$\boxed{\text{Outline}}$

I. Complex differential geometry

II. Classical uses of curvature

III. Singular metrics and curvature

Theme: The unreasonable effectiveness of curvature in algebraic geometry

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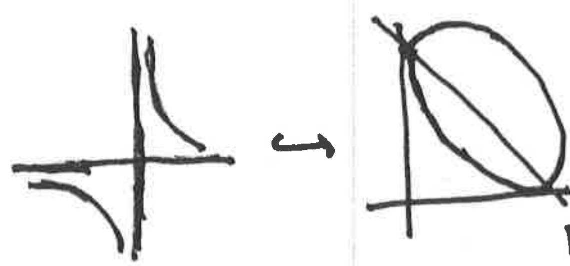
- complex algebraic geometry is study of

$$\begin{array}{l} X^0 = \{P_i(z_1,\dots,z_N)=0\} \subset \mathbb{C}^N \\ \cap \\ X \subset \mathbb{P}^N \quad \text{- use } \mathbb{P}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1}_\infty \\ \qquad\qquad\qquad \text{to have asymptotes} \end{array}$$



$\mathbb{P}^1_\infty$

- maps $X \to Y$ and equivalences

conic $\cong \mathbb{P}^1$

- classical methods
  - geometric
  - topology
  - differential geometry (curvature)
  - analysis {complex function theory, PDE's}

- current methods primarily algebraic (commutative and homological) - handle singularities on a par with smooth case - build on classical results
- recently use of singular metrics/curvature - leads to results not accessible algebraically

- Preview for how curvature enters for $\Sigma = \left\{\begin{array}{l}\text{algebraic curve} \\ \text{compact Riemann surface}\end{array}\right\}$

- existence: $f : \Sigma \to \mathbb{P}^1$ where

  degree $f^{-1}(p) \leq g+1$

  $D = \sum p_i$

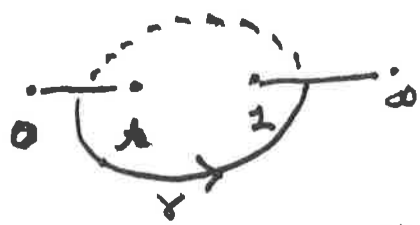
  $\deg D \geq g+1 \Rightarrow \left\{\begin{array}{l}\text{there exists } f \\ \text{with poles on } D\end{array}\right\}$

- $\left\{\begin{array}{l}\text{asymptotics} \\ \text{of periods}\end{array}\right\}$

  $f(x,y,t) = y^2 - x(x-1)(x-t)$

  $\pi(t) = \int_\gamma \frac{dx}{y}$

  $\pi(t) = A \frac{\log t}{2\pi i} + B$



curvature of resulting Hodge metrics
$\|\psi\|^2 = \int \psi_t \wedge \overline{\psi}_t$ measures the singularities

- <u>classification</u>
  - $G > 0$, $\mathbb{P}^1$, $\kappa(X) = -\infty$
  - $G = 0$, $\mathbb{C}/\Lambda$, $\kappa(X) = 0$
  - $G < 0$, $\mathcal{H}/\Gamma$, $\kappa(X) = 1$

curvature of Hodge bundles $\rightsquigarrow$ structural result

- <u>Schematic</u>: $L \to X$ line bundle

curvature positive $(L > 0)$ $\leftarrow$ differential geometric

$\downarrow$

harmonic forms are zero $\leftarrow$ analytic

$\downarrow$

cohomology vanishes $\leftarrow$ algebro-geometric

$\downarrow$

existence

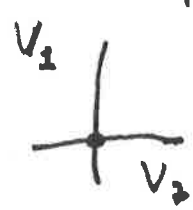
<u>Warmup</u>: some aspects of complex geometry

- $\mathbb{C}$ is oriented

$$(i/2)\, dz \wedge d\bar{z} = dx \wedge dy > 0$$

- holomorphic maps preserve orientation
  - $df = f_x\, dx + f_y\, dy = f_z\, dz + f_{\bar{z}}\, d\bar{z}$
  - f holomorphic $\Longleftrightarrow f_{\bar{z}} = 0$ (CR eqns)
  - $w = f(z) \Rightarrow (i/2)\, dw \wedge d\bar{w} = |f_z(z)|^2 (i/2)\, dz \wedge d\bar{z}$

- complex manifolds are oriented - complex submanifolds meeting in points have positive intersection number



$$V_1 \cdot V_2 = V_2 \cdot V_1 > 0$$

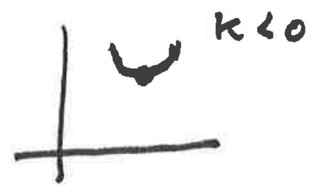
- in complex geometry things tend to have signs

