Period mappings for anti-canonical pairs *

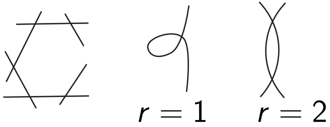
Phillip Griffiths

*Based on the works of Looijenga [L81], Friedman [F16], Engel-Friedman [EF21], and Gross-Hacking-Keel [GHK15]. Presentation, including notations, largely follows [F16].

Outline

I. Introduction

- Y = smooth rational surface and $D \in |-K_Y|$ is a connected cycle $(D_1, \dots, D_r)$ of $\mathbb{P}^1$'s; $r = r(D)$ is *length*, cases $r = 1, 2$ require special treatment;



- assume Y relatively minimal; i.e., $E \cap D \neq \emptyset$ for any -1 curve $E \subset Y$; recall that such an E satisfies $E^2 = E \cdot K_Y = -1 \implies Y = \mathrm{Bl}_p \overline{Y}$ where $\overline{Y}$ = smooth rational surface obtained by contracting $E \cong \mathbb{P}^1$ to $p \in \overline{Y}$; then $(\overline{Y}, \overline{D})$ is an anti-canonical pair where $\overline{D} \subset \overline{Y}$ is the image of D;
- well known that (Y, D) has a rich geometry; much of this is captured by the mixed Hodge structure (period matrix) on $H^2(Y - D)$; this is the main point we hope to illustrate in these talks;

- specific objective is to prove a global *Torelli theorem*; i.e., the pair (Y, D) is determined by the marked (to be defined) mixed Hodge structure on $H^2(Y - D)$; includes both *injectivity* from a suitably defined moduli space, and *surjectivity* onto a period domain, each of which is an existence result;
- the proof of this theorem will illustrate how Hodge theory may be used to help guide the study of the geometry of (Y, D);
- especially interested in the case when the intersection matrix

$$I := \|D_i \cdot D_j\| < 0 \qquad \text{(I.1)}$$

is negative definite; this is equivalent to

$$-d_i := D_i^2 \leqq -2 \text{ and some } D_j^2 \leqq -3; \qquad \text{(I.2)}$$

- in this case (Y, D) can be contracted to a normal cusp singularity (Y_0, p), and the referenced works discuss the

(I.3) **Q:** When can (Y_0, p) be smoothed?[†]

- for a simple elliptic singularity (Y, C) where $C =$ elliptic curve with $C^2 < 0$ the condition for smoothability is

$$-C^2 < 9; \tag{I.4}$$

- for cusps Looijenga conjectured that the condition be that the *dual* cycle $D' \in |-K_{Y'}|$ for a different rational surface Y'; he proved this for

$$m(D) := -D^2 = \text{multiplicity } \leqq 5,$$

and in the references it is established in general; proofs use ideas coming from mirror symmetry;

[†]This question really should be posed for a germ of isolated normal surface singularity; it turns out that it can be globalized both for simple elliptic singularities and for cusps.

- references also address the question of adjacency; which cusps can (Y_0, p) be partially smoothed to shorter cusps, simple elliptic singularities or to ordinary double points, and in how many ways can this be done;
- we will discuss the proof of the following result (the terms to be explained below).

Theorem

Let (Y, D) and (Y', D') be labeled anti-canonical pairs with $r(D) = r(D')$ and

$$\gamma : H^2(Y', \mathbb{Z}) \xrightarrow{\sim} H^2(Y, \mathbb{Z})$$

an integral isometry with

(i) $(\gamma[D_i']) = [D_i]$

$\implies \gamma : H^2(Y' - D') \xrightarrow{\sim} H^2(Y - D)$

(ii) $\gamma(\overline{\mathcal{A}}_{\mathrm{gen}}(Y')) = \overline{\mathcal{A}}_{\mathrm{gen}}(Y)$

(iii) $\Phi(Y, D) \circ \gamma = \Phi(Y', D')$ $\quad$ *(Φ = period mapping).*

Then there exist

$$\rho : (Y, D) \xrightarrow{\sim} (Y', D')$$

and $w \in W(\Delta_Y)$, unique up to $K(Y', D')$, with

$$\rho^* = w \circ \gamma.$$

We will also see that the period mapping $\Phi : \{\text{pairs } (Y, D) \text{ as above}\} \to \{\text{period domain}\}$ is surjective.

- here $W(\Delta_Y)$ is a Weyl group and $K(Y', D')$ is a finite group related to $\det I$ in the $I < 0$ case;
- period mapping involves choices to be taken into account in the global Torelli theorem; a basic issue is that in contrast to a traditional period mapping there is no intrinsic polarization corresponding to a natural ample line bundle associated to Y; anticipating what follows there is a natural generic ample cone $\mathcal{A}_{\text{gen}}(Y)$ whose closure $\overline{\mathcal{A}}_{\text{gen}}(Y)$ in $\{x \in H^2(Y, \mathbb{R}) : x^2 > 0\}$ appears in the above statement of the Torelli theorem;

- theme of this talk falls in the general topic of "Hodge theory and moduli." The Torelli theorem, as formulated and proved in the references, provides a beautiful example of the general theory. Typically simple elliptic singularities and cusps occur on the boundary of the KSBA moduli space of general type surfaces where they are normal singular loci. The Torelli theorem sometimes provides a suggestion on how to at least partially desingularize a boundary component;
- a first step is to show that the differential of the period mapping is an isomorphism, a result that might be hoped for since (Y, D) is a log-Calabi-Yau variety;
- further steps involve an analysis of the geometry of anti-canonical pairs (Y, D); this is an extensive subject that involves birational geometry; we will emphasize the part that has a Hodge-theoretic aspect.

