{array}{llllll} z^2P^1 & & & & & \\ z \cdot y^3 & zj^2P_2 & zyP_4 & zP_6 & & \\ y^5 \cdot P_1 & y^4P_3 & y^3P_3 & y^2P_7 & yP_9 & P_{11}. \end{array}$$

From this we observe that

> The image of the pairing $H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(10)) \to H^0(\mathcal{O}_{\mathbb{P}}(11))$ has codimension 1. In fact
>
> $$zy^3$$
>
> does not belong to the image.

Thus to prove local Torelli we need to show that

> *For a non-singular I-surface $X \subset \mathbb{P}(1,1,2,5)$ defined by the equation $F = z^2 - F_{10}(t_0, t_1, y) = 0$*
>
> $$\operatorname{Im} dF \text{ surjects onto } \frac{H^0(\mathcal{O}_{\mathbb{P}}(11))}{\operatorname{Im}\{H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(10))\}} \cong \mathbb{C}zy^3.$$

But this is clear since

$$\partial_z F = 2z + \{\text{terms not involving z}\}$$

and so that for $y^3 \in H^0(\mathcal{O}_{\mathbb{P}}(6))$ we see that $\partial_z F \cdot y^3$ is non-zero in the above quotient.

Remarks: By a suitable automorphism of $\mathbb{P}(1,1,2,5)$ we may assume that the equation of $\varphi_{2K_X}(X)$ has the above form. Calculations similar to the above give

$$\begin{aligned} T_X\mathcal{M}_I &\cong H^0(\mathcal{O}_{\mathbb{P}}(10))/\operatorname{Im} dF \text{ has dimension } = 28 \\ H^1(\Omega^1_X)_{\text{prim}} &\cong H^0(\mathcal{O}_{\mathbb{P}}(11))/\operatorname{Im} dF \text{ has dimension } = 28, \end{aligned}$$

confirming our earlier computation for the first and the consequence of Noether's theorem for the second.