• maximum principle: $|f(z)|$ has no interior maximum

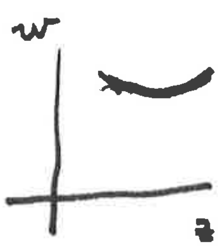
<u>Geometric argument</u>

$\mathbb{R}^2$

or



$\mathbb{C}^2$



- $w = f(z)$ gives $V \subset \mathbb{C}^2$
- $G(V) \leqq 0$ (curvatures decrease on submanifolds)

• conformal metric

$$H = h\, dz \otimes d\bar{z} = \underbrace{h\,(dx^2 + dy^2)}_{ds^2} - 2i\, \underbrace{h\, dx \wedge dy}_{\omega}$$

- curvature form

$$\Omega = (i/2)\, \partial\bar\partial \log h = G\omega$$

<u>Ex</u>
$$\frac{dz\, d\bar z}{(1-|z|^2)^2} \quad \text{on } \Delta, \quad G=-1$$

$$\frac{dt\, d\bar t}{|t|^2(\log|t|)^2} \quad \text{on } \Delta^* = \{0<|t|<1\}$$

$$\ell(x) \sim \log(1/x)$$

$$\ell(\theta) \sim \frac{\theta}{\log(1/x)}$$

- <u>Ahlfors' lemma</u>:

$f: \Delta \to I$ where $G_I \leq -1$

$\Downarrow$

$f^*(ds^2_I) \leq ds^2_\Delta$ (f is distance decreasing)

<u>Pf</u>: Maximum principle Schwarz

For $f: \Delta \to \Delta$, $f(0)=0$ this is $|f(z)| \leq |z|$

Hermitian differential geometry

- $E \xrightarrow{\pi} X$ holomorphic vector bundle

  $\pi^{-1}(U_\lambda) \cong U_\lambda \times \mathbb{C}^n$, $g_{\lambda\mu}(z) \in GL(n, \mathbb{C})$

- $A^{0,q}(E) = C^\infty$ E-valued $(0,q)$ forms

  - $\psi = (\psi_{\lambda\alpha} = \sum_I \psi_{\lambda\alpha I} \, d\bar{z}^{i_1} \wedge \ldots \wedge d\bar{z}^{i_q})$

  - $\bar\partial\psi = \left(\sum_I \bar\partial\psi_{\lambda\alpha I} \wedge d\bar{z}^I \right)$

  - $\bar\partial g_{\lambda\mu} = 0 \Rightarrow \bar\partial : A^{0,q}(E) \to A^{0,q+1}(E)$

  - $\bar\partial^2 = 0$, $H^{0,q}_{\bar\partial}(E) \cong H^q(\mathcal{O}_X(E))$

    Dolbeault

- Hermitian metric

  - (e, e') given by $h_{\lambda\alpha\bar\beta} = \overline{h_{\lambda\beta\bar\alpha}} > 0$

  - line bundle $h_\lambda > 0$, $h_\lambda = |g_{\lambda\mu}|^2 h_\mu$

- *Ex* $\dim X = 1$, $E = TX$, conformal metric

- Chern connection: There exists unique

$$D : A^0(E) \to A^1(E)$$

with

 - $D'' = \bar\partial$

 - $d(e, e') = (De, e') + (e, De')$

- connection $\theta_\lambda = h_\lambda^{-1} \partial h_\lambda$

 curvature $\Theta_E = \bar\partial \theta_\lambda \in A^{1,1}(\mathrm{End}\, E)$

- curvature form $\Theta_E(e, \xi) = (\Theta_E(e), e)(\xi \wedge \bar\xi)$

 where $e \in E_x$, $\xi \in T_x X$

- E positive if there exists Hermitian metric with $\Theta_E(\xi, \xi) > 0$

- Functoriality properties ($E^\vee < 0$, $E \oplus F$ etc)

- for line bundles usually use $\Omega = \Theta_L$

$$\Omega = -(i/2)\,\partial\bar\partial \log \|\sigma\|^2 \quad , \sigma \neq 0$$

$$= -(i/2)\,\partial\bar\partial \log (h_\lambda |\sigma_\lambda|^2)$$

- Two "natural" examples of positive vector bundles

  - $f: X \longrightarrow \mathbb{P}^N$

  - Hodge theory

Ex. $\mathbb{P}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1}$

$= $ lines through origin in $\mathbb{C}^{N+1}$

homogeneous coordinates $[z] = [z_0, .., z_N]$

- $H^* \to \mathbb{P}^N$ has fibre $H^*_{[z]} = \mathbb{C}z$

- choice of metric in $\mathbb{C}^{N+1}$ gives metric in $H^* \to \mathbb{P}^N$

- $\Omega^* = -(i/2)\,\partial\bar\partial \log \|z\|^2$

$$= -\left(\frac{1}{\|z\|^4}\right)\Big((z,z)(dz,dz) - (dz,z)\wedge(z,dz)\Big) = FS$$