II. Hodge theory (A): Mixed Hodge structure on $H^2(Y - D)$

- $D = (D_1, \ldots, D_r)$ is the *labeled anti-canonical divisor*; $D_i^* = D_i - \{0_i, \infty_i\} \cong \mathbb{C}^*$;
- *orientation* is $H_1(D) \cong \mathbb{Z} \cong H_1(\text{graph of } D)$; generator γ is the *circuit* around D; gives identifications $\infty_i = 0_{i+1}$;
- from the exact sequence of the pair $(Y, Y - D)$ using $H^i(Y - D) \cong H_i(Y, D)$ and $\chi_{\mathrm{top}}(Y, D) = \chi_{\mathrm{top}}(Y) - \chi_{\mathrm{top}}(D)$ we have

$$\chi_{\mathrm{top}}(Y - D) = b_2(Y) + 2 - r(D); \tag{II.1}$$

- the exact cohomology sequence of the same pair and $H^{\mathrm{odd}}(Y) = 0$ gives a short exact sequence

(II.2)

$$0 \to \underbrace{\mathrm{Im}\{H^2(Y, Y - D) \to H^2(Y)\}}_{2} \to H^2(Y - D)$$

$$\to \underbrace{H^3(Y, Y - D)}_{4} \to 0;$$

- using $I < 0$ the term over the first bracket has dimension $b_2(Y) - r(D)$;
- end pieces of (II.2) are pure Hodge structures with the weights written below; $H^2(Y - D)$ has a mixed Hodge structure with the indicated weight graded quotients;
- the last term is isomorphic to $\mathbb{Z}(-2)$ and is dual to $H_1(D)$ which has weight 0; it is generated by the Poincaré-Lefschetz dual to the circuit γ;

- since $D \in |-K_Y|$ there exists a 2-form Ω having as divisor a simple pole on D; normalize Ω by

$$\mathrm{Res}_{\infty_i}\left(\mathrm{Res}_{D_i}\left(\left(\frac{1}{2\pi\sqrt{-1}}\right)^2\Omega\right)\right)=1; \tag{II.3}$$

- will describe the mixed Hodge structure on $H^2(Y-D)$ in terms of periods

$$\int_\Sigma \Omega$$

where $\Sigma \in H_2(Y-D,\mathbb{Z})$;

- the standard identification $H^*(Y-D,\mathbb{C}) \cong \mathbb{H}^*(\Omega_Y^\bullet(\log D))$ gives

$$F^2H^2(Y-D) \cong \mathbb{C}\cdot\Omega; \tag{II.4}$$

- for the Σ's we use
(II.5)
$$\begin{array}{ccccccc} H_3(Y, Y-D) & \to & H_2(Y-D) & \to & H_2(Y) & \to & H_2(Y, Y-D); \\ \wr\| & & & & & & \wr\| \\ H^1(D) & & & & & & H^2(D) \end{array}$$
for the dual $\hat{\gamma} \in H^1(D)$ of the circuit this leads to describing $H_2(Y-D)/\mathbb{Z}\hat{\gamma}$ in terms of the $\Sigma \in H_2(Y)$ satisfying $\Sigma \cdot [D_i] = 0$ for all i;
- given Hodge structures A, B of weights a, b with $a < b$, the mixed Hodge structures C with
$$0 \to A \to C \to B \to 0$$
are classified by the elements in
(II.6)
$$\mathrm{Ext}^1_{\mathrm{MHS}}(B, A) \cong \mathrm{Hom}_{\mathbb{C}}(B, A) \backslash F^0 \,\mathrm{Hom}_{\mathbb{C}}(B, A) + \mathrm{Hom}_{\mathbb{Z}}(B, A);$$

- using (II.2), (II.4) and (II.5) we will use (II.6) to describe the mixed Hodge structure on $H^2(Y - D)$.
- if

$$A \cong \overset{k}{\oplus} \mathbb{Q}(-1),$$
$$B \cong \mathbb{Q}(-2)$$

where $\mathbb{Q}(-1)$ is the Hodge-Tate structure of weight 2 with $h^{2,0} = 0$, $h^{1,1} = 1$ ($\mathbb{Q}(-1) \cong H^2(\mathbb{P}^1, \mathbb{Q})$) and $\mathbb{Q}(-2) = \overset{2}{\otimes} \mathbb{Q}(-1)$, then

$$\mathrm{Ext}^1_{\mathrm{MHS}}(B, A) \cong \mathbb{C}^k/(2\pi i\mathbb{Z})^k \cong \mathbb{C}^{*k} := (\mathbb{G}_m)^k.$$