- $\Omega > 0$ on $\mathbb{P}^N$; $H = \mathcal{O}_{\mathbb{P}^N}(1) > 0$

- $f: X \hookrightarrow \mathbb{P}^N \Rightarrow L = f^*\mathcal{O}_{\mathbb{P}^N}(1) > 0$

- <u>Kodaira's theorem</u>: <u>$L \to X$ positive</u> $\Rightarrow$ $L^m = f^*\mathcal{O}_{\mathbb{P}^N}(1)$ <u>for</u> $m \geq m_0$

  (L is <u>ample</u>)

Have to use curvature assumption to construct something

- In more detail, there are no non-constant holomorphic functions on a compact complex manifold (maximum principle)
- So we want to construct meromorphic functions - do this by saying where poles are allowed
- Divisor D is given
  - $D \cap \mathcal{U}_\lambda = \{ f_\lambda = 0 \}$
  - $f_\lambda / f_\mu = g_{\lambda\mu} \in \mathcal{O}^*_X (\mathcal{U}_\lambda \cap \mathcal{U}_\mu)$
  - gives line bundle $\mathcal{O}_X(D)$

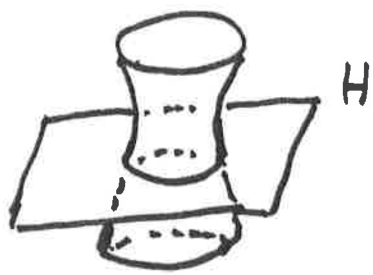
- sections given by holomorphic functions h_λ in U_λ such that

$$h_\lambda / f_\lambda = h_\mu / f_\mu \quad \text{in } U_\lambda \cap U_\mu$$

$$\Updownarrow$$

$$h_\lambda = g_{\lambda\mu} h_\mu$$

- $H^0(L)$ denotes space of sections of $L \to X$; then $H^0(\mathcal{O}_X(D)) =$ meromorphic functions with poles on D



<u>Ex</u> $X \subset \mathbb{P}^N$

$D = H \cap X$

$H^0(\mathcal{O}_X(1)) \cong$ linear functions $\ell(z)$

- construct $f : X \to \mathbb{P}^N$ by having $L \to X$ with sections; e.g., we want

$$H^0(L) \to L_x \to 0$$

for each $x \in X$

- most important bundle is the <u>canonical bundle</u> $K_X = \Lambda^n T^* X$ sections are locally

$$\psi = f(z)\, dz_1 \wedge \cdots \wedge dz_n$$

- more important than $L > 0$ will be

$$L + K_X > 0 \qquad (L + K_X = L \otimes K_X)$$

<u>Ex</u> $\mathcal{O}_{\mathbb{P}^1}(p_1+p_2+p_3) + K_{\mathbb{P}^1} > 0$ - sections are

$$\psi(z) = \frac{\ell(z)\, dz}{(z-a_1)(z-a_2)(z-a_3)} \quad , \quad d\left(\frac{1}{z}\right) = -\frac{dz}{z^2}$$

Gives metric with positive curvature in $K_X(p_1+p_2+p_3)$, which then gives metric on $\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$ with $G \leq -1$ - using Ahlfors lemma gives

$K_X = T^*X$

$$\begin{array}{l} f: \Delta^* \longrightarrow \mathbb{P}^1 \setminus \{p_1, p_2, p_3\} \\ \quad \cap \quad \nearrow \\ \quad \Delta \end{array}$$

<u>Corollary</u>: Picard's theorem

- argument extends to

$$f: \mathbb{C}^n \to \mathbb{P}^n \setminus \{H_1 \cup \dots \cup H_{n+1}\}$$

$\Rightarrow f(\mathbb{C}^n)$ lies in linear subspace

Beginning of Nevanlinna theory ~ non-compact algebraic geometry

- $E = TX$ - metric gives $TX \cong T^*X$
  - $\Theta_{TX}(\eta, \xi) =$ holomorphic bi-sectional curvature
  - $\Theta_{TX}(\xi, \xi) =$ holomorphic sectional curvature

- $S \subset E$ holomorphic sub-bundle,

$$\Theta_S(e,\xi) = \Theta_E(e,\xi) - (A(\xi)e, A(\xi)e)$$

where $A \in A^{1,0}(\mathrm{Hom}(S,E))$

"curvatures <u>decrease</u> on sub-bundles"

Ahlfors lemma applies to $f: \Delta \to X$ where $\Theta_{TX}(\xi,\xi) \leq -1$

- $\det\left(\left(\frac{1}{2\pi i}\right)\Theta_E + \lambda I\right) = \sum c_q(\Theta_E)\lambda^{r-q}$

$$\rightsquigarrow [c_q(\Theta_E)] = c_q(E)$$

<u>Ex</u> $\sigma \in H^0(L)$ holomorphic section of $L \to X$ with divisor $D = (\sigma)$

$$\underset{H^2(X)}{[D]} = \underset{H^2_{DR}(X)}{\left[\left(\frac{1}{2\pi i}\right)\Theta_L\right]}$$

II. Classical uses of curvature

- Want to use positivity of curvature to construct something geometric - e.g. meromorphic functions or sections of

$$L \to X$$

- ($\sigma_0, \sigma_1 \in H^0(L) \Rightarrow f = \sigma_0/\sigma_1$ is a meromorphic function)

- Projective realizations

$$f : X \to \mathbb{P}^N$$

are given by $\sigma_0, .., \sigma_N \in H^0(L)$ with no common zeroes - then

$$f(x) = [\sigma_0(x), .., \sigma_N(x)]$$

- Philosophy: "Positivity $\rightsquigarrow$ sections"

Will discuss two instances of this

- proof of Kodaira's theorem
- classification of algebraic varieties using Hodge theory

• Mechanism for first is

$$\left\{\begin{matrix}\text{positive}\\ \text{curvature}\end{matrix}\right\} \xrightarrow[\text{differential geometric}]{} \left\{\begin{matrix}\text{vanishing}\\ \text{of cohomology}\end{matrix}\right\} \xrightarrow[\text{algebro-geometric}]{} \left\{\begin{matrix}\text{existence}\\ \text{of sections}\end{matrix}\right\}$$

• $H^q(\mathcal{O}_X(E)) \cong H^{0,q}_{\bar\partial}(E) \cong \mathcal{H}^{0,q}_{\square}(E)$

curvature assumption $\Rightarrow$ RHS $= 0$

In some form, all vanishing theorems have this as a source

(weaker version of KVT in char p using Weil conjectures)

- <u>Kodaira vanishing theorem (KVT)</u>

$$L > 0 \Rightarrow H^q(L^*) = 0 \quad \text{for } q < n$$

- $\sigma \in H^0(L^*)$ - then

$$(i/2)\partial\bar\partial \log \|\sigma\|^2 = -\Theta_{L^*} > 0$$

- $\square = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$

$$= -\sum_i \nabla_i \nabla_{\bar i} + R(\Theta)$$

$$\Rightarrow 0 = (\square\psi, \psi) = \|\nabla\psi\|^2 + (R\psi, \psi)$$

- <u>Kodaira-Serre duality</u>

- $H^q(L^*) \cong H^{n-q}(K_X \otimes L)$

- $H^q(\Omega_X^p(L^*)) \cong H^{n-q}(\Omega_X^{n-p}(L))$

$$\begin{array}{ccc} H^q(\Omega_X^p(L^*)) & \cong & H^{n-q}(\Omega_X^{n-p}(L)) \\ \| & & \| \\ H^{p,q}(L^*) & & H^{n-p,n-q}(L) \end{array}$$

• Akizuki-Nakano-Kodaira V.T

(*) $$L > 0 \Rightarrow H^{p,q}(L^*) = 0 \quad \text{for } p+q < n$$

Will discuss later that if $\sigma \in H^0(L)$ with $Y = (\sigma)$ smooth

$$(*) \Leftrightarrow \left\{ \begin{array}{ll} H^q(X) \xrightarrow{\sim} H^q(Y), & q \leq n-2 \\ H^{n-1}(X) \hookrightarrow H^{n-1}(Y) & \end{array} \right\}$$

$$\uparrow$$

Lefschetz

Simplest case

$$0 \to L^* \xrightarrow{\sigma} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$$\Downarrow$$

$$0 \to H^0(L^*) \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_Y) \to H^1(L^*)$$

$$\Downarrow$$

$\dim X \geq 2 \Rightarrow Y$ is connected any coherent sheaf

- $L > 0 \Rightarrow H^q(L^m \otimes E) = 0$, $q > 0$ and $m \geq m_0$

Pf E = vector bundle

$$H^q(L^m \otimes E) \cong H^{n-q}(L^{*m} \otimes E^* \otimes K_X)$$

negative for $m \gg 0$

- How does vanishing produce sections?

$$0 \to I_x \otimes L^m \to L^m \to L^m_x \to 0$$

$$\Downarrow$$

$$H^0(L^m) \to H^0(L^m_x) \to H^1(I_x \otimes L^m)$$

For $\dim X \geq 2$ the sheaf I_x is not a vector bundle - Kodaira used blowing up to make it one - will give direct agument using singular metrics

Hodge theory

- vanishing and topology
- positivity of Hodge bundles

- $H^n(X) \cong \bigoplus_{p+q=n} H^{p,q}(X)$, $H^{p,q} = \overline{H^{q,p}}$

$$\begin{array}{ccc} \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ H^n_{DR}(X) & & \bigoplus_{p+q=n} H^q(\Omega^p_X) \end{array}$$