- **Definition:** Set $\Lambda(Y, D) = \{\mathrm{span}[D_i]\}^{\perp} \cap H^2(Y, \mathbb{Z})$. The *periods*

$$\Phi(Y, D) \in \mathrm{Hom}(\Lambda(Y, D), \mathbb{G}_m) \tag{II.7}$$

of (Y, D) are defined as follows: From $H^2(Y, \mathbb{Z}) \cong \mathrm{Pic}(Y)$ may represent classes in $H^2(Y, \mathbb{Z})$ as represented by algebraic 1-cycles Z. If Z represents a class in $\Lambda(Y, D)$ may assume Z meets each D_i^* in a 0-cycle of degree 0,

$$Z \cdot D = \sum_{i,j} p_{ij} - q_{ij}$$

where $p_{ij}, q_{ij} \in D_i^*$; for σ_{ij} a path in D_i^* connecting q_{ij} to p_{ij} we set

$$\begin{cases} \omega = \mathrm{Res}_D \, \Omega \\ \Phi(Y, D)(Z) = \exp\left(2\pi\sqrt{-1} \sum_{i,j} \int_{\sigma_{ij}} \omega\right). \end{cases} \tag{II.8}$$

A standard construction (cf. §I.5 in [L81]) interprets $\int_{\sigma_{ij}} \omega$ as $\int_{\sum_{ij}} \Omega$ where $\sum_{ij} \in H_2(Y - D)$ is a 2-cycle in $Y - D$;

- in fact given any 2-cycle $\sum \in \mathrm{Im}\{H_2(Y - D) \to H_2(Y)\}$ we note that $\sum$ maps to zero in $H_2(Y, Y - D) \cong H_2(U, U - D)$ where $U =$ tubular neighborhood of D in Y; thus $\sum \cap U = \partial\Gamma$ where $\Gamma =$ 3-chain in U and then $\sum - \partial\Gamma \subset Y - D$ and represents a cycle in $Y - D$ that gives the class of $\sum$; i.e., in homology we may pull $\sum$ away from D.

Theorem II.9

The periods give the extension class in (II.6) *that defines the mixed Hodge structure on* $H^2(Y - D)$.

Proof.

Setting $\Lambda = \Lambda(Y, D)$ we use that

$$\text{(II.10)} \qquad \mathrm{Hom}(\Lambda, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{G}_m, \Lambda^*)^*.$$

Here Λ is a sub-group of $H^2(Y, \mathbb{Z})$ while Λ^* is the quotient group $H^2(Y, \mathbb{Z}) / \mathrm{Im}\{H^2(Y, Y - D) \to H^2(Y)\}$ in (II.2). In the recipe (II.6) we have

$$\begin{cases} A \cong \Lambda^* \cong \overset{b_2 - r}{\oplus} \mathbb{Q}(-1) \\ B \cong \mathbb{Q}(-2) \text{ with generator the Poincaré-Lefschetz dual} \\ \qquad \text{of the circuit } \gamma. \end{cases}$$

Unwinding the dualities using that Poincaré duality is a unimodular pairing gives the result. □

- here we are using that from Poincaré duality over $\mathbb{Q}$ we have

$$0 \to \oplus \mathbb{Q}[D_i] \to H^2(Y, \mathbb{Q}) \to \Lambda^* \otimes \mathbb{Q} \to 0;$$

also the pairing on $H^2(Y, \mathbb{Z})$ induces one on $\Lambda(Y, D)$ that is unimodular $\iff$ $\det I = \pm 1$.

III. Hodge theory (B): $\mathrm{Pic}(D)$

- Periods of (Y, D) involve the Abel-Jacobi map on D; for this will give the Hodge-theoretic interpretation of $\mathrm{Pic}(D)$;
- for a line bundle $L \to D$ the *multi-degree* is

$$\left(\deg(L\big|_{D_1}), \ldots, \deg(L\big|_{D_r})\right);$$

- for $\nu : \widetilde{D} \to D$ the normalization we have

(III.1)

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathbb{G}_m^r \arrow[r] & \mathbb{G}_m^r \arrow[r] & \mathrm{Pic}(D) \arrow[r] & \mathrm{Pic}(\widetilde{D}) \arrow[r] \arrow[d, "\wr"] & 0 \\ & & & & & \mathbb{Z}^r & \end{tikzcd}$$

where the vertical isomorphism is the multi-degree; (III.1) is the cohomology sequence of

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{O}^*_D \arrow[r] & \nu_*\mathcal{O}^*_{\tilde{D}} \arrow[r] & \nu_*\mathcal{O}^*_{\tilde{D}}/\mathcal{O}^*_D \arrow[r] \arrow[d, phantom, "\wr\|"] & 1 ; \\ & & & (\mathbb{G}_m)^r & \end{tikzcd}$$

- for the kernel $\mathrm{Pic}^o(D)$ of the last map in (III.1) have

(III.2) $$\mathrm{AJ}_D : \mathrm{Pic}^o(D) \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}.$$

Here for $L \in \mathrm{Pic}^o(D)$ the line bundle associated to a divisor $\sum p_{ij} - q_{ij}$ as above

(III.3) $$\mathrm{AJ}_D(L) = \sum \int_{q_{ij}}^{p_{ij}} \omega;$$

- condition that $L \cong \mathcal{O}_D$, i.e., that there exists a rational function f on D with

$$f(\infty_i) = f(0_{i+1}) \in \mathbb{C}^*, \text{ and}$$

with divisor

$$(f) = \sum p_{ij} - q_{ij}$$

is $\mathrm{AJ}_D(L) = 0$ in $\mathbb{C}/\mathbb{Z}$;

- equivalently,

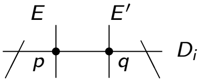
$$\text{(III.4)} \qquad \exp\left(2\pi\sqrt{-1}\sum_{i,j}\int_{q_{ij}}^{p_{ij}}\omega\right) = 1;$$

- in special case where the divisor is $p_i - q_i$ on D_i this is

$$\mathrm{CR}(0_i, p_i, q_i, \infty_i) = 1$$

where CR is the cross ratio;

- such divisors arise as $Z \cdot D$'s in a picture

(III.5) 

  where E, E' are curves on Y and $Z = E - E'$; we shall see that such configurations where E, E' are -1 curves form the basic building blocks of generic (Y, D)'s;
- since -1 curves are stable under deformations, such configurations map to open sets in the moduli space of (Y, D)'s;
- since distinct -1 curves on Y do not intersect, (III.5) is determined by the Abel-Jacobi image of $L = [p - q] \in \mathrm{Pic}^o(D)$, hence by the period point

$$\Phi_{(Y,D)}(Z) \in \mathbb{G}_m;$$

- the other basic curves are -2-curves C defined by $C \cdot K_Y = 0$, $C^2 = -2$; by adjunction it follows that $C \cong \mathbb{P}^1$ and from $D \in |-K_Y|$ that $C \cap D = \emptyset$ so that $C \in \Lambda(Y, D)$ and the period mapping does not "see" them; as a consequence of both injectivity and surjectivity of the period mapping we will find that such curves will only occur over proper subvarieties in moduli;
- this is a vintage example of how the period mapping can guide the geometry.

IV. Hodge theory (C): Differential of the period mapping

- From (III.3) see that if the algebraic 1-cycle Z with $Z \cdot D = \sum p_{ij} - q_{ij}$ moves non-trivially, then in general $\mathrm{AJ}_{(Y,D)}(Z)$ will vary; will make this precise;
- usual formulation of period mapping is

$$\text{(IV.1)} \qquad (Y, D) \to \underbrace{H^0(\Omega^2_Y(D)) \subset H^2(Y - D, \mathbb{C})}_{\text{line in a vector space}}$$

  where the term over brackets is a projective space;
- the integrals $\sum_j \int_{q_{ij}}^{p_{ij}} \omega$ for $i = 1, \ldots, r$ depend on $\dim \Lambda(I, D) = b_2(Y) - r(D)$ parameters;
- thus we need to show that the deformation space $\mathrm{Def}(Y; D_1, \ldots, D_r)$ has dimension $b_2 - r$ and locally maps 1-1 to the integrals (II.8);

- since the Z's move freely subject only to $Z \cdot D = 0$ this seems plausible;
- to make this precise use the standard identificaiton

$$T\,\mathrm{Def}(Y; D_1, \ldots, D_r) \cong H^1(T_Y(-\log D)); \tag{IV.2}$$

- importantly, since $D \in |-K_Y|$

$$\Omega^2_Y(D) \otimes T_Y(-\log D) \xrightarrow{\sim} \Omega^1_Y(\log D)$$

and using $H^0(\Omega^2_Y(D)) \cong \mathcal{O}_D$ we have

$$T\,\mathrm{Def}(Y; D_1, \ldots, D_r) \cong H^1(\Omega^1_Y(\log D)); \tag{IV.3}$$

- the cohomology sequence of

$$0 \to \Omega^1_Y \to \Omega^1_Y(\log D) \to \overset{i}{\oplus}\,\mathcal{O}_{D_i} \to 0$$

gives

$$0 \to \overset{i}{\oplus}\,H^0(\mathbb{C}_{D_i}) \to H^1(\Omega^1_Y) \to H^1(\Omega^1_Y(\log D)) \to 0$$

where the first arrow uses that the $[D_i]$ are linearly independent in $H^2(Y)$;

this gives

(IV.4) $$\dim T\,\mathrm{Def}(Y; Z_1, \ldots, Z_r) = b_2 - r;$$

- the differential of (IV.1) is

(IV.5)
$$T\,\mathrm{Def}(Y; D_1, \ldots, D_r) \to \mathrm{Hom}(\mathrm{Gr}_F^2 H^2(Y-D), \mathrm{Gr}_F^1 H^2(Y-D)).$$

We write this as

(IV.6)
$$\begin{array}{ccccc} T\,\mathrm{Def}(Y; D_1, \ldots, D_r) & \otimes & \mathrm{Gr}_F^2 H^2(Y-D) & \to & \mathrm{Gr}_F^1 H^2(Y-D) \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ H^1(T_Y(-\log D)) & \otimes & H^0(\Omega_Y^2(D)) & \to & \mathrm{Gr}_F^1 \mathrm{Gr}_2^W H^2(Y-D) \end{array}$$

where
$$\mathrm{Gr}_F^2 H^2(Y-D) = F^2 H^2(Y-D) \cong H^0(\Omega_Y^2(D)),$$

and where the vertical isomorphism on the right in (IV.6) uses
$$F^1 H^2(Y-D)/F^2 H^2(Y-D) = \mathrm{Gr}_F^1 \mathrm{Gr}_2^W H^2(Y-D).$$