Ex $H^{n,0}(X) = H^0(\Omega^n_X) = H^0(K_X)$

$$\begin{cases} 0 \to \Omega^p_X(L^*) \xrightarrow{\sigma} \Omega^p_X \to \Omega^p_X\big|_Y \to 0 \\ 0 \to \Omega^p_Y(L^*) \xrightarrow{d\sigma} \Omega^p_X\big|_Y \to \Omega^p_Y \to 0 \end{cases}$$

ANKVT $\Longleftrightarrow$ Lefschetz theorems

- $\mathcal{X} \xrightarrow{\pi} B$, $\pi^{-1}(t) = X_t$ smooth

$$\Downarrow$$

Hodge bundle $\longrightarrow$ $F \to B$ with $F_t = H^{n,0}(X_t)$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $$

$$H^0(K_{X_t})$$

Ex $y^2 = x(x-t)(x-a_1)\cdots(x-a_{2g})$

$$\begin{cases} \psi_t = \dfrac{p(x,t)\,dx}{y} = \dfrac{p(x,t)\,dx}{\sqrt{x(x-t)\cdots(x-a_{2g})}} \\ \deg_x p(x,t) \leq g-1 \end{cases}$$

- metric in Hodge bundle

$$\|\psi_t\|^2 = C \int_{X_t} \psi_t \wedge \overline{\psi}_t \quad (= L^2\text{-norm})$$

- $\Theta_F \geq 0$

  - $\Theta_{\det F} = \mathrm{Tr}\,\Theta_F > 0$ if local Torelli holds

- Application of positivity of Hodge bundle to classification of algebraic varieties

- $H^0(K_X^m)$ - sections are

  $$\psi = f(z)\,(dz_1 \wedge \cdots \wedge dz_n)^m$$

  - birational invariant

  - $h^0(K_X^m) = c\,m^d + \cdots$

    $d = \kappa(X) =$ Kodaira dimension

  - $d = n, n-1, \ldots, 0, -\infty$

- $\kappa(X) = n$ general type $(g \geq 2)$
- $\kappa(X) = 0$ — abelian varieties
  — Calabi-Yau varieties
- $\kappa(X) = -\infty$ rational, Fano $(K_X^* > 0)$

• General idea: $X \dashrightarrow Y$ given by $H^0(K_X^m)$ where $\kappa(X) = \kappa(Y) = \dim Y$ and $\kappa(X_y) = 0$

Ex: $\dim X = 2$, $\kappa(X) = 1 \rightsquigarrow$ elliptic surfaces

• The main result in pursuing the general idea is

<u>Theorem (Viewhweg, ...)</u>: $X \xrightarrow{f} Y$ <u>where</u>
- X_y <u>general type</u>
- $\mathrm{Var}\, f = \dim Y$

$\Rightarrow \kappa(X) \geq \kappa(X_y) + \kappa(Y)$

- All proofs use
  - $\left\{\begin{array}{l}\text{curvature property} \\ \text{of Hodge bundle}\end{array}\right\}$
  - $\left\{\begin{array}{l}\text{singularity structure} \\ \text{of Hodge metrics} \\ \text{and their curvatures}\end{array}\right\}$

. Idea: $H^0(K_X) \cong H^0(K_Y) \otimes H^0(f_* K_{X/Y})$

Results from

- $$\omega_X \cong f^* \omega_Y \otimes \omega_{X/Y}$$

$$\Rightarrow H^0(\omega_X) \cong H^0(\omega_Y) \otimes H^0(f_* \omega_{X/Y})$$

- $f_* \omega_{X/Y, y} = H^{n,0}(X_y)$; assume big

- Hodge bundle $\geqq 0$ and
  $\mathrm{var}\, f = \dim Y \Rightarrow \det(f_* \omega_{X/Y}) > 0$
  on a Zariski open set

- Singularities make Hodge bundles <u>more</u> positive

$\rightsquigarrow H^0(K_X)$ big if $H^0(K_{X_y})$ big

- X_y general type $\Rightarrow H^0(K_{X_y}^m)$ big

<u>Viewegs insight</u>: Can apply Hodge theory to $\psi = g(z)(dz_1 \wedge \cdots \wedge dz_n)^m$ by passing to
$$\psi^{1/m} = g(z)^{1/m} dz_1 \wedge \cdots \wedge dz_n \in H^0(K_{\tilde{X}_y})$$
on $\tilde{X}_y \to X_y$ branched over (ψ)

- Thus, even if $H^0(K_{X_y})$ not big by general type $H^0(K_{X_y}^m)$ will be big and can apply Hodge theory