- the above is the usual formulation of the differential of a period mapping. Here it is convenient to use the polarization on $\mathrm{Gr}_2^W H^2(Y-D)$ given by the form Q. Using Q to identify $\mathrm{Gr}_F^1 \mathrm{Gr}_2^W H^2(Y-D)$ with its dual (IV.6) dualizes to
(IV.7)
$H^1(T_Y(-\log D)) \otimes \mathrm{Gr}_F^1 \mathrm{Gr}_2^W H^2(Y-D) \to H^0(\Omega_Y^2(D))^*.$

We will show that

$$\text{(IV.8)} \qquad \mathrm{Gr}_F^1 \mathrm{Gr}_2^W H^2(Y-D) \cong H^1(\Omega_Y^1(\log D))(-D)$$
$$\text{(IV.9)} \qquad H^0(\Omega_Y^2(D))^* \cong H^2(\mathcal{O}_Y(-D));$$

using these (IV.7) then becomes the pairing
(IV.10)
$H^1(T_Y(-\log D)) \otimes H^1(\Omega_Y^1(\log D))(-D) \to H^2(\mathcal{O}_Y(-D)).$

The isomorphism (IV.9) is Kodaira-Serre duality, and (IV.10) is a duality pairing using the isomorphism $H^2(\mathcal{O}_Y(-D)) \cong \mathbb{C}$ given by Ω.

- Thus we need to prove (IV.8).
  For the normalization $\nu : \widetilde{D} \to D$ a local coordinate calculation gives

$$0 \to \Omega^1_Y(\log D)(-D) \to \Omega^1_Y \to \nu_*\Omega^1_{\widetilde{D}} \to 0.$$

From $H^0(\nu_*\Omega^1_{\widetilde{D}}) = 0$ and $H^1(\nu_*\Omega^1_{\widetilde{D}}) \cong \overset{i}{\oplus} H^1(\Omega^1_{D_i})$ we have
(IV.11)

$$0 \to H^1(\Omega^1_Y(\log D))(-D) \to H^1(\Omega^1_Y) \to \overset{i}{\oplus} H^1(\Omega^1_{D_i}) \to 0.$$

This gives

$$\begin{aligned} H^1(\Omega^1_Y(\log D))(-D) &\cong \ker\{F^1H^2(Y) \to F^1H^1(D)\} \\ &\cong \mathrm{Gr}^1_F \, \mathrm{Gr}^W_2 \, H^2(Y-D) \end{aligned}$$

as desired.

V. Hodge theory (D): Surjectivity of the period mapping

Theorem V.1

An anti-canonical pair (Y, D) has a deformation parametrized by some $\mathbb{G}_m^N$ and for which the differential of the period mapping on the parameter space is surjective at (Y, D).

Corollary V.2

The period mapping is surjective.

Proof of the corollary

The proof of the theorem will show that the period mapping $\Phi : \mathbb{G}_m^N \to \mathbb{G}_m^r$ induces a holomorphic algebraic mapping between connected abelian algebraic groups whose differential is surjective. This implies that Φ is a surjective homomorphism followed by a translation. $\square$

- for the proof of Theorem V.1 an argument similar to, but simpler than, the one in Section IV gives

$$H^2(T_Y(-\log D)) \cong H^2(\Omega^1_Y(\log D)) = 0;$$

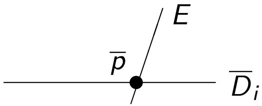
thus $H^1(T_Y(-\log D))$ is unobstructed and is isomorphic to the tangent space $T\,\mathrm{Def}(Y; D_1, \dots, D_r)$;
- this is an existence theorem; there exist algebraic cycles Z with $\deg Z \cdot D = 0$ that move with $\mathrm{rank}\,\Lambda(Y, D) = b_2(Y) - r(Y, D)$ parameters; will now discuss what such cycles can be.

- **Definition:** An *exceptional curve* $E \subset Y$ is an irreducible curve with $E^2 = -1$, $E \cdot K_Y = -1$.[‡]

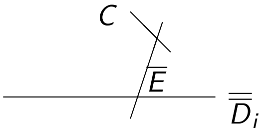
  Thus E is a $\mathbb{P}^1$ that can be contracted to a point p' on a smooth surface Y' such that $Y = \mathrm{Bl}_{p'} Y'$. We are assuming that any such E meets D; i.e., (Y, D) is *relatively minimal*;
- E is an *interior exceptional curve* if $E \neq D_i$ for any i; every exceptional curve is an interior one or a component of D;
- for $p \in D$ let $\overline{Y} = \mathrm{Bl}_p Y$ with exceptional curve E and $\overline{D} =$ proper transform of D;
  - if $p =$ smooth point of D; then $r(D) = r(\overline{D})$ and Y has a new interior exceptional curve;
  - if $p = D_{i-1} \cap D_i$, $\overline{D} =$ proper transform of D plus E with multiplicity 1, $r(\overline{D}) = r(D) + 1$ (this is a *corner blowup*);

[‡]Frequently called -1 curves.

– in general may blow up $p \in D_i^*$ to get



then blow up $\overline{p}$ to get



$$C^2 = -2, \overline{E}^2 = -1$$

etc. This leads to a *generalized exceptional curve*

$$C_1 + \cdots + C_{b-1} + E, \quad C_i^2 = -2, \quad E^2 = -1.$$

- Associated to any rational surface Y there is a minimal one $\overline{Y}$ and $Y \to \overline{Y}$ given by a sequence of blowdowns of -1 curves;

- associated to an anti-canonical pair (Y, D) there is an anti-canonical pair $(\overline{Y}, \overline{D})$ given by a sequence of blowdowns as described above; when we can go no further $\overline{Y} = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or a Hirzebruch surface $\mathbb{F}_N$ and there is a short list of possibilities for $(\overline{Y}, \overline{D})$ (cf. [F16]);
- to each interior or generalized interior blowup there is a $\mathbb{G}_{m,\alpha}$ and a point $p_\alpha \in \mathbb{G}_{m,\alpha}$;
- thus if we start with (Y, D) and serially contract interior exceptional curves until there are none left the remaining exceptional curves will be components of an anti-canonical cycle;

- as will be seen next such a $(\overline{Y}, \overline{D})$ will have no moduli; thus the moduli of (Y, D) will arise from the interior blowups;
- varying the centers of these interior blowups will thus locally map onto the local moduli space of (Y, D);
- moreover the parameter space for this process is a product of $\mathbb{G}_m$'s; this proves Theorem V.1;
- in summary Hodge theory plus deformation theory tells us how many algebraic cycles Z to look for; algebraic geometry then tells how to construct them.

VI. Proof of Torelli: first steps

(VI.1) • *Let* (Y, D) *and* (Y', D') *be labeled anti-canonical pairs with* $r(D) = r(D')$ *and*

$$\gamma : H^2(Y', \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

and integral isometry with

(i) $\gamma([D_i']) = [D_i]$
(ii) $\gamma(\overline{\mathcal{A}}_{\mathrm{gen}}(Y')) = \overline{\mathcal{A}}_{\mathrm{gen}}(Y)$
(iii) $\Phi(Y, D) \cdot \gamma = \Phi(Y', D')$.

There exists an isomorphism

(VI.2) $$\rho : (Y, D) \to (Y', D')$$

and $w \in W(\Delta_Y)$ *such that up to the action of* $K(Y', D')$

(VI.3) $$\rho^* \equiv w \cdot \gamma.$$

Notations will be explained below.

Basically we want to show that the data given in (i)–(iii) enables us to construct enough of the geometry of (Y, D) and (Y', D') to produce the map ρ and Weyl group element w satisfying (VI.3).
We begin with heuristic reasoning.

- Because of local Torelli, including the surjectivity part, it should suffice to prove the result when (Y, D) and (Y', D') are generic (to be explained below);
- (Y, D) is *taut* if D is determined by the intersection sequence D_i^2; taut pairs are rigid; i.e., $T\,\mathrm{Def}(Y; D_1, \ldots, D_r) = 0$;
- any (Y, D) may be constructed from a taut anti-canonical pair by the blowing up process described above;
- if (Y, D) and (Y', D') are taut, then the result is true; may be done by checking the (short) list of cases (cf. [F16]);

- from the construction of (Y, D) and (Y', D') from taut anti-canonical pairs, since blowing up increases $\mathrm{Pic}(Y)$ and $\mathrm{Pic}(Y')$, this suggests an inductive argument on the rank of $\mathrm{Pic}(Y)$ and focuses on the role of interior exceptional curves;
- this is reinforced since by the local Torelli theorem the moduli of anti-canonical pairs entirely arises from algebraic curves that meet the anti-canonical divisor; period mapping $\Phi(Y, D)$ does not see curves C with $C \cap D = \emptyset$;
- **Definition:** A -2-*curve* is a curve C with $K_Y \cdot C = 0$, $C^2 = -2$ and $C \neq D_i$ for any i; using adjunction

$$2p_a(C) - 2 = K_Y \cdot C + C^2 = -D \cdot C + C^2 \tag{VI.4}$$

  gives that $C \cong \mathbb{P}^1$ and $C \cap D = \emptyset$; the period mapping does not see -2-curves;

(VI.5) • if C is an irreducible curve with $C^2 < 0$, then C is smooth and is

$$\begin{cases} \text{a } -2\text{-curve,} \\ \text{an interior exceptional curve,} \\ \text{a component } D_i \text{ of } D; \end{cases}$$

proof again uses adjunction (VI.4).