- Finsler metric in $H^0(K^m_{X_y})$

$$\|\psi\|^2 = C\int(\psi\wedge\bar{\psi})^{1/m}$$

$$= C\int |g(z)|^{2/m}\, dz\wedge d\bar{z}$$

$$= \text{Hodge norm of } \psi^{1/m} \in H^0(K_{\tilde{X}_y})$$

- Thus has positive Finsler curvature - theory here seems not to have been worked out

III. Singular metrics and curvature

- Traditional limitation on the use of curvature (analytic methods) is that they have been restricted to
  - smooth case
  - positive definite case
- For many reasons geometry requires handling singularities
  - enumerative geometry
  - completions of moduli spaces
  - birational geometry
  - interest in their own right
- There exist algebraic varieties with $L \geqq 0$, $L > 0$ on open set but no L with $L > 0$ everywhere (Grauert-Riemenschneider)

- In recent years use of singular metrics and curvatures has arisen
  - metrics $h = h_0 e^{-2\varphi}$ where φ is a pluri-sub-harmonic (psh) weight function with analytic singularities, $h_0 =$ smooth metric
  - Hodge theory where the metrics and curvatures have logarithmic singularities
- Two simple models
  - $\varphi = \log|z|$, $(i/2)\partial\bar\partial \log \varphi = \delta_0$
  - $h = -\log|z|$, $(i/2)\partial\bar\partial \log h$

$$= (i/2)\partial\bar\partial\left(\log\left(\log \frac{1}{|z|}\right)\right)$$

$$= (i/2)\frac{dz \wedge d\bar z}{|z|^2(\log|z|)^2}$$

A. psh functions & analytic singularities

B. Nadel vanishing theorem

C. Hodge theory

<u>Theorem:</u> $\mathcal{M}$ = KSBA moduli space for varieties of general type with given numerical invariants. Then $\mathcal{M}$ has a canonical completion $\overline{\mathcal{M}}$, and

- the Hodge line bundle Λ_e lives on $\overline{\mathcal{M}}$
- Λ_e is nef and free
- $\operatorname{Proj}(\Lambda_e)$ is the minimal completion $\overline{\Sigma}$ of the image Σ of the period mapping

For curves, abelian varieties, K3's $\overline{\Sigma}$ is the Satake-Baily-Borel completion. Proof necessarily uses analysis/curvature

psh functions and analytic singularities

- $\mathcal{U} \subset \mathbb{C}^N$, $\varphi(z) \in L^1$ and $\varphi(z) \in [-\infty, +\infty)$

  - φ is psh $\iff$ $\varphi(z) \leqq \frac{1}{2\pi} \int \varphi(z + e^{i\theta}\xi)\, d\theta$

  sub-mean-value property on lines $\cap\ \mathcal{U}$

- $\varphi \in C^2(\mathcal{U})$ is psh $\iff$

$$(i/2)\, \partial\bar{\partial}\varphi \geqq 0$$

$$\Updownarrow$$

$$\sum_{j,k} \varphi_{z_j \bar{z}_k}\, \xi^j \bar{\xi}^k \geqq 0$$

- φ defines a distribution T_φ by

$$T_\varphi(\alpha) = \int \varphi\, \alpha\,, \qquad \alpha \in C_c^\infty(\mathcal{U})$$

$$\partial_{z_i} T_\varphi(\alpha) = -\int \varphi\, \partial_{z_i}\alpha \,.$$

Ex Heaviside has derivative Dirac

$$(i/2)\,\partial\bar\partial T_\varphi \geq 0 \iff \varphi \text{ is psh}$$

$\parallel$

(1,1) current

- will sometimes drop the T

Ex 1 $\varphi = \log|z|$, $(i/2)\partial\bar\partial\varphi = \delta_0\, dz\wedge d\bar z$

(residue theorem in disguise)

Ex 2 $\varphi = \log |f_1|^2 + \dots + |f_N|^2 = \log \|F\|^2$

- $(i/2)\partial\bar\partial\varphi = \dfrac{1}{\|F\|^4}\Big((F,F)(dF,dF) - (dF,F)\wedge(F,dF)\Big)$

where $F \neq 0$ (pullback of FS on $\mathbb{P}^N$)

- on complex lines reduces to Ex 1

Ex 3 $\varphi = -\log\left(\log\frac{1}{|z|}\right) \Rightarrow \partial\bar\partial\varphi = \dfrac{dz\wedge d\bar z}{|z|^2(\log|z|)^2}$

(*) $\left\{\begin{array}{l}\partial\varphi,\ \partial\bar\partial\varphi \text{ computed formally is} \\ \text{equal to } \partial\varphi, \partial\bar\partial\varphi \text{ computed as distribution}\end{array}\right\}$

(no residue terms)

- φ defines an ideal in $\mathcal{O}_{\mathcal{U}}$

$$\mathcal{I}(\varphi) = \left\{ f(z) \, ; \int |f|^2 e^{-2\varphi} < \infty \right\}$$

coherent sheaf of ideals

Ex 4 $\varphi = \log(|z_1|^{\alpha_1} + \cdots + |z_n|^{\alpha_n})$, $\alpha_i \geq 0$