- **Definition:** (Y, D) is *generic* if there are no -2-curves; to justify the definition note that

  $$\begin{cases} \Phi(Y,D)(C) = 1 \in \mathbb{G}_m \text{ for } C \text{ a } -2\text{-curve;} \\ \Phi(\mathrm{Def}(Y,D)) = \text{open set in } \mathrm{Hom}(\Lambda(Y,D), \mathbb{G}_m); \end{cases}$$

  together using surjectivity of the period mapping these imply that the (Y, D)'s having -2-curves lie in the complement of a proper subvariety in the deformation space;

- **Note:** As background, although we won't use it we note that in [EF21, Prop. 2.5] it is proved that if there are no -2 curves, then any effective divisor is linearly equivalent to $\sum a_i D_i + \sum b_\alpha E_\alpha$ where the E_α are interior exceptional curves.
- Set

$$\begin{aligned}
\mathcal{C} &= \{x \in H^2(Y, \mathbb{R}) : x^2 > 0\} \\
\cup & \\
\mathcal{C}^+ &= \text{component of } \mathcal{C} \text{ containing the ample cone } \mathcal{A}(Y), \\
\cup & \\
\overline{\mathcal{A}}(Y) &= \text{closure of } \mathcal{A}(Y) \text{ in } \mathcal{C}^+ \ (\textit{not} \text{ the closure in } H^2(Y, \mathbb{R}))
\end{aligned}$$

- **Provisional definition:** $\overline{\mathcal{A}}_{\mathrm{gen}}(Y)$ is $\overline{\mathcal{A}}(Y)$ for a generic (Y, D);
- this definition does not really tell us what $\overline{A}_{\mathrm{gen}}(Y)$ is; want a numerical description such as exists for usual ample cones.

VII. Proof of Torelli: next steps

- From the above want to show that for (Y, D) generic we can construct the interior exceptional curves from $\overline{\mathcal{A}}_{\mathrm{gen}}(Y)$;
- $\alpha \in H^2(Y, \mathbb{Z})$ is a *numerical exceptional curve if*

$$\begin{cases} \alpha^2 = -1 \\ \alpha \cdot K_Y = -\alpha \cdot D = -1; \end{cases}$$

  it is *effective* if α is the class of an effective divisor;
- α is an effective numerical exceptional curve if and only if there exists a nef $\mathbb{R}$-divisor x with

$$\begin{cases} x \cdot [D] > 0 \\ x \cdot \alpha \geqq 0. \end{cases}$$

Proof.

For $L_\alpha \to Y$ with $c_1(L_\alpha) = \alpha$, by Riemann-Roch

$$\chi(Y, L_\alpha) = 1.$$

Since $h^2(Y, L_\alpha) = h^0(Y, -D - \alpha)$ and $x \cdot (-D - \alpha) < 0$ this group is zero and then $h^0(Y, L_\alpha) > 0$. □

- Since a nef divisor has non-negative intersection number with every effective divisor we have the

Corollary

If there are no -2-curves on Y,

(VII.1)

$$\overline{\mathcal{A}}(Y) = \left\{ \begin{array}{c} x \in \mathcal{C}^+ : x \cdot [D_i] \geqq 0, x \cdot \alpha \geqq 0 \textit{ for all} \\ \textit{effective numerical exceptional curves } \alpha \end{array} \right\}.$$

This requires an argument (cf. [F16]); as motivation recall that the ample cone is

$$\{x \in H^2(Y, \mathbb{Q}) : x^2 > 0, x \cdot C > 0 \text{ for all curves } C \subset Y\}.$$

Definition

$$\overline{\mathcal{A}}_{\mathrm{gen}}(Y) = \left\{ \begin{matrix} \text{set of } x \in H^2(Y, \mathbb{R}) \text{ defined by} \\ \text{the right-hand side of (VII.1)} \end{matrix} \right\};$$

- since the conditions for an effective numerical exceptional curve are $\alpha^2 = \alpha \cdot K_Y = -1$ and $\alpha \cdot H \geqq 0$ for any ample H it follows that

$$\text{(VII.2)} \qquad \overline{\mathcal{A}}_{\mathrm{gen}}(Y) \text{ is invariant under monodromy.}$$

Thus this part of the criteria in the statement of the Torelli theorem is well defined.

From (VII.1) we may infer that

(VII.3)

$$\overline{\mathcal{A}}(Y) = \left\{ x \in \overline{\mathcal{A}}_{\mathrm{gen}}(Y) : x \cdot [C] \geqq 0 \text{ for all } -2\text{-curves } \mathcal{D} \right\}.$$

We now turn to the roots and the definition of $W(\Delta_Y)$.

- Given $\beta \in \Lambda(Y, D)$ with $\beta^2 = -2$ the map

$$\begin{cases} r_\beta : \Lambda \to \Lambda \\ r_\beta(\alpha) = \alpha - (\alpha \cdot \beta)\beta \end{cases}$$

  satisfies

$$r_\beta^2 = \mathrm{Id}$$

  and is called the *reflection in the class* β (or sometimes the reflection in the wall W_β);

- any -2 curve gives such a β; want to characterize numerically which β's are -2 curves; issue is subtle because could have $\beta = [E] - [E']$ where E, E' are interior -1 curves.

  (VII.4) • *for C a -2-curve,*

$$r_C(\overline{\mathcal{A}}_{\mathrm{gen}}(Y)) = \overline{\mathcal{A}}_{\mathrm{gen}}(Y).$$

Proof.

Since

- $r_C(\mathcal{C}^+) = \mathcal{C}^+$ $\quad$ (r_C preserves the intersection form)
- $r_C([D_i]) = [D_i]$

it suffices to show that r_C permutes the classes of effective numerical exceptional curves. For α with $\alpha^2 = -1$, $\alpha \cdot K_Y = -1$ using $r_C(K_Y) = K_Y$ we see that $r_C(\alpha)$ has these properties. For H a big and nef divisor with $H \cdot C = 0$, all $H \cdot D_i > 0$ we have $r_C(H) = H$. Thus $\alpha \cdot H \geqq 0 \iff r_C(\alpha) \cdot H \geqq 0$ so that α is effective $\iff r_C(\alpha)$ is effective. $\square$

(VII.5) **Definition:** $W(\Delta_Y)$ is the group generated by the reflections in the classes from the set Δ_Y of -2 curves on Y.

(VII.6) **Fact:** From (VII.3) it follows that $\overline{\mathcal{A}}(Y)$ is a fundamental domain for the action of $W(\Delta_Y)$ on $\overline{\mathcal{A}}_{\mathrm{gen}}(Y)$ (cf. 6.3 in [F16]).

- **Definition:** $\beta \in \Lambda$ with $\beta^2 = -2$ is a *root* if

  (VII.7) $$r_\beta(\overline{\mathcal{A}}_{\mathrm{gen}}(Y)) = \overline{\mathcal{A}}_{\mathrm{gen}}(Y);$$

- basic issue is to identify the -2-curves among the roots; result is

  (VII.8)
  $$\left\{\begin{array}{c}\beta \in \Lambda \text{ with} \\ \beta^2 = -2 \\ \text{is a root.}\end{array}\right\} \iff \left\{\begin{array}{l}\text{there is a trivial monodromy deforma-} \\ \text{tion } Y_1 \text{ of } Y \text{ with } \Phi(Y_1, D_1)(\beta) = 1, \\ Y_1 \text{ generic with this condition, and} \\ \pm\beta = [C] \text{ for a } -2\text{-curve } C\end{array}\right\}.$$

  This is an existence theorem that may be summarized by
  (VII.9)
  A root β satisfying (VII.7) *may be deformed into a -2-curve.*

  A proof of (VII.8) is in [F16, Th. 6.6];

- the final existence result is Theorem 6.12 in loc. cit.

  (VII.10) Given an anti-canonical pair (Y, D) and a root β, with two exceptions given there for every very general deformation (Y', D') of (Y, D) there are disjoint exceptional curves E, E' meeting the same D_i and such that $\beta = [E] - [E']$.

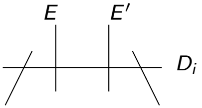
  In summary

  (VII.11) *-2-classes β satisfying $r_\beta(\overline{\mathcal{A}}_{\mathrm{gen}}(Y)) = \overline{\mathcal{A}}_{\mathrm{gen}}(Y)$ may be deformed into differences $E - E'$ of exceptional curves meeting the same D_i.*

  This is the existence result that provides the first step in the inductive proof of Torelli.

VIII. Completion of the argument

- idea is to give an inductive argument on rank $(\mathrm{Pic}(Y))$ by showing there is a configuration

(VIII.1) 

  where E, E' are interior exceptional curves meeting the same D_i and that may then be contracted to $(\overline{Y}, \overline{D})$ where $\mathrm{rank}(\mathrm{Pic}(\overline{Y})) < \mathrm{rank}(\mathrm{Pic}(Y))$;
- for this main step is to use that $\overline{\mathcal{A}}_{\mathrm{gen}}(Y)$ is a fundamental domain for the action of $W(\Delta_Y)$ on $\overline{\mathcal{A}}(Y)$ to produce a -2 class β satisfying the condition (VII.11) above and that may then be deformed into a configuration (VIII.1);

- schematically the skeleton of the argument is

$$
\begin{array}{c}
\text{curves } C \text{ with } C^2 < 0 \\
\downarrow \\
\text{analysis of these gives} \left\langle \begin{array}{l} -1 \text{ curves } \overline{E} \text{ meeting a } D_i^* \\ \left\{ \begin{array}{c} -2 \text{ curves } C \text{ in } Y - D \\ \text{with } C \cdot \overline{E} = 1 \end{array} \right. \end{array} \right. \\
\downarrow \\
\text{from these produce } E - E' \text{ above;}
\end{array}
$$

- in statement of theorem we are given isometry $H^2(Y', \mathbb{Z}) \to H^2(Y, \mathbb{Z})$;
- setting $\hat{\Lambda} = H^2(Y, \mathbb{Z})$, $\Lambda = (\mathrm{span}[D_i])^{\perp} \cap H^2(Y, \mathbb{Z})$ have

$$0 \to \Lambda \to \hat{\Lambda} \xrightarrow{f} \mathbb{Z}^r \to F \to 0 \tag{VIII.2}$$

where $f(\alpha) = (\ldots, \alpha \cdot [D_i], \ldots)$; $\det I = \pm 1 \implies F$ is trivial, $\det \neq 0 \implies F$ is finite;

- from (VIII.2) define $K = K(Y, D)$ by

$$K = \ker\{\mathrm{Aut}^+(Y, D) \to \mathrm{Aut}(\hat{\Lambda})\};$$

then we have
(VIII.3)
$$0 \to K \to \mathrm{Aut}^0 D \xrightarrow{f} \mathrm{Hom}(\hat{\Lambda}, \mathbb{G}_m) \to \mathrm{Hom}(\Lambda, \mathbb{G}_m) \to 0;$$

using $K' = K(Y', D')$ gives the non-uniqueness of ρ^* for map $\rho : Y \to Y'$ in the statement of the theorem.

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