$\mathcal{I}(\varphi)$ generated by monomials

$$z_1^{\beta_1} \cdots z_n^{\beta_n} \, ; \, \sum (\beta_j + 1)/\alpha_j > 1$$

Take $\alpha_j = 2 - \epsilon$ gives

(**) $$\mathcal{I}(\varphi) = \{ z_1, \ldots, z_n \} = \mathfrak{m}_0$$

- φ has analytic singularities if

$$\varphi = \text{Ex 4} + C^\infty \text{ function}$$

- Recall KVT: $L \otimes K_X > 0$ gives

$$H^q(L \otimes K_X) = 0 \, , \quad q > 0$$

Nadel vanishing theorem (NVT)

$$c_1(L) + c_1(K_X) + (\tfrac{i}{2})\partial\bar\partial\varphi > 0$$

$$\Downarrow$$

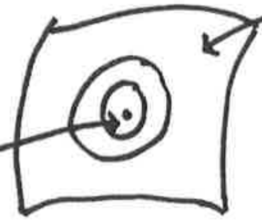
$$H^q(L\otimes K_X \otimes \mathcal{I}(\varphi)) = 0, \quad q > 0$$

Application: For $x \in X$

$$H^1(L^m \otimes I_x) = 0, \quad m \geq m_0$$

Proof: Choose φ as in (**) locally around $x \in X$

$\varphi = 0$



$\varphi = (**)$

$(i/2)\partial\bar\partial\varphi \geq 0$ near x

- $$L^m \otimes \mathcal{I}(\varphi) = \underbrace{L^m \otimes K_X^*}_{(i)} \otimes \underbrace{K_X}_{(ii)} \otimes \mathcal{I}(\varphi)$$

$$\Omega(i) + \Omega(ii) + (i/2)\partial\bar\partial\varphi > 0 \quad \text{for } m \gg 0$$

This gives from $0 \to L^m \otimes I_x \to L^m \to L^m_x \to 0$

$$H^0(L^m) \to H^0(L^m_x) \to 0$$
$$\begin{matrix} \| \\ L^m_x \end{matrix}$$

- method is very flexible - birational geometry / minimal model program

needs vanishing theorems for L's that are "close" to positive - here is one that is proved using singular weights

- $L \to X$ nef ($L \cdot C \geqq 0$ for all curves $C \subset X$)
- $L^n > 0$

$\Rightarrow H^q(K_X \otimes L) = 0$ for $q > 0$

(Kawamata - Viehweg)

Hodge theory

- Algebraic varieties X frequently have moduli spaces $\mathcal{M}$ with canonical completions $\overline{\mathcal{M}}$.

Ex X general type $(\kappa(X)=n)$

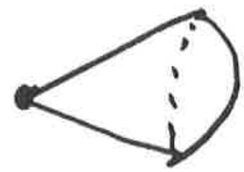
The local structure of the types of singularities on the specializations $X_0 \in \partial\mathcal{M}$ are understood (slc or semi-log-canonical). For curves get nodes

For surfaces

- double curve with pinch points

$$x^2 z = y^2 \quad \text{(swallowtail)}$$

- Gorenstein isolated singularities



cone over elliptic curve E with $-E^2 \leq 9$

- quotients by finite groups

• Global structure of X_0's and stratification of $\partial\mathcal{M}$ not understood in any example

• Basic invariant of X is its Hodge structure (period matrix) - Using Lie theory and complex analysis there is a good understanding of how Hodge structures degenerate

Suggests extend the period map

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Phi} & \Gamma\backslash D = \left\{ \begin{array}{c} \text{equivalence} \\ \text{classes of} \\ \text{PHS's} \end{array} \right\} \\ \cap & & \cap \\ \overline{\mathcal{M}} & \xrightarrow{\Phi_e} & \Gamma\backslash D_n \leftarrow \text{work in progress} \end{array}$$

where $\partial(\Gamma\backslash D)$ is roughly described as equivalence classes of LMHS's, and use this to help organize $\partial\mathcal{M}$. For this a main tool is the singularity structure of the Hodge metrics and their curvatures.

For algebraic curves we are looking asymptotics of periods

$$\cdots \quad \int_\gamma \frac{x^b\,dx}{y}, \quad b \le g-1$$

$$y = \prod_{i=1}^{2g} x(x-t)(x-a_i) \qquad \| \qquad c_b \frac{\log t}{2\pi i} + \cdots$$

- One works in situation

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] & \subset & \overline{\mathcal{X}} \arrow[d] \\ B & \subset & \overline{B} \end{tikzcd}$$

$\overline{B} \setminus B = Z$ is locally $\Delta^{*k} \times \Delta^{l}$

- $F = \pi_* \omega_{\mathcal{X}/B}$ has fibres $H^0(\Omega^n_{X_t}) = H^0(K_{X_t})$

- canonical extension $\pi_* \omega_{\overline{\mathcal{X}}/\overline{B}} = F_e \to \overline{B}$ where periods are polynomials in $\log t_j$ with holomorphic coefficients

Theorem on regular singular points

- Hodge norms are polynomials in $\log \frac{1}{|t_j|}$ - they have very special properties that result from Cattani-Kaplan-Schmid

- $\Lambda_e \to \overline{B}$ is Hodge line bundle
  $\parallel$
  $\det F_e$

General definition $\to$ • h has <u>logarithmic singularities</u> if

$$h = P\left(\log \frac{1}{|t_1|}, \ldots, \log \frac{1}{|t_k|}\right) + R$$

where P = homogeneous polynomial with the <u>very</u> special property (*)

$$\partial T_{\log h} = T_{\partial \log h}, \text{ same for } \partial\bar{\partial}$$

(no residue terms)

and where

- $P(x_1, \dots, x_n) > 0$ if $x_i > 0$
- the "remainder" terms R, ∂R, $\partial\bar{\partial} R$ are less singular than those for P

<u>**Theorem:** The Hodge metric in $\Lambda_e \to \bar{B}$ has logarithmic singularities</u>

Denote $\Omega = -(i/2)\partial\bar{\partial} \log h$ = Chern form

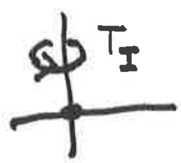
- On any ray $t_i = \lambda_i t$

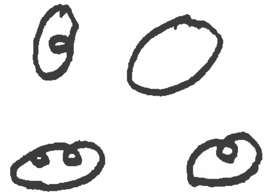
$$\Omega = C_\lambda \frac{dt \wedge d\bar{t}}{|t|^2 (\log |t|)^2} + \cdots \quad , \; C_\lambda > 0$$

- Ω is a closed $(1,1)$ current on $\bar{B}$ - represents $c_1(\Lambda_e)$

- Distributions T on a manifold M have singular support $\operatorname{sing} T = \left\{ \begin{array}{l} \text{closed subset where} \\ T \text{ is not } C^\infty \text{ fn} \end{array} \right\}$

Ex: Z_I^* = open strata of $Z = \cup Z_i$





$$T_I = \left\{ \begin{array}{l} \text{monodromy} \\ \text{around } Z_I^* \end{array} \right\} = \left\{ \begin{array}{c} \text{quasi-unipotent} \\ T_I = \underset{\uparrow}{T_I^s}\, T_I^u \end{array} \right\}$$

finite order

$$N_I = \log T_I^u$$

$$\rightsquigarrow \quad N_I \neq 0 \Rightarrow Z_I^* \subseteq \operatorname{sing} \Omega$$

- In general cannot restrict distributions to submanifolds - also cannot multiply them

Much more subtle than sing T is the <u>wave front set</u>

$$WF(T) \subset T^*M$$

- for $N \subset M$ with $TN \subset WF(T)^\perp$ the restriction

$$T\big|_N = \text{distribution on } N$$

is defined

co-normal bundle of Z_I^* in $\bar{B}$

<u>Ex</u> $WF(\Omega) = \bigcup_I N^*_{Z_I^*/\bar{B}}$

this implies that singularities make Ω more positive

- <u>What is Ω</u> $\Big|_{Z_I^*}$?

Along the strata we have an equi-singular family that gives a family of MHS's

The associated graded Gr(MHS) gives a direct sum of HS's. This gives a Hodge line bundle

$$\Lambda_I \to Z_I^*$$

with Chern form Ω_I. Then

- $\Lambda_e \big|_{Z_I^*} = \Lambda_I$

- $\Omega \big|_{Z_I^*} = \Omega_I$

- What does $\Omega\big|_{Z_I^*}$ mean in practice?

  - Ω has terms $\dfrac{dt_i \wedge d\bar{t}_i}{|t_i|^2(\log\frac{1}{|t_i|})^p}$, $p \geq 2$

    also cross terms $\dfrac{dt_i \wedge d\bar{t}_j}{(\cdots)}$,

    $\dfrac{dt_i \wedge d}{(\cdots)}$; first set $dt_i = 0$, $d\bar{t}_i = 0$

- Algebraic geometry tends to be a "1^{st} order" subject - for the two main sources of positive bundles, viz ample line bundles and Hodge bundles,

$$\Omega(z) = \| f_*(z) \|^2,$$

i.e. curvature measures the size of 1^{st} order information (need metric to talk about size)

- unity of geometry

——<o>——

A few curvature topics not discussed

- existence of intrinsic metrics in TX (Yau, Aubin, Tian, ...)